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A

ERNST MACH'S WORLD ELEMENTS: A STUDY IN NATURAL PHILOSOPHY

by

Erik Banks

A dissertation submitted to the Graduate Faculty in Philosophy in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2000

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This manuscript has been read and accepted for the Graduate Faculty In Philosophy in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

ERNST MACH'S WORLD ELEMENTS: A STUDY IN NATURAL PHILOSOPHY

by

Erik Banks

Advisor: Professor Arnold Koslow

This dissertation studies Mach's world-elements and his reduction of space and time to unextended intensities. The elements included not just human sensations, but mind-independent physical qualities in matter. Influenced by J.F. Herbart, Bernhard Riemann and Hermann von Helmholtz, Mach strove to develop a construction of space from these qualities. The study follows these ideas from Mach's intellectual struggles of the 1860s to his late writings, and relies upon extensive extracts from his scientific notebooks and other original documents.

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Thanks to my mother Laurene Buckley who unflaggingly financed my education, and to Frank Riessman who encouraged me to write for his magazine and helped nurture my writing skills. Above all, I dedicate this to Michelle Wilson, my dearest friend, whose example of a true philosopher guided me to the most consequential results of the following work.

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PREFACE

This dissertation studies Ernst Mach's world-elements and his reduction of space and time to unextended intensities.

Mach's elements included not only human sensations of color, sound and touch but also sensations in matter. Already in his Privatdozent work "Vorträge über Psychophysik" (1863)[1] Mach wrote that one could imagine the sensations of animals and then blur the boundary between very simple animals and plants, finally extending the analogy to inanimate material. In lectures of 1870-71 he said simply: *Die Materie empfindet*, "matter senses."[2] And he repeated his position later in letters to Friedrich Adler[3] and Gabriele Rabel.[4]

Mach thought that inquiries into the nature of the elements should be handled by natural science and not philosophy, since philosophical analogies tended to veer toward idealism or materialism. He pursued a neutral third way, treating elements as neither mental nor physical:

> Careful physical research will lead to an analysis of our sensations. We shall then discover that our hunger is not so essentially different from the tendency of sulphuric acid for zinc, and our will not so greatly different from the pressure of a stone, as it now appears. We shall again feel ourselves nearer nature, without its being necessary that we should resolve ourselves into a nebulous and mystical mass of molecules,

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1 (V), p. 364.

2 "Über einige Hauptfragen der Physik" Vortrag Sommer 1872 (NL 174/1/003).

3 (AU), Mach an Adler January 23, 1910 (Letter #49, pp 71-72).

4 G. Rabel, "Ernst Mach und die Realität der Außenwelt," *Physikalische Zeitschrift* **21** (1920): 433-434.

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or make nature a haunt of hobgoblins.[5]

Through advances in sense physiology, Mach thought the elements could become directly accessible to us, and spoke of a "Tunnel" to be built through psychophysics to other minds and then the natural world.[6] Thus, although he often denied having a philosophic Master Plan, he was driven by a monistic, well-nigh romantic, yearning for a reconciliation of human experience with nature. This first theme, which I will explore historico-critically, can be summed up: from sensations to world elements.

As a second theme, I will examine the nature of the elements themselves. Mach wrote many times, for example, that elements were intensive, instantaneous states not extended in space and time. His stock example of such a state was a pressure, pushing or pulling against others.[7] He wrote that if we consider the case of a weight on a table, where the forces are balanced, we may demonstrate pressures even when they produce no motion in space.[8] Philosophers have long debated whether states such as color or pressure must take up space in order to exist, or whether space itself is composed of simpler intensities. The problem of whether figure precedes color or vice versa appears already in Plato's

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[5] (M) p. 559, Cf. (PSL) p. 213.

[6] Rabel, p. 434.

[7] (R) p. 231, translated in (CE) pp. 88-89.

[8] (M) p. 319.

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dialogue *Meno*. It falls to those who defend the second position to explain how the extended world before us can be constructed.

Scientists of Mach's time regarded his ideas with extreme suspicion. Indeed the period c.1863-1900 is often called the flowering of the Mechanical World-View, when the scientific imagination had been seized by the programme of reducing all phenomena to quantitative matter and motion of particles. Sensations were considered metaphysically dispensable "secondary qualities," secondary to the properties described by mathematics. Entirely in this spirit Galileo once said:

> If ears, tongues and noses were removed, I am of the opinion that shape, quantity and motion would remain, but there would be an end of smells, tastes and sounds, which abstracted from the living creature I take to be mere words.[9]

The natural scientist of the seventeenth century was thus set free to ignore the individual qualities of objects and concentrate on abstract relationships. The most influential representative of the mechanical view in the nineteenth century was Hermann von Helmholtz, who used the law of the conservation of energy to argue that nature was composed solely of particles in empty space with forces emanating from their centers and dependent on spatial distance. All phenomena were thus reducible to mechanical phenomena and could be exactly described in dimensions of matter, space and

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[9] Quoted in Morris Kline, *Mathematical Thought From Ancient to Modern Times* Vol. 1 (Oxford: Oxford University Press, 1972) p. 329.

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time. Mach battled his whole career against this "metaphysical" view of nature, and his critique influenced many in the succeeding generation: Einstein, for one, acknowledged that while a student Mach's *Mechanics* had shaken his "dogmatic faith" in the certainty of mechanical conceptions as the basis for the rest of physics.[10]

Mach's criticisms of seventeenth century science were the advance guard of his deeper view of nature. There are abundant hints of it in his scientific notebooks, and Mach scholars like Erwin Hiebert have long urged that the arch *Antimetaphysiker* had a well-developed metaphysics of his own.[11] Very early on in this project I read an English translation of Mach's 1866 article on spatial representation reprinted in the *History and Root of the Principle of the Conservation of Energy* in which he spoke about such intensive "pressures" underneath extended matter.[12] This first encounter was followed by steady confirmation elsewhere.

Mach was by no means the first to consider the construction of extension from intensity, especially not in

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[10] "Ernst Mach war es, der in seiner Geschichte der Mechanik an diesem dogmatischen Glauben [d.h. dem Glauben an die Mechanik als gesicherte Basis der Physik] rüttelte; dies Buch hat gerade in dieser Beziehung einen tiefen Einfluß auf mich als Student ausgeübt." Einstein "Autobiographical Notes" in Schlipp ed. *Albert Einstein Philosopher-Scientist* (Evanston, IL: Open Court, 1949), p. 7. "It was Mach, in his history of Mechanics, who shook this dogmatic faith [i.e. the faith in mechanics as the sure basis of physics]; in this very connection, this book exercised a deep influence on me as a student."

[11] See Hiebert's Introduction to (KE).

[12] (CE) pp. 88-89.

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German-speaking Europe. Beginning with Leibniz, Wolff and Kant, the philosophers and more importantly the psychologists of Germany devoted more than a hundred years of unremitting effort to the problem. Mach found some of the analytical tools he needed in J.F. Herbart's work on space, Lotze and Wundt's theory of spatial vision, and later the geometry of Riemann and Helmholtz.

Mach's first attempts of the 1860s took the form of a "new theory of matter," inspired by Herbart's "intelligible space." In later years, inspired by Riemann, he worked up a realistic multiply-extended chemical manifold for physics. All told he spent fifty years on this classic problem. The goal of this dissertation is a reconstruction of the argument from intensity to extension, a subject of more than antiquarian interest.

A grant from the Deutscher Akademischer Austauschdienst (DAAD) for the summer of 1999 allowed me to study Mach's *Nachlaß*, at the Deutsches Museum in Munich, including his letters, lecture notes and notebooks. I am most grateful for DAAD's support and for the help of Dr. Wilhelm Füßl at the Archive. I was also able to study the secondary literature in Berlin, which greatly enriched this project.

Given Mach's enormous breadth of interests, this study of two ideas is but a slice from his banquet. The philosopher will be surprised to discover that Mach's ideas were often presented in scientific papers dealing with the specific concerns of physics and psychology, hence I have stuck with

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Mach's nineteenth century views and have not tried anachronistically to improve them. However, most of the problems of this dissertation are indeed of a lasting philosophical nature, and are independent of Mach's views on particular questions.

I have set as the goal of this project to follow in Mach's footsteps and use the historico-critical method, but not slavishly. History, as understood by Mach, is a dynamic process not only of interpretation but also of sharing the spirit of investigation, and an implied commitment to the historical figure to carry out ideas to their logical conclusions. Mach himself approached Newton, Riemann, Mayer and Müller in this spirit, and I think no less should be granted him.

Mach's personal qualities, above all his probity, impressed me during the course of this work. I found in Mach a Faustian character, who is not satisfied with knowing nature from the outside, but who wants to feel what nature feels, and remake science accordingly.

---Berlin and New York, 1999

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ABBREVIATIONS

(NL 174) *Ernst Mach Nachlaß*, Deutsches Museum, München

(C) *Compendium der Physik für Mediciner*. Wien: Wilhelm Braumüller, 1863.

(V) "Vorträge über Psychophysik." *Österreichische Zeitschrift für praktische Heilkunde* **9**(1863).

(R) "Bemerkungen über die Entwicklung der Raumvorstellungen." *Fichtes Zeitschrift für Philosophie und philosophische Kritik* **49** (1866): 227-232.

(SW) *Sitzungsberichte der kaiserlichen Akademie, Mathematisch-Naturwissenschaftliche Classe*, Wien.

(CR) *Carls Reperatorium für physikalische Technik*, München.

(CE) *The History and Root of the Principle of the Conservation of Energy*. Translated by P.E.B. Jourdain, Chicago: Open Court, 1911. First Edition: 1872.

(M) *The Science of Mechanics*. Sixth English Edition. Translated by Thomas McCormack, Chicago: Open Court, 1960. First Edition: 1883.

(AS) *The Analysis of Sensations*. Translated from the First German edition by C.M. Williams and from the Fifth German edition by Sidney Waterlow. New York: Dover, 1959. First Edition: 1886.

(H) *Principles of the Theory of Heat*. Translated by Thomas McCormack, P.E.B. Jourdain and A.E. Heath. Dordrecht: D.Reidel, 1986. First Edition: 1886.

(PSL) *Popular Scientific Lectures*. (1898) Third Edition, Translated by Thomas McCormack. Chicago: Open Court, 1898.

(PWV) *Populär- wissenschaftliche Vorlesungen*. Fifth Edition. Wien: Böhlau Verlag, 1987.

(SP) *Space and Geometry*. Translated by Thomas McCormack. La Salle, IL: Open Court, 1906.

(KE) *Knowledge and Error*. Translated by Thomas McCormack and Paul Foulkes, Dordrecht D. Reidel, 1976. First Edition: 1905.

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(AU) Blackmore, John and Hentschel, Klaus, eds., *Ernst Mach als Außenseiter*. Wien: Wilhelm Braumüller, 1985.

(DL) Blackmore, John, ed., *Ernst Mach: A Deeper Look: Documents and New Perspectives*. Dordrecht: Kluwer Academic Publishers, 1992.

(KU) Buzello, Herbert. *Kritische Untersuchung von Ernst Mach's Erkenntnistheorie, Kantstudien Ergänzungsheft No. 23*. Berlin: Verlag von Reuther & Reichard, 1911.

(MB) Ratliff, Floyd. *Mach Bands*. San Francisco: Holden Day, 1965.

(WK) Thiele, Joachim, ed., *Wissenschaftliche Kommunikation: Die Korrespondenz Ernst Machs*. Kastellaun: A. Henn, 1978.

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INTRODUCTION

Ernst Mach (1838-1916) the Viennese physicist refused the title of "philosopher" and insisted he was only a natural investigator with his own ideas,[1] a scientific rambler who drifted from one subject to another contributing a bit here and there. This modest self-portrait does not at all convey his importance. True, Mach often delved into curiosities, such as his study of supersonic bullets, which gave us the name "Mach 1," and the Mach Bands in sense-physiology, but he spent much of his time thinking about the deepest issues of science and philosophy: the mind-body problem, the nature of space and time, the principle of excluded perpetual motion and the second law of thermodynamics. Mach did not recognize a distinction between scientific and philosophical problems and moved easily between them. Through his books, *Die Mechanik in ihrer Entwicklung* (1883) and the *Analyse der Empfindungen* (1886) he inspired a world-wide audience beyond his Prague lecture room: physicists as different as Einstein, Pauli and Schrödinger, "logical positivist" philosophers of the Vienna Circle (Moritz Schlick, Rudolf Carnap, and Otto Neurath) and a host of other intellectuals including Robert Musil, the author of the novel *Der Mann ohne Eigenschaften*.

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[1] (AS) Introduction and pp. 47, 368; Mach to Wilhelm Schuppe February 7, 1905 in *Erkenntnis* **6** (1936), p. 79; (KE) Preface to the First Edition.

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Mach's legacy is not without its dark clouds. It is widely known that he opposed the existence of atoms, long after many scientists were convinced of them. In his 1913 Foreword to the *Optik*,[2] he also distanced himself from Einstein's theory of relativity, which he had helped inspire. Max Planck, after having been influenced by Mach in his youth, carried on a bitter polemic against him, in which both sides claimed to be misunderstood by the other.[3] Many historians consider these three episodes as three- strikes against Mach, dismissing him as an over-strict positivist who only believed in the reality of human sensations, and who opposed rather than fostered progress in science. Among philosophers Mach is so universally identified with the "logical positivism" of the Vienna Circle that whenever I would mention to a colleague what I was working on, the response came back like a recorded message: "Oh, Ernst Mach *and* the Wiener Kreis."

Mach's works engendered a vigorous critical reaction in his own time, a first-wave of secondary books and dissertations shortly after 1900. Like Richard Hönigswald's *Kritik der Machschen Philosophie* (1903) and Herbert Buzello's *Kritische Untersuchungen von Ernst Machs Erkenntnistheorie* (1911) some early trenchant criticism came from Neo-Kantian philosophers anxious to defend Kant from Mach. The most vitriolic attack however *Materialism and Empirio-Criticism*

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[2] (AU) pp. 169-170.

[3] (DL), Chapter 6.

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(1905) came from none other than Vladimir I. Lenin, the future Bolshevik leader, who feared Mach would seduce Marxist materialists into becoming ineffectual thumb-twiddling idealists.

Authors friendly to him such as Friedrich Adler and Josef Petzoldt published books and articles with the goal of elucidating or improving his theories. Also in this class are Hugo Dingler's *Die Grundgedanken der Mach'schen Philosophie* (1924) which included the first extracts from Mach's Notebooks to appear in print; Hans Henning's *Ernst Mach als Philosoph, Physiker und Psycholog* (1915) an exploration of Mach's unity of science thesis; and Kurt Gerhards' *Machs Erkenntnistheorie und der Realismus* (1914) an attempt at a realist construal of Mach's elements.

There have been two more recent waves of publication in the last thirty years. The second coincided roughly with the fiftieth anniversary of Mach's death in 1966, in which three commemorative volumes appeared,[4] the first from the Ernst Mach Institut in Freiburg, the second from the Boston Studies in the Philosophy of Science and the third in a special symposium in *Synthése*. During this time Floyd Ratliff translated Mach's articles of the 1860s on Mach Bands and neural inhibition

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[4] Frank Kerkhof, ed., *Symposium aus Anlass des 50. Todestages von Ernst Mach* (Freiburg im Breisgau: Ernst Mach Institut, 1967-68); Robert S. Cohen and Raymond J. Seeger, eds. *Ernst Mach: Physicist and Philosopher* (Dordrecht: Reidel, 1970); Jaako Hintikka, ed. *An Ernst Mach Symposium, Synthése* **18**(1968): 132-301.

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raising them to a central concern of sense physiology.

Meanwhile, much was accomplished on the historical front. The first complete bibliography of Mach's works was published by Joachim Thiele in 1963,[5] followed a bit later by a selection of letters from the Nachlaß under the title *Wissenschaftliche Kommunikation* (1978). Friedrich Herneck published Einstein's letters to Mach and revived the Mach-Einstein debate,[6] which centers on Einstein's coinage of "Mach's Principle" and its place in explaining inertia in general relativity. The debate has lasted the whole century without resolution. Some of the relevant scientific materials have been gathered in the recent volume *Mach's Principle* (1995).[7]

Two biographies topped the crest of the second wave: K.D. Heller's *Ernst Mach: Wegbereiter der modernen Physik* (1966) and John Blackmore's unflaggingly thorough *Ernst Mach: His Work, Life and Influence* (1973).

Some of the criticism in the second wave[8] came from self-

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5 "Ernst-Mach-Bibliographie" *Centaurus* **8** (1963): 189-237.

6 "Die Beziehung zwischen Einstein und Mach dokumentarisch dargestellt" *Naturwiss. Zeitschrift der Friedrich-Schiller-Universität Jena* **15** (1966), 45 ff.

7 Julian Barbour and Herbert Pfister eds. (Boston: Birkhäuser, 1995).

8 John Blackmore, *Ernst Mach: His Work Life and Influence* (Berkeley: University of California Press, 1973), Chapter 3 and especially Gerold Holton's "Mach, Einstein and the Search for Reality" in Cohen and Seeger eds. *Ernst Mach: Physicist and Philosopher* (Dordrecht: D. Reidel, 1970).

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styled "scientific realist" scholars of the 1960s. As Gereon Wolters perceptively remarked in his review of the literature in 1987, Mach scholarship was instrumentalized for a battle these scholars were waging against the institutionalized positivism of the three previous decades.

> It is to be regretted...that the "internal" arguments [by which Wolters means arguments others put in Mach's mouth], beginning with the second half of the 1960s, were presented exclusively by philosophers and historians of science of the 'anti-positivist' persuasion. These colleagues had an easy time of it, seeking to disavow positivism by hanging the sign 'Positivism' around Mach's neck, and dismissing him as an ignorant simpleton.[9]

Fortunately a third wave occurred around 1988, Mach's 150th birthday, when two new collections appeared in German, Rudolf Haller and Friedrich Stadler's collection *Ernst Mach: Werk und Wirkung* (1988) and Hubert Laitko and Dieter Hoffmann's *Ernst Mach: Studien und Dokumente zu Leben und Werk* (1991) including the release of new correspondence, biographical material and excerpts from the notebooks. Some of this material appears in English in John Blackmore's collection *Ernst Mach: A Deeper Look* (1992). A most interesting collection of letters is contained in Blackmore and Klaus Hentschel's *Ernst Mach als Außenseiter* (1985) including some gems from Mach's later years.

Highly influential in the third wave was Paul Feyerabend's interpretive article "Mach's Theory of Research

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[9] Gereon Wolters, *Mach I, Mach II, Einstein und die Relativitätstheorie* (Berlin: Walter de Gruyter) p. 9, see also pp. 204-205.

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and its Relation to Einstein"[10] in which he argued that Mach held a principle-driven (not empiricist) theory of inquiry and a theory of elements unlike that of logical positivism. In particular, Feyerabend disdained Vienna Circle empiricism as a corruption of Mach's nuanced and critical stance toward science into a dogmatic acceptance of whatever "science says":

> The first and most noticeable change is the transition from a critical philosophy to sense-data dogmatism. Elements are replaced by sensations not just temporarily, and as a matter of hypothesis, but once and for all...Secondly, criticism of science is replaced by logical reconstruction, which in plain English is nothing but a highly sophisticated form of conformism. As a result the idea of logical reconstruction became conformist. The task was now to correctly present rather than change science.[11]

Feyerabend set the tone for writers who tried to set Mach free from his positivist reputation, and from the 1960s vintage scholars. Another in this category is Wolter's book *Mach I, Mach II, Einstein und Die Relativitätstheorie* (1987) where it is argued, somewhat fancifully, that Mach's "anti-relativity" Foreword to his *Optik*, as well as a passage of his *Mechanik*, were forged by his son Ludwig. The second half of Mach's *Optik*, in which he intended to state his objections to relativity, has never been located and Wolters says he doesn't believe it was ever written. The *Optik II* remains the great mystery story of Mach scholarship.

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[10] *Studies in the History and Philosophy of Science* **15** (1) 1984: 1-22.

[11] "Philosophy of Science: A Subject with a Great Past" in *Minnesota Studies in the Philosophy of Science* Vol. V (1970) pp. 179-181.

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The most important result of the third wave was that the stereotype of Mach the arch-positivist could be challenged openly. Wolters for example thought Mach had "little to do" with the Vienna Circle and that Schlick's original group took the name *Verein Ernst Mach* to take advantage of the popularity of a local hero.[12] Rudolf Haller sought to distance Mach from the logical sense-data construction of Rudolf Carnap's *Der logische Aufbau der Welt*.[13] Another writer Keiichi Noé rightfully stressed the similarities between Mach and the movement *against* logical positivism led by historically-minded philosophers of science in the 1960s.[14]

Recent scholarship has been less ideological, seeking to characterize Mach as an independent thinker with his own ideas who still belongs in the historical context of twentieth century positivism. Two recent examples are Friedrich Stadler's *Studien zum Wiener Kreis* (1997) and Rudolf Haller's *Neopositivismus: Eine Historische Einführung in die Philosophie des Wiener Kreises* (1993). No one today seriously denies an affinity in temperament between Mach and philosophers like Carnap who were influenced by him. Slogans associated with positivism such as "the elimination of metaphysics," "pseudo-problem," and "unity of science" all

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[12] Wolters p. 205.

[13] "Poetic Imagination and Economy: Ernst Mach as Theorist of Science" in (DL) pp. 215-228.

[14] (DL), p. 240.

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stem from Mach, although he meant something different by them. Mach was also free of the tendency to recast all ontological problems as errors of language or logic. I must point out that the history of logical positivism itself is currently being rewritten, as significant differences emerge between the members of the *Wiener Kreis*.

The literature on Mach has reached an advanced state and it is now quite profitable to pursue specific themes or periods from the overall corpus. Many writers such as myself have been attracted to Mach's development in the 1860s, when he claimed to have developed most of his ideas.

The starting point of my dissertation is the period between 1863 and 1872, when Mach underwent an intense "intellectual struggle" to arrive at his theory of elements, at least ten years before the publication of his two best known works, the *Mechanik* (1883) and the *Beiträge zur Analyse der Empfindungen* (1886). In the Preface to the Fourth Edition (1902) of the *Analysis of Sensations* he wrote:

> ...about thirty-five years ago I succeeded by overcoming my own prejudices, in firmly establishing my present position and in setting myself free from the greatest intellectual discomfort of my life.

This passage dates the end of the struggle to about 1867, during his Graz and early Prague tenure. I cannot believe that the tumult was quieted in the Graz years from 1864-1867. Elsewhere, for example, Mach said 1871 was the year of his

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"greatest intellectual ferment"[15] when he "attained stability in his views."[16] I believe the composition of *History and Root of the Principle of the Conservation of Energy* in 1871 was the official end of the struggle. Mach told the story of his early development in a long note:

> I have always felt it as a stroke of good fortune that early in life, at about the age of fifteen, I lighted, in my father's library, on a copy of Kant's *Prolegomena to any Future Metaphysics*. The book made at the time a powerful and inextinguishable impression on me, the like of which I never afterwards experienced in any of my philosophical reading. Some two or three years later the superfluity of the thing in itself abruptly dawned upon me. On a bright summer day in the open air, the world with my ego suddenly appeared to me as one coherent mass of sensations, only more strongly coherent in the ego. Although the actual working out of this thought did not occur until a later period, yet the moment was decisive for my whole view. I still had to struggle long and hard before I was able to retain the new conception in my special subject. With the valuable parts of physical theories we necessarily assume a good dose of false metaphysics, which it is very difficult to sift out from what should be preserved, especially when those theories have become very familiar to us. At times, too, the traditional, instinctive views would arise with great power and place impediments in my way. Only by alternate studies in physics and the physiology of the senses, and by historico-physical investigations (since about 1863) and after having endeavored in vain to settle the conflict by a physico-psychological monadology (in my lectures on psychophysics...) have I attained to any considerable stability in my views.[17]

Elsewhere he tells us some of the steps from his idealist stage in the *Vorträge über Psychophysik* to his mature *Elementenlehre*. For example he said that after he rejected

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[15] (AS) p. 352.

[16] Draft Foreword to the *Analysis of Sensations*, Russian edition translated in (DL), p. 116.

[17] (AS) p. 30 n.

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the Kantian thing-in-itself, he also eliminated the unchanging I, the "simple subject" of Herbart.[18] In this he was influenced by the German aphorist Lichtenberg who suggested that instead of saying "I think" one should say *Es denkt*, to indicate that the thought simply *is*, in an unrestricted sense of being, without being confined to a subject. Mach then got rid of material objects by breaking them up into functions of elements:

> The older generation, especially the physicists and chemists, will be alarmed by this proposal not to treat matter as something absolutely constant, but to take as constant instead a fixed law of connexion among elements which in themselves seem extremely unstable. Even younger minds may find this conception difficult; but the view is inevitable, though I myself at one time went through a great struggle in order to arrive at it.[19]

By breaking up both complexes of the thing and the "I" Mach could now deal directly with the elements themselves. But there was still a lingering dualism to be dissolved between the psychological elements, liberated from the I, and the physical elements, liberated from the thing. Mach said that he struggled to a monistic conception of the elements by overcoming his own "dualist and panpsychist" tendencies:

> ...of all the approaches to my standpoint, the one by way of idealism seems to me the easiest and most natural. And connected with this is the fear of pan-psychism, which at the same time seizes my readers. Many are the victims that fall prey to pan-psychism in the desperate struggle between a monistic conception of the universe and instinctive dualistic prejudices. In my early youth

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[18] (DL) p. 26.

[19] (AS) p. 331.

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I had to work through these tendencies myself.[20] Finally he advanced to the "neutral monist" conception, which he presented synoptically in *History and Root* and more fully fourteen years later in the *Analysis of Sensations*. I will dwell on this early period at some length before moving on to Mach's later development.

Among other writers who cover the period of the struggle I mention first Wolfram Swoboda, whose exemplary dissertation *The Thought and Work of the Young Ernst Mach and the Antecedents to his Philosophy*[21] (1973) covered about half of these years. Swoboda was concerned with the Viennese milieu, Mach's work on the Doppler effect and sense physiology, the methodology of the *Compendium der Physik für Mediciner* and the *Vorträge über Psychophysik*. He ended his study before Mach's Graz period began in 1864.

More recently, Mach's relation to Herbart and Fechner in the Graz years has received some attention in German-language scholarship. Andreas Laaß in his article "Vom Sinn des Machschen Philosophierens" saw Mach's psycho-monadology of the "Vorträge" and his later neutral *Elementenlehre* as springing conceptually from his Herbartianism.[22] Michael Heidelberger

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[20] (AS) p. 362.

[21] doc. diss. Pittsburgh, 1973

[22] in Dieter Hoffmann and Hubert Laitko, eds., *Ernst Mach: Studien und Dokumente zu Leben und Werk* (Berlin: Deutscher Verlag der Wissenschaften, 1991), pp. 206-223.

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in his book on Fechner[23] devoted a chapter to the Graz struggle, pointing out the painful division of Mach and his mentor Fechner over Herbartian psychology.

Erwin N. Hiebert[24], Steven Brush, and Laurens Laudan have written more about the development of Mach's physical ideas during the struggle, in particular his growing dissatisfaction with mechanical atomism. Many scholars insist Mach may have accepted atomism, but only as an hypothesis for organizing the appearances as he purportedly told Einstein during their meeting in 1910.[25] Other suggest that even if Mach was wrong about the existence of atoms, he still had good methodological objections against entities he thought were not even mediately verifiable, as Feyerabend claimed.[26] To complicate the matter, there exists a story floated by Stephan Meyer that he convinced Mach of the existence of atoms in the laboratory,[27]

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[23] *Die Innere Seite der Natur: G.T. Fechners wissenschaftliche- philosophische Weltauffassung* (Frankfurt a.M.: Vittorio Klostermann, 1993).

[24] Erwin N. Hiebert "The Genesis of Mach's Early Views on Atomism" in Cohen and Seeger, eds., *Ernst Mach: Physicist and Philosopher* (Dordrecht: D. Reidel, 1970), pp. 79-106; Stephen G. Brush "Mach and Atomism" *Synthése* **18** (1968): 192-215; Laurens Laudan "The Methodological Foundations of Mach's Anti-Atomism and Their Historical Roots" in P.K. Machamer and R.G. Turnbull eds., *Motion and Time Space and Matter* (Ohio State University press, 1976), pp. 390-417.

[25] (DL) pp. 152-153.

[26] "Mach's Theory of Research..." *Studies in the History and Philosophy of Science* **15**(1) 1984: 19-20.

[27] (DL) p. 151.

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although, as John Blackmore rightly points out, this revelation did not effect any sea-change in Mach's ideas.

A final scholarly position is that there was a specifically Machian theory of matter at the sub-atomic level. Thus he merely doubted the *ultimacy* of atoms in comparison with his own deeper theory of world-elements. Feyerabend for example insisted: "Elements as envisaged by Mach are more fundamental than atoms."[28] If so it wouldn't have made any difference whether or not atoms existed, since they would be immediately replaced as soon as a new theory of matter was ripe. I have no doubt that Mach believed he could reduce the spatial atoms to non-spatial interior processes, and that the elimination of physiological sensory properties of spatiality and extended matter from physical science was the real problem that obsessed him under the name of atomism. As he explained much later in 1911:

> Even if the kinetic world-picture, which in any case I consider hypothetical without intending thereby to denigrate it, could "explain" all physical appearances, I would still hold that the manifoldness of the world had not been exhausted, because for me matter, space and time are still *problems* which the physicists (Lorentz, Einstein, Minkowski) are also approaching.[29]

Beyond the episode of Mach's "new theory of matter" much else needs to be added to our understanding of his physical thinking during the struggle. To be considered are the important influences on him coming out of psychology, his

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[28] "Mach's Theory of Research..." p. 11.

[29] (DL) p. 139.

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"neutral monism" as well as his historico-physical studies of Galileo, Newton, Lagrange, Gauss, Carnot, Clausius, Mayer and Helmholtz. The 1860s, and the ensuing years, marked a voyage of discovery for Mach. He found himself through a bewildering labyrinth of scientific problems and positions. His sheer variety of interests, in physics, sense physiology and psychology combined with an overwhelming tough-minded determination to trace each one to the bottom, make him a demanding personage to follow, but supremely rewarding of the effort. Many of Mach's ideas emerged as contributions to debates in the psychology and physics of his time that are no longer well known today. To some degree the historical content of chapters 3-7 is designed to supply this lack.

The 1860s were a magical decade for Mach. In addition to his work in psychology and philosophy, his space time theories, his definition of mass and his redefinition of the law of inertia all stem from that period, but the historical development of these ideas has received little coverage. I can't help thinking that the true secret of this time has yet to be revealed.

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Chapter One: The Viennese Background

Mach's ideas germinated in the Austro-Hungarian empire, whose monarch Franz Joseph ruled from 1848-1916, practically all of the scientist's long life. Mach was born in Chirlitz, Moravia in 1838 into a cultured, freethinking family. His father Johann was an eccentric who had studied science and philosophy in Prague, and was for a time a private tutor to the sons of the Baron Brethon in Vienna. He preferred to try farming and various "get rich quick" schemes, including the surprisingly half-successful idea to raise Asiatic silk-worms in Austria.[1] At any rate Mach owed to his father's instruction and garden-physics demonstrations his first education in science, as he says in the dedication of his *Einleitung in die Helmholtz'sche Musiktheorie* (Graz, 1866).

The Mach family was sympathetic to the calls for reform and self-determination that followed the Revolution of 1848. They even harbored a fugitive professor friend of Mach's father who had supported the Hungarian uprising.[2] The new regime of eighteen year-old Franz Joseph cracked down hard on the Hungarian revolt and its friends, and got off to an authoritarian start; Erwin Hiebert writes of the effect of these times on Mach's life-long distrust of authority for its

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[1] Hans Henning, *Ernst Mach as Philosoph, Physiker und Psycholog* (Leipzig 1915), Chapter 1.

[2] K.D. Heller, *Ernst Mach: Wegbereiter der modernen Physik* (Wien: Springer-Verlag, 1964), p. 6.

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own sake:

> Parental sympathies in the family home were certainly on the side of the Hungarian revolution and critical of the autocracy of the Hapsburg monarchy. Accordingly Mach was brought up in an environment that nurtured skepticism and unrestrained critical and stubborn inquisitiveness about the natural world, politics, religion and the status quo in general.[3]

In a surprising turn, the latter half of the reign of the bewhiskered and medal bedraped Franz Joseph was marked by increasing liberalization, in which the different ethnic groups within the empire (Czech, Slovak, Hungarian, Roumanian and Serbian) were allowed more cultural and linguistic autonomy, and in which dormant nationalisms emerged, with eventually disastrous consequences. Mach himself, as Rector of the Prague university, was plunged into a conflict between the Czech and German-speaking nationalities that divided the university in half.[4]

Franz Joseph's reign was also marked by a cultural enlightenment, tolerance of political criticism (from a spectrum that included Karl Kraus and Austro-Marxists such as Friedrich Adler), and a flourishing of art and science that is hard to overestimate. In addition to Mach, Sigmund Freud,

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[3] Erwin N. Hiebert "The Influence of Mach's Thought on Science," *Philosophia Naturalis* **21** (1984), p. 598.

[4] This unhappy episode is told in Blackmore's *Ernst Mach: His Work, Life and Influence* (Berkeley: University of California Press, 1973), and in Dieter Hoffmann's "Ernst Mach in Prag" pp. 141-178 in Hoffmann and Laitko eds., *Ernst Mach: Studien und Dokumente zu Leben und Werk* (Berlin: Deutscher Verlag der Wissenschaften, 1991), (DL) pp. 29-47.

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Josef Breuer, Ludwig Boltzmann, Franz Brentano, Robert Musil, Franz Kafka, Ludwig Wittgenstein, Gustav Klimt, Adolph Loos and even James Joyce in Trieste were all at some point under the wing of this empire--Katakania of Musil's novel. As Thomas Szasz writes in his Introduction to the *Analysis of Sensations*:

> Parallel to the political and social changes, an atmosphere developed which was very encouraging to the arts and sciences...The years between 1848 and the first world war in Austria-Hungary were characterized by a widespread belief in the stability and security of the government and the State coupled with a belief that forces promoting slow but steady changes toward ever better social conditions were constantly at work. This at least was the social Weltanschauung of the middle, upper and still higher classes of this society. Almost all of Mach's life (he died in 1916) was spent during this epoch which could be called the "Age of Security" (in Central Europe).

Among the post-revolutionary movements that affected Mach the most significant was the so-called Exner-Bonitz educational reform of the 1850s. These two ministers of education were disciples of the philosopher and pedagogue J.F. Herbart (1776-1841). They de-emphasized the rote of the cram-schools, and pitched education at the native critical and reasoning powers of the student. Given Mach's supreme critical powers and his early Herbartianism, this influence is not without consequence.

The most important sciences in Vienna at mid-century were medicine, anatomy and physiology, including the physiology of the sense-organs. Not coincidentally, it was the site of the *Allgemeines Krankenhaus*, the largest centralized hospital in

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Europe, which as Wolfram Swoboda remarks,[5] permitted physicians to observe the progress of diseases and to perform autopsies in large numbers of cases, since all the patients flowed through the same institution. The University of Vienna was geared towards the education of physicians and physiologists, which is evident from the high caliber of talent educated there--the young Freud and Breuer to name two who also worked with Mach's teacher Ernst Brücke. Mach's first job, in fact, was to teach a course of physics for medical students.

Swoboda has thoroughly researched the forces that led to a specific brand of Viennese empiricism and physicalism one generation *before* Mach. The Viennese doctors had rejected the vogue of *Naturphilosophie* from Germany and the various forms of vitalism that accompanied it. Vitalism was the idea that living organisms possess within a "vital force" (*Lebenskraft*) unexplainable by material means. *Naturphilosophie* was a principle- driven amalgam of science and philosophy based to some extent on the theories of Schelling and Hegel, and a bane to medical research since its abstract "polarities" and "oppositions" sanctioned no specific treatments.[6] The Viennese reaction was a race to the opposite pole of extreme empiricism on the model of English philosophy, as Hugo Dingler describes:

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[5] Swoboda pp. 93-94.

[6] Swoboda p. 90.

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> Mach's first beginnings were made during the time when the struggle against the groundless speculations of the Schelling- and Hegelian school still echoed faintly in the distance. In reaction against that groundless over-stretching of a completely unproven principle, a most extreme Empiricist cast of mind, just as unfounded in the last analysis, had established itself, and was supported by the influence of English philosophy.[7]

As Swoboda relates, Mach's two most important teachers Ernst Brücke and Carl Ludwig were students of Johannes Müller, who had founded an institute in Berlin for the study of human physiology. Müller, the author of the famous *Handbuch der menschlichen Physiologie* (1838), was an intriguingly complex personality: a hard-headed physiologist who remained a vitalist and a phenomenologist on the model of Goethe.

Ludwig and Brücke retained some of the complexity of their teacher. Mach relates of Ludwig for example that he believed physiological processes were only the "outside" of sensational processes.[8] But there were students of Müller such as Hermann von Helmholtz and Emil DuBois-Reymond who believed that even a good physiological account should be further reduced and *quantitatively* explained by the mechanical atomic theory. First materialism and then reductionism (in

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[7] *Die Grundgedanken der Machschen Philosophie* (Leipzig: Johann Barth, 1921), pp. 19-20: Machs erste Anfänge rangen noch in die Zeit, da der Kampf gegen die basislose Spekulation der Schelling-Hegelschen Schule sich dumpf nachhallend in der Ferne verlor. In Gegenwirkung gegen jene basis-lose Überspannung eines völlig unbewiesenen Prinzips hatte sich ein in seiner letzten Fundierung ebenso unklarer extremster Empirismus der Geister bemächtigt, der durch Einwirkung der englischen Philosophie unterstützt wurde.

[8] (V) p. 336.

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the formula "Physiology as applied physics") was the creed of these men. The view was widely influential in the medical community, as Michael Heidelberger writes:

> In the later fifties and in the sixties materialism enjoyed its greatest popularity and among doctors the materialistic *Weltanschauung*...was especially current.[9]

Dingler says that Anti-Metaphysics--which is so often associated with Mach himself--was a rallying cry of the generation *before* in their crusade against *Naturphilosophie* and vitalism. According to Dingler, anti-metaphysics was an *idée fixe* for Helmholtz, who often used "metaphysics" in the crudest sense to mean anything unverifiable with the human sense organs. "With Mach," Dingler added, "the concept of metaphysics is no longer so purely superficial."[10]

Helmholtz had no compunctions about the conflict between *atomism* (believing that quality-less atoms are the ultimate constituents of matter)[11] and *phenomenalism* (the position that sensations provide the most immediate data of science). He believed that sensory qualities and their elements were superficial and could be explained by recourse to the purely quantitative behavior of large numbers of atoms, similar to the way that a *pointelliste* painting is a mixture of colored

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[9] *Die Innere Seite der Natur* (Frankfurt a.M.: Vittorio Klostermann, 1993), p. 214.

[10] Dingler *Grundgedanken*, pp. 28-29.

[11] *Über die Erhaltung der Kraft, Ostwalds Klassiker der exakten Wissenschaften* (Leipzig: Wilhelm Engelmann, 1889, original 1847).

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dots.[12]

The conflict disquieted Mach greatly. His textbook discussions of Lebenskraft and teleology in his early *Compendium der Physik für Mediciner*, echoed the standard view:

> ...the phenomena of living organisms are only complicated physical phenomena. The medicine of today is long used to considering physiology as applied physics. A great number of phenomena which would otherwise be attributed to vitalism are already traced back to physical laws and it becomes increasingly probable that this will succeed more and more for all processes in organisms.[13]

But the young Mach realized early--and with more penetration than the preceding generation--the ontological conflict between sensations (or their tinier elements) and sensation-less atoms. He said later in the "Vorträge" that he simply couldn't believe sensory qualities were just "confused" properties emerging from a large collection of atoms that had no quality whatever in its parts. "How," he asked, "could the soul suddenly come into a combination of atoms in the

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[12] Helmholtz's ideas about the ultimacy of atoms may have changed over time. Heidelberger for example quotes Helmholtz as saying in 1871 that "the foundation of theoretical physics should not be derived from the hypothetical assumption of the atomic structure of natural bodies." p. 213. This is just the opposite of what Helmholtz said in 1847. Heidelberger claims Helmholtz's change of heart encouraged Mach to publish his anti-atomist views in 1872.

[13] (C), pp. 1-2. Cf. "Über das Sehen von Lagen und Winkeln durch die Bewegung des Auges" *SW* **41**(1861) 215-222, p. 224: "Sollten in Zukunft auch manche von den hier ausgesprochenen Ansichten modificiert, manche verworfen werden, so scheint mir Eins klar, dass gerade nur diese Art von Untersuchungen, jene Bausteine zu einer exacten Psychologie zu gewinnen sind, welche ganz ausserhalb der Seele, rein im körperlichen Organismus liegt."

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organism, if the kernel did not lie in the individual atoms?"[14]

This conflict was probably why young Mach set himself the problem of finding a common scientific foundation for the three sciences he learned at Vienna: sense psychology (phenomenology) physiology and physics. His inclusion of phenomenological data within natural science indicates that the spirit of Goethe and Müller still stirs in him.[15] Mach's theory of neutral elements is often viewed as his answer to this problem of the unity of science, his *Leitmotiv* all through the years of the struggle. He himself insisted that his ultimate goal was the unity of science, and not the creation of a fixed eternal philosophy, and said this often in his writings, for example:

> I do not claim the title of a philosopher. I want only to take, in physics, a standpoint which does not have to be abandoned immediately when we look over into the field of another science. For all of the sciences ultimately form one whole.[16]

As Swoboda has documented, while Mach was officially a physics student he went out of his way to immerse himself in the new sense-physiology with Brücke, chemistry, anatomy and even six hours per week in the dissecting room during the winter semester of 1859-1860. He failed to push through an

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[14] (V) p. 364.

[15] (AS), p. 1.

[16] (AS) Introduction

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application for his Habilitation as a "physicist-physiologist" in 1860, and had to settle for physics in an unrestricted sense, receiving his *venia legendi* on February 24, 1861.[17]

Two Early Publications

Mach's publications for the degree included a co-authored paper on electrical discharge and induction, the first of a series of papers on the Doppler effect and his first paper on psycho-physics.[18]

The Doppler controversy with Josef Petzval, one of his teachers at the Physical Institute, helped make Mach's early reputation. Swoboda treated this controversy in some detail[19] so I will merely summarize the main points. Everyone is familiar with the Doppler effect from daily experience. When a whistling locomotive goes by, the tone of the whistle changes; as it is speeding towards the observer the whistle is high-pitched, and as it passes and speeds away the pitch suddenly drops. The same phenomenon is responsible for the astronomical red-shift of the spectra of galaxies receding

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17 K.D. Heller, *Ernst Mach: Wegbereiter der modernen Physik*, pp. 11-12.

18 Blaserna-Mach-Peterin "Über elektrische Entladung und Induction" *SW* **37** (1859): 477-524; Mach "Über die Änderung des Tones und der Farbe durch Bewegung" *SW* **41** (1860): 543-560; "Über das Sehen von Lagen und Winkeln durch die Bewegung des Auges" *SW* **43** (1861): 215-224.

19 Swoboda pp. 20-74, Ratliff "On Mach's Contributions to the Analysis of Sensations" in Cohen and Seeger, eds., *Ernst Mach Physicist and Philosopher* (Dordrecht: D. Reidel, 1970).

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away from the observer. Mach was one of the first to propose that the speed of astronomical objects be measured by their Doppler shift.[20]

Petzval had argued against Doppler that, among other things, a wave motion in a uniform medium like the air must conserve its period of oscillation (the time it takes for one complete cycle of the wave to pass). A wave propagates through a medium by the flexing and relaxing of the internal connections holding the material together. Thus moving the medium will not get the wave through it any faster. As Mach added, a sound that reaches us through the air on a windy day does *not* alter in pitch. But Doppler's effect does not depend on the speed of the wave in a moving medium but on the movement of the source and the destination with respect to one another. In the case of a speeding siren for instance, the waves all move off at the same velocity but they are emitted from different points each time. As the first wave is getting off, the source moves and catches up with it a little before emitting the second wave. Hence they tend to "overlap" and superimpose to create a wave that seems to arrive with a higher frequency. The same effect is created by a motion of the observer towards the source because then conversely the waves are encountered with greater apparent frequency. As Mach proved by constructing an apparatus to separate the two

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[20] "Über die Änderung des Tones und der Farbe durch Bewegung" *SW* **41** (1860): 543-560.

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independent factors, a tone is heard as the same in pitch in both still air and moving air, but alters in pitch with any relative motion of the source and the observer. His article announcing the test was the conclusion of the controversy.

Mach's debut paper on psycho-physics "Über das Sehen von Lagen und Winkeln durch die Bewegung des Auges"[21] was also of lasting importance. Here he showed himself to be an ardent disciple of Gustav Fechner whose *Elemente der Psychophysik* had just appeared in 1860. Mach had written Fechner on January 14, 1861:

> For a long time I have busied myself with mathematical psychology; my goal being to seek out methods of experiment and observation for psychology similar to those long well-known in physics. I could find no correct basis for my ideas until, at last, your *Psychophysik* appeared. I read the book with affection, and found my expectations greatly exceeded.[22]

He goes on to say that he has extended these methods in experiments on spatial vision and the time sense, indicating already the drift of Mach's psychological researches toward the experience of space and time.

In his paper Mach argues that we estimate small just noticeable differences in lengths of lines and opening of angles by the strain of our eye muscles. It is already assumed implicitly that sensations of length are due to the muscular strain of scanning the visual field with the eye movements.

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[21] *SW* **43**(1861): 215-224.

[22] (AU), p. 1.

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Fechner's law of psychophysics, which we will discuss in more detail later, had been thoroughly tested for various types of muscular sensations, for example lifting weights, so Mach logically assumed it would also apply to the eye muscles and to sensations of small lengths (changes of position) and arcs estimated by muscular strain.[23]

> It appears that the hypothesis is in harmony with experience that the position (*Lage*) of a line is made known to us through the sensation of the eye muscles produced as the eye moves along this line.[24]

The system of eye muscles is such that their strain is not the same in every direction. Strain along the diagonal is greater, for example as it requires the work of two sets of ocular muscles. It is far easier to look straight along the vertical and horizontal. Thus if a line is situated at an angle with respect to the apparatus of the eye muscles, it is harder to scan it and it produces a sensation of greater

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[23] Fechner's law if literally applied to finite lengths would predict that a geometric series of distances 1,4,9,16,25... would be experienced as an arithmetic series of distance sensations 1,2,3,4,5... Obviously this would not be in accord with experience. But if the eye scans in little just noticeable jerking movements of arcs $d\theta$ and lengths ds, for all practical purposes a length can be considered the sum of small elements, rather than a constant strain. Hence Mach figures only the strain needed to budge the eye a little bit in one direction or another.

[24] "Über das Sehen von Lagen und Winkeln" p. 223: "Es scheint hiernach die Hypothese, dass die Lage einer gesehenen Linie für uns sich kenntlich mache durch die Empfindung der die Bewegung des Auges längs dieser Linie hervorrufenden Spannung der Augenmuskel, mit der Erfahrung in Einklange zu stehen."

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apparent length. Mach showed that indeed depending on the situation of a line, its length shrinks or grows, and angles open and close. He appealed to the familiar illusion of the geometrically congruent square and diamond, as evidence that the sides of the square look shorter than the diagonally situated sides of the diamond.

Within a few years Mach would extend these investigations in spatial vision to the phenomena of physiologically similar figures, symmetry sensations (independent "sensations of form" due to the vertically symmetrical arrangement of the eye muscles) and the physiological (as opposed to geometrical) significance of straight lines and curves. This issue is taken up again in chapter six.

It cannot be overestimated how important such psycho-physiological investigations were for Mach's *physical* thought about space. As it turns out, Mach came to many of his physical ideas about space and time by way of psychology.

Mach as Privatdozent in Vienna

After his degree, from 1861-1864 Mach taught as a *Privatdozent* in Vienna, where he was an active participant in the brilliant café discussion society and had many artists and musicians as friends. John Blackmore quotes a second-hand reminiscence by Ludwig Karpath about one of Mach's favorite haunts:

> A close relative has told me a lot about a Vienna coffeehouse which I can no longer remember, the Café

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> Elefant which was located in a narrow passage between Stefansplatz and the Graben.
>
> Every day scholars, artists and doctors of medicine and law would gather together. The regulars included such later famous people as Professor Mach, Lynkeus (Popper), a group of Wagner-oriented musicians...and many others.
>
> People wandered in about 2 P.M. and stayed until 2 in the morning, that is, some were always leaving while others were arriving. Unbroken wit and argument on philosophical, scientific and artistic matters kept the discussion sharp and stimulating.
>
> To a certain extent the young Dozent Ernst Mach presided over the gathering. His profound understanding and reflective manner impressed everyone. According to my relative he was one of the first to occupy himself with the recently published work of Helmholtz on tone perceptions about which he formed many interesting and instructive conclusions.[25]

His daughter in law, Frau Anna Karma Mach, relates that despite pecuniary difficulties he remembered this time with fondness:

> After his habilitation at the university began a time of great deprivation. His father came into financial troubles, his mother died and this sapped his father's energy. He had to earn his way by giving lessons and lectures. Yet he remembered this time with great joy because he presided over a merry company of writers, musicians, artists and painters. He was himself an excellent piano and organ player. Incidentally I should like to mention that a large painting hangs in the Prague museum, painted by a friend of his youth Prof. Lauffer. It represents the scene at Siegfried's funerary procession and in the middle is Ernst Mach in the guise of Haagens.[26]

Such was his typically Viennese love of music that even while

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[25] John Blackmore, *His Work, Life and Influence*, (Berkeley: University of California Press, 1973), p. 23.

[26] Frau Anna Karma Mach in Frank Kerkhof, ed., *Symposium aus Anlaß des 50. Todestages von Ernst Mach*, (Freiburg: Ernst Mach Institut, 1966) p. 10.

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on the rocks he managed to afford piano lessons.[27]

Mach was also prominent in the circle of students surrounding Brücke and Ludwig. In a letter to the Austrian Academy of Sciences of April 28, 1913 Mach says he was "less a student at the university than an apprentice of several members of the Academy."[28] In fact Mach "inherited" his first scientific problems from his teachers. For example it was Andreas Ritter von Ettingshausen who suggested he construct his Doppler apparatus, whereas through Ludwig's attempt to construct a machine to measure blood pressure Mach became involved in the analysis of fluid oscillating under resistance in a tube,[29] and the mechanics of oscillating systems under resistance that he later applied to the physiological acoustics of the ear.[30]

As a Privatdozent Mach was not paid to lecture at the university but had to earn his bread by giving private lectures: in 1862 and 1863 on psychophysics and in 1863 and 1864 on Helmholtz's theory of music. At the university, beginning in the fall of 1861, he taught his course "Physics for Medical Students" along with "Higher Physiological Physics" and "Methods of Physical Investigation." In the

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[27] Heller, p. 12.

[28] (AU), p. 114.

[29] "Zur Theorie der Pulswellenzeichner" *SW* **46**(1862): 157-174; "Über eine neue Einrichtung des Pulswellenzeichners" *SW* **47**(1863), 53-56.

[30] "Zur Theorie des Gehörorgans" *SW* **48** (1863): 283-300.

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summer of 1862 he taught "Mechanical Principles and Mechanistic Physics in its Historical Development."[31] As the title suggests, Mach had already been led to the historical presentation of material.[32] As Swoboda points out, however, Mach's use of the historical method was not unusual in Central Europe.[33] Even up to the time of Max Planck a working scientist would often have a strong command of the history of his subject.

Mach usually worked out his ideas in the lecture-room several years before he elaborated them in print. There he could commit himself to ideas that he worded more cautiously in published work for his professional colleagues. Mach said that after experiencing disappointment in discussions with other physicists, he concealed his views from his physicist colleagues for years.[34] It is a shame that we do not possess lecture notes from the 1860s, and that instead we must glean his views from his more cautiously worded publications. Fortunately there is enough in his published writings to guess at the physical *Weltanschauung* forming in his mind already in the early 1860s, including a philosophical-physical "new theory of matter" which will be the subject of the coming chapters.

