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RELATIVISTIC NEUTRON GAS INTERACTING THROUGH A SPIN 2 FIELD AND ITS ASTROPHYSICAL APPLICATIONS

by

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Abstract

RELATIVISTIC NEUTRON GAS INTERACTING THROUGH A SPIN 2 FIELD AND ITS ASTROPHYSICAL APPLICATIONS

by

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In this thesis we first present a Lagrangian formalism for an interacting massive spin 2 field. We then study, as a model for super-dense matter, a system of zero-temperature neutrons interacting via the exchange of scalar, vector and spin 2 mesons. The thermodynamic properties of the system are calculated in the relativistic Hartree approximation. The resulting equation of state is used to compute the bulk properties of a neutron star. The maximum mass and moment of inertia for a stable neutron star are found to be 1.75 $M_{\odot}$ ($M_{\odot}$ = solar mass) and 1.68 x 10^{45} gm cm^2, in very good agreement with the presently available observational bounds. The corresponding radius is found to be 10.7 km.

We find that the inclusion of the spin 2 interaction narrows the divergence of the existing relativistic and non-relativistic theories in their predictions of masses and moments of inertia for neutron stars.

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Chapter 1

INTRODUCTION

The recent great interest in the study of high-density matter has arisen because of the role played by neutron stars in current models of pulsars and compact binary X-ray sources. Specifically, the interest focuses on the allowed maximum mass of neutron stars due to the following reasons.

First, recent attempts to detect black holes rely entirely on the possibility of differentiating a black hole from a neutron star by its mass. For instance, observations[1] of the compact X-ray source Cygnus X-1 suggest it contains an accreting compact object of mass $\geqslant 4\ M_\odot$ ($M_\odot$ = solar mass), orbiting around a normal supergiant star. If one can show that a neutron star (even if rapidly rotating) cannot possibly have such a large mass, then the compact object in Cygnus X-1 could be identified with a black hole.

Second, the existence of high-mass neutron stars would put constraints on nuclear physics, the theory of gravitation, or both. Nuclear physics in principle provides the equation of state of matter at high densities, which then determines, given a specific theory of gravitation, the limiting mass of a stable neutron star. Existing many-body nuclear theories, together with Einstein's theory of gravitation, predict a limiting mass in the range 1.4 - 2.7 $M_\odot$ (Refs. 2-4). However, by modifying the theory of general relativity, one can construct a stable neutron star of almost any mass.[5,6]

Third, according to the generally accepted theory of stellar evolution, a highly-evolved massive star develops a degenerate core that grows by accreting mass from the surrounding envelope. As the core mass approaches the Chandrasekhar limit ($\sim$ 1.4 $M_{\odot}$), a thermonuclear runaway and/or dynamical collapse takes place which presumably leads to a supernova event and, in some cases, to the production of a neutron star remnant. A neutron star, formed in this process, will have a mass that is comparable to that of the core of the precursor star (i.e. $\sim$ 1.4 $M_{\odot}$). If the majority of observed neutron stars are found to have masses near 1.4 $M_{\odot}$, this will naturally lend support to conventional theories of advanced stellar evolution and neutron star formation.

The matter inside neutron stars is cold by microscopic standards. Within a few days of their formation, they cool by neutrino emission to interior temperatures of less than 1 MeV ($\simeq 10^{10}$ ^{0}K), and throughout most of their early life the temperature is $\sim$ 10 - 100 KeV (Refs. 9,10). The physics of the equation of state of such matter divides into four general density regions. (i) At the very lowest densities, there is a lattice of bare nuclei (^{56}Fe) immersed in a degenerate electron gas.[11] This state persists to densities up to 10^7 gm cm^{-3}. (ii) The next density region is characterized by neutron-rich nuclei[12] (up to ^{118}Kr) which come about because protons undergo inverse beta decay, in order to lower the electron fermi energy. (iii) At a density of 4.3×10^{11} gm cm^{-3} (called the "Neutron Drip" point), the nuclei become so neutron-rich that with increasing density neutrons start to "drip" out of the nuclei.[12,13] This region is characterized by neutron-rich nuclei permeated by a sea of superfluid neutrons.[14] (iv) At nuclear matter density, 2.4×10^{14} gm cm^{-3}, the nuclei merge into each other, losing their individual identities, and forming a fluid of uniform neutron matter, with a small percentage of protons and electrons and possibly pions and hyperons.[15,16]

The density regions (i) - (iii) are well-understood, qualitatively as well as quantitatively, in terms of the presently available models, and the results are fairly well-established. The same, however, cannot be said for region (iv). Firstly, the very short-range component of the nucleon-nucleon interaction plays an important role in this region.[17,18] None of the presently available computations has seriously focused on this problem. Secondly, at these densities conventional many-body calculations, which eliminate the explicit mesonic degrees of freedom in favor of a static potential description of the interaction, are of uncertain reliability. A fully relativistic theory is desirable. The few relativistic calculations available in the literature contain the assumption that the exchange of vector mesons between two nucleons provide the dominant interaction at high densities, and that this can be meaningfully extrapolated to obtain the behaviour of super-dense matter. In this dissertation we dispute the validity of this line of thought, and contend that when the nucleon-nucleon separation is very short ($\lesssim 0.4$ fermi), attractive forces due to the exchange of spin 2 mesons cannot be neglected.

Thus, this dissertation aims at presenting a relativistic many-body description of high-density neutrons in which, for the first time, forces deriving from the exchange of spin 2 mesons have been included in a consistent manner, in addition to the traditional scalar (spin 0) and vector (spin 1) interactions. The many-body calculations have been confined to zero temperature since neutron stars are essentially a $T = 0$ system. We find that the inclusion of spin 2 interaction brings strikingly new features in the behaviour of high-density matter. It provides the dominant force at high densities, and makes the equation of state (pressure-density relationship) rather soft. Moreover, beyond a critical density (which depends on the f^0 coupling constant), the pressure decreases monotonically with increasing density, approaching a negative asymptotic value.

The outline of this dissertation is as follows. In Chapter 2 we give a brief review of the previous work on high-density matter and underline the need to consider the f^0-meson interaction. The theory of higher spin fields, especially if they happen to be in interaction with external fields, presents non-trivial dynamical problems. In Chapter 3 we give a general review of a massive spin 2 field and an extended discussion of the problems that one encounters in formulating an interacting theory. The full formalism is developed in Chapter 4. Chapter 5 gives the detailed physical properties of a gas of neutrons interacting through a tensor field. The inclusion of scalar and vector fields and the full coupled field theory is presented in Chapter 7. The various theoretical models for dense matter that are presently available give substantially different values for the maximum mass of stable neutron stars. Our theory, by taking into account the spin-2 interaction in a relativistic manner, is able to reduce this discrepancy, and furthermore, it is in satisfactory agreement with the observational bounds.

Chapter 2

BACKGROUND

The basic input for the analysis of dense matter is the equation of state (i.e., pressure as a function of mass density $P(\rho)$). The earliest computation on dense matter and neutron star structure, due to Oppenheimer and Volkoff[19], was based on the assumption that dense neutrons are degenerate but non-interacting. Since then, several models for the equation of state have been proposed. However, at nuclear and super-nuclear densities, all versions of the equation of state differ because of different assumptions about the nucleon-nucleon interaction and different techniques employed to handle the many-body aspect of the problem.

Irrespective of detailed models, certain general statements can be made regarding the asymptotic behaviour of $P(\rho)$. For a system composed of relativistic free fermions, the energy per particle is proportional to $n^{1/3}$, where n is the fermion number density and one gets $P \to \rho c^2/3$. It was first shown by Zel'dovich[20] that if baryons interact with each other through a vector field, then the energy per particle will go like n, and one gets $P \to \rho c^2$. One can argue, on the basis of causality, that asymptotically P cannot exceed ρc^2; for if it did, the velocity of sound $c_s = (\partial P/\partial \rho)^{\frac{1}{2}}$ would exceed the velocity of light. However, this is not rigorously true, since it can be shown[21] that if the medium is an "active" or amplifying one (like in a laser), $c_s > c$ does not amount to a violation of causality; it does, however, if matter is in its ground state.

Specific attempts to describe high-density matter fall into the following general categories: (i) non-relativistic theories, (ii) statistical models and (iii) field theoretical models. A general review can be found in two articles by Canuto.[22,23] At high densities, mesonic degrees of freedom are inadequately represented by a static potential. Hence the non-relativistic theories are of uncertain reliability.

Methods[24-26] based on statistical models (essentially the statistical bootstrap models of Hagedorn[27] and Veneziano[28]) describe dense matter as a system of non-interacting zero-width baryon resonances, with a specific mass spectrum that supposedly takes care of all the interactions. Analysis of the mass spectrum however shows that it does not include the full repulsive component of the nucleon-nucleon interaction.[23]

The third approach aims at giving a relativistic microscopic description, by considering nucleons interacting through boson fields. One of the first such models was suggested by Zel'dovich[20], who considered baryons interacting through a classical massive vector field. Similar theories but involving purely scalar interactions have been discussed in the context of dense matter by Kalman.[29] Recently, Walecka[30] has proposed a relativistic mean-field calculation of high-density matter by considering the nucleon-nucleon scalar and vector interactions; the results indicate that asymptotically $P \rightarrow \rho c^2$. A similar model that includes non-linear interactions of the scalar field has been considered by Källman.[31] This approach is related to the theory of abnormal nuclear state proposed by Lee and Wick.[32] It suggests the possibility of a vanishing nucleon mass in symmetric nuclear matter. It is not clear whether one should expect such abnormal states in neutron star matter.

Recently, Canuto and Lodenquai[33] have considered ultra-dense matter in terms of Landau's hydrodynamical model[34] and attempted to deduce the velocity of sound in super-dense matter from high-energy p-p collision data.

From the theoretical standpoint, there is a priori no reason why one should stop with the scalar and vector interactions, especially as the density increases. Since the mass of the exchanged boson is inversely proportional to the range of the nuclear force, one must include the exchange of spin 2 mesons (mass 1260 MeV) in order to account for the very short-range ($\lesssim 0.4$ fermi) components of the nucleon-nucleon interaction.

Chapter 3

MASSIVE SPIN 2 FIELD: GENERAL REVIEW

3-1. Introduction.

The theory of higher spin fields was first proposed by Dirac[35], Fierz[36] and finally Fierz and Pauli[37]. It was later developed by Rarita and Schwinger[38] and various other workers. Generally speaking, there exist two different approaches to describe the field theory of higher spins. In the first approach,[39-41] the physically interacting field operators are the asymptotic field variables before and after the interactions have taken place. This approach has the advantage that it requires no complicated Lagrange function to describe the asymptotic field variables. Furthermore, these asymptotic field variables can be easily quantized by means of expansion in terms of creation and annihilation operators. Although this approach is simple and successful in perturbative applications, it has the disadvantage that the detailed structure of the interactions cannot be studied. Consequently, the dynamical aspects of the problem remain missing. The second approach, due mainly to Fierz and Pauli[37], gives emphasis to a Lagrangian formulation by requiring that all field equations and constraint conditions associated with higher spin fields be derivable from a generalized action principle. The advantage of this (classical) approach is that interactions can be introduced explicitly, and that Green functions can be computed. With a dynamical theory as our aim, we shall take this second approach. We shall first present, in Section 2, a discussion of the free spin 2 field together with a brief review of the earlier work. The theory of free spin 2 fields is a linear theory. That is. the equations of motion are linear in the field variables and their derivatives, and are derivable from a Lagrange function that is quadratic in these variables. The linear theory becomes inadequate when interactions are present. To incorporate source terms in the massive spin 2 field theory is a non-trivial problem, and in some sense there is no unique way of doing it. These points are discussed in Section 3 where we also present the criterion that we employ to develop the (non-linear) theory of interacting massive spin 2 fields.

3-2. Free Spin 2 Fields.

A massive spin 2 field can be described by symmetrical tensor of rank two, $\phi_{\mu\nu}$. Such a tensor field has ten linearly independent components, and in general will contain spin 2, spin 1 and two spin 0 components. It will describe spin 2 only after we have removed from it the ($3+1$)-component vector source ($\partial^{\mu}\phi_{\mu\nu}$) and the scalar source ($\mathrm{tr.}\,\phi_{\mu\nu}$) by means of the following constraint conditions:

$$\partial^{\mu}\phi_{\mu\nu} = 0 \tag{3.1}$$

$$\eta^{\mu\nu}\phi_{\mu\nu} = 0 \quad , \tag{3.2}$$

$\eta^{\mu\nu}$ = diag. (-1,1,1,1).

When the constraint conditions (3.1) and (3.2) are satisfied, the residual multiplicity of five will correspond to a spin 2 particle of non-zero mass. As long as the spin 2 field is a free field, a Lagrangian quadratic in the field variables and their derivatives can be devised that guarantees the constraint conditions (3.1) and (3.2), and gives a linear field equation. Single-parameter Lagrange functions having such properties have been devised by Rivers[42] and Nath[43]. Bhargava and Watanabe[44] have formulated a three-parameter Lagrange function from which they derived the field equations, the constraint conditions and also the condition that the field variable be symmetric in its indices. The significance of the parameters introduced by these authors is not clear; however, physical results such as the energy-momentum tensor, the free-field commutators, etc. are not dependent on these parameters. Prescription for constructing the generalized Lagrange function for a free system with arbitrary spin has also been given by Chang[45], who used the method of spin projection operators first introduced by Fronsdal[46]. This method gives non-local field equations. The non-localities are removed by introducing certain auxiliary fields. Chang has also constructed explicit Lagrangians for systems having spin $\leqslant 4$. For spin = $0, \frac{1}{2}, 1$, the results agree with the well-known local Lagrange functions; for spin = 3/2,2, Chang's results are equivalent to the ones previously obtained by Rarita and Schwinger[38] and by Fierz and Pauli[37].

The simplest form of the (linear) free spin 2 field equation, from which the constraint conditions (3.1) and (3.2) follow, can be written as follows:

$$(-\partial^2 + m^2)\phi_{\mu\nu} + \partial_\mu \partial^\lambda \phi_{\lambda\nu} + \partial_\nu \partial^\lambda \phi_{\mu\lambda} - \partial_\mu \partial_\nu \phi$$

$$-\eta_{\mu\nu}\left[(-\partial^2 + m^2)\phi + \partial_\kappa \partial_\lambda \phi^{\kappa\lambda}\right] = 0 \qquad (3.3)$$

where $\partial^2 \equiv \partial^\mu \partial_\mu$

$$\phi = \mathrm{tr.}\ \phi_{\mu\nu} = \eta^{\mu\nu}\phi_{\mu\nu}$$

m = mass of the spin 2 particle.

The Eq. (3.3) can be derived from the following quadratic Lagrangian:

$$\mathcal{L}^{(2)} = -\tfrac{1}{2}\left(\partial^\lambda \phi^{\mu\nu}\partial_\lambda \phi_{\mu\nu} + m^2 \phi^{\mu\nu}\phi_{\mu\nu} - \right.$$

$$\left.\partial^\lambda \phi\, \partial_\lambda \phi - m^2 \phi^2\right) - \partial_\mu \phi^{\mu\nu}\partial_\nu \phi + \partial_\mu \phi^{\mu\nu}\partial^\lambda \phi_{\lambda\nu} \qquad (3.4)$$

Now, taking the divergence of both sides of (3.3), we get

$$-\partial^2 \partial^\mu \phi_{\mu\nu} + m^2 \partial^\mu \phi_{\mu\nu} + \partial^2 \partial^\lambda \phi_{\lambda\nu} + \partial_\mu \partial_\nu \partial_\lambda \phi^{\mu\lambda}$$

$$-\partial^2 \partial_\nu \phi + \partial^2 \partial_\nu \phi - m^2 \partial_\nu \phi - \partial_\nu \partial_\kappa \partial_\lambda \phi^{\kappa\lambda} = 0$$

which can be re-written as

$$-\partial^2\partial^\mu\phi_{\mu\nu} + m^2\partial^\mu\phi_{\mu\nu} + \partial^2\partial^\lambda\phi_{\lambda\nu} + \partial_\mu\partial_\nu\partial_\lambda\phi^{\mu\lambda}$$
$$-\partial^2\partial_\nu\phi + \partial^2\partial_\nu\phi - m^2\partial_\nu\phi - \partial_\nu\partial_\kappa\partial_\lambda\phi^{\kappa\lambda} = 0$$

$$\therefore \qquad m^2(\partial^\mu\phi_{\mu\nu} - \partial_\nu\phi) = 0 \tag{3.5}$$

The trace of (3.3) is

$$(-\partial^2+m^2)\phi + \eta^{\mu\nu}\partial_\mu\partial^\lambda\phi_{\lambda\nu} + \eta^{\mu\nu}\partial_\nu\partial^\lambda\phi_{\mu\lambda}$$
$$-\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - 4\left[(-\partial^2+m^2)\phi + \partial_\kappa\partial_\lambda\phi^{\kappa\lambda}\right] = 0$$

Or,
$$-3m^2\phi + 2\partial^2\phi - 2\partial_\kappa\partial_\lambda\phi^{\kappa\lambda} = 0 \tag{3.6}$$

Solving (3.5) and (3.6) algebraically for $\partial^\mu\phi_{\mu\nu}$ and ϕ , we get

$$\partial^\mu\phi_{\mu\nu} = 0 \tag{3.1}$$

$$\phi = 0 \tag{3.2}$$

It is important to note here that in order to obtain (3.1) and (3.2), the mass term in the Lagrangian must be of the form given in Eq. (3.4), namely

$$-\tfrac{1}{2}m^2(\phi^{\mu\nu}\phi_{\mu\nu} - \phi^2) \tag{3.7}$$

This is the so-called Pauli-Fierz mass term for the spin 2 field. It is easy to verify that with other choices of mass terms:

$$-\frac{1}{2} m^2 \left(\phi^{\mu\nu} \phi_{\mu\nu} - a \phi^2 \right), \quad a \neq 1$$

one cannot retain the conditions (3.1) and (3.2) simultaneously. Hence, without the Pauli-Fierz structure there will be mixtures of lower spins, which may give rise to the negative-energy ghost properties.

3-3. Interacting Spin 2 Fields.

The source for an interacting spin 2 field must be a conserved tensor current, and the candidate for this is the total energy-momentum tensor of the system[47]. Coupling of this nature leads to a modification of field equations from linear to non-linear form. A familiar example is provided by the theory of gravitation, which corresponds to a massless spin 2 field. When one deals with interactions for a massive spin 2 field, the non-linear modifications of the mass term are not immediately clear. We shall consider the problem in two steps. First, we shall consider interactions for the massless case, and then we shall consider the problem of adding a mass to it. This is legitimate because the nature of the coupling to external fields is not influenced by the mass of the spin 2 field.

Setting m = 0 in Eq. (3.3), the equation of motion for a free massless spin 2 field is:

$$- \partial^2 \phi_{\mu\nu} + \partial_\mu \partial^\lambda \phi_{\lambda\nu} + \partial_\nu \partial^\lambda \phi_{\mu\lambda} - \partial_\mu \partial_\nu \phi$$

$$+ \eta_{\mu\nu} \left(\partial^2 \phi - \partial_\kappa \partial_\lambda \phi^{\kappa\lambda} \right) = 0$$

In the presence of interactions, this should become

$$- \partial^2 \phi_{\mu\nu} + \partial_\mu \partial^\lambda \phi_{\lambda\nu} + \partial_\nu \partial^\lambda \phi_{\mu\lambda} - \partial_\mu \partial_\nu \phi$$

$$+ \eta_{\mu\nu} \left(\partial^2 \phi - \partial_\kappa \partial_\lambda \phi^{\kappa\lambda} \right) = \theta_{\mu\nu} \qquad (3.8)$$

where $\theta_{\mu\nu}$ is some source function.

Now, the divergence of the left-hand side of (3.8) is identically zero. Therefore, for consistency, $\theta_{\mu\nu}$ must be a conserved quantity:

$$\partial^\mu \theta_{\mu\nu} = 0 \qquad (3.9)$$

The physical quantity that has this property is the total energy-momentum tensor of the system. The non-linearity of the problem now becomes obvious. In addition to the energy-momentum tensor of external fields ($t_{\mu\nu}$), $\theta_{\mu\nu}$ must contain the stress tensor $\theta^{(2)}_{\mu\nu}$ that arises from the Lagrangian $\mathcal{L}^{(2)}$ which is responsible for the left-hand side of Eq. (3.8). This means that the total Lagrangian should be

$$\mathcal{L}^{(2)} \rightarrow \mathcal{L}^{(3)} = \mathcal{L}^{(2)} + \phi^{\mu\nu}\,\theta^{(2)}_{\mu\nu}$$

Now $\mathcal{L}^{(3)}$, in turn, will give a cubic stress tensor $\theta^{(3)}_{\mu\nu}$, and the series continues to all orders making the final equations non-linear. When all the $\phi_{\mu\nu}$ -dependent terms in $\theta_{\mu\nu}$ are included by imposing the condition (3.9), one gets (as shown by Deser[48]) the well-known Einstein equations[49]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\kappa\, t_{\mu\nu} \qquad (3.10)$$

where κ characterizes the strength of spin 2 field's coupling. Eq. (3.10) is derivable from the following Lagrangian (ψ stands collectively for all the external fields):

$$\frac{1}{2\kappa}\mathcal{L}_E + \mathcal{L}(\psi, g) \qquad (3.11)$$

by noting the variations

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_E}{\delta g^{\mu\nu}} = G_{\mu\nu} \tag{3.12}$$

$$\frac{1}{2\sqrt{-g}} \frac{\delta \mathcal{L}(\psi, g)}{\delta g^{\mu\nu}} = t_{\mu\nu} \tag{3.13}$$

where $\delta/\delta g^{\mu\nu}$ stands for the Euler-Lagrange variation. Here,

$$g = \det.\ g^{\mu\nu} \tag{3.14}$$

$$\mathcal{L}_E = -\sqrt{-g}\ g^{\mu\nu} R_{\mu\nu} \tag{3.15}$$

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu} \Gamma^{\beta}_{\alpha\beta} + \partial_\nu \Gamma^{\alpha}_{\mu\alpha} - \partial_\alpha \Gamma^{\alpha}_{\mu\nu} \tag{3.16}$$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}) \tag{3.17}$$

We next turn to the problem of adding a mass in Eq. (3.10). The fully interacting theory with zero mass being non-linear, there is no reason for the mass term to be linear. The problem is further complicated by the fact that we do not have here an equivalent of the Pauli-Fierz mass criterion of the free field (or weak source) case. We shall think of the mass term in terms of a Lorentz-invariant theory, namely as a term $\frac{m^2}{2} H_{\mu\nu}$ to be added to $G_{\mu\nu}$ in Eq. (3.10).

Since the mass implies a range or asymptotic fall-off of forces, $H_{\mu\nu}$ must be independent of field derivatives. Thus we shall write the equation for the interacting massive spin 2 field as

$$G_{\mu\nu} + \frac{m^2}{2} H_{\mu\nu} = -K\, t_{\mu\nu} \qquad (3.18)$$

The corresponding Lagrangian will be of the following form:

$$\frac{1}{2K}\left(\mathcal{L}_E + m^2 \mathcal{L}_m \right) + \mathcal{L}(\psi, g) \qquad (3.19)$$

where $\mathcal{L}_m$ denotes the massive part of the Lagrangian, and is independent of field derivatives.

At this point we stress the essential difference between the massless and the massive cases. Unlike the massless case, there is no (Riemannian) geometric interpretation of the massive theory. Consequently, no general covariance is implied in the massive theory, and as far as the dynamics is concerned, the analogy to gravitation is purely formal. All tensor indices indicate Lorentz indices. Because of the above reasons we shall introduce different symbols $g_{\mu\nu}$ and its reciprocal $h^{\mu\nu}$. Raising or lowering of indices will be performed solely by using the Minkowski tensor $\eta_{\mu\nu}$, and not $h^{\mu\nu}$ or $g_{\mu\nu}$. Thus for example

$$g^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} g_{\alpha\beta}$$

and not $\quad g^{\mu\nu} = h^{\mu\nu}$.

Instead of (3.15), $\mathcal{L}_E$ will now be defined by

$$\mathcal{L}_E = -\sqrt{-g}\, h^{\mu\nu} R_{\mu\nu} \equiv -\bar{g}^{\mu\nu} R_{\mu\nu} \qquad (3.20)$$

$R_{\mu\nu}$ is still defined by (3.16), but now

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} h^{\alpha\beta} \left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu} \right) \qquad (3.21)$$

$$g_{\mu\alpha} h^{\alpha\nu} = \delta^{\nu}_{\mu} \qquad (3.22)$$

$$g = \det . g_{\mu\nu} = \frac{1}{\det . h^{\mu\nu}} \qquad (3.23)$$

$$g_{\mu\nu} = \frac{1}{h^{\mu\nu}} = -\frac{1}{\sqrt{-g}} \text{ minor } \bar{g}^{\mu\nu} \qquad (3.24)$$

Lastly, instead of (3.12) and (3.13) we now have

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_E}{\delta h^{\mu\nu}} = G_{\mu\nu} \qquad (3.25)$$

$$\frac{1}{2\sqrt{-g}} \frac{\delta \mathcal{L}(\psi, g)}{\delta h^{\mu\nu}} = t_{\mu\nu} \tag{3.26}$$

In addition,

$$\frac{1}{2} H_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta h^{\mu\nu}} \tag{3.27}$$

The determination of $\mathcal{L}_m$ is not entirely free of ambiguities. In the absence of any stronger guideline, the criterion that we shall adopt to determine $\mathcal{L}_m$ is the following. Since the (linear) theory of free spin 2 is uniquely known, we shall demand that $\mathcal{L}_m$ reduce to the Pauli-Fierz mass structure (3.7) in the linear approximation:

$$g_{\mu\nu} = \eta_{\mu\nu} + k\,\phi_{\mu\nu}$$

Use of the above criterion rules out the cosmological term $\lambda\sqrt{-g}$ of the Einstein theory as a possible candidate for $\mathcal{L}_m$.

Going back to the massive spin 2 equation (3.18), we note that for any dynamical source, $t_{\mu\nu}$ will be automatically "covariantly" conserved by virtue of the ψ-field equations alone (see Appendix A). Likewise, $G_{\mu\nu}$ is identically conserved: $G_{\mu\nu;\nu} \equiv 0$. This means that the "covariant" divergence of $H_{\mu\nu}$ must vanish. This divergence condition provides, as in the linear theory, the four conditions to remove the vector and one of the scalar fields present in an arbitrary 10-component $\phi_{\mu\nu}$. The trace of Eq. (3.18) implies

$$-h^{\mu\nu} R_{\mu\nu} + \frac{1}{2} m^2 h^{\mu\nu} H_{\mu\nu} = -\kappa\, h^{\mu\nu}\, t_{\mu\nu} \tag{3.28}$$

Now, the left-hand side of Eq. (3.28) involves terms which depend on field derivatives. Therefore, the trace condition (3.28) can no longer be considered as a constraint condition, as it was in the linear theory. The fully interacting theory will thus always contain a scalar mode. However, the important point is that notwithstanding the ambiguities of the massive non-linear theory, in our treatment we make sure that in the linear approximation we recover the unique Pauli-Fierz mass structure, so that at least in the linear limit the theory is free of ghost excitations.

Lastly, we mention two papers that have attempted to construct a general theory of interacting massive spin 2 fields. Ogievetsky and Polubarinov[50] have constructed Lagrangians, characterized by two parameters p and q , that lead to the subsidiary condition

$$\partial^{\mu} \phi_{\mu\nu} + q\, \partial_{\nu} \phi^{\mu}_{\mu} = 0$$

instead of (3.1) and (3.2). For a particular choice of the parameters ($q = 0$; $p = -2$), this theory corresponds to a field of pure spin 2. Their source term, however, is not the total energy-momentum tensor $\Theta_{\mu\nu}$ but rather

$$\Xi_{\mu\nu} \equiv \Theta_{\mu\nu} - \frac{1}{3m^2}\left(\partial_\mu \partial_\nu + \eta_{\mu\nu}\partial^2\right)\Theta^\lambda_\lambda$$

The physical interpretation of such a source term is not quite clear. Freund et al.[51] have tried to construct a massive version of Einstein's equations taking an approach similar to the one outlined in this section. However, their theory is unsatisfactory even at the linear level, as their mass term does not satisfy the Pauli-Fierz criterion.

To summarize, we have shown that when interactions are present, a massive spin 2 theory must necessarily be non-linear. For the mass term we adopt the criterion that in the linear limit, the theory should reproduce the unique Pauli-Fierz mass structure. The determination of the mass term is carried out in the next chapter.

Chapter 4

THE INTERACTING MASSIVE SPIN 2 FIELD

4-1. The Lagrangian Formulation.

The Lagrangian formulation of the interacting spin 2 field will be given based on the postulate (as in the case of gravitation) that the symmetric energy-momentum tensor constitutes the source. For gravitation (massless spin 2 field), the Lagrangian formalism can be summarized as follows:

(i) the Lagrangian for the field is

$$\mathcal{L} = \frac{1}{2K} \mathcal{L}_E = -\frac{1}{2K} \sqrt{-g}\, h^{\mu\nu} R_{\mu\nu} \qquad (4.1)$$

[see Eq. (3.20)].

(ii) let us choose the "contravariant" $h^{\alpha\beta}$ as the variational variable and calculate the variation of the Lagrangian:

$$\frac{\delta \mathcal{L}}{\delta h^{\alpha\beta}} \equiv \partial_\gamma \frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}{}_{,\gamma}} - \frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}} \qquad (4.2)$$

In terms of the variable $\bar{g}^{\alpha\beta} \equiv \sqrt{-g}\, h^{\alpha\beta}$,

$$\frac{\delta \mathcal{L}}{\delta h^{\alpha\beta}} = \frac{\delta \mathcal{L}}{\delta \bar{g}^{\gamma\delta}} \frac{d\, \bar{g}^{\gamma\delta}}{d\, h^{\alpha\beta}} \qquad (4.3)$$

Now, $$d\,\bar{g}^{\alpha\beta} = dh^{\alpha\beta}\sqrt{-g} + h^{\alpha\beta}\,d\sqrt{-g}$$

But

$$\begin{aligned}
d\sqrt{-g} &= -\frac{1}{2}\,\frac{dg}{\sqrt{-g}} \\
&= -\frac{1}{2}\,\frac{g}{\sqrt{-g}}\;h^{\lambda\kappa}\,dg_{\lambda\kappa} \\
&= \frac{1}{2}\sqrt{-g}\;h^{\lambda\kappa}\,dg_{\lambda\kappa} \\
&= -\frac{1}{2}\sqrt{-g}\;g_{\lambda\kappa}\,dh^{\lambda\kappa}\,,
\end{aligned} \tag{4.4}$$

since $\quad h^{\alpha\beta}\,g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

$$\therefore\quad d\,\bar{g}^{\gamma\delta} = \sqrt{-g}\left(dh^{\gamma\delta} - \frac{1}{2}h^{\gamma\delta}g_{\lambda\kappa}\,dh^{\lambda\kappa}\right) \tag{4.5}$$

Using (4.5), (4.3) is

$$\frac{\delta\mathcal{L}}{\delta h^{\alpha\beta}} = \sqrt{-g}\left\{\frac{\delta\mathcal{L}}{\delta\bar{g}^{\alpha\beta}} - \frac{1}{2}\,\frac{\delta\mathcal{L}}{\delta\bar{g}^{\gamma\delta}}\,h^{\gamma\delta}g_{\alpha\beta}\right\} \tag{4.6}$$

For convenience, we shall use the variable $\bar{g}^{\alpha\beta}$ instead of $h^{\alpha\beta}$.

