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ATOMIC MOTION IN A HIGH-INTENSITY STANDING WAVE LASER FIELD

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Atomic Motion in a High Intensity Standing Wave Laser Field.

by

Luis Felipe Saez

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Abstract

Atomic Motion in a High Intensity Standing Wave Laser Field.

by

Luis Felipe Saez

Adviser: Professor Marvin H. Mittleman

This work discusses the effect of a high intensity standing wave laser field on the motion of neutral atoms moving with a relatively high velocity. The analysis involves a detailed calculation of the force acting on the atoms and the calculation of the diffusion tensor associated with the fluctuations of the quantum force operator. The high intensity laser field limit corresponds to a Rabi frequency much greater than the natural rate of the atom. The general results are valid for any atomic velocity. We then specialize the results to the case of slow and fast atoms where the Doppler shift of the laser frequency due to the atomic motion is either smaller or larger than the natural decay rate of the atom.

The results obtained for the force and diffusion tensor are applied to a particular ideal experiment that studies the evolution of a fast atomic beam crossing a high intensity laser beam. The theories developed previous to ours, for a similar laser configuration, discuss only the low atomic velocities case and not the more realistic case of fast atoms. Here, an approximate solution of the equation for the distribution is obtained. Starting from the approximate distribution function, the deflection angle and dispersion angle for the atomic beam with respect to the free motion are calculated.

To my wife Veronica

Acknowledgements

I would like to thank to all those that helped me to reach this point in my career. Worthy of special mention are my advisor Professor Marvin H. Mittleman and my friend Dr. Walton R. Gutierrez.

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Introduction.

Since coherent optical radiation sources of light have been available, many researchers, both experimentalist and theoreticians, have investigated the effect of laser fields on atomic motion. Different limits with respect to a variety of parameters related to the atom or to the laser have been considered.

In general, two basic laser configurations are studied. One is a single traveling wave and the other is the standing wave configuration, the latter consisting of two opposite traveling waves. Most of the experimental results in this field are interpretated using the appropriate superpositions of the previously mentioned theoretical models.

The general physical picture.

The first effect affecting the motion of neutral atoms interacting with a laser field is the recoil or scattering force. This process is particularly large when the laser frequency is near resonance with one of the atomic transitions. Another effect is the force acting on the atom due to the inhomogeneity of the laser intensity. This part of the force is referred to as the intensity force.

In the case of the recoil force, the atom absorbs from the electromagnetic field a quantum, changing the internal state of the atom from a lower energy state to a higher energy state. This transition is a consequence of the energy conservation in the atom-photon collision. On the other hand, due to momentum conservation, the atoms gain or lose exactly the same amount of momentum as that which is carried by the colliding photon.

After the atom has been excited through the photon absorption, the atom re-emits a photon. In these processes, the atoms undergo a new change of momentum. This description of the effect of the scattering force is based on the rotating wave approximation [1,2,3] where atomic transitions far from resonance with the laser frequency are neglected. However, the re-emission process can occur in two different forms, spontaneous emission and induced emission. The atomic emission rate between these two processes depends strongly on the laser

intensity. For a large laser intensity, the probability of stimulated emission exceeds the spontaneous emission probability. For the spontaneous emission case, the atom may decay to a different state than the original state previous to the absorption. Therefore, to reproduce again the cycle of the atom absorbing a laser photon and then the re-emission of a photon with the atom subsequently returning to the original energy state, the lower atomic state has to be the atom's ground state or a sufficiently stable meta-state of the atom.

Comparing the relative magnitude between the momentum of the atom and that of the photons, this cycle must be repeated many times for the resulting force to have a significant effect on the atomic motion.

This force is generally called "resonance radiation pressure" [4] . There are qualitative differences between the effect on the atomic motion for the two basic laser configurations, i. e., for a single traveling wave and for the standing wave made out of two counter propagating traveling waves. In the standing wave case, we can predict that the induced absorption of a photon by the atom is from either of the two opposite traveling waves leading to an average recoil effect which vanishes at first order of approximation. The probability that the atom absorbs a laser photon is the same for each of the two waves when the Doppler shift is neglected. Thus, the stimulated emission of the atom is also in either of the two laser modes implying that the two recoil effects combined have a very small over-all effect on the motion of the atom. However, in the case of the single traveling wave, if the laser frequency can be tuned so as to be kept close to resonance (independent of the Doppler shift) the atoms could eventually be swept away.

The complete cycle of absorbing and then re-emitting a photon presents various random events. One of these random events is the direction of the fluorescent photons. Here, the excited atom can decay emitting a photon in a mode other than the laser mode without correlation between consecutive photon emissions. Another random effect comes from considering the time intervals that the atom expends between successive excitations and de-excitation in the excited and ground state of the atom during the Rabi cycle.

For the standing wave case, as was mentioned before, the atom absorbs a photon from either of the two traveling waves, and then re-emits the photon in the laser mode to either of the two waves. This is another random effect associated with the radiation force acting on the atomic motion. All of these random events are described by analyzing the quantum fluctuations of the operator force determining the atomic motion. These fluctuations are accounted for by studying the diffusion tensor associated with the laser-atom interaction.

General applications.

In general, the laser radiation pressure effect is applied to various problems: the trapping of neutral atoms [5,6] , the acceleration or deceleration of free atoms [7] , and the deflection of atomic beams [8] .

The trapping of neutral atoms eventually allows the confinement of atoms to a small spatial region for a long time. One of the suggested laser configurations obtains confinement of atoms for times as long as a minute [9]. The acceleration or deceleration of atomic beams is studied in relation to the process called cooling of atoms [10] and high resolution spectroscopy [11] . Some of the applications of the deflection of atomic beams are the isotopes separation [12] , and the separation of atoms in different quantum states [13] .

Reference to specific experiments and theoretical analysis of the previous applications of the phenomena of laser-atom interaction can be found in many of the articles listed in the reference section. In particular, a complete list of references appear in Stenholm [14] and in Letokhov-Minogin [15] .

Basic problem and method of work.

In this work, we consider the motion of atoms in a strong standing wave laser field. The analysis made here allows for a consistent calculation of the force and diffusion tensor starting from a fundamental Hamiltonian. The derivations include the velocity dependence of the diffusion tensor and force for two cases, slow motion or low atomic velocity regime and fast motion for relatively high atomic velocities. In the latter case, the analysis

includes the associated diffusion equation and approximated distribution function solution. The position dependence of the force and the diffusion tensor in both cases are retained for the subsequent analysis.

Basically, the physical model consist of an atom with only two levels and the laser is of the kind known as a continuous standing wave.

The calculation technique requires the extension for the standing wave laser field of several new commutation rules between the fields and atomic operators. Originally, similar relations were derived for the case of a single plane wave [23]. These commutators need to be evaluated for solving the two time expectation values associated with the calculation of the diffusion tensor.

The evaluation of the force and diffusion tensor is done for any atomic velocity and then specialized to the limits of slow-atom and fast-atom. The final expression of these two magnitudes differs appreciably in one of the limits with respect to the other; thus, emphasizing the strong dependence on the atomic velocity of the force and the force fluctuations. Our analysis allows us to separate the effect of the diffusion tensor into two parts: the first due to the stimulated emission and absorption of laser photons, and the second due to the spontaneous emission of photons.

The theory is described as a function of four parameters, the laser-atom coupling energy Λ, the natural decay rate of the atom, γ, the Doppler shift of the laser frequency $\eta = \vec{k}_L \cdot \vec{v}$, and the laser detuning $\Delta\omega$. For the strong laser intensity regime we have that $\Lambda \gg \gamma$ and $\Lambda \gg \eta$. Therefore, the Rabi frequency of the interaction, defined in Chapter 2, is much greater than the natural decay rate of the atom. In the particular case of a sodium atom and a laser intensity of $10 W/cm^2$, it is found that $\Lambda \sim 35\gamma$.

The Doppler shift parameter and its relation to the natural decay rate define what is called a fast or slow atom, depending upon whether the Doppler shift is larger or smaller than the natural decay rate. The case $\eta \ll \gamma$ defines the slow atom limit and $\gamma \ll \eta$ defines the fast atom limit. For $\gamma \sim 5\times10^7 sec^{-1}$ fast atoms move with a velocity in the $\vec{k}_L$ direction

of the order of 10^3–10^4cm/ sec corresponding to a Doppler shift of $\eta \sim 10\gamma$. The kinetic energy associated with this velocity range is 10^{-5}– 10^{-3}ev . The actual atomic kinetic energy is obtained by adding to the previous energy the kinetic energy associated with the atomic velocity in the direction perpendicular to the direction defined by $\vec{k}_L$.

The laser detuning is defined as the difference between the laser frequency and the frequency of the atomic transition in the laboratory frame. This parameter is important in the behavior of the force and diffusion tensor.

After the general expressions for the force and diffusion tensor have been calculated, these results can be applied to the evaluation of the deflection angle and dispersion of this angle for an atomic beam crossing the standing wave laser.

The atomic beam deflection is understood as the average deviation from the free motion path of the atoms. This deflection depends on the ratio between the laser-atom coupling energy to the initial kinetic energy of the atoms. It also depends on the frequency detuning, and the atomic beam incident angle with respect to the laser beam.

The atomic beam dispersion is measured as the standard deviation of the deflection angle with respect to the average deflection. In the same way as the deflection angle, the dispersion angle is, in principle, associated with two independent parts. First, atoms are dispersed by crossing the laser beam in different paths and "seeing" a different potential energy throughout their trajectories; and secondly, the atomic spreading coming from the force quantum fluctuations. This last effect is represented by the diffusion tensor.

Results

The results concerning the study of fast atoms may be applicable to problems such as isotope separation [12] , velocity selection [16] , beam cooling [17] , and acceleration of neutral atoms [18] . When the atomic beam is cooled to the point where the slow-atom limit may be considered, then confined atomic beams and trapped atoms [4,19] can be studied.

As in many of the previous results concerning radiative forces, the force changes

sign depending on the sign of the detuning. At the same time, the diffusion tensor is an even function of the detuning $\Delta\omega$ which is not zero at $\Delta\omega = 0$. Both of these results are valid for slow atoms and fast atoms. However, some fundamental details do differ especially for the diffusion tensor in the resonant case. In the resonant case, first the force is null and the diffusion tensor associated with the spontaneous emission is maximum for both the slow-atom and fast-atom cases. But for the slow-atom limit, the diffusion tensor coming from the stimulated emission and absorption is a smooth function of the position, as opposed to the fast-atom limit where this part of the diffusion tensor presents discontinuities which are explained in Chapter 3. These discontinuities do not represent a real physical phenomena.

In the present laser configuration, the force is an oscillatory function of the spatial position with its effect in the atomic motion changing from acceleration to deceleration (respect to the free trajectory of the atoms) depending on the location of the atom with respect to the standing wave profile. The potential energy associated to the force has the same kind of shape in both cases, slow atoms and fast atoms. As usual, the force acting on the atoms is directed with a direction such that the atoms are driven away from the maximum of the potential, toward the regions of lower potential energy. With respect to the laser field intensity, in the case of a negative detuning, the potential attracts the atoms to the regions of high laser intensities. Conversely, for a positive detuning, the potential attracts the atoms to the regions of low laser intensities.

Among the results obtained in this work, there is the first calculation of the diffusion tensor for fast atoms interacting with a high intensity standing wave laser field. An important feature of this effect is that the position dependence of the diffusion tensor is an oscillatory function, taking positives and negatives values, that could bring out spreading or antispreading of the atomic beam. In general, for systems in thermal equilibrium the diffusion tensor is a positive defined magnitude [25,26,27] . Here, the system formed by the fast atoms which quantum mechanically interact with the laser photons is not at thermal equilibrium. Consequently, the diffusion tensor does not need to be positive tensor everywhere as a func-

tion of the space position.

In the region where the diffusion tensor is positive, the usual spreading of the atomic beam occurs. When in the region in which the diffusion tensor is negative, we have antispreading. The superposition of the effect resulting from the two regions implies an accumulation of atoms in the regions of negative diffusion tensor. Comparing the phase of the diffusion tensor with respect to the potential associated with the force, it is obtained that the potential drives the atoms to the regions of zero diffusion tensor. This result is valid for a positive and a negative detuning. This result is posible because the oscillation frequency of the diffusion tensor is twice the oscillation frequency of the potential.

The presently described effect is associated with the diffusion tensor part accounting for the atom absorbing a photon from either of the two traveling waves and then the reemission of the photon also in a random way in either of the two laser modes. Thus, for the present laser intensities, the average effect (over one laser wave length) of this part of the diffusion tensor is very small because the second order in the ratio between the laser-atom coupling energy and the initial kinetic energy of the atoms is zero. The first contribution with respect to the previous ratio manifests at fourth order.

Comparing some of our results with previous calculations, we have found that for the fast-atom limit, there is a known result for the force [20] similar to the one obtained in the present work. In addition to the result for the fast-atom limit, it is interesting to mention that for the slow-atom limit the results obtained here for both the force and the diffusion tensor coincides with previous calculations in the appropriate limits [4,21,22] .

Conclusions.

The application of our calculation is to the problem of atomic beam deflection and dispersion. However, the results of the first two chapters can be used in the analysis of more general problems. The major restriction in the applicability of our results is related to the laser intensity because our results require a high laser-atom coupling energy. And, on the

other hand, the upper limit for the permitted atomic velocities along $\vec{k}_L$ is restricted by the relation $\Lambda \gg \eta$. For the present laser intensities, the laser atom interaction parameter is $\Lambda \sim 2\times10^{9}\text{sec}^{-1}$ and for the fast-atom limit the Doppler shift is $\eta \sim 5\times10^{8}\text{sec}^{-1}$. In Chapter 3 of this work, it is found that experimental precision in the order of 10^{-1}– 10^{-4} of a degree would allow measurement of the deflection and dispersion of a fast atomic beam crossing the standing wave laser field.

The magnitude of the deflection angle and the dispersion of the atomic beam is determined by the parameter $\epsilon_0 = \frac{\hbar\Lambda}{M_A v_{0_y}^2} \sim 5\times10^{-2}$. In the case of the deflection angle the proportipnalety is with respect to ϵ_0^2 while the dispersion of the beam is proportional to ϵ_0 .

These results are independent of the diffusion tensor. Because, on the one hand, the diffusion due to the spontaneous emission results proportional to the ratio γ/η which is being neglected, and on the other hand, the diffusion due to the induced absorption and emission contribute when terms proportional to ϵ_0^4 are considered (still terms proportional to γ/η are being neglected).

Considering the relative magnitude between the two parts of the diffusion tensor if terms proportional to γ/η are kept, then the diffusion part associated with the induced absorption and emission determine the behavior of the diffusion tensor.

Returning to the actual magnitude of the deflection angle and the dispersion of the beam, we note that the dispersion is larger than the deflection because $\epsilon_0 \ll 1$. Since the deflection of the beam is proportional to Λ^2 while the dispersion of the beam is proportional to Λ , we conclude that increasing the laser-atom interaction energy will increase the deflection of the beam more than its dispertion.

Finally, as presented in Chapter 3, the deflection angle and the dispertion angle as a function of the ratio between the detuning and the atom-laser energy ($\Delta\omega/\Lambda$) have a sharp maximum when this ratio is about .35 . This result suggests the previous maximum as an appropriated value for the ratio $\Delta\omega/\Lambda$ to be used in an experiment. With respect to the

incident angle, our results imply that the maximum deflection is obtained when the incident angle is approximatly 55° . However, the maximum dispersion is not obtained for 55° but for 45° suggesting two interesting angles for an experimental study.

The next stept in a future calculation should incorporate terms proportional to the ratio γ/η in the case of fast atoms for the force and diffusion tensor.

A more ambitious result may be attempting to see if the analysis of the force and the diffusion tensor can be carried out further independent of the magnitude of the ratio between γ/η . In this case, the more immediate approach will be a computational evaluation of the integrals involved in the calculation of the force and diffusion tensor.

Approximations.

In this work, the "pure-state" approach [24] is used in developing the theory of the intense laser interacting with an atomic beam. The calculations of many previous works for the slow-atoms regime used the density-matrix approach [4,21,22] .

The motion of the center of mass of the atom is considered to be a classical motion because the de Broglie wave length of the atom is much smaller than the laser wave length. A more restricting condition is to assume that the atomic center of mass motion is such that the atomic state adapts adiabatically to the local external field. For a two level atom, on the one hand, the internal characteristic atomic time is determine by γ^{-1} or an energy of $\hbar\gamma$ when, on the other hand, the change in translational atomic state is associated with the atomic recoil energy, $\hbar^2 k_L^2/\, 2M_A$. The atomic states then stay coupled with the external laser field after an absorption or emission if $\hbar\gamma \gg \hbar^2 k_L^2/\, 2M_A$; which for our case is $\hbar\gamma \approx 10^5 \hbar^2 k_L^2/\, 2M_A$. Both of these approximations are commonly used in the literature [15] .

Moreover, a well known approximation used in our work is to consider the effect on the system of the external laser field as a classical field. However, the fluorescent field is kept quantized. This approximation is valid as long as the number of fluorescent photons is small compared with the number of laser photons. In fact, the number of fluorescent photons

involved in the interaction can be in the order of thousands when the number of laser photons is in the order of 10^9 for high laser intensities [3].

For both fields, the gauge invariant interactions with the electric charges of the atom are provided by the usual electromagnetic minimal coupling. Then, the usual dipole approximation of the electromagnetic field is considered for the interaction of the electric charges with both photon fields. The corrections beyond the dipole approximation are about $(137)^{-1}$ times smaller than the dipolar term. After choosing a particular gauge, the interaction terms proportional to the ratio between the atomic velocity and the speed of light are neglected.

As was previously mentioned, the laser frequency is supposed to couple only two levels of the atom allowing the description of the internal state of the atom by the two levels involved in the transition. This assumption in conjunction with the rotating wave approximation (for the interaction of the atom with the external laser field) and the second level rotating wave approximation (for the interaction of the atom with the fluorescence field) permits the neglect of far off-shell atomic transitions and fast oscillating interaction terms. The previous approximations are corrected by terms in the order of $\Delta\omega/\ \omega_L$, where ω_L is the laser frequency [3].

In the present work, the interaction time between the laser and the atom is considered to be very long when compared with the natural life time of the atomic state. Therefore, the fluorescence effect is important in the description of the atomic motion because many spontaneous emissions occur during the interaction time. Even when we study the fluorescent field including a complete frequency range, it results that, for an interaction time much larger than the natural decay time, the fluorescent spectrum is sharply peaked at the laser frequency. For the subsequent analysis we consider only this peak. Corrections to this approximation considering a broadening fluorescent spectrum turn out to be approximately $v/\ c$ smaller than the leading term. On the other hand, the emission of fluorescent photons are supposed to be independent from one another, this implies that the whole process is Markovian

[25,26,27].

As mentioned before, the final dynamic results for the atomic motion are given in power series of a parameter representing the magnitude of the ratio between the energy of the laser atom interaction and the kinetic energy of the incident atom.

A comprehensive discussion and a more precise definition of all these approximations are given in the different sections of this work. The sequence of the approach followed for the interaction between the laser and the atom is outlined below.

Outline.

Starting from the Hamiltonian for an atom interacting through the minimal coupling with a standing wave laser in a general gauge, we go to the $\vec{E}\cdot\vec{R}$ gauge. A matrix representation for the atom considered as a two level system is then obtained. The rotating wave approximation is performed and all high frequency oscillating terms are neglected.

The second section of Chapter 1 presents the calculation of the time evolution for the operators associated with the fluorescent field. Here, renormalization effects are neglected and the emission and absorption processes are assumed to be Markovian.

Considering the initial condition of no fluorescent photons for the field and the atom being in the ground state, the action of some relevant quantum operators on the initial state of the system is described. These permit the solving of the time evolution of the expectation values associated with observables such as the force and the diffusion tensor. Here, various commutators are reduced to more useful forms.

In Section 4, the force and diffusion tensor are written in terms of the local expectation value $<\vec{\sigma}(t)>$ and the non-local expectation value $<\vec{\sigma}(t)\vec{\sigma}(t')>$. Here, $\vec{\sigma}(t)$ are the Pauli matrices associated with the two level description of the atom. The differential equations satisfied by their time evolution are obtained.

Chapter 2 is dedicated to the solution of the previous differential equations. By the elimination of fast oscillating terms, the time evolution of $<\vec{\sigma}(t)>$ and $<\vec{\sigma}(t)\vec{\sigma}(t')>$ is

solved. Thus, a final form of the force and diffusion tensor is presented for both limits $\gamma \ll \eta$ and $\eta \ll \gamma$. Here, γ is the natural decay rate and $\eta = \vec{k}_L \cdot \vec{v}$ is the Doppler shift of the laser frequency.

To arrive at these results, extended use of the conditions $\gamma \ll \Lambda$ and $\eta \ll \Lambda$ is necessary. This condition precludes low intensity laser fields.

Chapter 3 presents a more detailed analysis of the force and diffusion tensor. A specific application of the results of the previous sections is worked out in this chapter. This requires solving the diffusion equation in a particular case explained below.

There are several works that use in one way or another the same Fokker-Plank equation as we do. In fact, the ambiguity about the definition of the diffusion term is avoided following Stenholm [14] prescription. With the help of this equation, we study the distribution function for the description of the atomic motion crossing a monochromatic laser beam.

The solutions of this equation are investigated for some cases compatible with the domain of our parameters. In lowest order, the equation is a force free Newtonian equation. The next order is a Newtonian equation including the force field, this equation is solved exactly in terms of the implicit integral defining the trajectory of the atoms inside the laser beam. Perturbations around this solution allows us to solve the Fokker-Planck equation at first order in the diffusion tensor. In this context, we can calculate the average deflection angle and its standard deviation both in a power series of a parameter that compares the energy of the laser atom interaction to the kinetic energy of the incoming atoms.

1.1 - The Hamiltonian for the Atom Laser Interaction.

In this section, we shall present the basic approximation applied to the Hamiltonian used throughout our work. This Hamiltonian represents the interaction of an atom with an electromagnetic field in the so-called minimal coupling. We start with the Hamiltonian written in a general gauge from which we go to a more workable structure choosing a particular gauge which, in any case, is used in most of the literature relevant to this work. Furthermore, in this gauge many calculations are simpler than in others. In the next step, we change to the matrix representation of the Hamiltonian and simultaneously project it over a subsystem of only two coupled states. Here, the rotating wave approximation is performed in both the laser external field and the radiation field, this last approximation is called " second level rotating wave approximation ".

Following the treatment given by Mittleman [3] for an atomic Hamiltonian interacting with an electromagnetic field in minimal coupling.

$$H = \frac{\vec{P}_\rho^{\,2}}{2M_A} + \frac{1}{2\mu}\sum_{i=1}^{Z}\left|\vec{p}_i + \frac{e}{c}\vec{A}(\vec{\rho},t)\right|^2 + \frac{1}{M_N}\sum_{i>j=1}^{Z}\left|\vec{p}_i + \frac{e}{c}\vec{A}(\vec{\rho},t)\right|\cdot\left|\vec{p}_j + \frac{e}{c}\vec{A}(\vec{\rho},t)\right| + V(\vec{\chi}_1 \cdots \vec{\chi}_Z, \vec{\rho}) \quad , \tag{1.1.1}$$

where the coordinates transformation from the nuclear and electron coordinates $\vec{R}_N$, $\vec{r}_i$ to the center of mass and relative coordinates has been done,

$$\vec{\rho} = \frac{1}{M_A}\left(M_N \vec{R}_N + \sum_{i=1}^{Z} m\ \vec{r}_i\right)$$

$$\vec{\chi}_i = \vec{r}_i - \vec{R}_N \qquad ; \quad \mu = \frac{mM_N}{m + M_N} \quad . \tag{1.1.2}$$

The dipole approximation applied to the electromagnetic potential eliminates the small correction to the Hamiltonian coming from the fact that the electronic position is different from the center of mass position $\vec{r}_i = \vec{\rho} + \vec{\chi}_i + O(m / M_N)$. We neglect the corrections to the electromagnetic field coming from the evaluation at this point. Let us consider the

corrections to the expectation values of the field derived from neglecting $\vec{\chi}_i$ in the dipole approximation for the laser field. They are associated with $e^{\pm i\vec{k}\cdot\vec{r}_i} \approx e^{\pm i\vec{k}\cdot\vec{\rho}}(1 \pm i\vec{k}\cdot\vec{\chi}_i)$ where $\vec{\chi}_i$ is limited to the atom's size; let us say a Bohr radios a_0. Hence, the correction becomes of the order of magnitude of $ka_0 \approx (137)^{-1}$.

In the above expressions M_A, M_N and μ, are the atomic mass, nuclear mass, and reduced mass respectively. The number of electrons is Z. Thus, $V(\vec{\chi},\vec{\rho})$ is a potential which describes the free atom. The electromagnetic potential includes the laser field and the fluorescence field as well.

It is known [3] that the high number of laser photons makes it possible to treat the laser field as a classical field. The effect of the laser field on the atoms is to produce an oscillation of the electrons between the two coupled states with transition energy near resonance with the laser field. In fact, the atom's reaction to the laser field can be described as the stimulated absorption of a laser photon followed by a transition from the exited state, of the atom, back to the ground state emitting a photon in the laser mode. This description, two levels involved in the transition, is only good in the framework of the rotating wave approximation [1,28].

The radiation field accounts for the reaction of the atoms known as spontaneous emission. An atom in the exited state may spontaneously decay to the ground state after a given time with a natural decay rate independent of the laser field. The time required for this transition is called the natural decay time. Therefore, if the atom is in the laser field for a long time compared with the natural decay time many of the transitions will occur in modes different than the laser mode. In fact, the number of fluorescent photons involved in a typical experiment is of the order of thousands, which is a large number of photons but still small compered with the number of laser photons interacting in the induced oscillation with the laser field. We need to use a different approach in the treatment of the fewer fluorescent photons involved compared with the higher number of photons associated with the laser mode.

Our analysis of the fluorescence photons effect on the atom-laser interaction will follow the presentation given by Mittleman [3,20,23] which is a reformulation of the so-called pure state theory develop by Mollow [29] for this problem.

The freedom associated with the gauge invariance allows us to make a time and position dependent unitary transformation for the wave function and Hamiltonian. The physical results coming from the wave function and operators evaluated at different gauges are the same; this occurred even when calculations in a particular gauge might be more suitable than in others. The transformation is defined by:

$$\begin{aligned} \Psi' &= e^{i\phi}\Psi \\ H' &= e^{i\phi} H\ e^{-i\phi} - \hbar\,\partial_t \phi \\ \phi(\vec{r}_1 \cdots \vec{r}_Z, \vec{R}, t) &= \frac{e}{c\hbar}\sum_{i=1}^{Z} \vec{r}_i \cdot \vec{A}(\vec{R}, t) \end{aligned} \tag{1.1.3}$$

Under the above transformation, the Hamiltonian for the system is: (removing the prime)

$$\begin{aligned} H = {} & \frac{\vec{P}_\rho^{\,2}}{2M_A} + \frac{1}{2\mu}\sum_{i=1}^{Z}\vec{p}_i^{\,2} + \frac{1}{M_N}\sum_{i>j}^{Z}\vec{p}_i\cdot\vec{p}_j + V(\vec{r}_1 \cdots \vec{r}_Z, \vec{R}) + e\vec{E}(\vec{R}, t)\sum_{i=1}^{Z}\vec{r}_i \\ & + \frac{\left|\hbar\vec{\nabla}_\rho \phi\right|^2}{2M_A} - \left|\hbar\vec{\nabla}_\rho\phi\right|\frac{\vec{P}_\rho}{M_A} - i\,\frac{\hbar^2\vec{\nabla}_\rho^2\phi}{2M_A}\,, \end{aligned} \tag{1.1.4}$$

where we have used $e^{i\phi}\vec{p}_j\, e^{-i\phi} = \vec{p}_j - \frac{e}{c}\vec{A}(\vec{R}, t)$ removing completely from the Hamiltonian (1.1.1) both the ponderomotive potential, which is a term proportional to the time average of $\vec{A}^2(\vec{R}, t)$, and the interacting terms given by $\vec{A}(\vec{R}, t)\cdot\vec{p}_i$. The last three terms come from the transformation of the momentum for the center of mass $e^{i\phi}\vec{P}_\rho e^{-i\phi}$.

The interaction term between the electronic position and the electric field comes from the time derivative of the phase transformation, with the electric field, $\vec{E}(\vec{R}, t) = -\frac{1}{c}\dot{\vec{A}}(\vec{R}, t)$

However, the three last terms of the Hamiltonian (1.1.4) have an order of magnitude much smaller than the leading interacting term $\vec{E}\cdot\vec{r}$ because $m / M_A \ll 1$. The first of

them is proportional to $e^2E_0^2 / c^2M_A$ which is very small; the second one has an order of magnitude of v/c . This is the largest among these three terms. Finally, the last one is proportional to $e\hbar k_L E_0/ cM_A$.

The part of the vector potential associated with the standing wave laser field is:

$$\vec{A}_L(\rho,t) = -2\vec{E}_0\frac{c}{\omega}\cos\vec{k}_L\cdot\vec{\rho}\,\sin\omega(t-t') \qquad (1.1.5)$$

using this expression in the transformation (1.1.3) we find $(\vec{\nabla}_\rho\phi)\cdot\vec{v}_\rho \approx eE_0 v/c$, when the leading interacting term becomes eE_0 . The phase has been chosen according to the initial condition of no interaction at $t = t'$ between the laser field and the atomic states described by equation (1.1.1).

The first term of H is the free motion for the atom's center of mass . The next three terms are referred to as the atomic Hamiltonian, H_A . The fifth is the interaction term $(\vec{E}\cdot\vec{\chi})$ and all the other terms are neglected as we saw before. Here, the electric field includes both the laser field and the radiation field.

At this point, we assume that the standing wave laser couples one atomic transition defined by two states belonging to the free Hamiltonian $|u_0>$ and $|u_1>$ of energy W_0 and W_1 , respectively. This is achieved by tuning the laser field between two states in such a way that, on the one hand, the laser frequency is near resonance with the energy transition and, on the other hand, the laser frequency width is smaller than the splitting of the subshells involved in the transition. A unitary transformation that represents this effect will allow us to go to a (2×2) matrix representation of the Hamiltonian,

$$U = e^{i\omega(t-t')/2}|u_0>(1,0) + e^{-i\omega(t-t')/2}|u_1>(0,1) \quad . \qquad (1.1.6)$$

Using that the time derivative transform under the above transformation as:

$$U^{-1}i\hbar\partial_t U = -\hbar\frac{\omega}{2}\sigma_z + i\hbar\partial_t \quad ,$$

and the interaction term of the Hamiltonian transform to

$$U^{-1}e\vec{E}\cdot\sum_{i=1}^{Z}\vec{\chi}_i\,U = e^{-i\omega(t-t')}\vec{E}\cdot\vec{d}_{01}\,\sigma_+ + e^{i\omega(t-t')}\vec{E}\cdot\vec{d}_{10}\,\sigma_- \quad ,$$

where $\vec{d}_{01} \equiv \langle u_0 | e\sum_{i=1}^{Z}\vec{\chi}_i | u_1\rangle$ and $\vec{d}_{10}=\vec{d}_{01}^{\,*}$ are the dipole atomic momenta approximated for near resonant two-level atoms.

Here, $\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\,\sigma_y)$ with $\sigma_x,\sigma_y,\sigma_z$ being the Pauli matrices.

The Hamiltonian in the matrix representation is:

$$H_M = \left[T_{\vec{\rho}}+(W_1+W_0)/2\right] I + \hbar\frac{\Delta\omega}{2}\sigma_z + e^{-i\omega(t-t')}\vec{E}\cdot\vec{d}_{01}\sigma_+ + e^{i\omega(t-t')}\vec{E}\cdot\vec{d}_{10}\sigma_- \tag{1.1.7}$$

Here, W_1, W_0 and the detuning $\Delta\omega=\omega-(W_1-W_0)/\hbar$ are $\vec{\rho}$ independent since they are associated with the free atomic Hamiltonian and the laser frequency. The dressing of the eigenvalues is produced by the laser field in second order in the perturbation; the wave function dresses at first order in the interaction. Since, we are working with the free undressed states of the system $\langle u_i | T_{\vec{\rho}} | u_j\rangle = T_{\vec{\rho}}\delta_{ij}$, i. e., there is no $\vec{\rho}$ dependence in the atomic state of the system. Where $T_{\vec{\rho}}$ is the center of mass kinetic energy operator.

In order to completely add the radiation field which is described quantum electrodynamically, we need to discuss further the different terms of the fluorescence photons field in the previous Hamiltonian. The free Hamiltonian of the electromagnetic field has been removed through the unitary transformation, $\Psi^{(new)} = e^{iH_F(t-t')/\hbar}\Psi^{(old)}$. Where H_F is the free Hamiltonian for the photons field when there is no interaction, $\sum_{k\lambda}\hbar\omega_k\, a_{F\,k\lambda}^{\dagger}(t)a_{F\,k\lambda}(t)$

The electric field present in (1.1.7) is the superposition of the laser field and radiation field, $\vec{E} = \vec{E}_L + \vec{E}_R$, with the radiative field defined as:

$$\vec{E}_R = \vec{\mathcal{E}}^{(+)}(t,\vec{\rho}) + \vec{\mathcal{E}}^{(-)}(t,\vec{\rho}) \tag{1.1.8}$$

$$\vec{\mathcal{E}}^{(+)}(t,\vec{\rho})=\vec{\mathcal{E}}^{(-)^{\dagger}}(t,\vec{\rho})$$

$$\vec{\mathcal{E}}^{(-)}(t,\vec{\rho}) = \sum_{k\lambda}\left(2\pi\hbar\frac{\omega_k}{V}\right)^{1/2} a_{k\lambda}(t)e^{-i\omega_k(t-t')}e^{i\vec{k}\cdot\vec{\rho}}\hat{\epsilon}_{k\lambda} \tag{1.1.9}$$

Here, the second quantization of the electromagnetic field has been done on a finite volume V . Therefore, k is a vector defined by three integers n_{k_x} , n_{k_y} , n_{k_z} ; λ is also an index that takes the values 1 or 2 defining the two possible states of polarization. The sum runs for each one of the indexs n_{k_i} from minus infinity to plus infinity.

The meaning of the extra time dependence on the creation and destruction operators, $a_{k\lambda}$ ($a^{\dagger}_{k\lambda}$) in the mode (k ,λ) is the deviation of their time evolution from the exponential behavior associated with the free radiation fields due to this field interaction with the atomic electrons. Note that in the above expression for the field we have explicitly written the time dependence associated with the free time evolution of the creation and destruction operators.