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[31] Blackmore 1973, p. 14.

[32] (DL), p. 24.

[33] Swoboda, p. 136.

[34] (CE), p. 87.

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Chapter Two: A New Theory of Matter

Introduction

Mach's new theory of matter was advanced in three major stages. The first stage consisted predominantly of fault-finding with the spatial representations of atomism and physics generally, in his early papers on spectral lines and in his *Compendium der Physik*. During this stage, which is the subject of the present chapter, Mach perceived what he called "metaphysical" or philosophical difficulties with prevailing views, but he had not yet proposed an alternative. In the second stage Mach put forward a "psychomonadology," covered in chapters three and four. From Fechner, Mach gleaned a central idea of the psychomonadology, that all natural processes had a qualitative inner side akin to sensations. Finally in the third stage, Mach pulled away from Fechner and Herbart, uniting the various strands of his research from the 1860s in his mature *Elementenlehre*, to be discussed in chapters ten and eleven.

I will treat each strand as though it were a separate episode, whereas the reader can verify by noting the dates cited that they comprised one simultaneous development in an extremely compressed time frame, in agreement with Mach's story of his struggle.

Atomism in the Compendium

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Mach used the textbook of his physics course for medical students as an organ to communicate his physical views, especially on mechanical atomism:

> In the year 1862 I drew up a compendium of physics for medical men in which because I strove after a certain philosophical satisfaction, I carried out rigorously the mechanical atomic theory. This work first made me conscious of the insufficiency of this theory and this was clearly expressed in the preface and at the end of the book, where I spoke of a total reformation of our views on the foundations of physics.[1]

In the *Compendium* Mach speaks of his growing dissatisfaction with the state of physics. He approaches atomism cautiously, laying out the bare features he considers indispensable to his *Körperatome*: discreetness, impenetrability and obedience to the law of inertia.[2] They also exert on one another a force of some nature that falls off proportional to the square of the distance. There are also cloud-like *Aetheratome* which are attracted to and surround the *Körperatome* and which repel one another by forces that fall off faster than the attraction. Mach wasn't quite sure whether the atoms were internally different among themselves or whether chemical differences emerged from varying combinations of identical atoms.[3]

Mach gave several arguments as to why discrete particles are a more favorable view of matter than the competing theory of a material plenum. For example he says that light is a

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[1] (CE), p. 86.

[2] (C) pp. 11-12.

[3] (C) p. 15.

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transverse wave which can only exist in a medium that consists of parts capable of vibrating in the spaces between them. Likewise the conduction and radiation of heat through matter in wave motions implies that bodies consist of strata of discrete particulate matter capable of vibration. Such was Mach's skill in rehearsing the conventional arguments and explanations of the theory that Erwin N. Hiebert writes:

> He had mastered the arguments of the atomic theory very well...although he later denounced the atomic theory vehemently and resolutely, he manifestly was not engaging in such polemics because of blindness or ignorance of the strengths and weaknesses of these conceptions.[4]

Mach mimicked the reductionist at first and claimed the physics of the day *should* be reducible to the mechanics of atoms:

> The present system grows more troublesome from day to day. It is no longer possible to portray light phenomena or heat phenomena for themselves and this defect must become more palpable in the future. In a textbook of physics one will treat the different kinds of mechanical processes systematically and consider it merely incidental, whether they are seen as light or felt as heat or heard as sound. In fact today's arrangement is not physical but physiological.[5]

He then considered the possibility that failing a reduction of all physical processes, the hypothesis of atoms should give way to a deeper "metaphysical" theory of matter:

> Here I have let the atomic theory step into the

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[4] Erwin Hiebert, "Mach's Early Views on Atomism" in Cohen and Seeger, eds., *Ernst Mach: Physicist and Philosopher*, (Dordrecht: D. Reidel, 1970) p. 95.

[5] (C), pp. V-VI.

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> foreground everywhere, not indeed in the belief that it is the last and highest, and doesn't itself need further support, rather because it brings the appearances into a simple and visualizeable [*anschaulichen*] association. The atomic theory, when one may express it thus, can be considered as a formula [*Formel*], which has already led to many results and will lead further to still more. In fact whatever kind of metaphysical opinion may arise in the future [*künftig auftreten wird*[6]], the results won according to the atomic theory will be translatable into them, as one may express formulas in polar coordinates or in parallel coordinates.[7]

He inclines slightly to the latter standpoint at the end of the book:

> We have now become acquainted with a series of physical phenomena and laws, which we, so far as it was possible by elementary means, have embraced under the point of view of mechanics and the atomic theory. A complete and strict tracing back of physical laws to a few exact principles is no longer possible today. The gaps are still too great. For heat and light phenomena we have at least found the most general contour of a theory and at least this contour rests on fixed foundations. We have not yet won so much insight into the phenomena of electricity and magnetism. In these areas are more empirical rules, to find one's way about in a large quantity of facts, than a real theory. However in this lies a stimulus to further research, and we hope that this will attain the goal, granted perhaps after a total reformulation of some of our fundamental physical views. At least the peculiarities of electrical and magnetic forces seem to point to the necessity of such a transformation.[8]

In the *Compendium* Mach's main objection to the mechanical view concerned Weber's law of electrical force between two charged particles, which was also puzzling the world of physics at the

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[6] Probably a reference to the title of Kant's *Prolegomena zu jeder künftigen Metaphysik, die als Wissenschaft wird auftreten können.*

[7] Ibid, p. VI.

[8] (C), pp. 271-272.

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time. The law says that, like gravity, the electric force between two particles in motion is proportional to the charges and inversely proportional to the square of the distance. But it is *also* proportional to the velocity and the acceleration of the two charges. The problem, Mach says, is Newton's law of inertia, which requires that motion at a constant velocity should not indicate the action of forces due to the laws of physics:

> This dependency of forces on velocity is still a riddle at present; it never happens when two bodies exert gravitational forces on one another for example.[9]

Mach lived to see the problem solved by the special theory of relativity, but in the 1863 book, he made the customary "saving" assumption of some kind of viscous medium, an ether which resists motion even at a constant velocity. Later in his *Mechanics*, he believed he could eliminate the ether by reformulating electrodynamics in his own vocabulary of potential and potential function, about which more later.[10]

Replacing Atomism

By the time of the *Compendium*, Mach was proposing some tentative steps towards replacing atomism with a deeper theory. The issue here was the bright-line emission spectra of gasses. He said in 1872, reflecting back:

My attempts to explain mechanically the spectra of the

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[9] (C) p. 256.

[10] (M) p. 598.

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> chemical elements and the divergence of the theory with experience strengthened my view that we must not represent to ourselves the chemical elements in a space of three dimensions. I did not, however, speak of this candidly before orthodox physicists. My notices in Schlömilch's Zeitschrift of 1863 and 1864 contained only an indication of it.[11]

In his article "Über die Spektra verschiedener Körper" *Schlömilchs Zeitschrift für Mathematik und Physik* 7 (1862): 214-216, Mach suggested that the emission spectra of gasses could be explained by the distances between vibrating atoms in a gas molecule.[12] When the atoms were shaken, oscillations in these distances produced the bright lines. There was no skepticism about the existence of atoms in this 1862 article, but Mach limited himself in a tell-tale way to the observable facts of the spectral lines and the *distances*, without relying on the atoms as such. It was as if he proposed to consider distances without answering the question "distances between what and what?" Perhaps for this reason, Mach's article struck physicists as "zu naturphilosophisch."[13] Incidentally Dingler writes that due to his philosophical views Mach was at

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[11] (CE), p. 87.

[12] Mach attributed the emission spectra to "molecules" and not "atoms" because he thought that atoms themselves were all the same, and that they entered into combinations in molecules which in turn led to the chemical and spectral differences. The distances between these atoms *in a gas molecule* are the cause of the spectra and not the internal structure of the atoms themselves. We now know that the internal structure is the cause of the different chemical spectra for each type of atom.

[13] Hiebert, "Mach's Early Views on Atomism," p. 99.

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this time regarded by his physicist colleagues as a crank whose views they found difficult to fathom (*ein schwer verständlicher Eigenbrödler*).[14] Mach gave a philosophical gloss on this article in his "Vorträge über Psychophysik" where he said he wished to eliminate the hypothesis of "spatial atoms" in favor of *Wesen und Kräften*--where these *Wesen* were the unextended monads of his psycho-physical monadology announced the same year:

> In...(*Schömilch's Zeitschrift für Mathematik* VII. Jahrg. III. Heft) I have shown how one may deduce the nature and constitution of gas molecules from the results of the latest spectral investigations of light from glowing gasses. From the oscillations that shake a system, one may draw conclusions about its constitution...If we wish to mix in as few hypotheses as possible then we must say that physics leads always and everywhere to the idea that it is a finite number of Wesens and forces that are active in the phenomena.[15]

Perhaps more bold in front of an audience of medical men, he added that "we are not justified in thinking of these Wesen spatially."[16] As he went on to explain, a Wesen was composed of unextended qualities akin to our sensations and only became extended through external causal interaction with other Wesens that together made up extended space and matter:

> How do we have to think of these Atoms? Colored, lighted, sounding, hard?
> These are sensory properties which atoms only have

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[14] Hugo Dingler, *Die Grundgedanken der Machschen Philosophie*, (Leipzig: Johann Barth, 1924), pp. 24-25.

[15] (V), p. 364.

[16] (CE) p. 87: "In my lectures on psychophysics I already stated clearly that we are not justified in thinking of atoms spatially."

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> in community with one another, for all physical phenomena result from a plurality of atoms. We cannot attribute these properties to one atom alone. We cannot even think of atoms as spatially extended. For as we have seen space is nothing original and results very probably from a mediate interaction of a plurality of Wesens.--Physicists, too, have already felt the difficulty of imagining atoms concretely, and for that reason some consider the atoms as mere centers of force. But a center of force for itself is actually nothing. And what does it mean anyway to say one center of force acts on another? Let us simply admit it! We cannot reasonably succeed in giving any kind of outside to atoms; should we think anything at all we must attribute an inner side to it, an inner side in some respects analogous to our own soul.[17]

Mach's critics had indeed smelled out in his earlier paper a *naturphilosophische* construction of matter brewing behind Mach's positivistic proposal to examine only observable phenomena.

Mach followed up with a second article "Vorläufige Bemerkungen über das Licht glühender Gase"[18] in which he said to his critics:

> I must remind you that I by no means consider the atomic theory as something established in itself, but rather as a useful temporary empirical formula; as a kind of *regula falsi* to be used in drawing closer to the truth.[19]

He reprised his argument from the *Compendium* that the results of the atomic theory could be "translated into a different "metaphysical" view of matter which, henceforth would prove to

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[17] (V) p. 364.

[18] *Schlömilchs Zeitschrift für Mathematik und Physik* **9** (1864): 69-70.

[19] Hiebert, "Mach's Early Views on Atomism," p. 100.

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be more cogent."[20] And now he explicitly declared his support for an alternative:

> I personally am now already very much inclined toward such a translation. And I shortly hope to show that my remarks by no means hang together so necessarily with theories, but can be understood very readily as an expression of the facts.[21]

Throughout the early 1860s Mach taunted and tantalized his readers with hints of a theory of matter superior to atomism, but did not publish anything like a construction until his 1866 article "Über die Entwicklung der Raumvorstellungen." How did that alternative take shape in Mach's mind in the early 1860s? From Mach's physics one sees that like most physicists of the time, he saw the phenomena of light and heat radiation from bodies as the key to uncovering the constitution of matter. On September 26, 1860 Mach sent a letter to G.R. Kirchhoff describing a method for isolating the spectrum of a phosphorescing body from the spectrum of its reflected light by using a polarizing filter, which got a gruff answer from the Master, and not the promise of financial support that Mach clearly hoped for.[22] In his *Theory of Heat* Mach has high praise for Kirchhoff's theoretical law of the equality of emissive and absorptive powers for any substance, and the theoretical limit of black-body radiation, comparing

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[20] Ibid.

[21] Ibid.

[22] (WK), p. 32.

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both in their fundamental importance to the principle of the excluded perpetual motion.[23]

Unlike most scientists, however, Mach came to see wavelike transmissions of energy, action, not only as the experimental key to the constitution of matter, but as matter itself. His articles on gaseous spectra are a sign of that. In his notebook of 1874, he said that he considered the passage of matter through space to be a wave, a propagating "difference":

> The movement of matter is a wave. A difference that progresses forward...Light as periodic potential.[24]

Mach wrote in his philosophical article of 1866 "Über die Entwicklung der Raumvorstellungen" that he considered light a "proto-chemical" separation and combination, because light can interact with chemical solutions,[25] his idea here being that not only was light chemical, but chemical processes in matter

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[23] (H) Chapter 9 and especially p. 144. Kirchhoff's law of equal emissive and absorptive powers for any substance and any wave-length of radiation was based on a thought experiment. He considered two bodies at *equal* temperatures facing one another and radiating heat back and forth between them. If one of the bodies were to "keep" some, then heat would have to flow from a colder body (the one that loses it) to a warmer body (the one that gains it), a violation of the postulate of Clausius that heat cannot flow from colder to warmer bodies. The theoretical limit of a perfect emitter and absorber of radiation is called a black body, or cavity.

[24] Das Fortbewegung d. Materie ist eine Welle. Eine Differenz die fortschreitet...Das Licht als periodische Potential. Notebook dated 18 September 1874: (NL 174/02/05).

[25] (R), p. 232.

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were like light. He also pointed out that chemical or qualitative[26] changes in matter need not be chalked up to the motion of bodies lying in space "next to one another" but that the phenomena of motion might really be qualitative "chemical" changes.[27] Had he not been frustrated in his attempt to get the financial backing he needed to become a theoretical physicist,[28] it is likely that he would have pursued these interests in a physical framework. Instead he went in another direction: philosophy.

The Role of Philosophy in Mach's Physics

Mach's philosophical views do help to explain why he thought it necessary to oppose and even destroy a theory that fit the evidence marvellously across all of physics. The answer concerns Mach's conviction that atomism was a philosophically naive spatial theory of natural processes.

Mach says that the study of Herbart's "intelligible space" a construction in the *Allgemeine Metaphysik* of 1829 captured his attention in 1862:

> I was busied at the same time with psychophysics and with Herbart's works and so I became convinced that the

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[26] The term is rather dated. As in Mach's "Raumvorstellungen" paper (R, 232) the term "chemical manifold" means a manifold in which motional changes reduce to qualitative "chemical" changes.

[27] Ibid.

[28] Mach says in an autobiographical fragment that he wanted to study with Franz Neumann at Königsberg but had to give up for lack of funds, (DL), p. 24.

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> intuition of space is bound up with the organization of the senses, and, consequently that we are not justified in ascribing spatial properties to things which are not perceived with the senses...At the same time the quite arbitrary and, on this account, faulty limitation of the number of dimensions in Herbart's derivation of "intelligible" space struck me. By that, now, it became clear to me that, for the understanding, relations like those of space, and of any number of dimensions, are thinkable.[29]

In his *Conservation of Energy* of 1872 and in his 1871 article "Über die physikalische Bedeutung der Gesetze der Symmetrie"[30] Mach parlayed his critique of Herbart into an argument against three-dimensional molecules, writing that the "physiologically based" assumption of a three dimensional space for physics artificially limited the number of possible distances between atoms.[31] Since Mach attributed the spectral lines to oscillations in these distances, he implied that he had found more lines than distances possible between atoms suspended in three dimensional space. That threat seems to hang over his articles addressed to physicists. He did not want to consider alternatives such as the possibility that the lines originated inside the atom, which is actually the case. This indicates that in addition to his formal arguments, Mach had entrenched philosophical reasons for his opposition.

It is the consensus of scholars that Mach got his early "Vorträge" theory of unextended monads of matter from

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[29] (CE), pp. 86-87.

[30] *Lotos* **21**: 139-147.

[31] (CE), pp. 51-54.

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Herbart's *Metaphysik*. In Herbart's construction space is composed of pressures which become extended, as Mach echoes in the Vorträge, only first through a "*Zusammenwirkung einer Mehrheit von Wesen*." Herbart also calls his unextended monads *Wesens* as Mach does his unextended monads in the "Vorträge." The next chapter is devoted to Herbart's construction.

The idea that Mach was drawing his ideas of matter from a metaphysician may seem strange given Mach's supposed antipathy toward metaphysics generally. But even in the *Compendium* there occurs a passage where Mach acknowledges the role of philosophy in natural science. He alerts his readers that philosophical work continuous with science may be necessary to see physical problems from another side:

> Where philosophical questions are touched on, which was not completely to be avoided, I restricted myself to the ground of natural science, in part because this standpoint has a certain authority, in part also in the conviction that one must have tasted of this outlook for a time to feel the necessity that the investigation be pursued from another side; some people never get this far.[32]

In Mach's maturer works he frequently considers philosophical questions in the foundations of science from the "other side": questions such as discreetness versus continuity, locomotion

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[32] (C), p. VI: "Wo philosophische Fragen berührt werden, was nicht ganz zu vermeiden war, hielt ich mich auf dem Boden der Naturwissenschaft. Teils weil dieser Standpunkt eine gewisse Berechtigung hat, teils auch in der Überzeugung, dass man von diesem Standpunkte einige Zeit die Aussicht müsse genossen haben, um die Notwendigkeit zu fühlen, dass die Untersuchung auch von einer anderen Seite her geführt werden müsse; manche Menschen gelangen nie dahin."

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versus qualitative change, symmetry versus asymmetry. In his "Definition of Mass" article of 1868 and the 1872 *Conservation of Energy* Mach referred to the ostensibly "metaphysical" principles of causality and sufficient reason as the logical root for the excluded perpetuum mobile principle; although he indicated that such principles as "every event has a cause" were empty unless experience informs us as to what to put in for "causes." Mach did however allow for constructive speculation at a level of generality above natural science. He said in his late work *Knowledge and Error* that philosophy and natural science differ only in the scope of the experience they handle:

> The ultimate end of all research is the same. This shows itself also in the fact that the greatest philosophers such as Plato, Aristotle, Descartes, Leibniz and others have opened up new ways of specialist inquiry while scientists like Galileo, Newton, Darwin and others have furnished philosophic thought without being called philosophers. Yet it is true that what the philosopher regards as a possible starting point appears to the scientist as the distant goal of his work...Through its attempt to summarize the most general features of large areas, philosophy has gained ample experience in this line, even learning gradually to recognize and avoid some of its own mistakes that the philosophically untrained scientist is almost bound to commit even today. However philosophy has furnished science with some positive notions of value too, for example ideas of conservation.[33]

Mach's Early Philosophical Development

Mach's was led to Herbart during his "Idealist" period,[34] [35]

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[33] (KE), pp. 2-3.

[34] (AS), p. 361-362 n.

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after he had rejected Kant's thing in itself, and arrived at thoughts that he says were similar to Berkeley and Hume, although it turns out he really meant Herbartian psychology.[36] Mach later bitterly denied being a Berkeleyian[37] and said that it was not Hume but Lichtenberg who influenced him to give up the "I think."

The confusion of Mach's views with those of the British Empiricists still clouds the philosophical literature. In spite of the frequent association of Mach with Hume, he was unacquainted with the Scotsman until the end of the 1880s and did not read the *Treatise of Human Nature* until 1907-08.[38] I think the crucial difference between them is that Mach asserted a general principle of causality, a "functional dependence of elements on each other" that was distinctly un-Humean, since Hume had implied that sensations were logically atomic and independent "existences," meaning possibly that a single sensation could occur alone and isolated from the rest

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[35] Draft Foreword to Russian Edition of the *Analysis of Sensations* (DL), p. 116: "The author became acquainted with an ideal similar to that of physics as a student through Herbart's over-schematic attempt to form a mathematical psychology without any physiological foundation and through that he came very close to Hume."

[36] "A Draft Foreword to the Russian Edition of the *Analysis of Sensations*" (1906/07), (DL), p. 116.

[37] (AS), pp. 361-362 n.

[38] "Die Leitgedanken meiner naturwissenschaftlichen Erkenntnislehre und ihre Aufname durch die Zeitgenossen," *Physikalische Zeitschrift* **11** (1910), p. 604.

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of reality.[39] Mach denied this vehemently, given his instinct of a whole of nature which cannot be segmented.

Even in his *Compendium* he stated that we may take as a fact of experience that sensations always occur together in complexes with one another and not alone, although it is a matter of experience what, particularly, will occur with what[40]:

> We attain through our senses a quantity of perceptions, which in inexact experience, in children and the uneducated do not at the beginning conform to any order. Soon however we notice that certain groups of perceptions or appearances are always combined, stand in certain relationships to one another, always occur simultaneously or follow upon one another, in short that uniformities or laws exist in nature.[41]

Hans Henning also remarked that Mach's function concept separated him from Hume:

> Even if differences exist between Mach and Hume, the point of departure is still the same. Yet Mach is not a critic of knowledge in the sense of Hume because he introduces the function concept into the theory of knowledge. This is something fundamentally new, which

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[39] To my knowledge Hume himself does not go this far. But other nominalists have, for example Russell in "Our Knowledge of the External World" held that one sensation could logically occur by itself.

[40] There is nothing inconsistent in holding that a general causal claim holds although no particular causal claim always holds. For example let A be the set $\{a_1,\ldots,a_n\}$ and let B be the set $\{b_1,\ldots,b_m\}$. Then the class A may always be associated with the class B, in that some 'a' is always associated with some 'b' though no particular a is always associated with any particular b.

[41] (C), p. 4.

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burns all bridges to earlier authors.[42]

Like many German speakers Mach started off his intellectual life as a Kantian. Mach said that he owed to Kant the destruction of his "naive realism," by which he probably meant Kant's refusal to extend spatial properties to things in themselves. Dingler is also right that Kant stimulated Mach's life-long interest in questions of space, time, causality and conservation.[43] In his *Conservation of Energy* Mach credits *only* Kant and Herbart as philosophical influences on his thought up to 1871.[44]

Mach's Exposure to Herbart's Ideas

Along with Gustav Fechner, whose ideas will be treated in a later chapter, J.F. Herbart was the dominant philosophical influence on the young Mach after Kant. Swoboda relates that Herbart's ideas probably reached Mach through Franz Lott, a family friend of the Mach's and Professor of philosophy in Vienna from 1849-1872. Lott had studied with Herbart at Göttigen, and Mach in turn studied psychology with Lott in the

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[42] Hans Henning, *Ernst Mach als Philosoph, Physiker und Psycholog*, (Leipzig: 1915), p. 19.

[43] "Das Interesse an den fundamentalen Fragen der exakten Wissenschaften, and den Fragen nach Raum, Zeit, Kausalität, Erhaltung und nach den mit diesen zusammenhängenden Gründsätzen war von Studium Kants her unauslöschlich geblieben." Dingler, *Grundgedanken*, p. 31.

[44] (CE), p. 16.

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summer of 1858.[45] They continued to correspond on matters of Herbartian psychology until as late as 1864. Letters from Gustav Fechner of that year confirm that in Fechner's eyes, Mach remained a convinced Herbartian.[46]

Herbart himself was Fichte's student at Jena. In 1809 he was called to Kant's chair in Königsberg where he stayed twenty-four years before moving on to Göttigen. In 1824-25 he published the two volumes of his *Psychologie als Wissenschaft* and in 1828-29, after many years of labor, his *Allgemeine Metaphysik*. Mach was not the only one to be influenced by Herbart's construction of space. Bernhard Riemann in his celebrated *Probevorlesung* "Über die Hypothese welche der Geometrie zum Grunde liegen" explicitly credits "some philosophical investigations of Herbart"[47] and Hermann Grassmann's *Ausdehnungslehre* was also influenced by this thinker.[48]

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[45] Swoboda p. 171.

[46] Lott to Mach October 11 1864 (NL 174/5/1368); Fechner to Mach April 18, 1864 in (WK): "Über die Herbartsche Psychologie will ich mich nicht mit Ihnen streiten, vermöchte ich auch nicht, da es zu lange her ist, daß ich umsonst versucht habe, mich mit ihr zu befreunden; und unstreiting ist in unserer beiderseitigen Stellung dazu etwas Glaubensache. Ich will ihr also einfach Glück wünschen, daß sie in Ihnen einen neuen so gewiegten Vertreter gefunden hat..."

[47] "On the Hypotheses that Lie at the Foundations of Geometry" in D.E. Smith, ed., *A Source Book in Mathematics*, (New York: McGraw Hill, 1929).

[48] See Max Jammer, *Concepts of Space*, Third Edition (New York, Dover, 1994) p. 177: "Certain ideas in Herbart's philosophy seem to have had a great influence on Riemann

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Herbart considered himself a "scientific philosopher," and thought metaphysics was science at a higher level of generality, but denied that such greater generality gave metaphysics more certainty:

> It is not required that metaphysics be more certain and penetrate deeper than it can in following experience. It rests on this, as on its own proper [*eigentümlichen*] hypotheses. Should human experience be found too limited, too incomplete, not enough in accord with hopes and wishes on some points to ground a completely satisfactory conviction, the fault cannot be unjustly foisted on metaphysics, which does not light with its own light but merely renders up what it has received.[49]

It is exactly the kind of thinking that would have attracted the young Mach, and did.

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and H. Grassmann in their formulation of a manifold with an arbitrary number of dimensions."

[49] J.F. Herbart, *Allgemeine Metaphysik*, (Königsberg 1829) in *Herbarts Werke* **8.** Band (Herausgegeben von K. Kehrbach und O. Flügel), p. 12.

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Chapter Three: Kant, Herbart, Mach and the Metaphysics of Space

Herbart's Metaphysics

The similarities between Mach and Herbart's thought are so abundant it's remarkable that there has never been an extensive comparative study. Moreover despite its seminal importance in mathematics, Herbart's space construction is not a well-known chapter in the history of philosophy. Filling in this chapter will reap copious rewards in understanding one of Mach's most enduring influences.

Like Mach's later theory of elements and functions, Herbart's metaphysics started out as an *Elementenlehre* of both matter and form.[1] The matter of the world was sensation, while the forms of space, time and objects indicated orderings of those elements together. In ordinary perception the form was all mixed up. We cannot tell which forms of objects are due to our own psychological apprehension of experience, and which are due to the action of external things on our senses.[2] Space thus divided into a space due to psychological reproduction, which Herbart gave in the *Psychologie als Wissenschaft*, and a more abstract "intelligible space" for natural processes worked out in the *Metaphysik*. These two

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[1] J.F. Herbart, *Allgemeine Metaphysik* (Königsberg, 1829), Sec. 169.

[2] *Metaphysik*, Sec. 170.

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spaces were not separate realms of existence but merely two different orderings of one and the same unextended, instantaneous content, one psychological, one metaphysical; psychological analysis had to come first to sort out which is which.[3] We see here a prefiguring of Mach's position that sensations are psychological in one set of "functional" connections, and physical in another.[4]

> Complexes and fusions, shaded off in inexhaustible variety, interwoven and roused to activity, give our representations in part made-up, in part empirical forms. The Mechanics of mind, which not only gets into the representations but also the states of the representing process, displays the possible forms and modes of influence [*Wirkungsarten*] of complexes and fusions. It thus teaches the conditions under which spatial shapes, time intervals, and series of changes are represented. The fulfillment of this condition is nature's affair; we thus posses a knowledge of nature, subject indeed to error and improvement, but of which we cannot be robbed, and which emerges victorious from all difficulties. For in the connections [*Verknüpfungen*] of our ideas, in so far as they are formed by experience anyway, are reflected the connections of things to one another and with us; and this connection between that which is in us and that which is outside of us, becomes clear through psychology in such a way that redounds not insignificantly to the confirmation of the true realistic metaphysics.[5]

Herbart also insisted, like Mach, that sensations *always* occur together in functions of some kind:

> Are the forms of experience given? Answer: yes, indeed they are given, although only as determinations of the manner in which sensations are bound up together. Were they not given, we could not only sunder them from sensation in such a way that the sensed could occur

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[3] *Metaphysik*. Sec. 170, p. 21.

[4] (AS), p. 17.

[5] *Metaphysik*, Sec. 170. p. 21

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> without any connection [*Zusammenhang*], completely isolated; rather we could also, at pleasure, see different shapes, hear other time intervals; similarly we could put together things out of properties by choice and change them...[6]

Herbart's realism about form is not unusual among German philosophers or scientists, and he does not imply that particular causal connections can be known a priori. Considerations on the form of experience in fact mark a crucial divide between German and the older British idealism. To the nominalist British Empiricists, reality was a stream of separable sensations, which Hume called logically separate *existences*.[7] The Germans inclined toward the monist intuition that experience should form a whole, or a One, from which the particulars cannot be separated. This whole of nature demands that while any particular sensation may occur or fail to occur in association with another, none may occur isolated from the rest of reality. Some formal "relatedness" of sensations is thus upheld, and such an intuition of a whole of nature is at the bottom of Herbart's forms and Mach's function concept.

F.H. Jacobi's *Über die Lehre des Spinoza in Briefe an den Herrn Moses Mendelssohn* (1795), announced the return of the

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[6] *Metaphysik* Sec. 171 p. 21.

[7] To my knowledge Hume never said that a *single* sensation could occur alone. Carnap and Goodman, who were nominalists, still did not go over the brink and picture the world as a bunch of isolated qualities.

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monist intuition to post-Kantian philosophy in Germany.[8] Jacobi interpreted Spinoza to mean that we have greater immediate certainty of an unconditioned "whole of being" (*Ur-Sein*) than we do of our own determined existence. Mach himself was a convinced monist and even identified his views with Spinoza's.[9] In fact the intuition that nature must form a whole found some of its most beautiful expression in Mach's writings.[10] We will return to the role of Spinoza's One in Mach's thought in Chapter ten.

The question is how to put to work such an abstract intuition as the whole of nature. For Herbart, the fact that sensations always occur together guaranteed that each could be seen as a "function" of the others occurring with it. The order of things existing together moreover is more general than spatial or temporal order. Existence or being doesn't logically entail being in time and space. Rather spatial and temporal forms of combination could be a subclass of more

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[8] Kant himself held the intuition that there could conceivably be many separate worlds. "Gedanken von der wahren Schatzung der lebendigen Krafte," *Kants gesammelte Schriften*, (Berlin: Königliche Akademie der Wissenschaften, 1910), p. 22: "Es ist daher nicht richtig geredet, wenn man in der Hörsälen der Weltweisheit immer lehrt, es könnte im metaphysichen Verstande nicht mehr wie eine einzige Welt existieren. Es ist wirklich möglich, daß Gott viele Millionen Welten, auch in recht metaphysicher Bedeutung genommen, erschaffen habe; daher bleibt es unentschieden, ob sie auch wirklich existieren oder nicht."

[9] (AS), p. 46., Letter to Petzoldt Dec 12, 1914 (AU) Letter #113.

[10] In his critique of Newton for example (M) p. 288.

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abstract relations.

The Method of Relations

Herbart finds his way through the chaos of sensations to the real with the "method of relations to the ground" (*Beziehungsgrund*),[11] To illustrate the method, blue fish and red fish are both fish. Blue and red are relative terms that oppose and limit one another, since no blue fish is a red fish. However fish in general is the "ground," the determination common to both. This goes on and on, for example fish, reptiles, mammals and so on are animals; animals in turn are living things; living things are physical objects, etc. As we ascend the scale, the relative terms collapse into being in a less and less restricted sense. The intuition of the pure quality-less being (*Sein*) common to all becomes the fundament for all of the limiting determinations, or sensations, existing together.[12]

Like Kant and Fichte, Herbart conceived of sensation qualities as "opposing and resisting" one another by Newtonian action and reaction. To correctly analyze an array of qualities is to find others that exactly cancel it, the way the scientist analyzes a collision with the help of the law of conservation of momentum: the momentum of the closed system

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[11] *Metaphysik*, Sec. 182-188.

[12] Of course every existing fish is either red or blue, and fish-as-such is an abstraction, a universal.

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must equal zero, so the momenta of the parts must all cancel in direction and magnitude. Herbart, who was adept at mathematics, sees the analysis of quality as a problem of algebra,[13] of finding some function such that if {a,b,c} are sensations, the function is f(a,b,c)=0.[14]

Mach seized on the idea that sensations were "oppositions" with quality and magnitude set equal to zero in complexes. In the *Conservation of Energy* he stated as a fundamental principle that all the elements are functions of one another, and can be gathered into an expression that brings them to zero.[15] In the *Analysis of Sensations* he says that the job of specialized research in psychology is to find those equations.[16] In fact, all of Mach's theories for the analysis of tone, spatial vision, color, and even the Mach Bands, depended on the idea that sensations are opposing processes. He shared this belief with his Prague colleague

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[13] *Metaphysik* Sec. 180-181. Herbart attacks the formal subject-predicate logic of his time, arguing instead for an algebraic conception of relations between propositions. The predicate does not inhere in the subject, but rather a system of functional relations holds between both concepts just as between the terms of a quadratic equation of algebra.

[14] He didn't seem worried by the possibility of more than one choice for such functions. Indeed I suppose if there are psychological orderings and physical orderings, both types of functions must be simultaneously possible.

[15] (CE), p. 73.

[16] (AS), p. 45.

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the physiologist Ewald Hering, whose "oppositional" theories of color and spatial vision are still very much admired. The case-study of Mach and Hering, like that of Riemann and Grassmann, is a good example of the night trading in ideas *bei Nacht und Nebel* that goes on between working scientists and speculative metaphysicians.

Herbart's Unextended Community of Wesens

Instead of tracing all of the oppositions back to a single being, as Fichte and Schelling did, Herbart followed Leibniz and proposed a system of many simple Wesens. These are the reals behind the confused appearances. Herbart says that Wesens are in community with one another and that each conserves itself against the others by "pressing back" through its qualities. The community at any point in time is an dynamic equilibrium of such Wesens straining against one another.

> We could say through a sensory comparison what they [the Wesens] do. They *press* one another. For in the world of the senses we find resistance in pressure, where nothing gives way although each is supposed to move. Pressure is rest, through reciprocal endurance against another. But sensory comparison is dangerous here. We are not talking about spatial relations...Here we are only talking about a change in quality that each should suffer from the other, however against which it conserves itself as that which it is. Disturbance should result; self-conservation cancels the disturbance in such a way that it does not occur at all.[17]

Herbart emphasized over and over that the community he speaks

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[17] *Metaphysik*, Sec. 234, p. 103.

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of is only "together" in a causal, not a spatial, sense. He also suggests that we do *not* attempt mentally to visualize the situation with a picture. We are forced to consider it as a purely logical possibility. Obviously if he is constructing space, he cannot beg the question by introducting spatial concepts at the outset. For this very reason the community cannot be thought of as embedded in space to begin with.

The idea of an unextended community is not unprecedented. Leibniz, Wolffe and even the pre-critical Kant had adumbrated a community of unextended monads changing their states qualitatively.

Abstract thought may indeed contemplate extensionless qualitative relationships that exhibit the logical structure of a community. An extensionless point with a series of superimposed hues may exhibit these complex relations: for example, the changes of yellow hue might turn out to depend on changes of blue more directly than independent changes of red and green, even though there is total spatial collapse or superposition of the hues on top of one another.

Herbart admitted that we may never succeed in forming a mental image of what is going on; we have to use our faculty of imageless thought to grasp the construction. However he could have tried harder to assuage the reader's wrath in being told he cannot use his physiological powers of visualization. Indeed the idea will be that the community's causal relations eventually do generate space so that in the final analysis we

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will get our old space back again, but with the difference that spatial relations of near and far are then seen to depend on more general causal proximity and remoteness and not vice versa.

Herbart says the community of Wesens pushing against one another qualitatively gives pre-conditions for a construction of space within it:

> The togetherness and separateness of substances is subject to an alternation [*Wechsel*], which encloses immediately a time-determination. One easily sees that motion and space are also assumed within;...Provisionally we call that space, which we unavoidably think in addition with the coming and going of substance, intelligible space.[18]

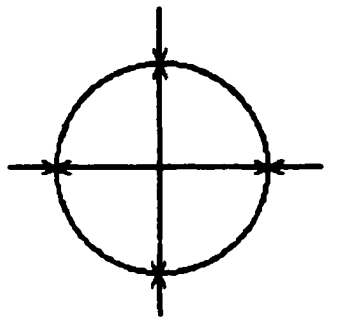
To get a rough idea of what is going on, one can imagine a wave-like pulsation of intensities, pressing one another and cancelling out at "nodes" where the point-like Wesens are situated.

The pulse directed from the node outward is what Herbart calls *Zusammen*, because here Wesens exert pressure on others, and the pulse directed inward on the node he calls *Nicht-Zusammen*, for here each Wesen pulls back and "conserves itself" (see Figure 3.1). Similar ideas had appeared both in Kant's *Metaphysical Foundations of Natural Science* and Schelling's *Ideas for a Philosophy of Nature*.

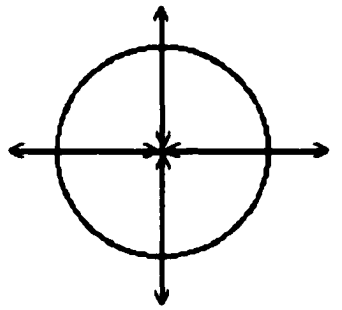
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[18] *Metaphysik*, Sec. 244, p. 118.

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Together



Not-together

FIG 3.1

The pushing and pulling of the qualities (at different intensities and in different directions) generates waves of action that undulate through the community and trace out the lines, surfaces and solids that make up spatial form. The community itself is a simpler order of qualities existing together in an instant according to abstract rules, a very difficult situation to imagine. Of course our capacity to imagine something has nothing to do with its possibility or even plausibility. Herbart reflected often on the need to

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eliminate psychological visualization from science. Mach too complained that the most general causal connections between his elements should not be pictured spatially, for these are the *conditions* for space:

> I think I must add and have already added in another publication[19] that the express drawing of space into consideration in the law of causality is at least superfluous. Since we only recognize what we call time and space by certain phenomena, spatial and temporal determinations are only determinations by means of other phenomena.[20]

In *Knowledge and Error* Mach voiced support for Herbart's aspatial construction in connection with his own later "chemical manifold":

> Our intuitions of space and time form the most important foundations of our sensory view of the world and as such cannot be eliminated. However this does not prevent us from trying to reduce the manifold of qualities of place-sensations to a physiological-chemical manifold. We might think of a system of mixtures in all proportions of a number of chemical qualities (processes). If such an attempt were one day to succeed, it would lead also to the question whether we might not give a physical sense to the speculations that Herbart, following Leibniz, conducted as regards the construction of intelligible space, so that we might reduce physical space to concepts of quality and magnitude. There is of course much to be objected to in Herbart's metaphysics. His tracking down of contradictions that are in part artificially contrived and his eleatic tendencies are none too attractive, but he will hardly have produced nothing but errors.[21]

In fact Mach's own views on physical space and time--

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[19] Probably (R).

[20] (CE), p. 60.

[21] (KE), p. 349.

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especially in later works--demand a Herbartian-style construction, as he realized in his notebooks. Thus as a philosopher of space and time, Mach carries on the intensive space construction in Germany stemming from Leibniz and Kant. We shall look into this problem now and provisionally discuss Herbart's and then Mach's solutions to some of the specific problems, although I must warn the reader that the exhaustive analysis will not be complete until we draw his late writings into consideration.

Kant and Herbart on Intensive Magnitudes

Kant had focused attention on intensity with his claim that the mind's original apprehension of sense impressions was intensive. For example at the *Antizipationen der Wahrnehmung*:

> Since sensation *an sich* is no objective representation, and in it neither the intuition of space, nor of time is met with, no extended magnitude is attributed to it, but to be sure a magnitude (and one in which empirical consciousness grows from nothing = 0 to a given measure in apprehending it in a certain time), that is an *intensive magnitude*. Correspondingly intensive magnitude must be attributed to all objects of perception, in so far as they contain sensation, i.e. a degree of influence on the senses.[22]

He believed that if we had impressions raw, they would appear in disjoint, strobe-like flashes. Each state would vanish when the next occurred. If that were so, we could not draw a line or collect up a series of points, because the points just drawn would constantly fall away. Neither would

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[22] *Kritik der reinen Vernunft* (Stuttgart: Reclam 1996), B 207 A 166.

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we be conscious of ourselves over time, since our past selves would be progressively annihilated and forgotten. Kant reasoned that the very possibility of experience in space and time must depend on our ability to reproduce past experiences in thought and recognize the present as a repetition of the past. When we experience, as opposed to passively sensing, our transcendental imagination or *Einbildungskraft* reproduces and adjoins memory images to our present impressions according to a rule:

> It is apparent that when I draw a line in thought, or think the time from one noon to another, or merely imagine a certain number, I must necessarily first conceive these manifold representations one after another in thought. Should I leave out and not reproduce the preceding (first parts of the line, the preceding parts of time or the imagined units after one another) in passing to what follows, then never would a whole representation, or the aforementioned thoughts, or even the purest and first fundamental representations of space and time be able to arise.[23]

Although I don't have exactly the same experiences when I return to a room I have just left, I do recognize new experiences as repetitions of old ones preserved in memory. This simple ability to recognize repeating experiences of the same was the foundation for scientific knowledge of objects for Kant, as it would be for Mach's theory of economy.

Kant thought intensive magnitudes occurred in all physical tendencies: instantaneous "moments" of gravity, temperature, and all pushes and pulls due to the laws of

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[23] A 101-102.

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nature.[24] These physical intensities were ontologically prior to their extension, since in one and the same instant of time or space, the real can communicate itself in any intensive degree:

> ...Since every reality has a degree, which can diminish to nothing (the void) through infinite gradations without in any way altering the extensive magnitude of the appearance there must be infinitely many degrees in which space and time may be filled. Intensive magnitude can in different appearances be smaller or greater, though the extensive magnitude of the intuition remain the same.[25]

Conversely, if intensities depended on their extension then ten times the force would require ten times longer, or ten times the space, to communicate; a greater quantity of matter would fill a greater volume since each unit quantity would have to be set aside in a separate space.[26]

If extended magnitudes are built up from intensities this meant to Kant that the intensities themselves had to satisfy certain rules, in order to "mark-off" units or quanta, and later more complex objects. The sum total of these rules describes Kant's *community* of intensities.

The first rule is that every intensity contains a manifold of degrees added up "within" and communicated in a single burst:

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[24] B 210-215.

[25] A 173, B 215.

[26] A 173, B 215 - A 175, B 217.

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> Every intuition contains a manifold in itself, which still would not be recognized as a one [*ein solches*] if the mind did not distinguish a time in the sequence of impressions upon one another. For a representation *contained in an instant* can never be anything other than absolute unity. Now so that unity of intuition arises out of this manifold (as for example in the representation of space), first the passage through the manifold and then the collecting up of it is necessary, which action I call the synthesis of apprehension, because it is, so to speak, directed at an intuition that presents a manifold, this however as one such manifold and as contained in one representation cannot be brought about without a preceding synthesis.[27]

By saying that an intensive sensation contains a "manifold within" Kant probably meant among other things that any intensive degree (for example of heat or pressure) includes every smaller. A pot of water at 70 degrees contains as it were a series of identical pots at every lower degree of temperature; any force may be thought of as a series of forces of lesser intensity all acting together in the same instant.

To "synthesize" intensities into a time-form (supposing we wanted to represent the intensities of two opposing pressures by how quickly they overcome one another in time) we first require a unit of time. We cannot base intensity on time. But we *can* base the unit instant of quickness on the instantaneous proportion to which a force overcomes a given force, and then use this as a standard of comparison for all the others. In the pre-critical "Gedanken von der wahren Schätzung der lebendigen Kräfte" Kant suggested that spatial

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[27] A 99-100.

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accelerations be understood intensively as a sequence of static moments, in which each moment contains the ratio of the impressed force to the resistance offered by a body instant-by-instant.[28] In the *K.d.r.V.* Kant represents all intensities schematically in time as a resultant or an "opposition" [*Entgegensetzung*] between reality (the maximum of intensity) and negation (the zero).

> Reality is that in a pure concept of the understanding which corresponds to a sensation, that, therefore, whose concept in itself points to a being (in time); negation is that whose concept represents non-being (in time). The opposition between the two happens in the distinguishing of the same time as a filled or empty time. Just as time is only the form of intuition, therefore of objects as appearances, so is that which corresponds to sensation in them the transcendental material of all objects as things in themselves (thing-hood, reality).
>
> Now every sensation has a degree or magnitude, through which it fills the same time, in respect to the self-same representation of an object, to a greater or lesser degree, until it ends in nothing (=0 negatio). Hence there is a relation and connection, or rather a transition [*Übergang*] from reality to negation, which makes every reality representable as a quantum; and the schema of a reality as the quantity of something in so far as it fills time is just its continuous and uniform production in time, in that one goes, in time, from a sensation with a certain degree downwards to its disappearance, or gradually climbs up from negation to the same magnitude.[29]

Kant had in mind a reality-to-negation ratio that could be used whenever intensities were set against one another. If two forces act on one another in the briefest imaginable

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[28] "Gedanken von der wahren Schätzung der lebendigen Kräfte," p. 19.

[29] A,B 182-183.

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instant, the ratio they fall into gives us a measure of the quickness with which the overcoming occurs and therefore a schema we can use for all the others. Say a force 3A is set against a force A and overcomes it in a ratio of 3:1. Then the instantaneous ratio for 3A is three parts full to one part empty, and the ratio for A three parts empty to one part full. The voided parts of one are the filled parts of the other.

Fichte seized on Kant's principle of opposition [*Entgegensetzung*] interpreting all qualities as oppositely signed magnitudes. And Herbart was even more sophisticated than Kant or Fichte in his understanding of how intensities oppose one another. For example, besides the "inner addition" of degree, Herbart also recognized a *capacity* factor, or "outer addition":

> Every intensive magnitude can be multiplied after two fashions inner and outer. For the intensity can climb, or be represented severally without climbing. The later occurs when not merely one object but several are present, all with the same intensity. The heat in a room is all the greater as the room is greater which has the same temperature throughout.[30]

As it turns out, the distinction of intensity from capacity was fundamental for Mach (he called them the potential function and potential, respectively)[31] For bodies their capacity for heat as well as their temperature determines the

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[30] *Metaphysik* Sec. 287, p. 180.

[31] "Eine Bemerkung über den zweiten Hauptgesetz der Wärmetheorie" *Lotos* 21 (February 1871): 17-18, and (CE) p. 47. The capacity of a body for heat describes the "quantity of heat" it is capable of sustaining.

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mean temperature of contact with another body. In a collision, the masses will be a factor in the distribution of the final velocities. We can recognize in the capacity a very dim presentiment of a "spatial" factor, just as intensity contains an inkling of time determination.

Objects and Community

We now turn our attention to a community of such intensities in the Herbartian system. Objects engaged in reciprocal causal relations with one another (Wesens for Herbart) were understood as points of zero quality, "nodes" at which the various qualities cancel in strength (Figure 3.2).

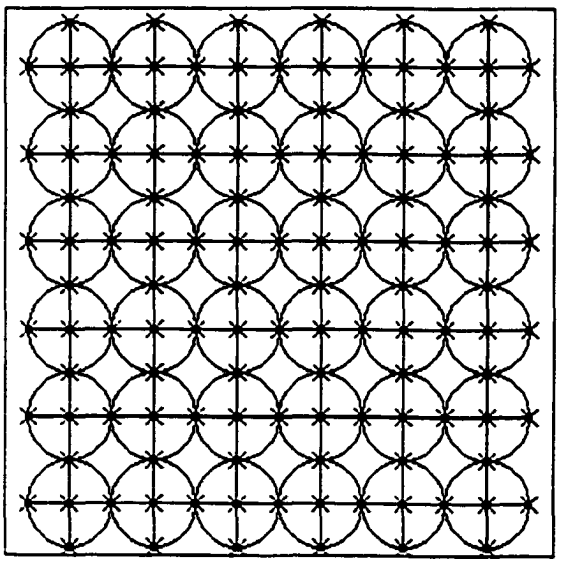


FIG 3.2

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The nodes are held in position by the mutually constraining forces acting there.

The diagram already hints at a feature of Herbart's monads Mach would later exploit. If they are only delimited by the cancellation of their qualities at a nodal point, then Wesens become more of a formal pattern in the flux or a function, rather than a subsisting object or substance that precedes its predicates.

As for the nature of the underlying qualities, we can see already that every quality making up a Wesen must have direction with which it opposes others. To explain their physical interaction they must also have magnitudes, both of intensity and capacity.

Andreas Laaß argues that for Mach the Herbartian community of monads, like the mechanical atoms, already contained the seeds of its own destruction since the monads were described as little more than mere functions of their qualities.[32] By his 1866 paper Mach could eliminate them and stick to functions and elements alone. Mach's philosophical move paralleled exactly his elimination of atoms and concentration on their potential to emit spectral lines.

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[32] "Vom Sinne des Machschen Philosophierens" in Hoffmann und Laitko, eds., *Ernst Mach: Studien und Dokumente zu Leben und Werk*, (Berlin: Deutscher Verlag der Wissenschaften, 1991), p. 213.

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Herbart and the Problem of Change

Kant, Fichte and Herbart were all agreed that the community was subject to constant change, an *Entstehen durch ein Vergehen*, and that enduring forms of objects were like patterns in the flux. But at the base-level of qualities, how could changes ever occur amid a set of rigidly fixed oppositions and cancellations? When one quality pushes, another must push back equally hard. How then could they rise or fall in intensity?