(iii) we note the (Eddington) identity[52]

$$\frac{\delta \mathcal{L}}{\delta \bar{g}^{\alpha\beta}} = \partial_\gamma \frac{\partial \mathcal{L}}{\partial \bar{g}^{\alpha\beta}{}_{,\gamma}} - \frac{\partial \mathcal{L}}{\partial \bar{g}^{\alpha\beta}}$$

$$= R_{\alpha\beta} \tag{4.7}$$

Then (4.6) becomes

$$\frac{\delta \mathcal{L}}{\delta h^{\alpha\beta}} = \sqrt{-g} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h^{\gamma\delta} R_{\gamma\delta} \right) \tag{4.8}$$

Setting

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h^{\gamma\delta} R_{\gamma\delta} \tag{4.9}$$

we get

$$\frac{\delta \mathcal{L}}{\delta h^{\alpha\beta}} = \sqrt{-g}\; G_{\alpha\beta} \tag{4.10}$$

(iv) the canonical stress tensor density of the gravitational field is (in mixed components)

$$\bar{T}^{\alpha}_{\beta} = h^{\mu\nu}{}_{,\beta} \frac{\partial \mathcal{L}}{\partial h^{\mu\nu}{}_{,\alpha}} - \delta^{\alpha}_{\beta} \mathcal{L} + \bar{S}^{\alpha}_{\beta}$$

$$= \bar{g}^{\mu\nu}{}_{,\beta} \frac{\partial \mathcal{L}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha}} - \delta^{\alpha}_{\beta} \mathcal{L} + \bar{S}^{\alpha}_{\beta} \tag{4.11}$$

Here $\bar{S}^{\alpha}_{\ \beta}$ stands for the symmetrizing Belinfante term. It depends on the derivatives of the field; but for the present, its detailed form is irrelevant.

(v) we can re-write Eq. (4.9) in mixed component form as

$$h^{\alpha\gamma} G_{\gamma\beta} = h^{\alpha\gamma} R_{\gamma\beta} - \frac{1}{2} \delta^{\alpha}_{\beta} h^{\gamma\delta} R_{\gamma\delta} \tag{4.12}$$

The quantity $h^{\alpha\gamma} G_{\gamma\beta}$ can be split into two parts: one linear in $\bar{g}^{\alpha\beta}$ and the other non-linear, which can be identified with the stress tensor of the gravitational field.[53] We write this as

$$h^{\alpha\gamma} G_{\gamma\beta} = \frac{1}{\sqrt{-g}} \left\{ \eta_{\beta\gamma} \bar{G}^{(L)\gamma\alpha} + \frac{1}{2} \bar{T}^{\alpha}_{\ \beta} \right\} \tag{4.13}$$

where $$\bar{G}^{(L)\gamma\alpha} = -\frac{1}{2} \left(\partial^2 \bar{g}^{\gamma\alpha} + D^{\gamma\alpha}_{\rho\sigma} \bar{g}^{\rho\sigma} \right), \tag{4.14}$$

$$D^{\gamma\alpha}_{\rho\sigma} \equiv \eta^{\gamma\alpha} \partial_{\rho} \partial_{\sigma} - \delta^{\alpha}_{\sigma} \partial^{\gamma} \partial_{\rho} - \delta^{\gamma}_{\rho} \partial^{\alpha} \partial_{\sigma} \tag{4.15}$$

(vi) the Einstein field equations then are

$$h^{\alpha\gamma} G_{\gamma\beta} = -\kappa\, h^{\alpha\gamma} t_{\gamma\beta} \tag{4.16}$$

where $t_{\gamma\beta}$ is the energy-momentum tensor of matter [given by Eq. (3.26)].

Separating the left-hand side of (4.16) into terms that are linear and non-linear in the variable $\bar{g}^{\mu\nu}$, we can re-write (4.16) as

$$\frac{1}{\sqrt{-g}}\left\{-\eta_{\beta\gamma}\left(\partial^2 \bar{g}^{\gamma\alpha} + D^{\gamma\alpha}_{\rho\sigma}\,\bar{g}^{\rho\sigma}\right) + \bar{T}^{\alpha}_{\beta}\right\}$$

$$= -2\kappa\, h^{\alpha\gamma}\, t_{\gamma\beta}\,. \tag{4.17}$$

4-2. Criterion for the Mass Part.

Following the above procedure, the massive spin 2 equation can be set up as follows:

(i) the equation should have the form

$$G_{\alpha\beta} + \frac{1}{2}m^2 H_{\alpha\beta} = -\kappa\, t_{\alpha\beta} \tag{4.18}$$

where m is the mass of the spin 2 particle. The corresponding Lagrangian should be of the form

$$\mathcal{L} = \frac{1}{2\kappa}\left(\mathcal{L}_E + m^2 \mathcal{L}_m\right) \tag{4.19}$$

where $\mathcal{L}_m$ denotes the massive part of the Lagrangian.

Being the mass term, $\mathcal{L}_m$ will be assumed to be free of field derivatives. Furthermore, like Eq. (4.10) for $G_{\alpha\beta}$, $H_{\alpha\beta}$ should be given by

$$\frac{1}{2} m^2 H_{\alpha\beta} = \frac{m^2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta h^{\alpha\beta}}$$

$$= \frac{m^2}{\sqrt{-g}} \frac{\partial \mathcal{L}_m}{\partial h^{\alpha\beta}} , \tag{4.20}$$

since there is no derivative coupling in $\mathcal{L}_m$.

(ii) like $G_{\alpha\beta}$, $H_{\alpha\beta}$ should be decomposable into a linear and a non-linear part in the following manner

$$\frac{1}{2} m^2 h^{\alpha\gamma} H_{\gamma\beta} = \frac{1}{\sqrt{-g}} \left\{ \frac{m^2}{2} \eta_{\beta\gamma} \overline{H}^{(L)\gamma\alpha} + \frac{1}{2} \overline{T}^{(m)\alpha}{}_{\beta} \right\} \tag{4.21}$$

where $\overline{T}^{(m)\alpha}{}_{\beta}$ = energy-momentum tensor arising from the massive part of the Lagrangian.

$$= - m^2 \delta^\alpha_\beta \mathcal{L}_m \tag{4.22}$$

The bar over $\overline{H}^{(L)\gamma\alpha}$ stands to indicate that $\overline{H}^{(L)\gamma\alpha}$ is a tensor density. We shall determine $\overline{H}^{(L)\gamma\alpha}$ by requiring that in the linear approximation

$$h^{\mu\nu} = \eta^{\mu\nu} - \kappa \phi^{\mu\nu}$$

$\overline{H}^{\gamma\alpha}_{(L)}$ should be identical to the Pauli-Fierz structure (with the appropriately redefined field variables). That is

$$\overline{H}^{\gamma\alpha}_{(L)} = \frac{\partial \mathcal{L}_m^{(L)}}{\partial \phi^{\gamma\alpha}} \tag{4.23}$$

where $\mathcal{L}_m^{(L)}$ represents the Pauli-Fierz form, (3.7), for the massive part of the Lagrangian.

4-3. The Determination of $\mathcal{L}_m$.

As discussed in Section 2 of Chapter 3, the Pauli-Fierz mass structure for the spin 2 field is

$$\mathcal{L}_m^{(L)} = -\frac{1}{2}\left(\phi^{\mu\nu}\phi_{\mu\nu} - \phi^2 \right) \tag{4.24}$$

Now, Eq. (4.21) is, using (4.20) and (4.22),

$$\frac{m^2}{\sqrt{-g}} h^{\alpha\gamma} \frac{\partial \mathcal{L}_m}{\partial h^{\beta\gamma}} = \frac{1}{\sqrt{-g}} \left\{ \frac{m^2}{2} \eta_{\beta\gamma} \overline{H}^{\gamma\alpha}_{(L)} - \frac{m^2}{2} \delta^{\alpha}_{\beta} \mathcal{L}_m \right\} \tag{4.25}$$

Using Eq. (4.24), Eq. (4.23) becomes

$$\overline{H}^{\gamma\alpha}_{(L)}(\phi) = - \left(\phi^{\gamma\alpha} - \eta^{\gamma\alpha}\phi \right)$$

Since we linearize according to

$$h^{\mu\nu} = \eta^{\mu\nu} - \kappa\, \phi^{\mu\nu} \tag{4.26}$$

therefore, for correct dimensionality, we shall write $\bar{H}^{\gamma\alpha}_{(L)}$ as

$$\bar{H}^{\gamma\alpha}_{(L)}(\phi) = -\kappa \left(\phi^{\gamma\alpha} - \eta^{\gamma\alpha}\phi \right) \tag{4.27}$$

Since we have been using $\bar{g}^{\mu\nu}$ as the variational variable, we shall use, instead of (4.26), the following (equivalent) linearization:

$$\bar{g}^{\mu\nu} = \eta^{\mu\nu} - \kappa\, \psi^{\mu\nu} \tag{4.28}$$

We note the following connection between $\psi^{\mu\nu}$ and $\phi^{\mu\nu}$:

$$\phi^{\mu\nu} = \psi^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\phi \tag{4.29}$$

where $$\psi \equiv \psi^{\mu\nu}\eta_{\mu\nu} = -\phi \tag{4.30}$$

In terms of ψ , Eq. (4.27) is

$$\bar{H}^{\gamma\alpha}_{(L)}(\psi) = -\kappa \left(\psi^{\gamma\alpha} + \tfrac{1}{2}\psi\, \eta^{\gamma\alpha} \right) \tag{4.31}$$

From (4.28), we have

$$\mathrm{tr.}\ \bar{g}^{\mu\nu} \equiv \bar{g} = 4 - \kappa\psi \tag{4.32}$$

Eq. (4.31) now yields the following tensor:

$$H^{(L)\gamma\alpha}(\bar{g}) \equiv \bar{H}^{(L)\gamma\alpha}(\bar{g}) \,/\, \sqrt{-g}$$

$$= \frac{1}{\sqrt{-g}}\left\{\bar{g}^{\gamma\alpha} - \eta^{\gamma\alpha} + \left(\tfrac{1}{2}\bar{g} - 2\right)\eta^{\gamma\alpha}\right\} \tag{4.33}$$

We can express the quantity $\partial \mathcal{L}_m / \partial h^{\alpha\beta}$ in terms of $\partial \mathcal{L}_m / \partial \bar{g}^{\alpha\beta}$ by noting (4.5):

$$\frac{h^{\alpha\gamma}}{\sqrt{-g}} \frac{\partial \mathcal{L}_m}{\partial h^{\beta\gamma}} = h^{\alpha\gamma} \frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\beta\gamma}} - \frac{1}{2}\delta^{\alpha}_{\beta} \frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\mu\nu}} h^{\mu\nu} \tag{4.34}$$

Then, Eq. (4.25) becomes

$$h^{\alpha\gamma} \frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\beta\gamma}} - \frac{1}{2}\delta^{\alpha}_{\beta} \frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\mu\nu}} h^{\mu\nu}$$

$$= \frac{1}{\sqrt{-g}}\left\{\frac{1}{2}\eta_{\beta\gamma}\bar{g}^{\gamma\alpha} - \frac{1}{2}\delta^{\alpha}_{\beta} + \frac{1}{2}\left(\frac{1}{2}\bar{g} - 2\right)\delta^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{L}_m\right\} \tag{4.35}$$

The trace of this gives

$$-h^{\mu\nu}\frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\mu\nu}} = \frac{1}{\sqrt{-g}}\left\{3\bar{g} - 8 - 8\mathcal{L}_m\right\} \tag{4.36}$$

Let us re-write the Einstein Eq. (4.17) as

$$\partial^2 \bar{g}^{\alpha\beta} + D^{\alpha\beta}_{\rho\sigma}\, \bar{g}^{\rho\sigma} = \eta^{\alpha\gamma}\, \bar{T}^{(0)\beta}{}_{\gamma} + 2\kappa\, \bar{t}^{\alpha\beta} \tag{4.37}$$

where the index (0) on $\bar{T}^{\alpha}_{\beta}$ has been put to mean that the equation refers to the massless spin 2 field, and

$$\bar{t}^{\alpha\beta} \equiv \sqrt{-g}\; \eta^{\alpha\gamma}\, h^{\beta\delta}\, t_{\gamma\delta}$$

Using (4.33), the following modification of (4.37) is suggestive when we have a massive spin 2 field:

$$\partial^2 \bar{g}^{\alpha\beta} + D^{\alpha\beta}_{\rho\sigma}\, \bar{g}^{\rho\sigma} - m^2 \left\{ \bar{g}^{\alpha\beta} - \eta^{\alpha\beta} + \tfrac{1}{2}\, \eta^{\alpha\beta} (\bar{g} - 4) \right\}$$

$$= \eta^{\alpha\gamma} \left(\bar{T}^{(0)\beta}{}_{\gamma} + \bar{T}^{(m)\beta}{}_{\gamma} \right) + 2\kappa\, \bar{t}^{\alpha\beta} \tag{4.38}$$

As for the massive part, we should have, using Eqs. (4.25) and (4.33),

$$2m^2 h^{\alpha\gamma} \frac{\partial \mathcal{L}_m}{\partial h^{\gamma\beta}} = m^2 \left[(\bar{g}^{\alpha}_{\beta} - \delta^{\alpha}_{\beta}) + \tfrac{1}{2}\, \delta^{\alpha}_{\beta} (\bar{g} - 4) \right] + \bar{T}^{(m)\alpha}_{\beta} \tag{4.39}$$

The divergence of (4.38) yields

$$\partial_{\alpha}\, \bar{g}^{\alpha\beta} + \tfrac{1}{2}\, \partial^{\beta}\, \bar{g} = 0 \tag{4.40}$$

Using (4.6) and (4.22), we can re-write Eq. (4.39) as

$$2\,\bar{g}^{\alpha\gamma}\,\frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\gamma\beta}} - \delta^{\alpha}_{\beta}\,\frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\rho\sigma}}\,\bar{g}^{\rho\sigma}$$

$$= \bar{g}^{\alpha}_{\beta} - 3\,\delta^{\alpha}_{\beta} + \tfrac{1}{2}\,\delta^{\alpha}_{\beta}\,\bar{g} - \delta^{\alpha}_{\beta}\,\mathcal{L}_m \tag{4.41}$$

Now all we have to do is to find an $\mathcal{L}_m$ obeying Eq. (4.41). In principle $\mathcal{L}_m$, which does not depend on the derivatives of the field, can depend on any invariant that we can build out of the field components. One such invariant is $p \equiv \sqrt{-\bar{g}}$. If $\mathcal{L}_m$ were to depend only on p, then Eq. (4.41) could not be obeyed [because of the presence of the term $\bar{g}^{\alpha}_{\beta}$]. In addition to p, $\mathcal{L}_m$ has to depend also on at least one more Lorentz-invariant quantity. The simplest such quantity is $q \equiv \eta_{\mu\nu}\,\bar{g}^{\mu\nu}$. We shall assume that $\mathcal{L}_m$ depends on $\bar{g}^{\mu\nu}$ only through the invariant combinations p and q.

Noting that

$$\frac{\partial p}{\partial \bar{g}^{\mu\nu}} = \frac{1}{2}\,g_{\mu\nu} \tag{4.42}$$

$$\frac{\partial q}{\partial \bar{g}^{\mu\nu}} = \eta_{\mu\nu} \tag{4.43}$$

we get

$$2\,\bar{g}^{\alpha\gamma}\frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\gamma\beta}} = 2\,\bar{g}^{\alpha\gamma}\left(\frac{1}{2}\,g_{\gamma\beta}\,\frac{\partial \mathcal{L}_m}{\partial p} + \eta_{\gamma\beta}\,\frac{\partial \mathcal{L}_m}{\partial q}\right)$$

$$= \delta^{\alpha}_{\beta}\,p\,\frac{\partial \mathcal{L}_m}{\partial p} + 2\,\bar{g}^{\alpha}_{\beta}\,\frac{\partial \mathcal{L}_m}{\partial q} \tag{4.44}$$

$$\frac{\partial \mathcal{L}_m}{\partial \bar{g}^{\rho\sigma}}\,\bar{g}^{\rho\sigma} = \left(\frac{1}{2}\,g_{\rho\sigma}\,\frac{\partial \mathcal{L}_m}{\partial p} + \eta_{\rho\sigma}\,\frac{\partial \mathcal{L}_m}{\partial q}\right)\bar{g}^{\rho\sigma}$$

$$= 2p\,\frac{\partial \mathcal{L}_m}{\partial p} + q\,\frac{\partial \mathcal{L}_m}{\partial q} \tag{4.45}$$

Substituting Eqs. (4.44) and (4.45) into Eq. (4.41), we get

$$\delta^{\alpha}_{\beta}\left\{\frac{1}{2}q - 3 - \mathcal{L}_m + q\,\frac{\partial \mathcal{L}_m}{\partial q} + 2p\,\frac{\partial \mathcal{L}_m}{\partial p} - p\,\frac{\partial \mathcal{L}_m}{\partial p}\right\}$$

$$+ \bar{g}^{\alpha}_{\beta}\left(1 - 2\,\frac{\partial \mathcal{L}_m}{\partial q}\right) = 0 \tag{4.46}$$

Unlike δ^{α}_{β} , $\bar{g}^{\alpha}_{\beta}$ in general does not have all its diagonal elements identical; hence Eq. (4.46) does not have a unique solution for $\mathcal{L}_m$.

In order to remedy this, we proceed as follows. We reconsider Eq. (4.38), in which we identified the part of

$\mathcal{L}_m$ that is non-linear in $\bar{g}^{\alpha\beta}$ as $\bar{T}^{(m)\alpha\beta}$, in analogy to gravitation. As we have just seen, such an analogy does not lead to a consistent solution for $\mathcal{L}_m$. We shall try the following replacement in Eq. (3.38):

$$\bar{T}^{(m)\alpha}{}_{\beta} \quad \rightarrow \quad (1+z) \quad \bar{T}^{(m)\alpha}{}_{\beta} \tag{4.47}$$

where z is a number to be fixed. To compensate for this added term, the divergence condition (4.40) gets modified to

$$m^2 \left(\partial_\alpha \bar{g}^{\alpha\beta} + \tfrac{1}{2} \partial^\beta \bar{g} \right) = - z \, \partial^\alpha \, \bar{T}^{(m)\beta}{}_{\alpha} \tag{4.48}$$

Instead of (4.46), we then have the following equation for $\mathcal{L}_m$:

$$\delta^\alpha_\beta \left\{ \tfrac{1}{2} q - 3 - (1+z) \mathcal{L}_m + q \frac{\partial \mathcal{L}_m}{\partial q} + p \frac{\partial \mathcal{L}_m}{\partial p} \right\}$$

$$+ \quad \bar{g}^\alpha{}_\beta \left(1 - 2 \frac{\partial \mathcal{L}_m}{\partial q} \right) = 0 \tag{4.49}$$

Eq. (4.49) suggests the following solution:

$$\mathcal{L}_m \equiv \tfrac{1}{2} q + K(p) \tag{4.50}$$

Substituting (4.50) into (4.49), we get

$$-3 + q \left\{ 1 - \tfrac{1}{2}(1+z) \right\} - (1+z) K(p) + p \frac{d K(p)}{dp} = 0 \tag{4.51}$$

It is now easy to see that (4.50) and (4.51) are consistent only if we set

$$z = 1 \tag{4.52}$$

Putting (4.51) into (4.51), we get

$$p \frac{dK(p)}{dp} - 2K(p) = 3 \tag{4.53}$$

Integrating,

$$K(p) = -\frac{3}{2} + \alpha p^2 \tag{4.54}$$

where α is an arbitrary constant of integration. Substi-titing for K(p) into (4.50), we get

$$\mathcal{L}_m = -\frac{3}{2} + \frac{q}{2} + \alpha p^2 \tag{4.55}$$

The constant of integration α can be chosen by requiring that $\qquad \mathcal{L}_m = 0$

when $\qquad \bar{g}^{\alpha\beta} = \eta^{\alpha\beta}$

i.e., when $\qquad \psi^{\alpha\beta} = 0$

Now, we note that when $\bar{g}^{\alpha\beta} = \eta^{\alpha\beta}$, we get the following values for p and q:

$$p = 1$$

$$q = 4$$

so that

$$0 = \mathcal{L}_m\left(\bar{g}^{\alpha\beta} = \eta^{\alpha\beta}\right) = -\tfrac{3}{2} + \tfrac{1}{2} - 4 + a$$

which determines a to be

$$a = -\tfrac{1}{2}$$

Substituting for a into Eq. (4.55), we finally get the following consistent solution for $\mathcal{L}_m$:

$$\mathcal{L}_m = \tfrac{1}{2}\left(g + \bar{g} - 3\right) \tag{4.56}$$

Going back to the divergence condition (4.48), we note that the right-hand side of (4.48) is (with z = 1):

$$\begin{aligned} -\partial^{\alpha}\,\bar{T}^{(m)}{}_{\alpha}^{\beta} &= m^2\,\delta^{\beta}_{\alpha}\,\partial^{\alpha}\mathcal{L}_m \\ &= m^2\,\partial^{\beta}\mathcal{L}_m \\ &= \frac{m^2}{2}\left(\partial^{\beta} g + \partial^{\beta}\bar{g}\right) \end{aligned} \tag{4.57}$$

using (4.56).

Using (4.57) in (4.48), we obtain

$$\partial_{\alpha}\,\bar{g}^{\alpha\beta} = \tfrac{1}{2}\,\partial^{\beta} g \tag{4.58}$$

On the other hand,

$$\partial_\alpha \bar{g}^{\alpha\beta} = \partial_\alpha \left(\sqrt{-g}\, h^{\alpha\beta} \right)$$

$$= \partial_\alpha \left(p\, h^{\alpha\beta} \right)$$

and $$\tfrac{1}{2}\, \partial^\beta g = \sqrt{-g}\; \partial^\beta \sqrt{-g}$$

$$= p\, \partial^\beta p \,,$$

so that Eq. (4.58) can be written as

$$\partial_\alpha \left\{ p \left(h^{\alpha\beta} - \tfrac{1}{2}\, p\, \eta^{\alpha\beta} \right) \right\}$$

$$\equiv \partial_\alpha \left(\bar{g}^{\alpha\beta} - \tfrac{1}{2}\, p^2\, \eta^{\alpha\beta} \right) = 0 \tag{4.59}$$

Eq. (4.59) is thus the divergence condition in the non-linear theory.

Turning now to the trace condition, the trace of Eq. (4.38) with $\bar{T}^{(m)}{}_\gamma^{\ \beta} \rightarrow (1+z)\bar{T}^{(m)}{}_\alpha^{\ \beta} = 2\bar{T}^{(m)}{}_\gamma^{\ \beta}$, is

$$\partial^2 \eta_{\alpha\beta}\, \bar{g}^{\alpha\beta} + \eta_{\alpha\beta}\, D^{\alpha\beta}_{\rho\sigma}\, \bar{g}^{\rho\sigma} - \eta_{\alpha\beta}\, m^2 \left\{ \bar{g}^{\alpha\beta} - \eta^{\alpha\beta} + \tfrac{1}{2}\eta^{\alpha\beta}(\bar{g}-4) \right\}$$

$$= \eta_{\alpha\beta}\, \eta^{\alpha\gamma} \left(\bar{T}^{(1)}{}_\gamma^{\ \beta} + 2\bar{T}^{(m)}{}_\gamma^{\ \beta} \right) + 2k\, \eta_{\alpha\beta}\, \bar{t}^{\alpha\beta} \tag{4.60}$$

Substituting for $\overline{T}^{(m)\beta}_{\ \gamma}$ which is now

$$\overline{T}^{(m)\beta}_{\ \gamma} = -m^2 \delta^\beta_\gamma \mathcal{L}_m$$

$$= -\frac{m^2}{2}\delta^\beta_\gamma \left(g + \bar{g} - 3 \right) \tag{4.61}$$

and carrying out the algebra, Eq. (4.60) becomes

$$\partial^2 \bar{g} + 2 \partial_\rho \partial_\sigma \bar{g}^{\rho\sigma} - m^2 \left(\bar{g} - 2g - 6 \right)$$

$$= \overline{T}^{(0)} + 2\kappa \bar{t} \tag{4.62}$$

where $\overline{T}^{(0)} \equiv \eta_{\alpha\beta} \overline{T}^{(0)\alpha\beta}$

$$\bar{t} \equiv \eta_{\alpha\beta} \bar{t}^{\alpha\beta}$$

Because Eq. (4.62) contains derivatives of the field, it is no longer a constraint condition. Thus in the interacting theory, the trace condition cannot be retained as a constraint condition.

To sum up, we have developed an expression for the Lagrangian of an interacting massive spin 2 field, which is given by

$$\frac{1}{2\kappa} \left(\mathcal{L}_E + m^2 \mathcal{L}_m \right) \tag{4.19}$$

where $\mathcal{L}_E$ and $\mathcal{L}_m$ are given by Eqs. (4.1) and (4.56).

The resulting equations of motion, which are non-linear, are

$$G_{\mu\nu} + \frac{1}{2} m^2 H_{\mu\nu} = -K\, t_{\mu\nu} \tag{4.18}$$

where $G_{\mu\nu}$ and $H_{\mu\nu}$ are given by Eqs. (4.10) and (4.20), and $t_{\mu\nu}$ represents the energy-momentum tensor of external fields. We have constructed the mass term in (4.19) in such a way that when the weak source or the linear limit is taken, we recover the uniquely known Pauli-Fierz structure of the free spin 2 theory.

Chapter 5

NEUTRON GAS INTERACTING THROUGH A SPIN 2 FIELD

5-1. Introduction.

In this chapter we present a relativistic description of a system of neutrons whose interactions are mediated via the exchange of massive spin 2 mesons. Thus, in the formalism developed in the preceding chapter we shall take fermions as providing the source term for the spin 2 field. We shall take the system to be at zero temperature, and by resorting to Hartree approximation, derive its thermodynamic properties. The Hartree approximation, in contrast to its non-relativistic counterpart, constitutes a non-trivial many-body problem. The parameters that determine the theory are (i) the density, or equivalently the fermi momentum, (ii) the spin 2 coupling constant and (iii) the mass of the spin 2 meson. However, in the Hartree approximation the latter two quantities do not enter independently; only their ratio is the relevant parameter. As we shall show, the properties of a fermi gas interacting through a spin 2 field are quite different from those of an ideal gas. At high densities, the pressure becomes a monotonically decreasing function of the density, eventually reaching a negative asymptotic value that is suggestive of collapse.

5-2. The Lagrangian.

As explained in Section 3 of Chapter 3, the total Lagrangian density of a system of fermions coupled to a massive spin 2 field can be written as

$$\mathcal{L}_{Tot} = \frac{1}{2K}\left(\mathcal{L}_E + m^2 \mathcal{L}_m \right) + \mathcal{L}(\psi, g) , \quad (5.1)$$

$\mathcal{L}_E$ and $\mathcal{L}_m$ being given by (4.1) and (4.56). To write down $\mathcal{L}(\psi, g)$, we shall use a formal analogy to gravitation. For ψ representing a fermion field, this is done by using the vierbein formalism. We shall introduce two sets of vierbein fields $d_{\mu a}$ and $e^{\mu a}$ ($\mu, a = 0,1,2,3$) defined by

$$d^a_\mu \, d^b_\nu \, \eta_{ab} = d^a_\mu \, d_{\nu a} = g_{\mu\nu} \quad (5.2)$$

$$e^{\mu a} \, e^{\nu b} \, \eta_{ab} = e^{\mu a} \, e^\nu_a = h^{\mu\nu} . \quad (5.3)$$

Then $\mathcal{L}(\psi, g)$ has the following structure (see Appendix B):

$$\mathcal{L}(\psi, g) = -\sqrt{-g}\, \bar{\psi} \left(\gamma^a e^\mu_a \frac{1}{i} \nabla_\mu + m_N \right) \psi \quad (5.4)$$

where m_N stands for the mass of the fermions. We have used natural units $\hbar = 1 = c$; γ^a are the usual Dirac matrices satisfying the anti-commutation rule

$$\left\{ \gamma^a, \gamma^b \right\} = -2\, \eta^{ab} \quad (5.5)$$

In our convention,

(i) $$\eta^{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) $$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} , \qquad k = 1, 2, 3,$$

where σ_k are the usual Pauli spin matrices.

(iii) $$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where I is 2 x 2 identity matrix.

(iv) $$\gamma_5 \equiv \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

The "covariant" derivative ∇_μ acting on a spinor field has the following form

$$\nabla_\mu \psi = \partial_\mu \psi - \omega_{b\mu c}\, \sigma^{bc} \tag{5.6}$$

where σ^{bc} are the generators of the Lorentz group belonging to a given representation and the $\omega_{b\mu c}$ are functions of the product of the vierbein field and its derivatives (see Appendix B).

Once we know the Lagrangian, the equations of motion follow easily. The fermion field satisfies the equation

$$\left(e^\mu_a \gamma^a \frac{1}{i} \nabla_\mu + m_N \right) \psi = 0 \tag{5.7}$$

The equation for the spin 2 field is obtained by noting the following variations

$$\delta \mathcal{L}_E / \delta h^{\mu\nu} = \sqrt{-g}\; G_{\mu\nu} \tag{5.8}$$

$$\delta \mathcal{L}_m / \delta h^{\mu\nu} = \frac{1}{2} \sqrt{-g}\; H_{\mu\nu} \tag{5.9}$$

$$\delta \mathcal{L}(\psi, g) / \delta h^{\mu\nu} = \frac{1}{2} \sqrt{-g}\; t_{\mu\nu} \tag{5.10}$$

so that the spin 2 equation is

$$G_{\mu\nu} + \frac{1}{2} m^2 H_{\mu\nu} = -\kappa\, t_{\mu\nu} \tag{5.11}$$

We shall write $m = m_f$ to indicate that it refers to the spin 2 meson, and shall set

$$K = 8\pi f^2 / m_N^2$$

where f^2 is the (universal) spin 2 coupling constant. Eq. (5.11) can be re-written as (see Appendix C):

$$R_{\mu\nu} - \frac{m_f^2}{2}\left(\sqrt{-g}\, g_{\mu\nu} - \eta_{\mu\nu}\right) = -\frac{8\pi f^2}{m_N^2}\left(t_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, t\right) \tag{5.12}$$

where

$$t \equiv h^{\alpha\beta}\, t_{\alpha\beta} \tag{5.13}$$

Straight-forward evaluation of (5.10) using (5.4) gives (see Appendix D):

$$t_{\mu\nu} = \bar{\Psi}\, d_{\mu a}\, \gamma^a \frac{1}{i} \nabla_\nu \psi \tag{5.14}$$

Eq. (5.13) then gives

$$t = -m_N \bar{\Psi} \psi \tag{5.15}$$

where we have made use of Eq. (5.7).