At this point, the rotating wave approximation in both the laser and the radiation fields is made. The rotating wave approximation in the laser field eliminates the fast oscillating term at a frequency of about twice the laser frequency compared with oscillating terms at the Rabi frequency. It can be seen that the neglected terms give corrections of the order $\Delta\omega/\omega$ [3,30]. Neglecting fast oscillating terms in the radiation field signifies discarding far off shell effects [23], this can be easily seen from the type of time evolution $e^{\pm i(\omega+\omega_k)t}$. Finally, the Hamiltonian used for our calculations is

$$H = \left[T_p + (W_1+W_0)/2\right] I + \frac{\Delta\omega}{2}\sigma_z + \vec{E}_0\cdot\vec{d}_{01}\cos\vec{k}_L\cdot\vec{\beta}\,\sigma_+ + \vec{E}_0\cdot\vec{d}_{10}\cos\vec{k}_L\cdot\vec{\beta}\,\sigma_-$$
$$+e^{-i\omega(t-t')}\vec{\epsilon}^{(+)}(t,\vec{\beta})\cdot\vec{d}_{01}\sigma_+ + e^{i\omega(t-t')}\vec{\epsilon}^{(-)}(t,\vec{\beta})\cdot\vec{d}_{10}\sigma_- \qquad (1.1.10)$$

The matrices $e^{i\omega(t-t')}\sigma_-$ and $e^{-i\omega(t-t')}\sigma_+$ are the excitation and deexcitation operators for the atom. The expectation value of $-\sigma_z$ is the population inversion [22].

The last interpretation follows directly from the result of applying the unitary transformation (1.1.6) to the respective operators in the bra-ket representation

$$
\begin{aligned}
U^{-1} \mid u_1\rangle\langle u_0 \mid U &= e^{i\omega(t-\tau)}\sigma_- \\
U^{-1} \mid u_0\rangle\langle u_1 \mid U &= e^{-i\omega(t-\tau)}\sigma_+ \\
U^{-1}\left\{\mid u_1\rangle\langle u_1 \mid - \mid u_0\rangle\langle u_0 \mid\right\}U &= -\sigma_z
\end{aligned}
\tag{1.1.11}
$$

1.2 - Markovian Approximation for the Radiation Field.

The equation for the time evolution of any operator, O , in the Heisenberg picture is

$$i\hbar\frac{d}{dt}O = \left[O, H \right] . \tag{1.2.1}$$

H is the Hamiltonian described in the previous section. For simplicity in the notation, it is convenient to use units where $\hbar=1$.

Basically, there are two types of operators for which we have to know their time evolution in order to describe the dynamic of the atoms in the standing wave laser. One class of operator are the photon creation and destruction operators and the other type are the operators related with the two states of the atom, i. e., the $\vec{\sigma}$ matrices.

The evolution equation for the σ_i matrices obtained from (1.2.1)

$$\begin{aligned}\dot{\sigma}_i(t) = {} & \Delta\omega(\hat{a}_z\times\vec{\sigma})_i + \vec{E}_0\cdot\vec{d}_{01}\cos\vec{k}_L\cdot\vec{\rho}\left[(\hat{a}_x\times\vec{\sigma})_i + i(\hat{a}_y\times\vec{\sigma})_i\right] + \\ & \vec{E}_0\cdot\vec{d}_{10}\cos\vec{k}_L\cdot\vec{\rho}\left[(\hat{a}_x\times\vec{\sigma})_i - i(\hat{a}_y\times\vec{\sigma})_i\right] \\ & + e^{-i\omega(t-\tau)}\vec{\mathcal{E}}^{(+)}(t,\vec{\rho})\cdot\vec{d}_{01}\left[(\hat{a}_x\times\vec{\sigma})_i + i(\hat{a}_y\times\vec{\sigma})_i\right] \\ & + e^{i\omega(t-\tau)}\vec{\mathcal{E}}^{(-)}(t,\vec{\rho})\cdot\vec{d}_{10}\left[(\hat{a}_x\times\vec{\sigma})_i - i(\hat{a}_y\times\vec{\sigma})_i\right]\end{aligned} \tag{1.2.2}$$

where, $\hat{a}_x,\hat{a}_y$, $\hat{a}_z$ are time independent unitary vectors in the x ,y ,z direction respectively in the 2×2 matrix space defined by the Pauli matrixes. These unitary vectors expand the space in which the orientation of the operator $\vec{\sigma}(t)$ evolves in time [28,31] .

In equation (1.2.2) the time ordering between the fields and the states related operators $\vec{\sigma}$ is irrelevant, since these operators commute when the evaluation is done at the same time for both.

Since the creation and destruction operators satisfies the commutator relation $\left[a_{k\lambda}(t), a^{\dagger}_{k'\lambda'}(t) \right] = \delta_{kk'}\delta_{\lambda\lambda'}$, we can write the time evolution equation:

$$\dot{a}_{k\lambda}(t) = -i\left[2\pi\frac{\omega_k}{V}\right]^{1/2} e^{i\omega_{kL}(t-\tau)}(\hat{\epsilon}^{*}_{k\lambda}\cdot\vec{d}_{01})\,e^{-i\vec{k}\cdot\vec{\rho}}\,\sigma_{+}(t) \tag{1.2.3}$$

where, it has been defined $\omega_{kL} = \omega_k - \omega$.

Our following step shall be the investigation of the solutions for these four coupled equations. It is not yet possible to find an exact solution for them. The solutions will be obtained through iteration in the interactions.

The solution for the operator $a_{k\lambda}(t)$ is divided in two parts, the first one is the solution of the homogeneous equation which describes the evolution of the operator when there is no interaction, and the second part is the inhomogeneous solution containing the interaction of the fluorescence photons with the atomic levels.

The homogeneous solution is $a_{k\lambda}(t) = a_{k\lambda}(t')$ constant in accord with the discussion about $\vec{E}^{(-)}(t,\vec{\beta})$ having already incorporated the time evolution for the free photons, with the initial condition established at $t = t'$.

The inhomogeneous part of the solution is directly obtained by integration, leading to:

$$a_{k\lambda}(t) = a^H_{k\lambda}(t) + a^I_{k\lambda}(t) \quad ; \quad a^H_{k\lambda}(t) = a_{k\lambda}(t')$$
$$a^I_{k\lambda}(t) = -i\left|2\pi\frac{\omega_k}{V}\right|^{1/2}(\hat{e}^{\,*}_{k\lambda}\cdot\vec{d}_{01})\int_{t'}^{t} ds\, e^{i\omega_{kL}(s-t')}\, e^{-i\vec{k}\cdot\vec{\beta}(s)}\sigma_+(s) \quad , \qquad (1.2.4)$$

substituting back into the equation (1.1.9), we obtain for the field:

$$e^{i\omega(t-t')}\vec{E}^{(-)}(t,\vec{\beta})\cdot\vec{d}_{10} = F^{(-)}(\vec{\beta},t,t') - \qquad (1.2.5)$$
$$i\sum_{k\lambda}\left|2\pi\frac{\omega_k}{V}\right| e^{-i\omega_{kL}(t-t')}\, e^{i\vec{k}\cdot\vec{\beta}}\,|\,\hat{e}_{k\lambda}\cdot\vec{d}_{10}\,|^2 \int_{t'}^{t} ds\, e^{i\omega_{kL}(s-t')}\, e^{-i\vec{k}\cdot\vec{\beta}(s)}\sigma_+(s)$$

with

$$F^{(-)}(\vec{\beta},t,t') = \sum_{k\lambda}\left|2\pi\frac{\omega_k}{V}\right|^{1/2} e^{-i\omega_{kL}(t-t')}\, a_{k\lambda}(t')\, e^{i\vec{k}\cdot\vec{\beta}}\,\hat{e}_{k\lambda}\cdot\vec{d}_{10} \quad . \qquad (1.2.6)$$

Considering that $\hat{k}, \hat{e}_{k1}, \hat{e}_{k2}$ form a three dimensional base for each k the sum over the polarization states reduces to:

$$\sum_{\lambda}|\,\hat{e}_{k\lambda}\cdot\vec{d}_{10}\,|^2 = |\,\vec{d}_{10}\,|^2\left|1 - |\,\hat{k}\cdot\hat{d}\,|^2\right| = |\,\vec{d}_{10}\,|^2\sin^2\theta$$

In the previous equation, the notation $\hat{d}$ represents the unit vector coming from the complex vectors $\vec{d}_{10} = \vec{d}^{\,*}_{01}$, thus we have chosen the z axis in the $\hat{d}$ direction and the summation represented by k will be solved in the usual way $\sum_k \left| \frac{2\pi}{V} \right| \to \int \frac{d^3k}{(2\pi)^3}$ in the limit $V \to \infty$.

In the integral (1.2.5) we used $e^{i\vec{k}\cdot(\vec{R}(t)-\vec{R}(s))} \approx 1$ because the corrections coming from the next term are of the order v/c . See discussion following equation (1.2.8).

The second term of (1.2.5) is

$$-i\frac{|\vec{d}_{10}|^2}{2\pi c^3}\int_0^\infty d\omega_k \ \omega_k^3 e^{-i\omega_{kL}(t-t')}\int_{t'}^{t} ds\ \sigma_+(s)e^{i\omega_{kL}(s-t')}$$

For solving the integral with respect to s , we need to know the time evolution of $\sigma_+(s)$. First, making a variable change defined by $s = x+t$, and second a Taylor series of $\sigma_+(x+t) = \sigma_+(t)+x\,\sigma_+(t)+\cdots$ in the limit of very large or " large enough " $|t-t'|$ we have [23]:

$$e^{-i\omega_{kL}(t-t')}\int_{t'}^{t} ds\ \sigma_+(s)e^{i\omega_{kL}(s-t')} = \left|\sigma_+(t)-i\,\sigma_+(t)\frac{\partial}{\partial\omega_{kL}}\right| \times \left|-i\frac{P.P.}{\omega_{kL}}+\pi sgn(t-t')\delta(\omega_{kL})\right| \qquad (1.2.7)$$

Up to this point we have made two approximations: a Taylor series and a large time difference between t and t' .

The second term in the Taylor series will give corrections to the field of the order of magnitude $\Delta\omega/\omega$ or Λ/ω smaller than the leading term, $\Lambda = \frac{e}{\hbar}|\vec{E}_0\cdot\sum_i \langle u_0|\chi_i|u_1\rangle|$, ($\hbar=1$) is the laser-atom coupling energy, see (1.2.2).

Therefore, the Taylor series is a series in the above parameters, $\Delta\omega/\omega$, Λ/ω justifying by itself the approximation done. Note that in our case $|\Delta\omega| \approx \Lambda$ and $\Lambda \approx 10^9\ sec^{-1}$ when $\omega \approx 10^{14}\ sec^{-1}$.

The meaning of a very large time approximation can be understood as $|t-t'|\omega \gg 1$ or a time long enough since the field is turned on. Working after a sufficient time elapse is equivalent to studying the interaction effect after many atomic transitions, ω is larger than the Rabi frequency, and not the details effect of a few transitions. An extended discussion of this point is given by Kimble and Mandel [32].

Since in this work we are not considering the renormalization of the energy $(W_1 - W_0)$ the term containing the principal part of the integral will be discarded. In a renormalizable theory, this part of the integral should be re-absorbed into the levels energy difference.

Using $\gamma = \frac{2\omega^3}{3c^3}|\vec{d}_{10}|^2$, which is related to the natural decay time of the atom through $\tau_0 = (2\gamma)^{-1}$, we can write for the field :

$$e^{i\omega(t-t')}\vec{\epsilon}^{(-)}(t,\vec{\beta})\cdot\vec{d}_{10} = F^{(-)}(\vec{\beta},t,t') - i\gamma sgn(t-t')\sigma_+(t) \quad . \qquad (1.2.8)$$

If we had not neglected the contributions coming from $e^{i\vec{k}\cdot(\vec{\beta}(t)-\vec{\beta}(s))} = e^{-i\eta_k(s-t)}$, $\eta_k = \vec{k}\cdot\vec{v}$ in the classical approximation for the atom's motion, we will have corrections to the delta function, of equation (1.2.7), associated to the Doppler shift of the fluorescence photos $\delta(\omega_{kL} - \vec{k}\cdot\vec{v})$, where $\vec{v}$ is the atomic velocity. The new γ will differ from the already defined by a order of v/c .

As the final result of the previous approximations we conclude that at the time, t the field and consequently their expectation values are completely determined by the knowledge of the field conditions at a previous time t'. Note that in the right hand side of (1.2.8) only t and t' are involved, and not any intermediate time, which is the case in equation (1.2.7).

Thus, as a consequence of the previous approximations we are left with a Markovian [25] problem and we might call these approximation a Markovian or " peaking " approximation.

Equation (1.2.8) is a very important result because it will allow us to evaluate the expectation values of the operators involved in the dynamics of our problem.

1.3 - Commutation Relations and Initial State Evaluation for the Radiation Field Operators.

First the notation can be reduced re-defining the electromagnetic field given by (1.1.9) as $\left|\vec{\epsilon}^{(-)}(t,\vec{\rho})\right\rangle(new) = e^{i\omega(t-t')}\left|\vec{\epsilon}^{(-)}(t,\vec{\rho})\right\rangle(old)$. Thus, from now on every time we mention the radiation field we are referring to the *new* field defined here.

We begin calculating the expectation values of the different operators based on the initial condition established for this problem that at $t = t'$ there are not any fluorescent photons.

Therefore, the following relations are a direct consequence of the initial conditions:

$$F^{(-)}(\vec{\rho},t,t')> = 0$$

The definition of the previous operator is given in equation (1.2.6), of section 2, and basically we could call it the " free negative frequency " part of the fluorescent field. In the previous section we also prove relation (1.2.8); namely, that when added to the initial condition specified previously allows us to write:

$$\vec{\epsilon}^{(-)}(t,\vec{\rho})\cdot\vec{d}_{10}> = -i\,\gamma sgn(t-t')\sigma_{+}(t)> \tag{1.3.1}$$

An equivalent relation to (1.3.1) and its Hermitian conjugated for a single plane wave was first derived by Mollow [29].

A relation equivalent to (1.2.5) for the derivative of the fields can be obtained in the same way as it was worked in the previous section, resulting in:

$$\vec{\nabla}_{\vec{\rho}}\left|\vec{\epsilon}^{(-)}(t,\vec{\rho})\cdot\vec{d}_{10}\right| = \vec{\nabla}_{\vec{\rho}}F^{(-)}(\vec{\rho},t,t') - i\sum_{k\lambda}\left|2\pi\frac{\omega_k}{V}\right|e^{-i\omega(t-t')}i\vec{k}\;|\,\hat{\epsilon}_{k\lambda}\cdot\vec{d}_{10}\,|^2\int_{t'}^{t}ds\,e^{i\omega_{kL}(s-t)}\sigma_{+}(s) \tag{1.3.2}$$

where, after taking the gradient, to the second term, with respect to $\vec{\rho}$ we have taken the

limit of the exponential of the position difference to one, $e^{\vec{k}\cdot(\vec{R}(t)-\vec{R}(s))}\approx 1$. The Doppler shift is, again, neglected for the fluorescence photons as was done in the previous section, see below equation (1.2.8). Here, the second term of (1.3.2) is an odd function of the vector $\vec{k}$ and therefore its contribution vanishes when the sum over k is performed.

Thus, in addition to the initial condition for the radiation field acting on the initial state equation (1.3.1), we have a condition for the derivative of the field,

$$\vec{\nabla}_{\vec{\beta}}\vec{\mathcal{E}}^{(-)}(t,\vec{\beta})\cdot\vec{d}_{10}>=\vec{\nabla}_{\vec{\beta}}F^{(-)}(\vec{\beta},t')>=0 \quad . \tag{1.3.3}$$

Moreover, using the Markovian approximation for the fluorescence fields, represented by (1.2.6), we can calculate several commutators that will be useful for later derivations.

First of all, the commutator evaluated at the same time between the creation and destruction operator, $a_{k\lambda}^{\dagger}$, $a_{k\lambda}$, with the operators $\vec{\sigma}$ is null,

$$\left[\vec{\sigma}(t),a_{k\lambda}(t)\right]=0 \quad . \tag{1.3.4}$$

This commutator vanishes since the operators refer to independent degrees of freedom when they are evaluated at the same time, representing different physical variables. Then, clearly:

$$\left[\vec{\nabla}_{\vec{\beta}}\vec{\mathcal{E}}^{(-)}(t,\vec{\beta})\cdot\vec{d}_{10},\vec{\sigma}(t')\right]=\left[\vec{\nabla}_{\vec{\beta}}F^{(-)}(\vec{\beta},t'),\vec{\sigma}(t')\right]=0 \tag{1.3.5}$$

even more general commutators can be calculated, as

$$\left[\vec{\nabla}_{\vec{\beta}_1}\vec{\mathcal{E}}^{(-)}(t_1,\vec{\beta}_1)\cdot\vec{d}_{10},\vec{\nabla}_{\vec{\beta}_2}\vec{\mathcal{E}}^{(+)}(t_2,\vec{\beta}_2)\cdot\vec{d}_{01}\right]=Z_{ij}(t_1-t_2,\vec{\beta}_1-\vec{\beta}_2)$$

$$=\frac{|\vec{d}_{10}|^2}{(2\pi)^2c^6}\int_0^{\infty}d\omega_k\,\omega_k^5e^{-i\omega_{kL}(t_1-t_2)}\int d\Omega_k\,\hat{k}_i\hat{k}_j(1-|\hat{k}\cdot\hat{d}|^2)e^{i\vec{k}\cdot(\vec{\beta}_1-\vec{\beta}_2)} \quad . \tag{1.3.6}$$

It is easy to see that the angular integral in the case $(\vec{\beta}_1-\vec{\beta}_2)=0$ becomes:

$$\int d\Omega_k\,\hat{k}_i\hat{k}_j\left[1-|\hat{k}\cdot\hat{d}|^2\right]=\frac{4\pi}{15}\left[2\delta_{ij}-\hat{E}_i\hat{E}_j\right]$$

where $\hat{E}_i$ is the unit vector component of the laser electric field in the i direction. This angular integral can also be solved when $\vec{\beta}_1\neq\vec{\beta}_2$ in that case the solution will depend upon

the angle between $\vec{\rho}_1$ with $\vec{\rho}_2$ and the k dependence will be carried out by the spherical Bessel function from order 0 up to the order 4.

For future calculations dealing with the evaluation of the diffusion tensor, $Z_{ij}(t_1-t_2 0)$ will be the important commutator:

$$Z_{ij}(t_1-t_2 0) = \frac{\gamma}{10\pi c^2\omega_L^3}\left[2\delta_{ij}-\hat{E}_i\hat{E}_j\right]\lim_{\epsilon\to 0}\int_0^\infty d\,\omega_k\,\omega_k^5 e^{-i\omega_{kL}(t_1-t_2)-\epsilon\omega_{kL}|t_1-t_2|} \tag{1.3.7}$$

and clearly it is a very sharp function of the time difference (t_1-t_2) near zero. The limit is used to insure the definition of the previous equation. We left the evaluation of the integral (1.3.7) for after a time integration of Z_{ij} . For our applications, as in the case of the diffusion tensor which is calculated in a following section, we only need to calculate the time integral of Z_{ij} and it is not need the knowledge of the more detail result represented by (1.3.7).

Until now, we have worked out most of the relevant commutators for the gradient of the radiation field, our next calculation shall be the commutators for the radiation electric field itself and those combining it with the gradient of the field. Directly from (1.2.8) and (1.3.4) we have:

$$\left[\vec{E}^{(-)}(t,\vec{\rho})\vec{d}_{10}\,,\,\sigma(t')\right] = -i\,\gamma\,sgn\,(t-t')\left[\sigma_+(t)\,,\,\sigma(t')\right] \tag{1.3.8}$$

with the equivalent commutator for the Hermitian conjugated

$$\left[\vec{E}^{(+)}(t,\vec{\rho})\vec{d}_{01}\,,\,\sigma(t')\right] = i\,\gamma\,sgn\,(t-t')\left[\sigma_-(t)\,,\,\sigma(t')\right]\,. \tag{1.3.9}$$

Another commutator useful for future calculations is one between the positive and negative frequencies of the radiation field. This commutator will be given in terms of the following commutator

$$\left[F^{(-)}(\vec{\rho}_1,t_1,t')\,,\,F^{(+)}(\vec{\rho}_2,t_2,t')\right] = \Gamma(t_1-t_2,\vec{\rho}_1-\vec{\rho}_2)$$

$$= \frac{|\vec{d}_{10}|^2 c}{(2\pi)^2}\int_0^\infty dk\,k^3 e^{-i\omega_{kL}(t_1-t_2)}\int d\,\Omega_k\,(1-|\hat{k}\cdot\hat{d}\,|^2)e^{i\vec{k}\cdot(\vec{\rho}_1-\vec{\rho}_2)} \tag{1.3.10}$$

Again, the angular integral can, in general, be easily solved. But as in the case of the integral in (1.3.6), we need only to know it for $\vec{\rho}_1 = \vec{\rho}_2$, in which case is

$$F(t_1 - t_2, 0) = \frac{\gamma}{\pi\omega^3}\int_0^\infty d\,\omega_k\,\omega_k^3 e^{-i\,\omega_{kL}(t_1 - t_2)} \tag{1.3.11}$$

which is also a very sharp function of $(t_1 - t_2)$ near zero. The calculation of either (1.3.6) or (1.3.10) for $\vec{\rho}_1 \neq \vec{\rho}_2$ and then taking the limit $\vec{\rho}_1 = \vec{\rho}_2$ gives the same result for the commutator as the one obtained in (1.3.7) and (1.3.11), respectively.

From relation (1.2.8), we can write:

$$\left[\vec{\mathcal{E}}^{(-)}(t_1,\vec{\rho}_1)\cdot\vec{d}_{10} + i\,\gamma\,sgn(t_1 - t')\sigma_+(t_1)\,,\,\vec{\mathcal{E}}^{(+)}(t_2,\vec{\rho}_2)\cdot\vec{d}_{01}\right.$$
$$\left. - i\,\gamma\,sgn(t_2 - t')\sigma_-(t_2)\right] = F(t_1 - t_2, \vec{\rho}_1 - \vec{\rho}_2)\ . \tag{1.3.12}$$

Using equation (1.3.8) and (1.3.9), we obtain:

$$\left[\vec{\mathcal{E}}^{(-)}(t_1,\vec{\rho}_1)\cdot\vec{d}_{10}\,,\,\vec{\mathcal{E}}^{(+)}(t_2,\vec{\rho}_2)\cdot\vec{d}_{01}\right] = \gamma^2 sgn(t_1 - t_2)\left[sgn(t_2 - t') - sgn(t_1 - t')\right]\times$$
$$\left[\sigma_+(t_1),\sigma_-(t_2)\right] = F(t_1 - t_2, \vec{\rho}_1 - \vec{\rho}_2) - \gamma^2 sgn(t_1 - t')\,sgn(t_2 - t')\left[\sigma_+(t_1),\sigma_-(t_2)\right] \tag{1.3.13}$$

Actually, the case of our interest is $t_1 > t'$ and $t_2 > t'$ without any condition for t_1 with respect to t_2; in which case

$$\left[\vec{\mathcal{E}}^{(-)}(t_1,\vec{\rho}_1)\cdot\vec{d}_{10}\,,\,\vec{\mathcal{E}}^{(+)}(t_2,\vec{\rho}_2)\cdot\vec{d}_{01}\right] = F(t_1 - t_2, \vec{\rho}_1 - \vec{\rho}_2) - \gamma^2\left[\sigma_+(t_1)\,,\,\sigma_-(t_2)\right] \tag{1.3.14}$$

The last commutator can be found with the use of relations (1.2.8), (1.3.2) and (1.3.5) is

$$\left[\vec{\nabla}_{\vec{\rho}_1}\vec{\mathcal{E}}^{(-)}(t_1,\vec{\rho}_1)\cdot\vec{d}_{10}\,,\,\vec{\mathcal{E}}^{(+)}(t_2,\vec{\rho}_2)\cdot\vec{d}_{01}\right] = \left[\vec{\nabla}_{\vec{\rho}_1}F^{(-)}(\vec{\rho}_1,t_1,t')\,,\,F^{(+)}(\vec{\rho}_2,t_2,t')\right] \tag{1.3.15}$$
$$= \frac{i\,|\vec{d}_{10}|^2}{(2\pi)^2}\int_0^\infty dk\,k^2 e^{-i\,\omega_{kL}(t_1 - t_2)}\int d\,\Omega_k\,k_i\,e^{i\vec{k}\cdot(\vec{\rho}_1 - \vec{\rho}_2)}(1 - |\hat{k}\cdot\hat{d}\,|^2) = Q_i(t_1 - t_2, \vec{\rho}_1 - \vec{\rho}_2)$$

and clearly this commutator vanishes for $\vec{\rho}_1 = \vec{\rho}_2$ since the angular integral has an integrand

which is an odd function. For $\vec{p}_1 \neq \vec{p}_2$, $Q_i(t_1 - t_2, \vec{p}_1 - \vec{p}_2)$ is a very sharp function of $t_1 - t_2$ in the neighborhood of $t_1 - t_2 = 0$.

Basically the same commutators were calculated for the first time by Mittleman [23] . In that work, the calculation is carried out for the case of only one plane wave interacting with the atom. Also, there, the calculation is worked out in what is called the $\vec{A} \cdot \vec{P}_{\vec{x}_i}$ gauge. The present work, however, shows the calculation for two plane waves moving in opposite directions. The gauge used here, as stated previously, is the $\vec{E} \cdot \vec{x}_i$ gauge. The variable $\vec{x}_i$ represents the electron position as defined in relation (1.1.2) of Section 1 and $\vec{P}_{\vec{x}_i} \equiv \vec{p}_i$ is the conjugated momentum of the coordinate $\vec{x}_i$.

1.4 - Differential Equations Governing the Time Evolution of the Force and Diffusion Tensor.

The calculation that we will present in this section will include both the laser force on the atom and the momentum diffusion tensor. The second one is the effect due to the quantum fluctuations of the force exerted by the laser on the atomic states.

The quantum equivalent to the Newton force in the Heisenberg picture, where as it is known, the quantum operators and not the wave functions are time dependent,

$$\vec{F} \equiv \frac{d\vec{P}}{dt} = -i\left[\vec{P}, H\right] = -\vec{\nabla} H \quad . \tag{1.4.1}$$

Equivalently the diffusion tensor is defined in terms of the quantum expectation value of the momenta fluctuations around the quantum average. As in the previous section we represent the expectation values in the initial states by $< >$.

$$D_{ij} = \frac{1}{4}\frac{d}{dt} < \Delta P_i(t)\Delta P_j(t) + \Delta P_j(t)\Delta P_i(t) > , \tag{1.4.2}$$

where $\Delta P_i(t) = P_i(t) - < P_i(t) >$ is the deviation of the operator P_i from its average value. Note, that D_{ij} defined in (1.4.2) is a symmetric tensor with respect to the index i and j .

The operator $\vec{P}$ to which we are referring in relations (1.4.1) and (1.4.2) is the momentum of the atom's center of mass. Thus, in our case, the gradient in (1.4.1) is with

respect to the $\vec{\beta}$ coordinate of the previous sections.

In the writing of the equation (1.4.1), for the last equality we have used the fact that the operator $\vec{P}(t)$ commutes, on the one hand, with the creation and destruction operators $a^{\dagger}_{k\lambda}(t)$, $a_{k\lambda}(t)$ and, on the other hand, with the $\vec{\sigma}(t)$ operators when these operators are evaluated at the same time. Therefore, the gradient here has the usual meaning, i. e., it acts only on the explicit $\vec{\beta}$ dependence of our Hamiltonian.

As in the previous section, the initial state of the system is assumed to be a fluorescent photon vacuum.

The action of the incident external electromagnetic field on this state results simply in the classical counter part of the electromagnetic field. Therefore, we are left with the investigation of the evaluation of the $\vec{\sigma}(t)$ operators acting on the initial state of the system.

With the help of the evolution equation (1.4.1), we can obtain an expression for the diffusion tensor, (1.4.2), in terms of the gradient of the Hamiltonian.

(1.4.3)

$$D_{ij} = \frac{1}{4}\left\{ <\dot{P}_i(t)P_j(t) + P_i(t)\dot{P}_j(t)> - 2<\dot{P}_i(t)><P_j(t)> + (i \longleftrightarrow j) \right\}$$

$$D_{ij} = \frac{1}{4}\int_0^t dt' \left\{ <\vec{\nabla}_{\vec{\beta}} H(t)\vec{\nabla}_{j\vec{\beta}'} H(t') + \vec{\nabla}_{\vec{\beta}'} H(t')\vec{\nabla}_{j\vec{\beta}} H(t)> \right.$$

$$\left. -2<\vec{\nabla}_{\vec{\beta}} H(t)><\vec{\nabla}_{j\vec{\beta}'} H(t')> + (i \longleftrightarrow j) \right\} . \qquad (1.4.4)$$

To derive this last line we have used the momentum defined as the time integral of the force. The initial time in the integral is any arbitrary time previous to t such that $\gamma t \gg 1$ allowing us to drop all the effects that are irrelevant for the large time behavior. Under this prescription any transitory effects can be eliminated.

The final and main goal of this section is to present the differential equation satisfied by the local expectation value $<\vec{\sigma}(t)>$ and the non-local expectation value $<\vec{\sigma}(t)\vec{\sigma}(t')>$.

The arbitrary initial time that we have put in the lower limit of the integral (1.4.4) comes from the integration, with respect to the time, of the momentum operator. This time is fixed as a time previous to the connection of the interaction. In which case, we write $P_i(t) = -\int_0^t \vec{\nabla}_{\vec{\beta}} H(t')\,dt' + C_i$, where C_i is a constant that represents the momentum expectation value at $t = 0$ multiplied by the identity operator. Obviously C_i does not appear in (1.4.4). Therefore, following this argument the lower limit of the integral (1.4.4) can be any time previous to the interaction, and the integral upper limit, t , is the time at which the atomic system interacting with the laser beam is being described. Moreover, we require of this time t to be much larger than the atomic life time. This method for studying the quantum fluctuation in our system is equivalent to the method used by Gordon and Ashkin [4].

Working with the Hamiltonian (1.1.10), of Section 1, we can derive the explicit form of the operator force defined in (1.4.1),

$$\begin{aligned} -\vec{F}(\vec{\beta})\,|_{\vec{k}_L\cdot\vec{\beta}=\eta t} = {} & \vec{\nabla}_{\vec{\beta}}\,\bar{\beta}(\vec{\beta})\,|_{\vec{k}_L\cdot\vec{\beta}=\eta t} + \sigma_z(t)\vec{\nabla}_{\vec{\beta}}\,\beta(\vec{\beta})\,|_{\vec{k}_L\cdot\vec{\beta}=\eta t} + \\ & \sigma_+(t)\vec{\nabla}_{\vec{\beta}}\,\alpha_{01}(\vec{\beta})\,|_{\vec{k}_L\cdot\vec{\beta}=\eta t} + \sigma_-(t)\vec{\nabla}_{\vec{\beta}}\,\alpha_{10}(\vec{\beta})\,|_{\vec{k}_L\cdot\vec{\beta}=\eta t} \\ & +\sigma_+(t)\vec{\nabla}_{\vec{\beta}}\,\vec{\mathcal{E}}^{(+)}(t,\vec{\beta})\vec{d}_{01}\,|_{\vec{k}_L\cdot\vec{\beta}=\eta t} + \sigma_-(t)\vec{\nabla}_{\vec{\beta}}\,\vec{\mathcal{E}}^{(-)}(t,\vec{\beta})\vec{d}_{10}\,|_{\vec{k}_L\cdot\vec{\beta}=\eta t} \end{aligned} \tag{1.4.5}$$

In writing equation (1.4.5) we have assumed a classical motion for the atom's center of mass. That allows us to evaluate the force (1.4.1) at the classical trajectory points defined by $\vec{k}_L\cdot\vec{\beta} = \eta t$, with $\vec{k}_L$ being the laser wave vector. The parameter η is the frequency Doppler shift for the two plane waves; with the parameter $\eta = \vec{k}_L\cdot\vec{v}$, where $\vec{v}$ is the atom's center of mass velocity.

It is interesting to mention that even when we are studying an accelerated motion we can still use $\vec{k}_L\cdot\vec{\beta} = \eta t$, which implies unaccelerated motion. At this point, we are assuming that the velocity remains almost constant in a period of time compared with the atomic time scale; this is so even when, for longer times the velocity does indeed change. What is used here can be referred to as adiabatic approximation for the atomic velocity.

This approximation is based on the assumption that the velocity stays basically constant during the time between atomic events. The largest time for this events is the natural decay time, $\sim 10^{-8}$ sec , which is very short compared with the total time that the atom takes crossing the laser beam, $\sim 10^{-6}$ sec for atoms not moving very slow; $\sim 10^{3}$–10^{5} cm/ sec .

In fact the velocity change is $\delta\vec{v} = [\vec{v}(t+\delta t) - \vec{v}(t)] = \delta t \vec{F}(t)/ M_A$, where $\vec{F}(t)$ is the force due to the interaction with the electromagnetic field, M_A is the atomic mass and $\delta t \approx 10^{-8}$ sec is the typical atomic time. This is the velocity correction to the atomic velocity in a time scale in the order of the atomic time, as was mentioned before this time correspond to the natural decay time of the atom.

Another presentation of this adiabatic approximation starts from requiring that the atomic momentum be much greater than the change of the momentum associated to the emission or absorption of a single laser photon, in such a way that after few of these events, $\Lambda/\gamma \sim 35$, [19,20] the atomic momentum is almost unchanged compared with the atom's total momentum.

A gross estimation of the ratio $\delta v/ v$ can be obtained from:

$$\frac{\delta v}{v} \approx \frac{\hbar k_L}{M_A v} \times \frac{\Lambda}{\gamma} \approx 8 \times 10^{-4} \ll 1 .$$

The meaning of evaluating the operator force at the classical points of the center of mass motion is that the atom as a whole behaves like a classical particle moving under the influence of an external force, (1.4.5), which is the result of the quantum interaction of the internal modes with the external electromagnetic field.

Each term in (1.4.5) can be explained separately as follows: first, in $\bar{\beta}(\vec{p}) = \frac{1}{2}[W_1(\vec{p}) + W_0(\vec{p})]$, which is a constant in the present calculation, the $\vec{p}$ dependence would be present only in the eventuality of an external field other than the laser field. Again, the characteristic internal atomic time for a free atom is very small compared with the time scale associated with the motion of the atomic center of mass. Therefore, the laser dressing of the atomic eigenvalues is a very small correction (second order in the perturbation).

An example of an external field could be a position dependent external magnetic field. The next term of (1.4.5) comes from $\beta(\vec{\rho})=\frac{1}{2}\Delta\omega(\vec{\rho})$ with an actual $\vec{\rho}$ dependence only under the same conditions as in the previous term.