Herbart, to his credit, was the first to realize that for one instant to give way to the next, the given ratio of reality and negation between every pair of qualities has to be suspended. That could only happen if the opposing moments of reality and negation could dissociate in-between instants. Herbart thus claimed that a time sequence, natural number line or series of degrees climbing and falling was not a series of points where the qualities rigidly coincide everywhere, but a constructed set of points *and* dissociating gaps. In the gaps the qualities are independent of one another (*Nicht-Zusammen*), until they coincide again at the next instant. This process is worth reconstructing in some detail. The figure below (3.3) represents let us say a rising intensity. That means the reality content (R) of the quality will increase at the expense of the negation content (N). At first R and N are rigidly connected to one another. Then they are allowed to slip out of dependence. In the dissociation of two qualities

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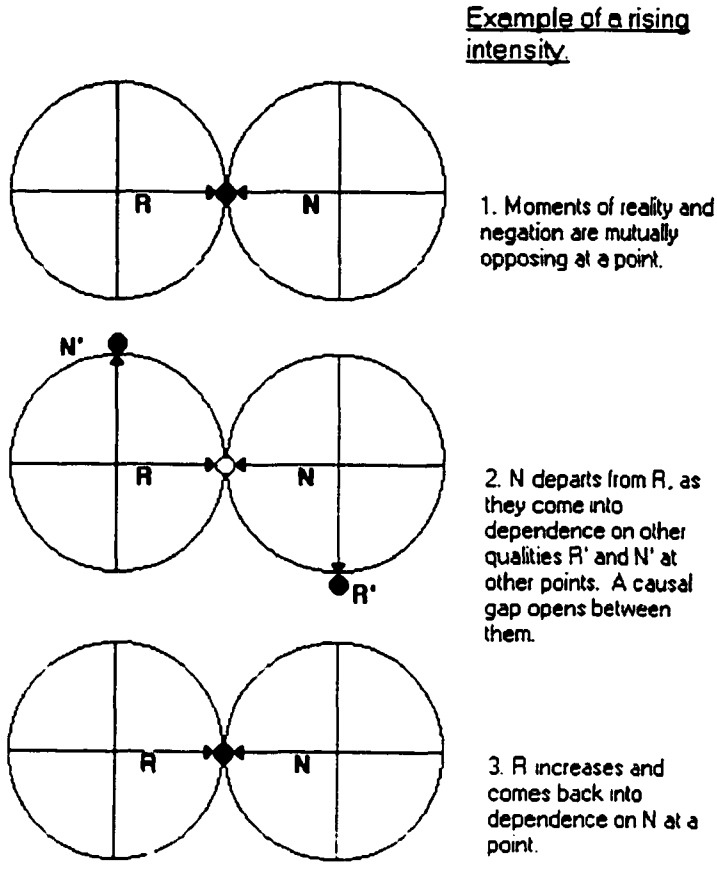


FIG. 3.3

R and N, they are free to change independently of each other and come into dependence instead on N′ and R′. So let N depend on some R′ that is greater than R, meaning the intensity of N will now be lower. Meanwhile R will drop out of connection with N, and into dependence on some N′ that is

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less than N, meaning that R's intensity will now be greater. After the switches have been made, we can join R and N together again at their new intensities. R now has a higher intensity and N a lower. I have represented the same process schematically in figure 3.4 and labeled the process along the left-side (N+R)(R-N)(N+R)(R-N)....(where + symbolizes "association" and - "dissociation")

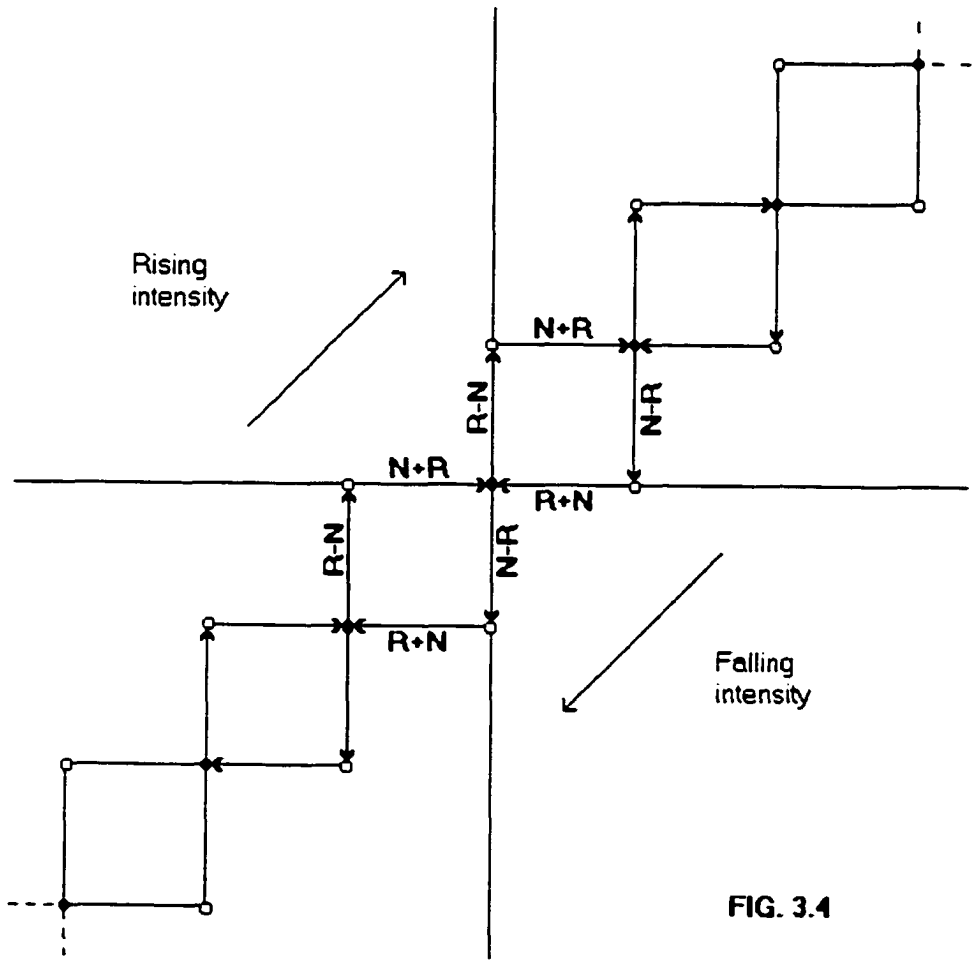


FIG. 3.4

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This notation will now be explained in connection with Herbart's ideas on natural number.

Herbart's Theory of Natural Number

Herbart applied his metaphysical ideas to his construction of "points and gaps" in natural numbers, not only explaining the process of preserving and collecting up points, as Kant had, but also how they were set apart from one another.

To discuss the tracing of a line in the community Herbart had to abstract to a level above that of the individual qualities represented by the figures above. Instead Herbart concentrated on the abstract patterns of flux spreading through the community, the way a water wave may spread from side to side while the individual molecules of the water move up and down but otherwise stay put.

Herbart posited an abstract constructing process that traced out spatial form. The constructing process, he reasoned, must be one of joining *and* separation (Zusammen and Nicht-Zusammen) to exhibit both the points and the spaces in the final line. By concentrating on the wave only and not the underlying Wesens,[33] he was able to represent the process abstractly as if it were produced by the contraction and expansion of two enduring Wesens **A** and **B**. Here I think it is best to picture Wesens as patterns of qualities rather than

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[33] *Metaphysik* Sec. 245, p. 119.

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subsisting objects. Compared to the real exchanges of qualities underneath, Herbart's line was largely an imaginary abstract schema, similar to the way a physical wave is considered schematically independent of the medium that supports it. This is how I represent the process moving from right to left in figure 3.5 with the Wesens as superimposed patterns over the community of qualities:

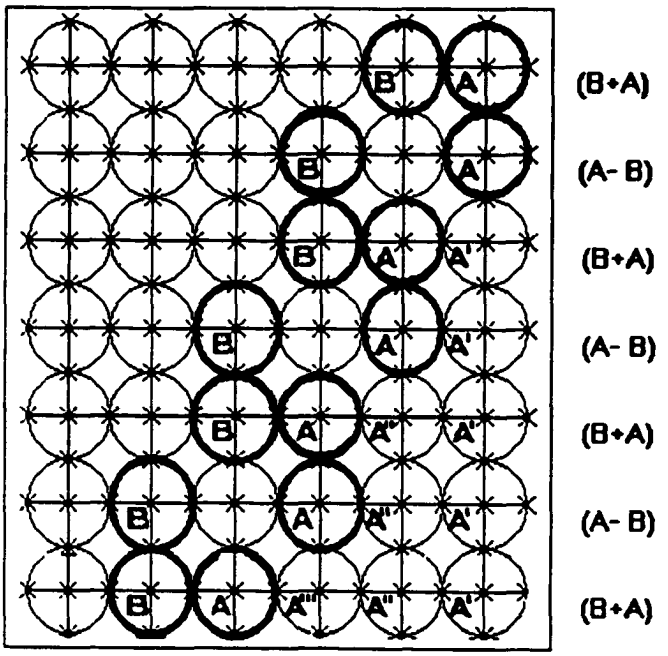


FIG. 3.5

Now for an explanation of the notation used in diagrams 3.5 and the schematic presentation of the same process in 3.6. The combinations **A+B** and **B+A** will both indicate that **A** and **B** are together. The Wesen "fixed" in its causal relations is

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named first, the Wesen that has changed to join the first is named second. In the first case a changing **B** comes into connection with a fixed **A**; in the second a changing **A** comes into connection with a fixed **B**.

Likewise **A-B** and **B-A** indicate that **A** and **B** are not-together. Again the first member is thought of as fixed, the second changing. Herbart observed that even when the two are not together, no *spatial* gap opens between A and B; they are still *Aneinander*, with no space in between.[34] The two independent qualities are spatially touching without a causal relation the same way a hand is together with a stone when it accelerates the stone at the same rate of its natural fall. There is no space between the hand and the stone yet their independent motions are due to different effective causes.[35]

But of two Wesens sitting up against one another, how can one remain fixed while another is changing? Wouldn't a change of one necessitate a change in the other to counter it? We know what to do in the case of dependent and independent qualities in time-synthesis: two paired qualities may come

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[34] *Metaphysik*, Sec. 243

[35] Herbart does insists that we are *not* to think of the particular qualities and their causal relations when we picture the abstract process of **A-B** (*Metaphysik* Sec. 246, p. 124). To me this distinction is entirely a matter of whether A and B are meant, in which case the opposition of qualities is signified, or whether **A** and **B** are meant, in which case an abstract Together and Not-Together is signified. A little more attention to such distinctions might have benefited Herbart.

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into dependence on each other while their previous partners change their intensities. In the production of spatial lines through such changes we may represent this underlying state of affairs schematically, according to Herbart, by interposing an imaginary *Bild* (an image or picture) to take the place of the changing Wesen and come Together with the fixed one. In fact this is just another pattern that we call by a special label to represent the place it occupies in the process. Say that **A** stays fixed and **B** changes. Then the pattern we call the "image" **B'** carries on the fixed relation with **A** that **B** once had. So by means of offsetting changes it is as if an object **B'** takes the place of **B**. The motion of the process leaves a trail of these images (or patterns) in its wake, and they are the imagined points of the number line, reproduced in thought.

As Herbart pointed out the images themselves are only abstractions (*leere Bilder*). They do not subsist after the process moves on, because all previous places touched upon will change in the interim;--thus "intelligible space" is indeed transcendentally ideal in Kant's sense. But if there are regularities in the causal structure of the community, the tracing process can return the way it has come and revisit, more or less, the same causal relations or patterns. The regularity of these causal relations makes possible the apparent tracing out of uniform lines, surfaces and solids in "space" over and above the particulars in it.

The possible ordered combinations of {+,-} and {**A**,**B**} give

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the "sense" of whether the process is moving forwards or back onto previously touched upon points:

Forwards: **(A-B),(B+A),(A-B),(B+A)...**

Backwards: **(A+B),(B-A),(A+B),(B-A)...**

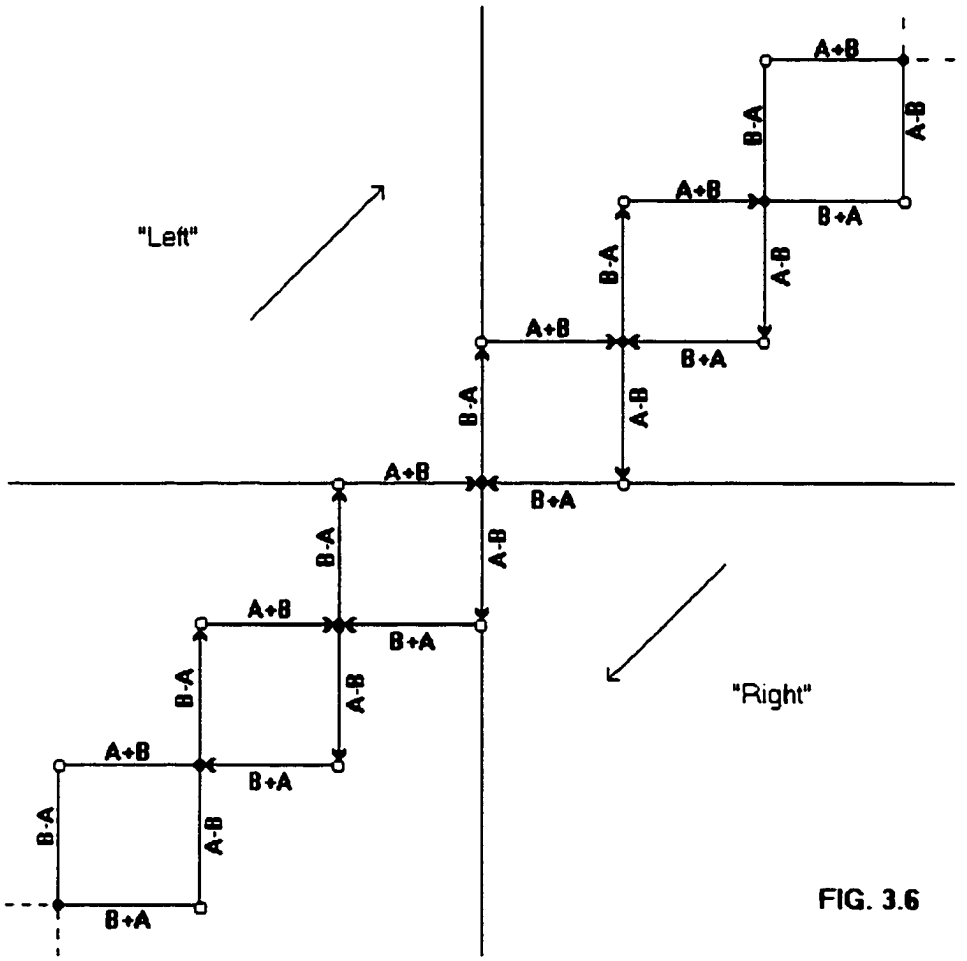


FIG. 3.6

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Herbart claimed that his construction of images gave every "thinkable distancing" that could occur between **A** and **B** in both directions.[36] A process could move forwards any number of spaces and then backwards retracing these spaces, or it could move infinitely in either direction. The line is the set of all possible combinations captured and collected in the natural numbers.

The process remains similar to itself in both directions forwards and backwards, as if it were constantly revisiting the same points. This is of course a matter of experience and depends on the underlying community of qualities and what we decide to count as the repetition of the same process. If so, then **A+B** (**B** coming to a fixed **A**) is the reverse of **B+A** opposite in sense (**A** coming to a fixed **B**). Likewise **A-B** could be the reverse of **B-A**.[37] Assuming all of the processes turn

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[36] *Metaphysik* Sec. 253. p. 136: "From the determinate beginning point A on, we find on the line AB two-fold every conceivable distancing that allows itself to be thought precisely and without contradiction. Namely right and left to indicate the opposing processes with well-known names."

[37] Grassmann (while keeping the idea that natural number lines are traced out by processes of "joining and separation") significantly clarified Herbart's ideas in the *Ausdehnungslehre* by introducing the symbols ∪ for joining (+) and ∩ for separation (-). The properties of communitivity, associativity, distribution, identity can then be added or taken away, in so far as the constructing process shows or fails to show them. *A New Branch of Mathematics: The Ausdehnungslehre of 1844 and Other Works*, Translated by Lloyd C. Kannenberg, (LaSalle, Illinois: Open Court, 1995), Introduction and Chapter One to Sec. 15.

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out symmetrical in both directions we have some of the tell-tale signs of spatial form: symmetry in the directions (isotropy) and homogeneity, or the indistinguishability of different positions.

Herbart went on to describe the tracing of surfaces and spherically propagating waves by adding independent Wesens **C**, **D** that can undergo processes of Together and Not-Together with **A** and **B**. For the four processes **A+B, B+A, A-B, B-A** there are sixteen independent combinations the process can undergo with a Wesen C.[38]

Herbart thus solved Kant and Fichte's problem of fixed oppositions, by introducing gaps of dissociation. There is still a universal dependence of all on all, but without a particular dependence that always holds between any two individual qualities. The general dependence without particular dependence is what allows these alternating independent gaps to subsist. Herbart was thus forced to admit certain standing differences or indeterminacies undulating through the community like a bump in the carpet. But these differences were a new and untested assumption: why were they

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(A+B)+C, C+(A+B)
(A+B)-C, C-(A+B)
(B+A)+C, C+(B+A)
(B+A)-C, C-(B+A)
(A-B)+C, C+(A-B)
(A-B)-C, C-(A-B)
(B-A)+C, C+(B-A)
(B-A)-C, C-(B-A)

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there, and were they as fundamental as oppositions and cancellations of quality?

Mach's Contributions in Outline

Mach's contribution to the space construction lay in articulating such standing "differences" in nature, and in describing with more physical precision the oscillating processes that equalize them, as they trace out lines and surfaces. In his *Conservation of Energy* he declared that differences were implied by the fact of the variability of nature:

> The law of causality is identical with the supposition that between the natural phenomena α, β, γ, δ,...ω, certain equations subsist. The law of causality says nothing about the number and form of these equations; it is the problem of positive natural investigation to determine this; but it is clear that if the number of the equations were equal to or greater than the number of the α, β, γ, δ,...ω, all of the α, β, γ, δ,...ω, would be thereby overdetermined or at least completely determined. The fact of the varying of nature therefore proves that the number of equations is less than that of the α, β, γ, δ,...ω.[39]

Mach must have encountered the problem sometime between 1866 and 1871. The "new theory of matter" he had been promising since 1863 was published in a philosophical article in 1866 "Bemerkungen über Die Entwicklung der Raumvorstellungen," and indeed, like his Vorträge theory, it is an analysis of spatial matter into intensive pressures:

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[39] (CE), pp. 61-62. Cf. (M), pp. 273-274.

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> It cannot be my intention here to criticize our conceptions of matter whose insufficiency is, indeed, generally felt. I will merely make my thoughts clear. Let us imagine, then a something behind (unter) matter in which different states can occur,[40] say for simplicity a pressure in it, which can become greater or smaller.

Mach further claimed that the law of force in nature holding between pieces of matter need not be thought of spatially:

> Physics has long been busied in expressing the mutual action, the mutual attraction (opposite accelerations, opposite pressures) of two material particles as a function of their distance from each other--therefore of a spatial relation. Forces are functions of the distance. But now, the spatial relations of material particles can, indeed, only be recognized by the forces which they exert on each other.
>
> Physics, then, does not strive , in the first place, after the discovery of the fundamental relations of the various pieces of matter, but after the derivation of relations from other, already given, ones. Now it seems to me that the fundamental law of force in nature need not contain the spatial relations of the pieces of matter, but must only state a dependence between the states of the pieces of matter.

Finally the states could be set down as mutually dependent functions of one another, a community of co-existing things which made up the forms of space and time:

> If the positions in space of the material parts of the whole universe and their forces as functions of these positions were once known, mechanics could give their motions completely,[41] that is to say, it could make all the positions discoverable at any time, or put down all

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[40] There is an ambiguity in the German: "Denken wir uns also unter Materie ein Etwas, in dem verschiedene Zustände eintreten können." It can mean "let us imagine under (i.e. under the rubric of) matter a something.." or "let us imagine underneath matter a something..." as Jourdain has done.

[41] [Jourdain's note: For this purpose it would be necessary also to know the velocities of the various parts at that instant.]

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positions as functions of time.

But, what does time mean when we consider the universe? This or that "is a function of time" means that it depends on the position of the vibrating pendulum, on the position of the rotating earth, and so on. Thus "all positions are functions of time" means, for the universe that all positions depend upon one another.

But since the positions in space of the material parts can be recognized only by their states, we can also say that all the states of the material universe *depend upon one another.*

The physical space which I have in mind--and which at the same time contains time in itself--is thus nothing other than *dependence of phenomena on one another.* A complete physics which would know this fundamental dependence, would have no more need of special considerations of space and time, for the latter considerations would be included in the former knowledge.[42]

Clearly the new fundamental concepts of science are qualitative "states" and functions that represent the oppositions of those states. With Herbart's Wesens discharged, nothing further is assumed by Mach. We shall see later exactly what he understood by them.

Mach's Herbartianism is quite evident in the 1866 paper, but the mutual dependence of states was rigid, and the problem of the generation of differences did not yet trouble him; only in 1872 did Mach feel forced to admit some indeterminacy in the relation of the elements.

There is no evidence why Mach would have suddenly admitted this indeterminacy in the functional relations of elements. We can only assume that his putative wave-theory of matter, comparing it with the "proto-chemical" separation and

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[42] (R) translated in (CE), pp. 88-89.

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combination of light, would have interested him in the question of how oscillations in the natural potentials could take place, especially if they are all overdetermined or uniquely determined. In any case, from 1863 to 1872, Mach changed his language from the philosophical vocabulary of the "Vorträge" and the "Raumvorstellungen" paper (*Wesens, Zustände*) to a physical vocabulary in the *Conservation of Energy* (elements, functions, potential, potential function), which suggests that he was serious about reconstructing his philosophical ideas in physics. In the 1860s Mach visibly infused the Herbartian schema of "intelligible space" with his own science, probably under the influence of J.R. Mayer's philosophical-scientific "universal outline of energies."

Mach devoted himself to "historico-critical" studies of physics in Prague during the summer of 1870, during a period of recuperation from an illness.[43] He also said[44] that he had spent some time putting himself into the intellectual position of Mayer, and this must have fallen in the period before 1871, for there Mach has fully absorbed it, adopting Mayer's principle of causality as the "logical root of the principle of the conservation of energy." Mayer, too, had reasoned to the theorem from general metaphysical principles, and Mach

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[43] K.D. Heller, *Ernst Mach: Wegbereiter der modernen Physik*, p. 23.

[44] (H), p. 232.

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applauded him for his constructive use of metaphysics.[45]

By the 1870s, Mach's description of the "qualities" or elements of the world was in terms of natural potentials, physical characteristics at a point. These may be gravitational potentials, temperatures, electro-magnetic potentials, or any other characteristic that provokes energy changes.[46] The natural processes always act to equalize any potential difference, between a point and a neighboring point, for example when a body cools to the mean temperature of neighboring bodies.[47] Herbart's waves of Together and Not-Together pass smoothly into Mach's vocabulary as equalizations of potential differences (Herbart's +) and the pre-existing differences themselves (Herbart's -). In Mach's language Herbart's space-tracing processes are waves of constant action.[48] In his Notebook of 1873 we already find ruminations about differences and equalizations:

> Equalization of differences.
> Only where there are differences, is a process possible, since here determination is not lacking.
> With the equalization of differences, necessarily equilibrium.
> Increase of differences. An increase determinative of a process? Change depends always upon a clearing away of the *cause of change*.
> A state will not pass over into another in which greater causes of change are included.

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[45] (H), p. 233 ff.

[46] (CE) p. 69.

[47] (M), p. 288.

[48] Action has dimensions erg/sec, energy per unit of time.

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> What does time have to do with the latter point. Diminution of a difference can draw an increase in another after it. But all in all a diminution must occur...[49]

Differences must exist, but the true natural processes for Mach only consist in their equalization, not their generation. When energetic processes create a difference it is always on the way to making another smaller. In the continuation of the same passage Mach says that an actual *increase* in the differences could have no determinate "rule":

> Equalities cannot carry in themselves any ground for inequalities. That would indicate no rule according to which it goes on. Inequalities can only become smaller.[50]

Mach returned to the problem of the generation of differences in a late edition of the *Analysis of Sensations* after he had been accused of being an "indeterminist." Here he admitted plainly that differences must enter into our environment

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[49] Ausgleich d. Differenzen
Nur wo Differenzen is ein Vorgang möglich, weil nur hier das Bestimmende nicht fehlt.
Mit d. Ausgleich der Differenzen nothwendig Gleichgewicht.
Vergrösserung d. Differenzen. [Den?] Vorgang bestimmende Vergrösserung? Die Veränderung geht immer auf Beseitigung *der Veränderungsursache*.
Ein Zustand wird nicht in einen anderen übergehen der noch grössere Ursachen der Veränderung in sich schliesst.
Was hat bei dem letzten Punkt die Zeit zu tun.
Verkleinerung der einen Differenz, kann eine Vergrösserung der anderen nach sich ziehen. Aber alles in allem muss doch eine Verkleinerung eintreten...
Notebook dated April 12 1874 (NL 174/02/03).

[50] ...Gleichheiten können keinen Grund zu Ungleicheiten in sich tragen. Da liesse sich keine Regel angeben nach welcher das vorginge. Ungleichheiten können nur kleiner werden.

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through "certain independent variables which are inaccessible to us":[51]

> The only assumption compatible with a general representation of our own limited environment is that of a tendency, on the whole, to a diminution of differences. But if circumstances that set up differences did not make themselves felt by forcing their way into our environment a time would soon come when nothing more would happen at all.[52]

He also wrote in a letter to Josef Popper:

> I agree that potential differences not only disappear but must also come into being. The latter point is uncommonly hard to understand. Do all potential-differences come from the outside, e.g. from the sun to the earth? Or do they arise from other quantities already on earth? The strangest case for me has always been the cooling of a liquid beneath the environing temperature through evaporation.[53]

He presented the generation of differences as a fact of experience, never speculating further on where they might come from. They became a standard assumption of his philosophy of space and time. Several other ideas for Mach's construction (for example metric properties) were sparked by his psycho-physical researches on the psychology of space. Moreover his notion of "states" or "elements" drew life from the psychology of Herbart and Fechner to which we now turn.

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[51] (AS), p. 350.

[52] (AS), p. 352.

[53] Mach to Popper November 13, 1894 (AU), p. 19.

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Chapter Four: The 1863 lectures on Psychophysics

Mach's (1863) "Vorträge über Psychophysik" contained the presentation of his first original theory of matter: a "psychomonadology" in which both psychical and physical qualities were united in material monads. The "Vorträge" also contained Mach's first explicit statement that nature had a Fechnerian inner side and that there were sensations or physical qualities in matter. The motivation for this early theory was to establish the new psychology of the 1860s on a rigorous basis and to unify it with physics. To that end Mach delved into several themes that became part of his mature thought.

The Goals of the Vorträge: Unity of Physics, Psychophysics and Psychology

Mach began these dense lectures by expressing his views on how psychology could become a science, and how the sciences of physics, psychology and physiology could all be unified.[12]

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1 (V), p. 147.

2 Andreas Laaß, "Vom Sinn des Machschen Philosophierens" in Hoffmann und Laitko, eds., *Ernst Mach: Studien und Dokumente zu Leben und Werk*, (Berlin: Deutscher Verlag der Wissenschaften, 1991) p. 210: "Wenn Mach seine Anschauungen in der ersten Hälfte der sechziger Jahre mit dem Terminus "physikalisch-psychologisch Monadologie" bezeichnete, so ist dies zuvörderst als ein Hinweis zu deuten, wie und warum Mach eine Grundlage dreier Wissenschaften anstrebte. Mit der Physik, der

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He claimed that the psychology of his time was divided between the phenomenological school of Herbart and that of the physiological psychologists (Fechner, Wundt and Helmholtz). It seemed to Mach that Herbart concentrated on the phenomenal relations between ideas to the exclusion of their physiological basis, while Fechner and Wundt pinned down a physiological correlate for each individual sensation, letting dangle the remaining phenomenal relations of sensations to one another.

> The newest physiological researches, namely those of Fechner, Helmholtz, Wundt and others have now already penetrated so deep that they leave off just where the Herbartian psychology picks up. This science does not just stand there isolated, for the physiologist its study is not in vain.[3]

Mach suggested that introspective psychology and physiology work in a "pincer movement." Phenomenological analysis would isolate the elements of perception, and physiology would locate the corresponding structures in the body. Such a parallelism is already at work in his article "Zur Theorie des Gehörorgans" (1863) where Mach determines the row of phenomenal tone sensations as fixed "places" in a one-dimensional manifold, and then works outward to match them to structures in the ear. By 1865 Mach articulated his principle of psychophysical parallelism:

> Sinnesphysiologie und der Psychologie sind äussere und innere Erfahrung und der Empfindungsprozess selber Gegenstand der wissenschaftlichen Untersuchung."

[3] (V), p. 168.

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> Every psychical event corresponds to a physical event and vice versa. Equal psychical processes correspond to equal physical processes, unequal to unequal ones. When a psychical process is analyzed in a purely psychological way, into a number of qualities a,b,c then there corresponds to them just as great a number of physical processes α, β, γ. To all of the details of psychological events correspond details of the physical events.[4]

This was his working principle of psychophysical research, to which he later added principles of continuity and sufficient determination in the *Analysis of Sensations* to rule out chance correlations between sensations and physiological structures that seemed to match but were really unrelated.[5]

Tempering this parallelism, however, Mach was aware that so long as sensations and their physiological counterparts remained heterogeneous (*ungleichartige*) magnitudes, no real psycho-physical law or function could exist. Mach was applying physical rigor to psychology here, specifically the theory of dimensions. According to a principle attributed to the physicist Fourier, all formally adequate natural laws must exhibit homogeneous dimensions on both sides.[6][7] As a physicist Mach knew that if one side of an equation has

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[4] "Über die Wirkung der räumlichen Vertheilung des Lichtreizes auf die Netzhaut" *SW* **52** (1865), translated in (MB) pp. 269-270.

[5] (AS), pp. 57-60.

[6] "Über die physiologische Wirkung räumlich vertheilter Lichtreize...Vierte Abhandlung" *SW* **57** (1868), translated in (MB) pp. 299-300.

[7] For Fourier's principle, see Percy Bridgman *Dimensional Analysis* (New Haven: Yale University Press, 1922), p. 51.

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dimensions of ml^2t^{-2} so must the other.[8] This ensures that both sides are scaled to the same units, can be added, cancelled, multiplied and so forth. A psychophysical law thus assumes a fundamental likeness in kind between the quantities to be compared, before they can even be correlated.

A theory of homogeneous dimensions for physics and psychology is a vital concern of the "Vorträge," since without it psychology could hardly stand next to physics as a science. Even while Mach is defending Fechner's use of heterogenous dimensions of sensation and stimulus, he says that psychophysics may have to do its own dimensional analysis, and not necessarily with the space, time and mass units of physical science:

> Of course we measure stimulus and sensation by different units, stimulus by stimulus, sensation by sensation, for both are heterogeneous magnitudes. This is no more surprising than that physics measures time and space by different means. And more! We must ourselves lay down different measures for different sensations, different for light, different for sound, different for vision. Physics currently measures everything with three buckets: mass, time and space. We haven't gotten so far in psychophysics, although not all prospects are lacking for an approach to a similar goal.[9]

At the end of the lectures, it turns out that the spatial dimensions of matter are derived from a simpler intensive unit.

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[8] (M), p. 368.

[9] (V), p. 227.

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Herbartian Psychology

In fact Herbart had already proposed such new dimensions for psychology in his *Psychologie als Wissenschaft*, by developing a rigorous mechanics of opposing and inhibiting psychical-physical qualities. (In his Preface Herbart crowed that the whole theory had withstood the scrutiny of a professional mathematician.) Mach adopted the "oppositional" theory of elements from Herbart, and came to the defense of mathematical psychology in the "Vorträge."

Herbart's *Mechanik des Geistes* was based on the fundamental principle that sensations leave traces (*Vorstellungen*) behind in the mind--faint copies of previous sensations, which Hume called ideas. Because they are confined qualities with direction and magnitude these ideas immediately act to push each another below the threshold of consciousness. At any given time only the strongest ideas are able to maintain their clearness, or force, against the others, which explains why consciousness is limited in scope.

What determines oppositions are the types of the qualities themselves, as if they were exactly analogous to directed magnitudes like forces. Some ideas like green and sweet do not conflict at all, some resist partways and some resist each other totally. When a conflict does occur, the two directed ideas strain to inhibit one another according to

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their respective strengths,[10] the idea half as strong being inhibited four times as fast.

As schematic as Herbart's psychology may seem, Mach defended it with all the arguments he could muster, even taking in stride its inaccessibility to independent measurement:

> Ideas, inner states, it is said are not quantities that admit of measurement; what doesn't admit of measurement cannot be the basis for calculations. This objection is unjust. If a body, acted on by a constant force receives the constant acceleration g in every unit of time, it covers during the time t the distance s, which is given in the formula $s=1/2\ gt^2$. The formula is found without knowing how large g may be, and without having measured it. It is learned that the distance laid-off is proportional to the square of the time, that in double the time four times the distance, in triple the time nine times the distance is covered. Such forms of laws with which mathematical psychology deals at present, are also, without measurement, derived from definite assumptions, if one only admits that inner states in general can be of different intensities.[11]

Mach believed Herbart's fundamental law of opposing qualities was connected through its consequences with verifiable facts of psychology, so as not to merit the pejorative of "metaphysics." Besides, as he added in a *Nachtrag* to the lectures, if the original development of a science from

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[10] As Mach explains, if A and B are two completely opposed ideas and A>B then the degree to which A overcomes B, or the diminution in B *caused* by A is to B as A is to the total force of A and B put together. From this proposition, switching the names, it follows that A is diminished by the same amount that B is, namely AB/A+B (V) p. 203 Cf. Edwin Boring, *A History of Experimental Psychology*, Second Edition (New York: Appleton-Century Crofts, 1950), pp. 247-249.

[11] (V), p. 168.

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metaphysical principles were to be ruled out, even mechanics would never have been established.[12] Mach was clearly willing to countenance indirect metaphysical principles[13] as long as they were continuous with natural science.

> The cases that Herbart treats are simpler than ever occur in reality; yet this is necessary for the beginning of an investigation. Mechanics proceeds from similar simple cases. The mathematical consequences of Herbart's assumptions however, a few of which we have derived, agree so often with experience that it is impossible to believe the fundamental assumptions should be far from the truth. For a good self-observer it is more than a mere image that the ideas reciprocally press one another. We feel this, so to speak, really working within us.[14]

Mach had another stake in defending Herbart, since he believed by 1863 that he had a physiological parallel worked out for Herbart's theory of consciousness. This was his theory of neural inhibition. Mach knew that nerves in proximity to one another tended to inhibit one another's activity, just like Herbart's ideas:

> ...it appears to me very plausible that generally speaking attention has its grounds in the mechanism of the body. If nerve energy is liberated in certain paths, then through this very mechanism other paths are cut off. And what is valid for sensory sensations might be valid for thought also. Thought can be conceived as a change of attention. Certain physical processes therefore exclude others. In the measure to which the first occur, the others withdraw. Physical processes, sensations press against one another. We have found on physiological grounds a law which governs the sensations

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[12] (V), p. 365.

[13] He says that he "avoids all metaphysical questions" by sticking solely with Herbart's intensities and not introducing a simple soul in which they inhere. (V), p. 170.

[14] (V) p. 204.

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similar to that which Herbart expressed for the ideas long ago.[15]

When Mach actually discovered evidence of neural inhibition in the eye in 1865, he quickly generalized the process from the sense organ to thought in the brain. This was a happy piece of comparative anatomy, since the neural tissue of the retina is literally a part of the brain living in the eye. In fact, Mach's own theory of the ego was that it was a kind of machine composed of innate devices of activation and inhibition, a conception that owed much to Herbart's *Mechanik des Geistes* and its oppositional mechanisms.

Fechner's Psychophysik

Mach now turns to the issue of measurement of the sensations. He originally thought that Fechner's 1860 psychophysical law could succeed where Herbart had failed to provide a direct measurement of sensation-units. Although Mach soon changed his mind after a closer analysis, his work on the Fechner law does reveal the philosophical conditions he thought psychophysical laws had to meet, particularly the homogeneity requirement.

The Fechner-Weber law is written:

$$y = a \log(x/b)$$ [16]

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[15] "Zur Theorie des Gehörorgans" *SW* **48**(1863): p. 297.

[16] The expression 'a' is a constant of sensitivity for the sense organ in question. The other constant b is known as the "absolute threshold" or the smallest noticeable

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which means that the sensation y is proportional to the logarithm of x, the stimulus. The formula involves several philosophical assumptions that would come to disturb Mach.

Weber's law, from which the above formula is derived, is empirical; it describes the familiar fact that our sensitivity to a stimulus--whether or not we notice a small increase or decrease--depends on the total amount of the stimulus present. For example we notice when a pound is added to a pound, but not when it is added to one hundred pounds.[17] Each sense organ has its own peculiar constant of sensitivity (a) which expresses the ratio of (Δx) the stimulus increase needed to produce a "just-noticeable difference" (j.n.d.) to (x) the total stimulus present:

$$\Delta x/x = a.$$

A very sensitive organ like the eye has a small constant of about 1/100, whereas muscular sensations in the arm are grosser and have a larger constant of about 1/17. Intuitively the law says that the more stimulus is present, the greater the amount of stimulus necessary to produce a just noticeable difference.

What Fechner added was the idea that this ratio can be

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stimulus still detectible by the subject. When x=b, for example, when the stimulus equals the threshold stimulus, in the above equation, it reduces to log(x/x) or log 1, which equals zero.

[17] Wilhelm Wundt Vorlesungen über Menschen und Tierseele, 1863, translated in William James' *Principles of Psychology* (New York: Encyclopedia Britannica, 1952), Chapter 13.

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interpreted as a direct measure of the sensation's intensity (y). For example, when the sensation intensity increases by one j.n.d. the stimulus needed increases by the Weber ratio for that sense organ. And as the amount present increases, the stimulus needed to get to the next j.n.d. increases in geometric ratio.

The correlation hangs on the idea that "adding-up" a number of same sized units of j.n.d. arithmetically yields up the total sensation intensity. But we don't know that the j.n.d.s are all the same sized unit. It is often argued that a scaling mistake is made in arguing from Weber's law to Fechner's formula. William James[18] (who took a dim view of Fechner's work) argued that the j.n.d. was not even a sensation-unit but a complex *judgment* that a sensation has occurred, and it has since been shown that the j.n.d. is indeed sensitive to "higher" processes of attentiveness, distraction and memory.

In 1865 Mach officially retracted his Vorträge defense of Fechner's law. In the interim the German physiologist Fick had demonstrated that a sensation's strength was directly proportional to the electrical stimulus induced in the nerve.[19] Mach continued to rate Weber's law as genuine, but

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[18] *Principles of Psychology*, Chapter 8 "Discrimination and Comparison."

[19] "Über die physiologische Wirkung räumlich vertheilte Lichtreize (Vierte Abhandlung)" *SW* **57** (1868), translated (MB), p. 299.

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limited, and he thought he could deduce it from his own law of neural inhibition. If nerve energies inhibit one another, only activity that originally stands out above the average of surrounding activity will remain; the rest will be leveled off by inhibition. Thus the eye senses only ratios of contrast *above* the average intensities. This indicates the sense organ's evolutionary adaptation to react to *ratios* of intensity rather than the absolute strengths:

> The remark, for example, that a visible object under varying intensity or illumination can be recognized as the same only when the sensation excited depends on the ratio of the illumination-intensities of object and surroundings, makes intelligible a whole train of organic properties of the eye. In this way we understand how the organism, in the interest of its survival, was obliged to adjust itself to feel the ratios of light-intensity. The so-called law of Weber, or the fundamental psychophysical formula of Fechner, thus appears not as something fundamental, but the explicable result of organic adjustments. The belief in the universal validity of the law is naturally herewith relinquished. I have given the arguments on this point in various papers. (*SW* **52** (1865); *Vierteljahresschrift* 1868; *SW* **57** (1868)).[20]

In his 1872 lectures "Über einige Hauptfragen der Physik" Mach also challenged Fechner's view that sensations were "sums" of unit j.n.d.s. Here he says that different intensities of sensation are actually different qualities, and as heterogeneous magnitudes cannot be added together:

> The sensations are different each according to its locality and therefore heterogeneous. Heterogeneous [magnitudes] are not to be added together...when the sensation goes over from gray to white, we do not sense more strongly, but rather completely differently. There are not homogeneous elements that compose a stronger

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[20] (AS), p. 81n.

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here, rather completely heterogeneous ones. These cannot be added.
It will not do to analyze an intensive sensation into such sensations that can be numbered. Thus Fechner's way of measuring intensity has no sense psychologically.
Measure and number too is only an exact description, which is inapplicable to the psychical. Here things must be described in an essentially different way. Thus the measure of sensations appears to me as something fairly illusory.[21]

Mach said later in the *Analysis* (a continuation of the second from the last quotation) that Fechner's technique is only a conventional proportion between sensations on one side and stimuli on the other:

> Strictly speaking I consider the expression "proportionality" also to be inappropriate, since there can be no question of an actual measurement of the sensations; all that can be done is to characterize them by numerical means. Compare what I have said about states of temperature (Prinzipien der Wärmelehre p. 56).[22]

It seems to me Mach was applying Fourier's principle to Fechner's law. While stimuli admit of ordinal scaling, sums

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[21] Die Empfindungen sind verschieden je nach der Stelle und daher ungleichartig. Ungleichartiges addirt man nicht...wenn die Empfindung von Grau zu Weiss übergeht, so empfinden wir nicht einfach stärker, sondern überhaupt anders. Es sind nicht gleichartige Elemente, die sich hier zur stärkern Empfindung zusammensetzen, sondern überhaupt ungleichartige. Solche addirt man aber nicht. Eine intensive Empfindung in solche abzählbare Empfindungen zu zerlegen geht nicht an.
Daher hat die Intensitätsmessung in der Fechner'schen Weise keinen psychologischen Sinn.
Das Messen und Abzählen is auch nur eine genaue Beschreibung, die sich aber auf das Psychische nicht anwenden lässt. Man muss also hier wohl wesentlich anders beschreiben. Und somit scheint mir das Messen der Empfindung ziemlich illusorisch. (NL 174/1/003)

[22] (AS), p. 81n.

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and differences, sensations do not. A seemingly more intense sensation does *not* include every smaller. Moreover Mach calls attention to the arbitrariness of setting equal units of j.n.d. against the log of equal stimulus units. In his *Theory of Heat* he asserts that the proportionality between temperature and changes of volume (in a thermometer for example) is of the same conventional nature.

> Concomitantly with the thermal sensation which a body provokes in us, other properties of the body also undergo alteration--, as for example, its electric resistance, its dielectric constant, its thermoelectric motive force, its index of refraction, etc. And not only might these properties be employed as indices of the thermal state, but they actually have found such employment. In the preferment of volume therefore as a test of states of heat, there is involved...a certain caprice; and in the general adoption of this choice a convention.[23]

Mach argues further that should some collection of say *three* mutually independent properties be chosen to represent the thermal state of a body, the state can be represented as a three-fold manifoldness (a vector) and not a simple manifold of numbers:

> The properties of the system of symbols we employ are not decisive of the properties of the states symbolized. If we were to take, for example, as our criterion of the state of a body K the pull exerted by K on an iron ball suspended from a balance, these pulls, the aggregate of which as symbols constitute a simple manifold, could be determined indifferently by the electric, magnetic and gravitational properties of K, and would be the symbolic correspondent consequently of a *threefold* manifold. Inquiry must determine in each case whether the symbolic system chosen is the appropriate one.[24]

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[23] (H), pp. 50-51

[24] (H), pp. 50-51.

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When Mach actually analyzed sensations of color and tone he held to the oppositional theory of Herbart rather than the j.n.d. of Fechner. An intermediary color between red and green is not obtained by "adding up" units of red, but is an opposition between "ungleichartige" elements, green and red, where green is not so much the absence of red but a different "opposing" sensation process.[25] The mixture of proportions of the red process to the green process gives a "numerical" catalogue of these intermediary states, all of which are *different* in quality, but none greater than another. In Mach's theory of tone, the same framework is proposed. He observes that every tone is a qualitatively different sensation and yet they do form a series in which we know which are higher and which lower. But it is absurd to think a middle C should "contain" all of the notes below it, rather he hypothesizes that the ordering comes from a proportion in every tone of just two opposing qualities which he calls Hell (Bright) and Dumpf (Dull).[26] Mach was not against the classification of sensations per se, but only insisted that the qualitative dimensional analysis precede the quantitative manipulation of symbols. And I must say the dimensions of Machian sensations looked much more like Herbart's than they did like Fechner's. This disagreement and others were soon to come to a head between the two men.

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[25] (AS), pp. 67-68.

[26] (AS), pp. 275-276.

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Mach and Fechner on "Psychophysical Relations"

By 1864 Mach had been rescued from the hardships of his Privatdozent work by an appointment as Professor of Mathematics at the "somewhat neglected" University of Graz. He used the time to recuperate, extend his still "threadbare" mathematical education, publish and conduct research in psychophysics and give lectures on the subject (1864/65), a follow up to his Vienna lectures.[27]

At Graz, Mach was in correspondence and personal contact with Fechner, who refused the dedication of what would be the *Analysis of Sensations* twenty years later. Fechner wrote Mach on 18 April 1864, graciously, that he would rather be recognized as the impetus that sent Mach on his own "self-standing way." He could not endorse this way personally:

> I can only take as an honor your intention to dedicate to me your soon to appear book on psychophysical relations and I sincerely take pleasure over this sign of recognition of beginnings, in which respect I too have moved on. You might perhaps think it over whether the self-standing path of inquiry you are travelling might not take you further from these beginnings than you might find convenient to acknowledge later, at least to the degree you would express the acknowledgement. In the meantime it can only be a stimulus that you need to acknowledge, and from this point of view I would have the least reservations of accepting a dedication, which you have so kindly offered.[28]

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[27] (DL), pp. 24-25.

[28] Fechner to Mach (WK):
Ihre Absicht, mir eine von Ihnen demnächst herauszugebende Schrift über psychophysiche Verhältnisse zu widmen, kann ich mir nur zu Ehre rechnen, und ich habe mich aufrichtig über dieß Zeichnen einer Anerkennung der

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Mach confided at seventy-three that Fechner's rejection still pursued him "in his dreams."[29] Their disagreement was over Mach's Herbartianism and Mach's new philosophical position, which Fechner elsewhere calls an "abweichender philosophischer Standpunkt."[30] Joachim Thiele speculated that the early draft of the *Analysis* would have resembled the final metaphysical section of the "Vorträge" where Mach discusses Fechner's view and then his own theory.[31]

Fechner's Inner Side of Nature

Fechner believed that mental and physical were two aspects of a single underlying substance, a *tertium quid* that had both inner "mental" attributes and outer "physical" attributes. We have an "interior" view of our own thoughts, feelings and intentions, and others observing the brain from

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> Anfänge, in denen ich mich selbst in dieser Hinsicht bewegt habe, gefreut. Mögen Sie aber nur recht überlegen, ob der selbstständige Forschungsweg, den Sie einschlagen, Sie nicht weiter von der Richtung dieser Anfänge abführen wird, als daß Ihnen die Anerkennung derselben, die Sie damit noch in gewissem Grade aussprechen würden, künftig noch genehm sein könnte. Inzwischen kann es nur eine Anregung sein, die Sie anzuerkennen brauchen, und aus diesem Gesichtspuncte würde ich am wenigsten Bedenken tragen, eine Widwung anzunehmen, die Sie mir mit so viel Freundlichkeit darbieten.

[29] Michael Heidelberger, *Die Innere Seite der Natur* (Frankfurt a.M.: Vittorio Klostermann, 1993), p. 208.

[30] Fechner to Mach December 11 1865 (WK), p. 44.

[31] (WK), p. 41.

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the outside have an "exterior" view of the physio-chemical processes, although even the external aspects of the observed brain are internal to the observers.[32] According to Fechner, we may break up all of these different observations, both interior and exterior types and pronounce them all phenomenal modes of appearance functionally related to one another and comprising a single underlying substance.

The reason why mind and body seem mutually exclusive, Fechner explained, was that substance can appear to itself in its inner mental aspect or appear to others in its physical aspect, but never both at once. The subject has his own idea of green and the observer has his own idea of the brain processes he observes. These cannot be experienced simultaneously, just as one cannot experience both aspects of "convexity" and "concavity" by standing inside a circle and outside it at the same time:

> At the moment when we are conscious of our feelings and thoughts, we are unable to perceive the activity of the brain that is associated with them and with which they are in turn associated--the material side is then hidden by the mental. Similarly although we are able to examine the bodies of other people, animals and the whole of nature directly in anatomical, physiological, physical and chemical terms, we are not able to know anything directly about the minds that belong to the former...for the spiritual side is here hidden by the material...And what can be the reason for this singular condition in which body and mind can be observed each for itself but never together, in spite of the fact that they belong to each other? Usually we can best observe things which belong together when they can occur together. The

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[32] A second qualification: even the exterior view is the observer's "interior" view of his own sensations of the working brain.

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> inviolability of this aspect of the relationship between the mental and material worlds makes us suspect that it is rooted in their basic natures.[33]

Heidelberger has very convincingly argued that Fechner did not in fact believe in a further underlying Spinozist *tertium quid* over and above the functional relation of the aspects, as was often supposed.[34] Rather an object for Fechner was the result of linking together all of its phenomenal aspects (interior and exterior) by functions. Apart from the functions there is no further substance necessary. This single object is half composed of mental aspects, the other half of physical aspects; just as a penny is a functional unity of a circular image, an ellipse, a rectangle, and so forth.

The "interior aspect" of all natural processes also explains why all aspects are mutually exclusive, even when they are inextricably united in substances.[35] Fechner's sophistical cleverness is admirable in showing this: invincible duality proves identity, since nothing would be so exclusive in its aspects unless they were interior and exterior aspects of the same thing; and identity demands

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[33] G.T. Fechner. *Elements of Psychophysics*. Helmut A. Adler trans. (NY: Holt, Reinhart and Winston, 1965), pp. 1-2.

[34] Michael Heidelberger *Die Innere Seite der Natur*...(Frankfurt a.M.: Vittorio Klostermann, 1993), pp. 123-142: Fechner's criterion for identity is the functional variability of the thing's inner "mental" aspects with its outer "physical" aspects.

[35] *Elements of Psychophysics*, p. 4.

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duality because nothing can be both itself and observe itself at the same time from the outside.

As the title of Heidelberger's book aptly illustrates, Fechner was a panpsychist, and something of a spiritualist, attributing an "inner side" to plants and inanimate matter--although presumably this was radically different from human experience. The inner side is only accessible *für sich*, by being the thing itself, and all natural things (not just minds) have this privileged access to themselves that is not granted to anything else.

Mach seems to have adopted many such views directly from Fechner; for example he too attacked the "simplistic" theory that identifies a particular green color with a particularly correlated brain-process:

> It cannot be said that thoughts are physical processes in the brain. When we investigate the processes in the brain more precisely, we find molecular motions but no thoughts. The electric currents in the brain are rather something essentially different from the idea of a green color, which perhaps only the investigated subject has. It is more likely *our* ideas that we find in investigating the brain and not those of the investigated brain.[36]

Mach also declared in the "Vorträge" that sensations were the interior natures of nerve processes, while their chemical electrolytes and electrical pulses represented external relations.[37]

Mach hoped until the end of his career to account for

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[36] (V), p. 363.

[37] (V), p. 335.

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differences in sensation quality by finding different chemical electrolytes carrying the electrical impulse, but it was not so. The sensory nerves do not seem to differ enough chemically to account for differences in the quality of sensations. The electrolytes are sodium and potassium ions all over. There is not even much of a difference between the nerve cells of different species. However the facts do cut the other way, too: quality does depend closely on the specific nerve activated, and *not* for example on the abstract intensity or frequency of the pulse it carries, which E.D. Adrian first confirmed in 1928.[38] It is still not known what it is about the *specific* locality in the brain, or the connections to local patterns of cells, that induces specific sensation qualities, but it is almost certain that the same abstract pattern of firings in *different* cells would occasion a different set of sensations. Thus there may be something to internality after all.

Specific Energies

Both Mach's and Fechner's notion of *Innerlichkeit* derived from the physiologically respectable doctrine of *specific energies* proposed first by Sir Charles Bell and then by Johannes Müller.[39] According to Edwin Boring, the doctrine

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[38] *The Basis of Sensation* (London: Christophers, 1928).

[39] Charles Bell, *Idea of a New Anatomy of the Brain* (monograph privately circulated, 1811); Johannes Müller, *Zur vergleichenden Physiologie des Gesichtssinnes*, 1826.

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states that we are aware not of external objects, but of the states of our nerves themselves. An object is not even needed to cause a sensation for we can cause colors and sounds by interfering with the nerves themselves, pressing the optic nerve produces sensations of light, pressing the auditory nerve sensations of sound, and so forth.[40]

But there is also a more substantive thesis to be found in Müller's formulation, which is that sensation qualities are what physical nerve energies really look like on the inside.[41] Mach wrote in a notebook entry of 1896, for probable inclusion in a new edition of the *Analysis of Sensations*,[42] that he considered himself and Ewald Hering the only two genuine heirs of Müller:

> I believe now that the thought expressed here, according to which just so many different kinds of physio-chemical nerve processes are suspected as there are sensation qualities to be distinguished, likewise has heuristic value, and that this thought can hope for some support from the physiological-chemical side. This thought which is merely a logical consequence of a monistic conception of Müller's principle of specific energies, corresponds little indeed to modern ideas. Following the lead of a great authority, one has gotten used to shoving away the explanation of different sensation qualities into as yet unknown regions, and regarding all nerve-processes as completely homogeneous in quality and only different in quantity. Already in 1863 (Lectures on Psychophysics) I had occasion to show how little this conception can lead to a deeper understanding of our sensations, and how little justified it is to hold that all nerve currents are qualitatively homogeneous physical processes. If I

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[40] Boring, *A History of Experimental Psychology*, p. 82.

[41] (V), p. 335.

[42] Perhaps the English edition of 1897? The passage did not see print.