Finally, we note from (5.7) and its hermitian conjugate that (see Appendix E)

$$\nabla_\mu \left(\bar{\Psi}\, e^\mu_a\, \gamma^a\, \psi\right) = 0 \tag{5.16}$$

which implies that (see Appendix F)

$$\partial_\mu \left(\sqrt{-g} \; \bar{\Psi} \, e^\mu_{\ a} \, \gamma^a \, \psi \right) = 0 \tag{5.17}$$

5-3. The Hartree Approximation.

The unambiguous approach to the study of the many-body system described by the Lagrangian (5.1) and the equations of motion (5.7) and (5.12) consists in applying the standard many-body Green's function technique. In the present case, however, the coupled nature of the equations make this a very difficult task. Further complicating the situation is the fact that the coupling constant involved is not small (unlike the situation in quantum electrodynamics), thus making useless any perturbation-type calculation. We can, nevertheless, introduce considerable simplification and at the same time gain a lot of insight into the problem by resorting to the Hartree approximation. Simply stated, the Hartree approximation consists in assuming that the fermions "see" an average, effective cloud of the spin 2 mesons. The meson field can then be replaced by its constant expectation value. Such an approximation can be justified when the fermions are very dense so that the source terms are large and therefore the quantum fluctuations about the expectation value of the meson field are small. This type of relativistic Hartree calculation for nucleons has been used by Kalman[29] for the scalar interaction, and

Zel'dovich[20] for the vector interaction. More recently, Walecka[30] has employed the same technique to study the properties of dense neutron matter interacting through a combination of scalar and vector fields.

We shall, therefore, make the assumption that $\bar{g}^{\mu\nu}$ has the following space-isotropic and diagonal structure:

$$\bar{g}^{\mu\nu} = \begin{pmatrix} -(1+\chi) & 0 & 0 & 0 \\ 0 & 1+\lambda & 0 & 0 \\ 0 & 0 & 1+\lambda & 0 \\ 0 & 0 & 0 & 1+\lambda \end{pmatrix} \tag{5.18}$$

where χ and λ are space-time independent.
For case of notation, we shall introduce the quantities x and y, defined as

$$x^2 \equiv 1+\chi \tag{5.19}$$

$$y^2 \equiv 1+\lambda \tag{5.20}$$

Then

$$g = \det g_{\mu\nu} = \det \bar{g}^{\mu\nu} = -x^2 y^6 \tag{5.21}$$

and,
$$g_{\mu\nu} = \frac{1}{h^{\mu\nu}} = -\frac{1}{\sqrt{-g}} \text{ minor } \bar{g}^{\mu\nu}$$

$$= \begin{pmatrix} -x^{-1}y^{3} & 0 & 0 & 0 \\ 0 & xy & 0 & 0 \\ 0 & 0 & xy & 0 \\ 0 & 0 & 0 & xy \end{pmatrix} \tag{5.22}$$

Eq. (5.2) and (5.3) then yield the following values for the vierbein field components:

$$d_{00}^{2} = x^{-1} y^{3} \tag{5.23}$$

$$d_{11}^{2} = d_{22}^{2} = d_{33}^{2} = xy \tag{5.24}$$

$$(e_{0}^{0})^{2} = x\, y^{-3} \tag{5.25}$$

$$(e_{1}^{1})^{2} = (e_{2}^{2})^{2} = (e_{3}^{3})^{2} = x^{-1} y^{-1} \tag{5.26}$$

Since the vierbein fields in the Hartree approximation are constants (i.e., independent of space-time), the functions that appear in Eq. (5.6) vanish, so that we can make the following replacement

$$\nabla_{\mu} \Psi \rightarrow \partial_{\mu} \Psi$$

in the fermion field equation (5.7). Eq. (5.7) then becomes

$$\left(e^{\mu}_{a} \gamma^{a} \frac{1}{i} \partial_{\mu} + m_N \right) \psi = 0 \tag{5.27}$$

Furthermore, (5.14) becomes

$$t_{\mu\nu} = \bar{\psi} \, d_{\mu a} \gamma^{a} \frac{1}{i} \partial_{\nu} \psi \tag{5.28}$$

The expression for t , namely (5.13), however remains the same.

Substituting (5.15) and (5.28) in (5.12) and remembering that $R_{\mu\nu}$ vanish in the Hartree approximation, the spin 2 equation becomes

$$\sqrt{-g}\, g_{\mu\nu} - \eta_{\mu\nu} = \frac{16\pi f^2}{m_f^2 m_N^2} \left(\bar{\psi} d_{\mu a} \gamma^{a} \frac{1}{i} \partial_{\nu} \psi + \frac{1}{2} g_{\mu\nu} m_N \bar{\psi}\psi \right) \tag{5.29}$$

Next, we evaluate the quantities $\bar{\psi} d_{\mu a} \gamma^{a} \frac{1}{i} \partial_{\nu} \psi$ and $\bar{\psi}\psi$ for the quantum ground state of the fermion system. The quantizing is done by expanding the fermion field wave function into normal modes:

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{k,r} \left\{ a_{kr} u_r(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + b_{kr} v_r(-\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \right\} \tag{5.30}$$

where the $\vec{k}$'s are momenta satisfying periodic boundary conditions, r is the helicity index, a^{+}_{kr} (b^{+}_{kr}) and a_{kr} (b_{kr}) are respectively the fermion (anti-fermion) creation and annihilation operators and u and v are respectively the positive and negative energy spinors. We choose the following normalization of the wave-function:

$$\sqrt{-g}\, e^{0}_{0}\, u^{+} u = 1 = \sqrt{-g}\, e^{0}_{0}\, v^{+} v \qquad (5.31)$$

From (5.17), it follows that the conserved fermion current is

$$\sqrt{-g}\;\; \bar{\Psi}\, e^{\mu}_{a}\, \gamma^{a}\, \psi$$

Therefore, the integral

$$\int_{V} \vec{dx}\;\; \sqrt{-g}\;\; \bar{\Psi}\; e^{0}_{a}\, \gamma^{a}\, \psi$$

is a constant of motion, and can be identified with the total fermion number operator B. The fermion number density is then

$$n = B / V$$

Evaulated for the ground state of a system of nucleons characterized by the wave function (5.30) and a fermi momentum k_F, we get

$$n = \frac{\gamma}{(2\pi)^3} \int_{0}^{k_F} \vec{dk} = \frac{\gamma}{(2\pi)^3}\; k_F^3 \quad , \qquad (5.32)$$

where γ is the spin degeneracy factor. (γ is 4 for nuclear matter and 2 for pure neutrons).

Let us re-write the fermion field equation (5.27) as

$$(\, e^{\mu}_{a}\, \gamma^{a}\, k_{\mu} + m_N\,)\;\; \psi = 0 \qquad (5.33)$$

where k_μ is the momentum 4-vector: $k_\mu = (\epsilon_k , \vec{k})$.
Eq. (5.33) then yields the following dispersion relation

$$\left(e^\nu_b \gamma^b k_\nu \; e^\mu_a \gamma^a k_\mu - m_N^2 \right) \psi = 0$$

or,
$$\left(- k_\mu h^{\mu\nu} k_\nu - m_N^2 \right) \psi = 0 \tag{5.34}$$

Written explicitly, (5.34) gives

$$\epsilon_k^2 h^{00} + k^2 h^{11} + m_N^2 = 0 , \tag{5.35}$$

since in the Hartree approximation $h^{11} = h^{22} = h^{33}$.
Thus we get the following eigenvalue equation for the single-particle energies:

$$\epsilon_k \rightarrow \epsilon_k^{\pm} = \pm \left(\vec{k}^{*2} + m_N^{*2} \right)^{\frac{1}{2}} \tag{5.36}$$

where
$$\vec{k}^* = \left(- \frac{h^{11}}{h^{00}} \right)^{\frac{1}{2}} \vec{k} = x^{-1} y \, \vec{k} \tag{5.37}$$

and
$$m_N^* = \frac{m_N}{\sqrt{-h^{00}}} = x^{-1/2} y^{3/2} m_N \tag{5.38}$$

Since m_N^* is independent of momentum, a fermi energy ϵ_F can be defined in the usual way. Furthermore, the identification between the chemical potential μ and ϵ_F can be made:

$$\mu = \left\{ \vec{k}_F^{*2} + m_N^{*2}(k_F) \right\}^{1/2} \tag{5.39}$$

Because of the implicit density dependence of m_N^*, it is not immediately apparent that μ, as defined by (5.39), satisfies the usual thermodynamic identities. We shall however show later that this is indeed the right definition.

We can regard $\vec{k}^*$ and m_N^* as the effective (field-dependent) momentum and mass, replacing $\vec{k}$ and m_N in the single-particle energy. However, it must be kept in mind that the physical momentum states are still $\vec{k}$'s, and not $\vec{k}^*$'s. For a given $\vec{k}$, the following equations

$$\left(\vec{\alpha}\cdot\vec{k}^* + \gamma^0 m_N^* \right) u_r(\vec{k}) = \epsilon_k^+ \, u_r(\vec{k}) \tag{5.40}$$

and

$$\left(\vec{\alpha}\cdot\vec{k}^* + \gamma^0 m_N^* \right) v_r(\vec{k}) = \epsilon_k^- \, v_r(\vec{k}) , \tag{5.41}$$

where $\alpha^i \equiv \gamma^0\gamma^i$, give a complete set of the fermion wave functions.

Since we are dealing with neutrons, and not anti-neutrons, we shall consider only the first of these above equations, namely (5.40), which we re-write as

$$\left(\vec{\alpha}\cdot\vec{k}^* + \gamma^0 m_N^* \right) u = \left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2} u \tag{5.42}$$

Taking the hermitian conjugate of (5.42), we get

$$u^{+}\left(\vec{\alpha}\cdot\vec{k}^{*}+\gamma^{0}m_{N}^{*}\right)=\left(\vec{k}^{*2}+m_{N}^{*2}\right)^{\frac{1}{2}}u^{+} \tag{5.43}$$

Multiplying (5.42) from the left by u^{+} and (5.43) from the right by u , and adding them together, we get

$$m_{N}^{*}\,u^{+}u=\left(\vec{k}^{*2}+m_{N}^{*2}\right)^{\frac{1}{2}}\bar{u}\,u\,, \tag{5.44}$$

where we have used

$$\left\{\vec{\alpha},\gamma^{0}\right\}=0$$

The average values of the quantities $\bar{\psi}\psi$ and $\bar{\psi}\,d_{\mu a}\gamma^{a}\frac{1}{i}\partial_{\nu}\psi$ that appear on the right hand side of (5.29) can now be evaluated for the nucleon ground state.

Using (5.44), (5.30) and (5.31), we get

$$\bar{\psi}\psi=\frac{\gamma}{(2\pi)^{3}}\frac{1}{\sqrt{-g}\,e_{0}^{0}}\int_{0}^{k_F}d\vec{k}\,\frac{m_{N}^{*}}{\left(\vec{k}^{*2}+m_{N}^{*2}\right)^{\frac{1}{2}}} \tag{5.45}$$

For $\bar{\psi}\,d_{\mu a}\gamma^{a}\frac{1}{i}\partial_{\nu}\psi$, we note that it will have only diagonal elements, and its space elements will be all equal.

The 0 - 0 component is

$$\bar{\Psi}\, d_{0a}\gamma^{a}\frac{1}{i}\partial_0\Psi = \bar{\Psi}\, d_{00}\gamma^{0}\frac{1}{i}\partial_0\Psi = -\bar{\Psi}\, d_{00}\gamma^{0}\epsilon_k\Psi$$

$$= -\frac{\gamma}{(2\pi)^3}\frac{d_{00}}{\sqrt{-g}\,e^0_0}\int_0^{k_F} d\vec{k}\,\left(k^{*2}+m_N^{*2}\right)^{1/2}$$

$$= -\frac{\gamma}{(2\pi)^3}\frac{g_{00}}{\sqrt{-g}}\int_0^{k_F} d\vec{k}\,\left(k^{*2}+m_N^{*2}\right)^{1/2}$$

$$= +\frac{\gamma}{(2\pi)^3}\frac{1}{x^2}\int_0^{k_F} d\vec{k}\left(k^{*2}+m_N^{*2}\right)^{1/2} \qquad (5.46)$$

using (5.30), (5.31), (5.36) and the following relation (see Appendix B):

$$d_{\mu a} = g_{\mu\nu}\, e^{\nu}_{a} \qquad (5.47)$$

The 1 - 1 component is then

$$\bar{\Psi}\, d_{1a}\gamma^{a}\frac{1}{i}\partial_1\Psi = \bar{\Psi}\, g_{1\alpha}\, e^{\alpha}_{a}\gamma^{a}\frac{1}{i}\partial_1\Psi$$

$$= \frac{1}{3}\bar{\Psi}\left(\sum_{j=1}^{3} g_{j\alpha}\, e^{\alpha}_{a}\gamma^{a}\frac{1}{i}\partial_j\right)\Psi$$

$$= \frac{1}{3}\bar{\Psi}\left\{g_{11}e^{1}_{a}\gamma^{a}\frac{1}{i}\partial_1 + g_{22}e^{2}_{a}\gamma^{a}\frac{1}{i}\partial_2 + g_{33}e^{3}_{a}\gamma^{a}\frac{1}{i}\partial_3\right\}\Psi$$

$$= \frac{1}{3}g_{11}\bar{\Psi}\left(\sum_{j=1}^{3} e^{j}_{a}\gamma^{a}\frac{1}{i}\partial_j\right)\Psi$$

$$= -\frac{1}{3}g_{11}\bar{\Psi}\left(e^{0}_{a}\gamma^{a}\frac{1}{i}\partial_0 + m_N\right)\Psi, \qquad (5.48)$$

where in the last step we have used (5.27).

From (5.47),

$$
\begin{aligned}
d_{oa} &= g_{o\nu}\, e^{\nu}_{a} \\
&= g_{oo}\, e^{o}_{a} \\
&= \frac{1}{h^{oo}}\, e^{o}_{a}
\end{aligned}
$$

so that $$e^{o}_{a} = h^{oo}\, d_{oa} \tag{5.49}$$

Using (5.45), (5.46) and (5.49), Eq. (5.48) is

$$
\begin{aligned}
\bar{\Psi}\, d_{1a} \gamma^{a} \frac{1}{i} \partial_1 \Psi &= -\frac{1}{3} g_{11} h^{oo} \bar{\Psi}\, d_{oa} \gamma^{a} \frac{1}{i} \partial_o \Psi - \frac{1}{3} g_{11} m_N \bar{\Psi} \Psi \\
&= -\frac{1}{3} g_{11} h^{oo} \frac{\gamma}{(2\pi)^3} \frac{1}{x^2} \int_0^{k_F} d\vec{k} \left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2} \\
&\quad - \frac{\gamma}{(2\pi)^3} \frac{g_{11}}{\sqrt{-g}} \frac{m_N}{e^{o}_{o}} \int_0^{k_F} \frac{d\vec{k}\; m_N^{*}}{\left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2}} \\
&= \frac{1}{3} \frac{\gamma}{(2\pi)^3} \frac{1}{y^2} \int_0^{k_F} \frac{d\vec{k}\; \vec{k}^{*2}}{\left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2}}
\end{aligned}
\tag{5.50}
$$

The integrals that appear in (5.45), (5.46) and (5.50) are easily evaluated, and we re-write them in the following fashion:

$$\int_0^{k_F} \frac{d\vec{k}}{\left(\vec{k}^{*2}+m_N^{*2}\right)^{1/2}} = \frac{2\pi x}{y}\left\{ k_F\left(k_F^2+xym_N^2\right)^{1/2} - xym^2\ \ln\left[\frac{k_F+\left(k_F^2+xym_N^2\right)^{\frac{1}{2}}}{\left(xym_N^2\right)^{1/2}}\right]\right\}$$

$$\equiv \frac{2\pi x}{y}\ I_1 \tag{5.51}$$

$$\int_0^{k_F} d\vec{k}\left(\vec{k}^{*2}+m_N^{*2}\right)^{\frac{1}{2}} = \frac{\pi y}{x}\left\{ 2k_F\left(k_F^2+xym_N^2\right)^{1/2} - k_F\,xym_N^2\left(k_F^2+xym_N^2\right)^{1/2} - x^2y^2m_N^4\ \ln\left[\frac{k_F+\left(k_F^2+xym_N^2\right)^{\frac{1}{2}}}{\left(xym_N^2\right)^{1/2}}\right]\right\}$$

$$\equiv \frac{\pi y}{x}\ I_2 \tag{5.52}$$

$$\int_0^{k_F} \frac{d\vec{k}\ \vec{k}^{*2}}{\left(\vec{k}^{*2}+m_N^{*2}\right)^{1/2}} = \frac{\pi y}{x}\left\{ 2k_F\left(k_F^2+xym_N^2\right)^{1/2} - 5k_F\,xym_N^2\left(k_F^2+xym_N^2\right)^{1/2} - 3x^2y^2m_N^4\ \ln\left[\frac{k_F+\left(k_F^2+xym_N^2\right)^{1/2}}{\left(xym_N^2\right)^{1/2}}\right]\right\}$$

$$\equiv \frac{\pi y}{x}\ I_3 \tag{5.53}$$

So, finally

$$\bar{\psi}\,\psi = \frac{\gamma}{(2\pi)^3}\ \frac{m_N}{xy}\ I_1 \tag{5.54}$$

$$\bar{\Psi}\, d_{0a}\gamma^{a}\frac{1}{i}\partial_{0}\Psi = \frac{\gamma}{16\pi^{2}}\,\frac{y}{x^{3}}\, I_{2} \tag{5.55}$$

$$\begin{aligned}\bar{\Psi}\, d_{1a}\gamma^{a}\frac{1}{i}\partial_{1}\Psi &= \frac{1}{3}\,\bar{\Psi}\sum_{j=1}^{3} d_{ja}\gamma^{a}\frac{1}{i}\partial_{j}\Psi \\ &= \frac{\gamma}{48\pi^{2}}\,\frac{1}{xy}\, I_{3}\end{aligned} \tag{5.56}$$

Using (5.54), (5.55) and (5.56), the equation (5.29) for the spin 2 field takes the following form:

1) 0-0 component:

$$F(x,y) \equiv x^{3} - x^{3}y^{6} - a y I_{2} + 2 a m_{N}^{2} x y^{2} I_{1} = 0 \tag{5.57}$$

2) 1-1 component:

$$G(x,y) \equiv 3x^{3}y^{5} - 3xy - a I_{3} - 6 a m_{N}^{2} x y I_{1} = 0 \tag{5.58}$$

where
$$a \equiv \frac{\gamma f^{2}}{\pi\, m_{N}^{2}\, m_{f}^{2}} \tag{5.59}$$

Equations (5.57) and (5.58) together determine the spin 2 field in the Hartree approximation. These equations are coupled and non-linear, and can be solved (numerically) to yield the values of x and y and hence χ and λ as functions of the fermi momentum k_F (or equivalently the fermion number density n), which we shall take as the independent parameter of the theory.

5-4. The Total Energy-Momentum Tensor and the Equation of State.

The conserved total energy-momentum tensor of the spin 2 field and the fermion field is given by (see Appendix G):

$$\theta^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + \sqrt{-g}\, h^{\mu\lambda}\, t_{\lambda\nu} \tag{5.60}$$

Here $\bar{T}^{\mu}_{\nu}$ is the energy-momentum tensor of the spin 2 field given by

$$\bar{T}^{\mu}_{\nu} = h^{\alpha\beta}_{,\nu} \frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}_{,\mu}} - \delta^{\mu}_{\nu}\mathcal{L} + \bar{S}^{\mu}_{\nu} \tag{5.61}$$

where $\mathcal{L}$ = spin 2 Lagrangian

$$= \frac{m_N^2}{16\pi f^2}\left(\mathcal{L}_E + m_f^2 \mathcal{L}_m\right) \tag{5.62}$$

$\mathcal{L}_E$ and $\mathcal{L}_m$ being given by (4.1) and (4.56). $\bar{S}^{\mu}_{\nu}$ is the symmetrizing Belinfante part of θ^{μ}_{ν}, and depends on the field derivatives. $t_{\lambda\nu}$ is given by (5.28) in the Hartree approximation.

In the Hartree approximation terms depending on the derivatives of the spin 2 field are taken to vanish. Then (5.61) reduces to

$$\bar{T}^{\mu}_{\nu} = -\delta^{\mu}_{\nu}\mathcal{L}$$

$$= -\frac{m_N^2 m_f^2}{16\pi f^2}\mathcal{L}_m \qquad (5.63)$$

Substituting for $\mathcal{L}_m$ from (4.56) and using (5.18) and (5.21), (5.63) becomes

$$\bar{T}^{\mu}_{\nu} = \delta^{\mu}_{\nu}\frac{m_f^2 m_N^2}{32\pi f^2}\left(3 + x^2y^6 - x^2 - 3y^2\right) \qquad (5.64)$$

We shall assume that the fermi gas we deal with can be regarded as a perfect fluid, with the ground state being an eigenstate of total energy E and vanishing three-momentum: $p^{\mu} = (E,0,0,0)$. For such a system the total energy-momentum tensor can be written as[54]

$$\theta_{\mu\nu} = P\eta_{\mu\nu} + \left(P/c^2 + \rho\right)u_{\mu}u_{\nu} \qquad (5.65)$$

where P and ρ are the pressure and the total mass density of the system and u_{μ} is the four-velocity satisfying

$$u^{\mu}u_{\mu} = -c^2$$

From (5.65), we can make the following identification for the total energy density (ϵ) and pressure (P):

$$\epsilon = \theta_{00} \qquad (5.66)$$

$$P = \theta_{11} = \theta_{22} = \theta_{33} \tag{5.67}$$

The underlying motivation of our theory is now clear. Once the spin 2 field equations (5.57) and (5.58) have been solved self-consistently as a function of the fermion number density n , the diagonal components of $\Theta_{\mu\nu}$ are easily evaluated. Eqs. (5.66) and (5.67) then yield a parametric form of the equation of state of the system, the parameter being the fermion density:

$$\epsilon = \epsilon(n) \tag{5.68}$$

$$P = P(n) \tag{5.69}$$

Explicit evaluation of the components of $\theta_{\mu\nu}$ in terms of the spin 2 field variables yields

$$\epsilon = \frac{m_N^2 m_f^2}{32\pi f^2}\left(x^2 + 3y^2 - x^2 y^6 - 3\right) + \frac{\gamma}{16\pi^2}\frac{y}{x} I_2 \tag{5.70}$$

$$P = -\frac{m_N^2 m_f^2}{32\pi f^2}\left(x^2 + 3y^2 - x^2 y^6 - 3\right) + \frac{\gamma}{48\pi^2}\frac{y}{x} I_3 \tag{5.71}$$

As a relevant point we note here that the first law of thermodynamics:

$$P\, dV = -dE \tag{5.72}$$

yields an alternate, thermodynamic definition of pressure which is

$$P = -\epsilon + n \, d\epsilon/dn \quad , \tag{5.73}$$

$\epsilon = E/N$ and $n = B/V$

For the Hartree Approximation scheme to be self-consistent, both the kinetic and the thermodynamic definitions of pressure namely Eqs. (5.71) and (5.73) must yield identical values. In other words, ϵ and P, as computed from (5.70) and (5.71) must in principle satisfy the thermodynamic consistency conditions (5.73). For non-relativistic, spatially uniform many-body systems, this type of self-consistency criterion is a trivially simple matter. However, for relativistic systems where the particle energy acquires complicated density dependence, this is not the case. We stress that it is an important feature of our treatment of the relativistic many-body spin 2 interaction that the above-mentioned self-consistency criterion is satisfied (see Appendix I). Eq. (5.73) can also be used to obtain the thermodynamic definition of the chemical potential. This is

$$\mu = \frac{1}{n} (P + \epsilon) \tag{5.74}$$

Both (5.39) and (5.74) are found to yield identical results.

5-5. Computation of the Equation of State.

The equation of state is computed by determining the density dependence of the pressure, energy and chemical potential. For this, we must first obtain a self-consistent solution for x and y from the field equations (5.57) and (5.58):

$$F(x,y) = 0 \tag{5.57}$$

$$G(x,y) = 0 \tag{5.58}$$

x and y being related to the spin 2 field variables χ and λ by means of (5.19) and (5.20). Inspection of the specific terms occurring in (5.57) and (5.58) shows that an analytic solution cannot be obtained, except in the following limiting cases:

(i) Low-density (i.e. non-relativistic) limit $k_F << m_N$: In this case, (5.57) and (5.58) give the following roots (See Appendix H):

$$\lambda = -\frac{8 f^2}{9\pi m_f^2 m_N} k_F^3 \tag{5.75}$$

$$\chi = \frac{40 f^2}{9\pi m_f^2 m_N} k_F^3 \tag{5.76}$$

(ii) High-density (i.e., relativistic) limit $k_F \gg m_N$:
In this case, we get (see Appendix H)

$$\lambda = 4^{-1/3} - 1 \tag{5.77}$$

$$\chi = \frac{2^{28/9}}{3}\left(\frac{f^2 m_N^2}{\pi m_f^2}\right)^{2/3}\left(\frac{k_F}{m_N}\right)^{8/3} \tag{5.78}$$

The general numerical solution

$$\chi = \chi(k_F)$$
$$\lambda = \lambda(k_F)$$

is presented in Figures 1 and 2. In performing the numerical calculations, we have made the following choice of the parameters involved:

Masses (see Particle Data Group[55]):

m_N = 939.5527 MeV (neutron)
m_f = 1260 MeV (f^0 meson)

The coupling constant of f^0 meson is not known exactly. Three experimental values are cited by Pilkuhn et al.[56], and these correspond to

$$f_{exp}^2 = \begin{cases} 2.91 \\ 6.55 \\ 7.44 \end{cases}$$

The chemical potential, (5.39), in terms of the field variables χ and λ is

$$\mu = \frac{1+\lambda}{1+\chi}\left\{k_F^2 + (1+\lambda)^{\frac{1}{2}}(1+\chi)^{\frac{1}{2}} m_N^2\right\}^{1/2} \qquad (5.79)$$

It is plotted against the density in Figure 3 for several values of the spin 2 coupling constant f^2. The total energy per particle ϵ/n is presented in Figure 4 as a function of the density. We see that both μ and ϵ/n exhibit non-monotonic dependences on the density. The next quantity of interest is the pressure, which is given by (5.71). Figure 5 shows the pressure-density relation for different values of f^2.

We can now construct a general picture of the equilibrium properties of the system. The behaviour is determined by two parameters: the density n (or the fermi momentum k_F) and the spin 2 coupling constant f^2. Let us first consider the low-density case. In this limit χ and λ are given by (5.75) and (5.76). Then (5.79), (5.70) and (5.71) yield (see Appendix H)

$$\mu = m_N\left\{1 + \frac{k_F^2}{2m_N^2} - 2\alpha k_F^3 - \frac{3\alpha}{2}\frac{k_F^5}{m_N^2} - 3\alpha^2 k_F^6 + \cdots\right\} \qquad (5.80)$$

$$\frac{\epsilon}{n} = m_N \left\{ 1 + \frac{3}{10} \frac{k_F^2}{m_N^2} - 2\alpha k_F^3 - \frac{9\alpha}{10} \frac{k_F^5}{m_N^2} - 3\alpha^2 k_F^6 + \cdots \right\} \tag{5.81}$$

$$P = \frac{m_N^4}{3\pi^2} \left\{ \frac{1}{5} \frac{k_F^5}{m_N^5} - \frac{1}{14} \frac{k_F^7}{m_N^7} - \frac{4\alpha}{5} \frac{k_F^8}{m_N^5} + \frac{3\alpha}{14} \frac{k_F^{10}}{m_N^7} + \frac{3\alpha^2}{5} \frac{k_F^{11}}{m_N^5} + \cdots \right\}, \tag{5.82}$$

where $\alpha \equiv \frac{8}{9\pi} \frac{f^2}{m_N m_f^2}$

The leading terms of the above expressions are identical to the corresponding quantities for an ideal non-relativistic fermion gas (except that μ and ϵ/n here include the rest energy).

More interesting results ensue from the evaluation of the high-density limit. The corresponding limiting values of χ and λ are provided by (5.77) and (5.78). One then gets the following asymptotic result (see Appendix H):

$$\mu = \frac{\pi^{1/3}\, 3^{1/3}\, m_f^{2/3}\, m_N^{2/3}}{2^{41/9}\, f^{2/3}}\, k_F^{-1/3} \tag{5.83}$$

$$\frac{\epsilon}{n} = \frac{\pi^{1/3}\, 3^{7/3}\, m_f^{2/3}\, m_N^{2/3}}{2^{41/9}\, f^{2/3}}\, k_F^{-1/3} \qquad (5.84)$$

$$P = -\frac{m_f^{2/3}\, m_N^{2/3}}{2^{41/9}\, 3^{2/3}\, \pi^{5/3}\, f^{2/3}}\, k_F^{8/3} \qquad (5.85)$$

The striking feature of the above is that the pressure becomes negative at high densities (as can be seen from Figure 5). Furthermore, (5.84) and (5.85) yield

$$P = -\frac{1}{9}\,\epsilon \qquad (5.86)$$

This is radically different from the characteristics of an ideal fermi gas whose pressure is always positive and which, in the ultra-relativistic limit, satisfies the relation[57]

$$P = \frac{1}{3}\,\epsilon \qquad (5.87)$$

Thus, left to itself, the spin 2 interactions will eventually bring about a collapse of the fermion matter.

Chapter 6

NEUTRON GAS INTERACTING THROUGH SCALAR, VECTOR AND SPIN 2 FIELDS

6-1. Introduction.

A neutron gas interacting exclusively via the exchange of spin 2 mesons does not correspond to a realistic physical situation. In order to have a complete description of dense matter, we must include interactions that arise from the exchanges of scalar and vector mesons. The importance of such interactions in theories of nuclear matter is well-established.[58] We shall not consider the pseudoscalar pion-exchange interaction as the pion has a relatively large Compton wavelength, and therefore, does not play a significant role in the description of dense matter.

The sections of this chapter are parallel to those of the previous chapter, with the difference that we now have, in addition to a fermion field, scalar and vector fields in interaction with the spin 2 field.

6-2. The Coupled Field Theory.

The source for the spin 2 field will now be provided by the fermion field (ψ), scalar field (σ), vector field (A_μ), and their interactions. For the fermion field the appropriate Lagrangian can be written using the vierbein formalism, as was done in the last chapter. The Lagrangian for the scalar and vector fields and their interactions can be written using similar formal analogy to gravitation. We recall that for the gravitational case this is done by using the principle of minimal coupling (see e.g. Weinberg[59]). It consists in taking the respective special-relativistic expressions for the Lagrangians, and making the following replacements: (i) $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and (ii) $\frac{\partial}{\partial x^\alpha}$ by the covariant derivative.