The next two terms are the forces on the atom due to the laser field. They are proportional to the laser wave vector $\vec{k}_L$, therefore the meaning of these force terms are the induced force on the atom producing a momentum change of magnitude $\hbar\vec{k}_L$, for each photon involved. Here, $\alpha_{01}(\vec{\rho})= \vec{E}_0\cdot\vec{d}_{01}\cos\vec{k}_L\cdot\vec{\rho}$ is associated with the re-emission of a laser photon, $\vec{d}_{01}\approx<u_0|\vec{\chi}|u_1>$, from either of the two plane waves $e^{-i\vec{k}_L\cdot\vec{\rho}}$ or $e^{i\vec{k}_L\cdot\vec{\rho}}$. Conversely, $\alpha_{10}(\vec{\rho})= \vec{E}_0\cdot\vec{d}_{10}\cos\vec{k}_L\cdot\vec{\rho}$ is associated with the absorption of a laser photon, $\vec{d}_{10}\approx<u_1|\vec{\chi}|u_0>$ and a jump to the exited state u_1 .

The last two terms are the forces due to the emission by the atom of fluorescent photons. These forces produce a momentum change to the atom in opposite direction to the total momentum carried by the fluorescent photons since, these force terms are proportional to the fluorescent photon wave vector $\vec{k}$.

Following the argument of the previous discussion, only the last two terms require a careful evaluation of the fluorescent field when we work on the quantum expectation value of the operator force. Due to the homogeneity of the fluorescent radiation, the expectation value of the last two terms are zero, since photons with momenta $\vec{k}$ and $-\vec{k}$ are equally probable. Remember that the operators $\sigma_+(t)$, $\sigma_-(t)$ and $\vec{\nabla}\vec{\epsilon}^{(+)}(t,\vec{\rho})\cdot\vec{d}_{01}$, $\vec{\nabla}\vec{\epsilon}^{(-)}(t,\vec{\rho})\cdot\vec{d}_{10}$ evaluated at the same time t commute. Therefore, using relation (1.3.3) and its Hermitian conjugated we obtain that the expectation values of the last two terms of the quantum force (1.4.5) vanishes. In fact, the expectation value of the operator force is simply:

$$-\vec{F}(t)=<-\vec{\mathbb{F}}(t)>=\vec{\nabla}_{\vec{\rho}}\bar{\beta}(\vec{\rho})|_{\vec{k}_L\cdot\vec{\rho}=\eta t}+S_z(t)\vec{\nabla}_{\vec{\rho}}\beta(\vec{\rho})|_{\vec{k}_L\cdot\vec{\rho}=\eta t}$$

$$+S_+(t)\vec{\nabla}_{\vec{\rho}}\alpha_{01}(\vec{\rho})|_{\vec{k}_L\cdot\vec{\rho}=\eta t}+S_-(t)\vec{\nabla}_{\vec{\rho}}\alpha_{10}(\vec{\rho})|_{\vec{k}_L\cdot\vec{\rho}=\eta t}\quad . \tag{1.4.6}$$

We are defining in (1.4.6) the real expectation values $S_i(t)\equiv<\sigma_i(t)>$ and also

$S_{\pm}(t) \equiv <\sigma_{\pm}(t)>$. The following notation will be useful for further developments: $\vec{\nabla}_{\vec{p}} B(\vec{p})|_{\vec{k}_L \cdot \vec{p} = \eta'} \equiv \vec{\nabla} B(t)$; where, B represents any of the quantum operators of (1.4.5) or any of the classical functions present at (1.4.6). Remember that as usual the quantum expectation values of the operators representing the external laser field become their classical counter parts.

The force (1.4.6) will be completely evaluated when the local expectation values $S_i(t)$ are known. After the time dependent solution of $S_i(t)$ is substituted into the equation (1.4.6) we go from the time dependent force to a position and velocity dependent force. This is again done using the classical prescription for the motion of the atom's center of mass.

For the explicit calculation of the diffusion tensor (1.4.4) using the operator force defined in (1.4.5) we need to make a more careful analysis which is the subject of the following section. However, we advance an important result: the integrand of (1.4.4) depended only on the expectation values $<\sigma_i(t)>$ and the non-local expectation values $<\sigma_i(t)\sigma_j(t')>$. In the non-local case there is also dependence on its complex conjugated expectation values, $<\sigma_j(t')\sigma_i(t)>$.

The last calculation performed in this section is directed to find the differential equation for the previous expectation values.

We start by finding the time evolution differential equation for the local expectation value $\dot{S}_i(t) = <\sigma_i(t)>$. This calculation is carried out by taking the expectation value with respect to the initial state of the system of equation (1.2.2). Then, it is followed by using the properties developed in Section 2 for evaluating the action of the radiation field acting over the initial state of the system. In particular, relation (1.2.8) and its Hermitian conjugated are used, leading to

$$\dot{S}_i(t) = \Delta\omega(\hat{a}_z \times \vec{S})_i + \vec{E}_0 \cdot \vec{d}_{01} \cos\vec{k}_L \cdot \vec{\rho} \left| (\hat{a}_x \times \vec{S})_i + i(\hat{a}_y \times \vec{S})_i \right| + \vec{E}_0 \cdot \vec{d}_{10} \cos\vec{k}_L \cdot \vec{\rho} \left| (\hat{a}_x \times \vec{S})_i - i(\hat{a}_y \times \vec{S})_i \right| + \left| (\hat{a}_x + i\hat{a}_y) \times \langle \vec{\mathcal{E}}^{(+)}(t, \vec{\rho}) \cdot \vec{d}_{01} \sigma(t) \rangle \right| + \left| (\hat{a}_x - i\hat{a}_y) \times \langle \sigma(t) \vec{\mathcal{E}}^{(-)}(t, \vec{\rho}) \cdot \vec{d}_{10} \rangle \right| . \quad (1.4.7)$$

Now using relations (1.3.1) and its complex conjugated for $t > t'$, we obtain the following equations for each component of the vector $\vec{S}(t)$:

$$\dot{S}_x(t) = -\Delta\omega S_y(t) + i S_z(t)[\alpha_{01}(t) - \alpha_{10}(t)] - \gamma \left| \langle \sigma_-(t) \sigma_z(t) \rangle + \langle \sigma_z(t) \sigma_+(t) \rangle \right| \quad (1.4.8)$$

$$\dot{S}_y(t) = \Delta\omega S_x(t) - S_z(t)[\alpha_{01}(t) + \alpha_{10}(t)] + i\gamma \left| \langle \sigma_z(t) \sigma_+(t) \rangle - \langle \sigma_-(t) \sigma_z(t) \rangle \right| \quad (1.4.9)$$

$$\dot{S}_z(t) = S_y(t)[\alpha_{01}(t) + \alpha_{10}(t)] + S_x(t)[\alpha_{10}(t) - \alpha_{01}(t)] + i\gamma \left| \langle \sigma_-(t) \sigma_y(t) \rangle - \langle \sigma_y(t) \sigma_+(t) \rangle \right| + \gamma \left| \langle \sigma_-(t) \sigma_x(t) \rangle + \langle \sigma_x(t) \sigma_+(t) \rangle \right| \quad (1.4.10)$$

In the previous equations, we choose the original atomic eigenstates u_1 and u_0 real [28] , in such a way that $\alpha_{01}(t) = \alpha_{10}(t) \equiv \frac{1}{2}\alpha(t) = \vec{E}_0 \cdot \vec{d}_{01} \cos\vec{k}_L \cdot \vec{\rho} \,|_{\vec{k}_L \cdot \vec{\rho} = \eta t} = \Lambda \cos\eta t$. Where, Λ has been defined after equation (1.2.7) on this chapter.

Applying the usual relations for the commutators and anti-commutators of the Pauli matrices

$$\left| \sigma_i, \sigma_j \right| = 2i \epsilon_{ijk} \sigma_k \quad ; \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij} ,$$

we can write the final form of the equations (1.4.8) to (1.4.10). The equation in a matrix form is

$$\dot{\vec{S}}(t) + \vec{M}(t) \vec{S}(t) = 2\gamma \hat{a}_z \quad (1.4.11)$$

where,

$$\vec{M}(t) = \begin{vmatrix} \gamma & \Delta\omega & 0 \\ -\Delta\omega & \gamma & \alpha(t) \\ 0 & -\alpha(t) & 2\gamma \end{vmatrix} \quad ; \quad \hat{a}_z = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \quad (1.4.12)$$

similarly, we define

$$A_{ij}(t,t') = <\sigma_i(t)\sigma_j(t')> = <\sigma_j(t')\sigma_i(t)>^* \qquad (1.4.13)$$

and $\dot{A}_{ij}(t,t') = <\dot{\sigma}_i(t)\sigma_j(t')>$.

To obtain the differential equation satisfied by the non local expectation values first, we have to multiply equation (1.2.2) by the right hand side for $\sigma_j(t')$ and second, we take the quantum average of the resulting operator.

The equivalent to equation (1.4.7) is

$$\begin{aligned}\dot{A}_{ij}(t,t') = {} & \Delta\omega\epsilon_{iln}(\hat{a}_z)_l A_{nj}(t,t') + \vec{E}_0\cdot\vec{d}_{01}\cos\vec{k}_L\cdot\vec{\rho}\epsilon_{iln}(\hat{a}_x+i\hat{a}_y)_l A_{nj}(t,t') \\ & + \vec{E}_0\cdot\vec{d}_{10}\cos\vec{k}_L\cdot\vec{\rho}\epsilon_{iln}(\hat{a}_x+i\hat{a}_y)_l A_{nj}(t,t') + \\ & \epsilon_{iln}(\hat{a}_x+i\hat{a}_y)_l <\vec{\mathcal{E}}^{(+)}(t,\vec{\rho})\cdot\vec{d}_{01}\sigma_n(t)\sigma_j(t')> + \\ & \epsilon_{iln}(\hat{a}_x-i\hat{a}_y)_l <\sigma_n(t)\vec{\mathcal{E}}^{(-)}(t,\vec{\rho})\cdot\vec{d}_{10}\sigma_j(t')> \quad ,\end{aligned} \qquad (1.4.14)$$

in this equation clearly the only term that needs special mention is the last. The other terms behave exactly like their equivalents for the $\dot{S}_i(t)$ equation.

The last term of this equation is evaluated using commutator (1.3.8) and relation (1.3.1); therefore, the final equation for the tensor $A_{ij}(t,t')$ is

$$\dot{\vec{A}}_j(t,t') + \vec{\vec{M}}(t)\vec{A}_j(t,t') = 2\gamma\hat{a}_z S_j(t') \qquad (1.4.15)$$

where $\left|\vec{A}_j(t,t')\right|_i \equiv A_{ij}(t,t')$ and the matrix $\vec{\vec{M}}(t)$ is the same one defined in (1.4.12).

Equation (1.4.11) is equivalent to equation (1.2.18) of Fiordilino-Mittleman [20]. As it was mentioned before they present the results in the $\vec{A}\cdot\vec{P}$ gauge formulation. The matrix obtained in Fiordilino-Mittleman work is in a time dependent coordinate system with time dependent unitary vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$. The same comments are applicable for the equation equivalent to (1.4.14) presented by Mittleman [23] for the case of one plane wave interacting with the atom.

In the problem of one plane wave interacting with the atomic states the explicit matrix $\vec{\vec{M}}$ is time independent allowing and straightforward solution of the differential equations. This is not so for the standing wave case which requires a more meticulous analysis.

The fact that the homogeneous part of equation (1.4.12) and (1.4.14) are equal implies that our problem is actually reduced to solve only three coupled differential equations. The solution for this system of coupled equations can not be obtained exactly. We will present an approximate solution for a time large compare with the natural atomic decay time. Also, fast oscillating terms will be neglected.

1.5 - The Diffusion Tensor.

We start with the force operator (1.4.5) written as,

$$-\vec{F}(t) = \vec{\nabla}\bar{\beta}(t) + \sigma_z(t)\vec{\nabla}\beta(t) + \frac{1}{2}\sigma_x(t)\vec{\nabla}\alpha(t) + \vec{\nabla}\vec{\epsilon}^{(+)}(t,\vec{\rho})\vec{d}_{01}\sigma_+(t) + \sigma_-(t)\vec{\nabla}\vec{\epsilon}^{(-)}(t,\vec{\rho})\vec{d}_{10} \tag{1.5.1}$$

and the integrand of equation (1.4.4) is:

$$I_{ij}(t,t') = <\vec{\nabla}_i H(t)\vec{\nabla}'_j H(t') + \vec{\nabla}'_i H(t')\vec{\nabla}_j H(t)> -2<\vec{\nabla}_i H(t)><\vec{\nabla}'_j H(t')> + (i \longleftrightarrow j) \tag{1.5.2}$$

The notation used in the previous equation defines $\vec{\nabla}_i$ as the gradient with respect to the i component of the position vector $\vec{\rho}$ and evaluated at $\vec{k}_L \cdot \vec{\rho} = \eta t$. This follows the classical prescription for the motion of the atomic's center of mass. Second, $\vec{\nabla}'_i$ is the equivalent to $\vec{\nabla}_i$ but now the derivative is with respect to the vector $\vec{\rho}'$ and consequently the evaluation is at $\vec{k}_L \cdot \vec{\rho} = \eta t'$.

Since, the integral (1.4.4) runs from 0 to t and the time t' is the integration variable; t' is always previous to t. Following equation (1.4.1) and using relation (1.3.3) of Section 3, we have for the gradient of the Hamiltonian acting on the initial ket,

$$\vec{\nabla}_j H(t)> = \vec{\nabla}_j \bar{\beta}(t)> + \vec{\nabla}_j \beta(t)\sigma_z(t)> + \tfrac{1}{2}\vec{\nabla}_j \alpha(t)\sigma_x(t)> + \vec{\nabla}_j \vec{\epsilon}^{(+)}(t,\vec{\rho})\vec{d}_{01}\sigma_+(t)> . \tag{1.5.3}$$

The first term gives contributions to the expectation value $<\vec{\nabla}_i H(t)\vec{\nabla}'_j H(t')>$ proportionals to the force average. However, at the end, this term does not contribute to the quantum fluctuations represented by the diffusion tensor because they are exactly cancel by the

contributions coming from $<\vec{\nabla}_i\ H(t)><\vec{\nabla}'_j\ H(t')>$.

Therefore:

$$<\vec{\nabla}_i\ H(t)\vec{\nabla}'_j\ H(t')> = \vec{\nabla}_i\ \beta(t)\vec{\nabla}'_j\ \beta(t')A_{zz}(t,t') + \tfrac{1}{2}\vec{\nabla}_i\ \beta(t)\vec{\nabla}'_j\ \alpha(t')A_{zz}(t,t') + \frac{1}{2}\vec{\nabla}_i\ \alpha(t)\vec{\nabla}'_j\ \beta(t')A_{zz}(t,t') + \frac{1}{4}\vec{\nabla}_i\ \alpha(t)\vec{\nabla}'_j\ \alpha(t')A_{zz}(t,t') + Z_{ij}(t-t',\vec{\beta}-\vec{\beta}')<\sigma_-(t)\sigma_+(t')>\ , \tag{1.5.4}$$

from which we can write the diffusion tensor as:

$$D_{ij}(t) = \frac{1}{4}\int_0^t dt' \Big\{ 2\Big|\vec{\nabla}_i\ \beta(t)\vec{\nabla}'_j\ \beta(t') + \vec{\nabla}_i\ \beta(t')\vec{\nabla}_j\ \beta(t)\Big|\Big|\mathrm{Re}A_{zz}(t,t') - S_z(t)S_z(t')\Big| + \Big|\vec{\nabla}_i\ \beta(t)\vec{\nabla}'_j\ \alpha(t') + \vec{\nabla}_j\ \beta(t)\vec{\nabla}'_i\ \alpha(t')\Big|\Big|\mathrm{Re}A_{zz}(t,t') - S_z(t)S_x(t')\Big| + \Big|\vec{\nabla}_i\ \alpha(t)\vec{\nabla}'_j\ \beta(t') + \vec{\nabla}_j\ \alpha(t)\vec{\nabla}'_i\ \beta(t')\Big|\Big|\mathrm{Re}A_{zz}(t,t') - S_x(t)S_z(t')\Big| + \frac{1}{2}\Big|\vec{\nabla}_i\ \alpha(t)\vec{\nabla}'_j\ \alpha(t') + \vec{\nabla}'_i\ \alpha(t')\vec{\nabla}_j\ \alpha(t)\Big|\Big|\mathrm{Re}A_{zz}(t,t') - S_x(t)S_x(t')\Big| 2Z_{ij}(t-t',\vec{\beta}-\vec{\beta}')<\sigma_-(t)\sigma_+(t')> + 2Z_{ij}(t'-t,\vec{\beta}'-\vec{\beta})<\sigma_-(t')\sigma_+(t)>\Big| \tag{1.5.5}$$

where $Z_{ij}(t-t',\vec{\beta}-\vec{\beta}')$ is the function defined in (1.3.6) of Section 3.

In the diffusion tensor the terms proportional to $\vec{\nabla}\beta$ can be dropped since, in our work we are not considering another external field except for the one associated with the laser.

Of the remaining terms, we can separate them in two categories one proportional to Z_{ij} and the other proportional to $\vec{k}_{L\,i}\ \vec{k}_{L\,j}$. The first one, proportional to Z_{ij}, comes from the commutator between the gradient of the fluorescence field evaluated at two different times, and it is related with the randomness of the natural decay process. The terms proportional to $\vec{k}_{L\,i}\ \vec{k}_{L\,j}$ are directly associated with the laser field and accounts for the atom's recoil when the atom absorbs a photon from one of the plane waves and then re-emits in the other laser mode.

Note that any external field that impose a position dependence on the energy levels will have also an effect on the diffusion tensor. This effect is being counted in the terms proportional to $\vec{\nabla}\beta$.

Now, we give the justification for evaluating the function $Z_{ij}(t-t',\vec{\beta}-\vec{\beta}')$ at $\vec{\beta}=\vec{\beta}'$. Suppose that we replace the Z_{ij} definition, (1.3.6), in the integral, (1.5.5), for the diffusion tensor. We first solve the time integral defined in (1.5.5) and then the angular integral of equation (1.3.6). Since Z_{ij} is a very sharp function of $t-t'$ around $t=t'$ this integral will be proportional to $\delta(\omega_{kL}-\eta_k)$, again $\eta_k=\omega_k\,\hat{k}\cdot\vec{v}/c$ is the Doppler shift of the fluorescence photons. Therefore, the $\hat{k}$ angular integral of (1.3.6) for $\vec{\beta}\neq\vec{\beta}'$ will be reduced to solve the case $\vec{\beta}=\vec{\beta}'$ plus correction of the order v/c , where v is the atomic velocity.

On the other hand, considering that $Z_{ij}(t-t',0)$ is a very sharp function of the argument around $t=t'$ the last terms of the integral (1.5.5) can be evaluated at $t=t'$. The final expression for the diffusion tensor is

$$D_{ij}(t)=\frac{1}{4}\int_0^t dt'\left\{\frac{1}{2}\left[\vec{\nabla}_i\,\alpha(t)\vec{\nabla}_j\,\alpha(t')+\vec{\nabla}_i\,\alpha(t')\vec{\nabla}_j\,\alpha(t)\right]\left[\mathrm{Re}A_{xx}(t,t')-S_x(t)S_x(t')\right]\right.$$
$$\left.+\,2Z_{ij}(t-t',0)\left[1-S_z(t)\right]\right\}. \qquad (1.5.6)$$

In the next chapter, we shall complete the solution of this integral which requires the knowledge of the time dependence of both kind of expectation values, $S_i(t)$ and $A_{ij}(t,t')$.

2.1 - Time Evolution and the Differential Equations.

In this section, the unitary matrix that governs the time evolution of the differential equations (1.4.11) and (1.4.13) of the previous chapter will be obtained. The method utilized in the present work was first used for the calculation of the force in the standing wave laser by Fiordilino and Mittleman [20].

The starting point is the matrix (1.4.12) of Chapter 1, written as

$$\vec{M}(t) = \frac{1}{2}\left[\vec{M}(t)+\vec{M}^{\dagger}(t)\right] + \frac{1}{2}\left[\vec{M}(t)-\vec{M}^{\dagger}(t)\right] \tag{2.1.1}$$

Defining $\frac{1}{2}(\vec{M}(t)+\vec{M}^{\dagger}(t)) = \vec{M}_R(t)$ and $(i/2)(\vec{M}(t)-\vec{M}^{\dagger}(t)) = \vec{M}_I(t)$ then

$$\vec{M}(t) = \vec{M}_R(t) - i\vec{M}_I(t)$$

here (2.1.2)

$$\vec{M}_R(t) = \gamma\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} \qquad \vec{M}_I(t) = i\begin{vmatrix} 0 & \Delta\omega & 0 \\ -\Delta\omega & 0 & \alpha(t) \\ 0 & \alpha(t) & 0 \end{vmatrix} .$$

Looking at the above matrices, we see that $\vec{M}_R$ is time independent while $\vec{M}_I(t)$ is explicitly time dependent. The time dependence of $\vec{M}_I$ is through the argument $\eta t = \vec{k}_L \cdot \vec{\beta}$ of $\alpha(t) = 2\Lambda\cos\eta t$. To absorb the extra factor 2, we redefine $\Lambda \to \Lambda(new) = 2\Lambda = \frac{2e}{\hbar}|\vec{E}_0 \cdot \sum_i \langle u_0|\chi_i|u_1\rangle|$. This redefinition will allow a more direct comparison between this work and that of Fiordilino-Mittleman [20].

The explicit time dependence associated with the matrix $\vec{M}_I(t)$ makes the differential equation very difficult to solve exactly in a closed algebraic form. In fact, these equations are equivalent to the Bloch equation [28,33] for this problem, and a general solution for them is not known.

Generally, the method for solving the differential equations (1.4.11) and (1.4.15) is to, first, solve for the evolution matrix obeying the differential equation

$$\dot{\vec{U}}(t) + \vec{M}(t)\vec{U} = \vec{0} \tag{2.1.3}$$

with the initial condition specified at $t = 0$, $\vec{U}(0) = \vec{I}$.

An approximate method for solving equation (2.1.3) consists of first solving for the reduced equation

$$i\dot{\vec{U}}_0 + \vec{M}_I(t)\vec{U}_0(t) = \vec{0}\ , \tag{2.1.4}$$

and, based on this solution, an approximated solution for $\vec{U}(t)$ is found.

The matrix $\vec{M}_I(t)$ is of the order of Λ and the matrix $\vec{M}_R$ is proportional to γ. Therefore, in writing (2.1.4), we are assuming $\Lambda \gg \gamma$, which is the case of our interest. The range of variation used in this work for the different parameters satisfies the previous relation $\Lambda \gg\gg \gamma$. This condition is called high laser intensity regime. Most of the known solutions for this problem do not include this limit for the interaction of a standing wave laser with an atomic beam [4,21].

The Hermitian matrix $\vec{M}_I(t)$ can be diagonalized using the standard method. The unitary matrix that performs the diagonalization is a time dependent matrix defined by

$$\vec{T}_0(t) = \epsilon^{-1}(t) \begin{vmatrix} \alpha(t) & i\,\Delta\omega/\sqrt{2} & -\Delta\omega/\sqrt{2} \\ 0 & \epsilon(t)/\sqrt{2} & -i\,\epsilon(t)/\sqrt{2} \\ \Delta\omega & -i\,\alpha(t)/\sqrt{2} & \alpha(t)/\sqrt{2} \end{vmatrix}\ . \tag{2.1.5}$$

In the above matrix, $\epsilon(t)$ is the Rabi frequency in the case of a two-level atom interacting with a standing wave laser,

$$\epsilon(t) = [(\Delta\omega)^2 + \Lambda^2\cos^2\eta z\,]^{1/2}\ . \tag{2.1.6}$$

The Rabi frequency is interpreted as the oscillation frequency for the atomic system probability of occupation between the ground and the excited state. In other words, we can say that due to the interaction with the laser field the electron oscillates between the ground and the excited state of the atom. The frequency of this oscillation is the Rabi frequency [2]. This interpretation is rigorously valid for the case in which the laser is resonant with the atomic transition between the two levels [28,33].

Since the unitary matrix $\vec{T}_0$ is a time dependent matrix, a diagonalization of the matrix $\vec{M}_I(t)$ does not diagonalize the second term of equation (2.1.4).

Defining

$$\vec{U}_0(t) = \vec{T}_0^{-1}\vec{U}_0(t) \ , \tag{2.1.7}$$

we shall demonstrate that $\vec{U}_0(t)$ satisfies a diagonal differential equation at order $(\eta/\epsilon(t))^0$. Neglecting higher order in the ratio $\eta/\epsilon(t)$ is equivalent to neglect order η/Λ, which also satisfies the conditions established for the relative value between the parameters of the present analysis. In fact, we are assuming that the Doppler shift of the laser frequency, due to the atomic velocity η, is much smaller than the coupling energy between the laser and the atom, $\Lambda \gg \eta$, here $\hbar = 1$.

Calling $\vec{D}_{I0}(t)$ the diagonal form of the matrix $\vec{M}_I(t)$ derived from the unitary transformation defined in (2.1.5), the equation satisfied by $\vec{U}_0(t)$ is:

$$i\dot{\vec{U}}_0(t) + \left[\vec{D}_{I0}(t) + i\dot{\vec{T}}_0^{-1}(t)\vec{T}_0(t)\right]\vec{U}_0(t) = \vec{0} \tag{2.1.8}$$

where

$$\vec{D}_{I0}(t) = \vec{T}_0^{-1}(t)\vec{M}_I(t)\vec{T}_0(t) = \begin{vmatrix} 0 & 0 & 0 \\ 0 & \epsilon(t) & 0 \\ 0 & 0 & -\epsilon(t) \end{vmatrix} .$$

Our following step is to neglect the term $\dot{\vec{T}}_0^{-1}(t)\vec{T}_0(t)$ in equation (2.1.8). First, the diagonal part of equation (2.1.8), i. e., matrix $\vec{D}_{I0}(t)$ is proportional to Λ because the Rabi frequency can be written as $\epsilon(t) = \Lambda\left[(\Delta\omega)^2/(\Lambda)^2 + \cos^2\eta t\right]^{1/2}$.

The dependence in the matrix $\vec{T}_0(t)$ appears only in two forms $\alpha(t)/\epsilon(t)$ and $\Delta\omega/\epsilon(t)$. Both terms are proportional to η when their time derivative is calculated, implying that $\dot{\vec{T}}_0(t)$ is also proportional to η. On the other hand, the comparative order of magnitude of the product $\dot{\vec{T}}_0^{-1}(t)\vec{T}_0(t)$ is the same has the order of magnitude of $\dot{\vec{T}}_0$. However, the matrix $\vec{D}_{I0}(t)$ is proportional to Λ compared with $\dot{\vec{T}}_0^{-1}(t)\vec{T}_0(t)$ which is proportional to η. Therefore, we have shown that the order of magnitude of the off-diagonal terms for the total matrix in equation (2.1.8) is η/Λ smaller than the leading term.

Up to this point, for solving equation (2.1.3), we have been using a power series for the matrix $\vec{M}(t)$; first in powers of γ/Λ and then, for each term of this series, a series

expansion in powers of η/Λ. The relative magnitude between the parameters γ and η remains undefined.

Suppose the first order in η/Λ is kept. The next term of the power series is obtained using the fact that equation (2.1.8) is of the same kind as equation (2.1.4). We begin by finding the time dependent unitary matrix that diagonalizes the Hermitian matrix $\tilde{D}_{I\,0}(t)+i\dot{\tilde{T}}_0^{-1}(t)\tilde{T}_0(t)\equiv\tilde{M}_{I\,1}(t)$. The second term of the "new" equation (2.1.8) will be $\tilde{D}_{I\,1}(t)+i\dot{\tilde{T}}_1^{-1}(t)\tilde{T}_1(t)$, where $\tilde{D}_{I\,1}(t)$ is the diagonalization of $\tilde{M}_{I\,1}(t)$ done by the unitary matrix $\tilde{T}_1(t)$. The previous procedure could be iterated for obtaining higher orders of the series expansion for the solution $\tilde{U}_0(t)$ of equation (2.1.4).

The term $\dot{\tilde{T}}_0^{-1}(t)\tilde{T}_0(t)$ in equation (2.1.8) can be neglected. This allows us to easily find the solution for $\tilde{U}_0(t)$ and from there to obtain the solution for $\tilde{U}_0(t)=\tilde{T}_0(t)\tilde{U}_0(t)$:

$$\tilde{U}_0(t)=\tilde{T}_0(t)\exp\left[i\int_0^t dt'\tilde{D}_{I\,0}(t')\right] \tag{2.1.9}$$

$$=\tilde{T}_0(t)\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\int_0^t dt'\,\epsilon(t')} & 0\\ 0 & 0 & e^{-i\int_0^t dt'\,\epsilon(t')}\end{pmatrix}$$

The equation (2.1.3) is reduced to a more workable form using the following function change [20]:

$$\tilde{U}(t)=e^{-\frac{4}{3}\gamma t}\tilde{U}_0(t)\tilde{W}(t)\tilde{T}_0^{-1}(t) \tag{2.1.10}$$

where the new unknown matrix $\tilde{W}(t)$ satisfies the differential equation

$$\dot{\tilde{W}}(t)-\frac{\gamma}{3}\tilde{U}_0^{-1}\tilde{C}\,\tilde{U}_0(t)\tilde{W}(t)=\tilde{0} \tag{2.1.11}$$

with $\tilde{C}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$ a traceless constant matrix.

Moreover, the explicit form of the matrix $\tilde{U}_0(t)$ is easily obtained from (2.1.9).

The second term of the matrix differential equation (2.1.11) can then be calculated, resulting in:

$$\vec{\vec{U}}_0^{-1}(t)\,\vec{\vec{C}}\,\vec{\vec{U}}_0(t) = \vec{\vec{C}}_0(t) + \vec{\vec{C}}_1(t) \tag{2.1.12}$$

where

$$\vec{\vec{C}}_0(t) = q(t)\vec{\vec{C}}' \quad ; \; q(t) = \frac{\Lambda^2\cos^2\eta t - 2\Delta\omega^2}{2\epsilon^2(t)} \tag{2.1.13}$$

with

$$\vec{\vec{C}}' = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \tag{2.1.14}$$

and $\vec{\vec{C}}_1(t)$ is a Hermitian matrix defined by:

(2.1.15)

$$\vec{\vec{C}}_1(t) = \frac{1}{\epsilon^2(t)} \begin{vmatrix} 0 & --- & --- \\ -\frac{3i}{\sqrt{2}}\Delta\omega\Lambda\cos\eta t\, e^{-i\int_0^t dt'\,\epsilon(t')} & 0 & --- \\ -\frac{3}{\sqrt{2}}\Delta\omega\Lambda\cos\eta t\, e^{i\int_0^t dt'\,\epsilon(t')} & \frac{3i}{2}\Lambda^2\cos^2\eta t\, e^{2i\int_0^t dt'\,\epsilon(t')} & 0 \end{vmatrix}$$

The solution of equation (2.1.11) using relations (2.1.12) to (2.1.15) is still a difficult task. However, the non diagonal part of (2.1.12), i. e. matrix (2.1.15), presents in each of its matrix elements a term proportional to $e^{\pm i\int_0^t dt'\,\epsilon(t')}$, with $\epsilon(t)$ the Rabi flopping frequency defined in the expression (2.1.6). These terms are fast oscillating terms, and their contribution to the solution for the matrix $\vec{\vec{U}}(t)$ are of the order $\gamma/\bar{\epsilon}$ smaller than the leading term. Here, $\bar{\epsilon}$ represents the time average of $\epsilon(t)$.

In fact, the solution of the equation (2.1.11), matrix $\vec{\vec{W}}(t)$, can be written as $\vec{\vec{W}}(t) = \vec{\vec{W}}_0(t)[1 + \vec{\vec{W}}_1(t) + \cdots]$ with $\vec{\vec{W}}_0(t)$ solution of (2.1.11) neglecting (2.1.15),

$$\vec{W}_0(t) = \exp\left[\frac{\gamma}{3}\int_0^t dt' q(t')\vec{C}\cdot\right] \tag{2.1.16}$$

$$= \begin{vmatrix} e^{\frac{2}{3}\gamma\int_0^t dt' q(t')} & 0 & 0 \\ 0 & e^{-\frac{\gamma}{3}\int_0^t dt' q(t')} & 0 \\ 0 & 0 & e^{-\frac{\gamma}{3}\int_0^t dt' q(t')} \end{vmatrix}$$

Substituting this solution in equation (2.1.11), we obtain a differential matrix equation for $\vec{W}_1(t)$, for which an approximated solution can be calculated. (See below). This is equivalent to solving the first order in $\gamma/\bar{\epsilon}$ and neglecting the next order, i. e., $(\gamma/\bar{\epsilon})^2$. The solution for $\vec{W}_1(t)$ in terms of the previous order $\vec{W}_0(t)$ is:

$$\vec{W}_1(t) = \frac{\gamma}{3}\int_0^t dt' \vec{W}_0^{-1}(t')\vec{C}_1(t')\vec{W}_0(t') \ . \tag{2.1.17}$$

This matrix is an off diagonal matrix where each one of the matrix elements is proportional to either of the factors $e^{\pm i\int_0^t ds\, \epsilon(s)}$. For the resonant case $\Delta\omega = 0$, a closed form of equation (2.1.17) can be presented. We show it here to illustrate the general behavior of $\vec{W}_1(t)$,

$$\vec{W}_1(t) = \frac{\gamma}{2\eta}\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\int_0^{\eta t} ds e^{-2i\frac{\Lambda}{\eta}\sin s} \\ 0 & i\int_0^{\eta t} ds e^{2i\frac{\Lambda}{\eta}\sin s} & 0 \end{vmatrix} . \tag{2.1.18}$$

The integrals for the matrix elements of (2.1.18) can be solved in the limit $\Lambda \gg \eta$.

We will use a more general result for studying the behavior of integrals with an integrand like $e^{g(x)/\nu}$ when the parameter ν is small. This formula will be useful for future calculations too.