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am not mistaken, only Hering still upholds Müller's original teaching.[43]

The "great authority" Mach speaks of above is, of course, Hermann von Helmholtz,[44] who as we have already seen wanted to reduce sensations qualities to quality-less matter. Helmholtz had extended Müller's specific energies principle, without the *Innerlichkeit* of course, by localizing particular sensations not only to the same sensory nerve but even to different fibers within the nerve.[45] Helmholtz associated a different sensation of tone to each of the 2,500 different fibers in the auditory nerve, or rather each row of them. Likewise the

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[43] Haller und Stadler eds. *Ernst Mach: Werk und Wirkung* (Wien: Hölder-Pichler-Tempsky, 1988), p. 190:
Ich glaube nun daß der hier ausgesprochene Gedanke, nach welchem so vielerlei physikalisch-chemische Nervenprocezesse zu vermuten sind, als man Empfindungsqualitäten zu unterscheiden vermag, ebenfalls heuristischen Wert hat, und das derselbe hoffen kann, einmal von physiologisch-chemischer Seite gefördert zu werden. Dieser Gedanke, welcher sich durch eine bloße konsequente monistische Auffassung des Müllerschen Prinzips der spezifischen Energien ergibt, entspricht ja wenig den modernen Vorstellungen. Man hat sich vielmehr unter Führung einer großen Autorität gewöhnt, die Erklärung verschiedener Empfindungsqualitäten in noch unbekannte Gebiete hinausschiebend, all Nervenprozesse als qualitativ durchaus gleichartig und nur quantitative verschieden anzusehen. Schon 1863 (Vortrag über Psychophysik Wien Sommersemester) hatte ich anlaß darzulegen, wie wenig diese Auffassung zu einem tieferen Verständnis unserer Empfindungen führen kann, und wie wenig es berechtigt ist, alle Nervenströme für qualitativ gleichartige physikalische Prozesse zu halten. Wenn ich nicht irre, hält nur Hering allein die ursprüngliche Müllersche Lehre noch aufrecht.

[44] (V), p. 335.

[45] (V), pp. 295-298; 316-318.

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three primary colors (red, green and blue) of the Young-Helmholtz theory of color correspond to three types of fibre postulated to exist within the optic nerve. Helmholtz broke with Müller however by claiming that the different sensation qualities were reducible to quantitative differences in the intensity and frequency of pulses carried by each nerve. The theory came to be known as the "telegraph-wire" model according to which the sensory nerves were all of a kind and only conduits for electrical vis viva that varies in signal, or in the number and type of connections between cells. The quantitative aspects of the message moving through the nerve is what matters, and not the internal properties or locality of the nerves themselves.

What worried Mach was Helmholtz's outright reduction of qualitative differences to quantitative ones, since he believed that quality logically preceded quantity. Müller's theory seemed to Mach to dictate that where there was a difference in the quality of sensation, there had to be some difference in the physical energy of the nerve.

> We find in the nerves only electrical currents, which can be thought of quite easily as being of different intensity, but not of different quality. The nerves may accordingly be similar in their role to telegraph wires, which release only certain signals corresponding to sensations...That for different sensations different nerves correspond is not to be disputed. However, what if the electrical processes in the nerves should be too simple to suffice as an explanation? Should it then be necessary to push explanation back into unfamiliar regions? How then when after more precise and thorough research of the whole brain, only electrical currents are found everywhere?
>
> My humble opinion is this. The electrical

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> investigations of the nerves are certainly of a very fine sort. In certain respects however they are also very crude. An electrical current of a given intensity tells us nothing, other than that a certain quantity of *vis viva* passes through a cross section of the current in unit time. Which processes and molecular motions convey the vis viva, we don't know. The most diverse processes could underlie the self-same intensity of current. Ludwig once spoke to this point directly: when the chemical elements of a nerve are given us, we will know nothing more of their constitution than we know of a telescope when someone tells us that it consists of glass, metal and lamp-black. The electrical current, too, shows us merely the outside of nerve processes.[46]

Mach favored a much closer examination into the interior nature of physical processes, even unto the sub-atomic level. The external, electrical side of these nerve processes is the side that is accessible through physical interference with other phenomena and other observers. Following out this thought, one might say that every natural event, in addition to its quantitative external aspects, also possesses a internal *monadic* aspect. Monadic aspects or properties do not translate into external relations, and cannot be directly represented by them. The external aspect, on the other hand, is a stable pattern of interference that ranges broadly across natural processes, giving them a uniform exterior. As we know, Mach interpreted spatial properties and motions as belonging to the set of external relations, including beats of electrical current moving through different media.

Mach declared these views clearly in a somewhat later lecture "Über einige Hauptfragen der Physik" (1870-71). Here

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[46] (V), p. 335.

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he stated that all matter possessed a monadic aspect similar to that of sensations:

> Sensation is a general property of matter, more general than motion. Let us seek to set down this proposition clearly. An organism is a system of molecules. Electrical currents run into the interior and come back again into the muscles. Everything is physically explicable. But not that the person should have sensations. What we can investigate physically is always merely physical. We find no sensation. And yet the human being senses. The material flows forward through and through him. The old depart. The new comes in. We therefore have the problem of finding something fundamentally new in the whole that is not in the parts. We escape this difficulty when we consider sensation as a general property of matter.
>
> We have therefore two sides, an inner and an outer side. According to analogy we assume these also in animals, though on weaker evidence. In plants still weaker, but the boundary blurs. With respect to the inorganic again the boundary blurs. Therefore no doubt everywhere an inner side. One can only be convinced of it from one's own soul. The rest is only conjectured.
>
> What scientific value this assumption of a general sensation of matter has, this can only be decided by how much better we can deduce and understand physical phenomena through them. Rules for deduction of our sensations with the help of other sensations added in thought and in causal relation to them. Thus, as the most immediate final goal of science the construction of the world out of sensations. The appearance of matter, in so far as it is sensation, built up similar to the way physics has built up out of material added in thought (the atom).[47]

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[47] In Haller und Stadler *Werk und Wirkung* p. 173:
Die Empfindung ist eine allgemeine Eigenschaft der Materie, allgemeiner als die Beweglichkeit. Suchen wir diesen Satz klar zu stellen. Ein Organismus ist ein System von Molekülen. Elektrische Ströme laufen hinein von aussen angeregt und kommen wieder zurück in die Muskel. Alles lässt sich physikalisch erklären. Nur daß diser Mensch empfindet, nicht. Was wir physikalisch erforschen können, ist immer nur physikalisch. Wir finden keine Empfindung. Nun Empfindet aber der Mensch. Der Stoff strömt fort durch ihn durch. Alter tritt aus. Neuer tritt ein. Wir haben also die Schwierigkeit, etwas

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In case there was any doubt, Mach indicated that these world-elements "sensations in matter" were not just *our* sensations considered in their physical variations, but new sensations not given to us but added in thought (*hinzugedacht*):

> Why do we assume a soul outside of our own?...This assumption: to certain of our own, to add in thought certain others, which make more comprehensible the triggering of our own...Animals and plants. Still more of such sensations which we add in thought to make others comprehensible. Finally we add them in thought everywhere.[48]

Further the sensations of matter are not *Dinge an sich*, but

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fundemental Neues in einer Gruppierung zu haben, was in den Theilen nicht ist. Dieser Schwierigkeit entgehen wir, wenn wir die Empfindung als eine allgemeine Eigenschaft der Materie betrachten.
Wir haben also zwei Seiten, eine Innen- und eine Außenseite. Nach der Analogie nehmen wir diese auch bei den Tieren an, wenngleich schon schwächere Gründe. Bei den Pflanzen noch schwächer, aber die Grenze verwischt. Gegen das Anorganische wieder verwischt. Also wohl überall eine Innenseite. Überzeugen kann man sich nur von seiner eigenen Seele. Die übrigen erschließt mann...Welchen wissenschaftlichen Wert diese Annahme einer allgemeinen Empfindung der Materie hat, das kann nur dadurch entschieden werden, ob wir physikalische Erscheinung durch dieselbe besser ableiten und verstehen können. Ableitungsregeln für unsere Empfindungeg mit Hilfe anderer hinzugedachter und im Kausalzusammenhang gedachter Empfindungen. Also Aufbau der Welt aus Empfindungen als des Unmittelbarsten, als letztes Ziel der Wissenschaft. Ähnlich wie die Physik die Erscheinungen der Materie, soweit sie empfunden, aus hinzugedachter Materie (Atom) aufbaut.

[48] Warum machen wir die Annahme einer Seele ausser der unsrigen?...Diese Annahme: zu gewissen eigenen gewisse andere Empfindungen hinzudenken, die die Erregung der unsrigen leichter begreiflich machen...Die Thiere und Pflanzen . Immer mehr solche Empfindungen mit welcher wir andere zum Verständnis hinzudenken. Zuletzt können wir sie bei allem hinzudenken. Ibid.

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stand in causal relations to our own.

We have already seen how Mach's new physical theory of matter attributed an interior of pressures and qualities to atoms similar to sensations.[49] Human sensations he says are only attributable to atoms in large groupings, and thus there can be no question of thinking a sensation of color or sound into an atom.[50] Rather what seems to exist there are extremely tiny, internal elements, which perhaps in *very* complex cells might superimpose internally to a whole sensation. The internal qualities in the atoms (similar to Leibniz's petites perceptions) were probably the part of Mach's "selbstständiger Weg," that Fechner rejected.

As we now know, central sensation energies and atomic-scale energies though comparable in kind are of vastly different orders of magnitude, and Mach was certainly wrong that different sensory nerves must contain different electrolytes.[51] But he was right about the localization of sensations to very specific structures. Perhaps he could have thought up some kind of mechanism to explain the quality of sensations by their localization. Such a mechanism of emerging properties was suggested in his later analysis of tone sensations, where he postulated small "supplementary

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[49] (V), p. 364.

[50] (V), p. 364.

[51] (AS), pp. 370-371.

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colorings" to tone sensations, called Z.[52] These are indistinguishable in single tones, but become conspicuous in combinations where they add up "just as the contrasts of faintly colored, almost white lights become vivid when such lights are brought together."[53] Mach was also aware, through his work on neural inhibition, that the internal states of single cells could be affected by the complex environment of neighboring cells. In his standard formula: no process in nature ever occurs in isolation from others. He probably also held that the local environment of a particular nerve process could conversely "drive its way inward" to alter the states of each individual in the group.

In any case, the small, practically infinitesimal, world elements, which may seem the most unrealistic feature of his "vain" monadology were not among the aspects that Mach changed in later work. The only features he did alter were the existence of the monads or atoms, and his tendency to think sensation-elements spatially "into" atoms or the cross-section of nerve-processes. It thus seems that a different type of causal relation or nexus of elements is responsible for the relations of world-elements to sensation elements, perhaps a many-to-one relation. When he spoke again about the psycho-physical relation all remaining spatial metaphors had been purged from his thinking and replaced by functions. This

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[52] "Zur Analyse der Tonempfindungen" *SW* **92** (December 1885).

[53] (AS), p. 295.

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followed an intense engagement with the psychology of space in the 1860s.

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Chapter Five: The Influence of German Psychology on Mach's Theory of Space

The psychology and physics of space were two of Mach's ruling passions. The Vorträge show him very conversant with theories of spatial representation in psychology, which by the 1860s had received much effort from German psychologists and philosophers. Some like Herbart had developed constructions of space from associations and dissociations of unextended content. This empiricist assumption was overturned in psychology, as nativism became ascendent, but the purely conceptual analysis of space in terms of the unextended always remained relevant to Mach, since he drew on these ideas for his physics.

Herbart's Spatial Reproduction Series

Mach reviewed three major theories in the Vorträge: Herbart's reproduction series, Lotze's local signs and Wundt's complex local sign theory.[1]

We have seen how Herbart separated psychological space-representation from his intelligible space of geometry and physics. In psychology, Herbart proposed that the (monocular) visual field becomes extended through experience, motion and the addition of memory images. Space becomes a phenomenon of the serial reproduction of past experiences by memory, which

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[1] (V), pp. 337-354.

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Herbart called a reproduction series.

As we know, for Herbart impressions are preserved in memory as ideas (*Vorstellungen*), until they are inhibited. However, supposing an impression recurs and is recognized as the same, it will enliven the memory image and any other ideas inhibited by the memory. The problem for the theory of inhibition is to explain why if A,B,C have occurred in that order, and A recurs, B and C will be remembered as following A. When we remember a boat going downstream we never remember it going backwards even though by rights the later impressions C and then B should be more recent and forcible than the proper order of images in the series, B and then C.

Herbart had an explanation. He said that when we recall the series in memory, we sense B as coming after A because B inhibits and becomes associated with A's slowly fading after-image in memory. Then when C comes along, it inhibits both B's after-image and whatever remains of A's after-image *annexed* to B. In this way a time-order of impressions is encoded in the after-images as their order of inhibition.[2]

Herbart conceived of inhibited ideas, like A and its after images A' and A'' as if they were floating blocks of wood, pressed down on top of each other under water. If you lift one up, the one underneath inhibited by it bobs up too. An idea becomes annexed to a second by pushing the second under the surface. For example, if A recurs, the impression

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[2] (V), p. 337; *Psychologie als Wissenschaft* §§ 109-116.

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of the present A will immediately call up the memory image A' and *then* B's image which presses down on A' through inhibition, and finally C' annexed both to B' and to the second after-image A''. The reawakening of ideas in sequence is like recalling the stanzas of a poem, only the first stanza can help you remember the second; the third is no help.

In the Vorträge, the immediate application of the reproduction series--a general psychological phenomenon--was to visual impressions and ideas. Mach explained:

> If the eye has run through the series a,b,c several times and in the opposite direction, the stimulation of one member of the series will have the consequence of stimulating the running through of the series in both directions from this member. From this member on, the members to each side of it will appear simultaneously, with descending clarity. If we consider that from an 'a' the eye can run through not just one but infinitely many series in different directions, and if we apply the law of reproduction in series, we attain something at least very similar to spatial perception.[3]

Herbart's space, like Kant's, is largely made up of memory images added to the present impression. Taking in and retaining a space is like exploring an Egyptian tomb with a torch and a notebook, receiving impressions one at a time, recording the order of parts in the notebook that are no longer present, and inferring the lay-out of the whole. In truth, every new pass of the eye is an experience of new points. When we recognize the particulars again they will not be exactly the same; they need only be similar enough to our memories to be recognized as repetitions of the previous

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[3] (V), p. 337.

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arrangement.

From Time Order to Extension

Because these ideas will be used later, there is one important fact to be pointed out that is only hinted at in Mach's sketch and which I will now elucidate for clarity. A linear reproduction series is just an ordering of A,B,C by means of their after-images. The trouble is the spaces between. It is true that the after-images of A continue to endure through the appearance of B, but this does not account for a true space in which A is shading off as B is arising. Memory images do not fill in these gaps to give a true extension. The transition is still abrupt and immediate.

In fact Herbart was sensitive to this problem. He said that only *two* overlapping and independent reproduction series can generate an extension, as opposed to a mere ordering, by setting points apart from one another.[4] Everyone can see intuitively what is meant by looking down the length of a thin wire. Head-on it looks like a point. Only when it is turned through an independent, dissociating, direction does it appears as a line. Herbart's psychological mechanisms of association and dissociation mirror exactly the causal dependencies and independencies of intelligible space. Two linear reproduction series crossed over one another in a double row produce the first linear extension in psychology.

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[4] *Psychologie als Wissenschaft* §§ 109-116.

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A better metaphor for a simple linear reproduction series like A,B,C above is an extensionless point that changes qualitatively in hue, or if you like in pitch. With the help of memory we can tell which qualities have come up in what order. Once we get used to the pattern, we may even come to recognize the point as travelling up and down a range of qualities. Once again the transitions are immediate without spaces in between. However, by superimposing on this very point an *independent* set of qualities, as it were another linear series sitting perpendicular to the first, the original point's progress may be seen as tracing-out an extended line.

The use of a double row to mark extension is an absolutely essential feature of Herbart's theory of space, and it seems to have influenced Riemann, who said in his *Probevorlesung* that extension first becomes possible when *two* independent modes of determination of the same concept cross-over one another.

The idea is so important for later work that it is important to clarify it further. For the construction to work we must imagine that every quality covered by the point in the original series is also a member of another series, intersecting it at that point (and dependent on it there) but also running perpendicular in a direction that is *not* dependent on the progress of A,B,C. For simplicity, and to stick with Herbart's idea of a spatial visual field, imagine *four* reproduction series, each point changing in two

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directions through its own range of qualities. Series (1) will run straight up and down the page, while (2), (3) and (4) run perpendicular to it left and right.[5]

(1)

(2) ... **A** B′ C′...

(3) ... A′′ **B** C′′...

(4) ... A′′′ B′′′ **C** ...

The first series (1) **A**,**B**,**C** includes the hues of our original qualitatively-changing extensionless point. The hues of the second series **A**, B′, C′ are associated with the first series only at **A**, and are otherwise independent in their changes. The third series meets the first only at **B** and is otherwise independent, and the same for the fourth at **C**.

Now for the individual associations. **A** is associated with **B** (and through **B** mediately with **C**), but **A** also calls up images of B′ and C′ and it is to this extent causally *dissociated* from **B** and **C**. Likewise **B** is associated with **A** and **C**, but *dissociated* from them by its connections to A′′ and B′′. Finally **C** is associated with **B** and mediately with **A**, but *dissociated* from them by its associations with A′′′ and B′′′.

If we set the machine working and think of all these series running at once, then in passing from hues **A** to **B** to **C**, **A** is both associated with **B** and dissociated from it, while **B**

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[5] For absolute accuracy, each of the dissociating series must be perpendicular to one another, which is impossible to represent on the page, but may be readily imagined.

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is associated with **C** and dissociated from it. We may think of these independent dissociations as "gaps" of extension between the members setting **A** apart from **B** apart from **C**. If we take any one of the series in fact, its members will be set apart and extended against the background of laterally intervening series.

Herbart's application of this model to the eye is somewhat fanciful. We would have to imagine that the eye had a practically infinite number of elements arranged in rows and an infinite number of reproduction series running in independent directions through each element at all times. This would require an enormous memory, even if unconscious, and thus Herbart's ideas seem much more interesting mathematically than they are psychologically.

In the Vorträge Mach was favorably disposed towards the reproduction series, which I don't think he or other German psychologists really ever gave up, but Mach thought they failed to explain why every sense is not spatial, since sounds, tastes and smells could conceivably give rise to similar series.[6] This shortcoming reflects Herbart's armchair philosophic contempt for physiology and his disposition to keep his analysis strictly phenomenal. But as Mach perceived with the help of his doctrine of psychophysical parallelism, the organ of sense must also have something to do with the phenomenal structure of what is sensed there. To this end

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[6] (V), p. 338.

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Mach turned his discussion to Lotze's theory of local signs.[7]

Lotze's Local-Signs

Lotze observed that when different parts of an organ like the skin are stimulated, in addition to the quality of the sensation felt there (heat, cold, pain, itching), the mind also receives a distinctively different local feeling. We learn from experience that the same local sign may be associated with any of a range of qualities, and that more or less any quality can be associated with each of the different local signs. (This is not precisely true since quality and local sign are always slightly dependent.)

According to Lotze, local signs were unextended feelings different only in their quality. As such they formed a mosaic of partitioned sensations but not a connected manifold of points. A mere collection has no dimensions, distances or directions from one point to another.[8] Literally, as William James remarked, local signs are places without space.

Local signs and qualities always occur together; every quality has a locality and every locality is sensed along with a quality that occurs there. Without a specific psychological organization to dissociate them, they would remain fused and

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[7] (V), p. 353.

[8] (V), p. 353: If for example two sensations that arise from two different spots on the skin meet together in the soul, spatial contiguity [Nebeneinander] is still not given.

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indistinguishable. Lotze believed that the mind was innately organized to construct spatial appearances by separating out this data. With repeated experience and recollection, it will be noticed (if not consciously then unconsciously) that the same quality Q is associated at one time or another with any of a series of local signs A,B,C. In memory we can dissociate the quality from the local sign, since the appearance of Q doesn't trigger the recollection of any of the A,B,C uniquely. In like fashion, a local sign A is dissociated from its qualities P,Q,R.

However qualities always strongly recall past qualities associated with them, and local signs strongly recall past local signs. So even if a quality Q changes its local sign the mind interprets this as an identical quality in each new location, moving against an independent background A,B,C. The other, completely equivalent, relative motion is that Q be sensed as fixed and A,B,C, (affixed to the body of the creature) be sensed as in motion. Mach pointed out in the *Analysis* twenty years later that it is very easy to get into confusion about which case is which, if there are no peripheral sensations to tell us whether our bodies are in motion or not. If we stand on a bridge and stare fixedly at the water below, soon the water is seen at rest and we and the bridge are felt to move.[9]

When motion does occur, instead of a mosaic of fused

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[9] (AS), p. 142.

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quality-and-local sign tiles together we sense groupings of qualities detached and moving against a fixed background of local signs on our bodies. The qualities that move together are sensed as a connected object, not a mosaic of little separable dots, and the background is sensed as a region of connected points. Both series first become connected manifolds by motion across the other.

The explanation draws on the reproduction series. Say that qualities, not yet connected, A,B,C,D,E, fall on three local signs K,L,M and move from left to right. At first, in (1), A is associated with K and they will recall one another on their reappearance; as B will recall L, and C, M.

(1)	(2)	(3)
ABC	BCD	CDE
KLM	KLM	KLM

Thus in (2), K reappears and recalls A, but now *with* K's present partner B. In Herbartian terms the fading after-image of A on the spot K becomes inhibited by the new B. Similarly, L recalls B's after-image, but now conjoined with L's new partner C, while M recalls C's after-image and conjoins it with D. A,B,C thus come into association with one another through their after-images, while gradually dissociating from K,L,M in the same motion. In (3) the process continues: K now recalls B's after-image with C, L recalls C's after-image with D, and M recalls D's after-image with E. A,B,C,D,E all end up associated with one another and connected, as they are

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progressively dissociated from KLM.

In the opposite direction, the qualities A,B,C,D,E also recall the local signs K,L,M and associate them with one another, while dissociating them from the quality row. If we add motion in the vertical, the fused mosaic has come to look like two connected surfaces sliding over each other, one invisible and fixed due to the arrangement of local signs on our bodies the other containing the world of colored and solid objects.

To account for the almost constant motion required by the theory, Lotze himself devoted special attention to the eye movements, especially a reflex that fixates all distal objects at the center of the visual field, as the mechanism for keeping the associative and dissociative processes going all the time.

Lotze's construction of the monocular visual field is thus more realistic than Herbart's because of the inclusion of physiologically-based local signs. In fact the orientation of German psychology was to start with general mathematical and metaphysical constructions (Lotze was himself a metaphysician-psychologist like Herbart) and to embody those constructions more and more concretely with the help of the latest physiological research. By the time of Mach, for example, philosophical reproduction series were increasingly seen as shorthand for unconscious nerve processes and their automatic after-images, and Vorstellungs- and Einbildungskraft were

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regarded as fainter central innervations.

Wundt's Theory of Spatial Vision

Wundt proceeded further in this realistic direction, modifying Lotze's local sign theory with further anatomical and psychological details. His theory of spatial vision is called the "complex local sign theory."[10] Like the others, Wundt held that visual space was a construction out of non-spatial elements, associated and dissociated by the eye-movements. Long after Mach and many other psychologists had become nativists Wundt wrote:

> Spatial ideas of sight cannot be considered original and immediately given...any more than can the spatial ideas of touch. The spatial order, here, too, is developed from the combination of certain sensational components, which, taken separately, have no spatial attributes.[11]

Wundt thought that Lotze had achieved half of the construction of monocular visual space by turning a fused mosaic into two connected manifolds of points sliding over each other. However Lotze's construction did not explain other spatial properties of distances traced between points and visual directions. We have with Lotze only a diffuse hang-togetherness of points, extension without metric properties:--more than a mosaic of places, but less than total spatial perception.

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[10] (V), pp. 353-354.

[11] Wilhelm Wundt *Outlines of Psychology* Translated by Chas. Hubbard Judd (New York: E. Stechert, 1897), p. 127.

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According to Wundt, the muscles of the eye accounted for the metrical properties of the visual field. He graded the muscular feelings from the center of the eye in direction and intensity. These feelings formed a second local-sign system associated with the retinal local signs. Whenever an image falls on a peripheral point of the visual field, a reflex moves the eye to fixate that point at the fovea, the center of the retina. In this way, Wundt explained, all of the local signs on the retina come to possess a direction, and an intensity of effort needed to fixate the given point at the center of the visual field. Through practice, these reflexes remain lively even when we try to stare dead-ahead. Direction and distance are thus interpreted, respectively, as the type and intensity of effort the ocular muscles *would* have to expend to fixate that point.

As Wundt said, and Mach repeated, the retinal mosaic and the system of the ocular muscles form a fixed register into which sensation qualities are entered with respect to place, direction and distances from one another. We bring this semi-innate spatial representation to all visual experiences, and are so used to it we think we see these properties fully formed solely by retinal images.

For support for his theory, Wundt called attention to a whole class of illusions of direction and length of lines due to the eye's musculature: for example because it is easier for the eyes to look down than up, vertical lines look longer,

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making true squares look like vertically stretched rectangles. Mach had already discovered some of the experimental proof in his 1861 article "Über das Sehen von Lagen und Winkeln durch die Bewegung des Auges."[12] As Wundt reported, patients with partly paralyzed eye muscles overestimate the distances to objects in the direction of the most difficult ocular movements.[13]

Wundt also had physiological evidence that distance in the visual field is not estimated on the surface of the retina. The retina itself looks like a mosaic of tiles, but unlike a real mosaic they are unevenly spaced: tightly packed at the fovea and spaced out more widely and irregularly toward the periphery. Yet we rate the distances between two spots the same, whether in the center of vision or the sides, even when the actual retinal arc between points is twice as great.[14] In the twentieth century it was discovered that the cells of area 17, where the visual field is mapped point by point onto the brain, are also unevenly distributed.[15] The center gets the majority of cells and the periphery less; yet this does not lead to a distortion of distance information. This does argue that distance is indeed a property

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[12] *SW* **43** (1861): 215-224.

[13] (V), p. 354.

[14] (AS), p. 127-128.

[15] H.B. Barlow and J.D. Mollon, eds., *The Senses* (Cambridge:1982) pp 29, 44-45.

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independently determined from locality. As we shall see in the next chapter, the psychological examples showed to Mach and his contemporaries that these properties were logically independent even in geometry.

Mach's Theory of Innate Psychological Space

Mach's based his mature theory of innate space sensations on Wundt's complex local sign theory, and wrote in the *Analysis of Sensations* twenty years later: "that space sensation is connected with motor processes has long since ceased to be disputed."[16] This was not exactly correct: Carl Stumpf and William James were two prominent dissenters.

Mach was so taken by Wundt's theory that he added to it, identifying the sensation of distance not just with the ocular motions, but with the mere will to move the eyes to a certain position. Now Mach was certainly no believer in a Kantian noumenal freedom; rather by willed eye motions, he simply meant that the otherwise involuntary tension of the eye muscles is caused by a central memory innervation. This memory image is of a previously performed involuntary motion that led to a desired result. Mach said quite bluntly: "the will to perform movements of the eyes, or the innervation to the act, is itself the space sensation."[17] Like Wundt, he had an experiment to support his view: if the eyes are fixed in

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[16] (AS), p. 127.

[17] (AS) p. 129.

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place with putty, and we merely will to turn them to the right, objects in the visual field suffer a strong rightward displacement, although the retinal images have not moved.[18]

Machian Sensations of Space and Relative Movement

Mach's use of the word "will" was deliberately provocative and brought many of his readers to the point of disbelief; for example Stumpf in his review of the first edition of [Contributions to the] *Analysis of Sensations* in 1886.[19] Mach introduced these central innervations of will to distinguish between cases of relative motion, where impressions move over our fixed bodies, and cases where the body moves against a seemingly fixed background of objects (as in the illusion of the bridge and the river).[20] According to Mach, the way we tell the difference between these equivalent relative motions is that voluntary innervations to our eye muscles tell us that we are moving our eyes; hence in these cases the world will appear fixed.[21]

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[18] (AS), pp. 128-129.

[19] *Deutsche Literaturzeitung* 7(1886): 947-948.

[20] (AS), p. 166.

[21] The true number of cases is somewhat larger. As Mach says (AS) p. 206, the eyes also carry out compensatory movements to stay locked on the same point in the visual field, compensating for motions of the head. This is due to the action of the otoliths and the semicircular canals which send messages to the ocular muscles. In a compensatory movement the head is sensed to be in motion and yet the surrounding objects are still seen at rest because the retinas manage to achieve rest through the

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Mach claimed all of our movements begin as involuntary reflexes, and yet indeed the involuntary movements of our bodies demonstrate considerable innate spatial sophistication, already at birth. For example, as Mach discovered along with Josef Breuer and Crum Browne, involuntary motions of the eyes caused by the otoliths and semicircular canals compensate for all motions of the head and keep images of objects fixed on the retinal surfaces. One other example that especially impressed him was that newly hatched chicks can peck accurately at what they see with practically no experience. According to Mach, memory images of these early reflexes endure and occasion the new, so-called voluntary motions.[22] In the case of the eye-musculature, the set of memory images of previously performed fixations constitutes a memory-space of possible directions and distances.

These remarks take on great plausibility in connection with Mach's work on sensations of movement;[23] yet his innervation theory for distinguishing relative motions came under attack from James and Hugo Munsterberg, who argued that the relative motion of the body is really sensed by peripheral

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motions of the head. Finally we can will to inhibit some of these compensatory motions by moving our eyes voluntarily, but then we still sense that our own bodies are in motion.

[22] "We can conceive voluntary movement at any rate in principle as reflex movement modified by reflexes." (AS), p. 172.

[23] The interested reader can begin with (AS), Chapter 7.

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kinaesthetic sensations of the skin and muscles and not through central innervations of willed movement.

Mach's Psychological Space

Although Mach was a nativist and Wundt an empiricist about psychological space, in practice the distinction seems to have made little difference. Mach was captivated by Wundt's notion that our outward perception of space is due to a fixed register of local signs on our bodies and musculature. For Wundt this register is developed with experience; for Mach it is inborn and due to the evolved experience of previous generations of organisms. Mach extended Wundt's register to sensations of time and tone, although under him the theory was thoroughly nativist.[24] Mach's admiration for the nativist theories of Hering for color and vision, the Darwinian theory and the example of the chick had convinced him that basic senses of direction and movement had to be present at birth, and he finally wrote in *Knowledge and Error*:

> The attempt which is frequently made of deriving tactual space psychologically...by the aid of the concept of time and on the assumption of space-less sensations is an altogether futile one.[25]

Mach's final position in his *Space and Geometry* articles was that spatial representation was wired into the organism.[26] He

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[24] (AS), p. 126.

[25] (SG), p. 19.

[26] (SG), p. 25.

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was especially impressed with the ready-made complicated involuntary muscular movements contained in the simple will to move from one place to another manifested in a very large class of organisms.

The Impact of These Views on Mach's Physics

Mach the physicist was of course a thorough empiricist about space, and his German predecessors in psychology had shown convincingly how a perception of space is really quite complex containing many separable properties: place, extension, direction and distance. For example an extended manifold of places contains no super-added metric information as yet. The psychologists also showed conceptually that the various conceptions involved in spatial extension could be built-up convincingly out of certain functional associations and dissociations of intensive content, which if we except Herbart's metaphysics, had yet to be attempted in geometry before Riemann and Grassmann.

Mach's view of *physical space* remained close to the functional associations and dissociations of the empiricist psychologists, and their constructions--or deconstructions--do seem to have encouraged him to carry out a wide-ranging physiological critique of physical and geometrical concepts involving space. As we will see in the next chapter this was what Mach really meant by the "elimination of metaphysics."

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Chapter Six: Space-Sensations, Geometry and Riemann's Definition of Extension

Antimetaphysics

For Mach shapes, similar figures and straight lines all had a physiological significance unrelated to their geometric properties. Physical science, for him, also harbored concepts of space derived from our own innate sensory imagery. This led him to the following antimetaphysical argument: if such concepts were only psychologically valid, there was no reason to extend them to the physical world uncritically. In Mach's view, our innate spatial and temporal psychological representations, over the long course of the history of science, had affected how we visualize mind-independent physical events. These representations constituted a kind of "residue" on the abstract structure of physical and mathematical theories which he thought should be extirpated. Mach saw an urgent need to separate psychological space-sensations from abstract physical and geometrical space in what he called an antimetaphysical critique of the sciences.

Mach's Analysis of Space Sensations

Before he could turn his attention to physics and geometry, Mach first needed a precise analysis of psychological space. In his article of 1865 "Bemerkungen über

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die Lehre vom räumlichen Sehen"[1], Mach asked how we perceive similarities of spatial *form* even when the individual qualities are different: for instance, we recognize the "circularity" of both a blue circle and a yellow circle. Herbartian reproduction series could not explain this because their inhibitions and reproductions all depended on the qualities involved. Mach pointed out that the yellow and blue circles *would* reproduce each other if they possessed identical "form-qualities" over and above their different colors. These form-qualities are in fact Wundt's register of muscular sensations:

> If two differently colored figures of equal size reproduce each other and are recognized as equal, the result can be due to nothing but the existence in both series of perceptions of an idea or percept that is qualitatively *the same*. The colors are different. Consequently, like or equal percepts must be connected with the colors which are yet independent of the muscular feelings of the eye when confronted by the two figures. We might say that we reach the vision of space by the registering of light-sensations in a schedule of graduated muscle sensations.[2]

He added that sensations of form were also to be found in ornamental motifs, and in music, where the same melody is recognized through different absolute pitches of the notes and the same rhythm is recognized independent of the absolute durations of the beats.[3]

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[1] *Fichtes Zeitschrift für Philosophie* 46(1865): 1-5.

[2] Ibid p. 3.

[3] Ibid. pp. 4-5. Because of his investigation of *Gestaltensehen* Edwin Boring gave Mach a place in the history of Gestalt psychology, although he objected to

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Mach thought that whenever two experiences are judged similar, there must be elements within those experiences that are *exactly* the same. Hence his formula: similarity is partial identity. Yet the mind must not be capable of inferring all possible patterns of similarity, for then everything would be similar to everything else in some respect. Thus Mach held that we are physiologically predisposed to sense only certain identities, in sound, rhythm, spatial form; the rest we ignore. He described the strategy in the *Analysis*:

> If I see figures which are the same in size and shape but differently colored, I seek, in connexion with the different color sensations, certain identical space sensations and correspondingly identical nerve processes. If two figures are similar (that is, if they yield partly identical space sensations) then the corresponding nerve processes also contain part identical components. If two different melodies have the same rhythm, then, side by side with the different tone-sensations there exists in both cases an identical time-sensation with identical corresponding nerve processes. If two melodies of different pitch are identical then the tone-sensations as well as their physiological conditions have, in spite of their different pitches, identical constituents.[4]

In his 1865 article "Üntersuchungen über den Zeitsinn des Ohres" Mach wrote that both the space and time sensations are a "register" in the body or a "contour outline" of experience in which the individual qualities are entered.

> One can completely liberate the rhythm of a melody from the sensations of tone, like the contour drawing of a

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Mach's calling space a "sensation" *History of Experimental Psychology* (New York: Appleton-Century Crofts, 1950) p. 442.

[4] (AS), pp. 62-63.

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> painting, in that one can tap out the melody without tones. This would not be possible if there were not, to some extent, self-standing series of rhythm sensations.[5]

As Mach pointed out in his next major article "Bemerkungen über die Entwicklung der Raumvorstellungen" (1866), the register has certain physiological peculiarities which show up prominently in space perception. Repetitions of an ornamental motif, even in various sizes, tends to condition a repetition of the muscular sensations over the shapes, and a pleasurable aesthetic feeling of "relief."

Symmetry

Psychological spatial symmetry, for Mach, was also to be explained as a repetition of identical sensations. For example, vertical symmetry is immediately noticed, and occurs when the plane of symmetry of objects divides the head on either side of the nose. We tend to line up vertically symmetrical objects with our noses; however horizontal symmetry--the symmetry of a house and its upside-down image in a lake--is not immediately noticed and requires a special intellectual effort.[6]

Mach claimed that geometrically similar figures were only psychologically similar if they were situated such that they

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[5] "Üntersuchungen über den Zeitsinn des Ohres" *SW* **51**: 146-147.

[6] "Bemerkungen über die Lehre von räumlichen Sehen" pp. 3-4.

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occasioned the *same* muscular sensations. He got into deep water here, since there seem to be too many similarities of form to keep up with. We sense the similarity of two straight lines of different sizes, inclined at different angles to one another, although none of these scanning motions measure out identical distances, or muscular strains. But on the other hand there are many opportunities to infer optical similarities based on identical scanning motions and muscular strains where we do not.

To keep the similarities from proliferating, Mach had to postulate certain innate preferences for the eye. For example, he thought that straight lines, when they are symmetrically (horizontally or even vertically) situated produce special physiological effects, which help us to recognize them as similar later when they are situated obliquely:

> The straight line is defined geometrically as the shortest distance between two points, which already for that reason is peculiar since all measurements of length presuppose the straight line. This definition certainly has no psychological meaning. I have already mentioned earlier that the straight line can be laid down (vertically or horizontally) so that both of its retinal images fall on identical or symmetrical points. This is what gives the straight line a preference before other curves and this psychological preference is not without significance for the history of geometry.[7]

Straight lines are all similar because identical sensations of direction are felt when the eye moves from one point on a line

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[7] (R), p. 228.

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to another, a pleasurable sensation. Further, the sensations produced by moving to a position on a straight line is always the average of the intensity of sensations of moving to neighboring points, which is not true of curves. Thus, a physiological preference for straightness emerges. A curve may cast the image of a straight line when it is seen edge-on and a long straight line tilted away from the eye may cast a short retinal image, but these possibilities are almost always disregarded. Whenever a retinal image can be viewed as a straight line, with all its points at the same depth, it will be.[8]

Mach held as a general physiological principle that sensory systems are geared to notice any above-average deviations of sensations from a resting position. The resting position is where the intensity of any given sensation intensity (whether of length, depth, light or muscular strain) is the average of the intensity of its neighbors and no especial contrasts occur.[9] This resting position is always preferred unless the sensory system is forced out of it.

Physiologically speaking, curves and their first and second derivatives are perceived as forced deviations from the resting position of the straight line.[10] According to Mach,

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[8] "Bemerkungen über monoculare Stereoscopie" *SW* **58** (1868) 731-736.

[9] (R), p. 229.

[10] (R), p. 229.

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this is why we can perceive directly both the slope of a curve (its first derivative) and its curvature (its second derivative) but not higher order derivatives. We have no organs for those higher order contrasts.

The purpose of Mach's physiological critique of geometry was not to breed cynicism--and certainly *not* to encourage geometers to stick exclusively to sensory properties. He merely pointed out that historically, geometry had exaggerated the importance of physiologically significant properties of straightness and symmetry, neglecting more abstract properties of affinity and collineation.[11]

Geometry

Given his antimetaphysical critique, Mach conceived of geometry as more than a science studying a previously given concept of space. For Mach, it was rather the task of geometry to develop the concept of space from more abstract concepts. In his "Raumvorstellungen" paper Mach called geometry the science of "metric space" because he thought geometrical axioms rested in the last analysis upon facts about measurement. He said that geometry lays down a standard measure, a fixed length, area or volume and institutes

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[11] (R), p. 229. One can imagine geometric properties as produced by different transformations on lines and figures. If a figure is shrunken but the relative distances between all points are preserved the new figure will retain its similarity. Affinity and collineation name two more abstract properties that are preserved even when similarity is not.

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procedures for carrying this measure into congruence with every other length, area or volume through motion. If these relations are upheld, it follows that every spatial magnitude can be compared with every other through their relations to the standard length.[12]

Gauss and Bernhard Riemann are probably the most famous representatives of this view that geometry is, at least in part, dependent on empirical metric procedures, and not on a priori certainties. Riemann wrote in his *Probevorlesung* delivered in 1854 that the critical empirical hypothesis to ensure the congruence of lines and figures is "that the length of lines be independent of their situation, that every line be measurable by every other."[13] Mach wrote in 1872 on the subject of metric conceptions of space that "in this external stimuli were entirely lacking, for Riemann's paper, which first appeared in 1867, was quite unknown to me."[14] In fact, the first mention of Riemann's name in Mach's papers occurred in 1865 while he was at Graz, administrating a hiring committee for the philosophical faculty. One of the prospectives was Hermann Hankel, a mathematician who had studied at Göttigen and been present on the few occasions when

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[12] (R), p. 230.

[13] Berhard Riemann "On the Hypotheses that Lie at the Foundation of Geometry" in D.W. Smith, ed., *A Source Book in Mathematics*, (New York: McGraw Hill, 1929), pp 415-416.

[14] (CE), p. 88.

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Riemann had treated of his new points of view exhaustively.[15] The report does not mention geometry per se. Instead Mach wrote in his report on Hankel that it was the theory of "complex functions" that he had absorbed from Riemann; his knowledge of the newer geometry came from Möbius and Steiner:

> It should be especially emphasized his familiarity with the Riemannean theory of complex functions and with the so-called new geometry as founded by Möbius and Steiner.[16]

In 1866, the same year as Mach's paper, Hermann von Helmholtz proposed his own response to Riemann's *Probevorlesung*, entitled "Über die tatsächlichen Grundlagen der Geometrie" (On the Factual (Empirical) Foundations of Geometry).[17] Helmholtz showed that the axioms of geometry could be deduced from the empirical assumption that a system of points such as a rigid body preserves its relative distances through all motions and rotation in place. This fact, he said, rested on physical laws:

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[15] See Carl Ludwig's recommendation letter on behalf of Hankel (NL 174/5/1379): "Alle stimmen darin überein, dass er ein Mann von reichen Bildung sei. und dass ihn bei seinem Studiengang das Glück geradezu begünstigt habe so zählt er u.a. zu den wenigen welche Riemann in seinen besten Tagen gehört haben; er hat einer der weniger Vorlesungen beigewohnt in denen jener hervorragende Mathematiker seine neuen Ansichten auf das erschöpfendste behandelt hat."

[16] Philosophische Fakultät der Universität Graz, Commissions Bericht von E. Mach den 21. Dec. 1865. The original document is in Graz, but Deutsches Museum possesses a photocopy, numbered (NL 174/3/13).

[17] *Wissenschaftliche Abhandlungen* Vol II. (Leipzig: Johann Barth, 1883).

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> The independence of the congruence of rigid systems of points from place, position (Lage) and relative rotation is the fact on which Geometry is founded...Like every physical measurement, that of space must also rest upon an unchangeable law of uniformity in the phenomena of Nature.[18]

Mach and Helmholtz were led to investigate metric properties because they were sense-physiologists and knew that in the visual field lengths shrink and grow from one place to another; the same is true on the skin. Thus it cannot be taken for granted that length is independent of position, as it is in geometry. Helmholtz said:

> In the [Handbook of] Physiological Optics I had before me two examples of other spatially representable manifolds capable of change in several senses, namely the system of colors, which Riemann also mentions, and the measuring out [*Ausmessung*] of the visual field with the glance [*Augenmaass*]. Both show fundamental differences from the measurement system of geometry and provide the stimulus for a comparison.[19]

Mach pursued the physiological side of the analysis further, showing how each of the sensory modes had its own kind of space, with local signs, dimensions, metrical properties, and invariant objects. He pointed out in his lecture "Über die physikalische Bedeutung der Gesetze der Symmetrie"[20] that the senses manifested a variety of spaces. There were senses with less symmetry than ordinary space, for example the one-

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[18] Ibid., p. 616.

[19] "Über die Tatsachen die der Geometrie zum Grunde liegen," p. 619.

[20] *Lotos* **21** (1871): 139-147. Translated as "On Symmetry" in PSL.

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dimensional tone row, in which there is no right-left symmetry in the directions, just a one-dimensional progression from low to high. Move a length along the straight line and it remains the same length; move a fixed relation of tone intervals like a melody up and down the tone row and it is no longer recognizable.

The modern conception of symmetry is the preservation of a given property through a group of transformations. For instance, a circle possesses centric symmetry because it remains the same figure when all of its points are rotated about the center. In his "Symmetrie" article Mach considered the deeper question: under what conditions do we even recognize the *repetition* of identical experiences, such as the "same" figure or property? By this stricter standard, the mere recognition of repeating events or qualities in time is the simplest symmetry of all. The psychological reproduction series rests on this. The next highest symmetry is the recognition of repeating complexes of sensations in objects like rigid bodies, and next the preservations of laws holding between those objects. Symmetry and preservation through change became in fact the foundation of science in Mach's economical method of description.

The complete title of Mach's 1871 article was "On the Physical Significance of the Laws of Symmetry," although there was not much talk about physics in it. Mach did however extensively explore the physiological comparison of 3-space

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with his one-dimensional *Tonraum*, in order to show that the properties of space were empirical matters that depended on the intrinsic structure of the manifold, not one single space for all.

In physics, where spatial properties should have rested on physical laws, Mach felt that the underlying structural questions that would determine the nature of space were undecided. Thus it could turn out that physical space had very different properties from the three-dimensional space of sight in which physical processes were often visualized as occurring. In the "Symmetrie" paper Mach went further and claimed as part of his antimetaphysical critique of science that three-dimensional physiological space and spherical atoms had been dragged into physics uncritically, without a derivation of spatial extension from natural laws:

> That theory of physics which traces back all phenomena to motion and equilibrium of the smallest parts, the so-called molecular theories are already set tottering through the progress of the theory of the senses and of space--and one may say their days are numbered. If someone who could only hear tried to develop a world view in his linear space he would fall considerably short in that his space would not be in the position to grasp the multiplicity of the real relationships. It is however no more justified of us to believe we can press the entire world into the space of our eyes, even parts of it we do not see. This is the predicament of all molecular theories.[21]

Thus Mach's attack on metaphysics and atoms were really of a piece with his physiological critique of science. For Mach,

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[21] "Über die physikalische Bedeutung der Gesetze der Symmetrie" *Lotos* **21** (1871), p. 146.

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metaphysical errors occurred when special physiological properties of the senses were attributed to realms where they did not extend. An example of such a metaphysical error familiar to everyone is the question why objects projected on the retina upside-down are seen right-side up.[22] The answer is that visual directions of "up" and "down" are applied in a mistaken "metaphysical" sense to retinal images outside the visual field.

Mach's philosophical hostility to corpuscularian theories and seventeenth century conceptions of space, time and matter, was thus provoked by what he saw as their readily visualized properties and dependence on human physiology, imagination and memory.

Objects

Besides his antimetaphysical critique, one of Mach's most important breakthroughs during the 1860s was his conception of a material object as an invariant function of simpler element-qualities.[23] Here it was chiefly geometric conceptions of Helmholtz and Riemann that influenced him.

In his already-mentioned 1866 article "Über die tatsächlichen Grundlagen der Geometrie," Helmholtz defined a rigid body as a system of simultaneous equations holding

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[22] (AS), pp 38-39.

[23] (AS), p. 331.

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between each pair of points in the body.[24] Taking all possible combinations in hand, there will be m(m-1)/2 simultaneous equations for all pairs. When the rigid body is rotated or slid, the coordinates of one of its points (x,y,z) change to new coordinates (x', y', z'), but the equations are recognized as the same pattern after a transformation and they are what is permanent about the body.

If Helmholtz's spatial coordinates are replaced by qualitative modes of determining points in a manifold, the Helmholtzian view of body as a set of equations becomes Mach's view of matter to the letter,[25] and Helmholtz's papers of 1866 and 1868 seem to have been Mach's primary influence. We will now discuss that replacement of spatial coordinates in more detail.

The main difference between Mach and Helmholtz concerned how to set up a manifold extension and a system of coordinates. These fundamental constructions precede invariant metric properties of the manifold like distances.

Let's start with Helmholtz's story and then move to Mach's refinements. Ordinarily to set up a coordinate system in three-space each individual x must be specified by giving three arbitrary fixed points, (a,b,c) and the three distances

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[24] Helmholtz, *Wissenschaftliche Abhandlungen* p. 622.

[25] (AS), pp. 191, 331-332

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from the fixed points to x, f(x,a), f(x,b) f(x,c).[26] The coordinates are functions of the fixed points and of the path from the fixed point to the point to be located. For example, if I want to locate any point in my room I have to give its distances from a fixed point on the floor, and two on the walls. If the space has n dimensions, we may pick n distances arbitrarily with which the n+1th is thereby determined.

When a rigid body is given as a set of simultaneous equations, if some of the coordinates of points on the body are given, others will be determined. The determining factors are the dimensions of the space (n) and the number of points on the body. In the *Conservation of Energy*[27] Mach worked out a table of the analytical relationships between pairs of points in a rigid body, such as a molecule, and the number of fixed points and distances needed to solve all of the equations in a manifold of an arbitrary number of dimensions. For example if we have a molecule of five points in 3-space we only need to know nine distances for the tenth to be determined.

The assumption that bodies retain their spatial rigidity on displacement gave Helmholtz the analytical means to solve for the form of the equations f(a, x) that held between the

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[26] In some cases distances alone do not distinguish between objects that are mirror images of one another, a right and a left hand for example. In this cases we have to draw a plane between them and give further information on which side of the plane the object is to be produced.

[27] (CE), pp. 51-52.

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pairs.[28] He found that in flat 3-space this function was the Pythagorean theorem $ds^2=dx^2+dy^2+dz^2$ taken in the infinitely small.[29] The function is like a distance in that, if the dx, dy and dz change sign, ds is unaffected. Secondly, the function is *homogeneous*[30] because changes of step in ds are related by a fixed rule to simultaneous changes of step in all three variables separately, right down to their first and second derivatives:

> It is to be remarked that by the continuity of change through motion, it is not only meant that in changing between two end-values, all values in between will be run through, but rather also that differential quotients exist, i.e. that the relations of the changes of coordinates that belong together approach a fixed relationship, as we take the size of the changes smaller and smaller.[31]

This last constraint on the line element is important, since we do not want a particle to cover different distances in each direction separately. We want the distance travelled to depend only upon the initial and final points and not

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[28] "Über die Tatsachen...," p. 623.

[29] If the coordinates are to be plotted on a surface only two variables (dx,dy) are necessary. Also a distance function applies in all spaces with constant curvature, where the angle between dx and dy is not 90 degrees, in which case it must be expressed in the general form: $ds^2 = dx^2 + 2dxdy\cos\theta + dy^2$.

[30] A function of two or more variables is *homogeneous* iff, for all real numbers m: $f(mx, my)=mf(x,y)$. In the case of the Pythagorean theorem, homogeneity proves that if all the dx,dy,dz increase in some ratio then the distance ds increases in that ratio.

[31] "Über die Tatsachen..." p. 622.

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explicitly upon path or direction. This geometric fact also plays a major role in defining a potential-energy function, $U(x,y,z)$,[32] and reveals a very interesting mathematical similarity between the distance function and the potential, which bears closer examination. Helmholtz may have been trying to call attention to this similarity by remarking that in the end geometric facts rested on natural laws. And Mach did take this further step in his notebooks, saying that the conservation of energy for potential functions translated into the conservation of length, area or volume for the distance function.[33] So by implication one could already interpret the transfer of energy between potential levels, action, as the tracing out of a distance, as in Mach's later view.

What was so interesting about Helmholtz's paper at the time was that he eliminated spatial intuition as much as possible and took a thoroughly analytical approach.[34] For

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[32] For example, to define a potential-function it is necessary that the changes of the so-called mixed partials are in step with one another, $\partial^2u/\partial x\partial y=\partial^2u/\partial y\partial x$, $\partial^2u/\partial x\partial z=\partial^2u/\partial z\partial x$, $\partial^2u/\partial y\partial z=\partial^2u/\partial z\partial y$.

[33] Notebook dated 5 July 1888 (NL 174/2/28): "Begriff Arbeit in Bewegungsfragen maassgebend. Energie allgemeines Maass." Cf. Notebook dated 9 November 1900 (NL 174/2/43) "Die Raumerhaltung dieselbe Rolle in der Geometrie, wie die Continuität u. Massenerhaltung in d. Physik."

[34] Sophus Lie eventually generalized Helmholtz's ideas using groups of transformations. Each transformation ("motion" of a point) is a function taking one coordinate-system to another. The transformations form a group if there is an identity transformation, inverses and associativity, the replacement of two transformations with a third single transformation. See Max Jammer, *Concepts of Space*, Third

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example, we cannot appeal a priori to the fact that space has three dimensions; we must consider instead how many fixed points and distances can be picked independently without determining the rest.