Thus the Lagrangian density describing nucleons interacting through scalar, vector and spin 2 mesons is given by ($\hbar = 1 = c$):

$$\begin{aligned} \mathcal{L}_{Tot} = {} & \frac{m_N^2}{16\pi f^2}\left(\mathcal{L}_E + m_f^2 \mathcal{L}_m\right) - \sqrt{-g}\,\bar{\Psi}\left(e^\mu_\alpha \gamma^\alpha \frac{1}{i}\nabla_\mu + m_N\right)\Psi \\ & - \frac{\sqrt{-g}}{8\pi}\left(\partial_\mu \sigma\, h^{\mu\nu} \partial_\nu \sigma + m_s^2 \sigma^2\right) \\ & - \frac{\sqrt{-g}}{4\pi}\left(\frac{1}{4} F_{\mu\nu} h^{\mu\lambda} h^{\nu\rho} F_{\lambda\rho} + \frac{1}{2} m_v^2 A_\mu h^{\mu\nu} A_\nu\right) \\ & + g_s \sqrt{-g}\,\bar{\Psi}\Psi\sigma + g_v \sqrt{-g}\,\bar{\Psi} e^\mu_a \gamma^a \Psi A_\mu \end{aligned} \tag{6.1}$$

In this expression the first two terms on the right-hand side represent the Lagrangian of the spin 2 field, $\mathcal{L}_E$ and $\mathcal{L}_m$ being given by (4.1) and (4.56). $e^\mu{}_\alpha$ are the vierbein fields given by (5.3). $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$, where ∇_μ is the covariant derivative. m_S and m_V are the masses of the scalar and vector fields, and g_S and g_V represent their coupling strengths to the nucleons.

The equations of motion that follow from (6.1) are

1) Fermion field:

$$\left\{ e^{\mu}_{a} \gamma^{a} \frac{1}{i} (\nabla_{\mu} - i g_{v} A_{\mu}) + m_{N} - g_{s} \sigma \right\} \psi = 0 \qquad (6.2)$$

2) Scalar field:

$$\left(- \partial^{2} + m_{s}^{2} \right) \sigma = 4 \pi g_{s} \bar{\psi} \psi \qquad (6.3)$$

3) Vector field:

$$\sqrt{-g}\, m_{v}^{2} A_{\mu} h^{\mu\nu} + \partial_{\mu} \left(\sqrt{-g}\, F_{\lambda\rho} h^{\lambda\nu} h^{\rho\mu} \right) = 4\pi g_{v} \sqrt{-g}\, \bar{\psi} e^{\nu}_{a} \gamma^{a} \psi \qquad (6.4)$$

4) Spin 2 field:

$$R_{\mu\nu} - \frac{m_f^{2}}{2} \left(\sqrt{-g}\, g_{\mu\nu} - \eta_{\mu\nu} \right) = - \frac{8\pi f^{2}}{m_f^{2}} \left(t_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} t \right) \qquad (6.5)$$

where $t_{\mu\nu}$ is the energy-momentum tensor of the source fields, and is given by

$$t_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_s}{\delta h^{\mu\nu}} \qquad (6.6)$$

where $$\mathcal{L}_s \equiv \mathcal{L}_{Tot} - \frac{m_N^{2}}{16\pi f^{2}} \left(\mathcal{L}_E + m_f^{2} \mathcal{L}_m \right) \qquad (6.7)$$

Lastly, $$t \equiv h^{\mu\nu} t_{\mu\nu} \qquad (6.8)$$

The evaluation of (6.6) yields (see Appendix D):

$$t_{\mu\nu} = \overline{\Psi}\, d_{\mu a} \gamma^{a} \frac{1}{i} (\nabla_{\nu} - i\, g_{v} A_{\nu}) \psi + \frac{1}{4\pi} \partial_{\mu}\sigma \partial_{\nu}\sigma$$

$$- \frac{1}{8\pi} g_{\mu\nu} (\partial_{\lambda}\sigma\, h^{\lambda\rho} \partial_{\rho}\sigma + m_{s}^{2} \sigma^{2}) + \frac{1}{4\pi} m_{v}^{2} A_{\mu} A_{\nu}$$

$$- \frac{1}{8\pi} m_{v}^{2} g_{\mu\nu} A_{\alpha} h^{\alpha\beta} A_{\beta} + \frac{1}{4\pi} F_{\mu\nu'} F_{\mu'\nu} h^{\mu'\nu'}$$

$$- \frac{1}{16\pi} g_{\mu\nu} F_{\lambda'\rho'} h^{\lambda'\lambda} h^{\rho'\rho} F_{\lambda\rho} \quad , \tag{6.9}$$

$d_{\mu a}$ being defined by (5.2).

As in the previous chapter, we shall take the source terms appearing in the meson field equations to be average quantities over the ground state of the nucleons. In the present case, however, we have to deal with three coupled partial differential equations corresponding to three meson fields. As before, we shall use the Hartree approximation to obtain the physical properties of the system.

We shall use the same operator representation for ψ , namely Eq. (5.30) with (5.31) as the normalization. We note, from (6.2) and its hermitian conjugate, that (see Appendix E)

$$\nabla_{\mu} \left(\overline{\Psi}\, e^{\mu}_{a}\, \gamma^{a}\, \psi \right) = 0 \tag{6.10}$$

This implies (see Appendix F)

$$\partial_{\mu} \left(\sqrt{-g}\; \overline{\Psi}\, e^{\mu}_{a}\, \gamma^{a}\, \psi \right) = 0 \tag{6.11}$$

so that

$$n = \sqrt{-g}\ \bar{\Psi}\ e^{o}_{\alpha}\gamma^{a}\psi \tag{6.12}$$

can be interpreted as the operator giving the fermion number density. Furthermore,

$$n = \frac{\gamma}{6\pi^2}\ k_F^3\ , \tag{6.13}$$

k_F = fermion momentum of the nucleon gas.

6-3. Hartree Approximation for the Coupled Fields.

The Hartree approximation provides a simple way of handling the coupled field theory presented in the preceding section. In such an approximation the meson field variables assume constant expectation values, and their space-time derivatives are not of physical significance. Thus, for the scalar and vector fields we shall make the following replacements

$$\sigma \to \sigma_o \tag{6.14}$$

$$A_\mu \to \delta^{o}_{\mu}\ A_o\ , \tag{6.15}$$

where σ_o and A_o are assumed space-time independent. For the spin 2 field we shall assume the same diagonal, space-isotropic structure as in the last chapter, namely Eq. (5.18).

Eqs. (6.2), (6.3) and (6.4) then become respectively

$$\left(e^{\mu}_{a} \gamma^{a} \frac{1}{i} D_{\mu} + m_N' \right) \psi = 0 \quad , \tag{6.16}$$

where
$$D_{\mu} \equiv \nabla_{\mu} - i g_{v} A_{\mu}$$

$$= \nabla_{\mu} - i g_{v} \delta^{0}_{\mu} A_{0} \tag{6.17}$$

and
$$m_N' \equiv m_N - g_s \sigma_0 \tag{6.18}$$

$$\sigma_0 = \frac{4\pi g_s}{m_s^2} \bar{\psi} \psi \tag{6.19}$$

$$A_0 h^{00} = \frac{4\pi g_v}{m_v^2} \bar{\psi} e^{0}_{a} \gamma^{a} \psi \tag{6.20}$$

Eq. (6.9) becomes

$$t_{\mu\nu} = \bar{\psi} d_{\mu a} \gamma^{a} \frac{1}{i} D_{\nu} \psi - \frac{m_s^2}{8\pi} g_{\mu\nu} \sigma_0^{2}$$

$$+ \frac{m_v^2}{4\pi} A_0 A_0 \delta^{0}_{\mu} \delta^{0}_{\nu} - \frac{m_v^2}{4\pi} g_{\mu\nu} A_0 h^{00} A_0 \tag{6.21}$$

Using (6.21) and (6.16), (6.8) becomes

$$t = - m_N' \bar{\psi} \psi - \frac{m_s^2}{2\pi} \sigma_0^{2} - \frac{m_v^2}{4\pi} A_0 h^{00} A_0 \tag{6.22}$$

The evaluation of the quantities $\bar{\Psi}\psi$ and $\bar{\Psi} d_{\mu a}\gamma^a \frac{1}{i} D_\nu \psi$ for the fermions' ground state can be performed in exactly the same way as in Chapter 5. This yields

$$\bar{\Psi}\psi = \frac{\gamma}{4\pi^2} \frac{m_N'}{xy} J_1 \tag{6.23}$$

$$\bar{\Psi} d_{0a}\gamma^a \frac{1}{i} D_0 \psi = \frac{\gamma}{16\pi^2} \frac{y}{x^3} J_2 \tag{6.24}$$

$$\bar{\Psi} d_{1a}\gamma^a \frac{1}{i} D_1 \psi = \frac{1}{3} \sum_{j=1}^{3} \bar{\Psi} d_{ja}\gamma^a \frac{1}{i} D_j \psi$$

$$= \frac{\gamma}{48\pi^2} \frac{1}{xy} J_3 \tag{6.25}$$

where x and y are given by (5.19) and (5.20) and J_1, J_2 and J_3 are equal to I_1, I_2 and I_3 [i.e. equations (5.51), (5.52) and (5.53)], but with the replacement

$$m_N \rightarrow m_N' = m_N - g_s \sigma_0 .$$

Using Eqs. (6.21) - (6.25) and remembering that in the Hartree approximation $R_{\mu\nu} = 0$, the equation for the spin 2 field, (6.5), becomes

1) 0-0 component:

$$x^4 - x^4 y^6 - a_1 xy J_2 + 2 a_1 m_N'^2 x^2 y^2 J_1 + a_2 x^3 y^3 \sigma_0^2 - a_3 k_F^6 = 0 \tag{6.26}$$

2) 1-1 component:

$$3x^3y^5 - 3xy - a_1 J_3 - 6\,a_1 m_N'^2\, xy\, J_1 - 3a_2 x^2 y^2 \sigma_0^2 = 0 \tag{6.27}$$

Here,

$$a_1 \equiv \frac{\gamma f^2}{\pi m_f^2 m_N^2} \tag{6.28}$$

$$a_2 \equiv \frac{2 f^2 m_s^2}{m_f^2 m_N^2} \tag{6.29}$$

$$a_3 \equiv \frac{16\gamma^2 f^2 g_v^2}{9\pi^3 m_f^2 m_N^2 m_s^2} \tag{6.30}$$

Substitution of (6.23) into (6.19) gives for the scalar field

$$xy\,\sigma_0 - a_4 m_N' J_1 = 0 \tag{6.31}$$

where $$a_4 \equiv \gamma g_s / \pi m_s^2 \tag{6.32}$$

Noting that

$$\frac{e_0^0}{h^{00}} = e_0^0 g_{00} = d_{00} \;,$$

the equation for the vector field (6.20) becomes

$$x^2 A_0 + a_5 k_F^3 = 0 \,, \tag{6.33}$$

where
$$a_5 \equiv \frac{2\gamma g_v}{3\pi m_v^2} \tag{6.34}$$

6-4. The Total Energy-Momentum Tensor and the Equation of State.

The conserved total energy-momentum tensor of the system is (see Appendix G):

$$\theta^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + \sqrt{-g}\, h^{\mu\lambda} t_{\lambda\nu} \tag{6.35}$$

where $\bar{T}^{\mu}_{\nu}$ and $t_{\lambda\nu}$ are given by (5.61) and (6.21). We note that x and y, which determine the spin 2 field components, have now implicit dependences on the scalar field σ_0. Writing the right-hand side of (6.35) in terms of the field variables, and identifying the total energy density (ϵ) and pressure (P) with the diagonal components of $\theta_{\mu\nu}$, we get

$$\epsilon = \frac{m_N^2 m_f^2}{32\pi f^2}\left(x^2 + 3y^2 - x^2 y^6 - 3\right) + \frac{\gamma}{16\pi^2}\frac{y}{x} J_2$$

$$+ \frac{m_s^2}{8\pi} x y^3 \sigma_0^2 + \frac{\gamma^2 g_v^2}{18\pi^3 m_v^2} \frac{k_F^6}{x^2} \tag{6.36}$$

$$P = -\frac{m_N^2 m_f^2}{32\pi f^2}\left(x^2 + 3y^2 - x^2 y^6 - 3\right) + \frac{\gamma}{16\pi^2}\frac{y}{x} J_3$$

$$- \frac{m_s^2}{8\pi} x y^3 \sigma_0^2 + \frac{\gamma^2 g_v^2}{18\pi^3 m_v^2}\frac{k_F^6}{x^2} \qquad (6.37)$$

The equation of state, namely the evaluation of (6.36) and (6.37), is determined by first solving (numerically) the set of equations (6.26), (6.27), (6.31) and (6.33). In doing the calculations we have kept the mass and the coupling constant of the spin 2 meson to be the same as in Chapter 5. For the scalar and vector mesons, we have chosen the following standard values:[55,56]

$$m_s = 700 \text{ MeV}$$
$$g_s^2 = 13.9$$
$$m_v = 784 \text{ MeV}$$
$$g_v^2 = 10.0$$

The energy per particle (ϵ/n) and the pressure (P) as functions of neutron density are presented in Figures 6 and 7. The chemical potential of the system, defined by Eq. (5.39), is plotted in Figure 8 as a function of the neutron density.

Because of the complication brought about by the introduction of scalar and vector fields, it is not possible to verify analytically that the values of pressure and energy density as given by (6.37) and (6.36) satisfy the thermodynamic consistency criterion (5.73):

$$P + \epsilon = n \frac{d\epsilon}{dn} \tag{6.38}$$

We have, however, made a numerical check and we find that (6.38) is indeed satisfied. Thus our coupled field theory is thermodynamically consistent.

Chapter 7

ASTROPHYSICAL APPLICATIONS

7-1. Introduction.

In this chapter we shall present the calculations of masses and moments of inertia of neutron stars using as input the equation of state derived in the preceding chapter. Historically, the earliest numerical computation of the structure of neutron stars was performed by Oppenheimer and Volkoff.[19] They assumed that the degenerate neutrons making up the star are non-interacting, and obtained the result that the maximum mass for a stable neutron star is 0.72 $M_\odot$. Of course, we know that at the densities prevalant inside a neutron star, the interactions among neutrons cannot be neglected. Since the pioneering work by Oppenheimer and Volkoff, numerous improved equations of state for matter at nuclear and super-nuclear densities have been proposed. A detailed account of the neutron star structures corresponding to the presently available models of the equation of state can be found in a survey by Arnett and Bowers.[60] Irrespective of detailed models, however, certain general conclusions are available. Assuming the correctness of the equation of state as given by Baym et al.[13] up to $\rho \sim 5 \times 10^{14}$ gm cm^{-3}, Sabbadini and Hartle[61] have shown, on general grounds, that the maximum mass of a neutron star cannot exceed 5 $M_\odot$, whatever the equation of state at higher densities. If one imposes the condition of causality, $P < \rho c^2$, on the equation of state, then, as Rhoades and Ruffini[62] have shown, the maximum mass is reduced to 3.2 $M_{\odot}$. Similar limits have also been found by Nauenberg and Chapline.[63]

We shall first briefly review the general astrophysical considerations regarding neutron stars, and then present the results that our equation of state [namely, Eqs. (6.36) and (3.37)] yields. For condensed objects such as neutron stars, the temperature is effectively zero on the scale of microscopic energies (unless the star is very young). So we shall confine our attention to zero temperature. For purpose of calculating the mass, we shall neclect the rotation of the star. (The effect of rotation is to cause changes in the moment of inertia and mass in the lowest and second order, respectively, of the angular velocity.) The moment of inertia will be calculated for slowly-rotating models using the non-rotating model for the mass. Finally, we shall present a comparison of the results of our theory with the presently available observational data on pulsars, and also with the predictions of the various other theoretical models.

7-2. The Mass, Radius and Moment of Inertia.

The calculation of the mass, radius and moment of inertia of a neutron star will be done assuming that the star is in its normal state, characterized by the conditions of thermal and hydrostatic equilibrium. Since a neutron star is essentially a zero temperature system, it is in a state of thermal equilibrium. The hydrostatic equilibrium is provided by equating the gravitational force to the pressure force that acts on each mass element. Non-relativistically, this is expressed by the following equation

$$\vec{\nabla}\, P(\vec{r}) = G\, \rho(\vec{r})\, \vec{\nabla} \int d\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \tag{7.1}$$

where $P(\vec{r})$ stands for the pressure and $\rho(\vec{r})$ for the mass density at the point $\vec{r}$ in the star. G is the gravitational constant = 6.6732×10^{-8} c.g.s. units. Neglecting the effects of rotation, we can put Eq. (7.1) in the spherically symmetric form:

$$\frac{\partial P(r)}{\partial r} = - G\, \rho(r)\, \frac{m(r)}{r^2} \tag{7.2}$$

where
$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr' \tag{7.3}$$

represents the mass contained within a radius vector $\vec{r}$.

The pressure P, in general, is a function of the density ρ, the entropy per nucleon and the chemical composition. Since in a neutron star the temperature is essentially 0 ^{0}K, the entropy per nucleon is constant throughout. We shall furthermore, assume that matter, in the region of our interest, has uniform chemical composition. We can then regard the pressure $P(r)$ as a function of $\rho(r)$ alone, with no explicit dependence on r. Physically, the significant point is to know the equation of state of the star. Given $P(r)$ as a function of $P(\rho(r))$, we can write

$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial r} \tag{7.4}$$

Substitution of (7.4) into (7.3) gives the differential equation that determines the density profile:

$$\frac{\partial \rho}{\partial r} = - \frac{G \rho}{(\partial P/\partial \rho)} \frac{m(r)}{r^2} \tag{7.5}$$

The initial conditions for solving the equations (7.3) and (7.5) are

$$m(0) = 0 \tag{7.6}$$

$$\rho(0) = \rho_c \tag{7.7}$$

where ρ_c is a specified central density. Eqs. (7.3) and (7.5) can be integrated out from the center of the star until P drops to zero at some value r = R. We can then interpret R as the radius and $m(R) \equiv M$ as the total mass of the star having a central density ρ_c. In general, the stiffer the equation of state, the larger the value of the maximum mass; the lower the central density, the higher the radius and the moment of inertia.

For actual calculations we shall use the relativistic generalization of Eq. (7.2), also known as the Tolman-Oppenheimer-Volkoff equation[64]:

$$\frac{\partial P}{\partial r} = - \frac{G\left[\rho(r) + P(r)/c^2\right]\left[m(r) + 4\pi r^3 P(r)/c^2\right]}{r^2\left[1 - 2G\, m(r)/rc^2\right]} \qquad (7.8)$$

In Eq. (7.8) the terms that are proportional to $1/c^2$ are the corrections due to general relativity. Inspection of this equation shows that the (Schwarzchild) singularity in the denominator prevents the existence of stars having

$$m(r) > \frac{rc^2}{2G}$$

Neutron stars of too large a mass are unstable against gravitational collapse, whereas those with too small a mass are unstable against radial oscillations. The locus of the points M versus ρ_c is a curve which is shown schematically in Figure 9. The region of stable equilibrium is given by

$$\frac{dM(\rho_c)}{d\rho_c} > 0$$

This is a necessary (but not sufficient) condition for stability. For sufficiency, one must, in addition, have[65]

$$\Gamma \equiv \frac{\rho}{P}\frac{\partial P}{\partial \rho} > \frac{4}{3}$$

We shall write the mass m(R) as

$$m(R) = M_G \tag{7.9}$$

the suffix G indicating it the gravitational mass of the star as measured by a distant observer in Keplerian orbit around the star. It represents the total energy of the star and its gravitational field. It is instructive to compare (7.9) with the quantity

$$M_B = m_B N \ , \tag{7.10}$$

where m_B is the baryon rest mass and N is the total number of baryons in the star: M_B represents the total energy that the stellar matter would have, if dispersed to infinity. In terms of the neutron number density n(r), (7.10) can be written as[66]

$$M_B = m_B \int_0^R \frac{4\pi r^2 \, n(r) \, dr}{\left[1 - 2G\, m(r)/rc^2\right]^{1/2}} \tag{7.11}$$

A qualitative consideration of the equations presented so far can already give us some feeling for the structure of a neutron star. For simplicity, let us consider the non-relativistic equation (7.5). If we approximate the quantities ρ and $\partial P/\partial \rho$, that appear on the right-hand side of (7.5), by their values ρ_c and $(\partial P/\partial \rho)_c$ at the center of the star, we obtain

$$\frac{\partial \rho(r)}{\partial r} = - \frac{4\pi G}{3} \frac{\rho_c^2 \, r}{(\partial P/\partial \rho)_c} \tag{7.12}$$

Integration of (7.12) gives

$$\rho(r) = \frac{2\pi}{3} G \frac{\rho_c^2}{(\partial P/\partial \rho)_c} \left(R^2 - r^2 \right) , \qquad (7.13)$$

where the density has been constructed to vanish for r $\geqslant$ R, R being the radius of the star. If we set r = 0 in Eq. (7.13), then $\rho(0) = \rho_c$, and we get the following expression for the radius

$$R = \left[\frac{3}{2\pi G} \frac{(\partial P/\partial \rho)_c}{\rho_c} \right]^{1/2} \qquad (7.14)$$

The total gravitational mass of the star is, using (7.3), (7.13) and (7.14),

$$M_G = \frac{8\pi}{15} \rho_c R^3 \qquad (7.15)$$

For a slowly-rotating star, the moment of inertia (I) can be determined from the structure of the non-rotating model, as shown by Thorne.[67] Let us use Ω and $\bar{\omega}(r)$ to denote respectively the angular velocity of uniform rotation measured at infinity and the angular velocity of the fluid relative to particles with zero angular momentum. Due to the dragging of inertial frames, $\bar{\omega}(r) \neq \Omega$ but satisfies the following differential equation

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j \frac{d\bar{\omega}}{dr}\right) + \frac{4}{r}\frac{dj}{dr}\bar{\omega} = 0 \tag{7.16}$$

where
$$j(r) = e^{-\phi(r)}\left[1 - \frac{2G\,m(r)}{rc^2}\right]^{1/2} \tag{7.17}$$

and ϕ is given by

$$\frac{d\phi}{dr} = \frac{G}{rc^2}\left\{\frac{m(r) + 4\pi r^3 P(r)/c^2}{1 - 2G\,m(r)/rc^2}\right\} \tag{7.18}$$

The boundary conditions are

$$\left(\frac{d\bar{\omega}}{dr}\right)_{r=0} = 0 \tag{7.19}$$

and
$$\bar{\omega}(\infty) = \Omega \tag{7.20}$$

Outside the star

$$\bar{\omega}(r) = \Omega - \frac{2GJ}{r^3c^2}, \tag{7.21}$$

where J is the angular momentum of the star. From (7.21) it follows that

$$J = \frac{c^2}{6G} R^4 \left(\frac{d\bar{\omega}}{dr} \right)_{r=R} \tag{7.22}$$

Eq. (7.17) implies that

$$j(R) = 1$$

However, we note that since Eq. (7.26) is linear in j, for purpose of integration $\phi(r)$ need only be defined to within an additive constant. Consequently, $\bar{\omega}(o)$ can be taken as any non-zero constant, since $\bar{\omega}(r)$, Ω and J can all be scaled after the integration. This scaling does not affect I, which is defined as

$$I = J/\Omega \tag{7.23}$$

To integrate (7.16) we introduce an auxiliary variable

$$u \equiv r^4 \, d\bar{\omega}/dr$$

This then yields two coupled first-order equations:

$$\frac{du}{dr} + \left(u + 4r^3 \bar{\omega} \right) \frac{d}{dr}(\ln j) = 0 \tag{7.24}$$

$$\frac{d\bar{\omega}}{dr} = \frac{u}{r^4} \tag{7.25}$$

Equations (7.3), (7.8), (7.11), (7.16), (7.17), (7.24) and (7.25) can be numerically integrated outward from r = 0, given the equation of state $P(\rho)$. From (7.21), (7.22) and

(7.23), it follows that

$$I = \frac{c^2 u(R)}{6G} \left[\bar{\omega}(R) + \frac{u(R)}{3R^3} \right]^{-1} \tag{7.26}$$

7-3. The Equation of State and Numerical Results.

The equation of state is the basic input in calculating the bulk properties of a neutron star. In Chapter 6, we have presented our equation of state for neutron matter for different values of the spin 2 coupling constant. In order to facilitate the comparison with other equations of state, we have presented in Figure 10 all the relevant equations of state proposed in recent years. In addition, in Table 1 we present the numerical values of P and ρ as functions of the baryonic density n for two values of the spin 2 coupling constant. For densities lower than the last entry in Table 1, our results join smoothly with the values of P as given by Malone, Bethe and Johnson[3] (Model V). For still lower densities we have used the equation of state given by (i) Baym, Pethick and Sutherland[12] for $1.262 \times 10^{14} \leqslant \rho \leqslant 1.044 \times 10^{14}$ gm cm^{-3} and (ii) that given by Feynman, Metropolis and Teller[11] for $\rho \leqslant 1.150 \times 10^{3}$ gm cm^{-3} .

The physical constants used are:

$$c = 2.9979 \times 10^{10} \text{ cm sec}^{-1}$$

$$G = 6.6732 \times 10^{-8} \text{ cm}^3 \text{ gm}^{-1} \text{ sec}^{-2}$$

$$m_B = 1.659 \times 10^{-24} \text{gm}$$

In converting masses in grams to solar units, we have used

$$M_\odot = 1.987 \times 10^{33} \text{ gm}$$

The equations of state have all been used in the form of discrete data points. Entries for P (dynes cm^{-2}), ρ (gm cm^{-3}) and n (cm^{-3}) are read in to four significant figures (as given in Table 1). Values between two tabulated points such as (ρ_i, P_i) and (ρ_{i+1} , P_{i+1}) are obtained by using the standard SPLINE interpolation subroutine. Eqs. (7.3), (7.8), (7.11), (7.16), (7.17), (7.24) and (7.25) are then simultaneously integrated from the center to the surface using the Euler method with a step size of one meter. The numerical integration was terminated with the last step before $\rho < 7.86$ gm cm^{-3}. This is the density of iron nuclei and corresponds to the surface of a neutron star. The FORTRAN computer code for the whole integration scheme is given in Appendix J. The overall accuracy of the program was checked by recalculating the results obtained by Arnett and Bowers[60] in their review article. Our results agree with their published values to within a per cent.

7-4. Results and Discussion.

The results for the calculation are shown in Figures 11-14. Figures 11 and 12 indicate that the maximum mass for stable neutron stars is sensitive to the spin 2 coupling constant. (This however is expected from an inspection of the slopes of the pressure-density curves, Figure 7, corresponding to the different spin 2 coupling constants.) The critical densities at which the maxima come about all lie close to the value 2×10^{15} gm cm^{-3}, and are not so sensitive to the spin 2 coupling constant. In Table 2 we have summarized our results.

We now present a critical comparison of our results with the available observational information on pulsars and also with the results predicted by relevant models available in the literature. Let us first consider the mass, and refer to Figure 15, which is taken from Canuto.[2] On the right-hand side of this figure are quoted two sets of recent observational results due to Joss and Rappaport[68] and Avni.[69] The ranges of masses given by these authors are respectively 1.4 $M_{\odot}$ - 1.84 $M_{\odot}$ and 1 $M_{\odot}$ - 2.3 $M_{\odot}$. Let us now look at the various theoretical models. Curves A-G are based on non-relativistic computations, and yield the maximum masses that are all less than 1.84 $M_{\odot}$. Although these values seem to be quite compatible with the observational bounds, we stress that at the densities prevalant in the core region of a neutron star, any non-relativistic theory is an inadequate description.

Curves L, N and O give somewhat higher values for the mass. Curve L is the result of computations by Pandharipande and Smith[70] who included pion tensor interaction in a non-relativistic way. A recent analysis by Brown[71] has indicated that much of the repulsion inherent in Pandharipande and Smith's calculation is actually largely cancelled if higher order terms are included in their many-body treatment; this will then lower the mass.

Curve O is from the work of Bowers et al.[72], who used perturbation method and an effective Lagrangian formalism to derive the equation of state.

Curve N corresponds to a relativistic calculation by Walecka[30] that takes into account scalar and vector interactions among nucleons.

Finally, the result of the present work has been indicated by the dashed curve. The maximum mass turns out to be 1.75 $M_{\odot}$, corresponding to the spin 2 coupling constant $f^2_{exp.} = 2.91$. Our result is in agreement with the observational bounds, and bunches together with the results of the models A-G. Thus, incorporation of spin 2 interactions in a relativistic manner narrows the disagreement between the sets of results A-G and L,N,O. We would like to point out that the closeness of our result with those of the non-relativistic models A-G is a coincidence, and cannot be taken to mean that the non-relativistic theories provide an adequate description of high-density matter. The reason for this closeness is due to the competition, in our theory, of two important physical aspects which are (i) the inclusion of relativistic effects and (ii) the treatment of the very short-range attractive NN interaction due to the exchange of spin 2 mesons. Hence, in so far as a fully relativistic theory is imperative to describe high-density matter, our theory clearly provides a more realistic and adequate description.

Turning now to the moment of inertia, the corresponding comparison is presented in Figure 16. The two arrows in this figure indicate the value of the moment of inertia that is required if a neutron star is to be held responsible for the luminosity of the Crab Nebula as well as the kinetic energy of the expanding gas.[73] We see that our theory, corresponding to the case (b), can satisfy this requirement for masses in excess of 1 $M_{\odot}$.

Recently, it has been suggested[74] that neutron matter will undergo phase transition to quark matter at very high densities. If such is indeed the case, then our theory can fill an important gap in the understanding of dense matter in the region of density starting from nuclear matter density up to densities where neutrons are expected to merge into a quark soup.

TABLE 1. Mass density (ρ), energy per particle (ϵ/n) and pressure (P) of neutron matter as a function of neutron number density for two values of the spin 2 coupling constant. The scalar and vector coupling constants are: $g_s^2 = 13.9$ and $g_v^2 = 10.0$.

n (fm^{-3})	$f^2 = 2.91$ ρ ($gm\ cm^{-3}$)	ϵ/n (MeV)	P (Dynes cm^{-2})	$f^2 = 6.55$ ρ ($gm\ cm^{-3}$)	ϵ/n (MeV)	P (Dynes cm^{-2})
7.295	1.837×10^{16}	1.412×10^{3}	1.076×10^{36}	1.317×10^{16}	1.012×10^{3}	2.442×10^{35}
6.500	1.647	1.402	1.068	1.186	1.010	2.572
5.931	1.471	1.391	1.057	1.065	1.007	2.679
5.318	1.307	1.378	1.040	9.519×10^{15}	1.004	2.762
4.749	1.155	1.364	1.017	8.466	1.000	2.817
4.222	1.014	1.348	9.890×10^{35}	7.492	0.995	2.845
3.735	8.849×10^{15}	1.329	9.545	6.592	0.990	2.844
3.287	7.663	1.308	9.132	5.764	0.983	2.810
2.877	6.583	1.283	8.648	5.006	0.976	2.742
2.502	5.604	1.256	8.089	4.314	0.967	2.638
2.161	4.724	1.226	7.452	3.688	0.957	2.495
1.853	3.937	1.192	6.738	3.124	0.946	2.312
1.576	3.242	1.154	5.949	2.620	0.932	2.088
1.327	2.635	1.113	5.095	2.172	0.918	1.822
1.107	2.111	1.070	4.193	1.780	0.902	1.519
0.912	1.667	1.025	3.271	1.441	0.886	1.186
0.741	1.297	0.981	2.367	1.150	0.870	8.367×10^{34}

TABLE 2. Properties of the maximum-mass stable neutron star, as predicted by the present theory. Cases (a), (b) and (c) refer to the cases where $f^2 = 0$, 1.91 and 6.55 respectively.*

Case	$\frac{(M_G)_{max}}{M_\odot}$	$\frac{M_B}{M_\odot}$	ρ_c (gm cm^{-3})	R (km)	I (gm cm^2)	$\frac{M_{Bind}}{M_\odot}$
(a)	2.48	3.03	2.0×10^{15}	11.49	3.38×10^{45}	0.55
(b)	1.75	2.09	2.4×10^{15}	10.72	1.68×10^{45}	0.34
(c)	1.00	1.14	2.4×10^{15}	10.30	6.68×10^{44}	0.14

* Here $(M_G)_{max}$ is the maximum gravitational mass of a stable neutron star, M_B is the corresponding baryonic mass, ρ_c is the central density, R is the radius, I is the moment of inertia and $M_{Bind} = M_B - M_G$ = binding mass of the maximum-mass star.