The asymptotic expansion of the integral $\lim_{\nu\to 0}\int_a^b dx\, f(x) e^{g(x)/\nu}$ can be obtained by integrating by parts. The first term of the asymptotic series is

$$\int_a^b dx\, f(x) e^{\frac{g(x)}{\nu}} = \frac{\nu f(x)}{g'(x)} e^{\frac{g(x)}{\nu}} \Big|_a^b - \int_a^b dx\, \nu e^{\frac{g(x)}{\nu}} \left| \frac{f'(x)}{g'(x)} - \frac{f(x) g''(x)}{g'^2(x)} \right|,$$

and after a second integration by part, we obtain the two leading terms:

$$\int_a^b dx\, f(x) e^{\frac{g(x)}{\nu}} = \frac{\nu f(x)}{g'(x)} e^{\frac{g(x)}{\nu}} \Big|_a^b - \frac{\nu^2 f(x)}{g'(x)} e^{\frac{g(x)}{\nu}} \left| \frac{f'(x)}{f(x) g'(x)} - \frac{g''(x)}{g'^2(x)} \right| \Big|_a^b + O(\nu^3) \tag{2.1.19}$$

Using the previous expansion (2.1.19), we can find the matrix elements of $\vec{\vec{W}}_1(t)$ as,

$$-\frac{i\gamma}{2\eta} \int_0^{\pi} ds\, e^{-2i\frac{\Lambda}{\eta}\sin s} = -\frac{\gamma}{4\Lambda} \left| 1 + \frac{e^{-2i\frac{\Lambda}{\eta}\sin \eta t}}{\cos \eta t} - i\frac{\eta}{2\Lambda} \frac{\sin \eta t}{\cos^3 \eta t} e^{-2i\frac{\Lambda}{\eta}\sin \eta t} + \cdots \right|. \tag{2.1.20}$$

and for the particular case $\eta t = \pi/2$ the asymptotic expansion is obtained using that the integral can be evaluated as $\int_0^{\pi/2} ds \exp\left[-2i\frac{\Lambda}{\eta}\sin s\right] = \frac{1}{2}\int_0^{\pi} ds \exp\left[-2i\frac{\Lambda}{\eta}\sin s\right]$.

This result shows that the matrix $\vec{\vec{W}}_1(t)$, and consequently the corrections generated to the matrix $\vec{\vec{W}}(t)$, are of the order of magnitude γ/Λ smaller than the leading term. For the general non resonant case, it is also clear that the corrections generated by $\vec{\vec{W}}_1(t)$ could be obtained in a similar form. In this general case, Λ will be substitute by $\bar{\epsilon}$, which is also of the order of Λ for a detuning of the same order of magnitude than Λ, $\Delta\omega \approx \Lambda$.

Using relation (2.1.10), the final form for the matrix $\vec{\vec{U}}(t, t')$ which represents a time translation from t' to t can be obtained,

$$\vec{\vec{U}}(t, t') \equiv \vec{\vec{U}}(t) \vec{\vec{U}}^{-1}(t') = e^{-\frac{4}{3}\gamma(t-t')} \vec{\vec{U}}_0(t) \vec{\vec{W}}_0(t) \vec{\vec{W}}_0^{-1}(t') \vec{\vec{U}}_0^{-1}(t') \tag{2.1.21}$$

where the initial condition has been translated from $t = 0$ to $t = t'$. At this point, it is clear that $\vec{\vec{U}}(t', t') = \vec{\vec{I}}$.

The above equation presents the time evolution matrix $\vec{U}(t,t')$ in terms of the different matrixes calculated in this section. This matrix, $\vec{U}(t,t')$, solves the dynamic problem represented by the matrix (2.1.1), $\vec{M}(t)$, and within the range of the approximations performed we will find in the next section the solutions of equations (1.4.11) and (1.4.15) of Chapter 1.

2.2 - Solution for the Evolution of the Vector $\vec{S}$.

The solution of equation (1.4.11) and (1.4.15) can be easily obtained in terms of the solution of equation (2.1.3) given by relation (2.1.21), of the previous section.

In fact, the homogeneous solution of equation (1.4.11) is simply $\vec{S}(t) = \vec{U}(t,t')\vec{S}(t')$ and simultaneously the homogeneous solution of equation (1.4.15) is $\vec{A}_j(t,t') = \vec{U}(t,t')\vec{A}_j(t',t')$, where t' is some initial time at which the local expectation value $\vec{S}(t')$ and the non-local expectation value $\vec{A}_j(t,t')$ are known.

Since, the operator for the time evolution of the system satisfies,

$$\vec{U}(t,t') = \vec{U}(t,t'')\vec{U}(t'',t') \tag{2.2.1}$$

the homogeneous solution of the equations (1.4.11) and (1.4.15) can be calculated from the knowledge of the vectors $\vec{S}(t)$ and $\vec{A}_j(t,t')$ at any other time.

Other expected properties of the matrix $\vec{U}(t,t')$ that can be immediately verified from relation (2.1.21) are:

$$\vec{U}(t',t') = \vec{I} \tag{2.2.2}$$

and

$$\vec{U}^{-1}(t,t') = \vec{U}(t',t) \quad . \tag{2.2.3}$$

The complete solution of the time evolution equations for the expectation values $\vec{S}(t) = \langle \vec{\sigma}(t) \rangle$ and $\vec{A}_j(t,t') = \langle \vec{\sigma}(t)\sigma_j(t') \rangle$ are:

$$\dot{\vec{S}}(t) + \vec{M}(t)\vec{S}(t) = 2\gamma\hat{a}_z \tag{1.4.11}$$

$$\vec{S}(t) = \vec{U}(t,0)\vec{S}(0) + 2\gamma\int_0^t ds\vec{U}(t,s)\hat{a}_z \tag{2.2.4}$$

and

$$\dot{\vec{A}}_j(t,t') + \vec{M}(t)\vec{A}_j(t,t') = 2\gamma\hat{a}_z S_j(t') \tag{1.4.15}$$

$$\vec{A}_j(t,t') = \vec{U}(t,t')\vec{A}_j(t',t') + 2\gamma\int_{t'}^t ds\vec{U}(t,s)\hat{a}_z S_j(t') \tag{2.2.5}$$

where $\vec{A}_j(t',t') = \langle\vec{\sigma}(t')\sigma_j(t')\rangle$ or $A_{ij}(t',t') = \delta_{ij} + i\epsilon_{ij\mu}S_k(t')$.

In the limit of large t the first term of (2.2.4) is negligible due to the over-all exponential factor of the matrix $\vec{U}(t,0)$ defined in (2.1.21). Therefore, the details of the initial conditions of the system are irrelevant for studying the long time behavior of the atoms interacting with a standing wave laser.

In any case, the initial state of the system in the Dirac representation can be written as:

$$|\psi\rangle = c_0|u_0,0\rangle + c_1|u_1,0\rangle, \tag{2.2.6}$$

where $|\psi\rangle$ represent a general state at $t=0$, that satisfies the initial condition imposed on the fluorescent photons; i. e., there are no fluorescent photons at $t=0$, vacuum state. Here, $|u_{\alpha(1)},0\rangle$ is the direct product between either of the atomic states, ground or exited state, with the vacuum state of the fluorescent photons.

In the Dirac representation and the Schroedinger picture for a time $t \neq 0$ the previous state, $|\psi\rangle$, has a complicated representation due to the different fluorescent photons states [1,31].

The initial state, $|\psi\rangle$, in the matrix representation mapped by the unitary transformation defined in the previous chapter, (1.1.6) becomes,

$$U^{-1}|\psi\rangle = \begin{pmatrix} c_0 e^{-i\omega t/2} \\ c_1 e^{i\omega t/2} \end{pmatrix} \tag{2.2.7}$$

from where the initial conditions for the expectation values $S_x(0)$, $S_y(0)$ and $S_z(0)$ can be directly obtained,

$$\begin{aligned} S_x(0) &= c_1 c^*_0 e^{i\omega t} + c^*_1 c_0 e^{-i\omega t} \\ S_y(0) &= -i\left[c_1 c^*_0 e^{i\omega t} - c^*_1 c_0 e^{-i\omega t}\right] \\ S_z(0) &= -(|c_1|^2 - |c_0|^2) \end{aligned} \tag{2.2.8}$$

and clearly,

$$S_x^2(0) + S_y^2(0) + S_z^2(0) = (|c_1|^2 + |c_0|^2)^2 = 1 \tag{2.2.9}$$

the last equality is a consequence of the normalization of (2.2.6).

For our purpose, we can choose arbitrarily c_0 and c_1. Let us fix $c_0 = 1$ and $c_1 = 0$, which means that the atom encounters the laser beam initially in the ground state. This initial condition means $S_i(0) = \delta_{iz}$. As it was mentioned below equation (2.2.5), the long term behavior of the atomic states interacting with the laser field is independent of the initial state of the atom.

Because of the properties of the matrix $\vec{U}(t,t')$ given between equations (2.2.1) and (2.2.3), the expectation value $\vec{S}(t)$ might be written in terms of the expectation value at $t = t'$ as

$$\vec{S}(t) = \vec{U}(t,t')\vec{S}(t') + 2\gamma\int_{t'}^{t} ds\vec{U}(t,s)\hat{u}_z \quad . \tag{2.2.10}$$

For the calculation of the diffusion tensor one of the important magnitudes is $\mathrm{Re}A_{ij}(t,t') - S_i(t)S_j(t')$ which becomes in terms of the solutions for the differential equations (2.2.4) and (2.2.5),

$$\begin{aligned} \mathrm{Re}A_{ij}(t,t') - S_i(t)S_j(t') &= \mathrm{Re}\left\{\vec{U}_{ik}(t,t')A_{kj}(t',t') + 2\gamma\int_{t'}^{t} ds\vec{U}_{iz}S_j(t')\right\} \\ &- \vec{U}_{ik}(t,t')S_k(t')S_j(t') - 2\gamma\int_{t'}^{t} ds\vec{U}_{iz}(t,s)S_j(t') \quad . \end{aligned} \tag{2.2.11}$$

The explicit representation of the matrix $\vec{U}(t,t')$, see below at equation (2.2.13),

clearly shows that $\vec{U}(t,t')$ and $\vec{S}(t)$ are real functions. Therefore, relation (2.2.11) can be reduced to

$$\mathrm{Re}A_{ij}(t,t') - S_i(t)S_j(t') = \vec{U}_{ij}(t,t') - \vec{U}_{ik}(t,t')S_k(t')S_j(t') \tag{2.2.12}$$

Up to this point the explicit form of the matrix $\vec{U}(t,t')$ given in (2.1.21) is written as

$$\vec{U}(t,t') = \frac{e^{-\frac{4}{3}\gamma(t-t')}}{\epsilon(t)\epsilon(t')} \begin{vmatrix} \alpha(t)\alpha(t')e^{2\zeta(t,t')} + \Delta\omega^2 e^{-\zeta(t,t')}\cos\xi(t,t') & \\ \epsilon(t)\Delta\omega e^{-\zeta(t,t')}\sin\xi(t,t') & \\ \Delta\omega[\alpha(t')e^{2\zeta(t,t')} - \alpha(t)e^{-\zeta(t,t')}\cos\xi(t,t')] & \end{vmatrix}$$

$$\begin{vmatrix} -\Delta\omega\epsilon(t')e^{-\zeta(t,t')}\sin\xi(t,t') & \Delta\omega[\alpha(t)e^{2\zeta(t,t')} - \alpha(t')\epsilon - \zeta(t,t')\cos\xi(t,t')] \\ \epsilon(t)\epsilon(t')e^{-\zeta(t,t')}\cos\xi(t,t') & -\epsilon(t)\alpha(t')e^{-\zeta(t,t')}\sin\xi(t,t') \\ \epsilon(t')\alpha(t)e^{2\zeta(t,t')}\sin\xi(t,t') & \Delta\omega^2 e^{2\zeta(t,t')} + \alpha(t)\alpha(t')e^{-\zeta(t,t')}\cos\xi(t,t') \end{vmatrix} \tag{2.2.13}$$

where we have defined:

$$\zeta(t,t') = \frac{\gamma}{3}\int_{t'}^{t} ds\ q(s) \tag{2.2.14}$$

with $q(s)$ defined in relation (2.1.13); and $\xi(t,t')$ defined by

$$\xi(t,t') = \int_{t'}^{t} ds\ \epsilon(s) \tag{2.2.15}$$

with $\epsilon(s)$, the Rabi frequency, given in relation (2.1.6) of the previous section.

The matrix (2.2.13) is reduced to a more workable expression when the fast oscillating terms, proportional to either $\cos\xi(t,t')$ or $\sin\xi(t,t')$, are neglected. As was discused before, the function $\xi(t,t')$ is an oscillating function of t and t' at a frequency of the order $\bar{\epsilon}$, where $\bar{\epsilon}$ is the time average value of $\epsilon(t)$ defined in (2.1.13) [20]. We proved in the previous section that these terms contribute with corrections of the order of γ/Λ compared to the leading term. Therefore, again $\cos\xi(t,t')$ and $\sin\xi(t,t')$ are rapidly oscillating functions which can be eliminated. The matrix $\vec{U}(t,t')$ takes the simpler form:

$$\vec{U}(t,t') = \frac{e^{\frac{-4}{3}\gamma(t-t')}}{\epsilon(t)\epsilon(t')} e^{2\zeta(t,t')}\vec{V}(t,t') \tag{2.2.16}$$

with

$$\vec{V}(t,t') = \begin{vmatrix} \alpha(t)\alpha(t') & 0 & \Delta\omega\alpha(t) \\ 0 & 0 & 0 \\ \Delta\omega\alpha(t') & 0 & \Delta\omega^2 \end{vmatrix} . \qquad (2.2.17)$$

According with the discussion between equation (2.2.5) and (2.2.10), now it is clearly seen that the first term of (2.2.4), associated with the solution of the homogeneous part of the differential equation (1.4.11), can be neglected when the time is large compare with the natural decay time γ^{-1}, due to the exponential factor present in relation (2.2.16).

Therefore, for a large time, such that $\gamma t \gg 1$, the solution (2.2.4) of the matrix differential equation becomes:

$$\vec{S}(t) \approx 2\gamma \int_0^t ds \vec{U}(t,s)\hat{a}_z \quad . \qquad (2.2.18)$$

Moreover, equation (2.2.12) can also be reduced when $\gamma t \gg 1$

$$\mathrm{Re}A_{ij}(t,t') - S_i(t)S_j(t') = \vec{U}_{ij}(t,t') \qquad (2.2.19)$$
$$-4\gamma^2 \int_0^t ds \vec{U}_{iz}(t,s) \int_0^{t'} ds \vec{U}_{jz}(t',s')$$

the last relation is obtained substituting (2.2.18) directly into relation (2.2.11).

In presenting relations (2.2.18) and (2.2.19) we have reached the main results of this section. Commencing with them and with the matrix (2.2.16) we can calculate both the classical force, obtained in the previous chapter (1.4.6), and the quantum fluctuations of the force represented by the diffusion tensor (1.5.6) also obtained in the previous chapter.

2.3 - Solution for the Integrals Involved in the Calculation of $\vec{S}(t)$ and $\vec{A}_j(t,t')$ in the Limit $\gamma \ll \eta$.

In this section, the solutions for the primary integrals involved in the calculation of the force and diffusion tensor near the limit $\gamma/\eta \ll 1$ shall be presented. It is also indicated the starting point for the calculation of these two magnitudes, force and diffusion tensor, in the opposite limit, $\eta/\gamma \ll 1$.

The expectation value of the force, equation (1.4.6), evaluated at the classical trajectory of the atom, $\eta t = \vec{k}_L \cdot \vec{\rho}$, leads to a position and velocity dependent force. We are assuming that γt is much greater than one such that any term proportional to $e^{-\gamma t}$ can be neglected in either limits, $\eta/\gamma \ll 1$ or $\gamma/\eta \ll 1$. In the present case, the parameters η and γ will be related by $\eta \gg \gamma$. The velocity dependence of the force will be manifested through the Doppler shift parameter $\eta = \vec{k}_L \cdot \vec{v}$.

After evaluating at $\eta t = \vec{k}_L \cdot \vec{\rho}$, there is not explicit time dependence in the classical force and diffusion tensor. However, there is dependence with respect to both the atom position, $\vec{\rho}$, and the atom velocity, $\vec{v}$, for the force and diffusion tensor.

The force will be obtained from relation (1.4.6) of the previous chapter. Considering that there is not external interaction with the atom other than the laser field and that $\alpha_{01} = \alpha_{10}$ relation (1.4.6) is reduced to:

$$\vec{F}(t)\,|_{\eta t = \vec{k}_L \cdot \vec{\rho}} = -\frac{1}{2} S_x(t)\, \vec{\nabla}_{\vec{\rho}}\, \Lambda \cos \vec{k}_L \cdot \vec{\rho}\,|_{\eta t = \vec{k}_L \cdot \vec{\rho}} \,, \tag{2.3.1}$$

and the diffusion tensor presented in equation (1.5.6) with $\alpha(t) = \Lambda \cos \eta t$, see discussion below equation (2.1.2), becomes

$$\begin{aligned} D_{ij}(t)\,|_{\eta t = \vec{k}_L \cdot \vec{\rho}} = -\frac{1}{8} \Big| \Lambda \vec{k}_{Lj}\, \vec{\nabla}_i \Lambda \cos \vec{k}_L \cdot \vec{\rho} + \Lambda \vec{k}_{Li}\, \vec{\nabla}_j \Lambda \cos \vec{k}_L \cdot \vec{\rho} \Big| \times \\ \int_0^t dt' \sin \eta t' \, [\mathrm{Re} A_{xx}(t, t') - S_x(t) S_x(t')]\,|_{\eta t = \vec{k}_L \cdot \vec{\rho}} \\ + \frac{1}{2}[1 - S_z(t)] \int_0^t dt'\, Z_{ij}(t - t', 0)\,|_{\eta t = \vec{k}_L \cdot \vec{\rho}} \,. \end{aligned} \tag{2.3.2}$$

In the last two equations, $\vec{S}(t)$ and $\mathrm{Re} A_{xx}(t, t') - S_x(t) S_x(t')$ are given by relations (2.2.18) and (2.2.19), respectively. On the other hand, the last integral appearing in the previous equation, (2.3.2), can be solved using the fact that $Z_{ij}(t - t', 0)$, defined in relation (1.3.7), is a very sharp function of $t - t'$.

At this point, it is convenient to separate the diffusion tensor in two parts. The first part is associated with the stimulated absorption and emission of photons by the atomic levels, and the second part related to the spontaneous emission of the atom.

Making this separation, we write:

$$D_{ij}(\vec{k}_L \cdot \vec{\rho}) = D_{ij}^{(1)}(\vec{k}_L \cdot \vec{\rho}) + D_{ij}^{(2)}(\vec{k}_L \cdot \vec{\rho}) \tag{2.3.3}$$

where (2.3.4)

$$D_{ij}^{(1)}(\vec{k}_L \cdot \vec{\rho}) = \frac{1}{8}\Lambda^2 \vec{k}_{Li}\, \vec{k}_{Lj} \sin\vec{k}_L \cdot \vec{\rho} \int_0^t dt' \sin\eta t' \left[\mathrm{Re}A_{xx}(t,t') - S_x(t) S_x(t')\right]\Big|_{\eta = \vec{k}_L \cdot \vec{\rho}}$$

and (2.3.5)

$$D_{ij}^{(2)}(\vec{k}_L \cdot \vec{\rho}) = \frac{1}{2}[1 - S_z(t)]\Big|_{\eta = \vec{k}_L \cdot \vec{\rho}} \int_0^t dt'\, Z_{ij}(t-t',0)\Big|_{\eta = \vec{k}_L \cdot \vec{\rho}} .$$

The last integral in (2.3.5) is solved using relation (1.3.7) of Chapter 1. The result of the time integral of $Z_{ij}(t-t',0)$ is proportional to $\delta(\omega_k - \omega_L)$, which means that the leading spontaneous emission effect is emission of fluorescent photons with a frequency equal to the laser frequency. In obtaining relation (1.3.7) very small effects of the order v/c were neglected, see Section 1.2. It has been proved that the fluorescent spectrum can have peaks at $\omega_k = \omega_L + \vec{k}_L \cdot \vec{v} + \vec{k}_L \cdot \vec{v} n$, where n is an integer [34]; however, as it was mentioned above, for the present calculation this effect is very small. Substituting this result in relation (2.3.5), we obtain,

$$D_{ij}^{(2)}(\vec{k}_L \cdot \vec{\rho}) = \frac{\gamma k_L^2}{20\pi}(2\delta_{ij} - \hat{E}_i \hat{E}_j)[1 - S_z(t)] . \tag{2.3.6}$$

The part of the diffusion tensor represented in equation (2.3.6), $D_{ij}^{(2)}$, is proportional to γ which is half of Einstein's A coefficient. Therefore, it is associated with the spontaneous emission of photons from the atom. In fact, reviewing the derivations done in the previous chapter $Z_{ij}(t-t',0)$ accounts for the radiation field. On the other hand, the part of the diffusion tensor given in equation (2.3.3), $D_{ij}^{(1)}$, is related with the emission and absorption of laser photons. This part of the diffusion tensor is proportional to $\vec{k}_{Li}\, \vec{k}_{Lj}$ with $\vec{k}_{Li}$ being the laser photon wave vector. This clear separation between the two effects, spontaneous emission and induced emission-absorption, will allow us to compare their relative effect on the atomic motion.

The generalization, for an arbitrary time [28], of equation (2.2.9) bounds the maximum value possible for the component of the vector $\vec{S}(t)$. In our case, for example, restrict

the value of $S_z(t)$ to be confined between -1 and 1. It follows that equation (2.3.6) is a positive definite matrix; when the diffusion tensor is considered a matrix in the index i and j.

Another form of obtaining this result is as follows: since the matrixes $\sigma_i(t)$ are Hermitian; then, $(\sigma_i(t)-<\sigma_i(t)>)^2$ is a positive definite matrix for each i. Consequently, the expectation value $<[\sigma_i(t)-<\sigma_i(t)>]^2> \geq 0$, from where, $<\sigma_i(t)\sigma_i(t)> \geq S_i^2(t)$ for each i. But the square of any of the Pauli matrixes is the identity matrix which together with the normalization of the state, involved in the calculation of the expectation value, implies that $1 \geq S_i^2(t)$. This relation in conjunction with $S_i(t)$ being real functions confines each of the components of the vector $S_i(t)$ to the range $-1 \leq S_i \leq 1$.

Let us indicate that looking at the equations (2.2.17) and (2.2.18) of the previous section, it is found that $S_y(t) = 0$.

The final result for equation (2.3.1), representing the force, and (2.3.6), being the part of the diffusion tensor associated with the fluorescence, is obtained after the integrals represented in relation (2.2.18), of the previous section, are evaluated. These integrals and the integral involved in the calculation of (2.3.4) are:

$$S_x(t) = 2\gamma\int_0^t ds\tilde{U}_{xx}(t,s) = \frac{2\gamma\alpha(t)\Delta\omega}{\epsilon(t)}\int_0^t \frac{ds}{\epsilon(s)} e^{-\frac{4}{3}\gamma(t-s)} e^{2\zeta(t,s)}, \tag{2.3.7}$$

$$S_z(t) = 2\gamma\int_0^t ds\tilde{U}_{zz}(t,s) = \frac{2\gamma\Delta\omega^2}{\epsilon(t)}\int_0^t \frac{ds}{\epsilon(s)} e^{-\frac{4}{3}\gamma(t-s)} e^{2\zeta(t,s)} \tag{2.3.8}$$

and

$$I \equiv \int_0^t dt' \sin\eta t' \left| \tilde{U}_{xx}(t,t') - 4\gamma^2\int_0^{t'} ds\tilde{U}_{xx}(t,s)\int_0^{t'} ds\tilde{U}_{xx}(t',s') \right| . \tag{2.3.9}$$

To solve these integrals we will follow the method introduced by Fiordilino and Mittleman [20].

We begin by introducing a Fourier series for the periodic function contained in the integrand. The function $\zeta(t,t')$ defined in (2.1.14) is:

$$2\zeta(t,s) = \frac{\gamma}{3\eta}\left[\int_{\eta s}^{\pi} d\theta[2q(\theta) - 2\bar{q}] + \int_{\eta s}^{\pi} d\theta\, 2\bar{q}\right] \tag{2.3.10}$$

where $\bar{q}$ is the average of $q(\theta)$ in one period. This average is defined in the usual form

$$2\bar{q} = \frac{1}{\pi}\int_{0}^{2\pi} d\theta\, q(\theta) = 1 - \frac{3}{\lambda}$$

with λ a positive defined magnitude defined as

$$\lambda = \left[1 + \frac{\Lambda^2}{\Delta\omega^2}\right]^{1/2} = \frac{\epsilon(0)}{|\Delta\omega|} \ . \tag{2.3.11}$$

The Fourier series for the integrand of relations (2.3.8) and (2.3.9) for an arbitrary α is:

$$\frac{e^{\frac{\gamma}{3\eta}\int_{\alpha}^{\pi} d\theta[2q(\theta)-2\bar{q}]}}{\epsilon(\alpha)}\begin{bmatrix}\sin\alpha\cos\alpha\\ \cos\alpha\\ 1\end{bmatrix} = \sum_{n=-\infty}^{\infty}\begin{bmatrix}X_n(\eta t,\sin\cos)\\ X_n(\eta t,\cos)\\ X_n(\eta t,1)\end{bmatrix} e^{-in\alpha} \tag{2.3.12}$$

with (2.3.13)

$$\begin{bmatrix}X_n(\eta t,\sin\cos)\\ X_n(\eta t,\cos)\\ X_n(\eta t,1)\end{bmatrix} = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\frac{e^{in\theta}}{\epsilon(\theta)}\begin{bmatrix}\sin\theta\cos\theta\\ \cos\theta\\ 1\end{bmatrix} e^{\frac{\gamma}{3\eta}\int_{0}^{\pi} d\theta'[2q(\theta')-2\bar{q}]}$$

where, the notation means the corresponding X_n representing each one of the different functions at the left hand side of the equation (2.3.12), one for $\sin\alpha\cos\alpha$, the other for $\cos\alpha$, and the bottom line multiplied only by the factor one. In other words, as the equality between two vectors.

The Fourier series of the integrand can be defined since the left hand side of (2.3.12) is a periodic function on both arguments, α and ηt. In fact, the integral $\int_{\alpha}^{\pi} d\theta[2q(\theta)-2\bar{q}]$ remains unchanged when either of the limits is translated by a factor of 2π. Therefore, after considering the long time behavior of the vector $\vec{S}(t)$ and $\vec{A}_j(t,t')$, the variation of $\eta t = \vec{k}_L \cdot \vec{p}$ is confined to only one period. Any result valid for the region of one period, can be extended periodically along the total region of interaction between the atoms and the laser field.

Therefore, using the above series expansion for the integrand of relations (2.3.7) and (2.3.8), they can be re-written as:

$$\begin{vmatrix} S_x(t) \\ S_z(t) \end{vmatrix} = \frac{2\gamma}{\eta\epsilon(t)} \begin{vmatrix} \Delta\omega\Lambda\cos\eta t \\ \Delta\omega^2 \end{vmatrix} \sum_{n=-\infty}^{\infty} \int_0^{\pi} d\alpha X_n(\eta t, 1) e^{\left| -\frac{\gamma}{\eta}(1+\frac{1}{\lambda})(\eta t - \alpha) \right|} e^{-in\alpha} . \tag{2.3.14}$$

The next step in the calculation of relation (2.3.14) follows after neglecting terms proportional to $e^{-\frac{\gamma}{\eta}(1+\frac{1}{\lambda})\eta t}$, for a time large enough such that $\gamma t \gg 1$, and then substituting back into equation (2.3.14) the Fourier coefficients, $X_n(\eta t, 1)$, as defined in relation (2.3.13). After these calculations the component of the vector $\vec{S}(t)$ are

$$\begin{vmatrix} S_x(t) \\ S_z(t) \end{vmatrix} = \frac{2\gamma}{\eta\epsilon(t)} \begin{vmatrix} \Delta\omega\Lambda\cos\eta t \\ \Delta\omega^2 \end{vmatrix} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi\epsilon(\theta)} e^{\frac{\gamma}{3\eta}\int_{\theta}^{\eta t} d\theta' [2\eta(\theta') - 2\bar{\eta}]} \sum_{n=-\infty}^{\infty} \frac{e^{-inx}}{[a - in]} \tag{2.3.15}$$

where $x = x(\theta) = \eta t - \theta$ and $a = \frac{\gamma}{\eta}(1 + 1/\lambda)$.

The sum, appearing in equation (2.3.15), can be evaluated by adding a regularization parameter ν to the integral representation of the ratio $\frac{e^{-inx}}{[a - in]}$ in such a way that the sum, s, becomes:

$$s = \sum_{n=-\infty}^{\infty} \frac{e^{-inx}}{a - in} = \lim_{\nu \to 0} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\tau e^{-\tau(a - in)} e^{-inx} e^{-\nu|n|} . \tag{2.3.16}$$

Assuming that the previous sum, integral, and limit can be interchanged the sum s is reduced to:

$$s = \int_0^{\infty} d\tau e^{-a\tau} \lim_{\nu \to 0} \frac{(1 - e^{-\nu}) e^{\nu/2}}{(1 - e^{-\nu})^2 e^{\nu} + 4\sin^2 \tfrac{1}{2}(x - \tau)} = \pi \int_0^{\infty} d\tau e^{-a\tau} \delta[2\sin \tfrac{1}{2}(\tau - x)] .$$

The argument of the $\delta-$ function vanishes for $\tau = x + 2m\pi$ since, $\tau \geq 0$ the integer m has to be positive or zero. The periodicity of the integrand of equation (2.3.15) implies that $-\pi < x < \pi$. The previous constraint, $-\pi < x < \pi$, requires that the minimal value of m, for which the argument of the $\delta-$ function vanishes, to be separated in two groups; one for

$x<0$, $m_{min}=1$ and one for $x\geq 0$, $m_{min}=0$. The final result for s is then,

$$s=\sum_{m=m_{min}}^{\infty}\pi\int_0^{\infty}d\tau\,\left|\frac{d}{d\tau}2\sin^{1}\!/_{2}(\tau-x)\right|^{-1}e^{-a\tau}\begin{vmatrix} m_{min}=1 & x<0\\ m_{min}=0 & x>0\end{vmatrix}$$

$$s=\frac{\pi e^{-ax}}{1-e^{-a\pi}}\begin{vmatrix} e^{-a\pi} & x<0\\ 1 & x>0\end{vmatrix}. \qquad (2.3.17)$$

This result for the sum s is the same as the result first presented by Fiordilino and Mittleman [20]. Substituting the previous result for the sum s into the integral (2.3.15) we are left with the following expression for $S_x(t)$ and $S_z(t)$,

(2.3.18)

$$\begin{vmatrix} S_x(t)\\ S_z(t)\end{vmatrix}=\frac{\gamma}{\eta\epsilon(t)}\begin{vmatrix}\Delta\omega\Lambda\cos\eta t\\ \Delta\omega^2\end{vmatrix}\int_{-\pi}^{\pi}\frac{d\theta}{\epsilon(\theta)}e^{\frac{\gamma}{3\eta}\int_{\theta}^{\pi}d\theta'[2q(\theta')-2\bar{q}]}\frac{e^{-a(\eta t-\theta)}}{1-e^{-a\pi}}\begin{vmatrix} e^{-a\pi} & \theta>\eta t\\ 1 & \theta<\eta t\end{vmatrix}$$

In arriving to these results, we have not compared the relative magnitude between the parameter γ with η and in principle this is a general result with respect to the ratio γ/η. Now, the calculation of the force and diffusion tensor in both limits $\gamma/\eta\gg 1$ and $\eta/\gamma\gg 1$ can be derived from equation (2.3.18).

Between equations (2.1.4) and (2.1.8), the kind of approximations being done for solving the differential equation for the time evolution of the vector $\vec{S}(t)$ has been discussed. There, the relations used among the physical parameters are: high laser intensity regime, $\Lambda\gg\gamma$, and very fast atoms are discarded, $\Lambda\gg\eta$. Furthermore, fast oscillating terms were neglected. Therefore, except for these restrictions equation (2.3.18) is a general result.

Presently, we are studying atoms with a relatively high speed, in which case $\gamma\ll\eta\ll\Lambda$. The actual numerical values [35,36] used for a typical sodium atom are $\gamma\sim 5\times10^{7}\ sec^{-1}$, $\eta\sim 5\times10^{8}\ sec^{-1}$ and $\Lambda\sim 2\times10^{9}\ sec^{-1}$.

The next step for the calculation of $\vec{S}(t)$ is to calculate the integral contained in the integrand of relation (2.3.18); we present this result in the following way:

$$-a(\eta t-\theta)+\frac{\gamma}{3\eta}\int_{\theta}^{\eta t}d\theta'[2q(\theta')-2\bar{q}]=-\frac{\gamma}{\eta}\phi(\eta t)+\frac{\gamma}{\eta}\phi(\theta)\ , \qquad (2.3.19)$$

where we have defined

$$\phi(\theta)= \theta-\frac{1}{\lambda}\arctan(\lambda\cot\theta) \ , \tag{2.3.20}$$

which is an odd function of θ .

Therefore, the solution for the vector $\vec{S}(t)$ is represented as

$$\begin{vmatrix} S_x(t) \\ S_z(t) \end{vmatrix}= \frac{\gamma}{\eta\epsilon(t)} \begin{vmatrix} \Delta\omega\Lambda\cos\eta t \\ \Delta\omega^2 \end{vmatrix} \frac{e^{-\frac{\gamma}{\eta}\phi(\pi)}}{1-e^{-\alpha\pi}} \left| \int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} e^{\frac{\gamma}{\eta}\phi(\theta)} + e^{-\alpha\pi}\int_{\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} e^{\frac{\gamma}{\eta}\phi(\theta)} \right| . \tag{2.3.21}$$

The expression (2.3.21) can be used for calculating both limits, $\gamma/\eta \gg 1$ and $\gamma/\eta \ll 1$. In this work, the calculation for the limit $\gamma/\eta \ll 1$ is presented. The calculation in the limit $\gamma/\eta \gg 1$ will be presented in the next section .

Again, we evaluate the atomic motion at the classical trajectory defined by $\vec{k}_L \cdot\vec{p}=\eta t$; and as it was explained before, the product $\vec{k}_L \cdot\vec{p}$ is constrained to one period, $-\pi < \vec{k}_L \cdot\vec{p} < \pi$, after the long time behavior approximation has been done, $\gamma t \gg 1$. The previous substitution replaces the time dependence of the atomic motion by a dependence in both the atomic position and the atomic velocity.

Looking at relation (2.3.21), we can see that the force and the diffusion tensor, in general, have a very complex velocity and position dependence in the region where both γ and η are smaller than Λ.