> The analytical treatment of the question, how space is differentiated from other measurable [*abmessbar*] multiply extended continuous manifolds, is to be recommended...for in just these circumstances visual intuition [*Anschaulichkeit*] is lacking, and is therefore not subject to illusions that are so difficult to avoid because of the special limitation of our intuitions.[35]

Mach was of one mind with Helmholtz on many of these questions during the 1860s. But in later writings he felt that Helmholtz had not gone far enough to eliminate spatial coordinates.

For one thing, the laying-out of the coordinate system, and especially the use of distances to determine coordinates, still seemed to assume the existence of a pre-existing space. Helmholtz's fundamental assumption also seems question-begging: the preservation of spatial distance depends on the preservation of rigid bodies through sliding and rotation, but bodies are recognized as the same through the very spatial permanence and congruence we are using them to prove. We must have another non-spatial criterion for the permanence of matter first.

Helmholtz seemed to use distance twice in his deduction:

Edition (New York: Dover, 1994), p. 157.

[35] *Wissenschaftliche Abhandlungen* p. 618-619.

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once to locate points in the coordinate system with n distances to fixed points, and a second time to determine the metric properties of the manifold with a standard invariant length. He wrote as if we already needed the conception of distance to set up the coordinate system and manifold extension. But there is no reason to think so, because there are extended manifolds of places, the psychological local sign manifolds of vision and touch, with localized coordinates but not invariant distances.

Mach was absolutely determined to set up even the basic coordinate-system and extension without spatial intuition. The *Conservation of Energy* gives several inklings of the idea. For example, Mach interpreted distances between the points of a molecule as the qualitatively different heats of combination of each pair of chemical elements, which he said "gives us a clearer idea of the stability and manner of combination than any pictorial representation."[36] The variety of different heats gives the number of equations that hold between pairs of atoms in the molecule. A physical body thus became a mass of simultaneous equations but defined over qualitative coordinates. The differences between these coordinates could rather be like the qualitative differences between hues of colors, or degrees of heat.

Also in the *Conservation of Energy* Mach rejected the "mechanical" idea which he attributed to Wundt, that all

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[36] (CE), pp. 53-54.

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natural changes had to be changes of place. Mach made clear that qualitative change was equally admissible for him:

> According to Wundt, all changes in nature are mere changes of place. All causes are motional causes. Any discussions of the philosophical grounds on which Wundt supports his theory would lead us deep into the speculations of the Eleatics and the Herbartians. Change of place, Wundt holds, is the only change of a thing in which a thing remains identical with itself. If a thing changed *qualitatively*, we should be obliged to imagine that something was annihilated and something else created it its place, which is not to be reconciled with our idea of the identity of the object observed and of the indestructibility of matter. But we have only to remember that the Eleatics encountered difficulties of exactly the same sort in motion. Can we not also imagine that a thing is destroyed in one place and in another an exactly similar thing is created?[37]

Mach's physical object of the 1860s thus seemed to be a functionally connected group of elements undergoing qualitative changes as they moved from one coordinate to another.[38] Even the mechanical principle of the conservation of energy, was replaced in his mind by a reciprocal dependence of qualities.

Riemann's Definition of Extension

I think that Mach's elimination of space in favor of a manifold of qualities had certainly occurred to Riemann[39] and perhaps also to Helmholtz.[40] Both had mentioned the project

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37 (CE), p. 50.

38 (AS), p. 29.

39 "On the Hypotheses" p. 413.

40 "Über die Tatsachen..." p. 619.

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of constructing a manifold of colors.

Mach took up the challenge of constructing the color manifold from hues, as part of his general catalogue mapping the sensory spaces. In these investigations, Mach further reduced the quasi-spatial language of coordinates, distances and extended objects to his bare-bones vocabulary of elements.

The immediate stimulus toward Mach's mature philosophy of space was probably a deep engagement with Riemann's *Probevoslesung*, published in 1867 and over time the standard text of geometry. Beginning with his investigations of the tone row, Mach came to interpret each sensory space as a Riemannean manifold of a different type. Riemann himself was extraordinarily sensitive to the need to analyze extension in terms of the unextended; a motivation that probably stemmed from his study of Herbart. In his "Plan der Untersuchung" Riemann said that space and the first fundamental notions for constructions in space could not be assumed a priori, as Euclid had done.[41] Riemann then defined extension by saying that a "concept of quantity" must first admit of two simpler "modes of determination" and that as there existed a discrete or continuous transition [*Übergang*] from one mode to another, there would then exist an extended manifold of points or elements.[42] The modes of determination cannot themselves be manifolds of points, since transitions between the two modes

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[41] Riemann "On the Hypotheses..." p. 411.

[42] p. 412.

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are what *result* in points. Riemann explained further on in his lecture that the alternations from one mode to another were possible in two directions:

> In a concept whose various modes of determination form a continuous manifold, if one passes in a definite way from one mode of determination to another, the modes of determination which are traversed constitute a simply extended manifold and its essential mark is this, that in it a continuous progress is possible from any point only in two directions, forward or backward.[43]

In comparison with Herbart's construction, these passages lose some of their opacity. There the two alternating processes **(N+R),(R-N)** make up a progression in the left direction, and **(R+N)** and **(N-R)** do the same in the rightward direction. In all, each of the two directions of right and left is composed of two simpler qualities or modes undergoing alternations in one sense or another, i.e., either the **R** is increasing and the **N** decreasing, or vice versa. The alternating process serves to locate any point (R'+ N') from a given fixed point (R+N) by constructing all of the points between them in either direction. Like Lotze's local sign manifold, the coordinate system is a set of places gathered up together.[44] We do not know from constructing a system of

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[43] p. 414.

[44] Edna Kramer gives an illuminating example in her discussion of Riemannean geometry: the extended manifold with its Gaussian coordinates is like a grid of city blocks. We can locate any point on the grid from its address, but since the blocks are different sizes we cannot tell distances from one point to another. *The Nature and Growth of Modern Mathematics* (Princeton, 1982), p. 455.

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coordinates whether points are unevenly spaced, or whether the spaces covered in the right direction are the same as those covered to the left, or whether the property of distance has any application at all. Perhaps the changes from one point to another are more like qualitative changes from blue to green.

Mach's Application of Riemann's Ideas to Sense Manifolds: The Tonraum

Mach's first attempt to apply Riemann's ideas to sensory spaces was extremely faithful to the definition of the *Probevorlesung* as I am interpreting it. For his tone row, Mach took as the qualitative "x and y coordinates" a combination of two qualities, two intermixed modes of Dull [*Dumpf*] and Bright [*Hell*].[45] Each tone sensation on the row could be located immediately by its different proportion of the two qualities, more Dull in the mix of a point near the low end, more Bright near the high end.[46] A tone sensation on the row thus has two qualitative modes of determination, and coordinates constructible according to the rule:

$$D + H = 1,$$

or if H goes up, D goes down and vice versa. In Herbartian terms, it would be easy to define four processes (H+D),(D-

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[45] (AS). p. 294; "Zur Analyse der Tonempfindungen" *SW* **92** (December 1885): 1283-1289.

[46] Another benefit of using qualities is that the fixed points can be specified by looking, or immediate qualitative identity, and without a cascading regress of relations to all other points.

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H), (D+H), (H-D) for getting from a fixed point on the tone row to every point to be located.

Apart from the properties and processes that set the coordinates, Mach claimed that a *separate* property determined an invariant interval in the tone row,[47] analogous to a distance. His expression for the interval (in terms of the rate of vibration) was:

$$\log (n'/n) = \text{Const.}$$[48]

which expresses "the constancy of the length on displacement"[49] just like a rigid meter stick moved about through space.

> The unalterable, substantial physical object which we sense as an interval is for the ear temporally determined, whereas the analogous object for the senses of sight and touch is spatially determined.[50]

Although the tone row does not have symmetry in the directions of high and low, it does exhibit clues that a tone-invariant exists, since the same "notes" repeat one octave higher in pitch, a kind of symmetry. In fact every tone has "partial tones" that occur at other intervals, the third, the fifth and so forth. When two tones have overlapping partials,

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[47] Mach suggests further down ((SG), p. 105) in his exposition of the color manifold that the function log (n'/n) can be used as a means of determining coordinates, and he probably has in mind the tone-row when he says this.

[48] Where n' is higher in pitch than n.

[49] (SG), p. 101.

[50] (SG), p. 101.

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the ear hears out the "distance" between them.

The Color Manifold

In his three *Monist* articles "Space and Geometry" (1901-03) Mach treated the color manifold according to Riemann's principles, similar to his treatment of tone. The system of color coordinates is achieved by pairing every two opposing color-processes, {White, Black} {Red, Green}, {Blue, Yellow}, along its own independent dimension.[51]

> Comparing sensations of space to sensations of color we discover that to the continuous series "above and below," "right and left," "near and far" correspond the three sensational series of mixed colors, black-white, red-green, blue-yellow. The system of sensed (seen) places is a triple continuous manifold like the system of color sensations.[52]

Along each axis, we can judge where a hue is from a fixed standard hue by successively constructing all hues in between according to a rule, such as (H)+(D)=1. These are once again the alternating transitions between two modes, as mentioned by Riemann. Because there are six qualities in all, a point can alter its hue in each of three different dimensions simultaneously. The actual psychological details suggest a different story, but the differences are unimportant for an abstract conceptual treatment of space such as we are pursuing here.

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[51] Mach of course follows Hering's "opposing processes" color theory and not the Helmholtz-Young theory.

[52] (SG), p. 99.

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Mach proceeded further in a section called "Another View of Riemann's Manifold." Here he represented all spaces (sensory and physical) as combinations of qualities:

> Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\ldots,\alpha_{n+1}$ be any elements whatsoever (sensational qualities, substances, etc). If we conceive these elements mingled in all of their possible relations, then each single composite will be represented by the expression
>
> $$\alpha_1\alpha_1,\ \alpha_2\alpha_2,\alpha_3\alpha_3,\alpha_4\alpha_4,\ldots,\alpha_{n+1}\alpha_{n+1} = 1.$$
>
> where the coefficients α satisfy the equation
>
> $$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\ldots,\alpha_{n+1} = 1.$$
>
> In as much, therefore, as n of these coefficients α may be selected at pleasure, the totality of the composites of the n+1 elements will represent an n-fold continuous manifold. (If the six fundamental color-sensations were totally independent of one another, the system of color sensations would represent a five-fold manifold. Since they are contrasted in pairs, the system corresponds to a three-fold manifold.) As coordinates of a point of this manifold we may regard expressions of the form:
>
> $$\alpha_m/\alpha_1 \text{ or } f(\alpha_m/\alpha_1), \text{ for example } \log(\alpha_m/\alpha_1).^{53}$$

The coordinate expression (α_m/α_1) is of course a function relating the fixed point (α_1) to a point to be located (α_m) according to some rule for constructing the coordinates. The actual coordinate value is always a function of the fixed point chosen, just as in Helmholtz's treatment.

There are no metric properties to a coordinate manifold, just manifold extension and a rule for constructing every

---

[53] (SG), p. 105. The third coordinate expression seems to suggest that Mach regarded the coordinates of the tone-row as constructed by the same logarithmic rule that determines the invariant distance. However he does not say it must be so.

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position from every other. The sensation qualities (the large alpha letters) of which he is speaking are the Riemannean modes. For example a possible interpretation for:

$$\alpha_1\alpha_1,\ \alpha_2\alpha_2,\ \alpha_3\alpha_3,\ \alpha_4\alpha_4,\ \alpha_5\alpha_5,\ \alpha_6\alpha_6 = 1.$$

is (α_1) units in the green direction, (α_2) units blue, (α_3) units white, (α_4) units red, (α_5) units yellow, (α_6) units black.

Assuming only two opposing colors, the expression reduces to:

$$\alpha_1\alpha_1 + \alpha_2\alpha_2 = 1, \text{ or } \alpha_1\alpha_1 = -\alpha_2\alpha_2$$

If one quality in one direction is picked the other is determined, hence the manifold is one-fold extended.[54] The coefficients of this two-color manifold (the small alpha letters) describe the qualitative mixture of one color process (in large alpha letters) and its opposing color process, adding, it says, to 1. If the mixture contains α_1 of α_1, it contains $(1-\alpha_1)$ or α_2 of α_2.

Troubles with the "Modes"

Now the "modes of determination" are supposed to combine

---

[54] It will always be the case that with the second to last coefficient chosen the last is determined. If only one coefficient can be picked arbitrarily the manifold is one-fold extended. If three can be picked arbitrarily and the fourth determined it is three fold extended and so on. The extension always refers to the number of arbitrary coefficients needed to determine a point there.

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in a proportion and yet always be mutually opposed, a seeming contradiction which raised an eyebrow with more than one of Mach's critics. Carl Stumpf in his 1886 review of the *Beiträge zur Analyse der Empfindungen*[55] complained that if a tone is made up of mutually opposing processes of *Hell* and *Dumpf*, related like + and -, why don't they cancel each other out, leaving a residue of only one of them? Brentano, Stumpf's teacher, and an opponent of Riemann, had a similar objection to the analysis of colors; if they are opposed, and if a bluish-yellow or a reddish-green are supposedly impossible, how could mixtures subsist?[56] Mach's direct responses to these criticisms are not known.[57]

His critics had a point. If the H and D qualities are always rigidly opposed how is it possible for them to alternate. It has to be explained how H and D can pass out of dependence on each other, and into dependence on other qualities. In the case of sensory manifolds at least the association of a quality with its local sign can serve to dissociate it from another quality, as in Lotze's construction. Perhaps Mach could have appealed to a second set of local signs to dissociate the hues.

---

[55] *Deutsche Literaturzeitung* 4 (no. 27) 947-948.

[56] Brentano to Mach 18 November 1903 in the Briefwechsel, of Franz Brentano *Über Ernst Machs "Erkenntnis und Irrtum"* (Amsterdam: Rodolpi, 1988).

[57] But for an argument see Kant's "Amphibolie der Reflexionsbegriffe," *K.d.r.V.* B 328/ A 272-B 329-A 273.

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In *Knowledge and Error* Chapter 24 "Space and Time Physically Considered" Mach put together his notion that space and time rest on fixed causal dependencies between the elements,[58] with his earlier Herbartian idea that independencies of the qualities in nature are due to "differences" in the natural potentials.[59] The Herbartian construction of qualities and lines depends on the idea that qualities can come into dependence and out of dependence in alternating fashion, and there is one mysterious excerpt from Mach's notebook that suggests such a connection in his mind:

> Indifference of dimensions. Quantity and quality concepts of Herbart.[60]

In fact, Mach came to see raw extension as due to the causal "between-ness" and independence of processes from one another, Herbart's Nicht-Zusammen:

> Space is the between, but the three-fold between. Time also the between but here simple, the between of steps of change.
>
> Spatial distancing, so many bodies between. Cancellation of immediate contact.[61]

---

[58] See also "Sensory Elements and Scientific Concepts" in (DL), pp. 119-120.

[59] (KE), pp. 340-341

[60] Indifferenz der Dimensionen
Quantitäts und Qualitätsbegriffe Herbarts. Notebook dated 8 June 1904 (NL 174/2/47).

[61] Der Raum das Zwischen, aber das dreifache Zwischen. Die Zeit auch das Zwischen aber hier einfach, das Zwischen der Änderungsstufen.

Raumentfernung, so viele Körper zwischen. Zeitentfernung so viele Vorgänge-Veränderungen zwischen. Aufhebung des

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The Riemann-Mach Definition of Extension: A Closer Look

In his notebooks Mach worked through his doubts about Riemann's "modes" and asked how a solid extension could ever really be composed of an alternating mixture of qualities:

> How does it happen that the diagonal is longer than the two sides of the square, if the length is a difference of properties of the elements of the manifold: $m=\alpha A+\beta B$. The third side applied perpendicularly must be the sum of both of the others. Is the distance the sum of differences? Homogeneous function $f[dx^2+dy^2+dz^2]^{1/2}$. But dx dy dz is itself distance.[62]
>
> Mixture of several elements. A member corresponds to a point of a space of several dimensions. But where is the volume? Where the derived distance.[63]
>
> Coordinates already presuppose distance. Distance. Straightness. Straightness rigid bodies. H in the plane.[64]

By 1907, Mach came to regard extension as a *Wechselbeziehung* (a relation of two alternating terms) of two qualities situated in independent directions, as it were along the "two

---

unmittelbaren Contakt. Notebook Dated 27 November 1908 (NL 174/2/52).

62 Wie kommt es dass die Diagonale länger, als die beiden Quadratseiten, wenn die Länge Differenz der Eigenschaften der Elemente der Mannigfaltigkeit ist: $m=\alpha A+\beta B$. Die dritte Seite müsste bei senkrechtem Ansatz die Summe der beiden anderen sein. Ist die Entfernung die *Summe* der Differenzen? Homogene Function $f[dx^2+dy^2+dz^2]^{1/2}$. Aber dx dy dz ist selbst Entfernung. Notebook dated 21 June 1893 (NL 174/2/36).

63 Mischung von mehrere Elementen. Ein Gleid entspricht einem Punkt eines Raumes von mehrere Dimensionen. Aber wo ist das Volum? Wo die abgeleitete Entfernung. Notebook dated 5 October 1902 (NL 174/2/45)

64 Coordinaten setzen schon Entfernung voraus. Entfernung Gerade. Gerade feste Körper. H in der Ebene. Notebook dated 29 May 1898 (NL 174/2/42).

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legs" of the right triangle:

> The hypotenuse through the exchanging relationship of two qualities. [He adjoins a picture of a right triangle E.B.] Inverse quadratic with the distance. Foundation of distance measurement.[65]

Riemann had written that the invariant length ds could be expressed as a function of the legs of an infinitesimal right triangle dx and dy. The question that perturbed Mach was: if this length ds is the first extended quantity, what are dx and dy? Riemann's use of the Pythagorean theorem for infinitely small quantities was confusing when taken as a definition of raw "extendedness" since the legs which he assigns the "lengths" dx and dy are still extended. True, they are limits of extended quantities, but they are never allowed to be of zero length.

The deeper question really concerned the nature of Riemann's qualitative modes. To make sense of both associations and dissociations between them when they are used to construct coordinates each mode must be sometimes independent, as well as sometimes dependent, on the other mode. Thus although blue and yellow are usually conceived as opposing and thus dependent properties, we may imagine that they could also be independent of each other. For example a person could produce a sequencing of objects by more or less

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[65] Die Schiefe durch die Wechselbeziehung zweier Qualitäten. Verkehrt quadratisch mit der Entfernung. [Here he adjoins a picture of a right triangle E.B.]. Grundlage der Entfernungsmessung. Notebook dated July 14 1907 (NL 174/2/51).

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blue by systematically applying a standard exemplar. He would not know of a counterpart doing the same with a standard yellow. They are two logically independent orderings; and if we like we can think of each of these independent blue and yellow sequencings as two "qualitative" x and y axes analogous to the legs of a right triangle in independent directions.

These modes must also be capable of passing into relations to other qualities. As Riemann maintained, quantity requires that the two modes are capable of alternating on and off with one another in a higher order concept. We can imagine the "hypotenuse" of the qualitative right triangle as produced by the mixture of the two dependent and independent modes, and constructing elements of a more complex, extended magnitude incommensurable by either mode alone.

An Schematic Example of Simple Extension

These ideas will seem less difficult with an example, and I hope my own schematic construction of a simple extended stretch out of colors will make this easier to follow. Mach held that blue and yellow colors were antagonistic processes, related like plus and minus and like Herbart's pressure-qualities. But let's assume instead that properties of blue and yellow can be both dependent and independent of one another, should they fall into dependence on other qualities, which we will call X and Z. Let us also assume that this dependence alternates in a regular way. The point of these

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switches is to allow the hues to change their intensities while upholding a universal dependence.

I am concerned specifically with the passage, or change, of an extensionless point from a certain qualitative coordinate (B,Y) to a certain other with a different proportion of blue to yellow, (B′,Y′). We start with B and Y hues dependent on one another, which I will represent as follows, building the concept of extension in stages:

1. {B,Y}

Now imagine that B and Y dissociate and suddenly change their values as they come into dependence on two background qualities X′ and Z′ as we mentioned:

2. {{B,Y}, {{B′,X′}, {Y′,Z′}}}.

In this notation I group the most closely associated qualities in brackets and reproduce along with them the vanished past stages of the process. At this point let us say that the new values B′ and Y′ also happen to add to 1 in a new proportion, in which case we need only fuse them back together at those new values, B′ and Y′:

3. {{{B,Y}, {{B′,X′}, {Y′,Z′}}}, {B′,Y′}}

That completes the passage from the coordinates (B,Y) to (B′,Y′). 3. is thus the simplest concept of quantity or

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extension, a progressive sequencing of associations and dissociations of two different modes B and Y with the help of certain background qualities.

There are some details I leave out. For example, the qualities in the background undergo the same process, so that the simple extension from one (B,Y) coordinate to another is really part of a double-row, including the (X,Z). The exchanges are also more complex than indicated, as for example when B remains fixed in magnitude and Y is allowed to depart in value, which I have elsewhere symbolized (B-Y). There are also combinations of movements we could program a point to carry out. We could even consider instructions for progressions from all points to all others through all paths. Standardized distances could then be defined as an equivalence class. Paths that are equal backwards and forward, are transitive in length, and equal zero from any point to itself, for example are the class of equal distances. Paths are extended already, but *only* such standardized paths qualify as distances.

Relation to Mach's View of Physical Space

As we have been saying, metric properties or invariants on the manifold are separate from the raw construction of extension and the laying down of a coordinate system. It is possible to consider manifolds in which properties change from one location to another and over time. Invariants like

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distance are defined as those properties that remain the same through a group of transformations from one set of coordinates to another. On Mach's metric conception this means that a standard length, or even a tone-interval, may be transformed by rigid motion and remain recognizably the same, and thus that all lengths in the manifold can be compared with one another through the transformations we allow.

Mach was criticized for his "metric" conception of space by Franz Brentano and Herbert Buzello, who pointed out against both him and Riemann that eliminations of space and time in favor of systems of measurement did not eliminate the intuition of space, since it was still presupposed that the standard metre stick was capable of moving about *in* space, and that the chronometer is capable of measuring intervals of time *over* different times.[66]

This is by now a common cavil. But I think these critics confounded the invariant with the properties used to construct the coordinates. They had in mind a construction like Helmholtz's, in which the property of distance plays both parts. It seems to me that Mach was aware of the distinction because of his work in sense-physiology, where the properties were customarily separated, and that his scheme did assign the

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[66] Franz Brentano *Über Ernst Machs "Erkenntnis und Irrtum"* R. M. Chisholm und Johann C. Marek, eds., *Studien zur österreichen Philosophie* B 16. Amsterdam: Rodolpi, 1988, p 208 ff.; Herbert Buzello *Kritische Untersuchung von Ernst Machs Erkenntnistheorie*, *Kantstudien* Ergänzungsheft No. 23 (Berlin: Verlag von Reuther & Reichard, 1911) pp. 28-33.

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standard units of measure by means of a second physical property.[67] When he said that space and time were absolutely "reducible" to the behavior of elements he probably had in mind a two stage reduction.[68]

Mach's psychological analysis of sensation qualities in the color manifold of *Knowledge and Error* remains his most penetrating attempt to get at the nature of space in qualitative terms. However he never really stopped thinking about these constructions. The issue will resurface once more in Chapter sixteen.

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[67] Mach wrote in *Knowledge and Error* that light rays and interferometry would play the role of the standard "line drawing" process: p. 348:

> We still need rigid bodies to build our equipment, but we can use light interference to mark points and measure stretches by wavelengths in seemingly empty space much more accurately than would be possible by means of rigid bodies that abut and touch one another. It is even likely that light waves will furnish future physical standards of length and time, in terms of wavelength and period of oscillation respectively and that these basic standards will be more appropriate and generally comparable than any others. Through such changes space and time increasingly lose their hyper-physical character.

[68] See for example His "Sensory Elements and Scientific Concepts" in (DL), pp. 119-120: What we call spatial and temporal order is abstracted from the behavior of elements. If we want to spatially determine an element, we will say, by and between which other known elements it is to be found. For the temporal determination of a change of an element we say, with which and between which known changes of other known elements does it take place. Every determination by means of spatial and temporal coordinates is just a convenient convention-based circumscription of this procedure.

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Chapter Seven: The Episode of the Monocular Depth Sensations

Introductory Remarks

Mach's belief that spatial length width and depth were sensations like any other hardened him further against taking these properties realistically. Space seemed to him biological in nature and thus ineliminable from our imaginations, but as we saw this very fact called even more urgently for an antimetaphysical elimination of space from physical thought. Reviewing the details, as difficult as they are, does make it clear why Mach thought space was a sensation.

Philosophers will often read the passage in the *Analysis of Sensations* to the effect that the world "consists of sensations, colors, spaces and times"[1] and assume that he is laying down these elements by fiat. Actually, these results lay at the end of a long analysis in Mach's scientific works. This chapter is thus an episode in the history of psychology which informed Mach's philosophy of space.

Mach's Space Sensations

Mach held that in addition to the muscular sensations and sensations of will, certain of the space sensations were due to the local sign system of the retina, particularly monocular and binocular sensations of depth. Mach followed Ewald

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[1] (AS), p.2.

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Hering's 1865 nativist theory of stereoscopic vision, but he also applied his work on neural inhibition to the retinal local signs, enhancing Hering's theory. Local sensations of place and depth mutually inhibit one another, bending, distorting and scaling to the average, just like light sensations. Mach's investigations of the 1860s on the relationship between Mach Bands and Monocular Depth Sensations are a lost episode in the history of psychology which shows just how much Mach was willing to consider space a sensation like light or color.

Mach Bands and Neural Inhibition

Mach's most important psychophysical researches in the Graz and early Prague period (1864-1868) centered on what are now called Mach Bands. These subjective bands are produced by certain brightness distributions on the retina that suffer changes in brightness above or below the average of neighboring points. Wherever a point is above average it appears unusually bright; when it is below average it appears unusually dark. In cases where the brightness increases uniformly, say along a cylindrical wall, Mach remarked that we do not notice a rise in illumination, but experience instead a sensation of depth.[2] Interestingly in the most obvious case of a sudden, discontinuous change of brightness, an edge of

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[2] (MB) p. 264. On the relation of uniformly increasing light sensations and depth sensations, see pp. 288-289.

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pure black against pure white, no Mach band is produced.[3] The best examples are provided by a black and a white sector separated by a penumbra of ever lighter grays, a so-called trapezoidal wave.

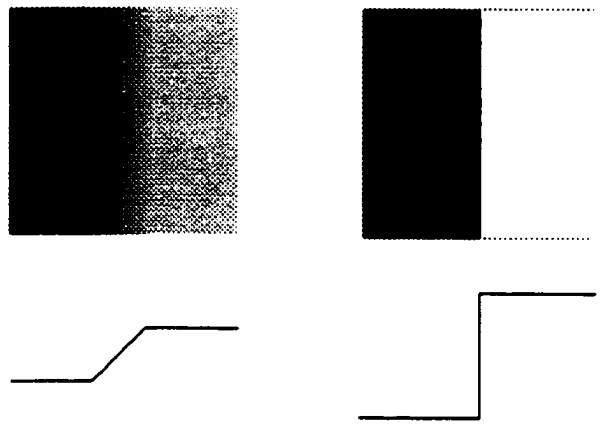


FIG. 7.1: Trapezoidal and Square Waves

Already in his first paper Mach offered an explanation that has since become standard.[4] He knew from physiology that the light receptors do not plug into the optic nerve directly,

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[3] M. Concetta Morrone et al. "Mach Bands are Phase Dependent" *Nature* **324**(20) November 1986: 250-253.

[4] (MB) pp. 267-271.

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but are connected with one another, so that one optic nerve fibre culls the responses of about 100 cross-connected receptors. Mach wrote that the receptors must undergo "a series of reciprocal actions" before their excitations are transmitted to the optic nerve.[5] Each receptor's activity laterally inhibits that of the neighboring receptors before they all become sensations of light.

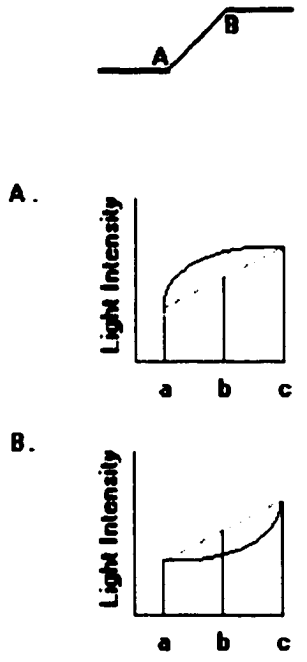


FIG. 7.2 Mach Bands Explanation

We still notice the absolute illuminance of course, the difference between candle light and a strongly lighted room; but if there are any contrasts they will be greatly

[5] (MB) pp. 268-269.

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exaggerated. Mach remarked: "The visual organ shows a definite unmistakable tendency to schematize, in that it merely notices striking irregularities. The same tendency manifests itself in details as well as the whole field."[6] The whole retina scales its impressions of brightness from the middle tone of the visual field, the way photographers do when they meter for the middle tone of a picture and not the absolute brightness, allowing absolute grays to overexpose to whites and dark tones to underexpose to black, so that the maximum detail of the image is attained. The evolutionary advantage of such an optic system is that an animal's perception of contour and shape remains unaffected by the level of light present.

The neural tissue of the retina is anatomically similar to brain tissue and, as we saw in a previous chapter, Mach extended his analysis of the retina to thought in the brain; "why," he said, "should the ganglion cells of these organs behave differently than those of the rest of the nervous system and the brain?"[7] For Mach the eye thinks the same way the brain does, and there is no need to refer all visual processes to central inferences, [*Unbewusster Schluss*] as Helmholtz had done. Helmholtz had maintained in the monumental third volume of his *Handbook of Physiological Optics* that many sensory phenomena and illusions could be

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[6] (MB) p. 265.

[7] (MB) p. 269.

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attributed to inferences by the central nervous system, and not the organization of the sense organs. Mach took exception to this:

> I now have grounds to believe that the majority of the phenomena which we have believed to be errors of judgment (i.e. essentially due to processes in the central nervous system) actually have their basis in a more autonomous behavior of the sense organs. I will describe a series of observations that led me to the view that is already becoming familiar in comparative anatomy, namely that the sense organs are to be considered as subordinate central organs.[8]

Indeed Mach believed that consciousness or attention was grounded in the mechanics of neural inhibition--a result he reached in his work on hearing in 1863.[9] Ratliff[10] firmly agreed with Mach's use of neural inhibition to describe central processes, as did neurophysiologist R. Jung at the Freiburg Symposium:

> The visual phenomena discovered by Mach are not only sensory mechanisms for the improvement of the perception of contrasts, but rather also general principles of distribution for excitation and inhibition: every stronger excitation in the central nervous system is surrounded by a wall of inhibition. This hinders unbounded widening of the excitation, which would otherwise lead to massive neural discharge and to epileptic seizures. The physiological bounding of excitation works with contrast mechanisms toward raising to prominence significant modules of excitation [*Erregungsmuster*] and at the same time serve a functional ordering which keeps the entire nervous system at a mean level of excitation. Mach had already expressed similar

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8 (MB) p. 308.

9 "Zur Theorie des Gehörorgans" *SW* **48** (1863) p. 297.

10 "On Mach's Contribution to the Analysis of Sensations" in Cohen and Seeger, eds., *Ernst Mach: Physicist and Philosopher*, (Dordrecht: D. Reidel 1970).

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thoughts in his economy principle.[11]

Georg von Békésy, the Nobel prize winner for his work on hearing, who rediscovered Mach's work on neural inhibition in 1928, attributed the neurological law of contrasts to Mach and mentioned its application to the nerve endings of the cochlea in the inner ear. The stimulus produced in the cochlea by travelling pressure waves, first demonstrated by von Békésy, was too far diffuse and required a mechanism for focusing the stimulus into localized sensations of tone:

> It appears that Mach's law of contrast should play a highly significant role in auditory theory. It achieves a certain patterning of sensation. A distribution of stimulation in the form of a single resonance curve will produce a high peaking of the maximum because at this point the stimulus intensity undergoes its greatest variation, whereas on either side the stimulus effect falls off and the curve there is convex downward.[12]

Von Békésy also produced an article for the Freiburg Symposium entitled "Die Erweiterung der Existenz der Mach-Bänder auf andere Sinnesgebiete" (Extending the Existence of Mach Bands to other Senses) in which he showed that the skin and even the taste buds exhibit the phenomenon.[13]

The Mach-Békésy-Ratliff theory of neural inhibition for

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[11] *Symposium aus Anlaß des 50. Todestages von Ernst Mach* (Freiburg: Ernst Mach Institut, 1966) p. 135.

[12] Georg von Békésy *Experiments in Hearing* (New York: McGraw Hill), p. 418.

[13] Ibid. pp. 161-169.

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vision was challenged more recently in 1986, by Morrone et al.[14] They started from the fact that Mach Bands disappear for square waves, and found instead that the bands actually occur only when the Fourier components (sine wave gratings of light and dark) of the light distribution are 180 degrees out of phase with one another, i.e. when the light distribution is a mixture of odd and even sine-waves.

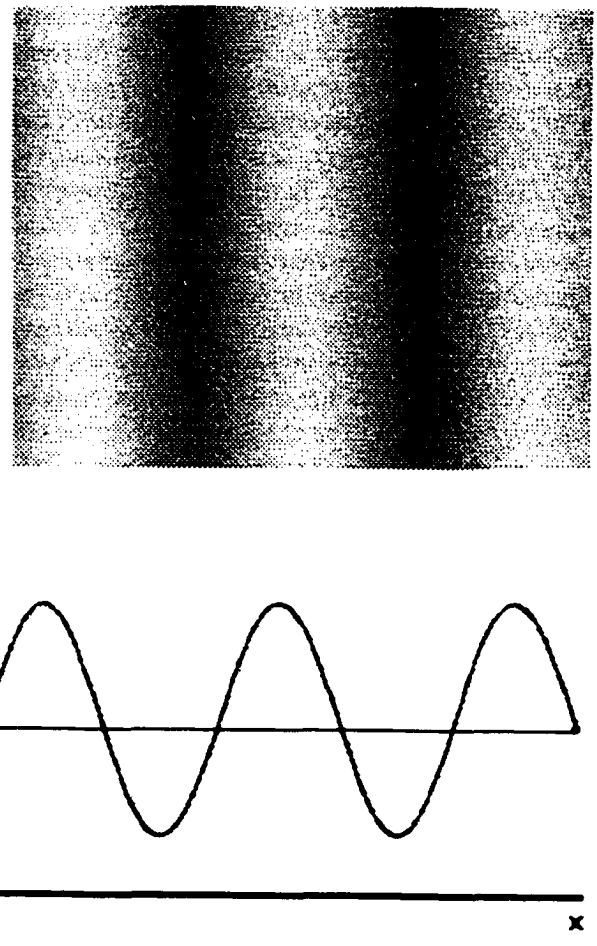


FIG. 7.3 Spatial Frequency

[14] M. Concetta Morrone et al. "Mach Bands are Phase Dependent" ***Nature*** *324* (20) November 1986: 250-253.

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Mach himself recommended this representation of light surfaces in Fourier components, or what is called spatial frequency, light distributed periodically over a distance or area in a fuzzy sine-wave pattern (see Figure 7.3).[15]

Trapezoidal waves are mixtures of odd and even sine wave components, and produce the strongest Mach bands. Square waves are a mixture of all positive components and produce an edge instead of a band. This gives strong evidence that both Mach bands and edges depend on phase and that the eye sees the simpler sine-wave gratings directly. There is existing evidence that the eye sees light in "channels" of spatial frequencies.[16] However as Barlow and Mollon point out, these facts can be explained by the neural inhibition model.[17] The issue is still very much alive.

Mach Bands and Monocular Depth Sensations

Mach's sallies against Helmholtz indicated that he was aligning himself with the "nativism" of Ewald Hering.[18] The issue came to a head in the middle 1860s when Hering claimed that spatial depth in stereopsis was sensed by means of an innate organization of the two retinas, and was not inferred

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[15] (MB), p. 292.

[16] H.B. Barlow and J.D. Mollon, eds., *The Senses*, (Cambridge: Cambridge University Press, 1982), pp. 148-151.

[17] Ibid.

[18] (MB) p. 298.

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from experience.

Hering modelled his theory of innate depth perception on Johannes Müller's theory of "corresponding points." Müller knew that the nerves from the visual field cross over one another at the optic chiasma on their way to the brain where the left retina's temporal half connects up with the right retina's nasal half and vice versa. This would be an odd state of affairs were it not useful for stereopsis.

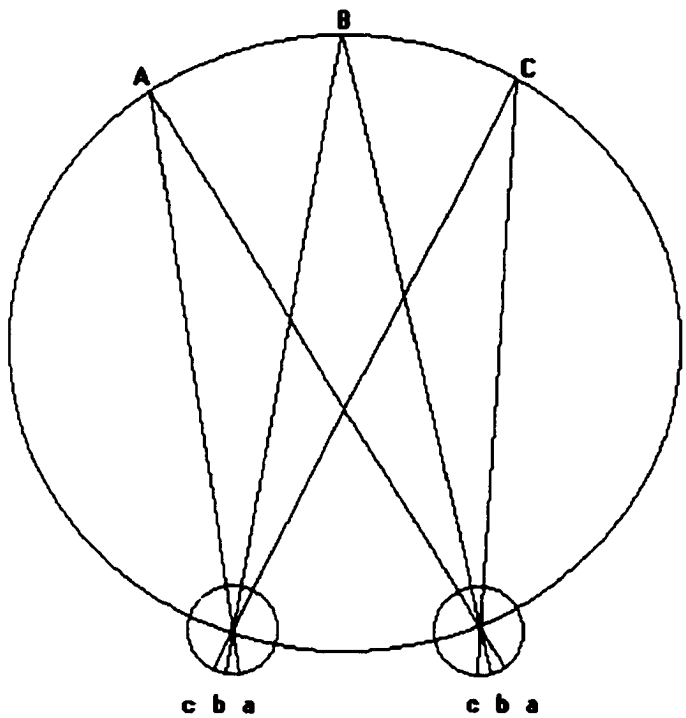


FIG. 7.4 Muller's Horopter

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For Müller, corresponding light sensations were carried along the same nerve and he hypothesized that such points would be seen single and fused in a sensation of depth. The other images along non-corresponding points would be seen double. The set of all corresponding points that would be carried along the same nerve from both eyes Müller called the horopter and drew it as a large circle passing through the point on which the two eyes are fixed and the optical centers of both eyes (see Figure 7.4). It was quickly pointed out that points not on the horopter could also be seen as fused and in depth. There is in fact a whole area of fused images surrounding the horopter called Panum's fusional area. The physicist Wheatstone used his newly-invented stereoscope to show that two images on non-corresponding points could be fused if they shared a similarity in spatial form.

The old rival theory of projection was thus revived, in which it is argued that the eye starts with two retinal images, somehow recognizes them as similar, and calculates from their retinal disparity, the distance between the two eyes, and their angle of convergence what depth an object would have to have in order to cast the two images. These inferences are unconscious and automatic, thus we see not the two images but a fused image in space external to our eyes. Helmholtz seems to have inclined toward a form of the empiricist projection theory, although he endorsed nativism for some elements of vision.

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Hering's theory of stereopsis, which appeared in Part Five of his *Beiträge zur Physiologie* in 1865,[19] revived Müller's theory of corresponding points. As R. Steven Turner writes in his book *In the Eye's Mind: Vision and the Helmholtz-Hering Controversy*:[20]

> Hering found it inherently implausible that the eye's mind should be constantly recalculating the distance of external objects and projecting their images appropriately. He charged that projection theories merely reify the geometrical abstractions of the physicist and substitute them for real subjective perceptions and their underlying physiological mechanisms. This criticism encapsulated his lifelong hostility to explanations that portrayed the processes of sensory perception as analogous to higher intellectualistic functions of the mind, rather than as immediate expressions of the life forces in the sense organs themselves.[21]

In contrast to Müller, who thought that the corresponding sensations "added" in fusion, Hering regarded the connections of the corresponding points as mutually inhibiting. For Hering, points connected up to the same nerves averaged-out one another when they were fused into a binocular perception.

To show how the process of inhibition worked, Hering demonstrated first that the direction of sight for both eyes is an "average" of the lines of sight of each individual eye. He proved this by an elegant experiment. Sit down in front of a window with your left eye shut and the right eye fixed on a

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[19] (Leipzig: Wilhelm Engelmann).

[20] (Princeton: 1994).

[21] Ibid. pp. 61-62.

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distant object, such as a flagpole. Mark the spot on the window with an X. Now close your right eye and open your left and note the object your left eye sees through the X. Say it is a tree. Now open both eyes and stare through the X. Surprisingly both the flagpole and the tree are seen in the same straight line through the X. Thus objects lie in the same visual direction for the fused visual field, that are in different, opposing directions for each eye. Hering explained these facts of visual perception with the following scheme:

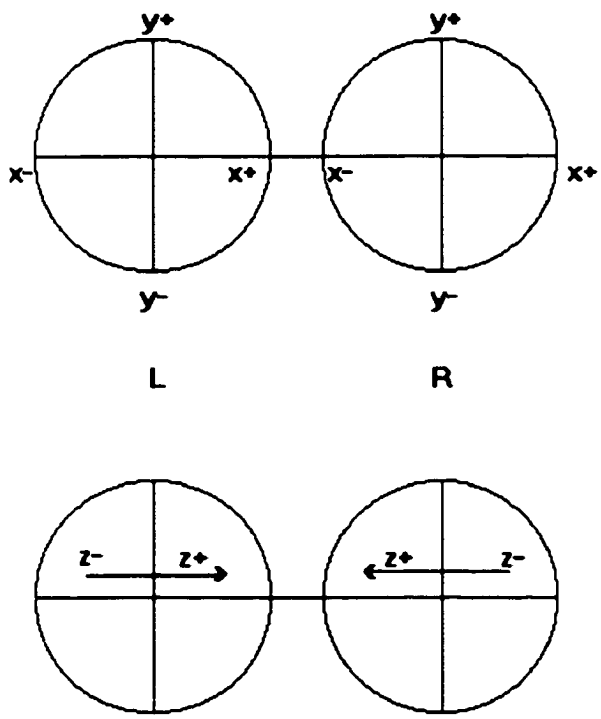


FIG. 7.5 Hering's Scheme of Length, Width and Depth Local Signs

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Each eye is to be divided into x and y axes, with the fovea at the origin. These conventions give us a full set of height and width sensations with positive and negative sign. Now Hering said that whenever we use one eye, the position of the object is immediately sensed on this grid of local signs. When we use both eyes the direction in the fused image is the algebraic sum (average) of the x and y values with signs taken into account. Images on corresponding points are equal in value and opposite in sign and are thus seen in the same direction.[22] Other points not undergoing inhibition may be seen double.

Hering's solution to the problem of depth was to postulate local depth signs on the retina. These depth values were so arranged on either side of the retina to increase in value from the nasal side to the temporal side of each eye, and were opposite in sign on either side of the middle.[23] Objects on corresponding points have equal and opposite depth values and are seen on a mean zero-depth surface that Hering called the *Kernfläche;* objects far and close are seen at positive or negative depth, figured from the baseline position of this surface. Hering showed that all points on the somewhat elliptical horopter are seen as if they lay on a flat wall at the same depth.

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[22] (AS), p. 124.

[23] (AS), p. 124.

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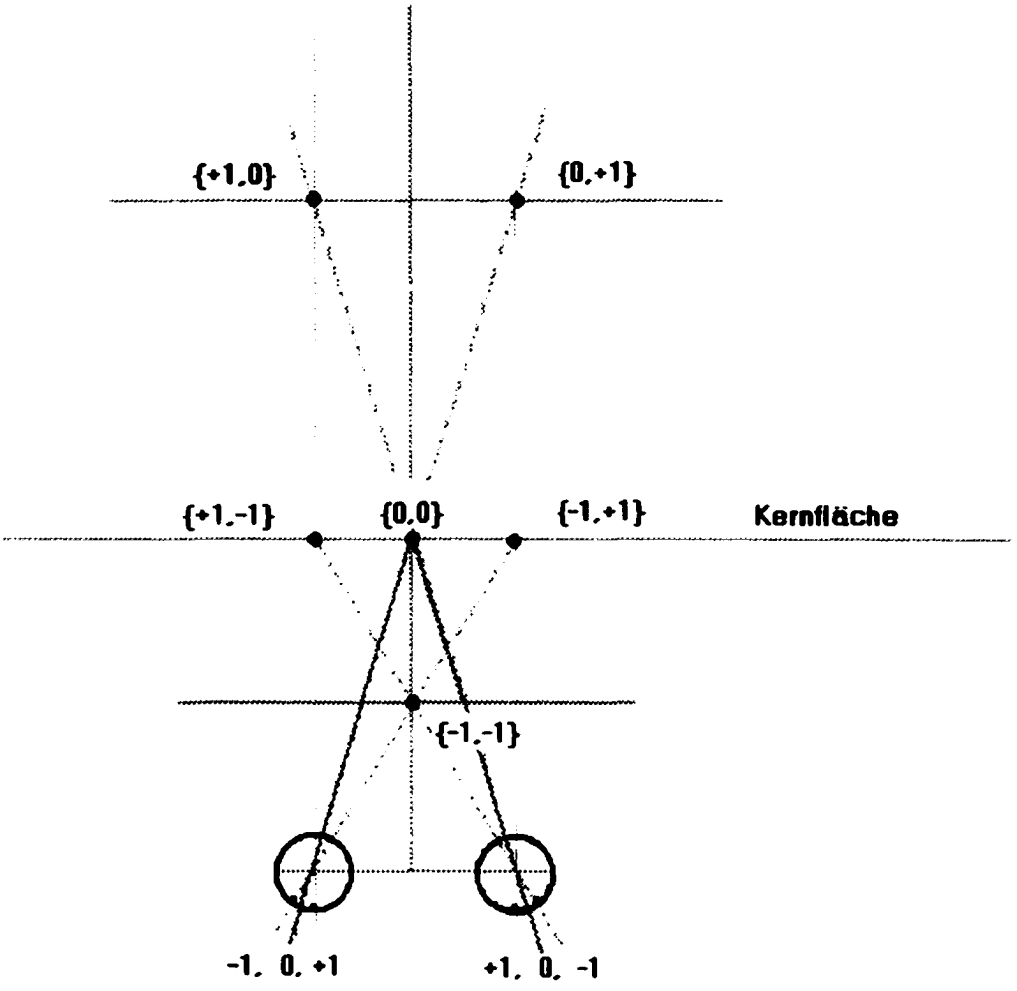


FIG. 7.6 Hering's Depth Sensations

The theory is simple, elegant and accounts for double images on non-corresponding points, giving them each a separate depth value, but no average since they do not inhibit and fuse. Mach declared himself a follower of Hering in

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1866,[24] even after the appearance of Helmholtz's attack in his *Handbook of Physiological Optics* Vol. III. Indeed, in the *Analysis of Sensations* one would never guess there was any other theory of vision but Hering's.[25]

Helmholtz's Critique of the Monocular Depth Sensations

Helmholtz's most trenchant criticism targeted the "monocular" depth sensations. He reasoned that if depth values were spread out along the retina from right to left, covering one eye or parts of one eye would leave only those monocular depth sensations active. For example if we cover one eye, a horizontal wall should be seen angled towards the one eye, since points that fall on the nasal side of the single eye should be seen as having greater depth than the points on the temporal side.

Hering himself had been forced to admit that certain experiential cues could override the depth sensations. And there were other arguments and responses concerning the "re-learning" of stereopsis by people with improperly aligned eyes, impossible if the corresponding points must be precisely connected, and nerves cannot regrow.[26] Turner writes that as of 1866 the "monocular" depth sensations became a dead issue:

Neither [Hering] nor any of his students ever again

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[24] (MB) p. 298.

[25] (AS) p. 124.

[26] Turner pp. 168-169.

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> defended the hypothesis of monocular depth-values following Helmholtz's attack of 1866.[27]

He overlooks however Mach's article of 1868 "Beobachtungen über Monoculare Stereoscopie"[28], his Mach Bands articles and the *Analysis of Sensations* in which Mach attempted to save monocular depth sensations for Hering. Indeed in the Mach band articles Mach turned immediately from his study of mutually inhibiting light sensations to monocular sensations of depth, a rather puzzling turn of events out of context. A contemporary however could not read Mach's 1868 paper on "monocular depth sensations" without thinking immediately of the Hering-Helmholtz controversy and the Mach Band papers are peppered with broadsides against Helmholtz and defenses of Hering.

Mach hoped to save Hering's theory by arguing first that monocular depth sensations could be activated directly by sensations of light. Second he applied his theory of neural inhibition to explain why steady increases in depth sensation went unnoticed, as in the example of the wall, while increases above the average of neighboring depth sensations were picked up. The extension of complex neural inhibition to depth sensations is entirely new and original with Mach.

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[27] Turner p. 86.

[28] *SW* **58**: 731-736.

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Light and Depth Sensation

In the third Mach band article "Über die physiologische Wirkung räumlich vertheilter Lichtreize (Dritte Abhandlung)"[29] Mach called attention to the relationship between light and depth sensation, using the phenomenon of the Mach Card:

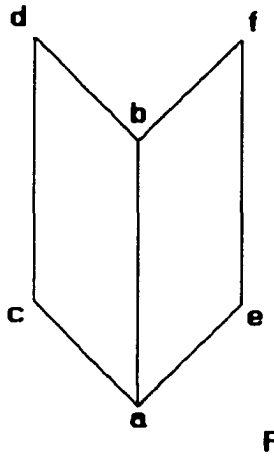


FIG. 7.7 **Mach Card**

> If I fold a rectangular piece of stiff paper along a straight line to form two congruent rectangular parts and place it on the table in front of me so that when vertical it has the projecting side turned toward me, I obtain the image shown in [the figure above E.B.]. The light should fall from the left and the right side will thus be somewhat darker. If then I close one eye, the sensation is almost completely unaltered. If I try to see the paper as recessed, however, I succeed after some effort. The left side thereupon appears to become much brighter and the right side much darker. Light and shade appear as if painted upon it. Moreover I notice a different color on the right and left sides which had not been noticeable to me earlier. Furthermore the form of the paper has changed. The lines bf, bd have become

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[29] SW **54** (1866): 393-408 translated in (MB), pp. 285-298.

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> longer than ae, ac, the lines cd and ef shorter than ab. Finally the corner ab no longer remains vertical, but the paper lies like an open book before me. These simultaneous alterations of the sensation of depth, the sensation of form and the sensations of light which occur without the slightest theoretical deliberation, indicate how quickly these things, through well-regulated reflexes transform themselves into one another. It thereby appears that the sensation through which another becomes elicited, itself diminishes in the same proportion as it elicits the other.[30]

The reader should try the experiment. An ordinary piece of white paper and a desk light will do, and the result is indeed fascinating. Instead of a white paper with shading, we get the opposite impression of black paper bathed in strong light. The reversal of the element of depth occasions a reversal of the light sensations, exactly as if the two were rigidly connected. As the light sensations of the paper reverse, depth sensations arise to take their place. In addition the lengths shrink as the rectangular dimensions of the paper contract into two trapezoidal shaped sections. There is thus a similar connection between depth and length. Mach thought so highly of this connection that he compared it to the conservation of energy, for example as the potential energy decreases the kinetic energy increases.[31] Mach used this invariant relationship as a basis for objecthood in the visual field, the idea being that when the surface of an object tilts away and reduces its area of illumination, the depth sensation

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[30] (MB), pp. 295-296.

[31] (AS), p. 211.

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increases and supplies the lack. We see a tilted surface in depth rather than a smaller surface.