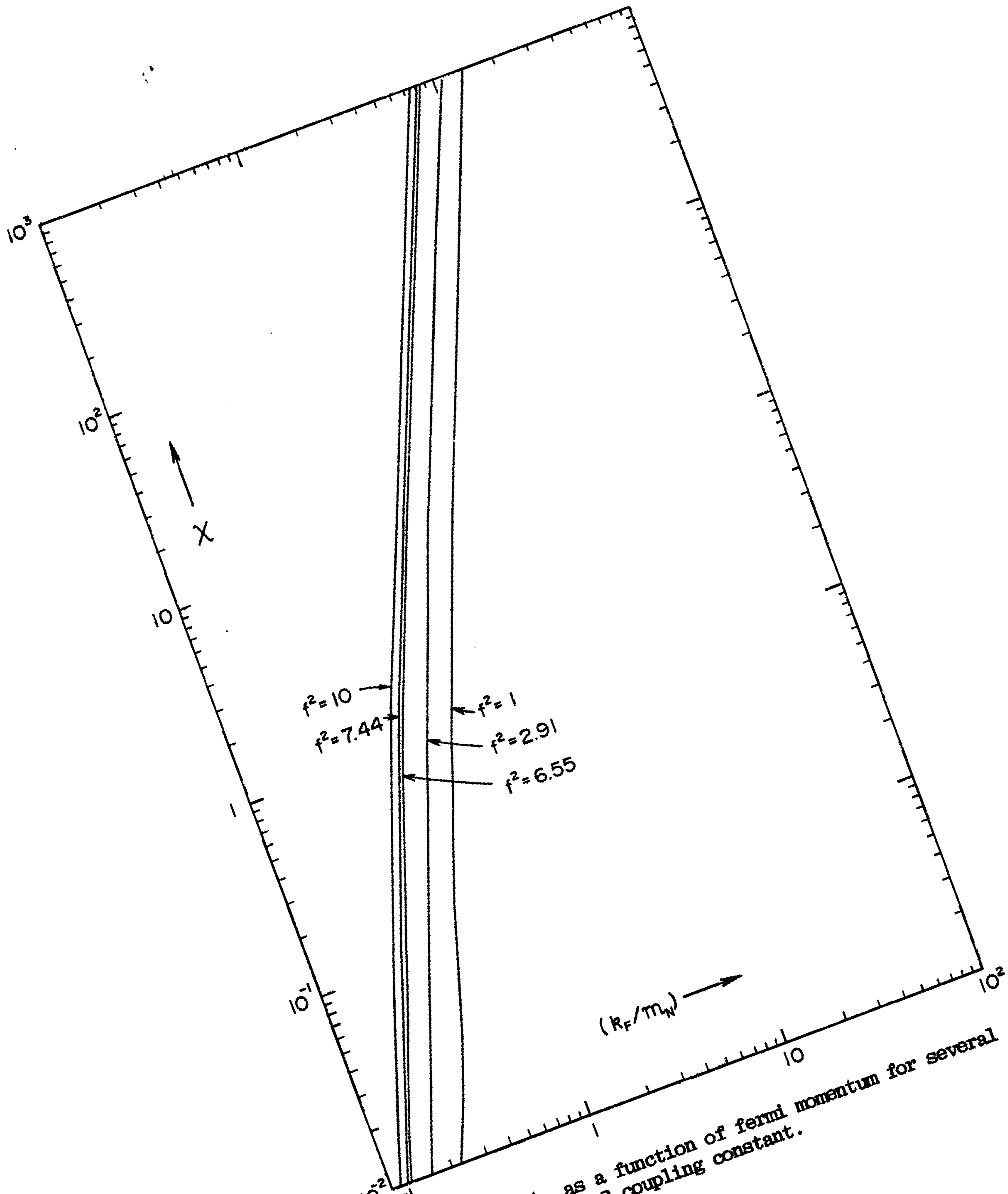

FIGURE 1. χ as a function of fermi momentum for several values of the spin 2 coupling constant.

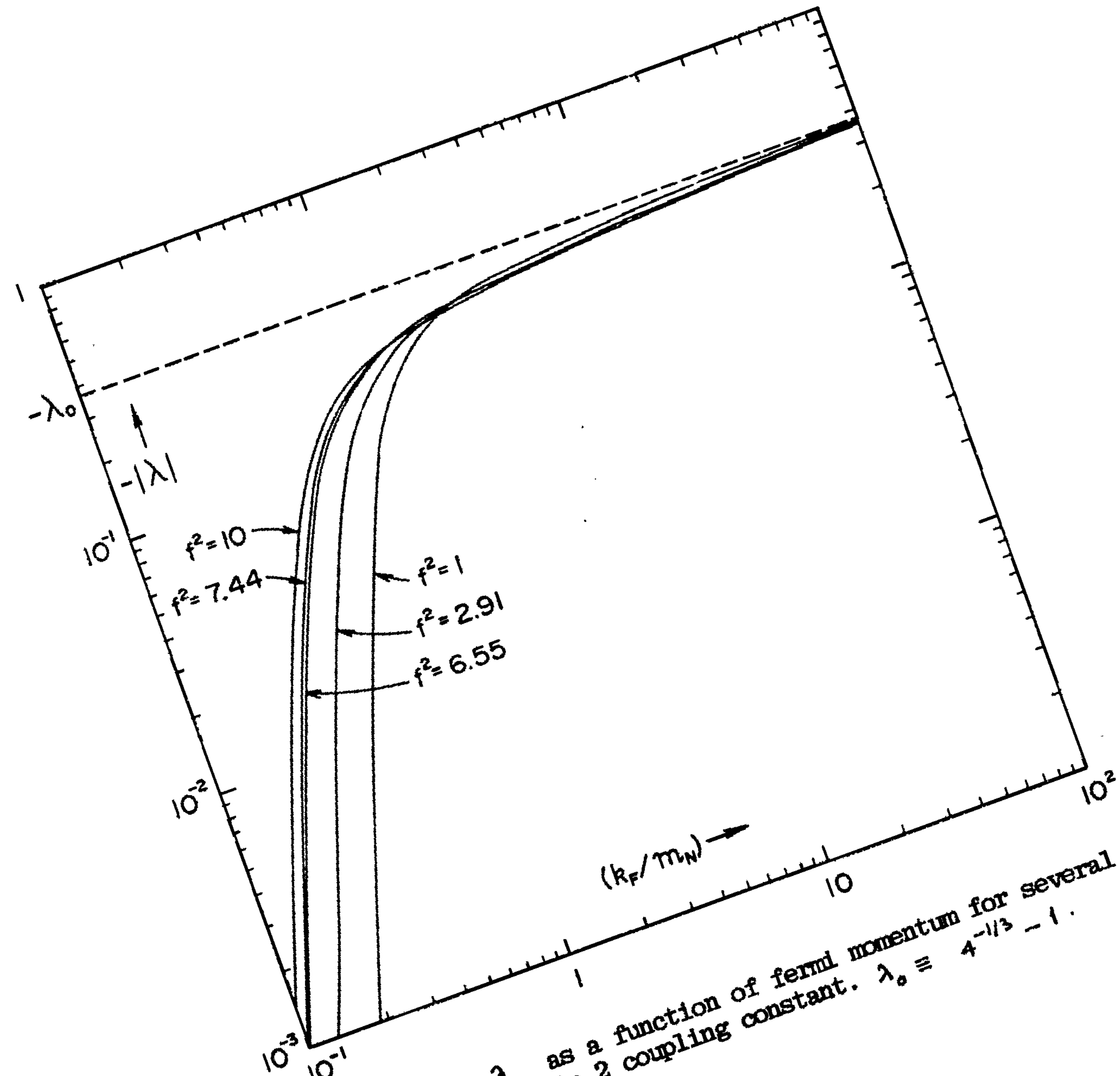

FIGURE 2. λ as a function of fermi momentum for several values of the spin 2 coupling constant. $\lambda_0 \equiv 4^{-1/3} - 1$.

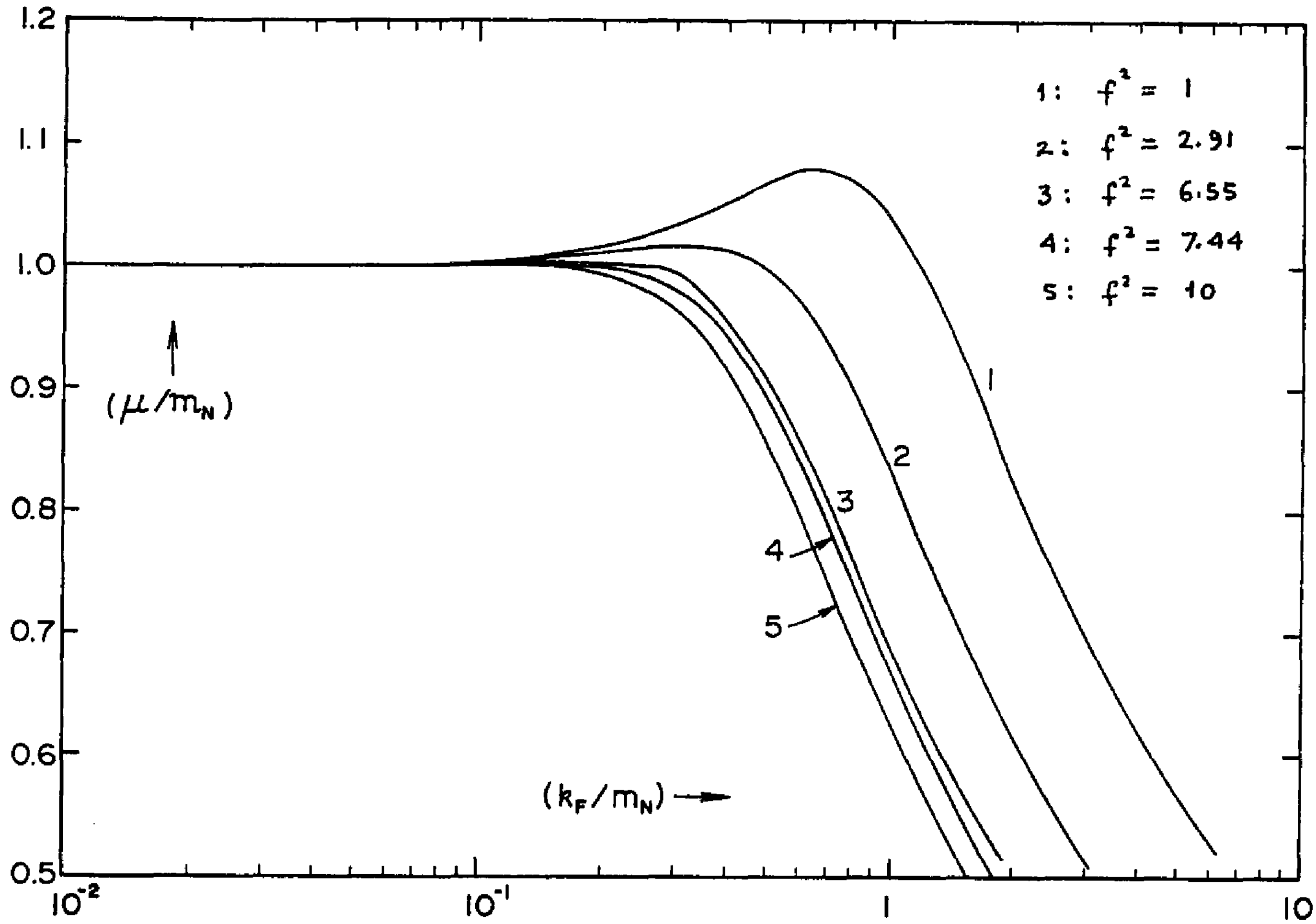

FIGURE 3. Chemical potential of neutron matter (in units of neutron rest mass) versus fermi momentum for several values of the spin 2 coupling constant. Scalar and vector interactions are absent.

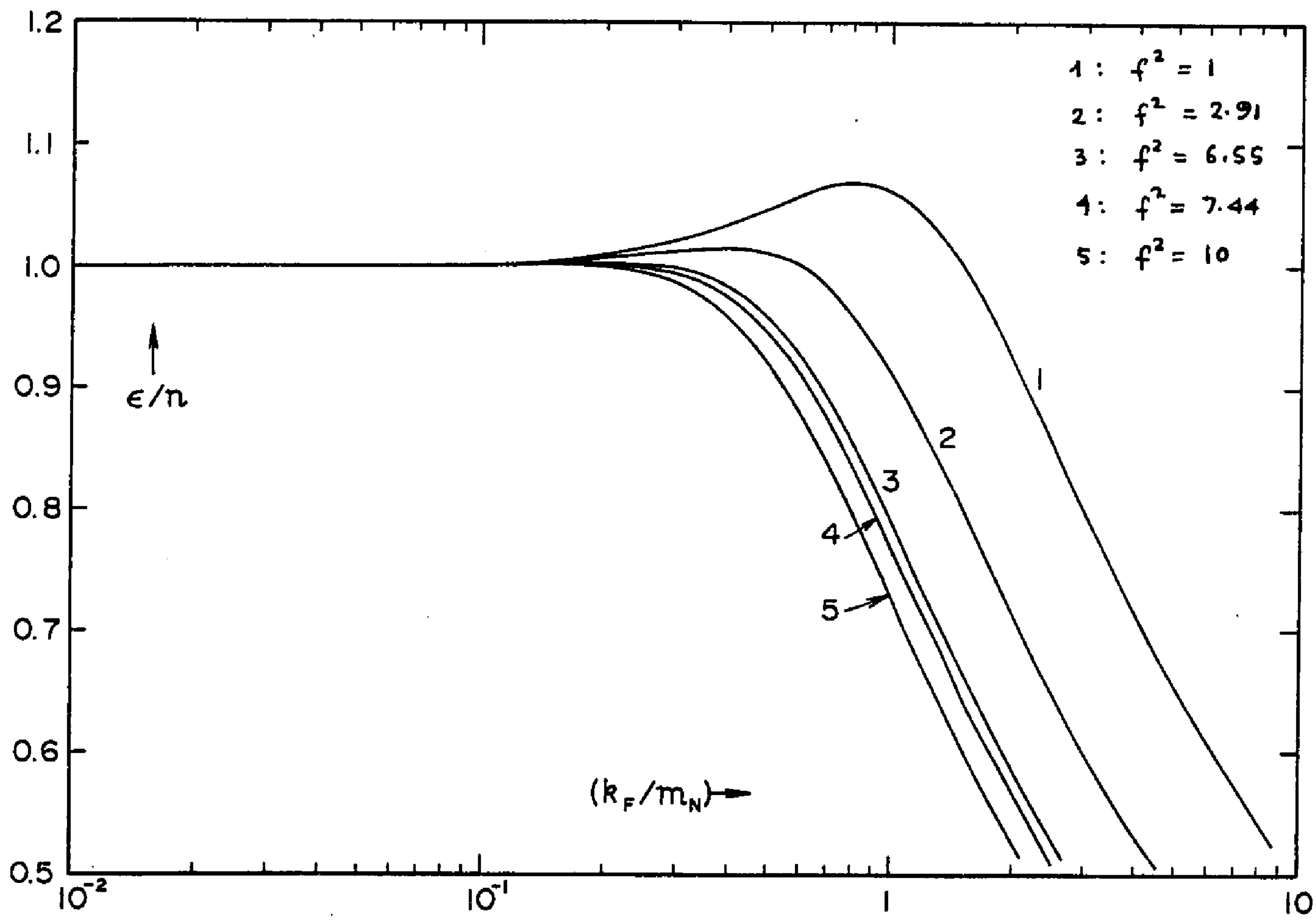

Figure 4. Energy per particle of neutron matter (in units of neutron rest mass) versus fermi momentum for several values of the spin 2 coupling constants. Scalar and vector interactions are absent.

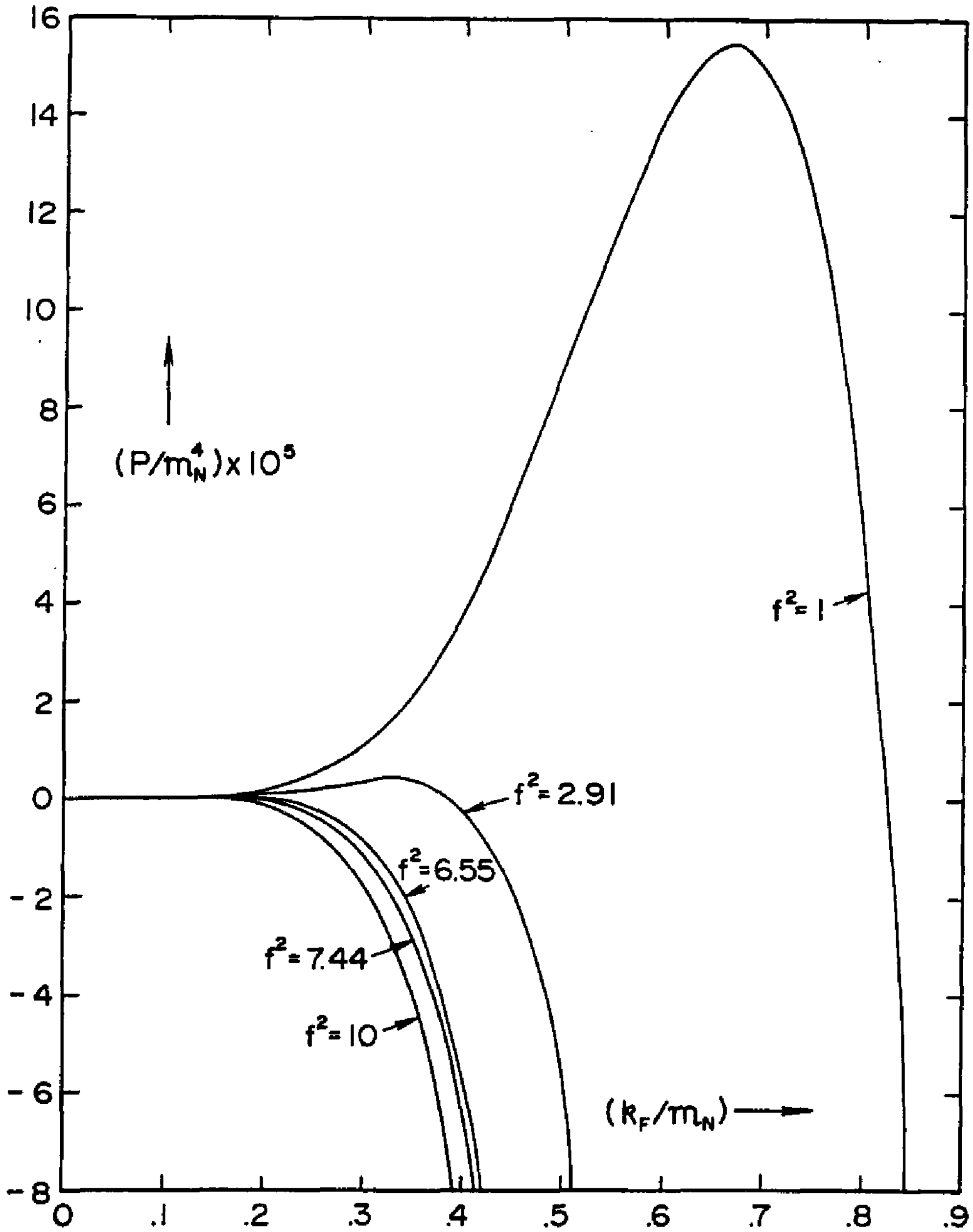

FIGURE 5. Pressure of neutron matter (in dimensionless units) versus fermi momentum for several values of the spin 2 coupling constant. Scalar and vector interactions are absent.

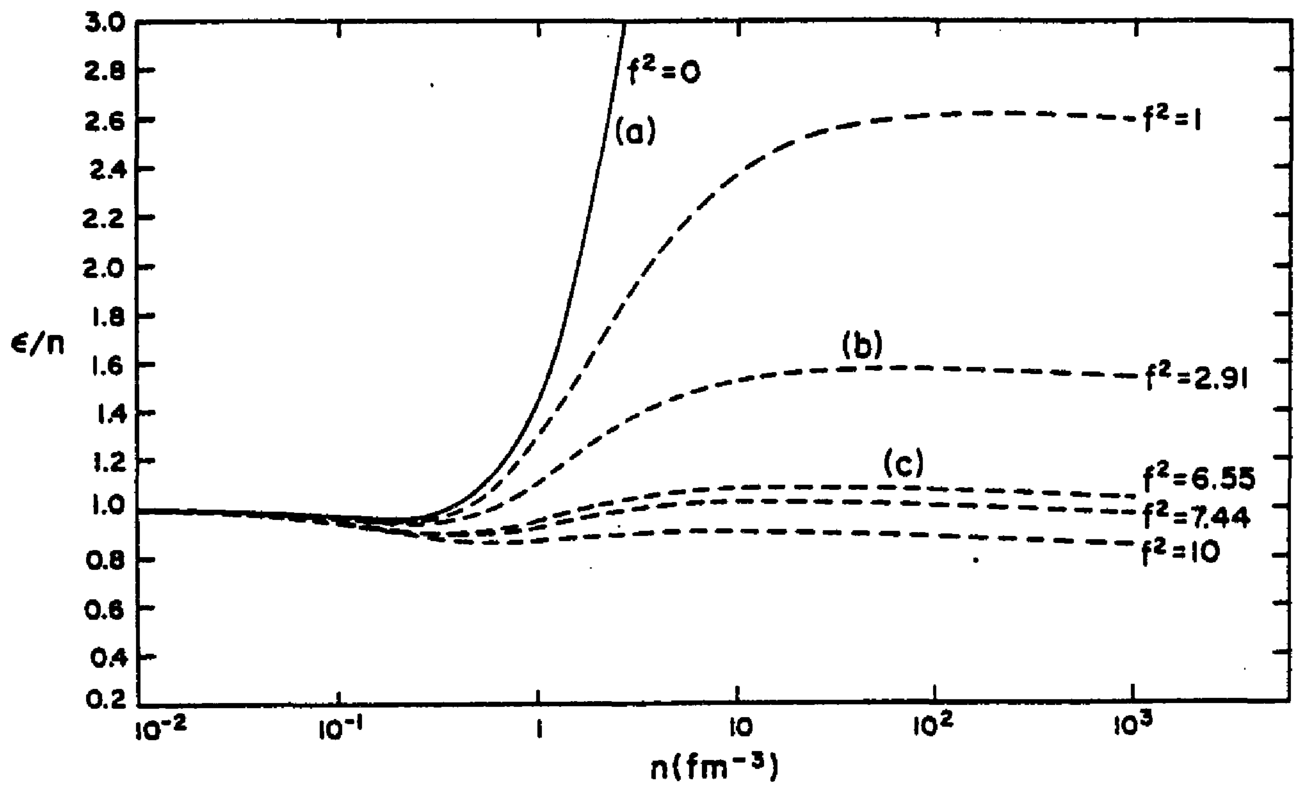

FIGURE 6. Energy per particle of neutron matter (in units of $m_N c^2$) versus neutron number density for several values of the spin 2 coupling constant. The solid curve corresponds to the case where spin 2 mesons are absent. The scalar and vector coupling constants have been taken to be: $g_s^2 = 13.9$ and $g_v^2 = 10.0$ (see text).

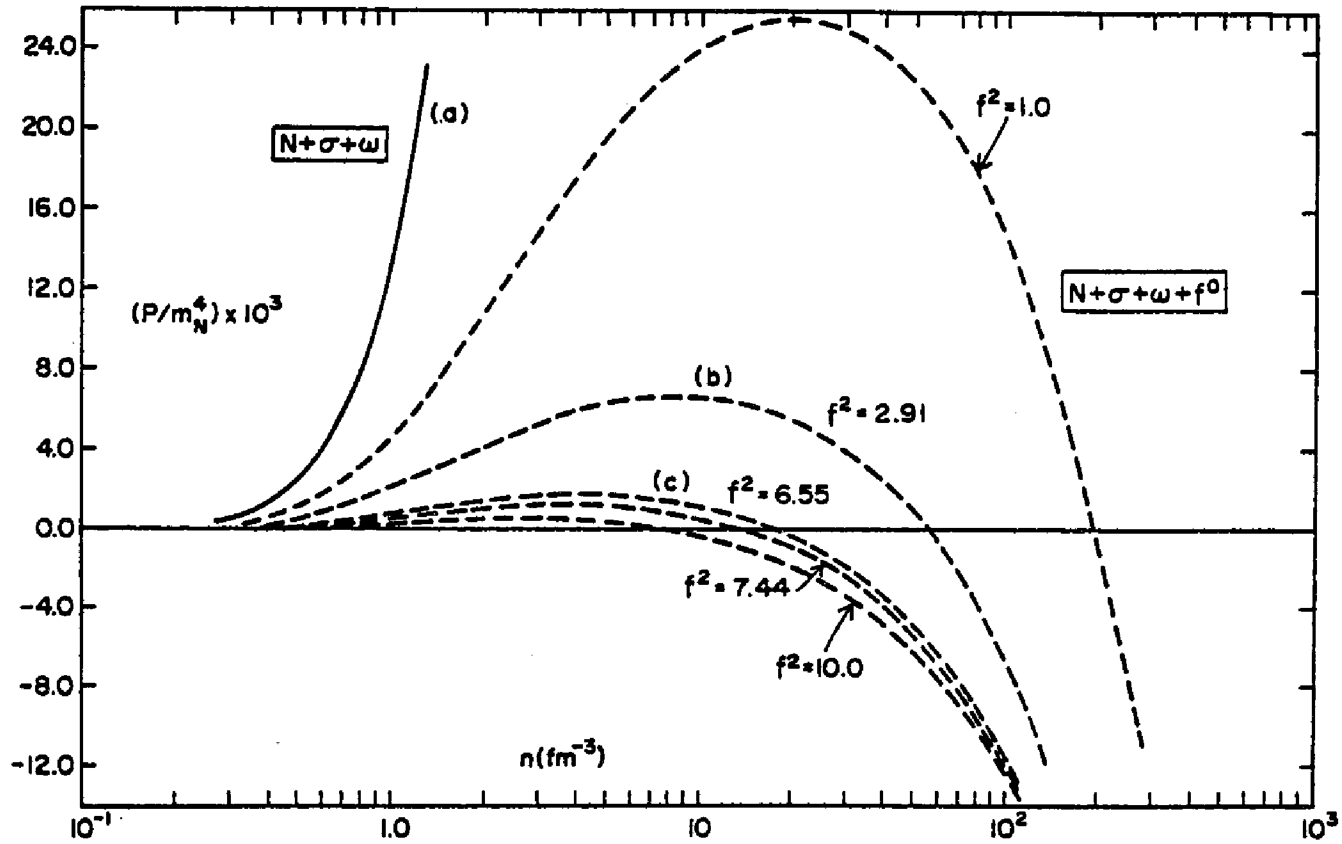

FIGURE 7. Pressure of neutron matter (in dimensionless units) versus neutron number density for several values of the spin 2 coupling constant. The solid curve corresponds to the case where spin 2 mesons are absent. The scalar and vector coupling constants have been taken to be: $g_s^2 = 13.9$ and $g_v^2 = 10.0$ (see text).

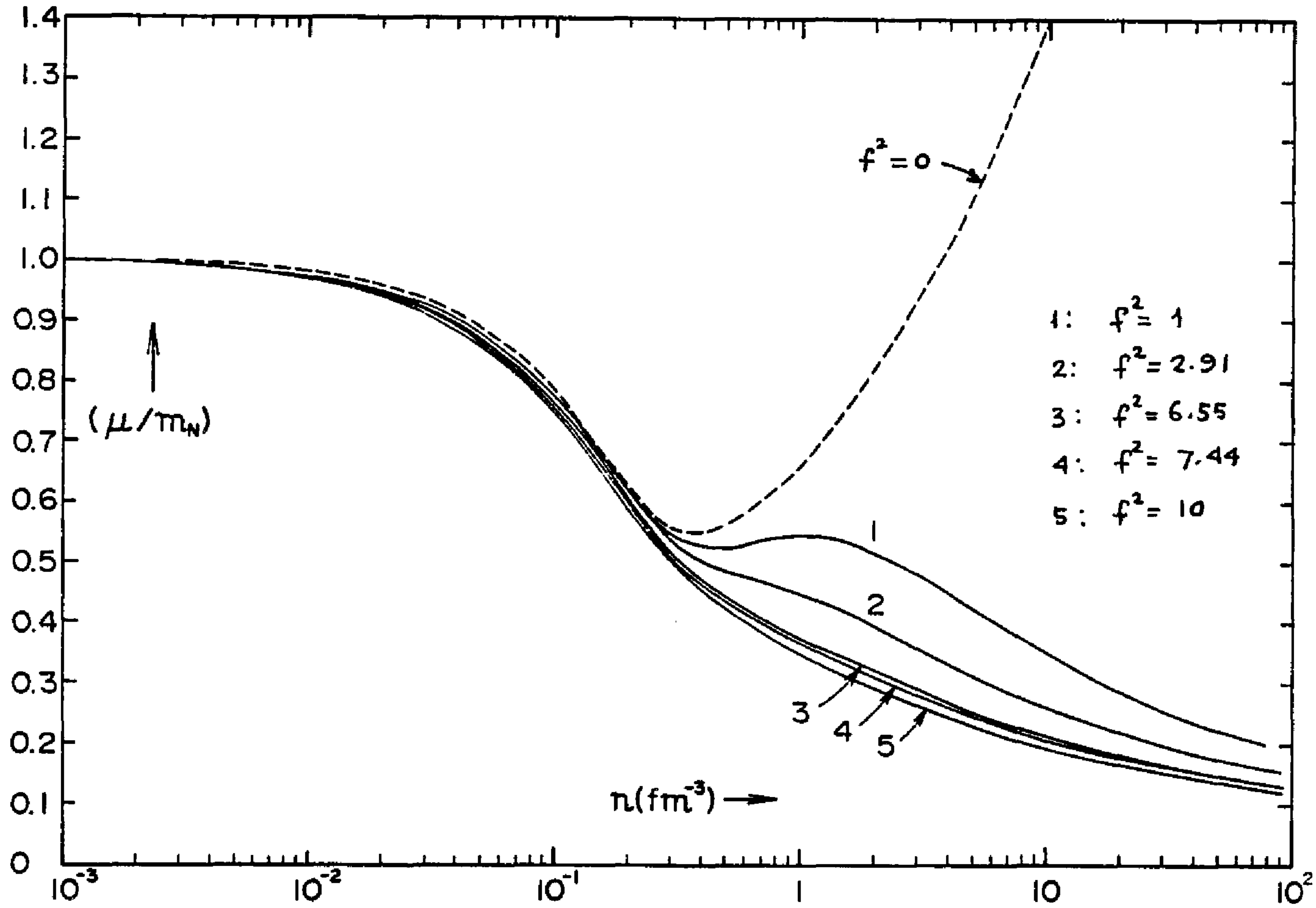

FIGURE 8. Chemical potential of neutron matter (in units of neutron rest mass) versus neutron number density for several values of the spin 2 coupling constant. The scalar and vector coupling constants have been taken to be: $g_s^2 = 13.9$ and $g_v^2 = 10.0$ (see text).