An expansion in a power series in the parameter γ/η is necessary for further developments. In fact, no analytic solution of the integral appearing in (2.3.21) is yet known. Keeping up to order γ/η in the series expansion of the integral (2.3.21), the result for the vector $\vec{S}(\vec{k}_L \cdot\vec{p})$ is reduced to:

$$\begin{vmatrix} S_x(\vec{k}_L\cdot\vec{\rho}) \\ S_z(\vec{k}_L\cdot\vec{\rho}) \end{vmatrix} = \frac{1}{\pi\epsilon(\vec{k}_L\cdot\vec{\rho})} \begin{vmatrix} \Delta\omega\Lambda\cos\vec{k}_L\cdot\vec{\rho} \\ \Delta\omega^2 \end{vmatrix} \frac{1}{1+1/\lambda} \left| \int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} + \right.$$

$$\frac{\gamma}{\eta} \left| -\phi(\vec{k}_L\cdot\vec{\rho}) \int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} + \frac{\pi}{2}(1+1/\lambda) \int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \right. \tag{2.3.22}$$

$$\left. \left. -\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)}\phi(\theta) - \pi(1+1/\lambda) \int_{\vec{k}_L\cdot\vec{\rho}}^{\pi} \frac{d\theta}{\epsilon(\theta)} \right| \right| .$$

To obtain the final result we use the solutions for the integrals given in Appendix A, in particular we use equations (A.3), (A.5), (A.6). The non zero component of the vector $\vec{S}(\vec{k}_L\cdot\vec{\rho})$ at first order in the parameter γ/η are,

$$\begin{vmatrix} S_x(\vec{k}_L\cdot\vec{\rho}) \\ S_z(\vec{k}_L\cdot\vec{\rho}) \end{vmatrix} = \frac{4K\left(\frac{|\Lambda|}{\epsilon(0)}\right)}{\pi\epsilon(0)\epsilon(\vec{k}_L\cdot\vec{\rho})(1+1/\lambda)} \begin{vmatrix} \Delta\omega\Lambda\cos\vec{k}_L\cdot\vec{\rho} \\ \Delta\omega^2 \end{vmatrix} \left| 1+\frac{\gamma}{\eta} \left| \frac{\pi}{2}(1+\frac{1}{\lambda}) \right. \right.$$

$$\left. \left. -\phi(\vec{k}_L\cdot\vec{\rho}) + \frac{\pi}{4}(1+\frac{1}{\lambda}) \frac{\mathbb{K}\left(\vec{k}_L\cdot\vec{\rho}-\pi\,|\,\frac{|\Lambda|}{\epsilon(0)}\right)}{K\left(\frac{|\Lambda|}{\epsilon(0)}\right)} \right| \right| , \tag{2.3.23}$$

where $K\left(\frac{|\Lambda|}{\epsilon(0)}\right)$ and $\mathbb{K}\left(\vec{k}_L\cdot\vec{\rho}-\pi\,|\,\frac{|\Lambda|}{\epsilon(0)}\right)$ are the complete elliptic integral of the first type and the incomplete elliptic integral of the first type, respectively [37].

The last calculation in this section is the evaluation of the integral (2.3.9) which is needed for the evaluation of the diffusion tensor,

$$I = I_1 + I_2 = \frac{\Lambda^2\cos\eta t}{\epsilon(t)} \int_0^t \frac{dt'}{\epsilon(t')} \sin\eta t' \cos\eta t' e^{\frac{-4}{3}\gamma(t-t')} e^{2\zeta(t,t')} \tag{2.3.24}$$

$$-\frac{\Delta\omega^2\Lambda^2}{\epsilon(t)}\cos\eta t \;\; 4\gamma^2 \int_0^t \frac{dt'}{\epsilon(t')} \sin\eta t' \cos\eta t' \int_0^{t'} \frac{ds}{\epsilon(s)} e^{-\frac{4}{3}\gamma(t-s)} e^{2\zeta(t,s)} \int_0^{t'} \frac{ds'}{\epsilon(s')} e^{-\frac{4}{3}\gamma(t'-s')} e^{2\zeta(t',s')}$$

The integral I_1 of the first line of equation (2.3.24), has been solved when $\vec{S}(t)$ was calculated. The result in the present case is:

(2.3.25)

$$I_1 = \frac{\Lambda^2 \cos\eta t}{2\eta\epsilon(t)} \frac{e^{-\frac{\gamma}{\eta}\phi(\eta t)}}{(1-e^{-\pi\frac{\gamma}{\eta}})} \left[\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\, e^{\frac{\gamma}{\eta}\phi(\theta)} + e^{-\pi\frac{\gamma}{\eta}} \int_{\eta t}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\, e^{\frac{\gamma}{\eta}\phi(\theta)} \right]$$

In this expression, we have not yet made the expansion in power of $\gamma/\eta \ll 1$. As in relation (2.3.21), this integral is a general result. When this expansion is done, we have to keep terms up to the order γ^2/η^2 in the integrals since (2.3.25) is not proportional to γ/η as in the case of the vector $\vec{S}$. In any case, the result for I_1 will be shown with terms proportional to the first order in γ/η only, all other higher order terms will be neglected. Expanding the integrals appearing in (2.3.25) in power series, we obtain

$$\begin{aligned} I_1 = \frac{\Lambda^2 \cos\vec{k}_L\cdot\vec{\beta}}{2\eta\epsilon(\vec{k}_L\cdot\vec{\beta})\pi(1+1/\lambda)} \Bigg[& \int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\,\phi(\theta) - \pi(1+1/\lambda) \int_{\vec{k}_L\cdot\vec{\beta}}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta \\ & - \frac{\gamma}{\eta} \Bigg[[\phi(\vec{k}_L\cdot\vec{\beta}) - \frac{\pi}{2}(1+1/\lambda)] \int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\,\phi(\theta) - \\ & \phi(\vec{k}_L\cdot\vec{\beta})\,\pi(1+1/\lambda) \int_{\vec{k}_L\cdot\vec{\beta}}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta + \pi(1+1/\lambda) \int_{\vec{k}_L\cdot\vec{\beta}}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\,\phi(\theta) \Bigg] \Bigg] , \end{aligned} \tag{2.3.26}$$

where the remaining integrals are given in Appendix A, see relations (A.7), (A.8), and (A.9). After substituting these integrals in the previous expression, we obtain:

$$\begin{aligned} I_1 = \frac{\cos\vec{k}_L\cdot\vec{\beta}}{2\eta\epsilon(\vec{k}_L\cdot\vec{\beta})\pi(1+1/\lambda)} \Bigg[& -\pi(1+1/\lambda)[\epsilon(\vec{k}_L\cdot\vec{\beta}) + \epsilon(0)] + 4\epsilon(0)E\left(\frac{|\Lambda|}{\epsilon(0)}\right) + \\ & \frac{4\Delta\omega^2}{\epsilon(0)} K\left(\frac{|\Lambda|}{\epsilon(0)}\right) + \frac{\gamma}{\eta} \Bigg[\pi(1+1/\lambda)\epsilon(0)[\phi(\vec{k}_L\cdot\vec{\beta}) + \pi(1+\frac{1}{2\lambda})] - \\ & 4[\phi(\vec{k}_L\cdot\vec{\beta}) - \frac{\pi}{2}(1+1/\lambda)] \left[\epsilon(0)E\left(\frac{|\Lambda|}{\epsilon(0)}\right) + \frac{\Delta\omega^2}{\epsilon(0)} K\left(\frac{|\Lambda|}{\epsilon(0)}\right) \right] + \\ & \pi(1+1/\lambda) \left[\epsilon(0)\mathbb{E}\left(\vec{k}_L\cdot\vec{\beta} - \pi\,|\,\frac{|\Lambda|}{\epsilon(0)}\right) + \frac{\Delta\omega^2}{\epsilon(0)} \mathbb{K}\left(\vec{k}_L\cdot\vec{\beta} - \pi\,|\,\frac{|\Lambda|}{\epsilon(0)}\right) \right] \Bigg] \Bigg] \end{aligned} \tag{2.3.27}$$

The next calculation is for the second part of I , I_2 where I_2 corresponds to the second line of equation (2.3.24). The result for I_2 after following a similar procedure as the

used for calculating (2.3.16) is

$$I_2 = -\frac{\Delta\omega^2\Lambda^2}{\eta^3\epsilon(t)}\cos\eta t \;\; 4\gamma^2\int_{-\pi}^{\pi}\frac{d\theta}{2\pi\epsilon(\theta)}\sum_{\substack{n=-\infty\\ m=-\infty\\ l=-\infty}}^{\infty}\frac{X_m(\eta t,1)e^{in\theta}}{(a-im)(a-in)}e^{-a\eta t}\times$$

$$X_l(\theta,\sin\cos)\int_0^{\eta t}dx e^{-ilx}\left[e^{(a-im)x}-1\right]\left[e^{-inx}-e^{-ax}\right] \tag{2.3.28}$$

where X is defined in equation (2.3.13). The previous integral, (2.3.28), in the limit ηt much larger than one, becomes

$$I_2 = -\frac{\Delta\omega^2\Lambda^2}{\eta^3\epsilon(t)}\cos\eta t \;\; \frac{4\gamma^2}{8\pi^3}\int_{-\pi}^{\pi}\frac{d\theta}{\epsilon(\theta)}\int_{-\pi}^{\pi}\frac{d\alpha}{\epsilon(\alpha)}e^{\frac{\gamma}{3\eta}\int_a^{\eta t}ds\,[2\eta(s)-2\bar{\eta}]}\times \tag{2.3.29}$$

$$\int_{-\pi}^{\pi}\frac{d\beta}{\epsilon(\beta)}\sin\beta\cos\beta\, e^{\frac{\gamma}{3\eta}\int_0^{\beta}ds'\,[2\eta(s')-2\bar{\eta}]}\sum_{\substack{n=-\infty\\ m=-\infty\\ l=-\infty}}^{\infty}\frac{e^{i(m\alpha+n\theta+l\beta)}e^{-i(m+n+l)\eta t}}{(a-im)(a-in)[a-i(m+n+l)]}\,,$$

where any term proportional to $e^{-\eta t}$ has been dropped. Here, a is defined as in relation (2.3.15), $a=\frac{\gamma}{\eta}(1+1/\lambda)$.

In a form similar to the sum (2.3.16), the previous sum can be solved. Its result is:

$$S_3 = \sum_{\substack{n=-\infty\\ m=-\infty\\ l=-\infty}}^{\infty}\frac{e^{i(m\alpha+n\theta+l\beta)}e^{-i(l+m+n)\eta t}}{(a-im)(a-in)[a-i(l+m+n)]} \tag{2.3.30}$$

$$= \frac{\pi^3}{(1-e^{-a\pi})^3}e^{-a(\eta t-\beta)}e^{-a(\beta-\theta)}e^{-a(\beta-\alpha)}\begin{cases}e^{-a\pi} & \beta>\eta t\\ 1 & \beta<\eta t\end{cases}\begin{cases}e^{-a\pi} & \theta>\beta\\ 1 & \theta<\beta\end{cases}\begin{cases}e^{-a\pi} & \alpha>\beta\\ 1 & \alpha<\beta\end{cases}$$

The result for the integral I_2 previous to the expansion in power series for either of the limits $\gamma/\eta \ll 1$ or $\eta/\gamma \ll 1$ is shown below. To obtain this result, we use the solution for the integral appearing in the integrand of relation (2.3.29) given at the relation (2.3.19). Therefore,

$$I_2 = -\frac{\Delta\omega^2\Lambda^2\cos\vec{k}_L\cdot\vec{\rho}}{2\eta^3\epsilon(\vec{k}_L\cdot\vec{\rho})(1-e^{-2\pi})^3}\gamma^2 e^{-\frac{\gamma}{\eta}\phi(\vec{k}_L\cdot\vec{\rho})}\int_{-\pi}^{\pi}\frac{d\beta}{\epsilon(\beta)}\sin\beta\cos\beta e^{-\frac{\gamma}{\eta}\phi(\beta)}\times \tag{2.3.31}$$

$$\left|\begin{matrix}1 & \beta<\vec{k}_L\cdot\vec{\rho}\\ e^{-2\pi} & \beta>\vec{k}_L\cdot\vec{\rho}\end{matrix}\right|\int_{-\pi}^{\pi}\frac{d\alpha}{\epsilon(\alpha)}e^{\frac{\gamma}{\eta}\phi(\alpha)}\left|\begin{matrix}1 & \alpha<\beta\\ e^{-2\pi} & \alpha>\beta\end{matrix}\right|\int_{-\pi}^{\pi}\frac{d\theta}{\epsilon(\theta)}e^{\frac{\gamma}{\eta}\phi(\theta)}\left|\begin{matrix}1 & \theta<\beta\\ e^{-2\pi} & \theta>\beta\end{matrix}\right|$$

Again, the previous result is general with respect to the ratio γ/η and we now evaluated it for the limit $\gamma/\eta \ll 1$

$$I_2 = \frac{\Delta\omega^2\cos\vec{k}_L\cdot\vec{\rho}}{\eta\epsilon(\vec{k}_L\cdot\vec{\rho})\epsilon(0)\pi^3(1+1/\lambda)^3}K\left(\frac{|\Lambda|}{\epsilon(0)}\right)\left|-2\pi^2(1+1/\lambda)+\frac{2}{\epsilon(0)}K\left(\frac{|\Lambda|}{\epsilon(0)}\right)\right|\pi(1+1/\lambda)\times$$

$$[\epsilon(\vec{k}_L\cdot\vec{\rho})-\epsilon(0)]+4\epsilon(0)E\left(\frac{|\Lambda|}{\epsilon(0)}\right)+\frac{4\Delta\omega^2}{\epsilon(0)}K\left(\frac{|\Lambda|}{\epsilon(0)}\right)\Bigg|- \tag{2.3.32}$$

$$\frac{\gamma}{\eta}\Bigg|\pi^2(1+1/\lambda)^2\vec{k}_L\cdot\vec{\rho}-2\pi^2(1+1/\lambda)\phi(\vec{k}_L\cdot\vec{\rho})-\pi^2(1+1/\lambda)^2\frac{\epsilon(\vec{k}_L\cdot\vec{\rho})}{\epsilon(0)}\mathbb{K}\left(\vec{k}_L\cdot\vec{\rho}-\pi\,|\,\frac{|\Lambda|}{\epsilon(0)}\right)$$

$$+\frac{2}{\epsilon(0)}K\left(\frac{|\Lambda|}{\epsilon(0)}\right)\Bigg|-2\pi\epsilon(0)\left(1+\frac{1}{4\lambda}\right)-\epsilon(0)\pi(1+1/\lambda)\left[\phi(\vec{k}_L\cdot\vec{\rho})-\frac{3\pi}{2}(1+1/\lambda)\right]+$$

$$\left[1+\phi(\vec{k}_L\cdot\vec{\rho})-\frac{3\pi}{2}(1+1/\lambda)\right]\Bigg|\epsilon(\vec{k}_L\cdot\vec{\rho})\pi(1+1/\lambda)+4\epsilon(0)E\left(\frac{|\Lambda|}{\epsilon(0)}\right)+\frac{4\Delta\omega^2}{\epsilon(0)}K\left(\frac{|\Lambda|}{\epsilon(0)}\right)\Bigg|+$$

$$\epsilon(\vec{k}_L\cdot\vec{\rho})\phi(\vec{k}_L\cdot\vec{\rho})-\frac{\Delta\omega^2}{\epsilon(0)}\mathbb{K}\left(\vec{k}_L\cdot\vec{\rho}-\pi\,|\,\frac{|\Lambda|}{\epsilon(0)}\right)-\epsilon(0)\mathbb{E}\left(\vec{k}_L\cdot\vec{\rho}-\pi\,|\,\frac{|\Lambda|}{\epsilon(0)}\right)\Bigg|\Bigg|\Bigg|\Bigg|$$

The previous result is obtained after some extended algebra and using the solutions for the integrals given in appendix A.

After calculating I_1 and I_2 we can join these two results to show the solution of the integral I necessary for the calculation of $D_{ij}^{(1)}$ of the diffusion tensor. Therefore, replacing the solutions (2.3.27) and (2.3.32) into (2.3.24) we obtain:

$$I = \frac{\cos\vec{k}_L\cdot\vec{\beta}}{2\eta\epsilon(\vec{k}_L\cdot\vec{\beta})\pi(1+1/\lambda)}\left\{-\pi(1+1/\lambda)[\epsilon(0)+\epsilon(\vec{k}_L\cdot\vec{\beta})]+4\epsilon(0)E\left[\frac{|\Lambda|}{\epsilon(0)}\right]+\frac{4\Delta\omega^2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right\}$$
$$+\frac{\Delta\omega^2\cos\vec{k}_L\cdot\vec{\beta}}{\eta\epsilon(\vec{k}_L\cdot\vec{\beta})\epsilon(0)\pi^3(1+1/\lambda)^3}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\left\{-2\pi^2(1+1/\lambda)+\frac{2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right\}\left\{\pi(1+1/\lambda)\times\right.$$
$$\left.[\epsilon(\vec{k}_L\cdot\vec{\beta})-\epsilon(0)]+4\epsilon(0)E\left[\frac{|\Lambda|}{\epsilon(0)}\right]+\frac{4\Delta\omega^2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right\}+ \qquad (2.3.33)$$
$$\frac{\gamma}{\eta}\frac{\cos\vec{k}_L\cdot\vec{\beta}}{2\eta\epsilon(\vec{k}_L\cdot\vec{\beta})\pi(1+1/\lambda)}\left\{\pi(1+1/\lambda)\,\epsilon(0)\,[\phi(\vec{k}_L\cdot\vec{\beta})+\pi(1+\frac{1}{2\lambda})]-\right.$$
$$4[\phi(\vec{k}_L\cdot\vec{\beta})-\frac{\pi}{2}(1+1/\lambda)]\left\{\epsilon(0)E\left[\frac{|\Lambda|}{\epsilon(0)}\right]+\frac{\Delta\omega^2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right\}+$$
$$\pi(1+1/\lambda)\left\{\epsilon(0)E\left[\vec{k}_L\cdot\vec{\beta}-\pi\,\frac{|\Lambda|}{\epsilon(0)}\right]+\frac{\Delta\omega^2}{\epsilon(0)}K\left[\vec{k}_L\cdot\vec{\beta}-\pi\,\frac{|\Lambda|}{\epsilon(0)}\right]\right\}\Big\}-$$
$$\frac{\gamma}{\eta}\frac{\Delta\omega^2\cos\vec{k}_L\cdot\vec{\beta}}{\eta\,\epsilon(\vec{k}_L\cdot\vec{\beta})\,\epsilon(0)\pi^3(1+1/\lambda)^3}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\left\{\pi^2(1+1/\lambda)^2\,\vec{k}_L\cdot\vec{\beta}-\right.$$
$$2\pi^2(1+1/\lambda)\phi(\vec{k}_L\cdot\vec{\beta})-\pi^2(1+1/\lambda)^2\frac{\epsilon(\vec{k}_L\cdot\vec{\beta})}{\epsilon(0)}K\left[\vec{k}_L\cdot\vec{\beta}-\pi\,\frac{|\Lambda|}{\epsilon(0)}\right]+\frac{2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\times$$
$$\left\{-2\pi\epsilon(0)\left[1+\frac{1}{4\lambda}\right),-\epsilon(0)\pi(1+1/\lambda)[\phi(\vec{k}_L\cdot\vec{\beta})-\frac{3\pi}{2}(1+1/\lambda)]+[1+\phi(\vec{k}_L\cdot\vec{\beta})-\frac{3\pi}{2}(1+1/\lambda)]\times\right.$$
$$\left\{\epsilon(\vec{k}_L\cdot\vec{\beta})\,\pi(1+1/\lambda)+4\epsilon(0)E\left[\frac{|\Lambda|}{\epsilon(0)}\right]+\frac{4\Delta\omega^2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right\}+\epsilon(\vec{k}_L\cdot\vec{\beta})\phi(\vec{k}_L\cdot\vec{\beta})-$$
$$\left.\left.\frac{\Delta\omega^2}{\epsilon(0)}K\left[\vec{k}_L\cdot\vec{\beta}-\pi\,\frac{|\Lambda|}{\epsilon(0)}\right]-\epsilon(0)E\left[\vec{k}_L\cdot\vec{\beta}-\pi\,\frac{|\Lambda|}{\epsilon(0)}\right]\right\}\right\}$$

For our practical applications, we neglect the order γ/η, in calculations involving the diffusion tensor. After the power series expansion of the integrals involved in the calculation of the force and diffusion tensor in the case $\gamma/\eta \ll 1$, the only velocity dependence is in the integral I through the factor $1/\eta$. The previous calculation requires that the atomic velocity component in the direction of the wave vector $\vec{k}_L$ can not be small, because the previous expansion is not in the ratio γ/v but in the parameter $\gamma/\vec{k}_L\cdot\nabla$ which depends upon the velocity component in the $\vec{k}_L$ direction.

2.4 - The Force and Diffusion Tensor in the Limits $\gamma \ll \eta$ and $\eta \ll \gamma$.

For the complete evaluation of the force and the diffusion tensor in the limit

$\gamma/\eta \ll 1$ we need only to substitute the result of the integrals, (2.3.23), and (2.3.33) into the equation (2.3.1), (2.3.4) and (2.3.6) corresponding to $\vec{F}(\vec{k}_L \cdot \vec{\rho})$, $D_i\}^{(1)}(\vec{k}_L \cdot \vec{\rho})$ and $D_i\}^{(2)}(\vec{k}_L \cdot \vec{\rho})$ respectively.

In this section, we will drop the terms proportional to γ/η. For a sodium atom coming out of an oven, i. e., with a thermal velocity of about $v = 2\times10^5$ cm/ sec is equivalent to neglect terms of the order $\gamma/\eta \sim .25$. This number is obtained assuming that the component of the velocity in the direction of the wave vector $\vec{k}_L$ can be reduced to $v_{\vec{k}_L} \sim 2\times10^3$ cm/ sec and considering that the optic active levels are associated to a transition with wave vector $k_L \sim 10^5$ cm^{-1}. Thus, we obtain for the Doppler shift parameter the value $\eta \sim 2\times10^8$ sec^{-1} and using for a sodium atom that $\gamma \sim 5\times10^7$ sec^{-1} [36] we have that the ratio $\gamma/\eta \sim .25 < 1$.

For our calculations, in the present section, we do not use higher velocities than $\vec{v} \sim 2\times10^5$cm/ sec since the maximum value of the parameter η is restricted by the upper limit condition $\eta \ll \Lambda$. The present highest laser intensities, of the order of 10 W / cm^2, $\Lambda \sim 2\times10^9$ sec^{-1} ; for which the actual ratio between η and Λ is $\eta/\Lambda \sim 10^{-1}$. As higher laser intensities becomes accessible, the present calculations will be applicable to a wider range of atomic velocities.

When the ratio γ/η is neglected the force , $\vec{F}(\vec{k}_L \cdot \vec{\rho})$, and the diffusion tensor part corresponding to the fluorescence emission, $D_i\}^{(2)}$, becomes velocity independent. Therefore, the only term that has a velocity dependence belongs to the diffusion tensor part corresponding to the stimulated emission and absorption of laser photons, $D_i\}^{(1)}$. Here, $\vec{F}(\vec{k}_L \cdot \vec{\rho})$ and $D_i\}^{(2)}$ are velocity independent in the range determined by $\gamma \ll \eta \ll \Lambda$.

Therefore, in the limit $\gamma \ll \eta \ll \Lambda$ the force acting on the atom due to the interaction with the laser field is obtained substituting the solution (2.3.23) into relation (2.3.1),

$$\vec{F}(\vec{k}_L \cdot \vec{\rho}) = -\frac{2\Delta\omega}{\pi\epsilon(0)(1+1/\lambda)} K\left|\frac{|\Lambda|}{\epsilon(0)}\right| \vec{\nabla}\epsilon(\vec{k}_L \cdot \vec{\rho}) \quad ; \qquad (2.4.1)$$

here, the terms proportional to the ratio γ/η has been dropped.

Looking at equation (2.4.1) we see that the force is an odd function of the detuning parameter $\Delta\omega$; in fact, this result is valid independent of the present approximations. The proportionality of the force to $\Delta\omega$ comes from the original differential equation defined in equation (1.4.12) of the previous chapter. We also note that $\vec{F}(\vec{k}_L \cdot \vec{p})$ can be either a damping force or an accelerating force depending on both the sign of the detuning $\Delta\omega$ and in the standing wave phase $\vec{k}_L \cdot \vec{p}$.

If we keep the first order in η/γ the previous result is:

$$\vec{F}(\vec{k}_L \cdot \vec{p}) = -\frac{2\Delta\omega}{\pi\epsilon(0)(1+1/\lambda)} K\left[\frac{|\Lambda|}{\epsilon(0)}\right] [\vec{\nabla}\epsilon(\vec{k}_L \cdot \vec{p})] \left\{ 1 + \frac{\gamma}{\eta} \left[-\phi(\vec{k}_L \cdot \vec{p}) + \frac{\pi}{4}(1+1/\lambda) \frac{K\left[\vec{k}_L \cdot \vec{p} \left| \frac{|\Lambda|}{\epsilon(0)} \right.\right]}{K\left[\frac{|\Lambda|}{\epsilon(0)}\right]} \right] \right\} \tag{2.4.2}$$

We present this result to compare with the final force calculated by Fiordilino-Mittleman [20] acting on the atom due to the interaction with the standing wave laser field. The result obtained in their work was calculated in a different gauge where the leading interacting term between the laser and the atom is proportional to $\vec{p} \cdot \vec{A}$. Here, $\vec{p}$ is the electronic momentum and $\vec{A}$ is the vector potential corresponding to the electromagnetic field. In the present calculation the leading interacting term is $\vec{\chi} \cdot \vec{E}$; where, $\vec{\chi}$ is the position for the electron and $\vec{E}$ is the electromagnetic field. As expected from gauge invariance both results are the same.

A more detail description of the force dependences will be presented in the next chapter, in connection to the approximated solutions of the Fokker-Plank equation for the motion of the atoms.

Other calculation completed in this work is the expression for the quantum fluctuations of the force, the so-called diffusion tensor. As in the previous section it is con-

venient to work with two separate part of the diffusion tensor, one associated with the stimulated absorption and emission of photons and a second part associated with the atoms fluorescence. For the limit $\gamma \ll \eta \ll \Lambda$ the diffusion tensor part associated with spontaneous emission, $D_i{}_j^{(2)}$ is:

$$D_i{}_j^{(2)}(\vec{k}_L \cdot \vec{\beta}) = \frac{\gamma k_L^2}{20\pi}(2\delta_{ij} - \hat{E}_i \hat{E}_j)\left\{1 - \frac{4\Delta\omega^2}{\pi\epsilon(0)\epsilon(\vec{k}_L \cdot \vec{\beta})(1+1/\lambda)} K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right\}\left\{1 + \frac{\gamma}{\eta}\left[-\phi(\vec{k}_L \cdot \vec{\beta}) + \frac{\pi}{4}(1+1/\lambda)\frac{K\left[\vec{k}_L \cdot \vec{\beta}\,\frac{|\Lambda|}{\epsilon(0)}\right]}{K\left[\frac{|\Lambda|}{\epsilon(0)}\right]}\right]\right\} \tag{2.4.3}$$

where, after neglecting terms of the order γ/η is reduced to,

$$D_i{}_j^{(2)}(\vec{k}_L \cdot \vec{\beta}) = \frac{\gamma k_L^2}{20\pi}(2\delta_{ij} - \hat{E}_i \hat{E}_j)\left\{1 - \frac{4\Delta\omega^2}{\pi\epsilon(0)\epsilon(\vec{k}_L \cdot \vec{\beta})(1+1/\lambda)} K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right\} \tag{2.4.4}$$

Let us mention here, that this term is maximum for a null detuning in comparison with the force which is zero for $\Delta\omega = 0$. As the force, in the present limit $\gamma/\eta \ll 1$, this term is independent of the atomic velocity. More exactly, we found that the dependence of the force $\vec{F}$ and $D_i{}_j^{(2)}$ with the velocity is very small for a velocity in the range satisfying the relation $\gamma \ll \vec{k}_L \cdot \vec{v} \ll \Lambda$. Again, using $\gamma \sim 5\times10^7 sec^{-1}$, $\Lambda \sim 2\times10^9 sec^{-1}$, and $k_L \sim 10^5 cm^{-1}$ the previous range is $5\times10^2 cm/sec \ll v_{\vec{k}_L} \ll 2\times10^4 cm/sec$.

The part of the diffusion tensor associated with the absorption and emission of laser photons, $D_i{}_j^{(1)}$, can be written as,

$$D_i{}_j^{(1)}(\vec{k}_L \cdot \vec{\beta}, \vec{k}_L \cdot \vec{v}) = \frac{1}{8}\Lambda^2 \vec{k}_{Li}\,\vec{k}_{Lj} \sin\vec{k}_L \cdot \vec{\beta} \times I \tag{2.4.5}$$

where I is given in relation (2.3.33) of the previous section. We will not attempt to write it here again. From now on, we will neglect γ/η terms. In such case, the previous equation is:

$$D_{ij}^{(1)}(\vec{k}_L\cdot\vec{\rho},\vec{k}_L\cdot\vec{v}) = -\frac{1}{32}\left[\vec{k}_{Li}\,\vec{\nabla}_j\,\epsilon(\vec{k}_L\cdot\vec{\rho}) + \vec{k}_{Lj}\,\vec{\nabla}\,\epsilon(\vec{k}_L\cdot\vec{\rho})\right]\frac{1}{\eta\pi(1+1/\lambda)}\times$$
$$\left\{-\pi(1+1/\lambda)[\epsilon(0)+\epsilon(\vec{k}_L\cdot\vec{\rho})] + 4\epsilon(0)E\left[\frac{|\Lambda|}{\epsilon(0)}\right] + \frac{4\Delta\omega^2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right] + \right.$$
$$\frac{2\Delta\omega^2}{\pi^2(1+1/\lambda)^2\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\left[-2\pi^2(1+1/\lambda) + \frac{2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\left[\pi(1+1/\lambda)[\epsilon(\vec{k}_L\cdot\vec{\rho}) - \epsilon(0)] + \right.\right.$$
$$\left.\left.\left. 4\epsilon(0)E\left[\frac{|\Lambda|}{\epsilon(0)}\right] + \frac{4\Delta\omega^2}{\epsilon(0)}K\left[\frac{|\Lambda|}{\epsilon(0)}\right]\right]\right]\right\} \qquad (2.4.6)$$

This term shows a very complicated dependence in $\vec{k}_L\cdot\vec{\rho}$ as in the ratio $\Delta\omega^2/\Lambda^2$. The analysis of this dependence will be left for the next chapter.

The second part of this section, and last part of this chapter, will present the force and diffusion tensor for the opposite limit than the one shown before, i. e., $\eta/\gamma \ll 1$. This limit include very slow atoms, but near atom trapping conditions this limit is the relevant one. In this limit the calculations can include the extreme case of atoms at rest.

In this case, the calculation of $S_x(t)$ and $S_z(t)$ are made starting from equation (2.3.21). The most important part of the calculation is carried out using relation (2.1.19) for the asymptotic expansion of the integrals involved in the solution of (2.3.21). In this limit $\eta \ll \gamma$ the second part of the integral (2.3.21) does not contribute to the final result because of the over-all exponential factor $e^{-\frac{\gamma}{\eta}(1+1/\lambda)\pi}$. Therefore, in the limit $\eta \ll \gamma \ll \Lambda$ we have that the force is

$$\vec{F}(\vec{k}_L\cdot\vec{\rho}) = -\frac{\Delta\omega}{4}\left[1 - \frac{\eta}{2\gamma}\frac{\Lambda^4\cos^2\vec{k}_L\cdot\vec{\rho}\sin 2\vec{k}_L\cdot\vec{\rho}}{(\epsilon^2(\vec{k}_L\cdot\vec{\rho}) + \Delta\omega^2)^2}\right]\vec{\nabla}\ln\left[\frac{\Delta\omega^2 + \epsilon^2(\vec{k}_L\cdot\vec{\rho})}{\epsilon^2(0)}\right] \qquad (2.4.7)$$

which is the same obtained by Fiordilino-Mittleman [20] and Gordon-Ashkin [4]. Using the same procedure than the used for the calculation of the force in the present limit, the second part of the diffusion tensor is:

(2.4.8)

$$D_{ij}^{(2)}(\vec{k}_L\cdot\vec{\rho},\eta)=\frac{\gamma k_L^2}{20\pi}(2\delta_{ij}-\hat{E}_i\hat{E}_j)\frac{\epsilon^2(\vec{k}_L\cdot\vec{\rho})}{\Delta\omega^2+\epsilon^2(\vec{k}_L\cdot\vec{\rho})}\left|1+\frac{\eta}{\gamma}\frac{\Delta\omega^2\Lambda^4\cos^2\vec{k}_L\cdot\vec{\rho}\sin 2\vec{k}_L\cdot\vec{\rho}}{2\epsilon^2(\vec{k}_L\cdot\vec{\rho})[\epsilon^2(\vec{k}_L\cdot\vec{\rho})+\Delta\omega^2]^2}\right|$$

In this case, the integral I_1 of equation (2.3.25) calculated under the limit $\eta \ll \gamma$ becomes,

$$I_1=\frac{\Lambda^2\cos\vec{k}_L\cdot\vec{\rho}}{4\gamma[\epsilon^2(\vec{k}_L\cdot\vec{\rho})+\Delta\omega^2]}\left|\sin 2\vec{k}_L\cdot\vec{\rho}-\frac{2\eta}{\gamma[\epsilon^2(\vec{k}_L\cdot\vec{\rho})+\Delta\omega^2]}\times\right. \tag{2.4.9}$$
$$\left.\left|\epsilon^2(\vec{k}_L\cdot\vec{\rho})\cos 2\vec{k}_L\cdot\vec{\rho}+\frac{\Lambda^4\cos^2\vec{k}_L\cdot\vec{\rho}\sin 2\vec{k}_L\cdot\vec{\rho}}{4[\epsilon^2(\vec{k}_L\cdot\vec{\rho})+\Delta\omega^2]}\right|\right|$$

The solution of the integral (2.3.31), I_2 in the present limit $\eta/\gamma \ll 1$ is more complicated than the previous three integrals shown above. However, the calculation for this integral are reduced by noting that the factor $e^{-\frac{\gamma}{\eta}(1+1/\lambda)\pi}$ eliminates the contribution to I_2 coming from the single integrals after their discontinuity, see (2.3.31). Using the previous property, the solution of (2.3.31) is reduced to the simple evaluation of the expression:

$$I_2=-\frac{\Delta\omega^2\Lambda^2\cos\vec{k}_L\cdot\vec{\rho}}{2\eta\epsilon(\vec{k}_L\cdot\vec{\rho})}\left|1-\frac{\eta}{\gamma}\frac{\Lambda^4\cos^2\vec{k}_L\cdot\vec{\rho}\sin 2\vec{k}_L\cdot\vec{\rho}}{[\epsilon^2(\vec{k}_L\cdot\vec{\rho})+\Delta\omega^2]^2}\right|e^{-\frac{\gamma}{\eta}\phi(\vec{k}_L\cdot\vec{\rho})} \tag{2.4.10}$$
$$\times\int_{-\pi}^{\vec{k}_L\cdot\vec{\rho}}\frac{d\theta\,\epsilon(\theta)\sin\theta\cos\theta}{[\epsilon^2(\theta)+\Delta\omega^2]^2}e^{\frac{\gamma}{\eta}\phi(\theta)}$$

Making the remaining integral in (2.4.10), and then together with the solution of I_1 it is easy to show that for the limit $\eta \ll \gamma \ll \Lambda$ the part of the diffusion tensor associated with the stimulated emission and absorption of laser photons becomes:

$$D_{ij}^{(1)} = \frac{\Lambda^4 \vec{k}_{Li}\,\vec{k}_{Lj}\sin^2 2\vec{k}_L\cdot\vec{\beta}}{64\gamma[\epsilon^2(\vec{k}_L\cdot\vec{\beta})+\Delta\omega^2]^3}\left|3\Delta\omega^4+3\Delta\omega^2\Lambda^2\cos^2\vec{k}_L\cdot\vec{\beta}+\Lambda^4\cos^4\vec{k}_L\cdot\vec{\beta}\right| \qquad (2.4.11)$$

$$-\frac{\eta}{\gamma^2}\frac{\Lambda^4\vec{k}_{Li}\,\vec{k}_{Lj}\sin 2\vec{k}_L\cdot\vec{\beta}}{32[\epsilon^2(\vec{k}_L\cdot\vec{\beta})+\Delta\omega]^4}\left\{\epsilon^2(\vec{k}_L\cdot\vec{\beta})\cos 2\vec{k}_L\cdot\vec{\beta}\left|3\Delta\omega^2\epsilon^2(\vec{k}_L\cdot\vec{\beta})+\Lambda^4\cos^4\vec{k}_L\cdot\vec{\beta}\right|+\right.$$

$$\left.\frac{1}{4}\Lambda^4\cos^2\vec{k}_L\cdot\vec{\beta}\sin 2\vec{k}_L\cdot\vec{\beta}\left|\epsilon^2(\vec{k}_L\cdot\vec{\beta})+\Delta\omega^2\right|+\frac{\Delta\omega^2\Lambda^2\epsilon^2(\vec{k}_L\cdot\vec{\beta})\sin^2 2\vec{k}_L\cdot\vec{\beta}\left|3\epsilon^2(\vec{k}_L\cdot\vec{\beta})-\Delta\omega^2\right|}{\epsilon^2(\vec{k}_L\cdot\vec{\beta})+\Delta\omega^2}\right\}$$

These results complete the calculation of the force and diffusion tensor for an atom interacting with a standing wave laser. One of the main results of this work are the one contain in this section. The next chapter will deal with the application of this result to the study of the motion of an atomic beam interacting with a high intensity standing wave laser field.