> ...we may expect that whenever a coherent mass of sensations, which, in virtue of its continuous transitions and its common coloring, merged into a unity, exhibits spatial alteration, the change will be seen preferably as the motion of a solid body. I must confess, however, that this way of looking at the matter does not satisfy me. I believe rather that here too an elementary habit of the organ of sight is at the root of the matter--a habit which did not originally arise through the conscious experience of the individual, but on the contrary, antecedently facilitated our apprehension of the movement of solid bodies. If we should assume, for example, that every diminution of the transverse dimension of an optical sensation mass to which the attention was directed had the tendency to induce a corresponding augmentation of the dimension of depth, and vice versa, we should have a process quite analogous to that which we have considered above [The Mach Card E.B.] and which was compared to the Conservation of Energy. This view is certainly much simpler and supplies an equally adequate explanation. Furthermore it allows us to comprehend easily how such an elementary habit could be acquired, how it could find expression in the organism and how the disposition toward it could be inherited.[32]

Mach also called attention to the fact that although a steady increase in illumination didn't create a sensation of contour in light, it *did* occasion a sensation of relief on a raised surface. In his third Mach Band article he argued that because of the physical laws of illumination (light intensity falls off not with the distance but with the square of the distance) a cylindrical surface will cast an image that rises only steadily in lightness. The direction of the light then determines whether a given surface is seen as convex or

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[32] (AS) pp. 233-234.

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concave.[33] Mach thus claimed that depth was automatically "triggered" by light sensations:

> Let us imagine for a moment that every retinal point could generate sensations of depth of different intensity in addition to light sensations of different quality and intensity. It would now require nothing more than a simple reflex mechanism, which would trigger the depth sensations simultaneously with the light sensation, for this phenomenon to be understood. Three successive retinal elements, from right to left, of which the left is always more strongly illuminated than the right, would always generate in themselves three depth sensations, which would increase in the opposite direction, but in such a way that the middle element would produce a somewhat greater depth sensation than the average depth sensation of the two neighboring elements.[34]

Adding detail to this theory, he remarked in his Notebook of 1904:

> Theory of plasticity and illumination. A constant increase of light corresponds to a constant curvature of the surface. The depth sensation integrates the light intensity.
>
> Light intensity---rise, fall in depth
> Rise in intensity--curvature of the surface.[35]

What this means is that a surface of constantly increasing illumination, following the law of a straight line with an upward slope, is assigned an upward *curve* of depth values, the

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[33] (MB) pp. 287-288.

[34] (MB) p. 289.

[35] Theorie d. Plastik und Beleuchtung. Einer Constanten Steigung d. Lichtes entspricht eine Constante Krümmung der Fläche. Die Tiefempfindung integriert die Lichtintensität.

Lichtintensität ---·-- Stei. Gefäll in d. Tiefe
Steig. d. Intens. --·-- Krümmung in d. Tiefe. Notebook dated 8 June 1904 (NL 174/2/47).

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"integral" of the straight line.

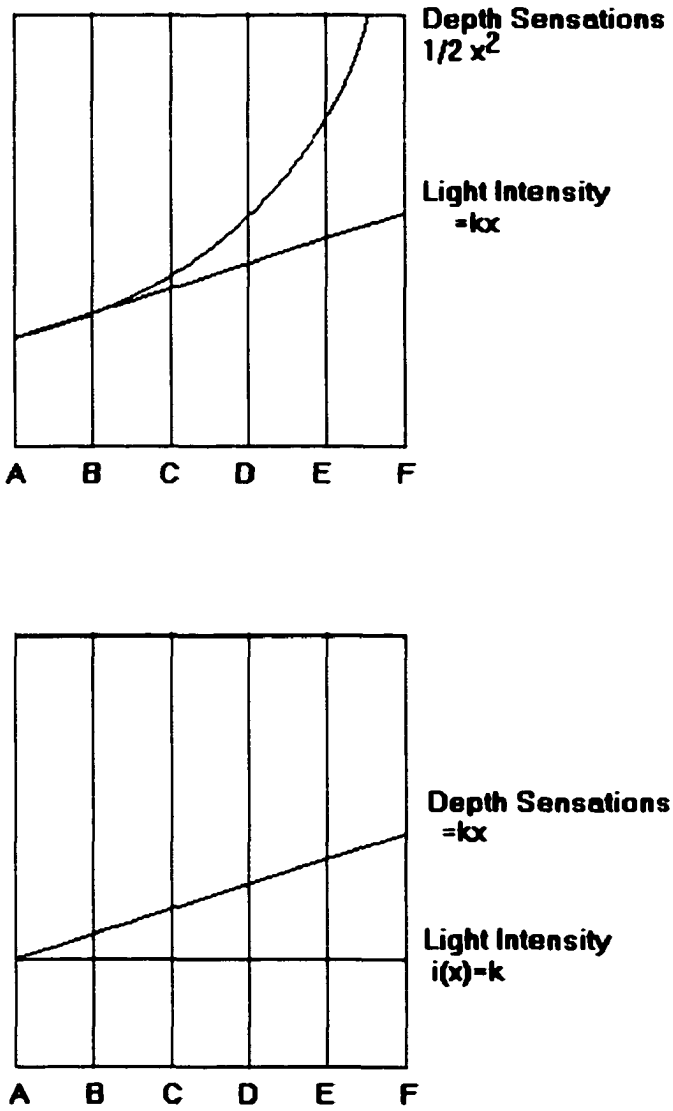


FIG. 7.8 Depth Sensations as Integral of Light Intensity

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Likewise a constant illumination would produce a straight line increase in depth sensations on the retina, increasing from right to left or left to right depending on which eye was used. However--as Mach would go on to show--depth sensations follow the law of neural inhibition, just as light sensations respond only to above average absolute illuminances. Thus, in the case of the wall, the steadily increasing depth sensations are ignored, since none exceeds the average value of its neighbors and thus the wall is sensed as flat, as it should be. This result, which is clearly implied by Mach's extensive work, seems to have been the kernel of his defense of Hering against Helmholtz.

Mach's "Monocular" Conception of Stereopsis

It occurred to Mach that if differences of illumination on one retina gave rise to automatically triggered monocular depth sensations, perhaps the same monocular principle generated stereopsis when both eyes were used. He remarked that stereopsis depends somehow on unequal light distributions on either retina.[36] He also pointed out that just as a uniformly rising illumination leads to a perception of curved relief for one eye, so two uniformly rising illuminations on both retinas give a still more plastic sensation in stereopsis.[37] Furthermore, detecting contours of objects is

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[36] (MB) p. 292.

[37] (MB) p. 290.

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not necessary for stereoscopic relief; we do not need to discern two shapes of similar objects on either retina and then compare their disparity as in the projection theory;--only the two unequal light distributions are needed:

> Each image alone gives the impression of the plastic cylindrical surface referred to above. They appear even more plastic when I superimpose them by crossing my eyes. In this way the stereoscopic images can be constructed without any contours.[38]

In the climate when this article was written, it was a direct attack on the projection theory that we must unconsciously attribute two light surfaces, which we pick out through contours, to an object. The fact that two light surfaces alone suffice for stereopsis was re-discovered in dramatic form by Bela Julesz in 1962, when he introduced the random dot stereogram.[39]

Mach speculated that the stereoscopic field of vision might behave like a monocular field, where spatial differences in the fused field of light sensations from the two eyes trigger sensations of depth, by the same mechanism the monocular depth sensations obey. He does point out that although the depth sensation originally results from "the reciprocal action of the two retinas it is possible to develop through the reflex connections of light and depth, depth

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[38] (MB), p. 290.

[39] Turner *In the Eye's Mind* (Princeton, 1994), pp. 271-272.

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sensations on the one retina."[40] In a Notebook entry of 1875 Mach speculated further:

> Both eyes together give stereoscopy. One eye can do it with the help of the other. It can be that the monocular is primary. Binocular the secondary thing.[41]

Mach also noticed that in monocular vision both moving objects, and objects that shrink or grow occasion depth sensations.

> Height and width grows with depth. Difference of height and breadth gives depth. [42]

In another place he made a more general remark:

> Light contrast depth. Space contrast depth.[43]

Changes in the spatial height and width sensations of an image are differences just as the static changes of illuminance are differences. Physiologically nothing distinguishes them from sensations of light. Mach draws an implicit analogy here between an object moving, or shrinking or growing, in the monocular field and the two views of an object presented simultaneously in the fused binocular field. Abstractly considered, both contain differences (in height, width and

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[40] (MB), p. 298.

[41] Beide Augen zusammen geben Stereoscopisches. Ein Auge kann es mit Hilfe des anderen geben. Es kann aber das Monoculare auch das Primäre sein. Das Binoculare das secondäre. Notebook dated 20 August 1875 (NL 174/02/07).

[42] Höhe und Breite wächst mit der Tiefe. Differenz von Höhe und Breite gibt Tiefe. Notebook dated 20 August 1875 (NL 174/2/7).

[43] Lichtcontrast Tiefe. Raumcontrast Tiefe. Notebook dated April 1882 (NL 174/2/19).

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light) that occasion depth sensation.

Stereopsis: the Phenomenological Mystery

The perennial mystery of stereopsis is how two disparate two-dimensional images of different light distribution, length and width fuse into a three dimensional image of homogeneous light, length, width and depth. The actual process of phenomenologically putting the two images together seems utterly baffling. As we now know from sense physiology, on either side of the brain the signals from the optic nerves of both eyes overlap in alternating, irregular strips in area 17 where a full mapping of the points of each half of the visual field onto the brain occurs.[44]

If we imagine that the visual cortex is a fused monocular surface, then it will consist of a row of single images near the foveae and the corresponding points, and a set of overlaid double images increasing in disparity and superimposed flat on top of one another. Yet from Mach's point of view, the lengths and widths of the double images are just sensations that can shrink or grow in place as their annexed depth sensations rise or fall. This is just a case of mutually influencing energies inhibiting one another in different ways. The example of the Mach card showed that sensations of a line's length, width and even orientation could all be altered

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[44] H.B. Barlow and J.D. Mollon, eds., *The Senses*, (Cambridge: 1982), pp. 44-45.

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by reversing the depth sensations of the figure. If these automated relationships existed in monocular vision, perhaps they also existed in stereoscopic vision. At the corresponding points of the *Kernfläche* the match is precise, but at diverging points due to retinal disparity the superimposed images gradually fall out of alignment and double. Some of these images will fuse in front of the flattened *Kernfläche*, with lesser depth values, others behind it with greater depth values, and the rest remain double, but possess their individual monocular depths.

The *Kernfläche* and the double images are easily explained by Hering's theory. It is the foreground and background of fused images that present the phenomenological problem. Mach had the idea that sensations of light, width and length on the monocular fused surface were all fluid, as in the example of the Mach Card. Hence by adding or taking away depth sensation, any two images close enough to one another on the surface (in light, as well as position) could distend and be made to coincide, so long as the retinal disparity was not too great. Even the spatial frequencies or sine-wave gratings of close enough double images would tend to become more fluid, darkening and lightening and undergoing spatial changes in distribution. No intellectual similarity between images is required if the adjustment is made automatically. Indeed this is quite in agreement with Mach's mechanical philosophy of mind.

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Mach's own roving eye soon turned to other topics and he did not write much more on stereoscopy. The Machian neural inhibitions involved, as he says, at all retinal points, would be much more complex than Hering's elegant +/- system for specific points. But Mach's theory would be more robust, more generally spread over areas, and would take light sensation into account. Perhaps Mach's more complex system of inhibitions would yield some of Hering's results as special cases when full binocular vision is in action.

The Law of Inhibition for Monocular Depth Sensations

In "Beobachtungen über monoculare Stereoscopie" Mach discussed how we see monocular perspective drawings in depth, for example simple line drawings of pyramids and cubes. Such perspective images, Mach pointed out, represent not just the two or three possibilities we usually think of (a cube with one side forward, then the other, as in the Necker cube) but an infinite number of possible figures in space, which the eye limits to two or three. For example a curve seen edge-on in space could cast two straight line images on the retinas. But two straight line images on the retinas are *always* interpreted as a straight line in space, never as a curve.[45]

A second fact about seeing lines is that wherever possible, inclined lines in space, say a flagpole that is bent into or away from the plane of the viewer, are seen as

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[45] (MB) p. 296

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standing straight up, turned into the plane perpendicular to the line of sight. Whenever possible, the angle of inclination between a line and the line of sight is always seen as a right angle.[46]

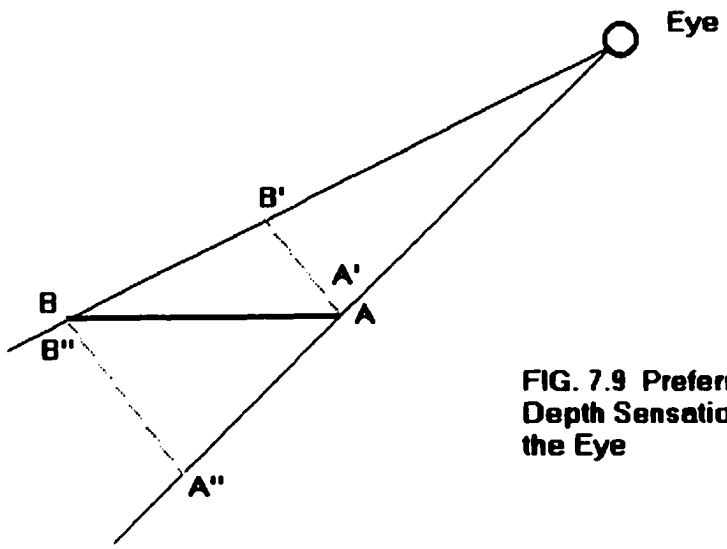


FIG. 7.9 Preferred Depth Sensations for the Eye

Mach attributed this innate preference to the principle that "the eye develops a *minimum* of depth differences." That is every depth sensation is assumed to be the mean of the neighboring depth sensations, the best case being all points at the same depth. Thus whenever we see a figure with lines

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[46] (MB), p. 297; "Beobachtungen über monoculare Stereoscopie" *SW* **58** (1868) p. 73.

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our first choice will be to see the line drawing as lying flat in a plane of equal depth. If we force ourselves to a sensation of a perspective drawing, the eye will try to see any angle as a right angle and interpret any deviations as due to the steadily increasing depth of points. But these depths will be the minimum needed to achieve the perception of a right angle in space, never curves or more complicated arrangements. For example, an angle ABC is regarded by the eye as cast by a right angle in a distance in space. This can happen in two ways: either B can be seen as pushed forward, or recessed back.

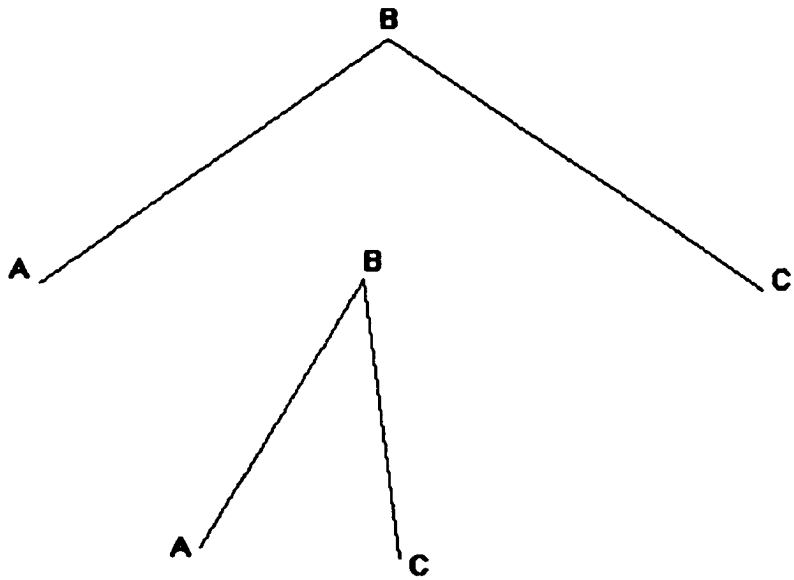


FIG. 7.10 All Angles Become Right Angles

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Drawings consisting of lines that already meet at right angles are not seen in perspective at all. For example children usually draw a table with all of its angles right and not in perspective, which is in some ways a more realistic representation than classical Renaissance perspective. Drawings with obtuse and acute angles appear to extend out of or into the page until the lines all meet at right angles in imagined three dimensional space. Mach thus wrote, speaking of the eye's preferences for steady changes of depth:

> It appears to me that a completely similar law governs the sensation of form, as holds for the reciprocal action of the retinal points, as I have laid it down for the light sensations.[47]

He elaborated in the *Analysis of Sensations*:

> ...every point of a straight line in space marks the mean of the depth sensations of the neighboring points. Thus the straight line in space gives the mean of the similar space values of the adjacent points; and the assumption forthwith presents itself that the straight line is seen with the least effort.[48]

Mach stated his law of illumination most clearly as: "the illumination of a position on the retina is felt in proportion to its deviation from the mean illuminations of the adjacent positions."[49] Transferring that formula to the depth sensations: the depth of a point is sensed as its deviation from the mean of the depths of adjacent points. Thus a line

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[47] "Beobachtungen über monoculare Stereoscopie" *SW* **58** (1868), p. 734.

[48] (AS) pp. 214-215.

[49] (AS) pp. 217-218.

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tilted away in space is sensed as a line with uniform depth. Figures are seen as flat as possible and if in depth, as made up of straight lines meeting at right angles. We would only expect such behavior from depth sensations, Mach thought, if they, too, were mutually inhibiting like the light sensations. Perhaps this explains why a steady rise in depth sensations induced by a flat, tilted surface are not noticed. Any steady rise of depth sensation falls victim to the eye's preference to see such rises as in a flat-as-possible plane perpendicular to the line of sight. Mach in fact stated that Hering's stereoscopic *Kernfläche* or horopter was this flat-as-possible surface, while objects in depth are seen at the minimum possible removal in front and in back of this surface.[50] Both Mach's and Hering's definition of the horopter is similar to Helmholtz's definition:--the set of points on which a deviation from the depth of neighboring points is most noticeable.[51] Mach's idea is that we estimate depth in general through such differences, triggered by light surfaces and retinal disparities.

I must emphasize that Mach never succeeded in putting these hypotheses together in a crucial experiment, and their psychological value is therefore indeterminate. However his views do clear up the mystery of why Mach thought space was properly a "sensation" and not necessarily a primary quality

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[50] (AS) pp. 223-225.

[51] Turner, pp 46-47.

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of the physical world impinging on consciousness. We will revisit these issues in greater philosophical depth in chapter eleven.

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Chapter Eight: Time Sensation

Mach's investigations of the time sensations paralleled his work on space. His first experiments on the time sense date to 1860 while he was preoccupied with verification of the Fechner-Weber law.[1] In crude trials he tested the just noticeable difference in the period of a pendulum using the length as stimulus and found that the law was confirmed. But in more detailed experiments reported in his "Untersuchungen über den Zeitsinn des Ohres" (1865)[2] he found that the Weber ratio ($\Delta t/t$) did not remain constant as the law demanded, but rather that it tended to increase with the time interval, (an increase of $\Delta t/t$ with t).

> Everywhere an unsteady but decided increase of $\Delta t/t$ with t is evident, which cannot be ascribed to chance and which is irreconcilable with the Fechner-Weber law, according to which, on the contrary $\Delta t/t$ would have to be constant.[3]

Now in general if the constant of sensitivity is growing that means the organ is becoming less sensitive. If the eye increases its Weber constant from 1/100 to 1/80 it will take a more intense light stimulus to generate a just noticeable difference. The decrease in the sensitivity to time intervals

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1 (V), p. 260.

2 *SW* **61**, p. 133.

3 "Untersuchungen über den Zeitsinn des Ohres" *SW* **61** (1865) p. 137.

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with increasing time suggested to Mach a fatiguing or flagging of attention of some organ of the time sense. Further evidence for such an organ of time was provided by the human ability to recognize similar rhythms and other time forms, just as there was an organ for the register of space sensations:

> It cannot be denied that one immediately recognizes the similarity in rhythm between two different melodies. Should one series of representations favor or reproduce the other, the one series must favor the other. But in the quality of the tone sensations the assistances cannot lie in different melodies. We must therefore consider again that every sensation of tone complicates itself with some other sensation and that in different melodies of the same rhythm the series of these sensations issues equally, as for instance in the same shapes of different colors the feelings of the eye-muscles are the same.[4]

Mach thus held that just as in the case of space sensations time sensations develop extended intervals between experiences by association and dissociation of a row of time sensations with a row of qualities. This is to be done by Herbart's mechanism of the reproduction series using a double row of time-qualities and qualities experienced in time.[5]

Mach believed the time sensations could be traced to the local feelings of an apparatus in the body; an empty register of intervals in which all of the individual experiences of the organism could be entered. Unlike space sensations, time sensations accompanied every experience and were correlated in

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[4] "Untersuchungen über den Zeitsinn des Ohres" *SW* **61** (1865) p. 146.

[5] Ibid., p. 146.

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an order with the flagging of the effort of attention that distinguished one local time from another, producing a sensation of different times:

> Perhaps the sensations of rhythm rest on the accommodation apparatus. Let us imagine the same tone repeats in measured-off pauses. By every new occurrence it draws attention to it, it is fixed, so to speak, and arouses besides a sensation of fixation. I can well imagine that in the accommodation apparatus from the first moment on, a constant change (fatigue, relaxation, recuperation) occurs. Should the tone occur for the second or third time it encounters the apparatus--according to the length of the pause--in a different state and arouses accordingly a corresponding fixation sensation. Those tones that repeat in the same times would always release the same sensations. The simplest *most pleasing* rhythms would rest at bottom in a natural way on the same sensations, like the symmetry of shapes and the consonance of tones.[6]

In his article "Zur Theorie des Gehörorgans"(1863) Mach applied his neural inhibition model to attention, treating attention-energy as work. It is literally work because the sensations are the specific energies of the nerves. Mach thought of attention as work done by the central organ, and time as the gradual consumption of that energy. In the *Analysis of Sensations* he felt confident enough in his theory to declare that we feel "the work of attention as time."[7]

Mach used the theory to explain why all of the different parallel time-sequences during the day are regarded as part of the same underlying time. It is because all of the different senses draw their energy of attention from the total energy of

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[6] Ibid., p. 148.

[7] (AS), p. 250.

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the central organ available.

Mach also tried to explain so-called "personal" phenomena: experiences that do not appear in consciousness in the physical time order of the events, but rather in an order that depends on the attention, memory and expectations of the observer. The "personal equation" was an attempt to explain such errors made by astronomical observers. For example, when observers were assigned the task of comparing the second hand of a clock to the appearance of a moon from behind a planet, some saw the events *reversed*, because the later event awaited with more heated expectation came to consciousness first. Mach argued that the event with the stronger energy of attention annexed to it appears first and the weaker after, because for small times, the time order tends to follow the weakening of attentive strength.[8] He also cited a case where surgeons see the blood spurt out before they see themselves inserting a scalpel.[9]

Mach's theory of time sensations as the weakening energy of attention was illuminating for short times but it was incomplete for explaining the memory of much longer time periods of days, months and years. For one thing, as Herbert Buzello objected,[10] the theory did not explain the

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[8] (AS), p. 253.

[9] (AS), p. 251.

[10] *Kritische Untersuchung von Ernst Machs Erkenntnistheorie. Kantstudien* Ergänzungsheft No. 23 (Berlin, 1911) pp. 26-27:

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recollection of one unbroken time over the life of an organism, since the attentive energy rises and falls irregularly from one day or year to another. I may do intensive work on Monday and sleep all day Tuesday but I do not confuse them in order. Sensing this weakness, Mach asked himself why we do not experience time running backwards as we recuperate our mental energy.[11]

To his credit, Mach did not fear to grapple with the great puzzle of time representation: we seem to be both aware of the change of time and we also seem to keep in memory a fixed timeline of events that does not change at all.

Buzello and Richard Hönigswald brought a Kantian argument to bear against Mach's notion of a changing ego. They maintained that a time series was only perceived as such if an unalterable ego endured through every stage, making separate times part of one fixed background time order, like a series of tree rings in which every earlier time is successively included and preserved in every later time. The permanent ego thus holds all previous times in memory as part of one same time. But if the ego itself alters and is not even recognized as the same later there is no such fixed background. For

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> Nun soll die Eindeutigkeit des Zeitverlaufes dadurch bedingt sein, daß die Arbeit der Aufmerksamkeit im wachen Leben stets zunimmt. Gerade das ist aber bei der Ermüdungsempfindung nicht der Fall. Diese ist von der Stimmung und besonders der Art der Beschäftigung abhängig. Sie nimmt oft ab statt zu;

[11] (AS), p. 253.

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instance Hönigswald says:

> What does Mach imagine by an economical "comprehension of the flux" through a changing consciousness? He forgets that the perception of the flux would be precluded to such a consciousness.[12]

Buzello echoes the critique:

> ...were time a sensation, I would have to assume a time in which to perceive it changing. Further, Mach himself describes the storing of memories, their connections, their reawakening by one another, memory and association as the fundamental determinations of psychic life. But between mere elements these relations are impossible, if one does not assume memory-feelings or "indicating marks" of memory representations. With this, naturally, the problem is just as little resolved that an previously occurring experience is regarded at "another point in time."[13]

These questions get right to the point of dispute between Kant and Mach: how can the ego be alterable (as Mach claimed) and yet contain the perceptions of previous times as parts of one timeline in memory?

Now Mach was very well aware of the difference between a changing awareness and the awareness of change.[14] In *Knowledge and Error* he even appears to take a pseudo-Kantian position:

> At the outset it is clear that a temporal course of mental elements, whether sensations or ideas, does not as such include consciousness of such a course. If our mental field of vision were always temporally confined to a sufficiently narrow present, the fact of change could

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[12] Richard Hönigswald, *Zur Kritik der Machschen Philosophie* (Berlin: 1903), p. 22.

[13] Herbert Buzello, *Kritische Untersuchungen von Ernst Machs Erkenntnistheorie*, *Kantstudien* Ergänzungsheft #23 (Berlin, 1911), p. 14.

[14] "Untersuchungen über den Zeitsinn des Ohres" *SW* **61** (1865), p. 145.

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> not be perceived at all. Therefore consciousness must always encompass a finite stretch of time in which there are fading sensations and ideas at the same time as newly emerging ones, for the first to be viewed as earlier and the second as later, if in addition we conceive of the relatively stable ego-complex, characterized by common feelings and so on, it constitutes a kind of rock with the temporally ordered stream of change flowing past it.[15]

But, if I may paraphrase, Mach also argued that the experience of permanence, against which temporal change is experienced, depends only on the enduring *disposition* to reawaken memories in the proper order, in the physiological direction from the past to the present. We thus do not require a fixed representation of an unchanging time or inalterable ego of stored memories. The disposition requires only the correct order of association between present sensations and memory traces (which themselves might even be changeable). The disposition to order representations in time is thus an enduring rule to be sure, but not an enduring representation. Thus we can have both changing contents of awareness and an awareness of change even without permanence.

Now for the evidence. Mach treated the question of time-representation in memory at length in his lectures of 1898 "Specialfragen der Psychologie," which reflections made their way into the 1900 Second Edition of the *Analysis of Sensations* in a new chapter "Sensation, Memory and Association." Mach hypothesized there that each and every process, both physical and psychical, leaves traces behind. Even as a thing becomes

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[15] (KE), pp. 332-333.

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a new thing and the past is irretrievably lost, these traces continue to represent how things used to stand. Otherwise of course we would not know anything had changed. Mach mentions the example of an old violin, which contains traces of the pieces it has played engraved on the instrument, however ephemeral.[16] He remarks that, like the violin, "a person that has had a certain experience is an essentially different person."[17] The traces both signify a change of state *and* give evidence of previous states, from which we conclude the endurance of objects. Permanence and change are therefore related, but not the way one thought.

The view is worth building out. One might think that the traces themselves are permanent survivors from past states of time, but this assumption does not hold. Traces, like ruined Greek temples and statues, age along with the rest of nature and do not remain *in* the past. However, these objects play an historical role as we try to reconstruct the past course of events with objects and traces existing entirely in the present.

I do not believe Mach's memory traces were enduring representations or energies that remained lively as part of a fixed ego-complex, but were more like crags and pits in the nervous system in which a present energy of attention could be

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[16] "Specialfragen der Psychologie" delivered 1898 (NL 174/1/015), Sections B-E; Cf. AS p. 237.

[17] "Specialfragen..." Section A.

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nicked off course.[18] This leads to the partial reawakening of older memories which are recognized as repetitions of past experiences although they are phenomena of the present. The important point is that the ruts and nicks are only diversions for a present energy, and do not possess an enduring energy of their own. We might call this the Odysseus's Scar theory, recalling that the nurse who bathed Odysseus as a child recognized him by his scar when he returned to Ithaca as a much-altered stranger.

Mach had already suggested the model of traces and present energies in 1872, as a physiological foundation for Herbart's reproduction-series.

> Memory and association [based] on the connection of nerve-centers between one another, and that the ways travelled are more easily travelled.[19]

It could be that memory reawakens images only in one direction for some physiological reason of asymmetry, as Mach said in the *Analysis of Sensations*.[20] For example it could be that an aged rut becomes deeper as it keeps getting run over, and any nick added to the main rut later will be called up annexed to it as a "later" modification of the first. Hence if the present energy of attention falls into a rut it falls into the

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[18] (AS), p. 237.

[19] (NL 174/1/003 Summer 1872): Das Gedächtniss und die Assoziation auf die Verbindung der Nervencentra untereinander und dass befahrene Wege leichter befahren werden.

[20] (AS), pp. 246-247.

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deepest rut first, awakening that memory, and then the tributaries.

We could think of the brain and nervous system becoming more cragged and pitted as the organism ages. The local sign register in the aging organism includes traces of the past time sensations, like the rings of a tree, and annexes to the trace of any quality the state of the organism at the time the quality was impressed on it. It's possible that the time sensations are like scars or ruts left by the energy of attention. Like tree rings themselves, the whole apparatus could be made up of aging cells that merely retain the proper dispositions to reconstruct the past from a present energy at any moment.

> Memory traces. Just as much a register as space. State of the organ. Course of life periodic and yet unidirectional on the whole.
>
> Register like space. Place of attention. Time of attention. The phase of attention leaves traces in memory. Law of the phase of attention indifferent.[21]

As for the extension of a timeline in memory, this could be resolved into a double row of traces of time-sensations and traces of qualities, awakened by present attention and associated and dissociated from one another. That would give

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[21] Notebook dated 28 November 1904 NL 174/2/48:
Gedächtnisspuren. Ebenso Register wie Raum. Zustand des Organs. Verlauf des Lebens periodisch und doch einsinnig im Ganzen.
Register wie Raum. Ort d. Aufmerksamkeit. Zeit d. Aufmersamkeit. Die Phase d. Aufmerksamkeit im Gedächtniss spuren zurücklassend. Gesetz der Phase d. Aufmerksamkeit gleichgiltig.

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the impression of an extended line into the past that could be reawakened at pleasure.

Mach added the important idea that psychological time order rested on the physiologically irreversible processes of energy consumption and the production of heat by the body. The physiological irreversibility of memory might thus have been related to some physiological change in heat production by the body over time. This theory unites both of Mach's notions of the time sense as the decaying attentive energy of the organism over both short and long periods. Because of the relation between the physiological order of memory and the physically irreversible production of heat, time sensations were objective for Mach in a way that space sensations were not.

Mach made a good start towards a theory of time representation, but even with all of the charitable reconstructions in this chapter, he cannot be credited with resolving all of the formidable difficulties. It is much more important that Mach suggests an alternative to Kant's view, involving a changing consciousness that reconstructs the past as a phenomenon of the present. He succeeds in suggesting a theory where even the traces of past time are capable of alteration without destroying the ability to construct the previous order of time in memory. Mach succeeded in playing Heraclitus to Kant's Parmenides, and indeed he came to fit the part better and better.

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Chapter Nine: The Economy of Thought

Mach's ideas on the economy of thought are some of his most famous and lasting contributions to science and philosophy, but in their impact on scientific objectivity they were also some of the most potentially destructive doctrines he ever produced. These ideas reveal a fundamental conflict in Mach's thinking about science: on the one hand he firmly believed in the functional connection of elements with one another, while on the other hand he came to doubt the repetition of identical experiences in space and time. Even the conservation of energy, a particularly cherished law, eventually fell as he matched it against his own conception that nature was but once given. The battle was joined between Mach's Heraclitean view of constant flux and his commitment as a scientist to the view that nature repeats itself with sufficient regularity to frame physical laws. Yet in the end the former won out without completely submerging the latter.

Mach's Early Ideas on Economy

Mach's ideas on economy grew out of his treatment of time and memory. He said that he first noticed the economical behavior of scientific investigators under the influence of his economist colleague at Graz Emmanuel Hermann.[1] Mach introduced economy as a psychological doctrine to explain how

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[1] (DL), p. 25.

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inquirers find their way through a mass of facts by economizing them in memory through laws and formulas. Indeed one of Mach's formulations of the Economy Principle was psychological in nature: "the accommodation of thoughts to facts and thoughts to one another."

During his early career, Mach even conceived of economy teleologically as the goal of physical processes. For example in an 1868 lecture "On the Forms of Liquids"[2] he showed that fluids assume a shape of minimal surface area for the volume they contain. He also saw economy in the mechanical Principle of Least Action, that in a system of heavy weights and rigid connections the heavy weights will always move downwards as much as they can with a minimum straining against their connections to other weights. The example of a bowed cable hanging from two poles supposedly illustrates this because the cable assumes a shape in which no section can fall further without raising some equally heavy section.[3] But Mach gave up the idea for good when he realized that heavy weights merely sink whenever possible, spontaneously transmuting their potential energy into energy of motion. In fact Mach wrote that natural processes aim at a *maximal* transmutation of energies whenever possible and thus a principle of the "maximum of energy" was just as appropriate. When the natural processes do not have a choice among alternatives they do not

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[2] (PSL) pp. 1-16.

[3] (M) p. 591.

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economize anything.[4]

> Economy cannot be predicated of physical processes since there is no choice between the actual happening and another. For this very reason I have not used the notion of economy in any way in this domain.[5]

There was yet another area where Mach strove to connect psychological economy with physics which I mention only in passing. In the 1860s he tried to equate his physiological discovery that sensations responded to above average stimuli, to his idea that physical changes are activated in order to equalize differences in natural potentials that jut out above the average of neighboring potentials; examples of which he noted for gravity and for temperature. Mach called attention to the coincidence several times,[6] although he did not say whether he regarded it as a case of thought accommodating itself to reality, or physical theory coming into agreement with our physiological basis for thought.

The Mature Theory of Economy

After these tentative beginnings, Mach's mature theory of economy appeared in his lecture *Über die Ökonomische Natur der*

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[4] (H), pp. 359-361.

[5] (H), p. 360

[6] "Beobachtungen über monoculare Stereoscopie" *SW* **58** (1868): 731-736.; "Über Herrn Guébhard's Darstellung der Aequipotentialcurven" *SW* **86** (1882): 8-14; "Über den Einfluss räumlich und zeitlich vertheilter Lichtreize auf die Gesichtswahrnehmung" *SW* **115** (1906):633-648.

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physikalischen Forschung,[7] a section of which was incorporated into the *Mechanik* of 1883.[8] Here he emphasized that the world does not consist of absolutely stable objects and laws but of changing elements (colors, sounds, tastes) which are sometimes, but only sometimes, human and animal sensations. Mach was convinced that the most realistic portrayal of the world was in terms of absolutely unique, individual and irrepeatable states. He said this in several places:

1. Die Natur widerholt sich nicht; (Nature does not repeat itself)
2. Die Natur ist nur *einmal* da; (Nature is but once there)
3. Nature has but an individual existence.

Even in their momentary existence, however, the elements of nature always occur *together* and form a whole, according to Mach's principle of universal functional dependence.[9] This instantaneous whole of Nature does not depend on the repetition of certain types of elements over time; we might call it an order of coexistence, but not cotemporality or spatial contiguity.

Mach mused in his notebooks that the form of connection

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[7] Published in book form (Wien 1882) reprinted and translated in (PSL).

[8] (M), pp.7-8.

[9] (AS), pp. 362-363.

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of the elements was also a given fact of experience:

> Connection also given.[10]
>
> The elements and the connection. The connection is also a fact. What sort of fact? An inner?[11]

He wrote in the *Mechanics* and *Theory of Heat* that scientific investigators could learn to perceive forms directly. Like Husserl and Brentano, who believed that logical forms (constants like "and," "or," "not," "all," "some") could be directly perceived by the well-trained intellectual intuition, Mach held that Newton, Fourier and Mayer could "see" the formal relations between individualized elements which he called "big facts."[12] In fact, Feyerabend cited Mach's

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[10] Verbindung auch gegeben. Notebook dated April 1882 (NL 174/2/22)

[11] Die Elemente und der Zusammenhang. Der Zusammenhang is auch eine Tatsache. Was für eine? Eine innere? Notebook dated 19 August 1884 (NL 174/2/22).

[12] (M), p. 356:
The considerations here developed will convince us that we can dispose by the Newtonian principles of every phenomenon of a mechanical kind which may arise, provided we only take pains to enter far enough into details. We literally *see through* the cases of equilibrium and motion which here occur, and behold the masses actually impressed with the accelerations they determine in one another. It is the same big fact, which we recognize in the most various phenomena, or at least can recognize if we make a point of so doing. Thus a unity, homogeneity, and economy of thought were produced, and a new and wide domain of physical conception opened which before Newton's time was unattainable.

(H), p. 113: In saying that every material point tends to the mean temperature of the surrounding points, the result of Fourier's theory is so expressed that it appears almost self-evident, and very close to our instinctive perception. It lies as close as the observation that all heavy bodies left to themselves

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principle-driven theory of research to distance him from empiricist inductive methodology.[13] Mach stood closer to empiricists who have gone on to espouse the reality of formal relations as well as individual elements. James, for example held a doctrine of felt-relations as part of his own "radical empiricism."

Kurt Gerhards and Hans Henning discussed the "connectedness" of experience for Mach. Gerhards for example talks about looking around a room with various objects in it:

> When I succeed in completely diverting my attention from the table, the window, the light streaming in through the window and reflecting off the table, the inkwell...and finally from myself, there still remain colors, brightnesses, darknesses and their shapes...left over, and I can take note that all of these stand in a certain connection with one another. This connectedness, just as much as its ingredients, is simply there, I have not added it in thought, rather merely noticed it.[14]

Henning argued that Mach's elements-in-forms were a new metaphysical entity: a fusion of element and function, rather than elements or functions alone, so that there is no separating the element from its functional role:

> An element or a sensation is not a sensation an sich,

> sink. Science confirms, in both cases, an obvious fact, only more exactly and completely in all respects than involuntary and undisciplined observation is able to do. In mechanics and in the theory of the conduction of heat it is, really only one great fact in each domain which is ascertained.

[13] "Mach's Theory of Research and its Relation to Einstein" *Studies in the History and Philosophy of Science* **15** (1) 1984: 1-22.

[14] Kurt Gerhards, *Machs Erkenntnistheorie und der Realismus*, (Stuttgart: W. Spemann, 1914), p. 166.

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> like the sensation of blue or warmth, but rather a *function*.[15]...One seeks behind the element nothing other than a function, behind the complex only a Riemannean manifold.[16]

These elucidations by his expositors find many echoes in Mach's own writings. For example he wrote in *Knowledge and Error* that the division of experience into segmented elements was only provisional. These divisions responded to our needs at the time, like the elements of alchemy in the middle ages. Thus there is also a practical and historical factor to the segmenting of experiences.

Mach probably believed that in the order of coexistence would be found a seamless fusion of elements in functions, only artificially segmented by practical need:

> Nature does not begin with elements as we are obliged to begin with them. It is certainly fortunate for us, that we can, from time to time, turn aside our eyes from the overpowering unity of the All, and allow them to rest on individual details. But we should not omit, ultimately to complete and correct our views by a thorough consideration of the things which for the time being we left out of account.[17]

Paul Carus, noting the abstractness of Mach's elements, wrote that "his elements are really the result of an analysis; they are of an artificial nature and can exist only in the mind of

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[15] Hans Henning, *Ernst Mach als Philosoph, Physiker und Psycholog*, (Leipzig: 1915), p. 18.

[16] Henning, p. 23.

[17] (M), p. 228.

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a highly trained scientist like Professor Mach."[18]

In his later writings, particularly the essay "Sensory Elements and Scientific Concepts"[19] Mach said that he regarded the instantaneous elements as opposing "reactions," against one another, which would explain why they arrange themselves into functions, i.e. "actions and reactions." By including functional opposition in the very essence of the element states, the form of combination follows of itself. What form experience will take will then depend on the standard opposing quality that is chosen to interact with the rest--a procedure that might be carried out in a number of ways depending on practical and economical needs.

Mach's Conception of Causality: Pro and Con

We have already seen that Mach did not want the concept of space introduced in causal relations. In the coexisting order, Mach rather viewed causation as the instantaneous functional dependence of all-on-all. This may seem extremely cavalier for several reasons.[20] In this conception practically everything becomes some possible function of everything else with it, and the forms of combination become almost unbearably arbitrary. No functional connection seems to go deeper than any other and the most superficially

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[18] "Professor Mach's Philosophy" *Monist* **21** (1911), p. 36.

[19] (DL), p. 120.

[20] (CE), pp. 59-62.

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associated circumstances become dependent on one another simply because they all comprise a nature once given. It turns out, as we shall see below, that when the elements are ordered temporally and spatially, certain functional relations will have to supersede others in importance in order to come to an economical ordering of repeating events. Indeed we are wont to emphasize as causes and effects those sets of circumstances that we recognize as repeating in time and space. But in Mach's instantaneous whole of nature, these considerations do not yet enter; everything is *with* everything else, and we are free to emphasize this or that functional connection over another at pleasure, according to what features interest us and how we choose to interfere with the world. The question remains open which elements are more closely associated with one another and which are dissociated "background" circumstances. For different economical purposes different associations might be brought to the fore and others shoved into the background.

There are two common counterexamples often brought against associationist conceptions of cause like Mach's, each of which I believe contains a *petitio principii*. The first is Russell's example of the Manchester horns that sound to the laborers to stop work. When the horns go off in Manchester, the workers in London cease. Are the two events therefore related as cause and effect? The two circumstances are presumed to be unrelated, yet we are told that both go off

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exactly at 5:00 P.M. Of course we are not told how we know it is 5:00 in both places, without relying on causal relations between the two events that we are trying to deny. The trivial association is part of the background, but it still rests on some real relationship between the two events. This relationship does not happen to be the one that is of present interest, but neither is a complete independence established. Rather the feature used to dissociate the two events is simply a more general association. If more general circumstances are chosen, such as cosmical events, the cotemporality between the two events might become the crucial associating circumstance, for example the meridian of Greenwich running through Manchester and London. In such a case, a seemingly trivial circumstance from the background might be of prime interest.

A second type of counterexample is the association between a hurricane reported in the newspaper and a simultaneously raging hurricane, or atoms in the brains of world leaders in 1939 and the outbreak of World War II, considered as an historical event. Here we say the two types of events are unrelated because they have incompatible descriptions or predicates. But even the association of the two events by way of time already presupposes that the two events must indeed possess a common basis for comparison, proving again that the events are somehow related through background circumstances, though at a much higher level of generality. Time is very often the relating circumstance.

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Probably, like his friend William James, Mach would have embraced a pluralism of overlapping categories and classifications, depending on the standards chosen, narrowed down by practical not theoretical considerations.

Time, Space and Cause as "Economical" Conceptions

For Mach, the conditions for spatial and temporal determination attended on specific metrical choices of certain functional dependencies in nature. We have seen that spatial and temporal orderings rest on some more primitive causal association and dissociation of unextended instantaneous elements. This point was not clear to his critics. Buzello, for example, accused Mach of begging the question in assuming causal dependencies prior to space and time:

> The Concept of "dependence" if it holds between real existing objects, the concept of change, demands the time conception. Mach still owes us a criterion for immediate dependence.[21]

Hönigswald refused to accept Mach's suggestion that even absolutely simple individual elements are in functional dependence and do not flow on unconnected and disjointed:

> The material side of the Machian Philosophy, his original theory of elements, culminates in the heraklitean

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[21] Herbert Buzello, *Kritische Untersuchung von Ernst Machs Erkenntnistheorie*, *Kantstudien* Ergänzungsheft No. 23 (Berlin: 1911) p. 32:
Schon der Begriff der "Abhängigkeit," falls sie zwischen real existierenden Gegenständen besteht, der Begriff der Veränderung, fordert den Zeitbegriff...Ein Kriterium für eine unmittelbare Abhängigkeit bleib er uns schuldig.

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proposition of eternal flux.[22]

There is no end to the arguments one can muster against Mach if one is determined to interpret his functional causation as involving a prior space and time order. We must at least deem it possible that a prior causal order exists: an order of coexistence for the individual irrepeatable elements. Because such a world would have been unmanageable for us to live in, Mach then says that we developed faculties for sensing the "repetition" of past experiences and with this the world of symmetries, objects, spaces and times comes into focus.

A World from Economy

Let us now look at this economical construction of the world in more detail. Mach affirmed in his notebook that it is part of experience that we continually reproduce memories along with our present sensations, and thus live with one foot in the past and one in the present:

> Representation of time
> The images of earlier times are tied to the images of present states. States in the memory field are bound to other states in the perceptual field. That which was, we see at the same time with that which is.[23]

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[22] Richard Hönigswald *Kritik der Machschen Philosophie.* (Berlin: 1903) p. 13:
Die materiale Seite der Machschen Philosophie, seine originelle Elementenlehre kulminiert in dem heraklitischen Satze vom *ewigen* Flusse.

[23] Vorstellung der Zeit
Die Bilde frühere Zeiten sind an der Bilder gegenwärtiger Zustände geknüpft. Zuständen in Errinnerungsfelde sind mit anderen Zuständen in Wahrnehmungsfelde verbunden.

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While sensations themselves never repeat, we carry with us images of them in memory; it becomes possible to think of some sensations as repeating instances of past experiences, by comparing the memory and the present sensation. It is here that we first possess *concepts* of colors like red, blue and green, or other elements, as Mach sometimes calls these repeating element-types. This concept or schema is much more "economical" than a separate record for every individual occurrence:

> When we look over a province of facts for the first time, it appears to us manifold, heterogeneous, confused and contradictory. At first we only succeed in holding fixed every individual fact or connection from the rest. The province is as we say *unclear*. Bit by bit we find the simple, self-identical elements of the mosaic, from which the whole province permits of being reconstructed in thought. If we get this far, to recognize the *same* facts again, we do not feel ourselves so foreign in this province, we look over it without exertion, it is explained for us.[24]

It is interesting that for Mach concepts do not make their first appearance with words. The most primitive conditions for science to be possible are the possession of a past-record (a notebook or a memory) and the recognition of "repeating" experiences. This was precisely what Kant had maintained so many years before,[25] and it is somewhat amusing that Mach's staunchest critics were Neo-Kantians. Of course Mach was

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> Das was war sehen wir zugleich mit dem was ist...Notebook dated 26 January 1881 (NL 174/2/17).

[24] (PWV) p. 225.

[25] *K.d.r.V.* (A 95-110).

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clear that the repeating schemata were only evolved devices of economical need, not synthetic a priori principles:

> At best the modest beginnings of science are the uncovering of what are to us simple, self-identical entities. Half aware and non-arbitrarily, man obtains his first knowledge of Nature, in that he pictures [*nachbildet*] the facts reproduced in thought, and projects them forward [*vorbildet*], in that he completes ponderous experience through his more quickly moving thoughts, at first only to his material advantage.[26]

There are a few stages in which the world takes shape: first, the originally coexisting flux, then the stage where we distinguish repeating individual elements that we recognize as after-images of past experiences [*Nachbilder*],[27] and finally the stage where it is permissible to discuss repeating complexes of the environment, such as bodies, and general causal connections between bodies or features of bodies.

In fact, recalling Mach's physiological researches, each and every recurrence is a symmetry in time, defined by his "Symmetrie" paper as the recognition of repetitions of the same experiences. Mach points out here in his "Economy" article that the connection of a certain type of spatial or temporal cause and effect depends on an *ideal* repetition of features, a falsification that never truly occurs:

> If we intended to ascribe the property to nature, that under equal conditions she produces the same effects, we would not know how to find these equal conditions. Nature is but *once* there. Only our schematizing

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[26] (PWV), pp. 220-221.

[27] *Nachbild* is a psychological term for a memory after-image just as in English.

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> imagination [*Nachbilden*] produces equal cases. The dependency of certain properties on one another exists only in this. All of our toil, to mirror the world in thought were fruitless, were it not possible to find something enduring in this brightly colored flux. Hence the craving for the concept of substance, whose source is not different from that of modern ideas of the conservation of energy.[28]

The argument applies to all symmetries. A past impression and its circumstances are gone. When they are said to recur, that is when we do the work of connecting a memory image of a past impression with present sensations, there is no guarantee our memories and imaginations are following in nature's tracks.

One caveat: Mach did hold that memory was a real physical phenomenon--extending beyond human and animal memory to inanimate matter, in which traces of past events are left behind.[29] Indeed Mach and Evald Hering both thought that something analogous to memory traces would be left behind in any organized material.[30] Mach also held that physical and psychical processes were in truth *irreversible* and that past

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[28] (PWV), p. 230.

[29] For example in his 1872 lecture "Über einige Hauptfragen der Physik" (NL 174/1/003) Mach wrote:

> Die Naturgesetze lassen sich aber auch auf eine Art Gedächtniss zurückzuführen, es braucht nicht unseres zu sein. Geistig Geschehens kann nicht ungeschehen gemacht werden. Eine Vorstellung die wir einmal erhalten haben. Das Trägheitsgesetz. Materiele Geschehenes kann nicht ungeschehen werden. Die Zeit kann nur in eine Richtung ablaufen. Die materiellen Vorgänge nur in einer Nicht umkehrbare Vorgänge. Der Sauerstoff hat kein Gedächtniss. Vielleicht nur nicht bemerkbar.

[30] Evald Hering, "Über das Gedächtniss als eine allgemeine Funktion der organisierten Materie," (Wien: 1870).

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stages never actually recur. Indeed the whole idea that processes leave traces depends on the fact that processes are not completely reversible, else the traces too would revert to their previous states. In a Lecture "Specialfragen der Psychologie" of 1898 Mach wrote at some length on the schematizing tendencies of physics to imitate memory and assume repeatable, cyclical processes:

> Memory. Remembrance. Experiences leave psychical traces and dispositions behind which can be *made* lively again. Completely characteristic (definitive) of psychical life. No memory, no consciousness. No connection....We schematize too much in physics. Even in physics the elimination of the past is just an ideal. Force and velocity. Non-reversible processes. In reality there aren't any. No restitution of original state. With that the fact of time is given.[31]

The fact that even physical objects and events have something analogous to memory indicates the very broad objectivity of the phenomenon and its usefulness for science. In Kantian language, it is empirically real, as well as transcendentally ideal. It may even be one valid way to interpret the world, just not the only way.

Mach gave no indication which experiences are to be

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[31] Das Gedächtniss. Die Erinnerung.
Erlebnisse lassen psychische Spuren, Dispositionen zurück, welche wieder lebendig werden können.
Ganz characteristisch (bezeichnend) für d. psychische Leben. Ohne Erinnerung kein Bewusstsein. Kein Zusammenhang.
Wir schematisieren zu viel in Physik...Auch in Physik ist die elimination der Vergangenheit nur ein Ideal. Kraft und Geschwindigkeit. Nicht umkehrbare Vorgänge. In wirklichkeit gibt es keine. Ursprunglicher Zustand nicht resitutierbar. Thatsache der Zeit damit gegeben. NL 174/1/015:

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privileged as time-symmetric repetitions of others, although our physiological sensory apparatus makes certain biological preferences inevitable. Logically, it is possible to imagine many--perhaps infinitely many--features of past experiences as enduring and repeating, and thus many symmetries, conservation laws and theories built upon them. Mach thought that the investigator could select in thought from all of the repeating patterns available those facts that were most easily memorable and logically interrelated with one another. He generally held that scientific investigators make economical decisions as to which properties to regard as enduring and which to consider evanescent. Mach pointed out in his *Conservation of Energy* that Riess's thermoelectrometer offered a conception of electrical energy analogous to heat and Coulomb's balance an electric force analogous to the pull of gravity. Which is regarded as the "quantity of electricity" depends on convention. Mach tended to regard ideas of conservation, such as the conservation of energy, as the most basic of all such repeating patterns.

"On the Economical Nature of Physical Inquiry" thus laid down Mach's teaching on economy in two overall stages: first, the emergence of repeating patterns based on memory's need for permanence over time; and second, the *sorting* of the multitudinous patterns generated in the first step, to maximize or economize a given repeating feature.