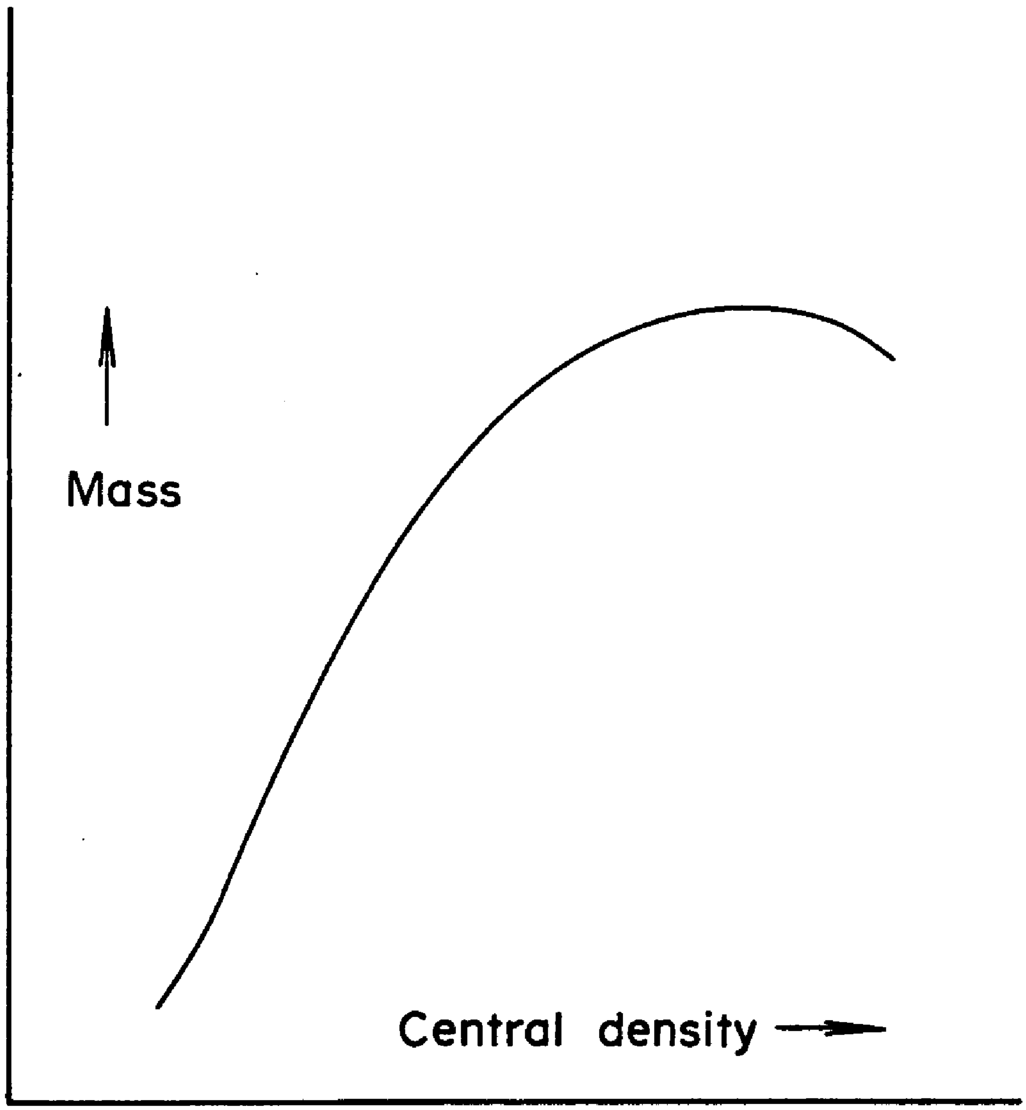

FIGURE 9. Schematic diagram showing mass-central density relationship for neutron stars.

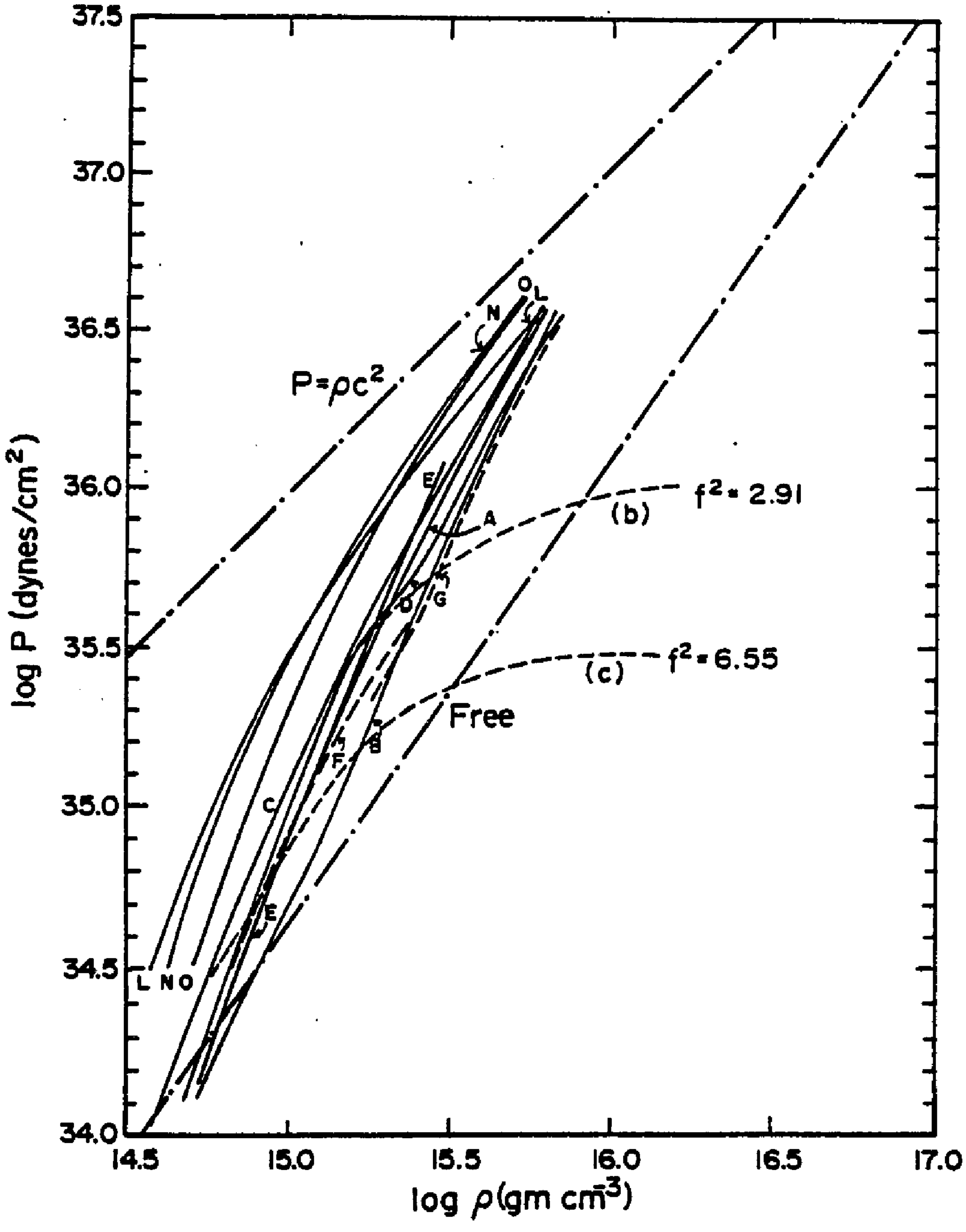

FIGURE 10. Pressure versus mass density. Curves (b) and (c) correspond to the present theory. Curves A - O correspond to other theoretical models (see Ref. 2).

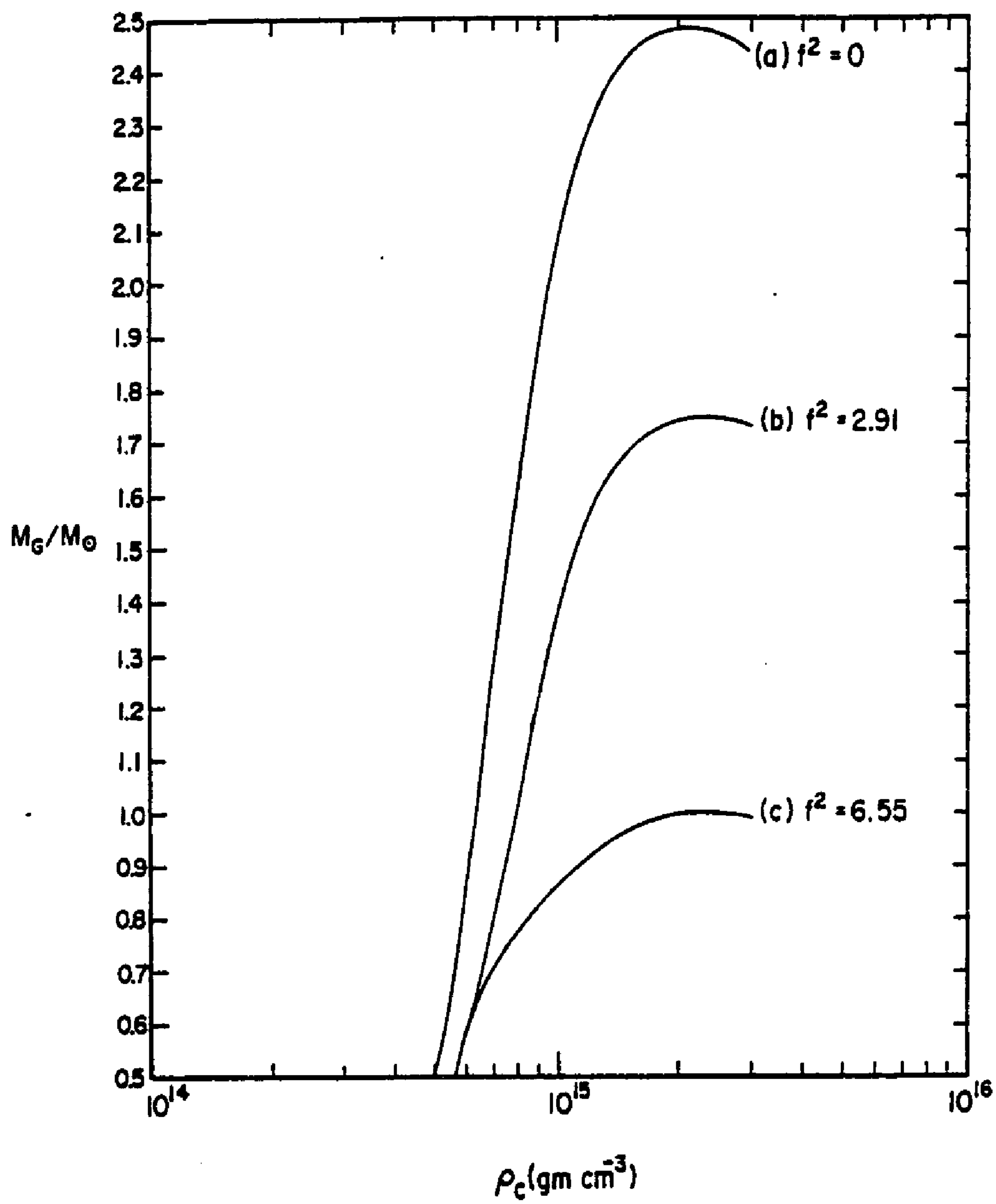

FIGURE 11. Neutron star gravitational mass, as predicted by the present theory, versus central density. The three curves correspond to three different values of the spin 2 coupling constant.

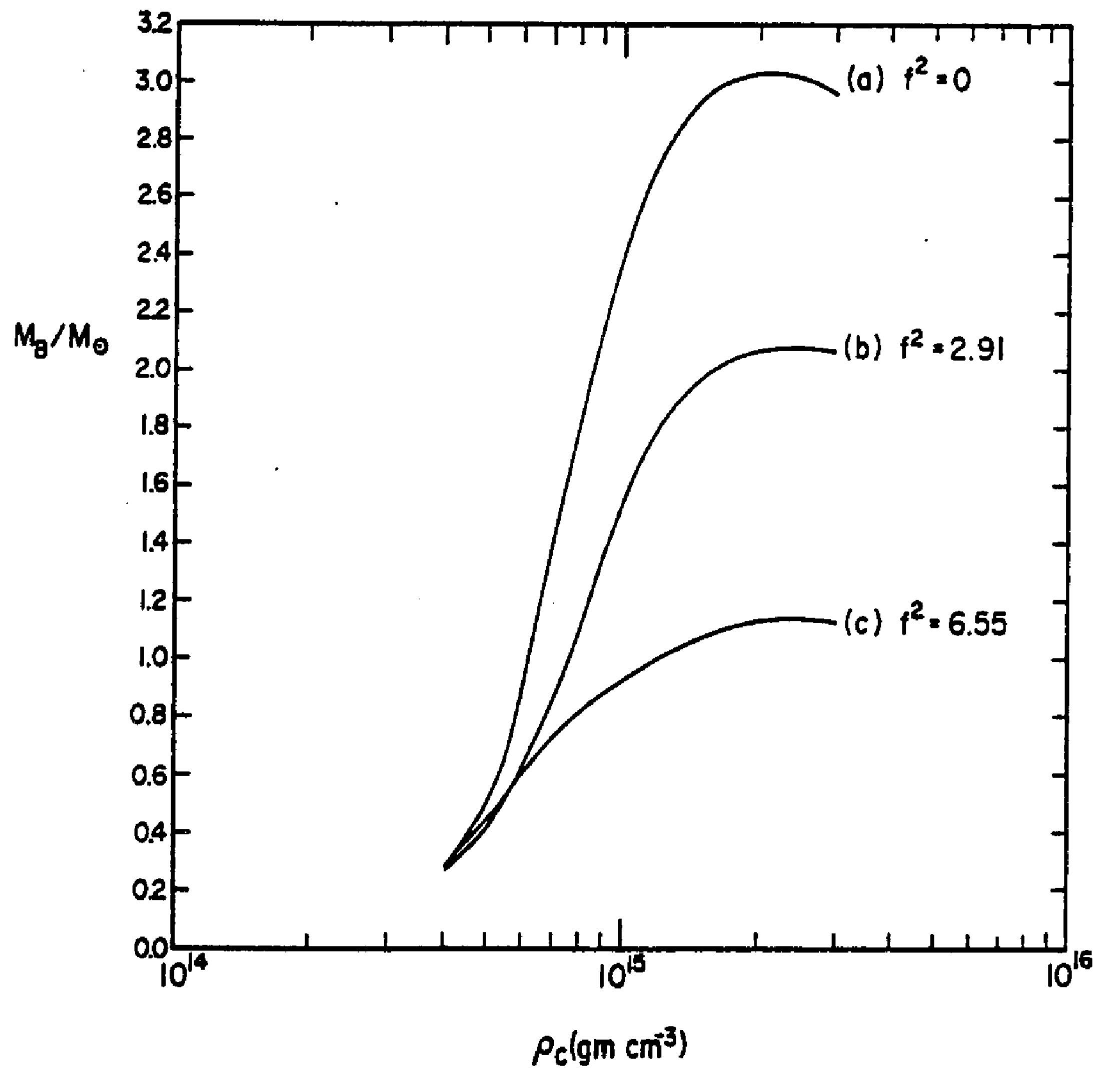

FIGURE 12. Neutron star baryonic mass, as predicted by the present theory, as a function of central density. The three curves correspond to three different values of the spin 2 coupling constant.

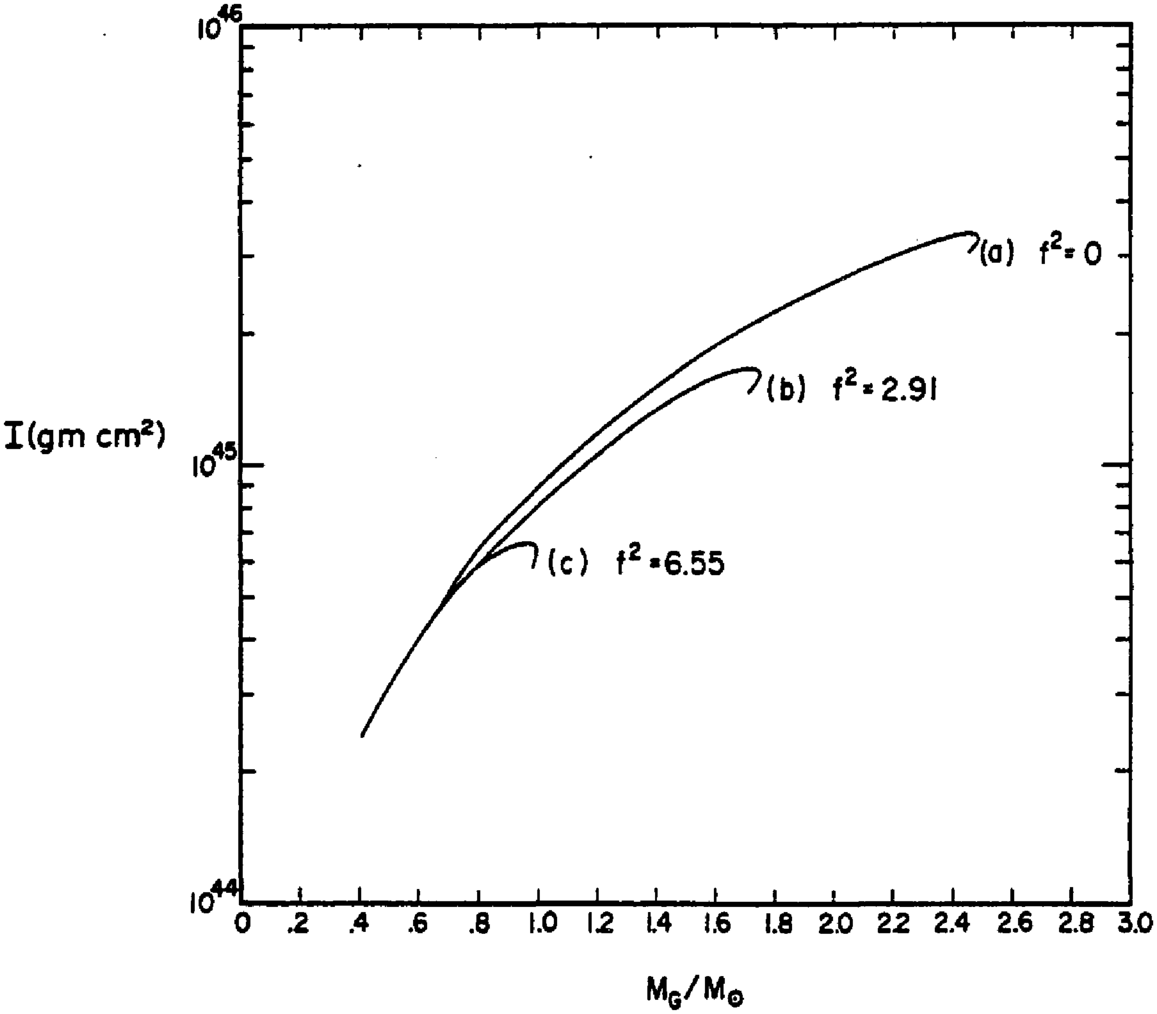

FIGURE 13. Neutron star moment of inertia versus gravitational mass, as predicted by the present theory. The three curves correspond to three different values of the spin 2 coupling constant.

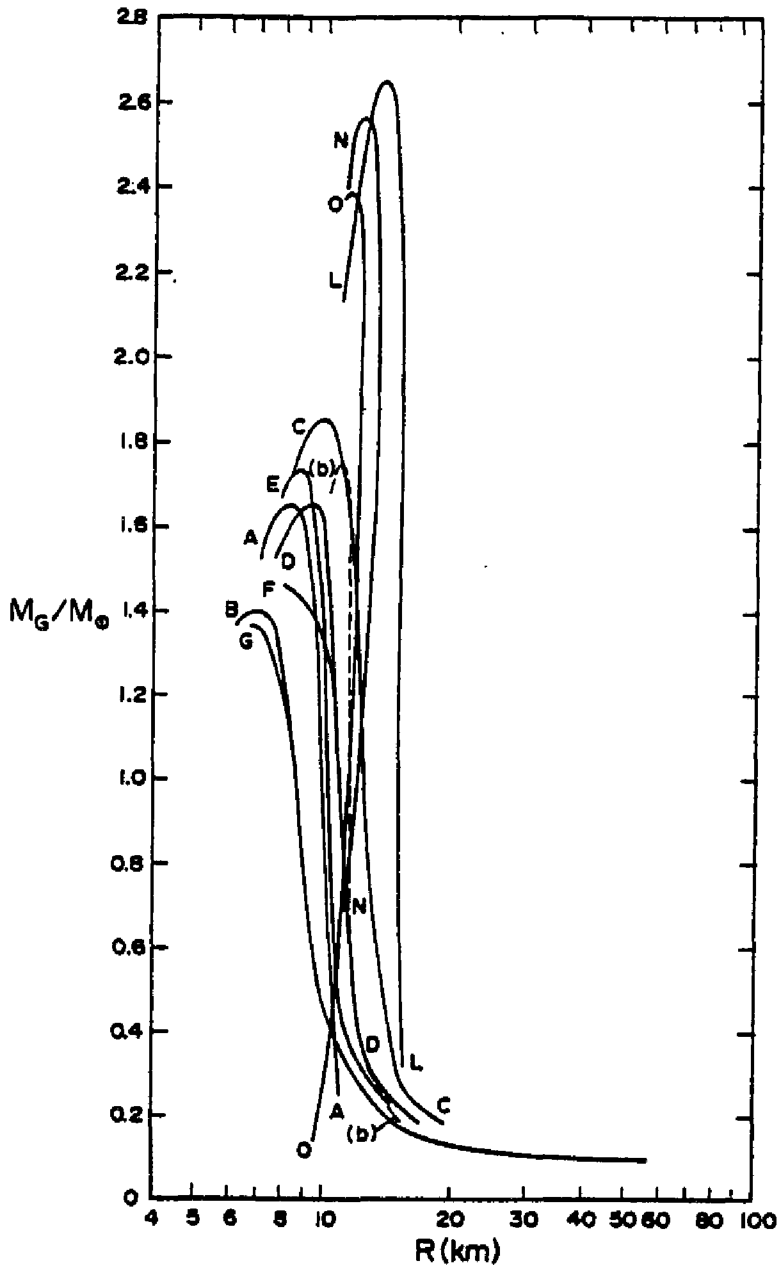

FIGURE 14. Neutron star gravitational mass versus radius. Curve (b) corresponds to the prediction of the present theory with f^2=2.91. Curves A - O correspond to other theoretical models (see Ref. 2).

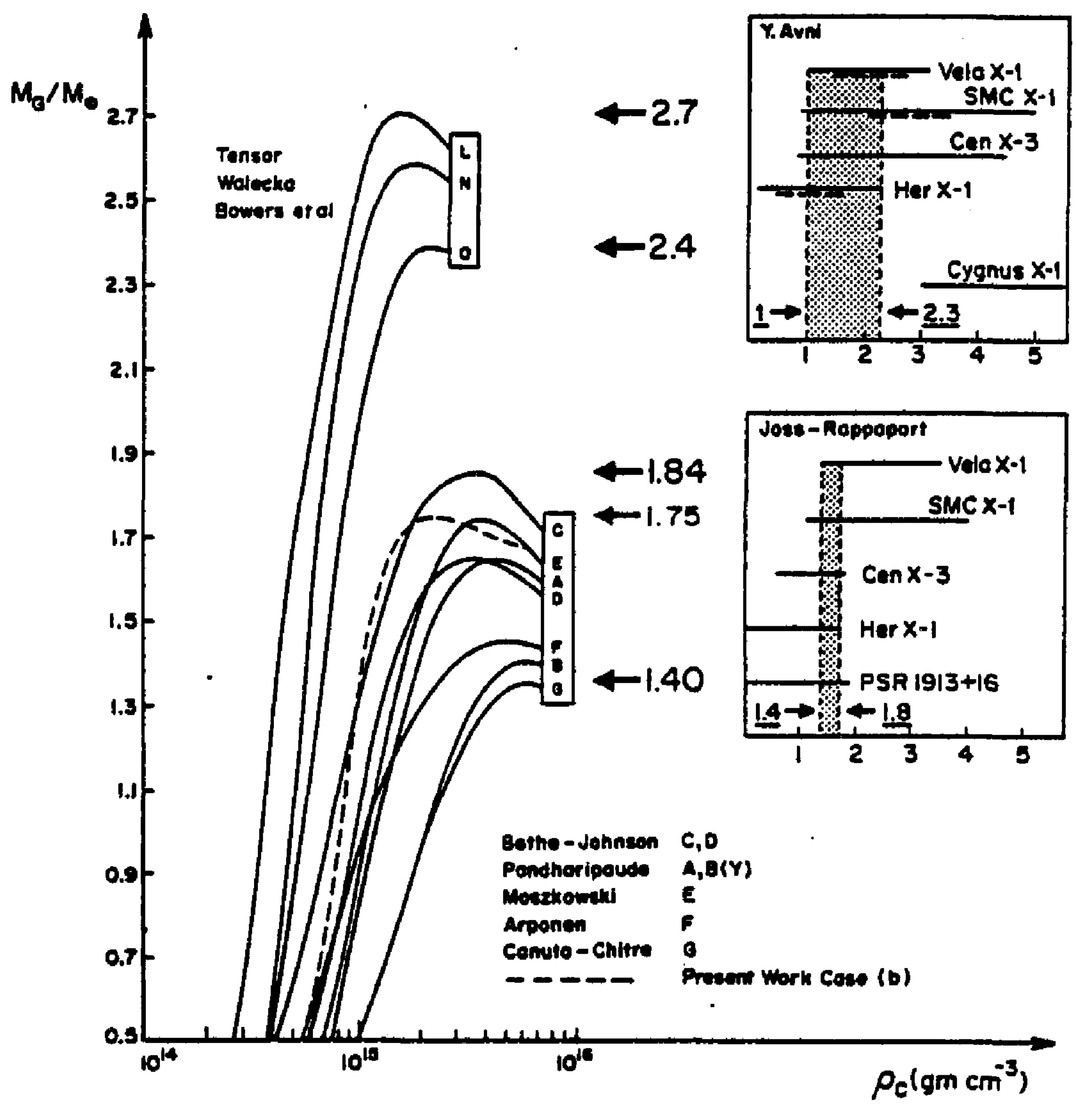

FIGURE 15. Comparison of various theoretical models for neutron star gravitational mass. The horizontal scale denotes central density. For curves A – O, see Ref. 2. Insets on the right refer to recent observational inferences for the mass (see Refs. 2, 68 and 69).

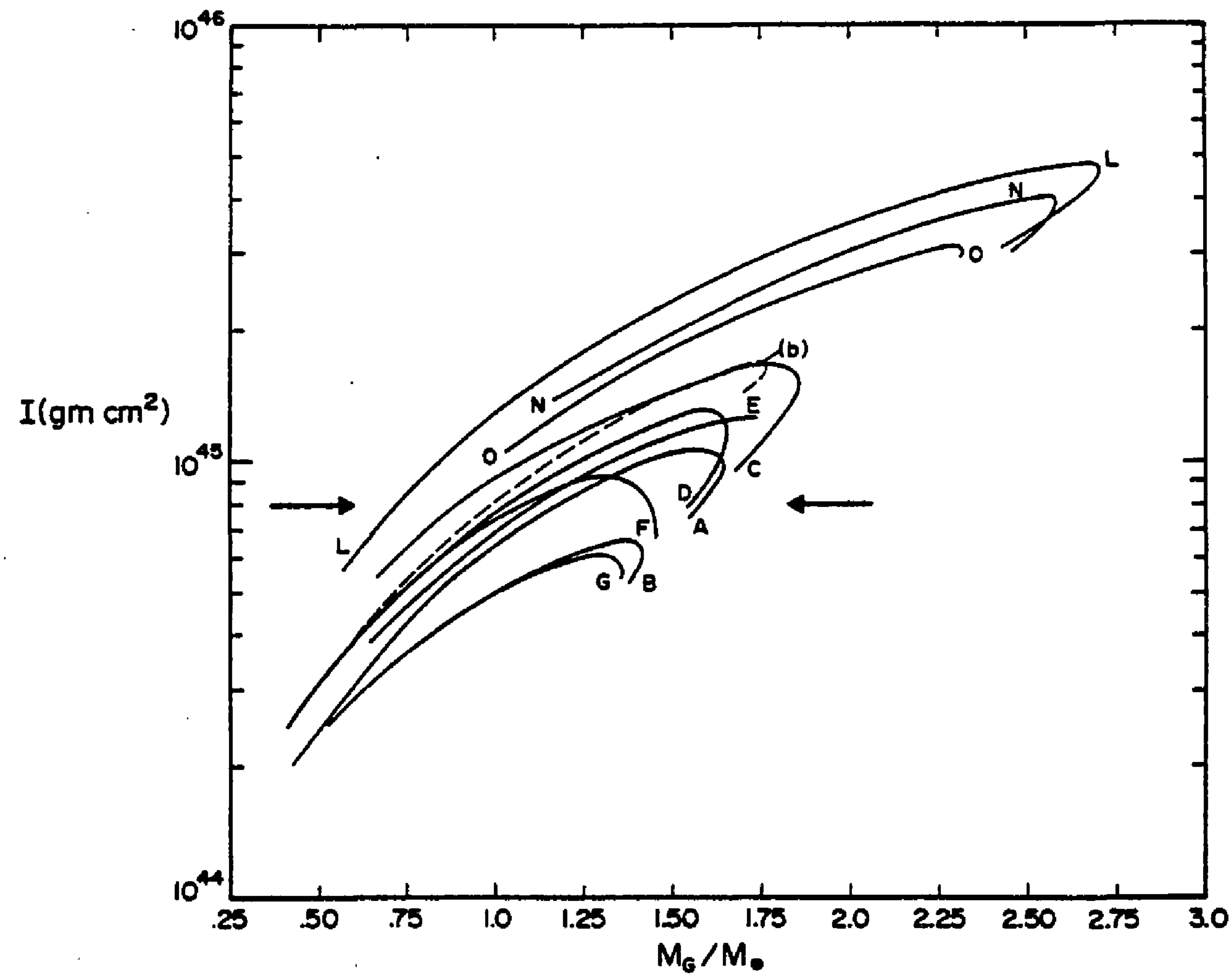

FIGURE 16. Neutron star moment of inertia versus gravitational mass. Curve (b) corresponds to the present theory with $f^2 = 2.91$. Curves A – O correspond to other theoretical models (see Ref. 2).