The results presented here in either of the limits $\gamma/\eta \ll 1$ or $\eta/\gamma \ll 1$ both the force and the diffusion tensor have a phase dependence $\vec{k}_L\cdot\vec{\beta}$ which actually depend on the location of the atom into the laser beam. The work of the Gordon-Ashkin [4] analyzes this problem only in the limit $\eta/\gamma \ll 1$. In the present calculations, the results obtained for the force and diffusion tensor in the limit $\eta/\gamma \ll 1$ are equivalent to the result of G-A [4]. In the opposite limit, as was mention before, the force is equivalent to the result of Fiordilino-Mittleman [20].

3.1 - Analysis of the Diffusion Tensor and Force. The Fokker-Planck Equation.

The first section of this chapter introduces a different notation that presents a more workable form of the force and diffusion tensor in terms of dimensionless quantities. At the end of this section we present the Fokker-Plank equation for the distribution function associated with the atoms interacting with the laser beam.

We start by changing to a dimensionless parameter, $w = \Delta\omega / \Lambda$ that represents the ratio between the laser detuning $\Delta\omega$, from the atomic transition, and the energy coupling energy Λ. The range of variation of w is determined below. For any of the successive calculations it will be considered that $\Lambda \equiv |\Lambda|$.

The possible restrictions on the variation of w , due to the approximations used for the calculations of the previous chapter, may come from the relation $S_i(\theta, w) \leq 1$. However, as will be shown later in this section, the previous condition does not impose any extra restriction on the range of variation of w . Nevertheless, most of the analysis refer to the case $\Delta\omega \sim \Lambda$. By its definition, w can be positive, negative, or zero, depending on the value of the detuning $\Delta\omega$. Certainly, Λ can not be considered a small number because for all the cases described in the previous chapter the laser beam was assumed a high intensity laser field with, $\gamma \ll \Lambda$ and $\eta \ll \Lambda$.

The other parameters introduced are $\theta = \vec{k}_L \cdot \vec{p}$ with $-\pi \leq \theta \leq \pi$, see Section 2.3, and $\overline{w} = \frac{1}{(1+w^2)^{1/2}}$ with $\overline{w}$ contained in the range $0 < \overline{w} \leq 1$.

We also re-write the Rabi frequency in terms of a new function defined below,

$$\epsilon(\vec{k}_L \cdot \vec{p}) = \Lambda (1+w^2)^{1/2} \Delta_w(\theta)$$

where it is defined

$$\Delta_w(\theta) = [1 - \overline{w}^2 \sin^2\theta]^{1/2} \qquad (3.1.1)$$

is a function of θ and w with a range of variation in the interval $0 \leq \Delta_w(\theta) \leq 1$. The value $\Delta_w(\theta) = 0$ occurs only when w is zero for the resonant case, at $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$.

In terms of the new notation, we have

$$\epsilon(0) = \Lambda(1+w^2)^{1/2} \quad , \quad \frac{|\Lambda|}{\epsilon(0)} = \bar{w}$$

$$\lambda = \frac{(1+w^2)^{1/2}}{|w|} \tag{3.1.2}$$

$$\left|1+\frac{1}{\lambda}\right| = \frac{\left[|w|+(1+w^2)^{1/2}\right]}{(1+w^2)^{1/2}}$$

and

$$\left|1+\frac{1}{\lambda}\right|^{-1} = (1+w^2)^{1/2}\left|(1+w^2)^{1/2} - |w|\right|$$

First, we check that the values of $S_i(\theta,w)$, as a function of w for any θ, are contained in the interval $(-1,1)$, as required by the general relation $|S_i(\theta,w)| \leq 1$, in both limits $\gamma/\eta \ll 1$ and $\eta/\gamma \ll 1$. The analysis starts with the case $\gamma \ll \eta$. Terms proportional to γ/η are being neglected. In this case, from equations (2.3.6) and (2.4.4), we have that

$$S_z(\theta,w) = \frac{4w^2 K(\bar{w})\left|(1+w^2)^{1/2} - |w|\right|}{\pi(1+w^2)^{1/2}\Delta_w(\theta)} \tag{3.1.3}$$

where $K(\bar{w})$ is the complete elliptic function of the first kind. In the same form, from equation (2.3.1) and (2.4.1) we get

$$S_x(\theta,w) = \frac{4wK(\bar{w})\left|(1+w^2)^{1/2} - |w|\right|\cos\theta}{\pi(1+w^2)^{1/2}\Delta_w(\theta)} \tag{3.1.4}$$

In general, in looking at relation (3.1.3), we see that $S_z(\theta,w)$ is always positive for any real value of w. Also $S_x(\theta,w)$ is the component of the vector $\vec{S}$ appearing in the definition of the $\vec{F}$ and basically represents the force dependence with respect to the parameter w, dimensionless detuning.

The maximum of $S_z(\theta,w)$ as a function of θ is at $\theta = \pm\frac{\pi}{2}$, where

$$S_z^{(max)}(w) = S_z(\pm\pi/2,w) = \frac{4}{\pi}|w|K(\bar{w})\left|(1+w^2)^{1/2} - |w|\right| \tag{3.1.5}$$

The general behavior of $S_z^{(max)}(w)$ as a function of w can be described as follows:

when $w = 0$, $S_z^{(max)}(0) = 0$. For w very large $S_z^{(max)}(w) \underset{w\to\infty}{\to} 1$ asymptotically. Starting from zero S_z grows until it reaches the maximum value of one; in the present case it results that for $w \sim 2$, $S_z^{(max)} \sim 1$. See Figure 1, where $S_z^{(max)}(w)$ is plotted as a function of w.

In the same form, $S_x(\theta,w)$ is maximum at $\theta = 0, \pm\pi$. For this case,

$$S_x^{(max)}(w) = S_x(0,w) = \frac{4wK(\bar{w})\left[(1+w^2)^{1/2} - |w|\right]}{\pi(1+w^2)^{1/2}} \tag{3.1.6}$$

The greater between $S_x^{(max)}$ and $S_z^{(max)}$ is determined from the ratio $|S_z^{(max)}/S_x^{(max)}| = (1+w^2)^{1/2}$. This ratio shows that $|S_z^{(max)}|$ is always larger than $|S_x^{(max)}|$ for any value of the parameter w. Therefore, both $|S_z(\theta,w)|$ and $|S_x(\theta,w)|$ never exceed 1 in the limit $\gamma/\eta \ll 1$.

The different quantities $\vec{F}$, $D_{ij}^{(1)}$ and $D_{ij}^{(2)}$ given in relation (2.4.2), (2.4.4) and (2.4.6) respectively can be written as:

$$\vec{F}(\theta,w) = \vec{k}_L \Lambda \frac{wK(\bar{w})\left[(1+w^2)^{1/2} - |w|\right]}{\pi(1+w^2)^{1/2}\Delta_w(\theta)} \sin 2\theta \tag{3.1.7}$$

$$= -\vec{k}_L \frac{\partial}{\partial\theta}\frac{\Lambda}{\pi} w(1+w^2)^{1/2} K(\bar{w})\left[(1+w^2)^{1/2} - |w|\right]\Delta_w(\theta)$$

$$D_{ij}^{(2)}(\theta,w) = \frac{\gamma k_L^2}{20\pi}(2\delta_{ij} - \hat{E}_i\hat{E}_j)\left[1 - \frac{4w^2K(\bar{w})\left[(1+w^2)^{1/2} - |w|\right]}{\pi(1+w^2)^{1/2}\Delta_w(\theta)}\right] \tag{3.1.8}$$

$$D_{ij}^{(1)}(\theta,w) = \frac{\vec{k}_{Li}\vec{k}_{Lj}\Lambda^2}{8\eta\pi}\frac{\sin 2\theta}{\Delta_w(\theta)}\left[(1+w^2)^{1/2} - |w|\right]\left[-\frac{\pi}{4}\left[(1+w^2)^{1/2} + |w|\right]\left[1+\Delta_w(\theta)\right]\right.$$

$$+ (1+w^2)^{1/2}E(\bar{w}) + \frac{|w|^3K(\bar{w})\left[(1+w^2)^{1/2} - |w|\right]}{(1+w^2)^{1/2}} \tag{3.1.9}$$

$$+ \frac{4}{\pi^2}w^2\left[(1+w^2)^{1/2} - |w|\right]^2K^2(\bar{w})\left[\frac{\pi}{4}\left[(1+w^2)^{1/2} + |w|\right]\left[\Delta_w(\theta) - 1\right]\right.$$

$$\left.\left. + (1+w^2)^{1/2}E(\bar{w}) + \frac{w^2K(\bar{w})}{(1+w^2)^{1/2}}\right]\right]$$

From the previous equations (3.1.7) to (3.1.9) we can also obtain a result for the case in which the laser beam is tuned to resonance with the atomic transition, $\Delta\omega = 0$ or

$w = 0$. This limit is found by using the fact that the Elliptic function of first kind [37] satisfies $\lim_{m \to 1} K(m) = \lim_{m \to 1} \frac{1}{2} \ln \frac{16}{1-m}$. Therefore, we conclude that $wK(\overline{w}) \approx w \ln w \to 0$ for $w \to 0$ for the resonant case. In this case, the force (3.1.7) vanishes. Also for the resonant case we have that the diffusion part coming from the spontaneous emission of fluorescent photons is maximum, $D_i^{(2)}(\theta, 0) = \frac{\gamma k_L^2}{20\pi}(2\delta_{ij} - \hat{E}_i \hat{E}_j)$. The significance of the previous limits is that even when there is no deflection of the atomic beam due to the laser field there is still spreading of the atomic beam as a consequence of the atoms' fluorescence.

Moreover, for a laser beam resonant with the atomic transition there is also spreading due to the diffusion induced by the random absorption and emission of laser photons. In fact, in the resonant regime

$$D_i^{(1)}(\theta) = \frac{\Lambda^2}{\eta} \vec{k}_{Li} \vec{k}_{Lj} \frac{(1-\pi/4)}{4\pi} \frac{\sin\theta \cos\theta}{|\cos\theta|} \left| 1 - \frac{\pi/4}{(1-\pi/4)} |\cos\theta| \right| . \qquad (3.1.10)$$

The non vanishing part of the diffusion tensor, $D_i^{(1)}$, becomes in the resonant case proportional to the spatial derivative of the Rabi frequency. The discontinuities of the diffusion tensor are associated with this derivative of the Rabi frequency. For the resonant case, the Rabi frequency becomes the absolute value of a function that depends on the phase of the standing wave in the different points along a laser wave length. This phase changes from positive to negative every half laser wave length with the corresponding jump in the derivative of the Rabi frequency. We should mention that for the fast-atom limit, if the force would not vanish at $\Delta\omega = 0$, it would present the same behavior as the diffusion tensor in the limit $\Delta\omega = 0$.

This discontinuity does not exist for the case of a real laser because, for a real laser, we can not have a laser frequency width equal to zero. In fact a realistic laser is composed of the superposition of many different modes which is represented by $\sum_i E_i \cos(\omega t + \delta\omega_i t)$.

For the limit Λ very large and $\Delta\omega$ fixes the parameter w is very small, $w \approx 0$. In this case, the force (3.1.7) grows like logarithm of Λ because the relevant factor is

$\Lambda w K(\overline{w}) = \Delta\omega K(\overline{w})$ which behaves as $K(\overline{w})$ near $\overline{w} = 1$ for large Λ.

The diffusion tensor associated with the spontaneous emission saturates for Λ large and it takes its maximum value. In this same limit, $D_i{}_j^{(1)}$ grows as Λ^2 implying that the associated spreading of the atoms also increase for large Λ. These results are also obtained for the standing wave configuration in the limit of atoms moving with very low velocity [4].

For the standing wave laser field, the atom can absorb a laser photon from one of the plane waves and then re-emits a photon in the other mode. This is the random process for which $D_i{}_j^{(1)}$ accounts for. Then the magnitude of $D_i{}_j^{(1)}$ depends on how many times the atoms stimulatively absorb and emit a photon. The number of these oscillations depends on the Rabi frequency which increases with Λ increasing.

It has to be mentioned here that as a consequence of the behavior of $S_z(\theta,w)$ it results that $D_i{}_j^{(2)}$ is always a positive definite magnitude. This is a property that can not be implied from the definition of the diffusion tensor, equation (1.4.2). This is not the behavior of the diffusion tensor part associated with the stimulated emission and absorption of laser photons, $D_i{}_j^{(1)}$. In fact, for the resonant case shown in equation (3.1.10) the diffusion tensor oscillates as a function of θ. A graph of the dimensionless part of $D_i{}_j^{(1)}(\theta,0)$ is presented in Figure 2. For the non-resonant case, $D_i{}_j^{(1)}$ also oscillates as a function of θ, see equation (3.1.9) and Figure 3.

In the limit $\gamma \ll \eta$ we have seen that the diffusion tensor has an oscillatory part, $D_i{}_j^{(1)}$, which for the present case $\gamma \ll \Lambda$ corresponds to the whole diffusion tensor becoming an oscillatory function of the position θ. In Section 3 of this chapter, we demonstrate that this oscillatory behavior of the diffusion tensor implies that there is not spreading of the atomic beam due to the stimulated absorption and emission of laser photons, at least up to third order in the ratio between the laser-atom interaction energy and the initial kinetic energy of the atoms.

The following discussion determines the behavior of $S_i(\theta,w)$ as a function of θ and w for the limit $\eta/\gamma \ll 1$, i. e, considering the case of atoms moving at very slow

velocity. This calculation starts by obtaining $S_z(\theta,w)$ from equation (2.4.8) neglecting the order η/γ, in which case

$$S_z(\theta,w)= \frac{w^2}{2w^2+\cos^2\theta} \quad . \tag{3.1.11}$$

This expression is always smaller than 1 as a function of θ and w reaching the maximum value at $\theta = \pm\pi/2$, where $S_z(\pm\pi,w)= \frac{1}{2}$.

Following equation (2.3.1) and (2.4.7), in the limit $\eta/\gamma \ll 1$, we obtain

$$S_x(\theta,w)= \frac{w\cos\theta}{2w^2+\cos^2\theta} \tag{3.1.12}$$

The extremes of the above expression are located at $\theta = 0, \pm\pi$ and $\cos^2\theta = 2w^2$. The last extreme is a maximum requiring that $w \leq \frac{\sqrt{2}}{2}\cos\theta$. On the other hand however, if $w > \frac{\sqrt{2}}{2}\cos\theta$ the maximum with respect to θ are located at $\theta = 0, \pi$, where the evaluation of (3.1.12) gives $S_x(0,w)= \frac{w}{1+2w^2}$. Note that, this expression has again the maximum at $w=\frac{\sqrt{2}}{2}$ and there $S_x(0,w)|_{w=\sqrt{2}/2}= \frac{\sqrt{2}}{4} < S_z^{(max)}= \frac{1}{2}$. Thus, in the case η/γ, it is also concluded that for any value of w both quantities $S_x(\theta,w)$ and $S_z(\theta,w)$ run into the prescribed limit defined by $|S_i(\theta,w)| \leq 1$.

In terms of the new notation, the force and diffusion tensor for the limit $\eta/\gamma \ll 1$, are

$$\vec{F}(\theta,w)= \frac{\Lambda\vec{k}_L}{4}\frac{w\sin 2\theta}{(2w^2+\cos^2\theta)} \tag{3.1.13}$$

$$= -\vec{k}_L \frac{\partial}{\partial\theta}\frac{\Lambda}{4} w \ln(2w^2+\cos^2\theta)$$

$$D_{ij}^{(2)}(\theta,w)= \frac{\gamma k_L^2}{20\pi}(2\delta_{ij} - \hat{E}_i\hat{E}_j)\frac{(w^2+\cos^2\theta)}{(2w^2+\cos^2\theta)} \tag{3.1.14}$$

$$D_{ij}^{(1)}(\theta,w)= \frac{\Lambda^2\vec{k}_{Li}\vec{k}_{Lj}}{64\gamma}\frac{\sin^2 2\theta\left|\cos^4\theta + 3w^2\cos^2\theta + 3w^4\right|}{\left|2w^2+\cos^2\theta\right|} \tag{3.1.15}$$

The force was obtained from equation (2.4.7) and the diffusion tensor from (2.4.8) and (2.4.11). Even when only the dependence on θ and w for the force and diffusion tensor

is explicitly expressed, they also have dependence in the vector $\vec{k}_L$ and in the parameters γ and Λ. For the limit $\gamma/\eta \ll 1$ neglecting first order in γ/η, there is a dependence on the velocity through the Doppler shift parameter η for the diffusion tensor part associated with the stimulated absorption and emission.

It is important to mention here that in the limit $\eta/\gamma \ll 1$ the two parts of the diffusion tensor $D_i{}^{(1)}_j$ and $D_i{}^{(2)}_j$ are positive defined tensors in the complete range defined between $-\pi$ and π. In this case, the velocity dependence of the diffusion tensor and the force is only through a power series of the ratio γ/η ; which means that at order zero in γ/η there is not velocity dependence. The only other dependence for the force and the diffusion tensor is with respect to the parameters Λ, $\vec{k}_{Li}$ and γ. As in the opposite limit, again, the force is an odd function of the detuning when the diffusion tensor is an even function of w. Thus, the force acting on the atom interacting with the standing wave laser oscillates as a function of the angle θ (i. e. the position in the wave).

For the resonant case, and $\eta \ll \gamma$, we again have that the force is zero. The diffusion part associated with the fluorescent photons becomes the same as the one obtained for the resonant case in the opposite limit $\gamma \ll \eta$.

In the case, $\eta/\gamma \ll 1$ and the laser beam tuned to resonance with the atomic transition, the diffusion tensor $D_i{}^{(1)}_j$ becomes,

$$D_i{}^{(1)}_j(\theta) = \frac{\vec{k}_{Li}\,\vec{k}_{Lj}\,\Lambda^2}{16\gamma}\sin^2\theta \qquad (3.1.16)$$

Most of the works referred in this presentation obtained the same kind of result for the limit $\eta/\gamma \ll 1$ [4,21,22,38].

The final part of this section introduces the Fokker-Planck equation used for the calculation of the position and velocity dependence of the distribution function associated with the motion of atoms inside the region of interaction with the standing wave laser field. The standard [21], see discussion below, form of the Fokker-Planck equation used for this calculations is:

(3.1.17)

$$\frac{\partial}{\partial t}f(\mathbf{x},\mathbf{v},t)+ v_i \frac{\partial}{\partial x_i}f(\mathbf{x},\mathbf{v},t)+ \frac{\partial}{\partial v_i}\frac{F_i(\mathbf{x},\mathbf{v},t)}{M_A}f(\mathbf{x},\mathbf{v},t)= \frac{\partial^2}{\partial v_i \partial v_j}\frac{D_{ij}(\mathbf{x},\mathbf{v},t)}{M_A^2}f(\mathbf{x},\mathbf{v},t)$$

where $F_i(\mathbf{x},\mathbf{v},t)$ is a position $\mathbf{x}$, velocity $\mathbf{v}$, and time dependent force. Here, $D_{ij}(\mathbf{x},\mathbf{v},t)$ is the diffusion tensor and $f(\mathbf{x},\mathbf{v},t)$ represent in our case the atoms distribution function.

The force $\vec{F}$ in the Fokker-Planck equation is the expectation value of the expression (1.4.1) of Chapter 1 and the diffusion tensor D_{ij} is obtained from relation (1.4.2). Nevertheless, these two magnitudes have been simplified throughout the calculations up to the reduced expressions (3.1.7) to (3.1.9) or (3.1.13) to (3.1.15) depending upon which one of the limits are being considered $\gamma/\eta \ll 1$ or $\eta/\gamma \ll 1$.

Many derivations using different approaches to equation (3.1.17) are known. In particular, very general presentations of this derivation are develop by Gardiner [25], Risken [26] and van Kampen [27].

Derivations of equation (3.1.17) with relation to the atom-laser interaction are given by Cook [22] and Cohen-Tannoudji [21]. The Ito-Strotonovich [26,27] ambiguity of the Fokker-Planck equation is discarded using the argument of Javanainen-Stenholm [39] later rediscussed by Stenholm [14]. This ambiguity is associated with the definition of the force term and the corresponding definition of the diffusion tensor term. In other words, the right hand side of equation (3.1.17) can be written as $\frac{\partial}{\partial v_i}D_{ij}(\mathbf{x},\mathbf{v},t)\frac{\partial}{\partial v_j}$; in which case, we have to modify the definition of the force in order to recuperate the form (3.1.17).

In essence, the argument is that in the Fokker-Planck equation, for the atoms interacting with the laser field, the terms involving the force and the diffusion tensor result from a series expansion. The expansion is in terms of the very small parameter $\hbar$ where, the force correspond to order $\hbar$ and the diffusion tensor correspond to order $\hbar^2$ respectively. Therefore, on the one hand, this expansion is being truncated at order $\hbar^2$, diffusion tensor, and on the other hand, adding terms to the force proportional to $\frac{D_{ij}}{M_A}\frac{\partial}{\partial v_i}$ is equivalent to mixing

terms of different order of magnitude which, at the end, do not change the form of the equation but the definition of the force. In fact, the force is derived from the average of a quantum operator, $\vec{\nabla}H$, when D_{ij} is a pure quantum effect because it is related to the quantum fluctuation of this operator.

Changing to standard units, $\hbar \sim 10^{-27}$erg—sec , the force and diffusion tensor, in either of the limits $\gamma \ll \eta$ or $\eta \ll \gamma$, are:

$$D_i^{(1)} = \Lambda^2 \hbar^2 \vec{k}_{Li}\, \vec{k}_{Lj}\, n_1(w,\theta) \times \begin{cases} \dfrac{1}{\eta} & \gamma \ll \eta \\ \dfrac{1}{\gamma} & \eta \ll \gamma \end{cases} \tag{3.1.18}$$

$$D_{ij}^{(2)} = \gamma \hbar^2 k_L^2 (2\delta_{ij} - \hat{E}_i\, \hat{E}_j)\, n_2(w,\theta) \tag{3.1.19}$$

and

$$\vec{F} = \hbar \vec{k}_L \Lambda n_3(w,\theta) \tag{3.1.20}$$

where n_1 , n_2 and n_3 are pure number factor depending on both the dimensionless detuning and the angle θ that can be easily obtained from (3.1.7) to (3.1.9) for the case $\gamma/\eta \ll 1$ or from (3.1.13) to (3.1.15) in the case $\eta/\gamma \ll 1$.

The comparison ratios for a moving atom between the force and the diffusion associated with the atoms' fluorescence is

$$\frac{D^{(2)}}{FM_A v} \approx \left(\frac{n_2}{n_3}\right)\left(\frac{\gamma}{\Lambda}\right)\left(\frac{\hbar k_L}{M_A v}\right) , \tag{3.1.21}$$

which is a very small number when the laser is out of resonance with the atomic transition because in the preceding calculations we have required $\gamma \ll \Lambda$ and, on the other hand, the momentum of the photons are much smaller than the momentum of the atoms. The ratio between the force and the diffusion due to the stimulated emission and absorption is

$$\frac{D^{(1)}}{FM_A v} \approx \left(\frac{n_1}{n_3}\right)\left(\frac{\hbar k_L}{M_A v}\right) \times \begin{cases} \dfrac{\Lambda}{\eta} & \gamma \ll \eta \\ \dfrac{\Lambda}{\gamma} & \eta \ll \gamma \end{cases} . \tag{3.1.22}$$

For this ratio a simple substitution of the typical numerical values of the parameters involved in relation (3.1.22) for a sodium atom leads to

$\frac{D^{(1)}}{FM_A v} \sim 1.25 \times (10^{-4} \text{ or } 10^{-3}) \frac{n_1}{n_3}$. Again, this result for a laser beam tuned out of resonance with the atomic transition implies that the above relation is much smaller than 1 .

Finally, the ratio between the two different parts of the diffusion tensor is

$$\frac{D^{(2)}}{D^{(1)}} \approx \left[\frac{n_2}{n_1}\right] \left[\frac{\gamma}{\Lambda}\right] \times \begin{cases} \frac{\eta}{\Lambda} & \gamma \ll \eta \\ \frac{\gamma}{\Lambda} & \eta \ll \gamma \end{cases} , \qquad (3.1.23)$$

proving that the fluorescence effects are very small compare with the stimulated emission and absorption of laser photons in the high laser intensities regime. This result is independent of the magnitude of the ratio γ/η for high laser intensities.

The conclusions reached for these ratios do not include those few points where n_3 or n_1 vanish.

Figures 3, 4 and 5 are plots of n_1, n_2 and n_3 for the limit $\gamma/\eta \ll 1$. The graphs 6, 7 and 8 represent the same magnitudes but for the opposite limit, $\eta/\gamma \ll 1$. The ratios $\frac{n_3}{n_2}$, $\frac{n_1}{n_3}$ and $\frac{n_1}{n_2}$ in both limits are plotted in Figure 9 to 14 .

3.2 - An Approximate Solution of the Fokker-Planck Equation.

In this section, an approximate solution of the distribution equation is found for the case of an atomic beam interacting with a standing wave laser field. The approximation is presented as a power series of a small parameter defined below.

The particular configuration of our problem assumes that the propagation direction of the standing wave laser field is along the y-direction, with the polarization vector directed in the x-direction in such a way that $\vec{k}_L = k_L \hat{y}$, $\hat{E}_2 = \hat{E}_3 = 0$ and $\hat{E}_1 = 1$. The idealized atomic beam strikes the laser field at the plane $z = 0$, with an unique initial velocity $\vec{v}$. The velocity at $z = 0$ is defined by $\vec{v}_0 = v_0 \cos\alpha \hat{z} + v_0 \sin\alpha \hat{y} = v_{0z} \hat{z} + v_{0y} \hat{y}$. The laser beam width, which defines the interaction region, is z_0 and therefore, the boundary condition for the distribution function representing the atomic motion is

$$f(\vec{x},\vec{v})\,|_{z=0}=\delta^{(3)}(\vec{v}-\vec{v}_0) \quad . \tag{3.2.1}$$

The coefficients for the distribution equation (3.1.17) are obtained from relations (3.1.18) to (3.1.20) of the previous section. In our case, the experimental regime implies that the solution of equation (3.1.17) has to be investigated for the so-called stationary solution limit that corresponds to a time independent equivalent of equation (3.1.17).

In fact, the experimental set up of the problem considers a constant flux of atoms interacting with a stationary standing wave. The width of the atomic beam is much larger than λ_L , which means that for our practical purpose, at the end, the average over the extension of one wave length for the spatial coordinate y has to be done. The experimental data are obtained far from the interaction region where we study the velocity distributions of the atoms as a function of the laser and atom parameters. The atoms trajectory, after leaving an ideal mono-velocity oven, begins with the free motion of the atoms with a unique velocity $\vec{v}_0$. This state of motion is preserved until the atoms reach the region of interaction with the laser beam. After crossing the laser beam, the atoms again move in a straight line with a constant velocity equal to the emerging velocity from the region of interaction, at the plane $z = z_0$.

It is also assumed that the laser intensity profile is constant with an amplitude independent of z in the region $0 \leqslant z \leqslant z_0$. Inside the interaction region, the atoms are deflected and dispersed. The deflection is attributed to the effect of the force and the dispersion is due to the diffusion tensor. However, as shown in the next section, the force also disperses the atomic beam due to the average over the position for the distance of one wave length.

For the present configuration, the force and diffusion tensor are obtained from relations (3.1.18) to (3.1.20) . Here, the analysis is confined to the limit fixed by the condition $\gamma \ll \eta \ll \Lambda$. This limit represents atoms of not so low velocity interacting with a high intensity laser beam.

The time of interaction t is constrained by the relation $\gamma t \gg 1$ and; simultaneously, our calculations remain valid only if a non-drastic change in the velocity occurs. We are assuming that the resulting change in the atomic velocity will not be so large that the formulas for the limit $\gamma \ll \eta$ can not be aplied. If the opposite limit has to be analyzed, then the force and diffusion tensor obtained in relations (2.4.7), (2.4.8) and (2.4.11) have to be used for the subsequent calculations.

Putting together the relations $1 \ll \gamma t$ and $\gamma \ll \eta$, it is obtained the condition $1 \ll \eta t$. This condition can be represented in terms of parameters associated with the laser and the initial velocity of the atom by replacing η for $k_L v_{0y}$ and t for z_0 / v_{0z} . Therefore, the laser beam width must satisfy the condition $1 \ll \frac{k_L z_0 v_{0y}}{v_{0z}}$. This condition is satisfied even for $z_0 \sim .1$mm and $v_{0y} / v_{0z} \sim 10^{-2}$. Also, Mandel [40] and Minogin [41] have shown that $\gamma t \geq 3$ sufficiently smooths the position dependence of the diffusion tensor and force for calculations similar to the one made in Chapter 2 .

For the previous set up, the force and diffusion tensor are:

$$\vec{F} = F\hat{y} = \hbar k_L n_3(w,\theta)\hat{y} \tag{3.2.2}$$

$$D_i^{(1)} = \delta_{i2}\delta_{j2} D_1 = \frac{\Lambda^2}{\eta} k_L^2 \hbar^2 n_1(w,\theta)\delta_{i2}\delta_{j2} \ , \tag{3.2.3}$$

$$D_i^{(2)} = (2\delta_{ij} - \delta_{i1}\delta_{j1}) D_2 = \gamma \hbar^2 k_L^2 n_2(w,\theta)(2\delta_{ij} - \delta_{i1}\delta_{j1}) \ . \tag{3.2.4}$$

In the following calculations, the limit $\gamma \ll \eta$ is considered. At the present we have $\eta = k_L v_{0y} = k_L v_0 \sin\alpha$. Moreover, $\theta = \vec{k}_L \cdot \vec{\rho}$ is reduced to $\theta = k_L y$ and the only position dependence of the force and diffusion tensor is with respect to the coordinate y . The velocity dependence for the limit $\gamma / \eta \ll 1$ appears only in the coefficient D_1 . As it is well understood, $D_i^{(2)}$ is associated with the randomness of the fluorescence process. This is the atom absorbing a laser photon jumping to the excited state. Later, a fluorescent photon is emitted with a random momentum direction but still with a magnitude k_L . For the standing wave, $D_i^{(1)}$ is associated with the random effect of first absorbing a laser photon from either of the opposing traveling waves forming the standing wave and, secondly, of the re-emission

of the photon in the other of the two modes closing the Rabi cycle.

As was discussed in the previous section wherever $w \neq 0$ (laser out of resonance case) the product $F \times M_A v$ is much larger than the two contributions to the diffusion tensor $D_i^{(1)}$ and $D_i^{(2)}$. Therefore, we will investigate the solutions of the time independent Fokker-Planck equation (3.1.17) as a power series of the parameter

$$\epsilon \equiv \frac{\hbar k_L}{M_A v_0} \quad . \tag{3.2.5}$$

This is a very small parameter $\epsilon \sim 10^{-4}$. It is defined as the ratio between the recoil momentum of the atom and the initial momentum of the atom. This parameter is not the best choice from the physical point of view, but it has the advantage of clearly appearing in each one of the terms of the distribution equation. It is later shown that a more realistic parameter should be proportional to the ratio between the atom-laser coupling interaction energy to the kinetic energy of the atom.

The Fokker-Planck equation for our problem following the previous prescription is reduced to

$$v_i \frac{\partial}{\partial x_i} f(\vec{x},\vec{v}) + v_0 \Lambda \epsilon n_3(w,k_L y) \frac{\partial}{\partial v_y} f(\vec{x},\vec{v}) = v_0^2 \gamma \epsilon^2 (2\delta_{ij} - \delta_{i1}\delta_{j1}) n_2(w,k_L y) \times$$
$$\frac{\partial^2}{\partial v_i \partial v_j} f(\vec{x},\vec{v}) + v_0^2 \Lambda \epsilon n_1(w,k_L y) \frac{\partial^2}{\partial v_y^2} \frac{\Lambda}{\eta} f(\vec{x},\vec{v}) \quad . \tag{3.2.6}$$

To solve the first order in ϵ implies eliminating the right hand side of the equation (3.2.6) , this correspond to the order zero in the diffusion tensor. The order zero in ϵ is equivalent to studying free atoms. First order in the parameter ϵ corresponds to solving the Newton equation for the atom moving under the influence of the force $\vec{F}$. We separate the distribution function in two parts: one corresponds to the solution of the Newton equation, and the second part is associated with the contribution to the distribution function coming from the right hand side of equation (3.2.6) , $f(\vec{x},\vec{v}) = f^{(1)}(\vec{x},\vec{v}) + f^{(2)}(\vec{x},\vec{v}) + \cdots$. It will be seen that $f^{(1)}(\vec{x},\vec{v})$ contains higher order in ϵ than the first. However, to obtain the physical results associated with the Newtonian motion, only up to the order ϵ^2 is kept. The equation

satisfied by $f^{(1)}(\mathbf{x},\mathbf{v})$ is

$$v_i \frac{\partial}{\partial x_i} f^{(1)}(\mathbf{x},\mathbf{v}) + \Lambda v_0 \epsilon n_3(w, k_L y) \frac{\partial}{\partial v_y} f^{(1)}(\mathbf{x},\mathbf{v}) = 0 \tag{3.2.7}$$

with the boundary condition defined in relation (3.2.1).