It is clear that in order to apply economy of the second

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kind, the set of all possible repeatable patterns in a domain of facts must be recognized first, and afterwards this or that arrangement of facts can then be chosen to maximize or minimize a given feature. Economy of the first kind, however, explains why there seem to be repetitions in experience at all.

Criticisms of the Economy of Thought: Buzello, Husserl and Carus

Most of the attacks on Mach's doctrine were directed at economy of the second, selective kind. Herbert Buzello argued:

> A teleological consideration in a science has only the value of a method or presentation, or of inquiry. It serves to order the individual facts for a given goal, in so far as they are suited to realize that goal. But it is continually assumed that the facts as such are already given.[32]

Edmund Husserl, in his treatment of Mach's "Denkökonomie" in the *Logische Untersuchungen*, made the same criticism, that the underlying logical forms of experience had to be discovered first before making an economical ordering of these contents.[33] Husserl and Hönigswald[34] both maintained, with some justice, that even if patterns in the phenomena were

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[32] Herbert Buzello *Untersuchung von Ernst Machs Erkenntnistheorie* (op. cit.) p. 16.

[33] pp. 196-213.

[34] Hönigswald, *Zur Kritik der Machschen Philosophie*, (Berlin: 1903), pp. 13-22.

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chosen to satisfy the desires of the ego, then at least an ego of stable desires towards permanence must endure as a factual pattern in experience. Mach seems to have agreed with this in remarks he added to a later edition of the *Mechanics* to answer Husserl:

> If my theory of mental economy be conceived merely as a teleological and provisional scheme for guidance, such a conception does not exclude its being based on deeper foundations, but goes toward making it so.[35]

What annoyed Mach's Neo-Kantian and Phenomenologist critics however was that he considered the disposition toward endurance to be a so-called irrational desire or need, rather than a rational faculty. Again Mach held the practical above the theoretical. He followed Schopenhauer in claiming that non-rational desires stood behind rational economy, which was absolute heresy to the neo-Kantians, Brentaneans and Phenomenologists, and allied him more with his contemporary Friedrich Nietzsche.[36]

Meantime, at a somewhat deeper level, Paul Carus argued

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[35] (M), p. 594.

[36] In fact Kant had prepared the way for this Schopenhauerian, Machian and Nietzschean line of thought by insisting that the synthetic a priori principles were not applicable to nature *an sich*, but stemmed from our own faculties for organizing experience. "It is finally time," Nietzsche declared, "to replace the Kantian question "how are synthetic a priori judgments possible?" with another question "why is it necessary to believe in such judgments at all?"--namely to grasp that for the purpose of preserving a species of our kind, such judgments must be *believed* to be true; for which reason of course they could still be *false* judgments." *Jenseits von Gut und Böse* (Stuttgart: Reclam, 1988), p. 17.

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against Mach that perhaps the repeating forms of experience should be taken realistically, or else science would not be the true or adequate description of the world:

> While I have no objection to Mach's description of science as economy of thought, I would hesitate to say that it is sufficient as a definition, because science is a correct (or adequate) description of facts in their essential features. Exactness, correctness, adequacy, truth, or whatever you may call it, is the main thing and comprehensiveness comes in second as a natural consequence whenever the essential features have been rightly understood...The difference between myself and Professor Mach as to which part economy plays in the realm of thought is mainly this: that while Professor Mach appears to rest satisfied with using it as a principle that has so far worked well in experience, I try to explain why economy of thought is possible at all, and my explanation is based on the idea that theory is not a purely subjective device to deal with experience but that there is a feature in experience itself which justifies the formulation of theories...the purely formal is not merely a matter of method, it is not purely subjective, but must be a feature of the objective world. It explains why there are uniformities.[37]

Mach inclined to aspects of this position himself, but did not favor realism about spatial and temporal form, only for an instantaneous mutual dependence of elements. The repeating forms of experiences in time are mostly schematic for Mach, the only exception being the physiological reality of memory traces in organized material. The schemata rest on an economic need to ignore the actual irrepeatability of experience, which may even overrule memory.

The conception of Nature of the most veracity for Mach was perhaps that of Heraclitus's sun that is new each day and

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[37] "Professor Mach's Philosophy" *Monist* **16** (1906): 336-337.

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once given. As Carus sensed, that conception poses a serious challenge to scientific objectivity, in particular to the conservation of properties through time. Mach's idealism about conserved substance, for instance, was very pronounced in his 1882 Lecture:

> Indeed one is used to regarding the conservation of weight or mass as a direct proof of the endurance of matter. Only this proof evaporates, when we get to the bottom of it, into such a mass of instrumental and intellectual operations, that in a certain sense it only constitutes an equation which our representations, picturing facts in thought, have to satisfy. The dark clump that we involuntarily add in thought, we seek in vain outside our own thought.[38]

The "Un-conservation" of Energy?

The conservation of energy was an exalted principle for Mach, which he interpreted as the deepest formal connection of the phenomena in one instantaneous whole. But it was bound to come to a showdown between the physical principle of the conservation of energy through *time* and Mach's idealism about temporally preserved properties.

This conflict finally came to a head in some remarks Mach wrote down in 1884 on J. R. Mayer. Mayer had suggested, in a passage Mach quoted with approval in his *Conservation of Energy*, that heat disappears when mechanical work appears, but that this mutual disappearance could be taken as identity or just the opposite.

> Just as little as, from the connexion between the tendency to fall [*Fallkraft*] and motion, we can conclude

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[38] (PWV) (Leipzig: Johann Barth, 1910) p. 234

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> that the essence of this tendency is motion, just so little does this conclusion hold for heat. Rather we might conclude the opposite, that, in order to become heat, motion, whether simple or vibrating, like light or radiant heat must cease to be motion.[39]

Mach now questions whether in irreversible processes where work is lost to heat there was still upheld an identity of the same energy, or whether it is simply an economical measure to assume the constancy of a certain something through change, having to do with our need to reproduce past circumstances in thought.

> The fact is, when things are made to go backwards, so far as can be, the same is again obtained, so far as experience reaches.
> Everything else however is just arbitrary determination. I need not conceive heat as equivalent. I *can* however. It is a pleasant, relaxing economical conception founded on the strongest, most visualizeable image of substance. The factual [or facticity, the notion of a thing's being a fact E.B.] is related to reversible processes (Clausius). What is not reversible, about this we can say nothing. The thought of equivalence is there completely arbitrary because it has no practical worth... An endurance for a certain type of reaction of our understanding. Not immediately visible. Time of no influence. Only nobody can secure a guarantee from time. For irreversible processes the thought of equivalence has no worth at all. It only becomes practical through reversibility.
> Equivalence of transformations.
> [Mayer] brings the factual into agreement with his concept and need.[40]

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[39] *Mechanik der Wärme* (Stuttgart: 1867) p. 9, quoted and translated in CE p. 37.

[40] Thatsächlich ist daran, wenn man rückgängig macht,--so weit man kann erhält man, so weit die Erfahrung reicht, wieder dasselbe.
Alles übrige ist aber nur willkührliche Bestimmung. Ich brauche die Wärme nicht als Aequivalent aufzufassen. Ich *kann* aber.
Es ist eine angenehme ruhige oeconomische Auffassung auf

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This *Gedankengang* led Mach to doubt the conservation of energy for irreversible processes, because work that is lost to heat through a transformation is no longer completely recoverable as mechanical work. He wrote further:

> When however heat at a lower [potential] level cannot be transformed, work is lost. There is thus no energy principle except in cases of simple reversibility.[41]

And finally:

> The energy principle has no unbounded validity.[42]
>
> Through irreversibility the whole energy principle is overthrown. That is a special experience. [43]

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das stärkste anschaulichste Stoffbild *gegründet*.
Das Thatsachliche bezieht sich auf umkehrbare Vorgänge (Clausius). Was nicht umkehrbar, darüber können wir gar nichts sagen. Der Aequivalenzgedanke ist da ganz willkührlich weil dasselbe auch gar keinen praktischen Werth hat...
Eine Beständigkeit für eine gewisse Art unserer *Verständesreaction*. Nicht unmittelbar sichtbar. Nicht mehr als, dass v bei gleicher Höhe immer dasselbe ist. Die Zeit keinen Einfluss. Allein für die Zeit kann niemand eine Garantie übernehmen. Für nicht umkehrbares hat der Aequivalenzgedanke *gar* keinen Werth. Er wird nur praktisch durch die Umkehrbarkeit.
Aequivalenz der Verwandlungen.
[Mayer] bringt das Thatsächliche mit seinem Begriff u. Bedürfnis in Übereinstimmung. Notebook dated 19 August 1884 (NL 174/2/22).

[41] Wenn aber die Wärme auf dem tieferen Niveau nicht umgewandelt werden kann ist die Arbeit doch verloren. Es gibt also kein Energieprincip ausser in Fällen einfacher Umkehrbarkeit. Notebook dated 4 December 1891 (NL 174/2/33).

[42] Das Energieprincip hat auch keine unbegrenzte Gültigkeit. Notebook dated 11 October 1892 (NL 174/2/35).

[43] Durch die Nichtumkehrbarkeit ist das ganze Energieprincip aufgehoben. Das ist eine besondere Erfahrung. Notebook dated 12 March 1894 (NL 174/2/37).

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Mach soon after converted himself to the idea that despite the appearance of some reversibility in nature, in truth all processes were irreversible, or had at least one irreversible component, in his 1892 article "Zur Geschichte und Kritik des Carnot'schen Wärmegesetztes"[44] and his 1896 *Theory of Heat*. We shall consider these questions further, along with Mach's theory of thermodynamics, in chapter fifteen.

Economy as a Challenge to Scientific Objectivity

Mach's criticism raises an interesting problem for the objectivity of science. Science depends on repetition of a property in time, but as such it cannot be fully confirmed or refuted, since testing requires that we compare a present circumstance to a past circumstance no longer extant. Present experience must be set equal, or unequal, to circumstances recorded in the past. Thus both scientific confirmation and disconfirmation rest on the simpler constraint of what features are identified through time and recognized in past, present and future experiences. But the criteria for repetition of an event are themselves normative constraints isolated from testing, if only because they set the very conditions for the test. The forms depend in no small way on our propensities, like our biological propensity for vertically standing lines.

1. Disposition toward the vertical.

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[44] *SW* **101**: 1589-1612.

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> 2. Disposition toward the future. Practically, only the future is of interest. Biological disposition in both cases. Not about knowledge in itself, rather about the biologically beneficial. Determined from the present, past and future are equal, disturbances excluded. Space and time in reality not just metric.
> The present can determine past and future in part, with permissible inexactness. Past and future are imprecisely known, hence the inference to the better known present is moot.
> Conscious goals must already be known from the past, and suspected to occur in the future to prepare for it. Time does not run backwards. Facts do not recur. Thought constructions can recur.[45]

Mach thus arrived at a Kantian-sounding type of idealism about time:

> We do not prescribe laws to nature, rather we prescribe to ourselves laws of expectation, guided by nature. Law restricts deed. Natural laws restriction of expectation.[46]

He thus came to doubt the reality of spatial and temporal forms of experience while upholding an instantaneous

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[45] 1. Einstellung auf die Verticale
2. Einstellung auf die Zukunft. Prakt. nur die Zukunft ein Interesse. Biologische Einstellung in beiden Fällen. Nicht um die Erkenntnis an sich sondern um die biologische förderliche. Bestimmt ist durch die Gegenwart Vergängenheit u. Zukunft gleich, unbekannte Störungen ausgeschlossen. R u Z [Raum und Zeit E.B.] in Wirk [-lichkeit] nicht bloss metrisch.
Gegenwart kann Vergangenheit u. Zukunft theilweise bestimmt werden mit zulässigen Ungenauigkeit. Vergangenheit und Zukunft ist ungenau bekannt, daher der Schluss auf die besser bekannte Gegenwart müssig.
Bewusster Zweck, man muss ihn schon aus der Vergangenheit kennen, zukünftig eintretend vermuthen, um ihn vorzubereiten. Zeit nicht rückläufig. Thatsachen kehren nicht ein. Gedankenconstructionen kann umkehren. Notebook dated 28 November 1904 (NL 174/2/48).

[46] Nicht wir schreiben d. Natur Gesetze vor, sondern wir schreiben uns Gesetze der Erwartung vor, durch die Natur geleitet. Gesetz einschränken d. Thun. Natur[gesetze] Einschränkung d. Erwartung. Ibid.

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unextended connection of elements in the coexisting order.

It is interesting to note that Mach's skepticism about repeating forms concerns some of the most exact principles of science. The conservation laws for energy and other exact properties admit of not the slightest imprecision, and as was discovered in the twentieth century, conservation principles in fact depend on exact symmetries of the fundamental equations in time and space. Any such "exactness" in physical theory, rather than heartening, was extremely suspect to Mach, and led him to believe that experience had been deliberately schematized in advance. As he showed in his work on physiological spaces, the various symmetries (such as isotropy and homogeneity) cannot be taken as basic but ultimately rest upon the recognition of repeating experiences. Mach's economical doubts thus cut extremely deep through symmetry arguments and threaten their foundations.

However without repetition practically none of science would be possible, only a sheer "experiencing" at odds with scientific thought. It is unclear whether Mach believed a science of pure experience was possible, or whether he thought we were doomed to live with our own creations because of the unmanageability of raw experience. It is a fascinating question whether science could be done without repeatability. In my opinion, Kant and Mach both got very close to answering the question "what makes science, as it is, possible?" without realizing that perhaps they were on the brink of proposing a

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way to do a new kind of science instead.

Was Mach Proposing a New Way to do Science?

Mach said in the *Mechanics*[47] that in the best case--assuming infinite memories and an ability to attend to all aspects of a phenomenon at once--science should be a mere tabulation of experiences, and should not involve laws at all. That claim is usually taken as proof that Mach had succumbed to pure sense-data positivism, for which he was soundly criticized, especially by Einstein, who in his 1922 Paris lectures called Mach's science "a catalogue but not a system."[48] We can now see that for Mach laws involve a falsification of the true stream of experience. But a non-repeating conception of science is psychologically impossible and practically unmanageable and almost fails to be science at all as we understand it. We must make abstraction from the particulars of each individual event if we are to have laws, while it may still be true that no law completely captures the peculiarities of each individual trial.

At this juncture we are faced with an either-or: do we admit of an irrepeatable science of pure experience or do we rest certain in the best possible science of laws, while

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[47] (M) pp. 6-7.

[48] Quoted in Gerald Holton "Mach, Einstein and the Search for Reality" in Cohen and Seeger eds. *Ernst Mach: Physicist and Philosopher* (Dordrecht: D. Reidel, 1970), p. 184.

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admitting of imperfection? A science of pure experiencing has never been seriously tried, and it is unclear what benefits it would bring, but the possibility has been raised. I don't think Mach had further ambitions than to suggest doubts, but as we shall see in the next chapter, Mach did consider a "future science" in close connection with psychophysics that could perhaps pave the way to pure experience without withdrawing the claims to knowledge that have distinguished the natural sciences for centuries. Indeed this is the most exciting consequence to draw from the economy of thought.

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Chapter Ten: From Sensations to World Elements

Introductory Remarks

We have seen that Mach's philosophical-scientific development advanced from sensations to world elements. He began in psychology by analyzing human sensations of color or sound into their less familiar constituent elements and processes. From there he considered sensations in their physical variations which are physical objects in a special immanent sense we shall consider below called neutral monism. Finally he made the push into the beyond by considering world elements fully independent of human minds. There are sufficiently many of these speculative leaps to warrant further treatment here, especially because Mach's fecund speculative instincts receive such short shrift in the literature. These writings will especially surprise those who think of Mach as an ultra-strict positivist.

It must be borne firmly in mind throughout this chapter--especially in the speculative parts--that Mach introduced the world elements chiefly to get on with science: to account for mind-independent objects and their elements. Mach had no interest in developing a speculative, philosophical system of elements without the possibility of testing them by some means, albeit in the very distant future. For example, in his "Über einige Hauptfragen der Physik" of 1872, he had simply proposed the most expedient hypothesis "matter has sensations"

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which he hoped would be useful for scientific problem-solving in the future.[1] In particular, he hoped to clear a path for psychology through "wrong-headed" philosophical pseudoproblems, such as: "how does the perception of a large tree get in the little head of a man?"[2] The point isn't to try and solve the problem, but to realize that at bottom it doesn't make any sense. Mach targeted only certain specific problems of the sciences for antimetaphysical critique and did not envision philosophy as wholly nonsensical wordplay.

Because this chapter will be more speculative than the others, a few more words about Mach's curious *odi et amo* relation to philosophy. He was of course deeply read in Kant, Herbart, Schopenhauer, Spinoza, Leibniz, Lichtenberg, Aristotle and other figures, but was cool to most contemporary philosophers. He said: "whether I shall ever succeed in making my fundamental ideas plausible to the philosophers, I must leave to time to decide. I do not attach much importance to this at present, though I have a deep reverence for the gigantic intellectual labors of the great philosophers of all ages."[3]

The philosopher will assume that Mach's "sensations in matter" involve some new postulate about nature and, being a member of a disputatious tribe, will naturally inquire as to

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[1] (NL 174/1/003).

[2] (AS), p. 27.

[3] (AS), pp. 368-369.

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how we could ever know that such things existed. Moreover, raising now the charge of logical contradiction, how could there be "unsensed sensations" or "perceptions separated from a consciousness" (such as Leibniz's petites perceptions). Do not all sensations have to be conscious? Now Mach relished intelligent criticism as much as anyone but these sorts of debates were wearisome to him. He often emphasized that his elements were a scientific hypothesis to be put to the test, not a postulate of supermental realities. The scientist--unlike the philosopher--enjoys a certain liberty to introduce new ideas without having to motivate them a priori. For example, Einstein once wrote that hypotheses were "free creations of the human mind." Unlike the philosopher, however, the scientist must do more work to isolate the hypotheses as much as possible and expose them to independent test. This was a bargain Mach fully embraced for his world elements.

Empfindungen: Physical or Psychical?

In developing his *Elementenlehre* during the 1860s Mach drew on his leading influences: Herbart and Fechner. From Herbart, Mach had adopted the notion of elements as opposing qualities and a machine-like ego consisting of excitation, inhibition and memory reproduction mechanisms. But where Herbart and the young Mach grouped qualities together into atom-like Monads, the riper Mach perceived that these could be

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functional patterns setting all of the qualities equal to zero.[4] He realized, at least by 1872, that qualities could be interpreted as physical or psychical indifferently, whether they occurred as parts of minds or physical bodies:

> Herbart assumed certain fundamental phenomena, and we find these upheld by all of his reals, thus both by the soul as well as by the rest of the monads which compose so called lifeless matter.[5]

In this next step Mach probably drew on Fechner's panpsychist conception that all natural process have an inner qualitative nature which does not appear when we consider only their external relations to one another in mechanics and physics. Through Fechner, Mach was encouraged to believe that the inanimate world was constituted out of qualities neither physical nor psychical in nature.

The way was now clear to come forward with his own psycho-physical *Elementenlehre* of neutral elements and functions, with no special distinction between those that are organized into physical functions, like bodies, and those that are collected in the sensory manifolds of the I. His first announcement of the new position was in the *Conservation of Energy*:

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[4] (AS), p. 45, (CE), p. 73.

[5] Herbart hat also gewisse Grundphänomene angenommen, und diese finden wir wohlgemerkt bei allen seinen Realen, also sowohl bei d. Seele wie bei den übrigen Monaden, welche die sogenannte unbelebte Materie zusammensetzen. "Über einige Hauptfragen der Physik" Vortrag, Sommer 1872 (NL 174/1/003).

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General Remarks

> We very soon learn to distinguish our presentations [*Vorstellungen*] from our sensations (perceptions). Now the problem of science can be split into three parts:
> 1. The determination of the connexion of presentations. This is psychology.
> 2. The discovery of the laws of the connexion of sensations (perceptions). This is physics.
> 3. The clear establishment of the laws of the connexion of sensations and presentations. This is psychophysics.[6]

The rather positivistic assignment of analysis of connexions among "sensations" to physics would not have raised an eyebrow in empiricist Vienna, or in Prague where the lecture was delivered. But it is strange considering that in the text Mach had designated physical energies associated with the "performance of work" as the elements of physical inquiry,[7] and had said nothing to suggest these were all human or animal sensations. Probably he meant the same thing by "sensation" as he did when he wrote "Die Materie empfindet" (matter senses), namely that sensation is just a word standing for any natural quality.

Mach already considered the elements as akin to sensations in the Vorträge, introducing mind-independent elements within matter in that work, and as we have seen, in his lecture "Über einige Hauptfragen der Physik" of 1872. Thus, the table above should be understood as a specialization of tasks for each science, within a general set of functional relations across all natural phenomena.

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[6] (CE), p. 91.

[7] (CE), p. 69.

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In the "Hauptfragen," Mach added that foreign sensations could be inferred through their causal interactions to our own sensations, and left no doubt that the sensations of matter are part of a real external world [*Außenwelt*]. The idea that our sensations are already interwoven in causal relations with mind-independent elements of the external world already suggested world elements, of which human sensations were only a special case.

Neutral Monism in the *Analysis of Sensations*

In the *Analysis of Sensations*, a book addressed "primarily to biologists,"[8] the world-elements receded into the background. Mach only remarked in passing that certain other elements had to be "added in thought" to those in human consciousness to complete objects and the contents of other minds. Mach gave a clue in the "Hauptfragen," as to why he was so cagey, where, after he stated that sensation was a "general property of matter," he added: "Mir selbst, da ich es ausgesprochen, übel genommen." (I have bourn ill-will for saying it.) He adjoined later that he did not want to excite the old "Gespensterfurcht" (fear of ghosts) among his colleagues. Michael Heidelberger concludes that their objections to his psycho-monadology had caused him to conceal his views:

It should probably be assumed that the doctors before

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[8] (CE), pp. 9-10.

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> whom Mach held his lectures, were not exactly enthusiastic about Fechner's world-soul phantasies and that Mach had to swallow a few insults for this.[9]

Mach may have decided to formulate a tough-minded positivism to chew the tough meat of the reigning materialist view, pushing his own tender-mindedness to the rear. Unfortunately, the position presented in the "Antimetaphysical" first chapter of the *Analysis of Sensations* has become the *locus classicus* for all of Mach's views, and for a good deal of twentieth-century positivism. The two quotes most often quoted from Mach are the following:

> The assertion...is correct that the world consists only of our sensations. In which case we have knowledge only of sensations, and the assumption of nuclei...or of a reciprocal action between them from which sensations proceed, turns out to be quite idle and superfluous.[10]

> Properly speaking the world is not composed of "things" as its elements, but of colors, tones, pressures, spaces, times, in short what we ordinarily call individual sensations.[11]

These certainly suggest that everything in the natural world must be limited to human sensations, in either their psychical or physical relations. But to get at the nuanced position beneath these brusque phrases requires our own analysis of the *Analysis*.

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[9] Heidelberger *Die Innere Seite der Natur*, p. 214.

[10] (AS), p. 12.

[11] (M), p. 579.

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Mach's Basic Division of Elements

Mach explained that human experience begins as a flooded agglomeration of elements, similar to James' "blooming buzzing confusion." Gradually we learn to divide them into three classes. The first class, labeled A,B,C...refers to our sensations of *physical objects*. The second, labeled K,L,M...refers to sensations we have of our own bodies (such as local signs), and finally in the third class labeled α, β, γ... are after-images and memory representations.

It turns out that quite often the A,B,C will vary with the elements of our own bodies K,L,M. If the retina suffers a chemical change, the colors of objects will also change. However the A,B,C may also undergo their own independent variations, for example when an object moves across the stationary retina. When this latter case occurs Mach calls the A,B,C physical objects or elements:

> A color is a physical object as soon as we consider its dependence for instance upon its luminous source, upon other colors, upon temperature, upon spaces, and so forth. When we consider, however, its dependence upon the retina (the elements K,L,M), it is a psychological object, a sensation. Not the subject matter but the direction of our investigation is different in the two domains.[12]

The elements themselves are neither psychical nor physical but neutral, separated only by the different types of functional variations they undergo. These variations are not absolutely separate either. A thing may enter into physical variations

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[12] (AS), p. 16.

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while still being subject to a whole range of psychological ones. The sensation of the yellow sun is affected by physical events, but also by the condition of the retina. If the retina were damaged nothing would appear and even the physical variations of shape, extension and radiant energy would be blocked to sight. Instead of hard and fast divisions between psychical and physical functions, it is more likely Mach thought the two different orderings of elements were different economical patterns chosen from among a plurality of coequal relations.

Criticisms of Neutral Monism

Neutral monism was subject to two kinds of attacks in the early twentieth century. The first was: if *no* psychical variation of elements is a physical variation, and vice versa, then how could a neutral sensation partake of both orderings? Would that not be tantamount to a thing's having two contradictory predicates? The criticism was put by Arthur O. Lovejoy in his 1930 *Revolt Against Dualism*,[13] where his target was the neutral monism of James and Russell. But the question was put to Mach earlier by Gabiele Rabel.[14] He answered that the division between psychical and physical variations was temporary and depended on the present state of science. They were not mutually exclusive as Lovejoy imagined. Rather

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[13] (NY: Harper and Row), pp. 56-59.

[14] Rabel op. cit., p. 434.

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psychological and physical "predicates" are homogeneous determinations, psychology being just a very specific form of overall physical (or natural) variation.

Mach aimed to eventually abolish the distinction of psychical and physical variations altogether, making for only one underlying plurality of natural variations between homogeneous elements, separated only by economical considerations. Indeed as we have seen the elements are part of any number of formal orderings into objects, depending on which functional variations and repetitions of experience we emphasize. The issue Lovejoy raises is an economical one and he is himself making the underlying error of assuming that elements are classified by the different types of objects they belong to. It is really the reverse: objects are selected as various patterns chosen from the set of homogeneous elements.

A second criticism of neutralism came from Franz Brentano and his school. Brentano believed that acts of the mind could be distinguished from physical events by what he called their immanent objectivity or intentionality, the act's internal reference to an object. As Brentano colorfully illustrated in a famous passage of his *Psychologie vom empirischen Standpunkt*, when we desire, love, hate, wish, there is always a certain object desired, loved, hated or wished for, included as it were "internally" in the mental act.[15] Brentano maintained that physical elements or events contained no such

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[15] *Psychologie vom empirischen Standpunkt*, (Leipzig: 1874).

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internal references. Thus for Brentano, and his students, psychical acts are irreducibly mental and not physical under any interpretation.

Brentano's student Antony Marty wrote Mach to dispute the point with him.[16] He said that Mach had failed to distinguish between act and content.[17] For example tone and the hearing of a tone were not the same.[18] The tone itself is a sensation, the hearing of the tone is an act with an "internal reference" to that sensational content.

Mach was seeking to join mind and nature instead of driving them apart and thus did not distinguish psychical capabilities from physical ones. Memory for example was not a psychical, intentional (and incidentally for the Austrian school totally unexplained) relationship to the object remembered. It was rather a capability to record and order traces in functions with present sensations, a capability that inanimate matter shares to some extent. Moreover, I believe Mach's distinction between sensations and their functional relations already captured the logic of content and act without implying a mental-physical divide. Mach believed that *every* element in nature, not just the psychical ones, belonged to some formal complex; thus all complexes consisted of their

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[16] Marty to Mach 3 April 1896 in (WK), pp. 137-141.

[17] See Edwin Boring *A History of Experimental Psychology* (NY: Appleton-Century Crofts, 1950), Chapter 19 "Act Psychology and the Austrian School."

[18] Ibid., p. 138.

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constituent "internal" elements, just as every intensional act takes its content for an object.

Sensation versus Element: Immanent Realism

Mach emphasized the objectivity of the physical variations by calling them elements:

> The elements A,B,C...are not only connected with one another, but also with K,L,M. To this extent and to this extent only do we call A,B,C...sensations and regard A,B,C as belonging to the ego...it is only in the connexion and relation in questions, only in their functional dependence , that the elements are sensations. In another functional relation they are at the same time physical objects. We only use the *additional* term "sensations" to describe the elements because most people are more familiar with the elements in question as sensations (colors, sounds, pressures, spaces, times).[19]

In so far as they are elements, and not in the specialized class of sensations, Mach thought the qualities were actual parts of physical objects immanent in consciousness. To explain immanence, one could say that the physical object of perception reaches all the way from the outer thing into consciousness, if we consider its "physical" variations to include all intermediate contact with the sense organs and even the specific energies excited in the brain, with which the sensations A,B,C are in turn *physically* connected. Hans Kleinpeter gave an apt description of Mach's realism:

> Mach explained as the object of the naturalist's occupation "sense perceptions," a name which he chose under compulsion in lieu of a more suitable one, and which he later replaced by "elements"; for the sense in

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[19] (AS), p. 16.

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> which he wished to have the word taken deviated to some extent from the customary usage. By the sound of the word we are inclined to put too much emphasis on the perceiving subject. But it was far from Mach's intention to emphasize this connection from only one point of view; on the contrary he saw at the time in sensations the material of the actual world. In this his fundamental views are essentially different from those of idealistic philosophers, Berkeley's among others. We may even call them realistic, but their realism is essentially different from so-called philosophical realism.[20]

According to Mach, these objective qualities are not only human sensations that happen to vary independently, they are already a proper part of the external world. When we draw the circle around a pattern of elements, we may draw it to include those elements that extend into consciousness:

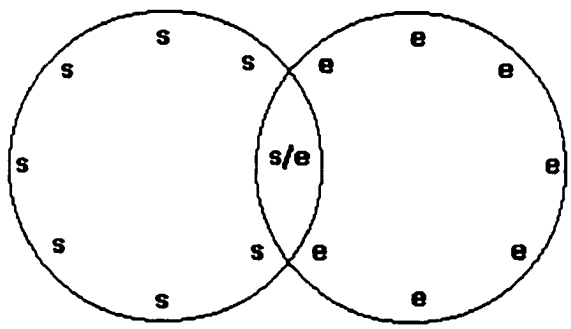


FIG. 10.1 Sensations and Elements

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[20] Hans Kleinpeter "On the Monism of Professor Mach" *Monist* **16**(2) April 1906: 161-168.

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The s-elements are sensations annexed to the K,L,M of the body and to memory images. The e-elements making up the object are mostly unsensed, but a piece of the object, designated with an 'e/s,' is common property between external object and ego. This immanent object is partly a mind-dependent sensation and partly a mind-independent world element.

Reading the *Analysis of Sensations*, it often seems that Mach wanted to leave out the fully mind-independent world elements designated only with an 'e' not an 'e/s.' This has become one of the thorniest questions of Mach scholarship:--must every physical element also be a human sensation under some interpretation of its variations? When Mach was pressed to answer he said clearly: no, sensations are a special, proper, subset of the world elements and not the other way around.[21]

The scholarly literature overplays the *Analysis of Sensations* and either omits or dismisses the world elements that appear in Mach's other works.[22] The consequence of

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[21] Bertrand Russell, when he came to develop his own Mach-inspired neutral monism, also felt compelled to introduce "unsensed sensibilia," an appellation with an unfortunate Millian ring to it. "Our Knowledge of the External World" *The Analysis of Mind* (London: 1921); *The Analysis of Matter* (New York: Dover 1954, orig. 1927).

[22] V.I. Lenin *Materialism and Empirio-Criticism* (Moscow: World Languages Publishing House, 1952), M. Schlick *General Theory of Knowledge* (Chicago: Open Court), Manjulekha Bhattacharya "Ernst Mach: Neutral Monism" *Stud. Int. Filosofia* 4(1972): 145-182; Andy Hamilton "Ernst Mach and the Elimination of Subjectivity" *Ratio* 3(2) December 1990: 117-135. A different view is attempted in Hazim Murad's dissertation *Mach Revisited*

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adopting such a position is that elements that are not also the sensations of a human being under some interpretation do not exist. In fact no neutral monist believed this, not Mach, not James, not Russell.

However, in Mach's other writings there are signs of a different position. For example in the article "Sensory Elements and Scientific Concepts" he wrote:

> If we pay attention not to the dependence of the elements, but to the interrelations or *reactions* of the elements in general, then we can generally say that the space-time region where the reactions of elements are connected may be called matter..The connection of perceptions is only a special case of the connections of reactions...of elements, of which the connection of the perceptions of people is only a special case.[23]

And in the *Theory of Heat* he said there were complexes of elements (physical equations like Helmholtz's physical objects) external to the mind:

> A definite activity of the organs of sense is not the only condition of a definite permanency. In order that something definite may be seen in a definite place, a definite tangible something must also be there; and thus a condition lying outside (foreign to) the organ of sight must be fulfilled...If now one wants to regard the said physical equations, in contradistinction to sensuous elements, as noumena, or on account of their importance in our knowledge of the actual world, as the expression of realities, such a quarrel about words is of little consequence.[24]

(Pittsburgh 1990) in which he quoted a passage from the *Theory of Heat* where Mach speaks of objects "foreign to the organ of sight" ((H), 384) Murad pp. 91-99. I am indebted to Murad for calling attention to this passage.

[23] (DL), pp. 119-120.

[24] (H), p. 384.

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The true neutral monist picture that emerges is of a "cosmic soup" of world-elements, including animal sensations and all of the physical qualities on the inside of nature, causally linked together in functions. There is no restriction to elements of the human ego.

In the *Analysis of Sensations*, however, Mach spoke only of a limited objectivity that sensation-elements could achieve by artificially isolating the variations of the A,B,C from the K,L,M:

> The ascertainment of the dependence of the elements ABC on one another, KLM being disregarded, is the task of natural science, or of physics in the broadest sense. But in reality the ABC are always dependent on the KLM. There are always equations of the form f(ABC...KLM)=0. Now since many different observers KLM, K'L'M', K''L''M'' are treated like physical instruments, each with its peculiarities, its special constants, and so forth, from which the results, as finally indicated have to be set free, we succeed in eliminating the accidental influence of the variation of KLM, and we thus obtain only the element that can be stated as common property, namely the pure dependence of the ABC's on one another.[25]

He said further that the pure variations of the ABC will lack a relation to any particular qualities, which their contact with the sense organs of the human observer gives them, and will assume the form of "spatial identities."[26] This is of course no conception of a mind-independent external object at all, not even an abstract spatial identity, since for all we know the spatial properties may also be due to their causal

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[25] (AS), p. 344.

[26] (AS), p. 345.

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contact with the human sense apparatus. The method of variations only gives a kind of standardized conception of the A,B,C in relation to the average human observer. This purely immanent objectivity says nothing about what the relations of the ABC would be without any observer present, which makes the ABC more dependent than ever on the human body. Mach seemed to realize this in the above passage ("But in reality the ABC are always dependent on KLM") and in his notebook:

> Holding constant, excluding of the body
> freeing it from chance incidentals
> Not absolute, quant. and qualit...
> KLM cannot be excluded, but held constant. Then conceptual relations of the ABC. By conceptually fixed ABC the pure influence of KLM on ABC.[27]

The *Analysis of Sensations* shipwrecks on this problem: the elements ABC could not be severed from their connection with the human body KLM, and indeed this is quite in order, so long as the ABC stand only for immanent elements that can also be sensations.

Philosophers call the restricted position of the *Analysis* "immanent realism" and trace it back to Kant and Fichte.[28] It

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[27] Constanthalten, Ausschaltung des Leibes von zufälligkeiten befreien.
Nicht absolut, quant. und qualit...
Mann kann KLM nicht ausschalten, aber constant halten. Dann Begriffliche Beziehungen von ABC. Bei begrifflich constanten ABC den reinen Einfluss von KLM auf ABC. Notebook dated 4 September 1901 (NL 174/2/44).

[28] The case can be made that Aristotle and neo-Aristotelians as various as Hegel and Franz Brentano have also represented the position. Brentano's form of immanent realism is admirably drawn by Barry Smith, *Austrian Philosophy: The Legacy of Franz Brentano* (Chicago and La Salle, IL: Open Court, 1994). Hegel described the

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is realism because the object has an independent physical existence (the physical variations of the ABC), but immanent because the object is always causally connected with the elements of the conscious ego and thus immanent in consciousness directly or mediately. The relation will be mediate if the chain of causes and effects extends not only to the immanent object but also to other objects causally related to it.

By the method of variations we can isolate abstract "spatial identities" which represent the pure causal relations of the external object to human sensibility. The human sense-qualities may or may not be abstracted away from this object,

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"appropriation" of the immanent object by the empirical I in his *Grundlinien der Philosophie des Rechts* (Stuttgart: Reclam, 1994) §.4 Zusatz: "Indem ich einen Gegenstand denke, mache ich ihn zum Gedanken und nehme ihm das Sinnliche: ich mache ihn zu etwas, das wesentlich und unmittelbar das Meinige ist...Wie Adam zu Eva sagt, du bist Fleisch von meinem Fleisch und Bein von meinem Bein, so sagt der Geist, dies is Geist von meinem Geist und die Fremdheit ist verschwunden." Mach was intrigued by Hegel and sensed an affinity even if Hegel's writings were impenetrable to him, a problem to which we can all relate. See H. Laitko und D. Hoffmann "Positionen zu Ernst Mach" in Laitko und Hoffmann, eds. *Ernst Mach: Studien und Dokumente zu Leben und Werk*, (Berlin: Deutscher Verlag der Wissenschaften, 1991), p. 28. They quote a letter from Mach to Adler of July 11, 1906: "I have begun to understand Hegel from Dietzgen, which I had not been able to do before. It appears that one arrives at the same position from any side and any method, when one goes at the matter consistently and without any prejudice. This is the most remarkable result of this article. This is also very worthwhile for me. Hegel didn't suit my psychical organ, and yet I was convinced that he is a thinker and not a fool."). Cf. Mach's letter to Adler February 18, 1907 in (AU), p. 49. and to William James, May 6, 1909 (WK), p. 176.

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but because it is linked through causal relations to an I its form is still indirectly determined by human sensibilities, not the object an sich. Remove the I from immanent realism and the form of the world falls to pieces. Mach himself declared that his position "bordered closely" on that of the immanentists', particularly Wilhelm Schuppe's. Gabriele Rabel argues that Mach's realism was of Kant's immanent sort.[29]

The Solipsism Charge

Because of the close dependence of objects on the ego there was some tendency to think that immanent realists like the Mach of the *Analysis* were solipsists who believed only in the I and perhaps only the immanent objects that overlap into consciousness. However there is an important distinction already made in immanent realism which leads the way towards a new position.

The idea goes back to Spinoza and Kant, who distinguished between the "empirical consciousness" of human awareness and an impersonal "unconscious ego" called the Absolute I. The Absolute I includes both the empirical awareness and the immanent objects of the environment. Empirical awareness is first achieved by separating the Empirical I from the environmental objects, both contained within the Absolute I.

The distinction is fundamental for Mach, who called the

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[29] *Physikalische Zeitschrift* **21** (1920): pp 433-437. Cf. *K.d.r.V.* Von der Amphibolie der Reflexionsbegriffe.

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Empirical I the "ego in the restricted sense."[30] It includes only the KLM, α β γ and the psychical variations of the ABC. Mach called the impersonal Absolute I the "ego in the unrestricted sense," since it included the physical variations of the ABC as immanent objects in the environment of the ego. Experience broadly conceived thus already contains material of the external world in the physical variations of the A,B,Cs. Not all of this environment can possibly be limited to the restricted empirical ego itself, for this would erase the very act that underlies empirical awareness.

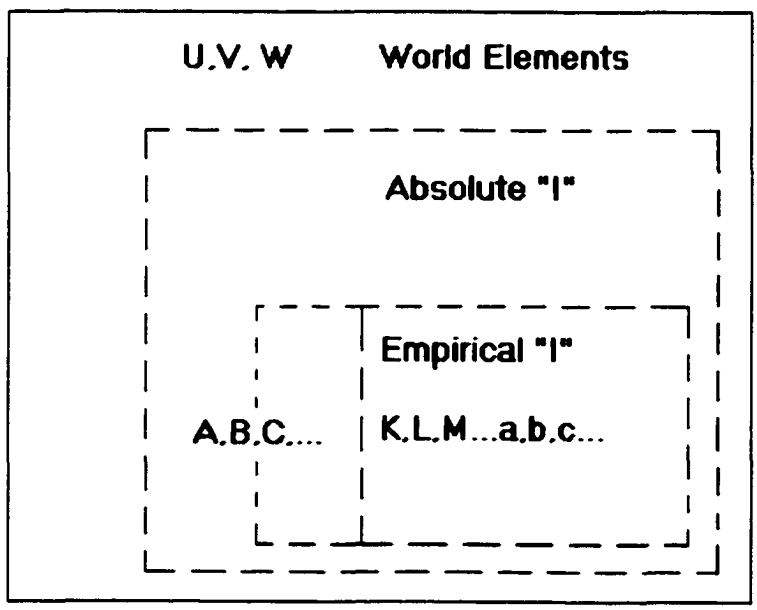


FIG. 10.2

[30] (KE), pp. 6-7.

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Mach expanded this conception into his own early form of "developmental psychology," claiming that consciousness, far from an inherent human attribute, was in fact a humble, learned ability to separate our bodies and psychical elements from the environment of the physical A,B,Cs. Until we can differentiate them we do not achieve the restricted "empirical" awareness of the ego. Consciousness thus emerges at the end of a historical development in early life. We grope our way unconsciously at first in the cosmic soup of the Absolute I and only gradually learn to associate the elements of our bodies and imagination with the Empirical I. The awareness of the self and the awareness of the environing objects occurs together.

Presumably, if we made the environment consist of the ideas of the ego--and unlike Descartes really succeeded in believing it--we would plunge back into that primitive unconsciousness whence we came. An illustrative and apt portrayal of that oscillation between the Absolute I and the Empirical I can be found in Poe's story, "The Pit and the Pendulum," where the narrator is not yet sure whether or not he has been executed by the Inquisition:

> Very suddenly there came back to my soul motion and sound--the tumultuous motion of my heart, and in my ears, the sound of its beating. Then a pause in which all is blank. Then again, sound and motion, and touch--a tingling sensation pervading my frame. Then the mere consciousness of existence, without thought--a condition which lasted long. Then, very suddenly, thought, and shuddering terror, and earnest endeavor to comprehend my true state. Then a strong desire to lapse into insensibility. Then a rushing revival of soul and a

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successful effort to move.

Mach wrote excruciatingly about the very first stages of differentiation, showing that we only *gradually* learn through specific experiences to regard the ABC as objects external to our bodily surfaces KLM:

> We see an object having a point S. If we touch S, that is bring it into connection with our body, we receive a prick. We can see S without feeling the prick but as soon as we feel the prick we find S on the skin. The visible point, therefore, is a permanent nucleus, to which the prick is annexed, according to circumstances, as something accidental. From the frequency of analogous occurrences we ultimately accustom ourselves to regard all properties of bodies as "effects" proceeding from permanent nuclei and conveyed to the ego through the medium of the body; which effects we call sensations.[31]

He then showed how the conceptions of ego and external world--including the seventeenth-century empiricist "theory of ideas" and representative theory of perception--both arise *within* the differentiation of the Absolute I from the Empirical I and not outside, for example in a further division between the Absolute I and the elements of the world. Mach claimed that he wrote the *Analysis of Sensations* from the standpoint of absolute nâiveté, before the distinction between the ego and world had been drawn. He wrote to Rabel for example:

> The kernel of my exposition lies in the elucidation of Fig 1. of the *Analysis* [the famous "headless body" picture[32] E.B.]. We arrive at it by observing children and putting ourselves back into our early childhoods, as we were just learning to differentiate our bodies from

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[31] (AS), p. 12.

[32] (AS), p. 19.

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> the environment. At that point we know nothing of matter and soul, of physical and psychical, of object and subject, stimulus and sensation. Everything still consists of homogeneous ingredients ABCD..., which are in themselves neither psychical nor physical, but rather indifferently neutral. They first become physical or psychical through the particular type of dependency we have in view.[33]

And in a letter to Friedrich Adler on the same subject:

> The point of departure of my natural world view when I wrote the *Analysis of Sensations* was more primitive. I placed myself artificially back in the position of a child, who has just begun to separate his body from the environment. There for the first time results the two-fold type of dependence of the elements on one another; not until then do these lead to the concept of a subject, an object...[34]

In both the *Analysis of Sensations* and *Knowledge and Error* Mach admitted that one could speak of the "ego in the unrestricted sense" by including the environment within the ego so that "my ego includes the world as its idea and representation."[35] But in this case Mach insisted that awareness, in the sense of a restricted ego, would only be possible if we then restricted the Absolute I against the external world of mind-independent elements and other minds at its boundaries:[36]

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[33] Rabel, p. 434.

[34] Mach to Adler January 23, 1910 (AU).

[35] (KE), pp. 6-7.

[36] Schelling was the first to devote attention to this form of Absolute or "transcendental" self-consciousness which includes the *Ding an sich* as the environing objects of awareness, in his *System des tranzendentalen Idealismus*.

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> If now I call the sum of my mental aspect, sensations included, my ego in the widest sense (in contrast with the restricted ego), then in this sense I could say that my ego contains the world (as sensation and idea). Still we must not overlook that this conception does not exclude others equally legitimate. This solipsist position seems to abolish the world as independent, blurring the contrast between it and the ego. The boundary nevertheless remains, only it no longer runs round the restricted ego but through the extended one, that is through "consciousness." Indeed we could not have derived the solipsist position without observing the boundary between my own and others' egos. Those who say we cannot go beyond the ego therefore mean the extended ego, which already contains a recognition of the world and other minds.[37]

The solipsist is faced with a Hobson's choice: accept an unconscious, unlimited set of elements (which, because no explicit limit is drawn surreptitiously includes any other elements that may exist), or accept an Absolute I delimited against other elements and minds, whose existence we must then admit anyway. Thus in either case it is impossible to rule out mind-independent elements, or to separate the ego from the world as a whole.

The first to point out that the elements of the Absolute I must be taken in an unrestricted sense of existence was the aphorist Lichtenberg, whose remarks on Descartes' cogito are quoted in full in the *Analysis of Sensations*.[38] Like Mach, Lichtenberg considered the Absolute I the sum of sensations, memories and bodily experiences, and claimed that instead of saying "I think" we should say "Es denkt" (there is a

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[37] (KE), pp. 6-7.

[38] (AS), pp. 28-29.

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thought). Lichtenberg is often credited with the first statement of an ego-less view of the world, which indeed he suggests, but the force of his argument runs a bit deeper. The idea is that if we must regard all sensation, thought and imagination as predicates of something, that something is not an I but an "It." Only by first granting existence unrestrictedly, can we then restrict some elements to a consciousness. Like Lichtenberg, Mach held that the elements existed unrestrictedly first before they were divided up between the ego and the environment.

The Absolute I and the World Elements

So just how unrestricted is the set of elements? Should it include only those in the Absolute I? Or is this limitation also suspect? Mach spoke of the need to "add in thought" additional elements when he dealt with the problem of other people's sensations as if he accepted such a limit. But he must have seen that the existence of sensations of other minds cannot be finessed through the distinction between the Absolute I and the Empirical I alone.

The situation would be different if we were allowed to begin with the unrestricted elements of the cosmic soup including the elements of other minds, animals and the world elements of matter. Experience would then help us restrict the elements of the soup to the patterns of egos and objects. The inference to my own ego apart from others would then be

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the same inference used to separate my own ego from the environment. Only here the environment is all of the elements together, not just those of the impersonal Absolute I.

There is good evidence that Mach thought himself justified in beginning with the completely unrestricted set. He said for example that the elements of egos existed together before we limit them to individuals:

> From the standpoint which I here take up for purposes of general orientation, I no more draw an essential distinction between my sensations and the sensations of another person than I regard red or green as belonging to an individual body. The same elements are connected at different points of attachment, namely the egos. But these points of attachment are not anything constant. They arise, they perish, and are incessantly being modified. But where there is no connexion at a given moment, there is also no perceptible reciprocal influence. Whether it may or may not prove possible to transfer someone else's sensation to me by means of nervous connexions, my view is not affected one way or the other.[39]

Avenarius and Schuppe also started from a set of elements that included the elements of other minds.[40]

Mach thus saw fit to begin with the broadened cosmic soup and not with the contents of the Absolute I, pushing considerably beyond immanent realism. It does not seem to me likely that he began with the Absolute ego of one person and

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[39] (AS), p. 361.

[40] Richard Avenarius *Der Menschliche Weltbegriff* Third Edition (Leipzig: O.R. Reisland, 1912), pp. 4-5; Wilhelm Schuppe "Der Bestätigung des naiven Realismus" Offner Brief an den Herrn Prof. R. Avenarius *Vierteljahresschrift für wissenschaftliche Philosophie und Soziologie* XVII(1893), p. 390 ff.

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tried to "infer" other minds and other sensations or elements. He rather started with all of the elements, which is why it is so easy for him to make the analogy to other minds, and objects, since he has already admitted their existence and the problem is simply to segregate them from the rest of the environment by dividing up what is already present:

> When I speak of the sensations of another person, those sensations are, of course, not exhibited in my optical or physical space; they are mentally added and I conceive them causally, not spatially, attached to the brain observed or rather functionally presented.[41]

Mach did not believe there were egos and objects first and then sensational content, but rather contents and then owners. This conception parallels Mach's idea of making elements prior to their formal arrangement into objects. Likewise, an element of blue or red does not belong to its owner in any Brentanesque intensional way:

> If the understanding of the relations of the elements (sensations) does not satisfy us and we ask "who has this relation of sensations, who senses?" we still are subject to the old habit to attribute every element (every sensation) to an unanalyzed complex; we sink thereby unnoticeably to an older, lower and more limited standpoint.[42]

Even the division of the elements "given" in the Absolute I from the rest of the cosmic soup is simply not made except as a causal or economical necessity.

Mach's standpoint is actually similar to that of

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[41] (AS), p. 27.

[42] (AS), p. 201.

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Schelling, who argued that the division between the Absolute I and the *Ding an sich* is produced by the unconscious "limiting" activity of the Absolute I within an absolutely unrestricted set that contains them both. Either way you slice it, a limit must be drawn within a greater totality of elements. The more limits we draw, the more elements we must admit into the original unrestricted set, else the idea of drawing limits falls to pieces. Thus when Mach said that nothing existed but the elements, it was probably the greatest totality he meant.

In a later attack on the given-ness of elements, in *Knowledge and Error*, Mach wrote that elements are thinkable separate from consciousness (given or not-given to an ego) just as long as they belong to some complex, which need not be a consciousness at all:

> ...a sensation will indeed always occur in a complex, but that this latter should always be a complete and wakeful human ego is doubtful; after all there is consciousness in dreams, hypnosis, ecstacy and there is animal consciousness, all in different degrees. Even a body, a lump of lead, the crudest item known to us always belongs to a complex and so to the world; nothing exists in isolation.[43]

He finally confirms what I think was his real view all along, that every element exists in a "cosmic stream." As it turns out--and indeed this was inevitable--this cosmic stream is the starting point, not the ego and the given:

If the ego is not a monad isolated from the world but a

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[43] (KE), p. 359.

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> part of it, in the midst of the cosmic stream from which it has emerged and into which it is ready to dissolve back again, then we shall no longer be inclined to regard the world as an unknowable something and we are then close enough to ourselves and in sufficient affinity to other parts of the world to hope for real knowledge.[44]

Real Knowledge

By "real knowledge" Mach meant a direct experiential knowledge of nature. He saw this as a distant goal of science and an impetus to present inquiry for us, restricted by our egos only as a practical matter. The future of science would be to discover and break through the link between human sensations and the cosmic stream of world elements. Only direct experience of the world would qualify as "real knowledge" *not* just the quantitative structure of quality-less relations. Mach thought the pursuit of quantitative relations only gave half the story, which is why physical science so badly needed sense-physiology and psycho-physics. With the psychophysical tunnel built through we will not only know nature second-hand, but also first-hand. We will know what nature really looks and feels like *an und für sich*, experience we already have in our narrow but direct access to the specific energies of our nervous systems.

Mach threw out a number of hints that the investigation of the cosmic soup was the future of science, for example in his essay "On the Economical Nature of Physical Inquiry":

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[44] (KE), p. 361.