Appendix A

CONSERVATION OF THE SOURCE STRESS TENSOR

Consider the action corresponding to the source Lagrangian:

$$S = \int \sqrt{-g}\, \mathcal{L}(\psi, g)\, d^4x$$

The variation of S under infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

vanishes because S is an invariant. The variation of S however yields

$$\left(\frac{\delta S}{\delta \psi}\right) \delta\psi + \left(\frac{\delta S}{\delta g_{\mu\nu}}\right) \delta g_{\mu\nu}$$

The first term vanishes by virtue of the Euler-Lagrange equations. The second term is proportional to $\xi_\nu\, t^{\mu\nu}{}_{;\mu}$ for arbitrary ξ_ν . Hence

$$t^{\mu\nu}{}_{;\mu} = 0$$

Appendix B

THE VIERBEIN FORMALISM

As discussed in the text (Chapter 3), interaction of a massive spin 2 field with external fields can be written in formal analogy to gravitation. For an integer system as the external field, this is done by taking the special relativistic equations that govern the system in the absence of gravitation, and then replacing (i) all the Lorentz tensors with quantities that are tensors (or tensor densities) under general coordinate transformation, (ii) ordinary coordinate derivatives with covariant derivatives and (iii) $\eta_{\mu\nu}$ with $g_{\mu\nu}$. However, when the field in question is a spinor field, this prescription does not work. This is because there exists no representation of the group GL(4) of general linear 4 x 4 matrices which behaves like a spinor under the subgroup of Lorentz transformations. The formalism of vierbein fields is introduced to "remedy" this situation.

The vierbein field d^a_μ is defined by

$$d^a_\mu(x=X) = \left(\frac{\partial \xi^a_X(x)}{\partial x^\mu}\right)_{x=X} \tag{B.1}$$

where $\xi^a_X(x)$ (a = 0,1,2,3) characterize a locally inertial Lorentz frame. Under coordinate transformations $x^\mu \to x'^\mu$, the vierbein field transforms as a covariant world vector. Under homogeneous local Lorentz transformations L, it transforms as a Lorentz vector:

$$d^{a\prime}_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} d^{a}_{\nu}(x) \tag{B.2}$$

$$d^{a\prime}_{\mu}(x) = L^{a}_{\ b}(x)\ d^{b}_{\ \mu}(x) \tag{B.3}$$

Fermion fields ψ can be introduced into general relativity[75] by describing them with respect to local Lorentz frames. They are defined to be world scalars and transform as ordinary spinors under local Lorentz transformations of the vierbein frames. Now, the ordinary derivative $\partial_\mu \Psi$ is a covariant world vector, but not a proper Lorentz spinor. A covariant derivative $\nabla_\mu \Psi$ can be introduced such that $\nabla_\mu \Psi$ is a covariant world vector and a Lorentz spinor:

$$\nabla_\mu \Psi = \partial_\mu \psi - \frac{i}{4} \omega_{a\mu b} \sigma^{ab} \psi , \tag{B.4}$$

where $$\sigma^{ab} = \frac{i}{2} [\gamma^a, \gamma^b] \tag{B.5}$$

Using Eq. (B.1) and the definition of the line element

$$(ds)^2 = d\xi^a\, d\xi^b\, \eta_{ab}$$

one gets $$d^a_\mu\ d^b_\nu\ \eta_{ab} = g_{\mu\nu} \tag{B.6}$$

The Latin index in d^a_μ is associated with local Minkowski coordinate system and the Greek index with general coordinate system. Hence

$$d_\mu^a \eta_{ab} = d_{\mu b} \tag{B.7}$$

$$d_{\mu a} = g_{\mu\nu} d_a^\nu \tag{B.8}$$

The structure of the spin connection $\omega_{a\mu b}$ can be determined by requiring that the operations of raising and lowering of indices and the operations of changing index-type commute with covariant differentiation.[76] Just as the condition $g_{\mu\nu;\lambda}=0$, in Riemannian geometry, leads to the identification of affine connection with the Christoffel symbols, similarly the condition

$$d^a_{\mu;\nu} = 0 \tag{B.9}$$

leads to the following identification of $\omega_{a\mu b}$:

$$\begin{aligned}\omega_{a\mu b} = \frac{1}{2} d^c_\mu \Big\{ & d_{\lambda a}(d_b^\nu \partial_\nu d_c^\lambda - d_c^\nu \partial_\nu d_b^\lambda) \\ & + d_{\lambda b}(d_c^\nu \partial_\nu d_a^\lambda - d_a^\nu \partial_\nu d_c^\lambda) \\ & - d_{\lambda c}(d_a^\nu \partial_\nu d_b^\lambda - d_b^\nu \partial_\nu d_a^\lambda) \Big\}\end{aligned} \tag{B.10}$$

We can now write down the Lagrangian density for a spin 2 field that is invariant under (i) arbitrary coordinate transformations and (ii) local Lorentz transformations. It is given by[77]

$$\mathcal{L}(\psi, g) = -d\,\overline{\Psi}\left\{\gamma^a d^{\mu}_{a}\frac{1}{i}\nabla_\mu + m\right\}\psi \tag{B.11}$$

where $$d \equiv \det d_{\mu a} = \sqrt{-g}\ , \tag{B.12}$$

$$g \equiv \det g_{\mu\nu} \tag{B.13}$$

In the formalism for a massive spin 2 theory no general covariance, as understood in general relativity, is implied. Hence, we shall distinguish d^a_μ from its "contravariant" counterpart by denoting the latter by $e^{\mu a}$. We then have the following equivalent of Eq. (B.5) for $e^{\mu a}$:

$$e^{\mu a}\ e^{\nu a}\ \eta_{ab} = h^{\mu\nu} \tag{B.14}$$

where $h^{\mu\nu}$ is the reciprocal of $g_{\mu\nu}$.

Correspondingly, we re-write Eq. (B.8), (B.10) and (B.11) as

$$d_{\mu a} = g_{\mu\nu}\ e^{\nu}_{a} \tag{B.15}$$

$$\begin{aligned}\omega_{a\mu b} = \tfrac{1}{2} d^c_\mu \Big\{ & d_{\lambda a}\left(e^{\nu}_{b}\partial_\nu e^{\lambda}_{c} - e^{\nu}_{c}\partial_\nu e^{\lambda}_{c}\right) \\ & + d_{\lambda b}\left(e^{\nu}_{c}\partial_\nu e^{\lambda}_{a} - e^{\nu}_{a}\partial_\nu e^{\lambda}_{c}\right) \\ & - d_{\lambda c}\left(e^{\nu}_{a}\partial_\nu e^{\lambda}_{b} - e^{\nu}_{b}\partial_\nu e^{\lambda}_{a}\right)\Big\}\end{aligned} \tag{B.16}$$

$$\mathcal{L}(\psi, g) = -d\ \bar{\psi}\left(\gamma^{a} e^{\mu}_{a} \frac{1}{i} \nabla_{\mu} + m\right) \psi \tag{B.17}$$

Appendix C

EQUATION OF MOTION FOR THE SPIN 2 FIELD

In this appendix we derive the form (5.12) for the equation of motion for the spin 2 field. We note that Eqs. (5.9) and (5.11) give

$$G_{\mu\nu} + \frac{m^2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta h^{\mu\nu}} = -\kappa\, t_{\mu\nu} \tag{C.1}$$

Writing $\delta \mathcal{L}_m / \delta h^{\mu\nu}$ in terms of $\delta \mathcal{L}_m / \delta \bar{g}^{\mu\nu}$ [see the relation (4.5)], we get

$$G_{\mu\nu} + m^2 \left\{ \frac{\delta \mathcal{L}_m}{\delta \bar{g}^{\mu\nu}} - \frac{1}{2} \frac{\delta \mathcal{L}_m}{\delta \bar{g}^{\alpha\beta}} h^{\alpha\beta} g_{\mu\nu} \right\} = -\kappa\, t_{\mu\nu} \tag{C.2}$$

Let us put

$$\frac{\delta \mathcal{L}_m}{\delta \bar{g}^{\mu\nu}} \equiv B_{\mu\nu} \tag{C.3}$$

and

$$B_{\mu\nu} h^{\mu\nu} \equiv B \tag{C.4}$$

Then, Eq. (C.2) becomes

$$G_{\mu\nu} + m^2 \left(B_{\mu\nu} - \frac{1}{2} g_{\mu\nu} B \right) = -\kappa\, t_{\mu\nu} \tag{C.5}$$

Taking the trace of both sides,

$$h^{\mu\nu} G_{\mu\nu} - m^2 B = -K t \tag{C.6}$$

where
$$t \equiv h^{\mu\nu} t_{\mu\nu} \tag{C.7}$$

From the definition (4.9) of $G_{\mu\nu}$:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{C.8}$$

we get

$$h^{\mu\nu} G_{\mu\nu} = -R \tag{C.9}$$

Substituting this in Eq. (C.6), we get

$$R + m^2 B = K t \tag{C.10}$$

Using Eqs. (C.8) and (C.10), Eq. (C.5) now becomes

$$R_{\mu\nu} + m^2 B_{\mu\nu} = -K\left(t_{\mu\nu} - \frac{1}{2} g_{\mu\nu} t\right) \tag{C.11}$$

From the expression (4.56) for $\mathcal{L}_m$, we get, using (C.3),

$$B_{\mu\nu} = \frac{1}{2}\left(\frac{\partial g}{\partial \bar{g}^{\mu\nu}} + \eta_{\mu\nu}\right)$$

Since

$$\frac{\partial \sqrt{-g}}{\partial \bar{g}^{\mu\nu}} = \frac{1}{2} g_{\mu\nu} \quad ,$$

$$B_{\mu\nu} = -\frac{1}{2}\left(\sqrt{-g}\; g_{\mu\nu} - \eta_{\mu\nu}\right) \tag{C.12}$$

Therefore, Eq. (C.11) becomes

$$R_{\mu\nu} - \frac{m^2}{2}\left(\sqrt{-g}\, g_{\mu\nu} - \eta_{\mu\nu}\right) = -K\left(t_{\mu\nu} - \frac{1}{2} g_{\mu\nu} t\right) \tag{C.13}$$

With $$m = m_f$$

and $$K = \frac{8\pi f^2}{m_N^2}$$

the above equation is identical to Eq. (5.12), presented in Chapter 5.

Appendix D

THE SOURCE TERM FOR THE SPIN 2 FIELD

The Lagrangian appropriate for a system of coupled fermion, scalar and vector fields that provide the source term for the spin 2 field is [see Eq. (6.1) in Chapter 6]:

$$\begin{aligned}
\mathcal{L} &= \sqrt{-g}\ \bar{\Psi}\left(\gamma^{a} e^{\mu}_{a} \frac{1}{i} \nabla_{\mu} + m_N\right)\Psi \\
&- \frac{1}{8\pi}\sqrt{-g}\left(\partial_{\mu}\sigma\, h^{\mu\nu}\partial_{\nu}\sigma + m_s^2\sigma^2\right) \\
&- \frac{1}{4\pi}\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}h^{\mu\lambda}h^{\nu\rho}F_{\lambda\rho} + \frac{1}{2}m_v^2 A_{\mu}h^{\mu\nu}A_{\nu}\right) \\
&+ g_s\sqrt{-g}\ \bar{\Psi}\Psi\sigma + g_v\sqrt{-g}\ \bar{\Psi}e^{\mu}_{a}\gamma^{a}\Psi A_{\mu}
\end{aligned} \tag{D.1}$$

where $\nabla_{\mu}\Psi \equiv (\partial_{\mu} - \frac{i}{4}\omega_{b\mu c})\Psi$.
The expression for $\omega_{b\mu c}$ is given by Eq. (B.16) in Appendix B.

The Lagrangian is always hermitian; therefore, in the second term on the right hand side of Eq. (D.1) only the anti-symmetric part of the matrices $\gamma^{a}\sigma^{bc}$ will contribute. This removes the terms with $a = b$ or $a = c$, which are proportional to the symmetrical matrices $\gamma^{0}\gamma^{a}$. For $a \neq b, c$, we can write

$$\gamma^a \sigma^{bc} = \frac{i\gamma^a}{2} [\gamma^b, \gamma^c]$$

$$= i \epsilon^{abcd} \gamma_d \gamma_5 \qquad \text{(D.2)}$$

where ϵ^{abcd} is the Levi-Civita tensor.

Setting

$$\bar{\omega}^d \equiv \frac{1}{4} \epsilon^{abcd} \omega_{a\nu c} e^\nu_b \qquad \text{(D.3)}$$

we re-write $\mathcal{L}$ as

$$\mathcal{L} = -\sqrt{-g}\, \bar{\Psi} \left(\gamma^a e^\mu_a \frac{1}{i} \partial_\mu + \frac{i}{4} \bar{\omega}^a \gamma_a \gamma_5 + m_N \right) \Psi$$

$$- \frac{\sqrt{-g}}{8\pi} \left(\partial_\mu \sigma\, h^{\mu\nu} \partial_\nu \sigma + m_s^2 \sigma^2 \right)$$

$$- \frac{\sqrt{-g}}{4\pi} \left(\frac{1}{4} F_{\mu\nu} h^{\mu\lambda} h^{\nu\rho} F_{\lambda\rho} + \frac{1}{2} m_v^2 A_\mu h^{\mu\nu} A_\nu \right)$$

$$+ g_s \sqrt{-g}\, \bar{\Psi} \Psi \sigma + g_v \sqrt{-g}\, \bar{\Psi} e^\mu_a \gamma^a \Psi A_\mu \qquad \text{(D.4)}$$

Equations of motion (the Euler-Lagrange equations) that follow from the above Lagrangian are:

1) Fermion field:

$$\left(\gamma^a e^\mu_a \frac{1}{i}\partial_\mu - \frac{1}{4i}\bar{\omega}^a \gamma_a \gamma_5 - g_v \gamma^a e^\mu_a A_\mu + m_N - g_s \sigma\right)\Psi = 0 \quad (D.5)$$

2) Scalar field:

$$\left(-\partial^2 + m_s^2\right)\sigma = 4\pi g_s \bar{\Psi}\Psi \quad (D.6)$$

3) Vector field:

$$\sqrt{-g}\, m_v^2 A_\mu h^{\mu\nu} + \partial_\mu\left(\sqrt{-g}\, F_{\lambda\rho} h^{\mu\lambda} h^{\nu\rho}\right) = 4\pi g_v \sqrt{-g}\, \bar{\Psi}\gamma^a e^\nu_a \Psi \quad (D.7)$$

The stress tensor $t_{\mu\nu}$ corresponding to the Lagrangian $\mathcal{L}$ is given by[78]

$$\frac{1}{2}\sqrt{-g}\; t_{\mu\nu} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial h^{\mu\nu}{}_{,\alpha}} - \frac{\partial \mathcal{L}}{\partial h^{\mu\nu}}$$

$$= -\frac{\partial \mathcal{L}}{\partial h^{\mu\nu}} , \quad (D.8)$$

since $\mathcal{L}$ does not involve derivatives of $h^{\mu\nu}$. To evaluate $t_{\mu\nu}$, we note the following variations[79]

$$\delta g = -g\, g_{\mu\nu}\, \delta h^{\mu\nu} \quad (D.9)$$

$$\delta e^\mu_a = \frac{1}{2}\, d_{\nu a}\, \delta h^{\mu\nu} \quad (D.10)$$

$$\delta \bar{\omega}^d = -\frac{1}{8}\, \epsilon^{abcd}\, d_{\mu a}\, \omega_{b\nu c}\, \delta h^{\mu\nu} \quad (D.11)$$

Using Eqs. (D.4) and (D.9)-(D.11), (D.8) yields

$$t_{\mu\nu} = -g_{\mu\nu}\,\bar{\Psi}\,\gamma^a e^{\alpha}_{a}\,\frac{1}{i}\,\partial_\alpha \psi \;+\; \bar{\Psi}\,\gamma^a d_{\nu a}\,\frac{1}{i}\,\partial_\mu \psi$$

$$+\frac{1}{4i}\,g_{\mu\nu}\,\bar{\Psi}\psi\,\bar{\omega}^{d}\gamma_d\gamma_5 \;+\; \frac{1}{4i}\,\bar{\Psi}\psi\,\epsilon^{abcd}\,d_{\mu a}\,\omega_{b\nu c}\,\gamma_d\gamma_5$$

$$-g_{\mu\nu}\,m_N\,\bar{\Psi}\psi \;-\; \frac{1}{8\pi}\,g_{\mu\nu}\,\partial_\alpha\sigma\,h^{\alpha\beta}\,\partial_\beta\sigma \;-\; \frac{1}{4\pi}\,\partial_\mu\sigma\,\partial_\nu\sigma$$

$$-\frac{m_s^2}{8\pi}\,g_{\mu\nu}\,\sigma^2 \;-\; \frac{1}{16\pi}\,g_{\mu\nu}\,F_{\alpha\beta}\,h^{\alpha\lambda}\,h^{\beta\rho}\,F_{\lambda\rho} \;+\; \frac{1}{4\pi}\,F_{\mu\alpha}\,h^{\alpha\beta}\,F_{\nu\beta}$$

$$-\frac{m_v^2}{8\pi}\,g_{\mu\nu}\,A_\alpha\,h^{\alpha\beta}A_\beta \;+\; \frac{m_v^2}{4\pi}\,A_\mu A_\nu \;+\; g_s\,g_{\mu\nu}\,\bar{\Psi}\psi\sigma$$

$$+g_v\,g_{\mu\nu}\,\bar{\Psi}\,e^{\alpha}_{a}\gamma^a\psi\,A_\alpha \;-\; g_v\sqrt{-g}\;\bar{\Psi}\,d_{\nu a}\gamma^a\psi\,A_\mu \qquad \text{(D.12)}$$

Re-arranging terms,

$$t_{\mu\nu} = -g_{\mu\nu}\bar{\Psi}\Big\{e^{\alpha}_{a}\gamma^a\frac{1}{i}\partial_\alpha - \frac{1}{4i}\,\bar{\omega}^{d}\gamma_d\gamma_5 - g_v\,e^{\alpha}_{a}\gamma^a A_\alpha$$

$$+\, m_N - g_s\sigma\Big\}\psi \;+\; \bar{\Psi}\Big(\gamma^a d_{\nu a}\frac{1}{i}\partial_\mu \;+$$

$$\epsilon^{abcd}d_{\mu a}\frac{1}{4i}\,\omega_{b\nu c}\,\gamma_d\gamma_5 \;-\; g_v\,d_{\nu a}\gamma^a A_\mu\Big)\psi$$

$$-\frac{1}{4\pi}\,\partial_\mu\sigma\,\partial_\nu\sigma \;-\; \frac{1}{8\pi}\,g_{\mu\nu}\,\partial_\alpha\sigma\,h^{\alpha\beta}\,\partial_\beta\sigma \;-$$

$$\frac{m_s^2}{8\pi}\,g_{\mu\nu}\,\sigma^2 \;+\; \frac{m_v^2}{4\pi}\,A_\mu A_\nu \;-\; \frac{m_v^2}{8\pi}\,g_{\mu\nu}\,A_\alpha\,h^{\alpha\beta}A_\beta$$

$$-\frac{1}{16\pi}\,g_{\mu\nu}\,F_{\alpha\beta}\,h^{\alpha\lambda}\,h^{\beta\rho}F_{\lambda\rho} \;+\; \frac{1}{4\pi}\,F_{\mu\alpha}\,h^{\alpha\beta}\,F_{\nu\beta} \qquad \text{(D.13)}$$

Because Ψ satisfies Eq. (D.5), the first four terms on the right hand-side of Eq. (D.13) do not contribute to $t_{\mu\nu}$. Also, since the stress tensor is always symmetric in its indices, we can re-write the fifth, sixth and seventh terms as

$$\bar{\Psi} d_{\mu a} \left\{ \frac{1}{i} \gamma^a \partial_\nu + \frac{1}{4i} \epsilon^{abcd} \omega_{b\nu c} \gamma_d \gamma_5 - g_v \gamma^a A_\nu \right\} \Psi$$

which is equal to

$$\bar{\Psi} \gamma^a d_{\mu a} \left\{ \frac{1}{i} \nabla_\nu - g_v A_\nu \right\} \Psi$$

where $$\nabla_\nu \equiv \partial_\mu - \frac{i}{4} \omega_{b\mu c} \sigma^{bc}$$

Therefore, Eq. (D.13) becomes

$$\begin{aligned} t_{\mu\nu} = {} & \bar{\Psi} \gamma^a d_{\mu a} \left(\frac{1}{i} \nabla_\nu - g_v A_\nu \right) \Psi - \frac{1}{4\pi} \partial_\mu \sigma \, \partial_\nu \sigma \\ & - \frac{1}{8\pi} g_{\mu\nu} \partial_\alpha \sigma \, h^{\alpha\beta} \partial_\beta \sigma - \frac{m_v^2}{8\pi} g_{\mu\nu} \sigma^2 \\ & + \frac{m_v^2}{4\pi} A_\mu A_\nu - \frac{m_v^2}{8\pi} g_{\mu\nu} A_\alpha h^{\alpha\beta} A_\beta \\ & - \frac{1}{16\pi} g_{\mu\nu} F_{\alpha\beta} h^{\alpha\lambda} h^{\beta\rho} F_{\lambda\rho} + \frac{1}{4\pi} F_{\mu\alpha} h^{\alpha\beta} F_{\nu\beta} \end{aligned} \tag{D.14}$$

When the scalar and vector fields are absent, this becomes

$$t_{\mu\nu} = \bar{\Psi} \gamma^a d_{\mu a} \frac{1}{i} \nabla_\nu \Psi \tag{D.15}$$

Appendix E

"COVARIANT" CONSERVATION OF FERMION CURRENT

The fermion field equation is given by [see Eq. (5.7)]

$$\left(\Gamma^{\mu} \frac{1}{i} \nabla_{\mu} + m_N \right) \Psi = 0 \qquad \text{(E.1)}$$

where
$$\Gamma^{\mu} \equiv e^{\mu}_{\ a} \, \gamma^{a} \qquad \text{(E.2)}$$

To prove $\nabla_{\mu} \left(\bar{\Psi} \Gamma^{\mu} \Psi \right) = 0$, we first obtain the hermitian conjugate of Eq. (E.1) by proceeding as follows. We re-write Eq. (E.1) as [see Eq. (D.5) of Appendix D]:

$$\left(\Gamma^{\mu} \frac{1}{i} \partial_{\mu} - \frac{1}{4i} \bar{\omega}^{d} \gamma_{d} \gamma_{5} + m_N \right) \Psi = 0 , \qquad \text{(E.3)}$$

the expression for $\bar{\omega}^{d}$ being given by Eq. (B.10) in Appendix B. We choose the Dirac-Pauli representation for γ^{a} , namely:

$$\gamma^{0} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \text{(E.4)}$$

$$\gamma^{k} = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \qquad \text{(E.5)}$$

where σ_k are the usual Pauli spin matrices.
We note that

$$(\gamma^0)^+ = \gamma^0$$

$$(\gamma^k)^+ = -\gamma^k$$

$$(\gamma_5)^+ = -\gamma_5$$

Eq. (E.3) is

$$\left(\gamma^k e^\mu_k \frac{1}{i} \partial_\mu + \gamma^0 e^\mu_0 \frac{1}{i} \partial_\mu - \frac{1}{4i} \bar{\omega}^d \gamma_d \gamma_5 + m_N \right) \psi = 0 \qquad \text{(E.6)}$$

Taking hermitian conjugate of Eq. (E.6), we get

$$-\frac{1}{i} \left\{ - \frac{\partial \psi^+}{\partial x^j} e^j_k \gamma^k - \frac{\partial \psi^+}{\partial x^0} e^0_k \gamma^k + \frac{\partial \psi^+}{\partial x^\ell} e^\ell_0 \gamma^0 \right.$$

$$\left. + \frac{\partial \psi^+}{\partial x^0} e^0_0 \gamma^0 \right\} + \frac{1}{4i} \psi^+ \gamma_5 (\gamma_d)^+ \bar{\omega}^d + m_N \psi^+ = 0$$

Multiplying this equation by γ^0 from the right, and noting that

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab}, \qquad \eta^{ab} = \mathrm{diag.}(-1, 1, 1, 1)$$

we get

$$-\frac{1}{i} \left(\frac{\partial \bar{\Psi}}{\partial x^\mu} e^\mu_k \gamma^k + \frac{\partial \bar{\Psi}}{\partial x^\mu} e^\mu_0 \gamma^0 \right) +$$

$$\frac{1}{4i} \psi^+ \gamma_5 (\gamma_d)^+ \bar{\omega}^d \gamma^0 + m_N \bar{\Psi} = 0$$

Or,

$$-\frac{1}{i}(\partial_\mu \bar{\Psi})\Gamma^\mu + \frac{1}{4i}\psi^+ \gamma_5 \gamma_d^+ \bar{\omega}^d \gamma^0 + m_N \bar{\Psi} = 0 \qquad \text{(E.7)}$$

Noting that

$$\gamma_5 (\gamma_d)^+ \bar{\omega}^d \gamma^0 = \gamma^0 \bar{\omega}^d \gamma_d \gamma_5 ,$$

Eq. (E.7) becomes

$$-\frac{1}{i}(\partial_\mu \bar{\Psi})\Gamma^\mu + \frac{1}{4i}\bar{\Psi}\bar{\omega}^d \gamma_d \gamma_5 + m_N \bar{\Psi} = 0$$

which is

$$-\frac{1}{i}(\nabla_\mu \bar{\Psi})\Gamma^\mu + m_N \bar{\Psi} = 0 \qquad \text{(E.8)}$$

This is the hermitian conjugate of Eq. (E.1). Now, we multiply Eq. (E.1) by $\bar{\Psi}$ from the left and Eq. (E.8) by Ψ from the right. This gives

$$\bar{\Psi}\left(\Gamma^\mu \frac{1}{i}\nabla_\mu + m_N\right)\Psi = 0 \qquad \text{(E.9)}$$

and $$-\frac{1}{i}(\nabla_\mu \bar{\Psi})\Gamma^\mu \Psi + m_N \bar{\Psi}\Psi = 0 \qquad \text{(E.10)}$$

Subtracting (E.10) from (E.9), we get

$$\bar{\Psi}\Gamma^\mu \frac{1}{i}\nabla_\mu \Psi + \frac{1}{i}(\nabla_\mu \bar{\Psi})\Gamma^\mu \Psi = 0$$

or,

$$\frac{1}{i}\nabla_\mu(\bar{\Psi}\Gamma^\mu\psi) - \bar{\Psi}\frac{1}{i}(\nabla_\mu\Gamma^\mu)\psi = 0 \qquad \text{(E.11)}$$

Now we note that

$$e^{\mu a} e^{\nu b} \eta_{ab} = h^{\mu\nu}$$

This gives

$$\nabla_\lambda h^{\mu\nu} = 2(\nabla_\lambda e^{\mu a}) e^\nu_a$$

Also, we know from general relativity that the covariant derivative of the metric vanishes. Therefore, since $e^\nu_a \neq 0$, we have

$$\nabla_\lambda e^{\mu a} = 0$$

Hence,

$$\nabla_\mu \Gamma^\mu = 0$$

Eq. (E.11) then gives

$$\nabla_\mu(\bar{\Psi}\Gamma^\mu\psi) = 0$$

Note that if, instead of (E.1), we have Eq. (6.2), namely:

$$\left\{ e^\mu_a \gamma^a \frac{1}{i}(\nabla_\mu - ig_v A_\mu) + m_N - g_s\sigma \right\}\psi = 0$$

we once again get

$$\nabla_\mu \left(\bar{\psi}\, \Gamma^\mu \psi \right) = 0$$

Since A_μ and σ are hermitian.