The solution of equation (3.2.7) can be obtained by the method of the characteristic. After a straightforward calculation the exact solution of equation (3.2.7) is

$$f^{(1)}(\mathbf{x},\mathbf{v}) = \delta(v_x)\,\delta\left(\left|2v_0 \Lambda \epsilon \int_{\bar{y}}^{y} dy'\, n_3(w, k_L y') + v_y^2\right|^{1/2} - v_{0y}\right)\delta(v_z - v_{0z}), \tag{3.2.8}$$

where $\bar{y}(y, v_y, z, v_z)$ is implicitly defined by

$$-\frac{z}{v_z} = \int_{\bar{y}}^{y} dy' \left|v_y^2 - 2\Lambda v_0 \epsilon \int_{y'}^{y} dy''\, n_3(w, k_L y'')\right|^{-1/2}. \tag{3.2.9}$$

and $\bar{y}$ represents the initial position of an atom for which the present position is defined by, z, y. The atom has evolved in deterministic motion to the coordinate location defined by y and z. The interpretation of (3.2.9) is the usual, the integrand is proportional to the work done by the atoms moving against the external field from the position $\bar{y}$ where the velocity is v_{0y} to the final position y where the velocity is v_y.

Note that this integral vanishes at $z = 0$ if $\bar{y} = y$ in which case, the solution (3.2.8) becomes (3.2.1) satisfying the initial condition for the motion of the atomic beam.

The integral of the force can be evaluated analytically

$$\int_{y'}^{y} dy''\, n_3(w, k_L y'') = -\frac{n(w)}{k_L}\left[\Delta_w(k_L y) - \Delta_w(k_L y')\right] \tag{3.2.10}$$

with $\Delta_w(k_L y)$ defined at equation (3.1.1) of the previous section. The number n is

$$n(w) \equiv \frac{w}{\pi} K(\bar{w}) \left|(1+w^2)^{1/2} - |w|\right| (1+w^2)^{1/2} \tag{3.2.11}$$

as in Section 1 of this chapter, $K(\bar{w})$ is the elliptic function of the first kind [37].

Our next calculations show the solution (3.2.8) at second order in the parameter ϵ.

To obtain the power series of the solution (3.2.8) we assume that

$$\bar{y} = a_0 + a_1\epsilon + \frac{1}{2}a_2\epsilon^2 + O(\epsilon^3)$$

and (3.2.12)

$$v_y = b_0 + b_1\epsilon + \frac{1}{2}b_2\epsilon^2 + O(\epsilon^3)$$

each one of the unknown coefficients a_0, a_1, a_2 and b_0, b_1, b_2 of the series expansion for $\bar{y}$ and v_y can be obtained solving the following two coupled equations up to second order in ϵ:

$$-\frac{z}{v_z} = \int_y^{\bar{y}} dy' \left\{ v_y^2 + \frac{2\Lambda v_0 \epsilon}{k_L} n(w) \left| \Delta_w(k_L y) - \Delta_w(k_L y') \right| \right\}^{-1/2}$$

and (3.2.13)

$$v_y^2 = v_{0y}^2 + \frac{2v_0\Lambda\epsilon}{k_L} n(w) \left| \Delta_w(k_L \bar{y}) - \Delta_w(k_L y) \right| .$$

In general, independent of the approximation, these two equations allow to find v_y as a function of all the other variables and initial conditions. The procedure consists of starting from the top equation (3.2.13), and in principle to solve $\bar{y}$ as a function of y, z, v_y and v_z. Then, this solution is substituted into the bottom equation (3.2.13) obtaining a transcendental equation for v_y. However, this approach can not be accomplished exactly. Since the first contribution to the distribution function coming from the diffusion tensor is proportional to ϵ^2, the previous equation has to be solved at least up to this order in such a way that the addition of the diffusion is made consistantly. This order of approximation is represented by the equations (3.2.12).

After some long but straightforward algebra, we obtain that the coefficients of the power series for v_y and $\bar{y}$ are

$$b_0 = v_{0y}$$

$$b_1 = v_{0y}\left[\frac{v_0 \Lambda n(w)}{k_L v_{0y}^2}\right]\left|\Delta_w\left|k_L y - \frac{k_L z v_{0y}}{v_z}\right| - \Delta_w\left|k_L y\right|\right| \qquad (3.2.14)$$

$$b_2 = -v_{0y}\left[\frac{v_0 \Lambda n(w)}{k_L v_{0y}^2}\right]^2\left\{\left|\Delta_w\left|k_L y - \frac{k_L z v_{0y}}{v_z}\right| - \Delta_w\left|k_L y\right|\right|^2 - \frac{\bar{w}^2 \sin 2(k_L y - \frac{k_L z v_{0y}}{v_z})}{\Delta_w(k_L y - \frac{k_L z v_{0y}}{v_z})}\right.$$

$$\left.\left|E(\bar{w}, y - \frac{z v_{0y}}{v_z}) - E(\bar{w}, y) + \frac{k_L z v_{0y}}{v_z}\Delta_w\left|k_L y - \frac{k_L z v_{0y}}{v_z}\right|\right|\right\}.$$

Let us recall that v_y represents the atomic velocity in the y direction at the point with coordinates y, z and $\bar{y}$ is the initial y coordinate of the atom trajectory. The coefficients associated with $\bar{y}$ are

$$a_0 = y - \frac{z v_{0y}}{v_z} \qquad (3.2.15)$$

$$a_1 = -\frac{1}{k_L}\left[\frac{v_0 \Lambda n(w)}{k_L v_{0y}^2}\right]\left|E(\bar{w}, a_0) - E(\bar{w}, y) + \frac{k_L z v_{0y}}{v_z}\Delta_w(k_L a_0)\right|$$

$$a_2 = \frac{1}{k_L}\left[\frac{v_0 \Lambda n(w)}{k_L v_{0y}^2}\right]^2\left\{2\left|E(\bar{w}, a_0) - E(\bar{w}, y)\right|\left|1 - \left|4\Delta_w(k_L a_0) - \Delta_w(k_L y)\right|\right|\right.$$

$$+ \frac{3}{4}\bar{w}^2(\sin 2k_L y - \sin 2k_L a_0) + \frac{k_L z v_{0y}}{v_z}\left|-2\Delta_w^2(k_L a_0)\left|1 - \left|\frac{5}{2}\Delta_w(k_L a_0) - \Delta_w(k_L y)\right|\right|\right.$$

$$\left. - \frac{\bar{w}^2 \sin 2k_L a_0}{\Delta_w(k_L a_0)}\left|E(\bar{w}, a_0) - E(\bar{w}, y)\right| + 3\left|1 - \frac{\bar{w}^2}{2}\right|\right| + \left|\frac{k_L z v_{0y}}{v_z}\right|^2 \Delta_w(k_L a_0)\Bigg|$$

where E is the incomplete elliptic function of the second kind [37]. The first part of the approximated distribution function becomes at order ϵ^2,

$$f^{(1)}(\bar{r}, \bar{v}) = \delta(v_x)\,\delta[v_y - (b_0 + b_1\epsilon + \frac{1}{2}b_2\epsilon^2)]\,\delta(v_z - v_{0z}). \qquad (3.2.16)$$

Because of the function $\delta(v_z - v_{0z})$, the coefficients b_0, b_1 and b_2 defined in relation (3.2.14) are evaluated at $v_z = v_{0z}$. Therefore, considering the remaining dependence in the parameters b_0, b_1 and b_2, it is found that v_y is a function of the position y and z. Also, v_y depends upon the laser parameters Λ, k_L, w and on the initial condition $\bar{v}_0$.

Looking at the dependence of the coefficients a_i and b_i we see that a more realistic expansion parameter is determined by the product $\epsilon \times \left| \frac{v_0 \Lambda n(w)}{k_L v_{0y}^2} \right| = \frac{\hbar \Lambda}{M_A v_{0y}^2} n(w) \sim 5\times 10^{-2} n(w)$ which, on the one side, proves that the very large time effect of the force is much larger than the individual recoil associated with the single emission or absorption of a photon. The other factors appearing in the previous coefficients are dependent on the electromagnetic field geometry and will determine the overall effect for the present configuration.

The partial differential equation satisfied by the next order of the distribution function, $f^{(2)}(\mathbf{x},\mathbf{v})$, is:

$$v_i \frac{\partial}{\partial x_i} f^{(2)}(\mathbf{x},\mathbf{v}) + \Lambda v_0 \epsilon n_3(w,k_L y) \frac{\partial}{\partial v_i} f^{(2)}(\mathbf{x},\mathbf{v}) = v_0^2 \gamma \epsilon^2 n_2(w,k_L y) \times \tag{3.2.17}$$

$$\delta(v_y - v_{0y})\delta(v_z - vz\,0) \frac{\partial^2}{\partial v_x^2} \delta(v_x) + 2v_0^2 \gamma \epsilon^2 n_2(w,k_L y) \delta(v_x) \delta(v_y - v_{0y}) \frac{\partial^2}{\partial v_z^2} \delta(v_z - v_{0z}) +$$

$$v_0^2 \epsilon^2 \left| 2\gamma n_2(w,k_L) + \frac{\Lambda^2}{k_L v_{0y}} n_1(w,k_L y) \right| \delta(v_x) \delta(v_z - v_{0z}) \frac{\partial^2}{\partial v_y^2} \delta(v_y - v_{0y}) \,.$$

The right hand side of the above equation has the support of the δ–function evaluated at the initial velocity $\mathbf{v}_0$ because $f^{(2)}(\mathbf{x},\mathbf{v})$ satisfies the Fokker-Planck equation only up to order ϵ^2 and therefore $f^{(1)}(\mathbf{x},\mathbf{v})$ has to be substituted with $\epsilon = 0$. Including corrections to the support of the δ–function will add higher order corrections than ϵ^2 to the distribution function at the end.

The boundary condition for the distribution function $f^{(2)}(\mathbf{x},\mathbf{v})$ is given at $z = 0$ where this function vanishes. The previous equation can also be solved by the method of the characteristic. In the present case, the solution of the partial differential equation (3.2.17) is largely simplified because the second member of the equation is already proportional to ϵ^2. Therefore, the actual calculations are made considering that any other term proportional to ϵ will add corrections of higher order to the solution. The solution is

$$f^{(2)}(\vec{x},\vec{v}) = v_0^2\gamma\epsilon^2\left|\frac{1}{v_{0y}}\delta(v_z - v_{0z})\delta(v_y - v_{0y})\frac{\partial^2}{\partial v_x^2}\delta(v_x) + \right. \tag{3.2.18}$$

$$\left.\frac{2}{v_{0y}}\delta(v_x)\delta(v_y - v_{0y})\frac{\partial^2}{\partial v_z^2}\delta(v_z - v_{0z}) + \frac{2}{v_y}\delta(v_x)\delta(v_z - v_{0z})\frac{\partial^2}{\partial v_y^2}\delta(v_y - v_{0y})\right|$$

$$\int_{\bar{y}}^{y} dy'\, n_2(w, k_L\, y') + \frac{v_0^2\epsilon^2\Lambda^2}{k_L\, v_{0y}\, v_y}\left|\frac{\partial^2}{\partial v_y^2}\delta(v_y - v_{0y})\right|\int_{\bar{y}}^{y} dy'\, n_1(w, k_L\, y')$$

where $\bar{y}$ is defined in relation (3.2.9). Nevertheless, for the case of the solution (3.2.18) we have to study the integral (3.2.9) at order zero in the parameter ϵ. Neglecting terms proportional to the parameter ϵ the integral that defines $\bar{y}$ can be solved. The solution is $\bar{y} = y - \frac{z\, v_y}{v_z}$, which reduces the equation (3.2.18) to a closed form.

Here, we find that each one of the remaining integrals can be calculated,

$$\int_{\bar{y}}^{y} dy' n_2(w, k_L\, y') = \frac{1}{20\pi}\left|\frac{z\, v_y}{v_z}\right| - \frac{n'(w)}{20\pi k_L}\left|K(\bar{w}, k_L\, y) - K(\bar{w}, k_L\, \bar{y})\right|$$

with (3.2.19)

$$n'(w) = \frac{4w^2 K(\bar{w})\left|(1+w^2)^{1/2} - |w|\right|}{\pi(1+w^2)^{1/2}}$$

the previous integral accounts for the spontaneous emission term of the diffusion equation. The next integral is associated with the diffusion due to the stimulated absorption and emission of photons,

$$\int_{\bar{y}}^{y} dy'\, n_1(w, k_L y') = -\frac{n_{11}(w)}{k_L}\left|\Delta_w(k_L y) - \Delta_w(k_L \bar{y})\right| - \frac{n_{12}(w)}{k_L}\left|\cos 2k_L y - \cos 2k_L \bar{y}\right|$$

with (3.2.20)

$$n_{11} = \frac{1}{4\pi}(1+w^2)\left|(1+w^2)^{1/2} - |w|\right|\left|-\frac{\pi}{4}\left|(1+w^2)^{1/2} + |w|\right| + (1+w^2)^{1/2}E(\bar{w}) + \frac{|w|^3 K(\bar{w})}{(1+w^2)^{1/2}}\left|(1+w^2)^{1/2} - |w|\right| + \frac{4}{\pi^2}w^2\left|(1+w^2)^{1/2} - |w|\right|^2 K^2(\bar{w})\left|-\frac{\pi}{4}\left|(1+w^2)^{1/2} + |w|\right| + (1+w^2)^{1/2}E(\bar{w}) + \frac{w^2 K(\bar{w})}{(1+w^2)^{1/2}}\right|\right|$$

and

$$n_{12}(w) = \frac{1}{16\pi}\left|-\frac{\pi}{4}\left|(1+w^2)^{1/2} + |w|\right| + \frac{w^2}{\pi}\left|(1+w^2)^{1/2} - |w|\right|^2 K^2(\bar{w})\left|(1+w^2)^{1/2} + |w|\right|\right| .$$

Therefore, the distribution function at second order in the parameter ϵ is given by:

$$f(\underline{x},\underline{v}) = \delta(v_x)\,\delta[v_y - (b_0 + b_1\epsilon + \tfrac{1}{2}b_2\epsilon^2)]\,\delta(v_z - v_{0z}) + \frac{v_0^2\gamma\epsilon^2}{20\pi k_L}\left|\frac{1}{v_{0y}}\delta(v_z - v_{0z})\delta(v_y - v_{0y})\frac{\partial^2}{\partial v_x^2}\delta(v_x) + \frac{2}{v_{0y}}\delta(v_x)\delta(v_y - v_{0y})\frac{\partial^2}{\partial v_z^2}\delta(v_z - v_{0z}) + \frac{2}{v_y}\delta(v_x)\delta(v_z - v_{0z})\frac{\partial^2}{\partial v_y^2}\delta(v_y - v_{0y})\right|\left|\frac{k_L z v_y}{v_z} - n'(w)\left|K(\bar{w}, k_L y) - K(\bar{w}, k_L)\right|\right| - \frac{v_0^2\epsilon^2\Lambda^2}{k_L^2 v_{0y} v_y}\left|\frac{\partial^2}{\partial v_y^2}\delta(v_y - v_{0y})\right| \times \left|n_{11}(w)\left|\Delta_w(k_L y) - \Delta_w(k_L \bar{y})\right| + n_{12}(w)\left|\cos 2k_L y - \cos 2k_L \bar{y}\right|\right| \quad (3.2.21)$$

where $\bar{y} = y - z\,v_y / v_z$ is evaluated at the velocity at $z \neq 0$ and b_0, b_1 and b_2 are defined in equation (3.2.14) but for equation (3.2.21) they are evaluated at $v_z = v_{0z}$.

The application of this approximated distribution function for the calculations of certain average is given in the next section.

Any anomaly of the previous solution comes from the fact that a power series of the Fokker-Planck equation implies that at two different orders ϵ and ϵ^2 we are actually solving partial differential equations of different order in the derivatives [25,26]. Following the

indications of the previous references the power series solution of the Fokker-Planck equation does indeed converge to the exact solution and adding higher orders provides a better estimation of the exact solution.

3.3 - Atomic Deflection and Dispersion by a Strong Standing Wave Laser Field.

In the present section, the application shall be made of the obtained distribution function to the calculation of the deflection angle for the atomic beam. First, it is presented the distribution function as a y and z function only and secondly as a function of the position z . However, this result is not full analyzed for the present problem configuration. In the second part of this section the deflection angle and standard deviation are calculated.

The distribution function in the coordinates y and z is obtained integrating the relation (3.2.21) with respect to the velocity $\vec{v}$,

$$f(\vec{x}) = \int d^3v f(\vec{x},\vec{v}) = 1 + \frac{v_0^2 \gamma \epsilon^2}{20\pi k_L v_{0y}^3} \left\{ 4n'(w) K(\bar{w}, k_L a) + \frac{k_L z v_{0y}}{v_{0z}} \left| \frac{4v_{0y}^2}{v_{0z}^2} - \right. \right.$$

$$\left. \frac{2v_{0y}^2 n'(w)}{v_{0z}^2 \Delta_w(k_L a)} + \frac{4n'(w)}{\Delta_w(k_L a)} \right| + \left. \left| \frac{k_L z v_{0y}}{v_{0z}} \right|^2 \frac{n'(w) \sin 2k_L a}{\Delta_w^3(k_L a)} \left| 1 + \frac{v_{0y}^2}{v_{0z}^2} \right| \right\}$$

$$- \frac{2v_0^2 \epsilon^2 \Lambda^2}{k_L^2 v_{0y}^4} \left\{ n_{11} \left| \Delta_w(k_L y) - \Delta_w(k_L a) \right| + n_{12}(w) \left| \cos 2k_L y - \cos 2k_L a \right| + \right.$$

$$\left. \frac{k_L z v_{0y}}{v_{0z}} \sin 2k_L a \left| 2n_{12}(w) + \frac{\bar{w}^2 n_{11}(w)}{2\Delta_w(k_L a)} \right| + \right. \tag{3.3.1}$$

$$\left. \left| \frac{k_L z v_{0y}}{v_{0z}} \right|^2 \left| \frac{\bar{w}^2 n_{11}(w)}{2\Delta_w(k_L a)} \cos 2k_L a + \frac{\bar{w}^4 n_{11}(w)}{8\Delta_w^3(k_L a)} \sin^2 2k_L a + 2n_{12}(w) \cos 2k_L a \right| \right\}$$

where $a = y - \frac{z v_{0y}}{v_{0z}}$ account for the free displacement of the atoms.

Because the beam is initially independent of y we can put $\theta = k_L y$ and take the average with respect to θ between $-\pi$ and π obtaining

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, f(z,\theta) = 1 + \frac{v_0^2 \gamma \epsilon^2 z}{5\pi v_{0z}^3} \left| 1 - \frac{n'(w)}{\pi} K(\bar{w}) \right| \tag{3.3.2}$$

This distribution implies that other than the free motion of the atoms toward the region outside the laser beam there is an effect represented by the factor $\left|1-\frac{n'(w)}{w}K(\vec{w})\right|$ due to the over-all effect of the fluorescent photons.

An important magnitude to investigate is the average deflection of the atomic beam crossing the standing wave laser field. Suppose that the atoms reach the surface at $z = 0$ with a velocity $\vec{v}_0$ at an angle α as mentioned before. After crossing the laser field the angle of the vector velocity is β such that $\delta\alpha = \beta - \alpha$ can be understood as the deflection due to the interaction with the laser beam. In this configuration, we have that $\tan\alpha = v_{0y} / v_{0z}$ and $\tan\beta = v_y / v_z$. Assuming the deflection with respect to the angle α is small we can approximate $\tan\delta\alpha \approx \delta\alpha$, in which case

$$\delta\alpha \approx \frac{v_y - v_z \tan\alpha}{v_z + v_y \tan\alpha} \quad . \tag{3.3.3}$$

Starting from $\delta\alpha$ we calculate the average of the deflection with respect to α and also the standard deviation from the average. We write for the average $\langle\overline{\delta\alpha}\rangle$ and the notation for the standard deviation is $\Delta\langle\overline{\delta\alpha}\rangle = \left|\langle\overline{\delta\alpha^2}\rangle^2 - \langle\overline{\delta\alpha}\rangle^2\right|^{1/2}$, where the velocity average is defined by

$$\overline{\delta\alpha} = \int d^3v f(\vec{x},\vec{v}) \left|\frac{v_y - v_z \tan\alpha}{v_z + v_y \tan\alpha}\right| \tag{3.3.4}$$

and

$$\overline{\delta\alpha^2} = \int d^3v f(\vec{x},\vec{v}) \left|\frac{v_y - v_z \tan\alpha}{v_z + v_y \tan\alpha}\right|^2 \tag{3.3.5}$$

and the distribution function $f(\vec{x},\vec{v})$ is defined at relation (3.2.21).

The contribution to the average of the deflection angle coming from $f^{(1)}(\vec{x},\vec{v})$ is

$$\overline{\delta\alpha^{(1)}}= \left|\frac{v_0\Lambda n(w)}{k_L v_{0y}^2}\right|\frac{v_{0y}v_{0z}}{v_0^2}\epsilon\left|\Delta_w(k_L a)-\Delta_w(k_L y)-\epsilon\left|\frac{v_0\Lambda n(w)}{k_L v_{0y}^2}\right|\right|\frac{1}{2}\left|1+2\frac{v_{0y}^2}{v_0^2}\times\right.$$

$$\left.\left|\Delta_w(k_L a)-\Delta_w(k_L y)\right|^2-\frac{\overline{w}^2\sin 2k_L a}{2\Delta_w(k_L a)}\left|E(\overline{w},a)-E(\overline{w},y)+\frac{k_L z v_{0y}}{v_{0z}}\Delta_w(k_L a)\right|\right|\right| \tag{3.3.6}$$

The preceding is the contribution to the deflection due to the action of the force alone on the motion of the atoms. The contribution coming from both parts of the diffusion tensor comes from the calculation of $\overline{\delta\alpha^{(2)}}$, see below. The next result is to consider the $\theta = k_L y$ average of (3.3.6), this average defines $<\overline{\delta\alpha^{(1)}}>$ as

$$<\overline{\delta\alpha^{(1)}}>= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\,\overline{\delta\alpha^{(1)}}= \left|\frac{v_0\Lambda n(w)}{k_L v_{0y}^2}\right|^2\epsilon^2\frac{5v_{0y}v_{0z}}{4v_0^2}\left|\left|1+\frac{8v_{0y}^2}{5v_0^2}\right|\right.$$
$$\left.\times\left|1-\frac{\overline{w}^2}{2}-\frac{1}{2\pi}I\left(\frac{k_L z v_{0y}}{v_{0z}}\right)\right|\right| \tag{3.3.7}$$

where the function $I(\theta_0)= \int_{-\pi}^{\pi} d\theta\,\Delta_w(\theta-\theta_0)\Delta_w(\theta)$ and $\theta_0= \frac{k_L z v_{0y}}{v_{0z}} \gg 1$.

An immediate consequence of (3.3.7) is that the average deflecting angle at order ϵ^2 is independent of the laser beam width. In fact, for $\theta_0= l\pi$, with l an integer, there is no deflection because at this point $\frac{1}{2\pi}I(l\pi)=1-\frac{\overline{w}^2}{2}$. On the other hand, we can write $\theta_0= \frac{2\pi z}{\lambda_L}\tan\alpha= \pi(l+\delta l)$ with l the largest integer smaller than $\frac{\theta_0}{\pi}$ and δl the non integer remaining part. Substituting this result back into the relation (3.3.7), the resultant expression for the deflection angle average depends only on $0\leqslant\delta l\leqslant 1$; which is actually independent of the real laser beam width.

To obtain this result, previous approximations were done as neglecting terms proportional to γ/η for the force. The dependence of the force with the detuning w occurs in the function $n(w)$; this function vanishes at $w=0$ for the resonant case, implying that there is not deflection of the atomic beam for this case independent of the value of $I(\theta_0)$. Expres-

sion (3.3.7) implies that the standing wave laser can be focused to a very small region increasing with this the avialable effective intensity and consequently Λ. Since the deflecting angle obtained above is proportional to Λ^2 then the deflection is proportional to the laser intensity.

Nevertheless, the average of the deflection angle depends critically on the integral $I(\theta_0)$. In particular, as mention previously, the case $\theta_0 = \pi l$ brings out a zero deflection. The actual integral $I(\theta_0)$ is a function of only $I(\pi\delta l)$ which at order zero in δl implies zero deflection. The effect can be described as follows: when the atoms cross the laser beam they " see " different intensities depending on the local standing wave profile. Therefore, the atoms oscillate around the straight line free trajectory depending on the local laser intensity. As a whole, the atoms are not deflected except by the last incomplete cycle of the oscillation.

In a typical laser we may expect that δl is not known exactly but rather δl has a random value along the laser ray and time. The estimated value of $I(\pi\delta l)$ can be, in this case, obtained from the average over a period. The average value of $I(\pi\delta l)$ is easily calculated to be $I_0 = \frac{8}{\pi} E^2(\bar{w})$; in which case a close form of the deflection angle as a function of the laser and atomic parameters is obtained,

(3.3.8)

$$\langle \overline{\delta\alpha^{(1)}} \rangle \approx \epsilon^2 \frac{5 v_{0x} v_{0y}}{4 v_0^2} \left| \frac{v_0 \Lambda n(w)}{k_L v_{0y}^2} \right|^2 \left| 1 + \frac{8 v_{0y}^2}{5 v_0^2} \right| \left| 1 - \frac{\bar{w}^2}{2} - \frac{4}{\pi^2} E^2(\bar{w}) \right|$$

$$\approx \frac{5}{8} \epsilon_0 \sin 2\alpha \left| 1 + \frac{8}{5} \sin^2\alpha \right| n^2(w) \left| 1 - \frac{\bar{w}^2}{2} - \frac{4}{\pi^2} E^2(\bar{w}) \right|$$

where it has been defined the parameter $\epsilon_0 = \frac{\hbar\Lambda}{M_A v_{0y}^2} \sim 5\times10^{-2}$. The last number is obtained when the values of the parameters defining ϵ_0 are replaced in the previous expression.

The angle α is the atomic beam incidence angle. The only restriction to the value of the angle α has come from the relation $\gamma < k_L v_0 \sin\alpha < \Lambda$. Therefore, if v_0 is adjusted by other procedures such as laser cooling in a way that the previous inequality is satisfied; then, the angle α can be fixed independently.

In the last expression, the deflection angle is shown as a function of the incident angle α and the dimensionless detuning w . The number ϵ_0 becomes a scale factor which requires an experimental accuracy in the order of a tenth of a degree.

The corrections to this result comes from the next term of the Fourier series for the integral I . Actually, I_0 corresponds to the first, $m = 0$, term of the Fourier series. The next non-null correction to I comes from $m = 2$.

The factor $\left|1-\frac{\overline{w}^2}{2}-\frac{4}{\pi^2}E^2(\overline{w})\right|$, which is an even function of w , is always positive for any value of the parameter w . For $w = 0$ then $\overline{w} = 1$ and this factor is $\frac{1}{2}-\frac{4}{\pi^2}$. When $w \to \infty$ then $\overline{w} \to 0$ and also the referred factor goes to zero.

Finally with respect to (3.3.8) a plot of the angular dependence $\sin 2\alpha\left|1+\frac{8}{5}\sin^2\alpha\right|$ is presented in Figure 15 . In Figure 16 is shown the deflection angle (3.3.8) as a function of the detuning w . Looking at the previous two figures, we see that the maximum deflection as a function of the incident angle α is for $\alpha \sim 55^\circ$ and with respect to the detuning this maximum occurs at $w \sim .35$. The resulting deflection angle for $\alpha \sim 55^\circ$ and $w \sim .35$ is approximately $.0002^\circ$.

Our next step is to calculate the effect of $f^{(2)}(\mathbf{X},\mathbf{V})$ on the deflection angle. This part of the distribution function is associated with the effect of the diffusion tensor. That accounts for either of the effects; namely, spontaneous emission or stimulated absorption and emission. The calculations involved here are long but straightforward, leading to the result previous to the spatial average with respect to y ,

$$\overline{\delta\alpha^{(2)}} = \left| \frac{v_0^2 \gamma \epsilon^2}{5\pi k_L} \right| \frac{1}{v_{0z}\, v_0^2} \left\{ \frac{v_{0z}^2}{v_{0y}^2} n'(w) \left| \mathbb{K}(\overline{w}, k_L y) - \mathbb{K}(\overline{w}, k_L a) \right| + \frac{k_L z\, v_{0y}}{v_{0z}} \left| 1 - \right. \right. \tag{3.3.9}$$

$$\left| 1 + \frac{v_{0z}^2}{v_{0y}^2} \right| \frac{n'(w)}{\Delta_w(k_L a)} \left| \right| + \frac{2 v_{0z}\, \epsilon^2 \Lambda^2}{k_L^2 v_{0y}^3} \left\{ \left| \left| 1 + \frac{v_{0y}^2}{v_0^2} \right| \left| n_{11}(w) \left| \Delta_w(k_L y) - \Delta_w(k_L a) \right| + \right. \right.$$

$$\left. \left. n_{12}(w) \left| \cos 2k_L y - \cos 2k_L a \right| \right| + \frac{k_L z\, v_{0y}}{v_{0z}} \left| \frac{n_{11}(w) \overline{w}^2 \sin 2k_L a}{2\Delta_w(k_L a)} + 2 n_{12}(w) \sin 2k_L a \right| \right|$$

The $\theta = k_L y$ average of (3.3.9) is

$$\langle \overline{\delta\alpha^{(2)}} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \overline{\delta\alpha^{(2)}} = \frac{\gamma \epsilon^2}{5\pi k_L\, v_{0z}} \frac{k_L z\, v_{0y}}{v_{0z}} \left| 1 - \frac{2n'(w)}{\pi} K(\overline{w}) \right| \tag{3.3.10}$$

$$= \frac{\gamma \epsilon^2}{5\pi\eta} k_L z \tan^2\alpha \left| 1 - \frac{2n'(w)}{\pi} K(\overline{w}) \right|$$

An analysis of the above relation, reveals that there is not contribution to the angle of deflection coming from the stimulated emission and absorption of laser photons. This result implies that even when the relative magnitude of $D_i^{(1)}$ is much larger than $D_i^{(2)}$, see Section 3 of this chapter, the random effect of absorbing a laser photon (from either of the traveling waves) and then the stimulated re-emission of the photon (also in either of the two mode) does not give contributions to the deflection angle. The complete term is associated with the spontaneous emission of fluorescent photons. This deflection is proportional to the laser width because of the factor $k_L z$.

It is interesting to note that at this order of approximation for the resonant case there is not deflection due to the scattering force or recoil force, $\langle \overline{\delta\alpha^{(1)}} \rangle = 0$, when deflection due to the fluorescent effect still exists. The oscillating part of the diffusion tensor will first manifest on the atomic motion at fourth order in the parameter ϵ .

However, expression (3.3.10) is proportional to γ/η implying that, for the present order of approximation, the fluorescence effect does not affect the atomic motion.

Next we make the calculation for $\langle \overline{\delta\alpha^2} \rangle$,

(3.3.11)

$$<\overline{\delta\alpha^{(1)2}}>=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\,\overline{\delta\alpha^{(1)2}}=\frac{2v_{0x}^2v_{0y}^2}{v_0^4}\epsilon^2\left|\frac{v_0\Lambda n(w)}{k_L v_{0y}^2}\right|^2\left|1-\frac{\bar{w}^2}{2}-\frac{4}{\pi^2}E^2(\bar{w})\right|$$
$$=\frac{\epsilon_0^2}{2}\sin^2 2\alpha n^2(w)\left|1-\frac{\bar{w}^2}{2}-\frac{4}{\pi^2}E^2(\bar{w})\right|$$

here, the approximation concerning the evaluation of the integral $I\left|\frac{k_L z v_{0y}}{v_{0x}}\right|$ has been done as discussed below equation (3.3.7).

$$<\overline{\delta\alpha^{(2)2}}>=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\,\overline{\delta\alpha^{(2)2}}=\frac{\gamma\epsilon^2}{5\pi k_L v_{0y}}\frac{k_L z v_{0y}}{v_{0x}}\left|1-\frac{2n'(w)}{\pi}K(\bar{w})\right| \qquad (3.3.12)$$
$$=\frac{\gamma\epsilon^2}{5\pi\eta}k_L z\tan\alpha\left|1-\frac{2n'(w)}{\pi}K(\bar{w})\right|$$

The standard deviation is obtained from equations (3.3.8), (3.3.10), (3.3.11) and (3.3.12). The result at first order in ϵ is:

$$\Delta<\overline{\delta\alpha}>=\epsilon\left|\frac{2v_{0x}^2v_{0y}^2}{v_0^4}\left|\frac{v_0\Lambda n(w)}{k_L v_{0y}^2}\right|^2\left|1-\frac{\bar{w}^2}{2}-\frac{4}{\pi^2}E^2(\bar{w})\right|+\right. \qquad (3.3.13)$$
$$\left.\frac{\gamma}{5\pi\eta}\frac{k_L z v_{0y}}{v_{0x}}\left|1-\frac{2n'(w)}{\pi}K(\bar{w})\right|\right|^{1/2}$$

Since the second term is proportional to γ/η, it can be negligible compared with the leading term. If $v_{0x}\geqslant v_{0y}$ the deflection angle associated with the spontaneous emission is also negligible for the case $\gamma/\eta\ll 1$. To obtain the standard deviation up to second order in ϵ we need to calculate relations (3.3.11) and (3.3.12) at order ϵ^3.

$$\Delta<\overline{\delta\alpha}>=\frac{\epsilon v_{0x}v_{0y}\sqrt{2}}{v_0^2}\left|\frac{v_0\Lambda n(w)}{k_L v_{0y}^2}\right|\left|1-\frac{\bar{w}^2}{2}-\frac{4}{\pi^2}E^2(\bar{w})\right|^{1/2} \qquad (3.3.14)$$
$$=\frac{\sqrt{2}}{2}\epsilon_0\sin 2\alpha\,|n(w)|\left|1-\frac{\bar{w}^2}{2}-\frac{4}{\pi^2}E^2(\bar{w})\right|^{1/2}$$

Looking at equation (3.3.8) and (3.3.10) it is seen that the deflection angle of the atomic beam coming from the effect of the force is proportional to v_{0x}, when the deflection due to the spontaneous emission is inversely proportional to v_{0x}.