Appendix F

CONTINUITY EQUATION FOR THE FERMION CURRENT

From Eq. (3.21), we note

$$\Gamma^{\alpha}_{\mu\alpha} = \tfrac{1}{2} h^{\alpha\beta} \left(\partial_{\alpha} g_{\beta\mu} + \partial_{\mu} g_{\beta\alpha} - \partial_{\beta} g_{\mu\alpha} \right) \tag{F.1}$$

Changing the positions of the indices β and α in the third and first terms in parentheses, we see that these two terms cancel each other, so that

$$\Gamma^{\alpha}_{\mu\alpha} = \tfrac{1}{2} h^{\alpha\beta} \partial_{\mu} g_{\alpha\beta} \tag{F.2}$$

We also note

$$\begin{aligned} dg &= g\, h^{\alpha\beta}\, dg_{\alpha\beta} \\ &= -g\, g_{\alpha\beta}\, dh^{\alpha\beta} \end{aligned} \tag{F.3}$$

$\therefore$ From (F.2) and (F.3), we get

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2g} \partial_{\mu} g = \partial_{\mu} \left(\ln \sqrt{-g} \right) \tag{F.4}$$

Now, the "covariant" divergence acting on a vector field A^{μ} is

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\nu\mu} A^\nu \tag{F.5}$$

Using Eq. (F.4), this becomes

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + A^\nu \partial_\nu (\ln \sqrt{-g})$$

$$= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, A^\mu) \tag{F.6}$$

Therefore, if one has

$$\nabla_\mu A^\mu = 0 \tag{F.7}$$

it follows that

$$\partial_\mu (\sqrt{-g}\, A^\mu) = 0 \tag{F.8}$$

Since $\nabla_\mu (\overline{\Psi}\, e^\mu_\alpha \gamma^\alpha \psi) = 0$, the above reasoning gives

$$\partial_\mu (\sqrt{-g}\, \overline{\Psi}\, e^\mu_\alpha \gamma^\alpha \psi) = 0 \tag{F.9}$$

Appendix G

THE CONSERVED TOTAL ENERGY-MOMENTUM TENSOR

Given a Lagrange function $\Lambda\left(q^{(\ell)}, \partial q^{(\ell)}/\partial x^{\mu} \right)$, where $q^{(\ell)}$ ($\ell = 1,2,3,\ldots$) denote the field variables, the energy-momentum tensor is given by[80]

$$\theta^{\mu}_{\nu} = \sum_{\ell} q^{(\ell)}{}_{,}{}^{\mu} \frac{\partial \Lambda}{\partial q^{(\ell)}_{,\nu}} - \delta^{\mu}_{\nu} \Lambda \tag{G.1}$$

and it is a conserved quantity:[80]

$$\partial_{\mu} \theta^{\mu}_{\nu} = 0 \tag{G.2}$$

Evaluation of this conserved quantity (G.1) for a system in which a spin 2 field is in interaction with external fields can be done in formal analogy to gravitation (which is simply a massless spin 2 field). In dealing with a spin 2 field, Eq. (G.1) will get modified into

$$\theta^{\mu}_{\nu} = \sum_{\ell} q^{(\ell)}{}_{,}{}^{\mu} \frac{\partial(\sqrt{-g}\Lambda)}{\partial q^{(\ell)}_{,\nu}} - \delta^{\mu}_{\nu} \sqrt{-g}\, \Lambda \tag{G.3}$$

In applying this formula to external fields which serve as the source for the spin 2 field, the quantities $q^{(\ell)}$ are different from $g_{\mu\nu}$; then we can take $\sqrt{-g}$ out from under the sign of differentiation, and the right-hand side of (G.3) becomes equal to $\sqrt{-g}\; t^{\mu}_{\nu}$, where t^{μ}_{ν} is the energy-momentum tensor of the external fields. This t^{μ}_{ν} and the definition given in the text, namely Eq. (6.6), are identical (see reference 78). When applying Eq. (G.3) to the spin 2 field, we must set $\sqrt{-g}\,\Lambda = \mathcal{L}$, where $\mathcal{L}$ is given by Eq. (4.19), and take the quantities $q^{(\ell)}$ as the components of $g_{\mu\nu}$; this gives us $\bar{T}^{\mu}_{\nu}$ [see Eq. (4.11)]. In our notation, $t^{\mu}_{\nu} = h^{\mu\lambda}\, t_{\lambda\nu}$. Hence the conserved total energy-momentum tensor is given by

$$\theta^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + \sqrt{-g}\; h^{\mu\lambda}\, t_{\lambda\nu} \tag{G.4}$$

Appendix H

LOW AND HIGH DENSITY LIMITS OF THE EQUATION OF STATE OF NEUTRONS INTERACTING THROUGH A SPIN 2 FIELD

First we obtain the analytical solution for the components of the spin 2 field in the limiting cases of low and high densities. For neutrons interacting through a spin 2 field, the equations that determine the spin 2 field are [see Eqs. (5.57) and (5.58) in Chapter 5]:

$$x^3 - x^3 y^6 - ay\, I_2 + 2a m_N^2\, xy^2\, I_1 = 0 \qquad (H.1)$$

$$3x^3 - 3xy^2 - ay\, I_3 - 6a m_N^2\, xy^2\, I_1 = 0 \qquad (H.2)$$

where $$a = 2f^2/\pi m_f^2 m_N^2 \qquad (H.3)$$

$$I_1 = \frac{y}{2\pi x} \int_0^{k_F} d\vec{k} / \left(\vec{k}^{x^2} + m_N^{*2}\right)^{1/2} \qquad (H.4)$$

$$I_2 = \frac{2x}{\pi y} \int_0^{k_F} d\vec{k} \left(\vec{k}^{x^2} + m_N^{x2}\right)^{1/2} \qquad (H.5)$$

$$I_3 = \frac{2x}{\pi y} \int_0^{k_F} d\vec{k}\; \vec{k}^{x^2} / \left(\vec{k}^{x^2} + m_N^{*2}\right)^{1/2} \qquad (H.6)$$

$$\vec{k}^{x^2} \equiv \frac{y^2}{x^2} \vec{k}^2 \qquad (H.7)$$

$$m_N^{*2} = \frac{y^3}{x} m_N^2 \qquad \text{(H.8)}$$

(i) Low density limit ($k_F << m_N$): In this limit I_1, I_2, and I_3 take the following form:

$$I_1 = \frac{2}{3m_N} x^{-1/2} y^{-1/2} k_F^3 \qquad \text{(H.9)}$$

$$I_2 = \frac{8m_N}{3} x^{1/2} y^{1/2} k_F^3 \qquad \text{(H.10)}$$

$$I_3 = \frac{8m_N}{5} x^{-1/2} y^{-1/2} k_F^5 \qquad \text{(H.11)}$$

Eqs. (H.1) and (H.2) then become

$$x^3 - x^3 y^6 - \frac{4}{3} a m_N x^{1/2} y^{3/2} k_F^3 = 0 \qquad \text{(H.12)}$$

$$3x^3 y^6 - 3xy^2 - \frac{8a}{5m_N} x^{-1/2} y^{1/2} k_F^5 - 4am_N x^{1/2} y^{3/2} k_F^3 = 0 \qquad \text{(H.13)}$$

Now, x and y are related to the spin 2 field components χ and λ by

$$x = (1+\chi)^{1/2} \qquad \text{(H.14)}$$

$$y = (1+\lambda)^{1/2} \qquad \text{(H.15)}$$

Using Eqs. (H.14) and (H.15) and remembering that $\chi, \lambda \ll 1$ in the low density limit, Eqs. (H.12) and (H.13) become

$$-3\lambda - \frac{4a}{3} m_N k_F^3 = 0 \tag{H.16}$$

$$3\chi + 6\lambda - \frac{8a}{5 m_N} k_F^3 - 4a m_N k_F^3 = 0 \tag{H.17}$$

These yield

$$\lambda = -\frac{8 f^2 k_F^3}{9\pi m_N m_f^2} \tag{H.18}$$

$$\begin{aligned} \chi &= \left\{ \frac{40 f^2}{9\pi m_N m_f^2} + \frac{16 f^2}{5\pi m_N^2 m_f^2} \right\} k_F^3 \\ &\simeq \frac{40 f^2}{9\pi m_N m_f^2} k_F^3 \\ &= -5\lambda \end{aligned} \tag{H.19}$$

(ii) High density limit ($k_F \gg m_N$): In this limit, we have

$$I_1 \simeq k_F^2 \tag{H.20}$$

$$I_2 \simeq 2\, k_F^4 \tag{H.21}$$

$$I_3 \simeq 2\, k_F^4 \tag{H.22}$$

Eqs. (H.1) and (H.2) then become

$$x^3 - x^3 y^6 - 2ay\, k_F^4 + 2a m_N^2 x y^2 k_F^2 = 0 \tag{H.23}$$

$$3x^3 y^6 - 3xy - 2ay\, k_F^4 - 6a m_N^2 x y^2 k_F^2 = 0 \tag{H.24}$$

Examination of these equations shows that the solutions will be of the following form:

$$y \equiv y_0 = \text{constant, independent of } k_F \tag{H.25}$$

$$x \propto k_F^{4/3}$$

Let $\quad x \equiv A\, k_F^{4/3}$ (H.26)

where A is the constant of proportionality.

Keeping the leading terms, Eqs. (H.23) and (H.24) become respectively (since $k_F \neq 0$):

$$A^3 (1 - y_0^6) = 2ay_0$$

$$3 A^3 y_0^6 = 2ay_0$$

These give $\quad y_0 = 2^{-1/3}$ (H.27) x is then evaluated using Eq. (H.27) in Eq. (H.23). We get

$$x = \frac{2^{11/9}}{3^{1/3}} \left(\frac{f^2 m_N^2}{\pi m_f^2} \right)^{1/3} \left(\frac{k_F}{m_N} \right)^{4/3} \tag{H.28}$$

Therefore, we get

$$\lambda \equiv \lambda_0 = y_0^2 - 1 = 2^{-2/3} - 1 \tag{H.29}$$

$$\begin{aligned} \chi &= x^2 - 1 \\ &\simeq x^2 \\ &= \frac{2^{22/9}}{3^{2/3}} \left(\frac{f^2 m_N^2}{\pi m_f^2} \right)^{2/3} \left(\frac{k_F}{m_N} \right)^{8/3} \end{aligned} \tag{H.30}$$

We now get the limiting expressions (corresponding to low and high densities) for the equation of state. The chemical potential (μ), energy density (ϵ) and pressure (P) are given by [see Eqs. (5.39), (5.70) and (5.71) in Chapter 5]:

$$\mu = \left(k_F^{*2} + m_N^{*2} \right)^{1/2} = \frac{y}{x} \left(k_F^2 + xy m_N^2 \right)^{1/2} \tag{H.31}$$

$$\epsilon = \frac{m_f^2 m_N^2}{32 \pi f^2} \left(x^2 + 3y^2 - x^2 y^6 - 3 \right) + \frac{2}{(2\pi)^3} \int_0^{k_F} d\vec{k} \left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2} \tag{H.32}$$

$$\mathcal{P} = -\frac{m_f^2 m_N^2}{32\pi f^2}\left(x^2+3y^2-x^2y^6-3\right) + \frac{2}{3(2\pi)^3}\int_0^{k_F}\frac{d\vec{k}\;\vec{k}^{*2}}{\left(\vec{k}^{*2}+m_N^{*2}\right)^{1/2}} \qquad \text{(H.33)}$$

(i) Low density limit: In this limit, the solution for χ and λ are given by (H.18) and (H.19):

$$\lambda = -\alpha k_F^3 \qquad \text{(H.34)}$$

$$\chi = 5\alpha k_F^3 \qquad \text{(H.35)}$$

Where $$\alpha \equiv \frac{8f^2}{9\pi m_f^2 m_N} \qquad \text{(H.36)}$$

We note that

$$y/x \simeq 1 - 3\alpha k_F^3$$

and $$x\,y \simeq 1 + 2\alpha k_F^3$$

Then, Eq. (H.31) gives

$$\mu = \left(1 - 3\alpha\, k_F^3\right)\left\{ k_F^2 + \left(1 + 2\alpha k_F^3\right) m_N^2 \right\}^{1/2}$$

$$\simeq m_N \left\{ 1 + \frac{k_F^2}{2m_N^2} - 2\alpha\, k_F^3 - \frac{3\alpha k_F^5}{2m_N^2} - 3\alpha^2\, k_F^6 \right\} \tag{H.37}$$

To evaluate ϵ and P , we note that in the low density limit we have

$$x^2 + 3y^2 - x^2 y^6 - 3 \simeq 0 \tag{H.38}$$

$$\int_0^{k_F} d\vec{k} \left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2} \simeq 4\pi\, m_N^4 \left\{ \frac{k_F^3}{3m_N^3} + \frac{k_F^5}{10\, m_N^5} - \frac{2\alpha k_F^6}{3m_N^3} - \frac{3\alpha\, k_F^8}{10\, m_N^8} - \frac{\alpha^2\, k_F^9}{m_N^3} \right\} \tag{H.39}$$

$$\int_0^{k_F} \frac{d\vec{k}\; \vec{k}^{*2}}{\left(\vec{k}^{*2} + m_N^{*2}\right)^{1/2}} \simeq 4\pi\, m_N^4 \left\{ \frac{k_F^5}{5\, m_N^5} - \frac{k_F^7}{14 m_N^7} - \frac{4\alpha\, k_F^8}{5\, m_N^5} + \frac{3\alpha\, k_F^{10}}{14\, m_N^7} + \frac{3\alpha^2\, k_F^{11}}{5\, m_N^5} \right\} \tag{H.40}$$

Therefore, Eqs. (H.32) and (H.33) give

$$\epsilon = \frac{m_N}{\pi^2}\left\{\frac{k_F^3}{3} + \frac{k_F^5}{10 m_N^2} - \frac{2\alpha k_F^6}{3} - \frac{3\alpha k_F^8}{10\, m_N^2} - \alpha^2 k_F^9\right\} \qquad \text{(H.41)}$$

$$P = \frac{m_N^4}{3\pi^2}\left\{\frac{k_F^5}{5 m_N^5} - \frac{k_F^7}{14 m_N^7} - \frac{4\alpha k_F^8}{5 m_N^5} + \frac{3\alpha k_F^{10}}{14 m_N^7} + \frac{3\alpha^2 k_F^{11}}{5\, m_N^5}\right\} \qquad \text{(H.42)}$$

The energy per particle is

$$\frac{\epsilon}{n} = m_N\left\{1 + \frac{3\, k_F^2}{10\, m_N^2} - 2\alpha k_F^3 - \frac{9\alpha k_F^5}{10\, m_N^2} - 3\alpha^2 k_F^6\right\} \qquad \text{(H.43)}$$

(ii) High density limit: In this limit, we have

$$\lambda = \lambda_0 = 2^{-2/3} - 1 \qquad \text{(H.44)}$$

$$\chi = \beta\, k_F^{8/3} \qquad \text{(H.45)}$$

where $$\beta \equiv \frac{2^{22/9}}{3^{2/3}}\left(\frac{f^2}{\pi\, m_f^2 m_N^2}\right)^{2/3} \qquad \text{(H.46)}$$

We note that, in this limit,

$$y/x \simeq (1+\lambda_0)^{1/2}\ \beta^{-1/2}\ k_F^{-4/3}$$

$$xy \simeq (1+\lambda_0)^{1/2}\ \beta^{1/2}\ k_F^{4/3}$$

Eq. (H.31) then gives

$$\mu = \left(1+\lambda_0\right)^{1/2} \beta^{-1/2}\, k_F^{-4/3} \left\{ k_F^2 + (1+\lambda_0)^{1/2} \beta^{1/2}\, k_F^{4/3} \right\}^{1/2}$$

$$\simeq \frac{3^{1/3}\ \pi^{1/3}\ m_f^{2/3}\ m_N^{2/3}}{2^{14/9}\ f^{2/3}}\ k_F^{-1/3} \qquad \text{(H.47)}$$

To evaluate ϵ and P, we note that in the high density limit we have

$$\left(x^2+3y^2-x^2y^6-3\right) = \beta\, k_F^{8/3} + 3y_0^2 - \beta\, y_0^6\, k_F^{8/3} - 3$$

$$\simeq \beta\left(1-y_0^6\right) k_F^{8/3}$$

$$= \frac{3}{4}\ \beta\ k_F^{8/3} \qquad \text{(H.48)}$$

$$\int_0^{k_F} d\vec{k} \left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2} \simeq \frac{\pi \, (1+\lambda_0)^{1/2}}{\beta^{1/2}} \, k_F^{8/3} \tag{H.49}$$

$$\int_0^{k_F} \frac{d\vec{k} \;\; \vec{k}^{*2}}{\left(\vec{k}^{*2} + m_N^{*2} \right)^{1/2}} \simeq \frac{\pi \, (1+\lambda_0)^{1/2}}{\beta^{1/2}} \, k_F^{8/3} \tag{H.50}$$

Then, Eq. (H.32) and (H.33) give

$$\epsilon = \frac{3^{4/3} \, m_f^{2/3} \, m_N^{2/3}}{2^{41/9} \, \pi^{5/3} \, f^{2/3}} \, k_F^{8/3} \tag{H.51}$$

$$P = - \frac{m_f^{2/3} \, m_N^{2/3}}{3^{2/3} \, 2^{41/9} \, \pi^{5/3} \, f^{2/3}} \, k_F^{8/3} \tag{H.52}$$

Finally, the energy per particle is

$$\frac{\epsilon}{n} = \frac{3^{7/2} \, \pi^{1/3} \, m_f^{2/3} \, m_N^{2/3}}{2^{41/9} \, f^{2/3}} \, k_F^{-1/3} \tag{H.53}$$

We note that in the high density limit we get the following relationships between μ , ϵ and $\mathcal{P}$:

$$\mu = \frac{8}{9}(\epsilon/n) \tag{H.54}$$

$$\mathcal{P} = -\frac{1}{9}\,\epsilon \tag{H.55}$$

Appendix I

SELF-CONSISTENCY OF THE HARTREE APPROXIMATION SCHEME

Consider the total energy of a system of neutrons interacting through spin 2 mesons. In the Hartree scheme, this is given by (V=volume)

$$E = V\epsilon \tag{I.1}$$

where ϵ is the total energy density, given by Eq. (5.70). We can re-write (I.1) as

$$E = \sum_k n_k \epsilon_k + V W \tag{I.2}$$

where

$$\epsilon_k = \left\{ \frac{1+\lambda}{1+\chi} \vec{k}^2 + \frac{(1+\lambda)^{3/2}}{(1+\chi)^{1/2}} m_N^2 \right\}^{1/2} \tag{I.3}$$

$$W = -z \, F(\chi, \lambda) \tag{I.4}$$

$$z \equiv \frac{3 m_f^2 m_N^2}{32 \pi f^2} \tag{I.5}$$

$$F(\chi, \lambda) = \chi \lambda + \chi \lambda^2 + \frac{1}{3} \chi \lambda^3 + \lambda^2 + \frac{1}{3} \lambda^3 \tag{I.6}$$

If Eq. (I.2) is thermodynamically correct expression for energy, appropriate for the ground state of the neutrons, then we should have

$$\frac{\partial E}{\partial \chi} = 0 \tag{I.7}$$

$$\frac{\partial E}{\partial \lambda} = 0 \tag{I.8}$$

Field equations: In the Hartree scheme, χ and λ are determined by the 0 - 0 and 1 - 1 components by Eq. (5.12) with $R_{\mu\nu} = 0$:

$$\frac{1}{2} m_f^2 \left(\sqrt{-g}\, g_{\mu\nu} - \eta_{\mu\nu}\right) = + 8\pi f^2 \left(t_{\mu\nu} - \frac{1}{2} g_{\mu\nu} t \right) \tag{I.9}$$

We can re-write this as

$$-\frac{1}{2} m_f^2 \sqrt{-g}\, g_{\mu\nu} - \frac{m_f^2}{2} \eta_{\mu\nu} + \frac{m_f^2}{4} h^{\alpha\beta} g_{\alpha\beta}\, g_{\mu\nu} = + \frac{8\pi f^2}{m_N^2} t_{\mu\nu} \tag{I.10}$$

Now, we note, from Eqs. (5.28), (5.46) and (5.50), the following relations:

$$t_{00} = \frac{n}{1+\chi} \langle \epsilon_k \rangle \tag{I.11}$$

$$t_{11} = \frac{n}{1+\lambda} \left\langle \frac{k^2}{3\epsilon_k} \right\rangle \tag{I.12}$$

Using Eqs. (I.10) and (I.11), and remembering that in the Hartree approximation $g_{\mu\nu}$ has the form (5.22), Eq. (I.10) becomes

0 - 0 component:

$$\frac{3\chi}{2} + \frac{3\lambda}{2} + 3\chi\lambda + 3\lambda^2 + \lambda^3 + 3\chi\lambda^2 + \chi\lambda^3 = \frac{16\pi f^2}{m_f^2 m_N^2} n \langle \epsilon_k \rangle \quad (I.13)$$

1 - 1 component:

$$\frac{\chi}{2} + \frac{5\lambda}{2} + 3\chi\lambda + 3\lambda^2 + \lambda^3 + 3\chi\lambda^2 + \chi\lambda^3 = -\frac{16\pi f^2}{m_f^2 m_N^2} n \left\langle \frac{k^{*2}}{3\epsilon_k} \right\rangle \quad (I.14)$$

Let us now go back to Eqs. (I.7) and (I.8). From Eq. (I.2) we note

$$\frac{\partial \epsilon_k}{\partial \chi} = -\frac{1}{2\epsilon_k}\left\{\frac{1+\lambda}{(1+\chi)^2} k^2 + \frac{1}{2}\left(\frac{1+\lambda}{1+\chi}\right)^{3/2} m_N^2\right\} \quad (I.15)$$

$$\frac{\partial \epsilon_k}{\partial \lambda} = \frac{1}{2\epsilon_k}\left\{\frac{1}{1+\chi} k^2 + \frac{3}{2}\left(\frac{1+\lambda}{1+\chi}\right)^{1/2} m_N^2\right\} \quad (I.16)$$

From Eq. (I.2), we also have

$$\left(\frac{1+\lambda}{1+\chi}\right)^{\frac{1}{2}} m_N^2 = \frac{\epsilon_k^2}{1+\lambda} - \frac{k^2}{1+\chi} \quad (I.17)$$

Using (I.17), (I.15) and (I.16) become

$$\frac{\partial \epsilon_k}{\partial \chi} = -\frac{1}{4(1+\chi)}(\epsilon_k + q_k) \quad (I.18)$$

$$\frac{\partial \epsilon_k}{\partial \lambda} = \frac{3}{4(1+\lambda)}\left(\epsilon_k - \frac{1}{3} q_k\right) \quad (I.19)$$

where

$$q_k \equiv \frac{1+\lambda}{1+\chi} \, \frac{k^2}{\epsilon_k} \tag{I.20}$$

Also, from (I.6),

$$\frac{\partial F}{\partial \chi} = \lambda + \lambda^2 + \frac{1}{3}\lambda^3 \tag{I.21}$$

$$\frac{\partial F}{\partial \lambda} = \chi + 2\lambda + 2\chi\lambda + \lambda^2 + \chi\lambda^2 \tag{I.22}$$

Then, Eqs. (I.7) and (I.8) become

$$\sum_k n_k \frac{3}{4(1+\chi)} \left(\epsilon_k - \frac{1}{3} q_k \right) - zV\left(\chi + 2\lambda + 2\chi\lambda + \lambda^2 + \chi\lambda^2 \right) = 0 \tag{I.23}$$

$$-\sum_k n_k \frac{1}{4(1+\chi)} \left(\epsilon_k + q_k \right) - zV\left(\lambda + \lambda^2 + \frac{1}{3}\lambda^3 \right) = 0 \tag{I.24}$$

Solving the above equations for $\sum_k n_k \epsilon_k$ and $\sum_k n_k q_k$, we get

$$\frac{1}{zV} \sum_k n_k \epsilon_k = \chi + \lambda + 2F(\chi, \lambda) \tag{I.25}$$

$$-\frac{1}{zV} \sum_k n_k q_k = \chi + 5\lambda + 6F(\chi, \lambda) \tag{I.26}$$

These equations can be re-written as

$$\frac{3\chi}{2} + \frac{3\lambda}{2} + 3F(\chi,\lambda) = \frac{16\pi f^2}{m_f^2 m_N^2}\; n \,\langle \epsilon_k \rangle \qquad (I.27)$$

$$\frac{\chi}{2} + \frac{5\lambda}{2} + 3F(\chi,\lambda) = -\frac{16\pi f^2}{m_N^2 m_f^2}\,\frac{n(1+\lambda)}{(1+\chi)}\left\langle \frac{k^2}{3\epsilon_k}\right\rangle \qquad (I.28)$$

We see that Eqs. (I.27) and (I.28), derived from the minimization of the total energy, are identical to the field equations (I.13) and (I.14), derived as the Euler-Lagrange equations from the Lagrangian. Hence, our approximation scheme is self-consistent, and should, therefore, comply with the first law of thermodynamics. To give a specific demonstration of this, we consider the case of high density limit ($k_F >> m_N$). In this case we have (see Appendix H):

$$\epsilon \sim k_F^{8/3} \sim n^{8/9} \qquad (I.29)$$

Then, the thermodynamic consistency criterion:

$$P = -\epsilon + n\,\frac{d\epsilon}{dn} \qquad (I.30)$$

gives the pressure to be

$$P = -\frac{1}{9}\,\epsilon \qquad (I.31)$$

We have shown in Appendix H [see Eq. (H.55)] that (I.31) is indeed true in the high density limit. Turning now to the chemical potential (μ), the thermodynamic consistency criterion gives [see Eq. (5.74) of Chapter 5]:

$$\mu = \frac{1}{n} (P + \epsilon) \tag{I.32}$$

With ϵ and P given by (I.29) and (I.31) in the high density limit, it follows that

$$\mu = \frac{8}{9} (\epsilon / n) \tag{I.33}$$

In Appendix H, we have shown that this is indeed valid in the high density limit.

Appendix J

FORTRAN CODE FOR THE MASS, RADIUS AND MOMENT OF INERTIA OF A NEUTRON STAR

```
C       MAIN
C       CØMPUTING MASS,RADIUS, MOMENT OF INERTIA OF A NEUTRON STAR
        IMPLICIT REAL*8(A-H,M,Ø-Z)
C       XN=MASS DENSITY,FN=PRESSURE,EN=NUMBER DENSITY
C       N IS THE NUMBER ØF XN,FN,EN PØINTS
        DIMENSIØN XN(100),FN(100),INDEX(100),EN(100),S1(100),
       1INDEX(100)
        N=100
        READ(2,50) (XN(J),J=1,N)
        READ(2,50) (FN(J),J=1,N)
        READ(2,50) (EN(J),J=1,N)
        CALL SPCØEF(N,XN,FN,S,INDEX)
        CALL SPCØEF(N,XN,EN,S1,INDEX1)
C       CENTRAL DENSITY AND INITIAL INCREMENT
     10 READ(2,70) RØ
        IF (RØ.EQ.0.D0) GØ TØ 99
        WRITE(3,80) RØ
        P=SPLINE(N,XN,FN,S,INDEX,RØ)
        DERY=DERIV(N,XN,FN,S,INDEX,RØ)
        DELR=1.D2
C       M IS THE GRAVITATIØNAL MASS
        M=12.56637062D0*DELR*DELR*DELR*RØ
        DEN=SPLINE(N,XN,EN,S1,INDEX1,RØ)
```

```
      FACTOR=DELR-1.484986855D-14*M
      DENØM=DSQRT(FACTØR)
C     MB IS THE BARYØNIC MASS
      MB=20.84760886D0*(DELR**3.5)*DEN/DENØM
C     SPECIFY ARBITRARY ØMEGA AT CENTER
      ØMEG=1.82342D0
      DFDR=7.424934275D-15*(M+12.56637062D0*(DELR**3)*P)/
     1(DELR*FACTØR)
      U=4.D0*(DELR**4)*ØMEG*(DFDR+9.933044759D-14*(DELR**2)*
     1RØ/FACTØR-7.42493427D-15*M/(DELR*FACTOR)
      R=DELR
      NLØØP=100
      DØ 30 I=1,150
      DØ 20 L=1,NLØØP
      A=RØ+P
      B=M/R+12.56637002D0*R*R*P
      C=R-1.484986855D-14*M
      DRØ=-7.424934275D-15*A*B/(C*DERY)
      RØ=RØ+DRØ*DELR
      IF (RØ.LE.7.86D-14) GØ TØ 40
      M=M+12.56637062D0*R*R*RØ*DELR
      DEN=SPLINE(N,XN,EN,S1,INDEX1,RØ)
      FACTOR=R-1.484986855D-14*M
      DENØM=DSQRT(FACTØR)
      MB=MB+20.84760886D0*(R**2.5)*DEN*DELR/DENØM
      P=SPLINE(N,XN,FN,S,INDEX,RØ)
      DERY=DERIV(N,XN,FN,S,INDEX,RØ)
```

```
      DFDR=7.424934275D-15*(M+12.56637062D0*(R**3)*P)/
     1(R*FACTØR)
      ØMEG=ØMEG+DELR*U/(R**4)
      U=U+DELR*(U+4.D0*(R**3)*ØMEG*(DFDR+9.33044759D-14*
     1(R**2)*RØ/FACTOR-7.424934275D-15*M/(R*FACTOR))
      R=R+DELR
   20 CØNTINUE
   30 CØNTINUE
   40 R=R-DELR
C        MASSES IN SØLAR UNITS
      M2=M/1.987D0
      M2=MB/1.987D0
C        AJ IS ANGULAR MOMENTUM
      AJ=2.244688781D13*U
      ØM=ØMEG+U/(3.D0*(R**3))
C        AI IS MØMENT ØF INERTIA
      AI=AJ/ØM
      WRITE(3,60) R,M1,M2,ØM,AJ,AI
      GØ TØ 10
   50 FØRMAT(6D12.6)
   60 FØRMAT(1X,'R=',D14.6,3X,'M1=',D14.6,3X,'M2=',D14.6,
     13X,'OM=',D14.6,3X,'J=',D14.6,3X,'I=',D14.6//)
   70 FØRMAT(D12.6)
   80 FØRMAT(1X,'DENSITY=',D12.6/)
   99 STØP
      END

      SUBRØUTINE SPCØEF(N,XN,FN,S,INDEX)
```

```
      IMPLICIT REAL*8(A-H,Ø-Z)
      DIMENSION XN(N),FN(N),S(N),INDEX(N),RHØ(150),TAU(150)
      NM1=N-1
      DØ 1 I=1,N
    1 INDEX(I)=I
      DØ 3 I=1,NM1
      IP1=I+1
      DØ 2 J=IP1,N
      II=INDEX(I)
      IJ=INDEX(J)
      IF (XN(II).LE.XN(IJ)) GØ TØ 2
      ITEMP=INDEX(I)
      INDEX(I)=INDEX(J)
      INDEX(J)=ITEMP
    2 CØNTINUE
    3 CØNTINUE
      NM2=N-2
      RHØ(2)=0.D0
      TAU(2)=0.D0
C
      DØ 4 I=2,NM1
      IIM1=INDEX(I-1)
      II=INDEX(I)
      IIP1=INDEX(I+1)
      HIM1=XN(II)-XN(IIM1)
      HI=XN(IIP1)-XN(II)
      TEMP=(HIM1/HI)*(RHØ(I)+2.D0)+2.D0
```

```
      RHØ=-1.D0/TEMP
      D=6.D0*((FN(IIP1)-FN(II))/HI-(FN(II)-FN(IIM1))/HIM1)/HI
    4 TAU(I+1)=(D-HIM1*TAU(I)/HI)/TEMP
      S(N)=0.D0
      DØ 5 I-1,NM1
      IB=N-I
      S(IB)=RHØ(IB=1)*S(IB+1)+TAU(IB+1)
    5 CØNTINUE
      RETURN
      END

      FUNCTIØN DERIV(N,XN,FN,S,INDEX,X)
      IMPLICIT REAL*8(A-H,Ø-Z)
      DIMENSIØN XN(N),FN(N),S(N),INDEX(N)
      I1=INDEX(1)
      IF (X.GE.XN(I1)) GØ TØ 1
      I2=INDEX(2)
      H1=SN(I2)-XN(I1)
      DERIV=((FN(I2)-FN(I1))/H1-H1*S(2)/6.D0)
      RETURN
    1 IN=INDEX(N)
      IF (X.LE.XN(IN)) GØ TØ 2
      INM1=INDEX(N-1)
      HNM1=XN(IN)-XN(INM1)
      DERIV=((FN(IN)-FN(INM1))/HNM1+HNM1*S(N-1)/6.D0)
      RETURN
    2 DØ 3 I=2,N
```

```
      II=INDEX(I)
      IF (X.LE.XN(II)) GØ TØ 4
    3 CØNTINUE
    4 L=I-1
      IL=INDEX(L)
      ILP1=INDEX(L+1)
      A=XN(ILP1)-X
      B=X-XN(IL)
      HL=XN(ILP1)-XN(IL)
      DERIV=-S(L)*(3.D0*A*A/HL-HL)/6.D0+S(L+1)*(3.D0*B*B/
     1HL-HL)/6.D0+(FN(ILP1)-FN(IL))/HL
      RETURN
      END

      FUNCTION SPLINE(N,XN,FN,S,INDEX,X)
      IMPLICIT REAL*8(A-H,Ø-Z)
      DIMENSIØN XN(N),FN(N),S(N),INDEX(N)
      I1=INDEX(1)
      IF (X.GE.XN(I1)) GØ TØ 1
      I2=INDEX(2)
      H1=XN(I2)-XN(I1)
      SPLINE=FN(I1)+(X-XN(I1))*((FN(I2)-FN(I1))/H1-H1*S(2)/
     16.D0)
      RETURN
    1 IN=INDEX(N)
      IF (X.LE.XN(IN)) GØ TØ 2
      INM1=INDEX(N-1)
```

```
      HNM1=XN(IN)-XN(INM1)
      SPLINE=FN(IN)+(X-XN(IN))*((FN(IN)-FN(INM1))/HNM1+HNM1*
     1S(N-1)/6.D0)
      RETURN
    2 DØ 3 I=2,N
      II=INDEX(I)
      IF (X.LE.XN(II)) GØ TØ 4
    3 CØNTINUE
    4 L=I-1
      IL=INDEX(L)
      ILP1=INDEX(L+1)
      A=XN(ILP1)-X
      B=X-XN(IL)
      HL=XN(ILP1)-XN(IL)
      SPLINE=A*S(L)*(A*A/HL-HL)/6.D0+B*S(L+1)*(B*B/HL-HL)/
     16.D0+(A*FN(IL)+B*FN(ILP1))/HL
      RETURN
      END
```

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