It is interesting to mention here that as a function of the incident angle α the maximum dispersion occurs at $\alpha = 45^{\circ}$, which is different than the value obtained for the maximum deflection which is at $\alpha \sim 55^{\circ}$. Here, the maximum for the standard deviation as a function of the detuning is at the same point as that obtained for the deflection. Considering that the standard deviation is proportional to ϵ_0 when the deflection angle is proportional to ϵ_0^2, we conclude that the over-all dispersion effect is much larger than the deflection even in the present results where there is not accounting for the spreading due to the diffusion tensor.

The maximum deflection angle is $.0002^{\circ}$ for $\alpha \sim 55^{\circ}$ and $w \sim .35$ while the maximum standard deviation is about $.1^{\circ}$ for $\alpha = 45^{\circ}$ and $w \sim .35$.

From the result of this section, it can be concluded that for the present laser intensities the effect of the recoil force in the atomic motion for fast atoms is very small, see (3.3.8), requiring great experimental accuracy for its systematic analysis.

Appendix A

This appendix present the solution of several integrals used for the evaluation of the force and diffusion tensor. All these formulas are referred to in Chapter 2 . The first two formulas are a repetition of definitions given in that chapter.

$$\phi(\theta) = \theta - \frac{1}{\lambda}\arctan(\lambda\cot\theta) \tag{A.1}$$

$$\epsilon(\theta) = \left| \Delta\omega^2 + \Lambda^2\cos^2\theta \right|^{1/2} \tag{A.2}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} = \frac{4}{\epsilon(0)} K\left(\frac{|\Lambda|}{\epsilon(0)}\right) \tag{A.3}$$

$$\int_{-\pi}^{\pi} d\theta\, \epsilon(\theta) = 4\epsilon(0) E\left(\frac{|\Lambda|}{\epsilon(0)}\right) \tag{A.4}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \phi(\theta) = 0 \tag{A.5}$$

$$\int_{\vec{k}_L\cdot\vec{\rho}}^{\pi} \frac{d\theta}{\epsilon(\theta)} = -\frac{1}{\epsilon(0)} \mathbb{K}\left(\vec{k}_L\cdot\vec{\rho} - \pi \,|\, \frac{|\Lambda|}{\epsilon(0)}\right) \tag{A.6}$$

(A.7)

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\,\phi(\theta) = -\frac{2\pi}{\Lambda^2}\epsilon(0)(1+1/\lambda) + \frac{4\epsilon(0)}{\Lambda^2} E\left(\frac{|\Lambda|}{\epsilon(0)}\right) + \frac{4\Delta\omega^2}{\Lambda^2\epsilon(0)} K\left(\frac{|\Lambda|}{\epsilon(0)}\right)$$

$$\int_{\vec{k}_L\cdot\vec{\rho}}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta = \frac{1}{\Lambda^2}[\epsilon(\vec{k}_L\cdot\vec{\rho}) - \epsilon(0)] \tag{A.8}$$

$$\int_{\vec{k}_L\cdot\vec{\rho}}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\,\phi(\theta) = -\frac{\pi}{\Lambda^2}\epsilon(0)\left(1+\frac{1}{2\lambda}\right) + \frac{1}{\Lambda^2}\epsilon(\vec{k}_L\cdot\vec{\rho})\,\phi(\vec{k}_L\cdot\vec{\rho}) - \tag{A.9}$$

$$\frac{\Delta\omega^2}{\Lambda^2\epsilon(0)} \mathbb{K}\left(\vec{k}_L\cdot\vec{\rho} - \pi \,|\, \frac{|\Lambda|}{\epsilon(0)}\right) - \frac{\epsilon(0)}{\Lambda^2} \mathbb{E}\left(\vec{k}_L\cdot\vec{\rho} - \pi \,|\, \frac{|\Lambda|}{\epsilon(0)}\right)$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta \int_{\theta}^{\pi} \frac{d\alpha}{\epsilon(\alpha)} = \frac{4}{\Lambda^2} K\left(\frac{|\Lambda|}{\epsilon(0)}\right) - \frac{2\pi}{\Lambda^2} \tag{A.10}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\,\phi(\theta) \int_{\theta}^{\pi} \frac{d\alpha}{\epsilon(\alpha)} = -\frac{4\pi}{\Lambda^2}(1+\frac{1}{2\lambda})K\left(\frac{|\Lambda|}{\epsilon(0)}\right) + \tag{A.11}$$

$$\frac{8}{\Lambda^2} E\left(\frac{|\Lambda|}{\epsilon(0)}\right) K\left(\frac{|\Lambda|}{\epsilon(0)}\right) + \frac{8\Delta\omega^2}{\Lambda^2\epsilon^2(0)} K^2\left(\frac{|\Lambda|}{\epsilon(0)}\right)$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta\,\phi^2(\theta) = 0 \tag{A.12}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta \int_{\theta}^{\pi} \frac{d\alpha}{\epsilon(\alpha)} \phi(\alpha) = 0 \tag{A.13}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta \int_{\theta}^{\pi} \frac{d\alpha}{\epsilon(\alpha)} \int_{\theta}^{\pi} \frac{d\beta}{\epsilon(\beta)} = \frac{16}{\Lambda^2\epsilon(0)} K^2\left(\frac{|\Lambda|}{\epsilon(0)}\right) - \frac{8\pi}{\Lambda^2\epsilon(0)} K\left(\frac{|\Lambda|}{\epsilon(0)}\right) \tag{A.14}$$

$$\int_{\vec{k}_L\cdot\vec{\rho}}^{\pi} \frac{d\theta}{\epsilon(\theta)} \sin\theta\cos\theta \int_0^{\pi} \frac{d\alpha}{\epsilon(\alpha)} = -\frac{\epsilon(\vec{k}_L\cdot\vec{\rho})}{\Lambda^2\epsilon(0)} \mathbb{K}\left[\vec{k}_L\cdot\vec{\rho}-\pi \mid \frac{|\Lambda|}{\epsilon(0)}\right] - \frac{(\pi-\vec{k}_L\cdot\vec{\rho})}{\Lambda^2} \qquad \text{(A.15)}$$

$$\mathbb{K}\left[\vec{k}_L\cdot\vec{\rho}-\pi \mid \frac{|\Lambda|}{\epsilon(0)}\right] = \frac{-2}{\epsilon(0)} K\left[\frac{|\Lambda|}{\epsilon(0)}\right] + \mathbb{K}\left[\vec{k}_L\cdot\vec{\rho} \mid \frac{|\Lambda|}{\epsilon(0)}\right] \qquad \text{(A.16)}$$

Figure 1 Sz max. vs w

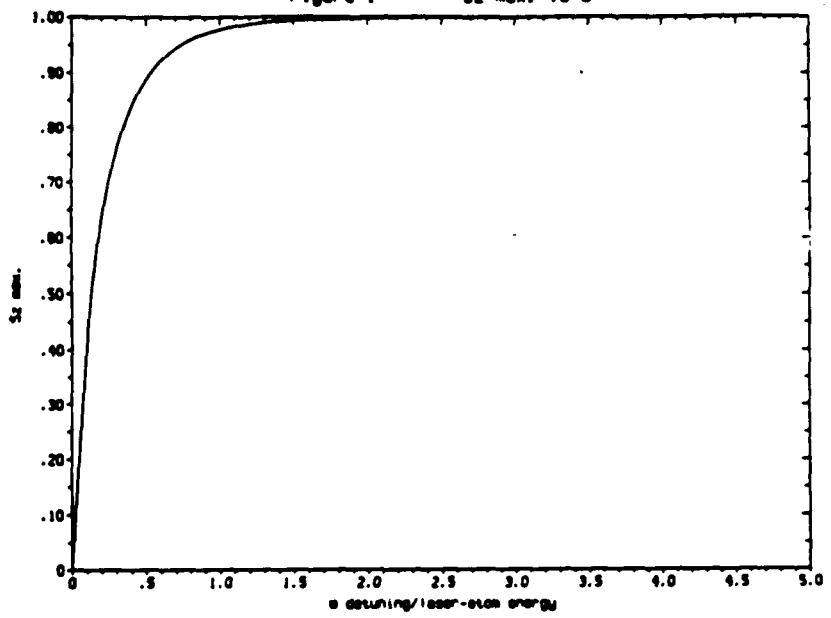


Figure2 D1 vs theta, w=0, eta>gamma

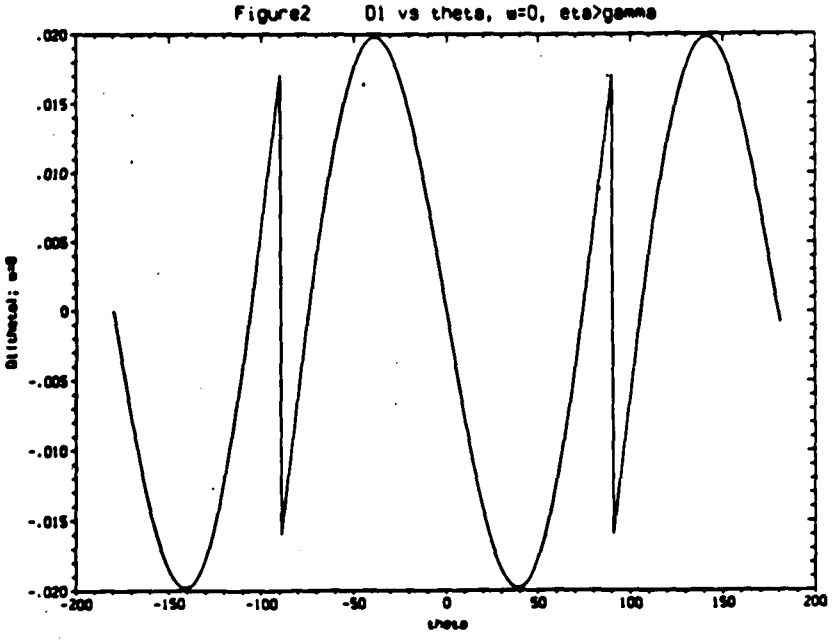


Figure 3 D1~n1 vs theta, gamma<eta

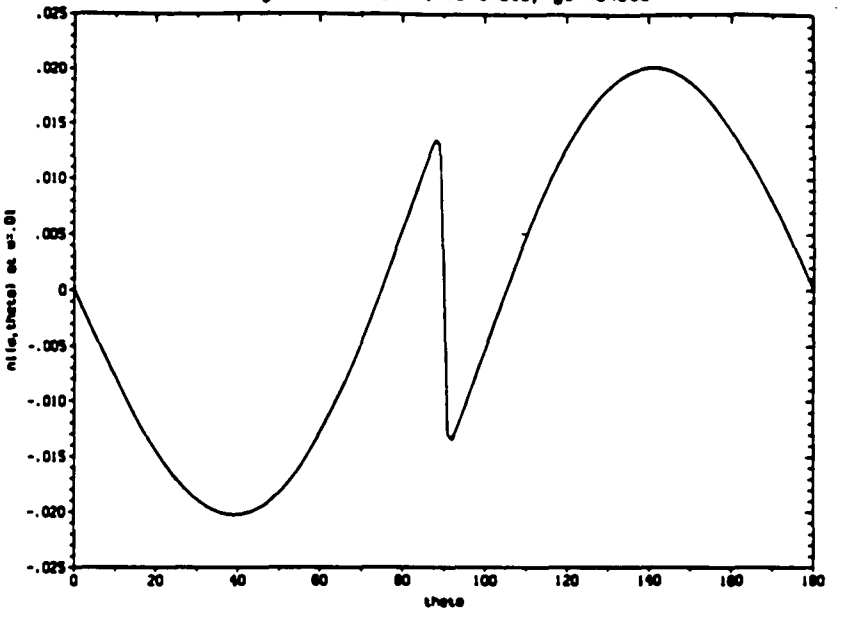
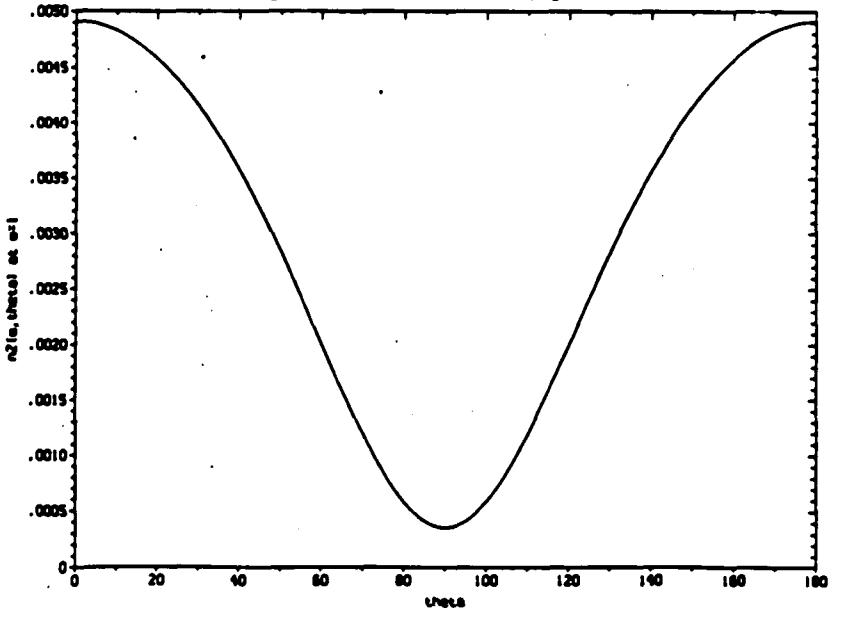


Figure 4 D2~n2 vs theta, gamma<eta



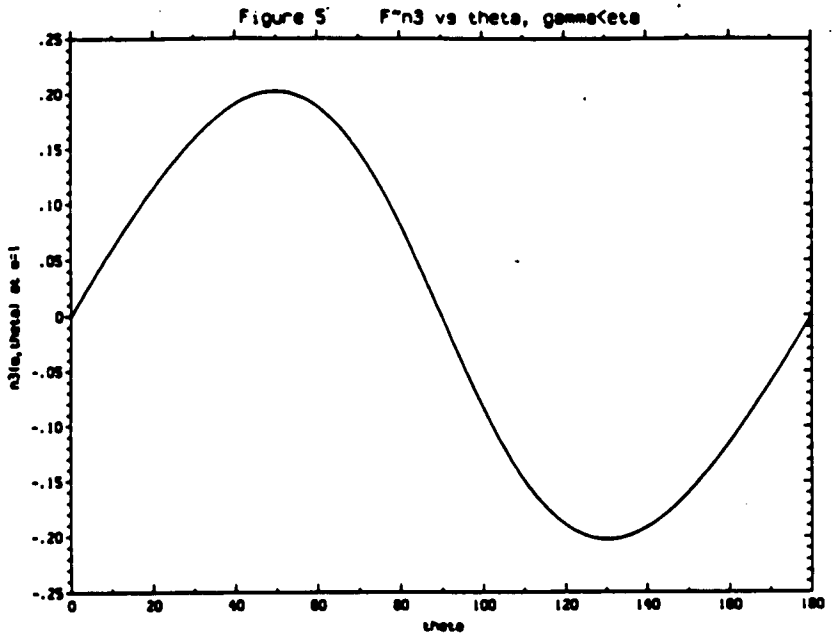


Figure 5 F"n3 vs theta, gamma<eta
.25
.20
.15
.10
.05
0
-.05
-.10
-.15
-.20
-.25
0
20
40
60
80
100
120
140
160
180
theta

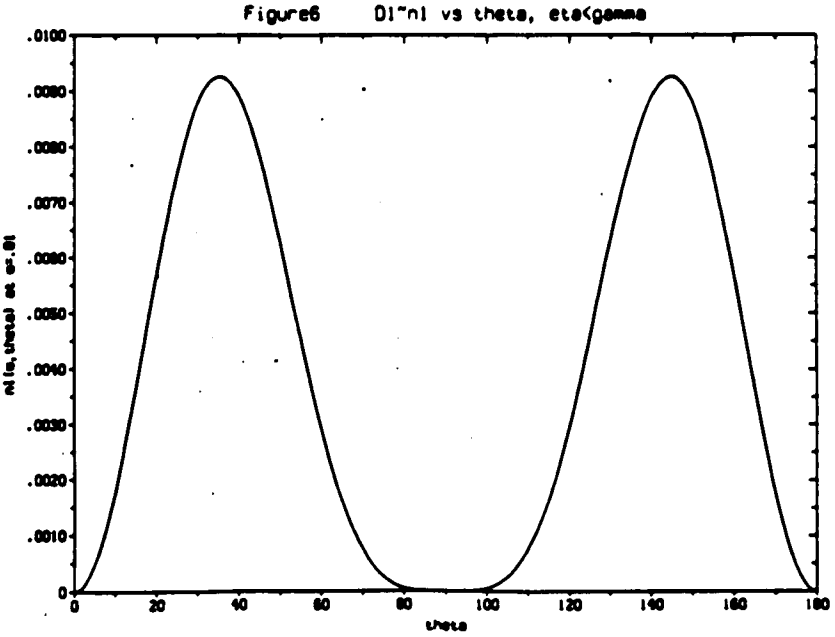


Figure6 D1"n1 vs theta, eta<gamma
.0100
.0090
.0080
.0070
.0060
.0050
.0040
.0030
.0020
.0010
0
0
20
40
60
80
100
120
140
160
180
theta

Figure 7 D2^n2 vs theta, eta<gamma

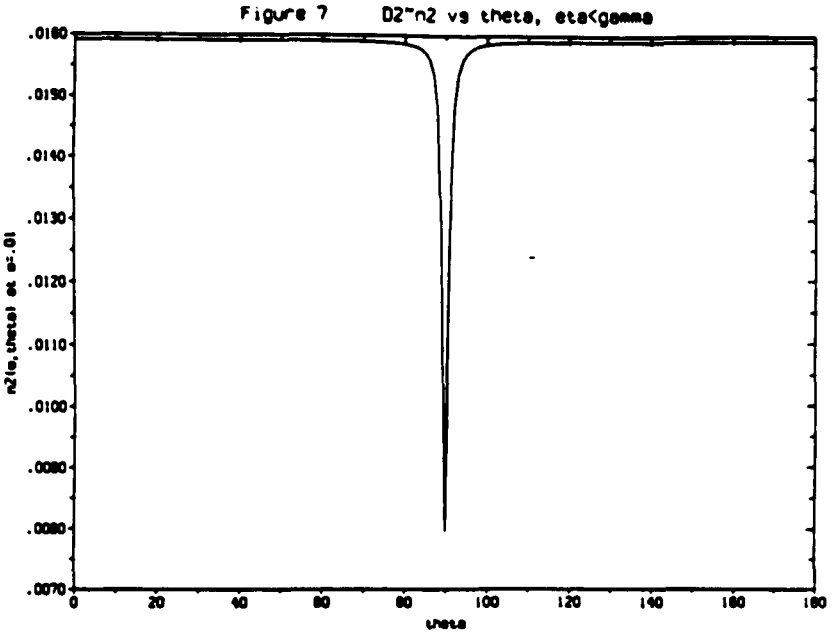
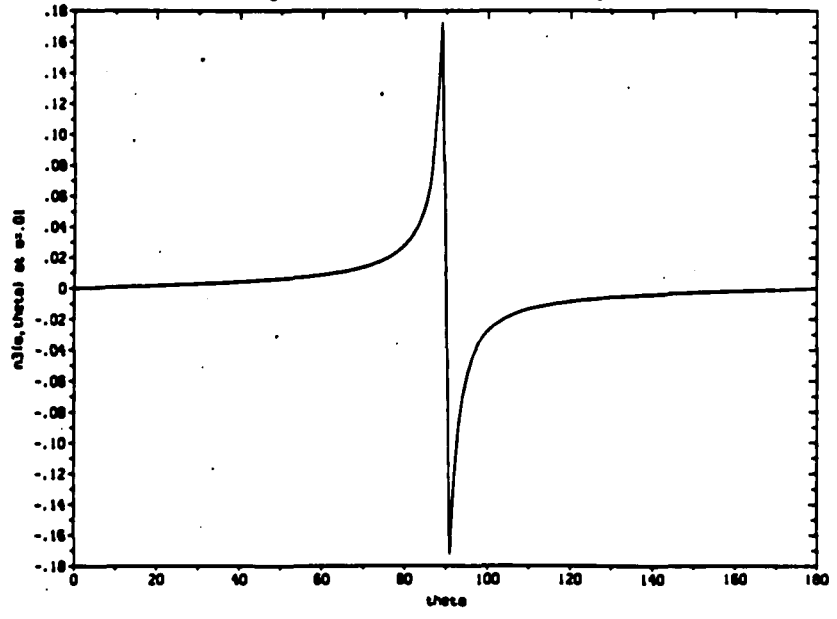


Figure 8 F^n3 vs theta, eta<gamma



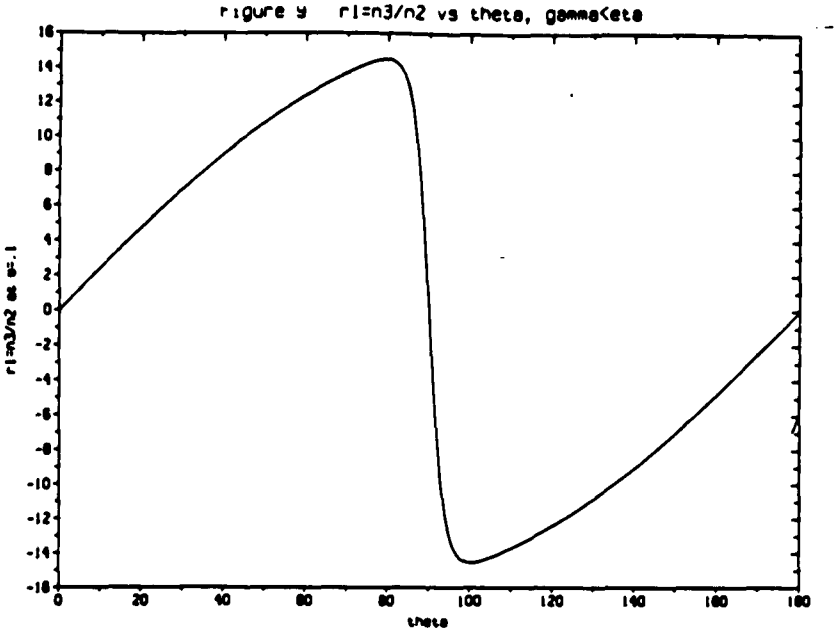


Figure 9 r1=n3/n2 vs theta, gamma<eta
r1=n3/n2 at e=.1
theta

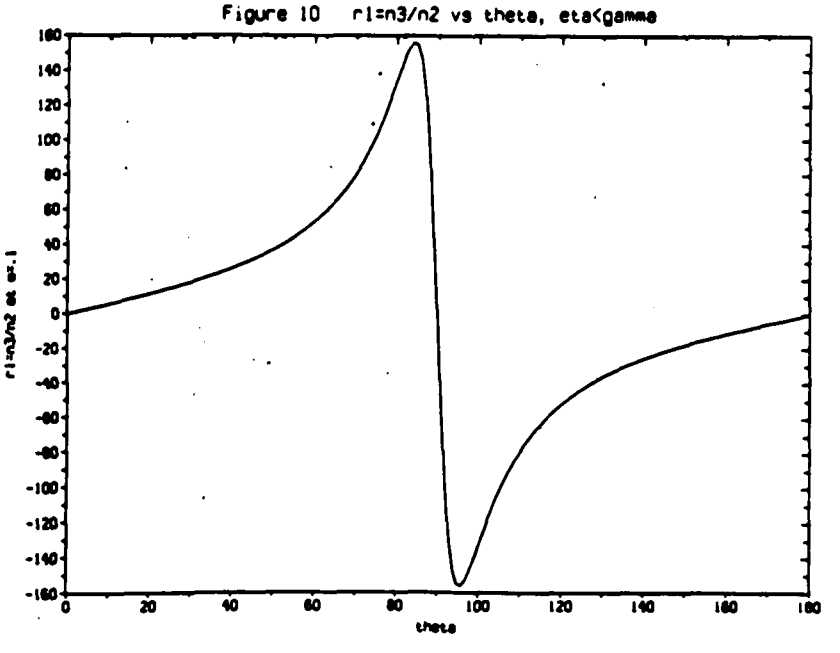


Figure 10 r1=n3/n2 vs theta, eta<gamma
r1=n3/n2 at e=.1
theta

Figure 11 r2=n1/n3 vs theta, gamma<eta

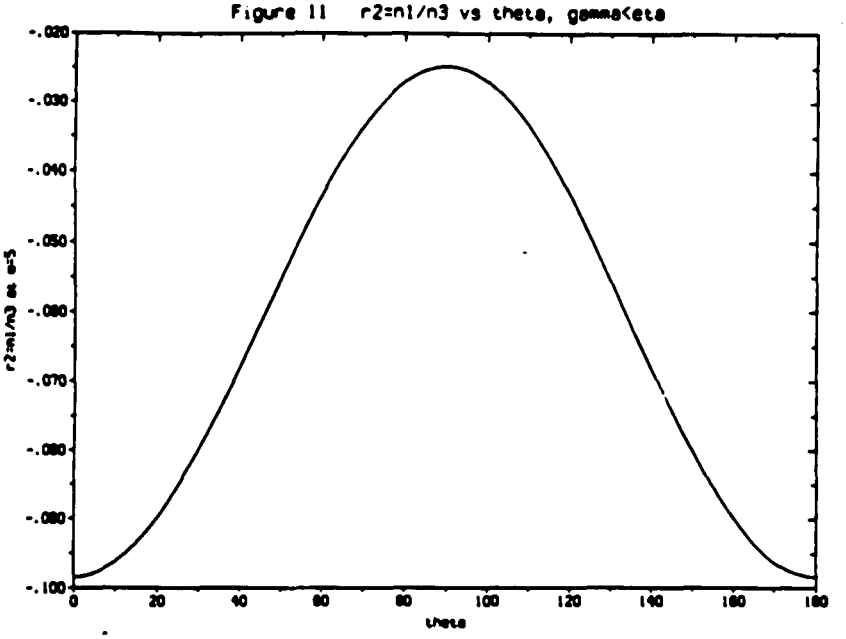


Figure 12 r2=n1/n3 vs theta, eta<gamma

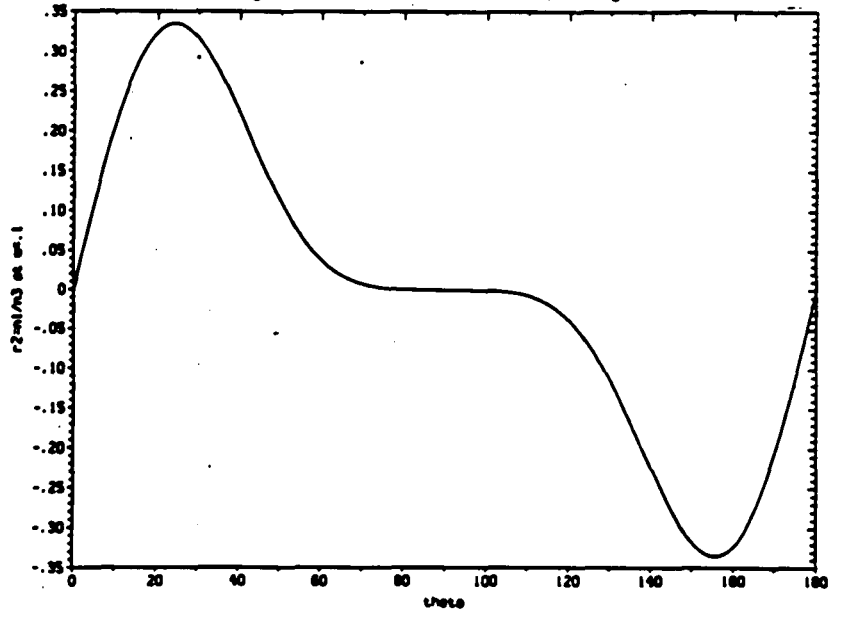


Figure 13 r3=n1/n2 vs theta, gamma<eta

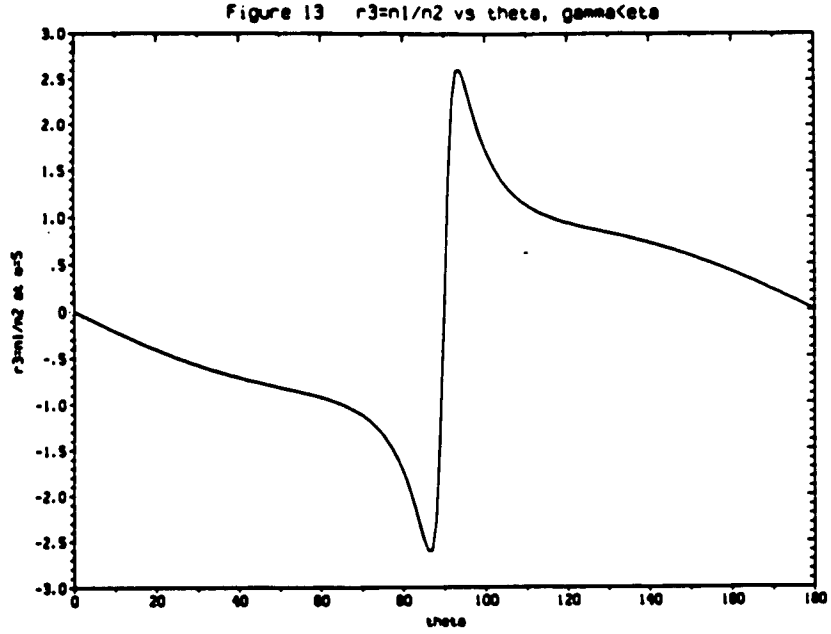


Figure 14 r3=n1/n2 vs theta, eta<gamma

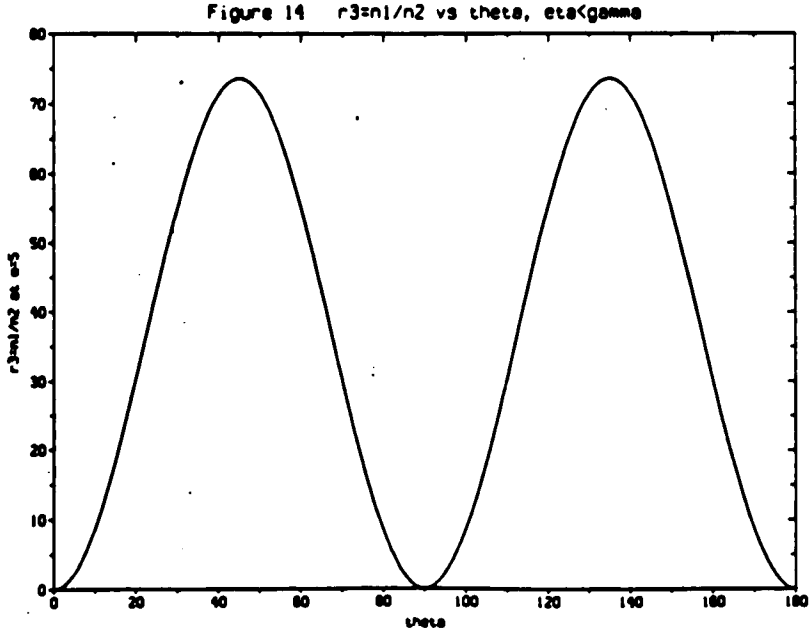


Figure 15 deflection vs alpha

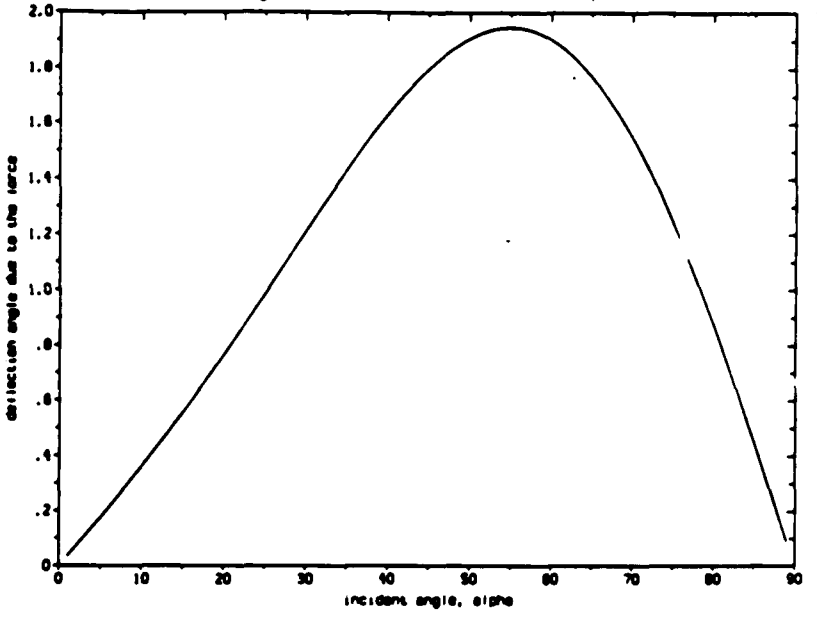
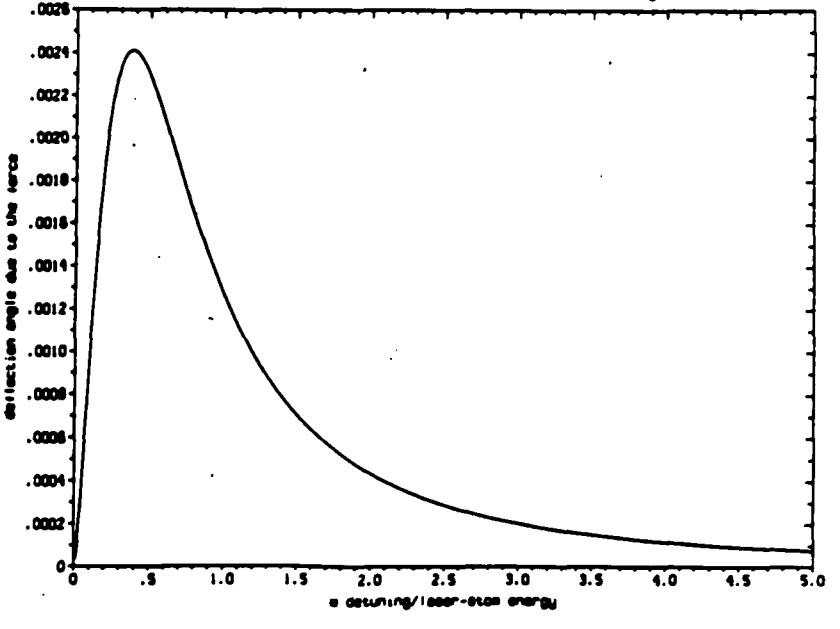


Figure 16 deflection vs detuning



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