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> We cannot draw the science of the future in sharp contours. We can only intimate that then the hard separating wall between man and the world will gradually disappear, than men will not only approach one another, but also the whole organic and so-called life-less Nature with less self-seeking and a warmer feeling.[45]

And the *Mechanics*:

> Careful physical research will lead to an analysis of our sensations. We shall then discover that hunger is not so essentially different from the tendency of sulfuric acid for zinc, and our will not so different from the pressure of a stone, as now appears. We shall again feel ourselves nearer nature, without its being necessary that we should resolve ourselves into a nebulous and mystical mass of molecules, or make nature a haunt of hobgoblins.[46]

This latter passage, incidentally goes back to the first drafts of the *Mechanics* in 1870-71.[47]

In Mach's correspondence with Rabel, which she quotes in full in her *Physikalische Zeitschrift* article, Mach wrote about what would happen when the psychophysical tunnel was completed through:

> Now we must investigate everything from either the physical or the psychical side. When, however, the Tunnel between the physical and the psychical is completed through or almost through, no longer will we be restricted to this. Then it will be granted us to ask how animals sense. Of course, this still lies quite far off. But for those who come after us and are cleverer there will be no restrictions, no *Ignorabimus* -- The astronomers of today know that the Ptolemaic and the Copernican world-views are both practical restrictions and that one is allowed freer room for questioning.[48]

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[45] (PWV), p. 244.

[46] (M), p. 559.

[47] Mach's Lectures of 1870-71 "Mechanik" (NL 174/1/001).

[48] Rabel, p. 434.

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Here he envisions a time when actual world-elements (non-human sensations) will be demonstrable in the laboratory by a future science.

Mach's Controversy with Adler

Friedrich Adler, who reckoned Mach a positivist in the sense that only human sensations existed (as sensations or physical elements), was disturbed by passages in the *Analysis* that suggested that partially observed facts had to be "completed in thought" by adding unexperienced elements. Finally he wrote to Mach on January 15, 1910 and asked pointedly: *do mind-independent elements exist?*

> ...I will stick to the explanations in my Ostwald article and indeed above all to the remark p. 14-15.[49] In this and other places I defend the standpoint: Elements that do not belong to any "I" do not occur and are extraneous, not to say metaphysical, for our world-view.
>
> My question is now: *Do you consider necessary the assumption of elements that belong to an object, but not to a subject?* Obviously such elements are not directly given, but do you assume them hypothetically, as it appears to me you do from the places cited in your *Analyse der Empfindungen*.
>
> Clifford answers this question the opposite of how I did at the time (*Natur der Dinge an sich* p. 42)[50]. He asks there: "can a sensation exist for itself, without forming a part of a consciousness?" He answers "Yes" and indeed this question is to be decided by the theory of

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[49] Friedrich Adler "Bemerkungen über die Metaphysik in der Ostwald'schen Energetik" *Vierteljahresschrift für wissenschaftliche Philosophie und Soziologie* **29**(1905): 282-333.

[50] William Kingdon Clifford (1845-1879) was also the translator and defender of Riemann in England. The article Adler refers to is "On the Nature of Things-In-Themselves" *Mind* **3** (1878):57-67, German 1903.

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evolution.[51]

Mach wrote back on the 23rd of January:

> If one already knows the elements but not yet an I, there is no reason to differentiate the elements that belong to the I from the others. Not until the analogy with other persons' bodies, a scientific, not a metaphysical conclusion, is the other's I and one's own I created. From your standpoint beginning with the principal coordination, you actually had to say: "Elements that do not belong to *my* I do not occur."...
>
> Because I only seek a standpoint on which physics can rationally build itself, I can halt by the double-dependence of elements, for these are experimentally demonstrable, and have nothing to do with a philosophy or metaphysics. From such a standpoint I may also assume analogous elements as contents of another I, and this approach can even extend to the lower animals, plants and inorganic bodies. This hypothesis serves only to round off the world view provisionally, and in hope of the future construction of biology...Healthy biological research must teach if this hypothesis has worth, and if so, what worth. Speculation cannot manage this. Provisionally it appears to me that we completely ignore a side of our experience when we overlook this hypothesis.
>
> ...I have not further cultivated all of these matters, for I always feared the nearness of the metaphysical abyss, where there is no experiential foundation.[52]

Clearly Mach would support further research into the hypothesis of world elements if it were pursued in the course of experimental science.

Mach and W.K. Clifford's Cosmic Monism

Mach had an important predecessor, W.K. Clifford, a grand eccentric in the English style, whom I will now mention. Both

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[51] (AU), p. 68.

[52] Ibid., pp. 71-72.

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Adler and Paul Carus[53] wrote to Mach regarding Clifford's article "On the Nature of Things in Themselves" (1878) in which it was argued that inanimate objects in themselves are composed of "mind-stuff," and that we may infer its existence from a long line of analogy between human and animal minds, plants and finally matter.

Mach's correspondents wanted to know if he also accepted such mind-independent elements, and what his opinion was of Clifford. Mach generally held him in high regard, calling him in the Preface to the 1901 Third Edition of the *Analysis* "a writer with an extremely close affinity to myself in the direction of his thought." Mach however disliked "Clifford's conception of elements as things in themselves,"[54] and his willingness to speculate where there was no hard evidence for mind-stuff in simple animal life and inanimate matter.

Clifford's article was in fact a bone fide forerunner to Mach's *Analysis*, although Mach did not learn of it until he had already formulated his own position. Like Mach, Clifford distinguished two orders elements may be entered into: a subjective, or psychical order, and an objective, or physical, order. The self-same sensations may be either my sensations or physical objects immanent in consciousness. From this

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[53] Carus to Mach 12 February, 1890 (WK), p 183. Carus thought Clifford's theory too subjective and that it was an attempt to swallow up the object in the subjective "as the nine lean cows devoured the fat in Pharaoh's dream."

[54] Mach to Adler 23 January 1910 (AU).

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standpoint, Clifford asked how there could be elements not given to my consciousness, which he called "ejects."

Like Mach, Clifford found that the inference to ejects was deceptively simple, since there were already elements in experience that may be regarded as "ejects" from my consciousness and part of the environment of the ego. Thus Clifford also possessed the distinction between the ego "in the restricted sense" and the absolute ego "in the unrestricted sense." Clifford pointed out that the immanent elements in their physical variations are ejects already since they are part of the environing world external to my restricted awareness. Any element considered without its relation to memory images and elements of the body existed outside the mind. He says: "a feeling at the instant when it exists, exists *an und für sich* and not as my feeling."[55]

Like Mach, Clifford saw the conscious faculties of reason, intellect and volition as "properties of a complex which is made up of elements themselves not rational, not intelligent, not conscious."[56] Clifford's merit was to have seen that once the ego is dissolved into smaller non-conscious elements, there is no further reason at all to restrict those elements to an ego. We may consider not only those elements liberated from the ego as elements *an sich* but also any other elements annexed to them in the cosmic stream:

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[55] Clifford, p. 62.

[56] Clifford, p. 67.

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> Consciousness is a complex of ejective facts--of elementary feelings, or rather of those remoter elements which cannot even be felt but of which the simplest feeling is built up. Such elementary ejective facts go along with the action of every organism, however simple; but it is only when the material organism has reached a certain complexity of nervous structure (not now to be specified) that the complex of ejective facts reaches that mode of complication which is called consciousness. But as the line of ascent is unbroken, and must end at last in inorganic matter, we have no choice but to admit that every motion of matter is simultaneous with some ejective fact or event which might be part of a consciousness.[57]

Thus we simply annex downward, following the line of analogy from present sensations to ejective physical objects to the completely mind-independent elements annexed to them: the "mind-stuff" making up simpler animals and matter.

Leibniz as Precursor to Mach's Elements and Clifford's Mind Stuff

Historically astute readers will have recognized that the real forerunner of both Mach's elements and Clifford's ejects was Leibniz, who argued that conscious ideas in the sense of Descartes and Locke could be divided at pleasure into elements so tiny they were unconscious, and thus lay already outside the boundaries of the ego. These petites perceptions, as Leibniz called them, were tiny enough to get under the threshold of awareness and in this sense already belonged to the external world. Leibniz gave as an example a sound which begins small but gradually breaks the threshold of attention

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[57] Clifford, p. 65.

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of a sleeping person and wakes him. He also spoke of the perception of a wave as a confused agglomeration of tiny petites perceptions too minute to be apprehended separately by consciousness.[58]

Clifford's ejects and Mach's elements, the smallest divisions of ideas, are clearly the descendants of the petites perceptions. Leibniz also anticipated Lichtenberg's argument in saying that the petites perceptions were not predicates of the I but rather of the universe itself. In the *Nouveaux Essais* he has his advocate Théophile say:

> I have read the book of the celebrated Englishman of whom you were just speaking. I esteem it well and have found many beautiful things there. But it appears to me that we must go much further and separate ourselves from our sentiments, when it is discovered that they bound us more than they should, and reveal a bit not only the condition of man, but even that of the universe.[59]

It is not always clear from the expressions he uses that Clifford regarded mind-stuff as the actual interior of matter and motion, as Fechner, Mach and Leibniz did, or whether he thought mind-stuff ran in parallel with material changes. At the end of the article however it does appear that Clifford thought that physical objects, too, were completely constituted of "mind-stuff" of a very low order.[60]

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[58] Leibniz, *Nouveaux Essais*...Préface in *Sämtliche Schriften und Briefe, Philosophische Schriften* Bd. IV pp. 53-57.

[59] Ibid., p 73.

[60] Clifford, p. 66: "The universe, then, consists entirely of mind-stuff."

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World Elements and Human Knowledge: Spinoza

The lingering problem with speculative metaphysical views such as the one presented above concerns standards of argument and proof. With our miserably limited human knowledge, how could we possibly feel justified in starting our inquiries with a grand total of *all* the elements of the world, especially if it is admitted that in our present state of knowledge we cannot know the mind-independent elements of the world a priori? I think some of the answers can be found in the historical antecedents to Mach's position, which I do think is an epistemic as well as a metaphysical standpoint.[61]

In his mood of commencing with the whole and not the part, the philosopher Mach most resembles is not Kant or even Leibniz but his fellow monist Spinoza, perhaps via Fechner who was a very ardent Spinozist. Mach mentioned in the *Analysis* that "a full enumeration of my points of contact with philosophy would require me to begin with Spinoza." Presumably he is required to begin with him because they both begin in the same place. And in a letter to Josef Petzoldt, Mach wrote:

> The Spinozist concept *natura sive deus* is not at all repellent to me, any more than the Schopenhauerian will in nature. Only when one thinks to have solved all

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[61] I don't mean to drag Wittgenstein's *Tractatus* into the discussion, but there is a great similarity between Mach's ultimate standpoint and the position taken there of beginning with elements and formal relations (combinatorial principles) without apparent concern for epistemic questions.

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> problems through these words do they strike one as empty.[62]

Spinoza began his *Ethics* with the insight that instead of beginning with ideas restricted to a mind, as Descartes and Locke did, we can take them in an absolutely simpler and epistemically prior sense as predicates of unrestricted being-as-such, which Spinoza called "God, or nature" *deus sive natura*. So when we think we are introspectively examining the contents of our own minds, we are really surveying a part of being in general. This sense of simple unrestricted existence for the ideas is not only prior in the order of being, but indeed also in the order of knowing once we lift the restriction. F.H. Jacobi, Spinoza's interpreter in Germany, expressed the thought thus:

> We need not first search for the unconditioned, rather we have a greater certainty of its existence than we do of our own determined existence.[63]

Descartes and Locke are thus forced to assume that ideas exist as predicates of unconditioned being and then the self. They must first admit an unrestricted existence for ideas and then limit this to the restricted spheres of extended things or mental things. But "simple existence" is completely neutral between the two. The division between thinking and extended things can only be made in fact if a common ground of being-

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[62] December 12, 1914 (AU), #113.

[63] F.H. Jacobi, *Über die Lehre des Spinoza in Briefe an den Herrn Moses Mendelssohn*. *Werke* Band 1.1 (Hamburg: Felix Meiner Verlag), 1998.

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as-such precedes them both:-- they are thus like two rooms divided within the same house. Again, Jacobi wrote:

> The first thing--not in extended and not in thinking things alone--rather what is first in the one as well as in the other, and in the same way in all things, original being [*Ur-Sein*], the omnipresent, unchangeable reality, which cannot itself be a predicate of anything which it has, this single infinite being Spinoza calls God or Substance.[64]

The idea that such an abstract being-as-such could be more immediately known than ideas or minds had an electrifying effect on philosophy in the early nineteenth century, and single-handedly rescued metaphysical thought from Kant's withering attack. Clearly the more certain we are of this primordial sense of being, the emptier it becomes; which must be what Mach meant when he said that the *deus sive natura* could be so many "empty words." Spinoza offers us access to limitless realms of being but only in a more and more vacuous sense. As Feste the jester would have it, even blind old hermits who have never seen pen and ink can still accredit their learning by citing: "That that is, is. For what is 'that' but that and 'is' but is?"

The useful thought Mach did apparently glean from Spinoza (perhaps indirectly) was the idea that human sensations could be taken to directly represent realities and were not illusions, secondary qualities or purely mental things. Mach himself claimed that even dreams, hallucinations, phantasies

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[64] Ibid., p. 18.

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and sense-illusions were all realities in their own right and not false representations of other facts.[65] "Even the wildest dream," he said, "is a fact like any other." In one of his most intriguing side-lines, he held free-form hallucinations in very high regard indeed as traces of the independent life of the sense organs and physiological central processes.[66] And he once wrote about sense-illusions that they give us real knowledge of how the sense organs work and are thus fully-fledged facts of science:

> The expression "sense illusion" shows that we have not yet become conscious of the fact that the senses neither falsely nor correctly manifest external events...The only exact thing we can say about the sense organs is that under different circumstances they produce different sensations and perceptions. The more unusual effects we are accustomed to call illusions.[67]

Needless to say, it is rare to find a philosopher who shows such respect and interest for hallucinations and sense illusions, which most take for non-veridical experiences. But to a monist like Mach these phenomena simply are, in the same one sense of being as anything else. Mach remarked once in conversation to Paul Carus that he preferred to say the elements "are" rather than "are given." Even his youthful monistic experience suggests Spinoza's One:

> On a bright summer day in the open air, the world with my

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[65] (AS), pp. 10-12.

[66] (AS), pp. 202-208.

[67] "Über die Abhängigkeit der Netzhautstellen..." 1868 (MB), p. 307.

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> ego suddenly appeared to me as one coherent mass of sensations, only more strongly coherent in the ego.[68]

Mind-and-Body as an Empirical Problem

For all that, it is metaphysically possible that the Absolute I of an individual might happen to possess all of the elements that existed. But if so it would have no boundary and would not constitute a delimited consciousness or mind; it would simply be, *simpliciter*. If we were solipsists, metaphysically speaking, we would certainly never know it epistemically, and would never succeed in attributing all of the world elements that existed to our own consciousness, for where then would the limit be drawn?

Fortunately, it wasn't Mach's goal to convince philosophers; he believed that philosophical conviction would eventually follow, if first science could demonstrate other elements of the cosmic soup empirically, perhaps by altering human sensory capacities or making world elements "sensible" by some means, building through a "tunnel" to the sensations of other minds, animals and objects, as Mach mused in his letters to Rabel.

The existence of the *Weltelemente*--a seemingly philosophical issue--thus transformed in Mach's capable hands into a practical question of scientific, and perhaps medical, possibility, not a theoretical one to be decided a priori. In fact, I think that Mach's investigations might reach their end

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[68] (AS), p. 30n.

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with the empirical demonstration of the first world element (non-human sensation) in the laboratory. If the Nobel prize were some day awarded for such a discovery, it would go to those who actually did the work, not those who merely imagined the results.

Proving the existence of the world elements empirically was *the* burning scientific question for Mach, and I think this is a live option today. The possibility of opening up a world of sensation and feeling in nature, real knowledge as Mach called it, was to drive the engine of science. In the next chapter we will see how this engine was put to work to dissolve lingering metaphysical problems in psychology.

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Chapter Eleven: Anti-Metaphysics, Mind and Body and Perception

In this chapter we shall see how Mach used the *Weltelemente* for solving metaphysical "pseudo-problems" in psychology. One of the most important of these was the relation between mental sensations and physical elements.

Mach believed that psychologists had been thinking sensations spatially into the brain, i.e. imagining that sensations are somehow inside, on top of, or in some other spatial relationship to brain tissue, whereas he held that sensations were functionally, but not necessarily spatially, attached to brain processes. Moreover this functional relation between psychical and physical elements was not of a special nature; the psychophysical relation is a natural causal relation between two elements, just like the relationship of any other two.

In this chapter, and in ensuing chapters, I will be doing more explicit reconstruction of Mach's ideas, sometimes even adding ideas of my own to complete the picture. This is necessary since Mach often indicated his drift in a short paragraph, without working out his idea completely and systematically. I will make a special point to indicate where I am engaging in explicit reconstruction of my own, and where I am reporting or interpreting what Mach said.

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Avenarius and Introjection

Mach's antimetaphysical attack on the use of spatial metaphor in psychology was not unusual in the nineteenth century. Richard Avenarius and William James both upheld versions of Mach's critique. Avenarius, whose views are similar but not identical to Mach's, exposed what he called the "error of introjection" in Chapter Two of his book *Der Menschliche Weltbegriff*.[1] Avenarius asks us to imagine we are looking at one of our fellow human beings, observing his brain down to the last detail, while our friend looks at some object like a tree. Avenarius says that we naturally assume our friend sees the same tree we do, and we "introject" or spatially superimpose our colored mental image into the brain processes we observe in him. Yet if we look in his brain, the cross-sections of the nerves, etc. we find nothing like the color green, the brown trunk, etc., but merely our own impressions of his brain tissue. We find only representations of the external electrochemical processes in his nerve cells and their physical properties. Even if we could get into every detail of the spatial interiors of the physical nerves we observe we would never discover the friend's sensations.

Avenarius claimed that once I have assumed that mental images are inside of, or in some spatial relationship to,

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[1] Third Edition (Leipzig: O.R. Reisland, 1912). Cf. Avenarius "Bemerkungen zum Begriff des Gegenstandes der Psychologie" *Vierteljahresschrift für wissenschaftliche Philosophie* Bd. XVIII (1894): 137 ff. and 400 ff.; Bd XIX (1895): 1 ff. 129 ff.

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brain tissue, I proceed to draw the faulty inference that my own ideas of color and shape are processes in my brain. According to Avenarius, this has the consequence that the environing objects of my ego making up the impersonal Absolute I implicitly become contents of my consciousness, rather than mind-independent physical elements.[2]

In his *Principles of Psychology*[3] James, who called the metaphysical error the "psychologist's fallacy," mentioned that just because an idea of an object is triangular, or colored to the observer, there is no reason the brain process of our friend must be triangular or colored too. Most sense-physiologists follow this precept carefully. There are of course cases like area 17 of the visual cortex where the objects of the visual field do seem to be mapped virtually point for point onto brain tissue. Even dots and bars in the visual field seem to look like dot- and bar-shaped patterns of cells. But there are also cases in which two features that coincide spatially in a perception are actually processed in spatially different regions of the brain, and are only causally linked to one another. The most obvious example is called cross-modality location: we locate a sound where we see the object that makes the sound, even though sound and vision are separately processed.

Besides, even the closest spatial "similarity" between a

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[2] *Der Menschliche Weltbegriff*, pp. 4-5.

[3] (Chicago: Encyclopedia Britannica 1952), p. 128

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brain state and a sensation does not prove that sensations or sensation complexes are in any way spatially superimposed on or inside the externally observed brain processes, merely that they are very closely causally related.

Causal Interactions between Sensations and Physical Elements

According to Mach, when I observe a person's brain I may "functionally" add his sensations to the brain processes I observe, but these sensations are causally, not spatially, annexed to the brain in front of me, which is still after all my perception and only mediately connected with the elements of the object an sich.

Furthermore, the physically observed brain tissue and the brain processes an sich (which appear to the friend as his own sensations) are different particulars. Mach did not think that sensations were the spatial interiors of the physically observed nerve energies. Rather the nerve processes as we observe them or interfere with them externally are made up through and through of their own elements.

It may still be that the sensations of the sensory nerves *an sich* are how those physical energies look on the inside. The only difference is that we cannot expect to locate them in the spatial cross section of the nerve we observe in the laboratory or in a friend's brain. That object is a perception, the result of mediate interference with the elements of the nerve *an sich* and does not contain the same

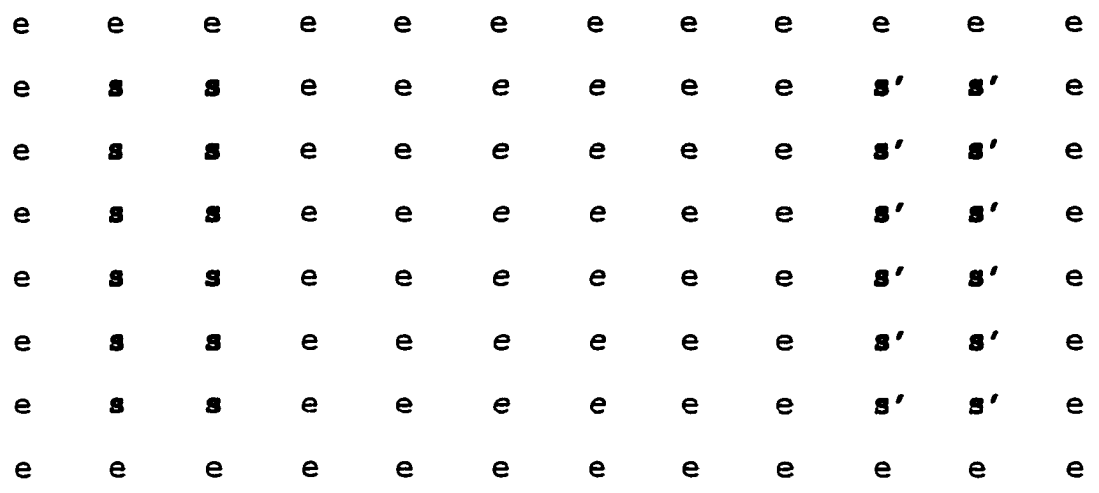
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individual qualities.

All of the elements, however, have the homogeneous dimensions of physical energies: each particular energy can act on the other and enter into the closest causal connection to the other while remaining distinct and non-identical: green remains green and the physical elements of the observed brain tissue remain different, though they are in the closest contact.

Mach's suggestion of a functional presentation to replace a spatial one is, if I may presume here to elaborate on it, that we simply collect up all of the different individual elements (the physical elements of the brain tissue, the elements of the sensations, and the elements of the observer's sensations) and flatten them all out side by side onto a flat functional map where closeness or remoteness has only to do with an element's functional connection to others. This is what I think Mach meant by a functional presentation of the sensation elements and the brain together. For example, taking the case of the friend's brain under observation, let e-elements be the physical elements of his brain, let e-elements be the elements of the object an sich, let s-elements be the friend's sensations, and let s'-elements be the observer's sensations, including both a perception of the object and a perception of the friend's brain tissue. Then the causal map will look like this:

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Now we can use our knowledge of the causal relations to separate out the different factors, and draw lines around the objects, such as the ego, the peripheral stuff of the brain tissue and the elements of the objects being perceived. For example the object being observed, say a candlestick, divides into: 1. the *e* elements of the object as it is to itself an sich; 2. the e-elements of the friend's brain tissue; 3. the **s** elements of the friend's sensations; 4. the **s'** elements of the observer when looking at the friend's brain tissue, *and* 5. the **s'** elements of the observer looking at the candlestick.[4] We may if we wish unite all of these aspects functionally into

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[4] Such causal maps with "multiple perspectives" of an object were also outlined by Clifford. Russell also argued that objects be broken up into multiple "perspectives." In the *Analysis of Mind* (London:1921) for example he represented objects as concentric rings of perspectives, some of which, as in Mach's and Clifford's causal maps, included unsensed sensibilia. However Russell thought that an object is all perspectives and that there was an "empty center" where the object and its elements *an und für sich* would be. This is out of line with both Mach's and Clifford's versions.

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a single distended object, as in Fechner's approach.

Each sensation or element is an individual quality connected into a complex of others by a causal relation. We might think of a sensation element s being annexed to a physical element e as the closest psychophysical relation between particulars like a green sensation-element and the last physical element(s) of a nerve-energy that goes along with it.

s e

The relation is no more mysterious that between other energies: potential energy (pressure) to vis viva, or electrical energy to heat, two different forms, which no one could predict would be equivalent, each of which appears when the other disappears and both of which are in the most intimate causal contact possible with one another, with no intermediary:

$$1/2\ mv^2 = mgh$$

One might well ask: what has become of Müller's doctrine that sensation-elements *are* the interior of nerve-energies sensed directly? I believe, from several of his later utterances quoted below, that Mach still held this view, and that the elements of brain cells *an sich* are indeed the

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directly-sensed sensation elements. However there must be some care for the wording: "interior" does not refer to a spatial relationship that would make sensations physically observable by delving into the interior of represented brain tissue.

There is nothing truly special about the relationship between brain tissue and its interior nature of sensation, since *every* physical element is an interior part of some complex in nature. The e-elements of physical nervous tissue we observe in the laboratory have their interior aspect and an appearance to themselves an sich, as colors, tastes, sounds and pains (or tinier elements of these) just like any other physical object.

When we observe another's brain tissue with our sense organs or with instruments, we are interfering with it and producing elements which we see as the external relations of the brain tissue or the nerve-cell carrying the sensation process. These physically observed processes do not reveal the nature of the elements an sich, but do preserve their external formal patterns of relation to other elements. So when we observe brain tissues with instruments or our own senses, the qualities of the elements an sich are not retained but all of the properties of their form, consistency, shape and physical properties are unchanged. This was what Mach and Fechner meant by the remark that observing someone's brain will not reveal the sensations of *that* brain, but rather our

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own sensations in *our* brains. The reason why we do not see the sensations of the brain itself is because of this mediate interference, not because sensations are any more inherently private than physical elements.[5] Like any element, sensations are absolutely individual existences that can come into causal contact with other elements, but can never be directly represented by another. One could never say a sensation of green was identical with a brain process observed with some instruments, only that it (the sensation) was closely related to certain physical elements, the physical substrates of the nervous system, and the observer's sensations or interferences closely related to those substrates.

No particular individual element is ever identical with another, thus the so-called "inaccessibility" of sensations is not a hallmark of the mental but is a property of every particular element. The reason why we do not notice this is because we are so used to ignoring the qualitative uniqueness of elements in material objects and concentrating on their quantitative relations in physics. We do not consider that a brain image observed by a doctor or on a machine is not the actual brain, because we do not ordinarily consider the

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[5] Russell, in his *Analysis of Matter*, elaborated on this point that we know nothing of the intrinsic quality of natural events, besides those of our own nervous systems, but only have a very precise knowledge of their quantitative relations and physically invariant properties.

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individuality of physical objects and their elements.

Psychophysical Parallelism

One of the traditional questions any theory of psychophysical parallelism is expected to answer is the following: do identical sensations always accompany identical brain processes and do identical brain processes always accompany identical sensations? Mach's 1865 statement of his principle of parallelism suggested there was a double-sided implication.

I think he made a misstep here, partially corrected later by a remark in *Knowledge and Error*. In light of the fact that both sensations and brain processes are physically homogeneous, there do not seem to be any special conditions for one to accompany another besides the conditions for ordinary causal relations between elements. Not just *any* old bundle of physical elements can annex to human sensation-elements of blue or the sound of a violin. But likewise there is no reason why those sensation-elements should not also be annexed to other physical elements besides brain tissue. Since it is a real physical quality, a human sensation of blue might certainly occur in complexes other than a human ego, as Mach implied in *Knowledge and Error*.[6]

The double-sided parallel between a specific complex of sensation-elements and a specific complex of physical elements might be upset, especially if the elements undergo reciprocal

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[6] (KE), p. 359.

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actions among themselves and then causally annex to others:

Case 1: e e e s

Case 2: e′ s

It is also possible that sensation-elements undergo certain interactions among themselves that alter their interactions with physical elements, thus making it possible for the same physical elements to become annexed to different sensation-elements.

Case 3: s s s e

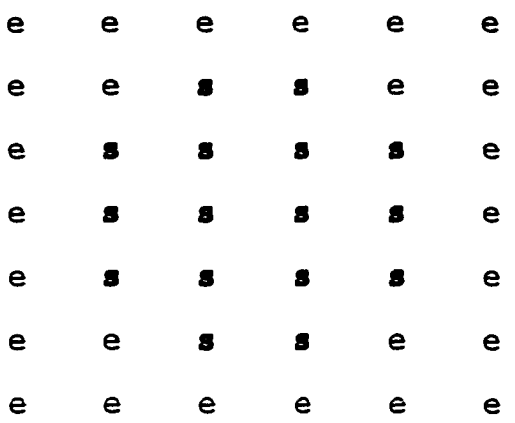
Case 4: s′ e

Polyadic interactions between elements open up further possibilities. This is one sign that the true causal map of the world, including the psychophysical relations can only be filled in by natural science. There is thus no reason to frame *any* a priori, metaphysical law of parallelism; all of the relevant questions are empirical ones.

The Ego-Complex

In the *Analysis of Sensations* it often seems that the ego is a closed bundle of sensation elements, with the following causal map:

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For example Mach wrote:

> I cannot refrain from...expressing my surprise that the principle of the conservation of energy has so often been dragged in in connexion with the question of whether there is a special psychical agent. On the assumption that energy is constant the course of physical processes is limited but not necessarily determined with perfect uniqueness. That the principle of conservation of energy is satisfied in all physiological cases merely tells us that the soul neither uses up work nor performs it. For all that, the soul may still be a partly determinant factor.[7]

What he has in mind is that sensation-energies can fill intermediary links between physical elements and leave the total energy entering and leaving the sensation complex the same. In the *Conservation of Energy* Mach admitted little islands (closed systems) of elements shut up in their own reciprocal interactions, and it is very tempting to think that these islets were egos and objects.[8] Their existence opens up contingencies in the causal map that cannot be predicted a priori.

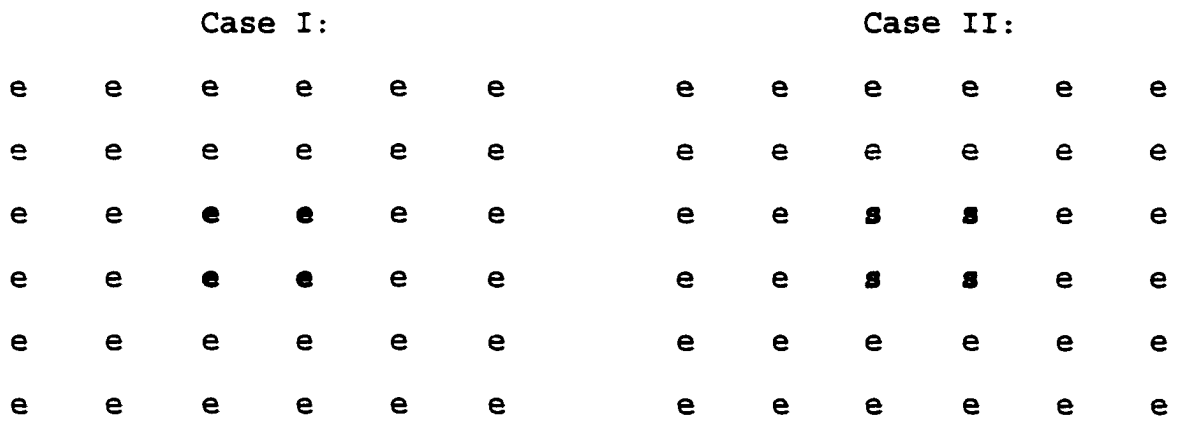
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[7] (AS), p. 55 n.

[8] (CE), pp. 70-71.

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To expand on this thought, consider the following two cases in which there are elements in the first and a closed set of mutually interacting sensations in the second:



The existence of the peculiar qualities of the ego is a contingent matter, just like the existence of particular physical qualities, and is not to be determined by the environment. This does not mean however that there could be a brain with all of the normal physical processes and no interior nature whatever. The physical elements themselves will always be the interiors of any object. That the interior nature of externally observed brain processes must be a human ego is a contingent truth, but we cannot rule out a priori the possibility that these same external processes could be annexed to different internal physical elements. It is however impossible, under this conception of the world at least, to find nothing corresponding the interior nature of physical elements an sich, and merely a set of external

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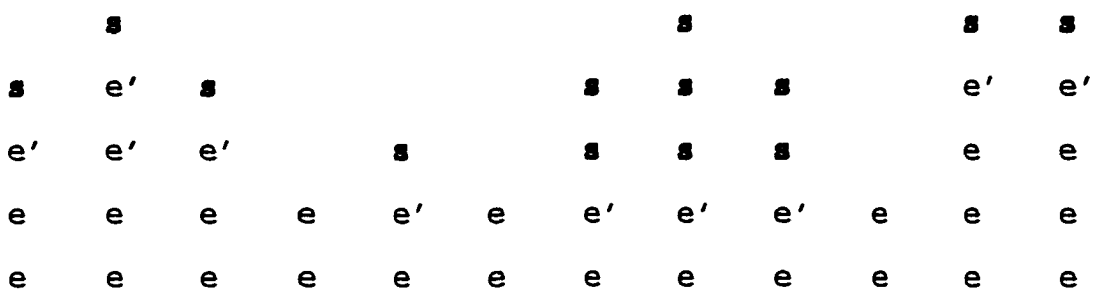
relations with no interior nature at all:-- that would be tantamount to asserting their non-existence.

Returning to Mach's notion of the ego, he also insisted on the divisibility of consciousness with the organ annexed to it. If we cut the associative fields of the brain, the sensory fields will also be severed from one another. It is therefore probable that each sense modality is like an island of elements protruding separately into consciousness, with unconscious reflexes and intermediary linking elements underneath the parts of the conscious ego and holding them together in one mass.

In *Knowledge and Error* Mach paid more heed to these intermediary mechanisms than he had in the *Analysis* attributing them to deeper physiological processes which can be understood on a purely chemical basis, like growth in plants. The sensations that get above the threshold into conscious sensory modalities are the peaks of a mountain range protruding over the clouds. They extend into awareness for pragmatic reasons of control.

The best way to understand Mach's ego is as a set of tiny "islets" of sensed elements (s), surrounded by habitual reflexes or elements that may or may not come to awareness, like the control of the breath (e'). Surrounding the reflexes are automatic, physiological processes like digestion, and beyond these actual chemical and physical elements (e):

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Mach was much criticized for thinking that all psychological phenomena could be analyzed into conscious bite-sized sensations immediately accessible to awareness. Stumpf, in particular, argued that the few sensations actually present to consciousness are only like the surface of the ocean, insufficient to explain the underlying currents and eddies. In his Notebook however--in a passage probably about the sensations of movement--Mach remarked that sensations that come to consciousness are only the "released" energy of the nerve and not the energy that physically (externally) affects other nerves, which remains unconscious:

> Sensations, ideas, only the releasing, not acting, energy of the organism. However both are of the same nature. Consciousness of the connection. It [the connection] is to be sure not physio-chemically "accessible." Consciousness a connection of energies. The size unimportant. Ridiculously small. But the consequences priceless.[9]

Mach thought sensations hang together causally with the other

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[9] Empfindungen, Vorstellungen nur die auslösende, nicht die handelnde Energie des Organismus. Aber beide von derselben Natur. Bewusstsein der Zusammenhang. Der ist allerdings nicht physiochemisch "geläufig." Bewusstsein Zusammenhang der Energien. Kommt auf die Grösse nicht an. Lächerlich klein. Aber Folgen unbezahlbar gross. Notebook dated 14 July, 1907 (NL 174/2/51).

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physical elements of the nerve, and the external physical energies released by a nerve process, which, unlike sensations themselves, are recorded by physio-chemical means and become the physical correlates of sensations:

> Physio-chemical until the central nervous system is reached a part is sensation, but not such that they are deliverable to our sense-organs, the microscope, galvanometer or reagents. Sensations release other analogous ones in the nerve and muscle. There they appear as physico-chemical.[10]

Perception

With a causal map, a reconstructed Machian theory of perception of external objects can be handled quite nicely. This I will now do, indicating points of connection with Mach's written statements. As we know, in the *Theory of Heat* Mach spoke of external objects of perception which we interact with mediately. These are made up of mind-independent world elements in matter. However in the *Analysis of Sensations* Mach spoke mostly of immanent objects of consciousness, causing no end of confusion. I am going to close this chapter by resolving this conflict.

It is true that when I observe a star I see light that left an object many years before (call this Object #1). That is why we say that the star we think we see now is no longer

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[10] Physio-chemisch bis ins central Nervensystem wenigstens ein Theil davon Empfindung aber doch nicht solche wie durch Sinnesorgane, Mikroscop, Galvanometer, Reagenz vermittelt werden. Empfindung löst andere analoge im Nerv und Muskel aus. Dort erscheinen sie als physikalisch-chemische. Notebook dated 14 July 1907 (NL 174/2/51).

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there. There is a mediate chain of action between the star existing at a great distance and my retina and a very mediate connection between my retina and the visual sensation itself. The fact that the action contains so many intermediary linkages indicates that many of the properties of the star as I see it do not belong to Object #1. All this is quite true if the star is taken to be a locally restricted set of world elements in its immediate causal neighborhood.

However, because we have a certain freedom to draw objects around different groups of elements we can consider a distended object that includes the elements in the neighborhood of the star as well as all of the effects that stream out to distant observers. We may call this distended thing Object #2, a boundary surface of effects from the star all the way to the telescope the retina, the brain and the annexed sensations of the A,B,C. This last mentioned part is the directly perceived immanent object.

Mach opened the *Analysis of Sensations* with the latter position of immanent realism (Object = Object #2). Similar to naive realists, Mach claimed that the objects perceived with the senses are the real objects and are not to be thought of as the result of a long chain of perception from an external Ding an sich to the central organ.[11]

To resolve the conflict between the object of mediate perception and immanent realism, which is really no conflict

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[11] (AS), p. 12.

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at all, we must understand why the boundaries of objects can be defined so indifferently in the Machian framework.

As we have seen to some extent in Chapter six, the Machian physics of objects always involves a set of systematic "metric" comparisons within a manifold of simpler qualities or elements. In Helmholtz's conception for example a rigid body was just a system of equations that remained the same as they moved from one set of coordinates to another.

We will explore the Machian conception of body in depth in Chapter sixteen. However for now we can give a light overview. An object is defined as a constant boundary surface or volume measured out over the set of qualitative coordinates. It will turn out that this volume has the special physical sense of a capacity measured at a certain potential level, where the level-value is indicated by the coordinates.

The metric relations can be induced by systematically comparing a standard capacity and level with an array of others that it resists and opposes. It may be helpful to picture two masses as capacities inducing qualities of pressure with magnitude and direction against one another. As yet however, we do not know if these oppositions of capacity and intensity conform to any stable pattern which we could use to pick out the enduring objects.

The capacities and their level-values will admit of measurement if we can find a standard capacity at a given

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level and then see if the comparison of this with the others forms some kind of constant system. The interactions of the capacities A,B with a standard C are direct comparisons A/C B/C, and mediate comparisons between A and B, such as A/B, B/A can be handled by the relation of both to the standard C. Thus the interactions with the standard become the direct interactions by hypothesis and the others assume values based on transitivity. Roughly speaking, an enduring object could be said to exist where there is a boundary across which the energy value measured leaving the surface is equal to the energy value measured going in.

Now for Mach's brand of realism "object-hood" in a pattern of qualities (such as our immanent sensations of Object #1) is not determined by how close they come to the pattern of the object in-itself (Object #2) for it too is made up of elements. As Mach says in "the Economical Nature of Physical Inquiry" what is important in defining the object-hood of a group of qualities is to find in the flux of elements certain groups that tend to combine together into conserved, invariant patterns. This will require that we pick some standard of interference from within, some measuring device that participates in this preserved pattern.

This is how objects are delineated around their constituent elements. Obviously depending on which standards are chosen and where we draw the boundaries there are many candidates for invariant patterns. Some of these are useful

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and some are not, but all are admissible as candidates. As we saw, Mach thought there was considerable leeway in delineating the most "economical" arrangement of objects. He did not believe that any natural fact or causal connection among elements went deeper than any other, although he did recognize the economical need to make some facts and causal connections more basic than others in a deductive arrangement.

Using an electron microscope or a particle accelerator, we like to say, gets us closer to objects in themselves, since a more precise device gives a way to interfere with natural phenomena at smaller spatial and temporal scales, and thus to employ very precise standards of comparison. But on the Machian view what really becomes clearer are better invariant patterns of elements, not a class of ultimate things. All physics and psychophysics for Mach was variation and interference among the elements, all surface, never the isolation of things in themselves behind the veil of appearance. For him appearance was not a veil, but the fabric from which the world is cut. In the next chapters we will look more closely at this comparative picture of the physical world.

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PART II: From Intensity to Extension

Chapter Twelve: Machian Physics: A Bird's Eye View

Introductory Remarks

This chapter is designed to provide a primer of the basic scientific knowledge needed to understand Mach's ideas about physics. I hope its length and textbook style will be excused by its usefulness in illuminating the physical context of Mach's work, the meaning of the scientist's terminology, and in bridging the distance to some of Mach's most breathtaking conjectures.

Mach's Philosophies of Science

Mach pursued his philosophy of science on two different fronts. On one hand his philosophy consisted of numerous sketches of the methodology and techniques of the sciences, in his *Popular Scientific Lectures*, *Theory of Heat* and especially *Knowledge and Error*. Here he treated such topics as scientific explanation, hypotheses, thought experiments, induction, analogy and economy. These writings have received a great deal of attention from philosophers of science and historians of methodology. Of course for those seeking the tried and true canons by which science achieves progress, truth and success they must prove disappointing, for as Mach emphasized there are no such methods. In fact he concluded his review of methodology in *Knowledge and Error* by saying:

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> Since there is no adequate method to guide us toward scientific discovery, successful discoveries appear in the light of artistic achievements.[1]

Mach also thought scientific explanation something of a misnomer. For him it meant a gathering of lesser known facts under one big fact, as mechanics is comprised under the inexplicable principles of the excluded perpetuum mobile and the *Beschleunigungsprincip* that masses tend to induce accelerations in one another. Likewise the theory of heat comes down to the first and second laws of thermodynamics and Fourier's (Newton's) principle of conduction, i.e. that the velocity of the flow of heat is proportional to the temperature differences.

Mach called the big facts "unintelligibilities" because we can find none more fundamental. Explanation of laws from one another or from big facts is merely a deductive subordination of certain unintelligibilities to others, which can usually be done in more than one way.[2] It is only a misleading familiarity of certain accustomed facts of nature that makes such explanations psychologically enlightening or inevitable.[3]

Moreover, Mach felt that when *an* explanation had been found for some phenomenon, it was usually only one of many paths from the fundamental unintelligibilities to the problem

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[1] (KE), p. 236.

[2] (CE), pp. 55-56.

[3] (M), p. 41.

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at hand. For the working scientist the historical progression along which phenomena are successively explained tends to become the *only* path. By contrast, the historian of science could explore other possible histories obscured by the success of just one view.

Mach also held a second philosophy of science which is sometimes called "phenomenological physics" or "energetics." As Erwin N. Hiebert writes for his *Dictionary of Scientific Biography* article, however, Mach is not to be grouped with the energetics movement of Ostwald, Rankine and Helm, from which he held aloof, mainly because their view of energy as an ultimate substance disquieted him. Ostwald, for example wanted to introduce fundamental dimensions of space, time and energy (s, t, e).

It's no accident that Mach's first mature work was called the *History and Root of the Principle of the Conservation of Energy*. He seized on the newly-discovered principle since energy provided a standard dimension of comparison for electricity, heat, mechanical work, light and so on, and could be used to unify the phenomena. Hiebert gives an instructive précis of the energy law in a little book whose title echoes Mach's.[4] He remarks that in addition to conservation laws for each domain of physics there is also a general conservation law by which energy values in different domains may be equated

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[4] *Historical Roots of the Principle of the Conservation of Energy* (Madison: University of Wisconsin State Historical Society, 1962).

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across their transformations into one another:

> If a one-to-one correspondence is maintained between any two forms of energy then there are any number of paths along which the transformation may take place. For example if a certain quantity of heat energy H is found to be equivalent to an amount of electrical energy E, being compared along the same numerical scale, and this amount of electrical energy is found also to be equivalent to an amount of mechanical energy M along the same numerical scale, then since H=E and E=M, then H=M also. Thus the same result obtains whether the transformation from H to M is achieved directly or indirectly by way of E. This rule can be summarized by saying that the energy changes for any given process are completely independent of the path or mechanism for the process; the energy changes are solely dependent upon the initial and the final states.[5]

This mechanism-independence of energy transformations is what marks out the generalized law as phenomenological. It is not necessary to know anything about the underlying structure of matter or the mechanisms by which the energy is transformed--they may be continuous, discontinuous, conveyed by contact, by radiation in empty space or through a medium. Such generality may seem wondrous, and indeed some arguments based on the principle of the excluded perpetual motion in physics, such as Stevinus' endless chain and Carnot's heat-engine do seem to have veritably magical powers. This rough magic is soon dispelled, however, since the energy law cannot predict when or how an energy transformation will take place, or along which path selected from any number of possible paths with the same initial and final states.

As a physicist, Mach was under no illusions about the

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[5] Ibid., p. 2.

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limits of energy data:

> If different levels of energies of the same kind meet together, it depends wholly on special physical circumstance whether and what transformations of energy occur...The principle of energy only determines the amounts of the transformation, not the circumstances under which the transformation takes place. It is a matter for the special domains of physics to determine the circumstances.[6]

He pointed out in particular that the peculiar irreversibility of heat transformations to other energies is not a consequence of the energy law.[7]

It was often thought at the time that the energy law captured the essence of the mechanical view of the world. Helmholtz thought that the incapacity of the law to specify the mechanisms of energy transfer showed it was less fundamental than the underlying description of all natural processes in terms of particles of matter and forces emanating from fixed centers. Meanwhile the universal applicability of the law showed him that all natural phenomena could be analyzed into underlying central forces and particles.

Mach believed contrariwise that spatial and temporal extension depended on the behavior of energies, and that the underlying transfers should not be conceived of spatially. Thus for Mach, both the energeticists and Helmholtz were wrong to develop a framework of energy *and* space and time.

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[6] (H), p. 313.

[7] (H), p. 313.

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Mach's Analysis of Energy: Capacity and Intensity

As Hiebert writes, following Mach, it is possible to analyze the energy (E) into a capacity factor (C) and an intensity factor (I), such that E=CI:

> The general equation to which every measurable potential energy manifestation can be fitted is given by the product of two factors which specify how great an agent is acting and how intensely that agent is acting.[8]

For example when a force acts through a distance the greatness of the force, or the attracting mass, is the capacity factor and the distance it acts through in the direction of the force is the intensity factor. Similarly, the amount of heat energy required to raise the temperature of a body depends on its capacity for heat (its specific heat times its mass) while the intensity factor is the temperature the body acquires.

Mach called the capacity factor "the potential" and the intensity factor the "potential function,"[9] the reason being that the potential to act, however great, is always a dependent function of the intensity factor. A massive body can attract another, but *only* as a function of the distance between them; a massive capacity for heat can absorb a quantity of heat from another body but *only* if its temperature is lower; two bodies can exchange vis viva through collision

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[8] Hiebert, p. 3.

[9] (CE), pp. 47, 85.

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but *only* if they have different velocities.[10]

Energetic Analogies

If the proper care is taken, some fascinating analogies can be made. For example two bodies m_1 and m_2 and temperatures T_1 and T_2 equalize to a mean temperature U:

$$U=\frac{m_1T_1 + m_2T_2}{m_1+m_2}$$

Likewise, two masses m_1 and m_2 at velocities v_1 and v_2 which collide and stick together go off at a velocity V:

$$V=\frac{m_1v_1 + m_2v_2}{m_1+m_2}$$[11]

The velocities (really v^2) behave like temperatures (numbered positive and negative from the mean) and masses behave like capacities for heat. Mach compared his definition of mass to a definition he proposed for the capacity for heat.[12] His notebooks are full to bursting with such analogies.

The Potential-Function U(x,y,z)

It is now time to look at potential more closely. And here I will elaborate to some degree, though as the reader is

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[10] This example must be altered because the square of the velocity is the potential function for vis viva. (CE), p. 94.

[11] (H), p. 180.

[12] (H), p. 178.

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invited to verify, nothing I say here goes beyond elementary knowledge. My intention here is to say what Mach *qua* physicist would not have deemed necessary to demonstrate.

When we speak formally of the gravitational potential of the earth we mean a function from a triple of spatial coordinates surrounding the earth to a single number representing the potential energy at that point. Specifically, the potential function $\mathbf{U}(x,y,z)=n$ takes the set of all points in space around the center of the earth (x,y,z), and maps them into concentric spherical shells of equipotential level, each shell with its own number n. Unlike the force which falls off with the square of the distance from the center, the values of the potential function decrease directly with distance. The expression for the gravitational potential for the field of a large body is M/r, where r is laid off directly to the center. In Mach's language, the mass (M) at the center is the capacity factor in determining the potential energy and the distance (r) is the intensity or potential function.

For somewhat pedantic clarity, I should distinguish the potential function $\mathbf{U}(x,y,z)=n$ from the potential value n at a point. The function $\mathbf{U}(x,y,z)$ gives as its scalar result the magnitude of some natural property at a certain point called the potential energy. The differences between values at any two points are differences in level-values or intensity. The attracting mass at the center through its capacity to

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accelerate other bodies is also sometimes called the potential by Mach. In context it is clear which is meant.

What we will call the "energy value" is the capacity factor at a given level value. Mach often represented the energy value in the general case as a quotient W/V. The energy value is a number n, but as the value of a potential energy function it expressly contains both the capacity and intensity factors the first as a function of the second W(V).

If we consider a small test-body m accelerating through the potential field of a large body M, then potential energy is generated, or destroyed, when mass m drops, or rises, between levels r and r'. Thus the mass m is the quantity of stuff transferred between levels r and r' tracing out a distance and accelerating to generate energy of motion (vis viva). Thus, the amount of potential energy finally converted to motion will depend not on position alone but also on the masses and the *difference* traversed between the initial and final positions.

There are some mathematical-philosophical oddities to the potential, which I mention in order to whet the reader's appetite for the Machian critique to come. First, although the set of points around a mass is three dimensional, the potential energy function is scalar-valued. It takes a triple to a single number, a sphere into a straight line paradoxically transforming a more complex concept to a simpler one. One would imagine that in the most natural treatment the

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simpler concept should come before the complex.

Second, at the center of force where the capacity sits, the potential energy function (m/r) has no well-defined value. Since the value gets larger and larger toward the center, it should be infinite there, which is absurd since a finite capacity (m) should not have an infinite potential energy. The potential function (r) grows infinitely small as the center is approached but no finite value is reached. In fact, Mach wrote in the "Vorträge" that he regarded the idea of the center of force as a reductio ad absurdum of atoms.

> Physicists have already felt the difficulty of imagining atoms concretely, and some therefore imagine atoms as mere centers of force. But a center of force as such is actually nothing. And what does it mean, pray, when it is said that one center of force acts on another?[13]

At great distances, the potential of masses must drop to zero, or otherwise cancel out somehow, since if they did not, the masses of the universe would attract and collapse on each other.

Forces and Potential

The forces that act in a conservative system are written as derivatives (instantaneous rates of change) of a potential energy function. If a body falls from a higher to a lower level, a force will accelerate it through the change of potential level accomplishing work. Therefore in general the

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[13] (V), p. 364.

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force varies continuously from point to point and the distance through which it works is a curve. The mathematical expression that permits of such generality is called a line integral:

$$(d)W = \int_a^b \mathbf{F}.d\mathbf{r}$$

Here the force $\mathbf{F}(\mathbf{r})$ is a dependent function of $\mathbf{r}$, and $\mathbf{r}$ is a vector function of the coordinates at a certain time: $\mathbf{r}(x(t),y(t),z(t))$. The work element is a "dot-product" of the two vectors $\mathbf{F}$ and $\mathbf{r}$, which will be a scalar. The dot product makes two vectors into a scalar by multiplying their components in x, y, z separately and adding them all up together. Thus only the distances covered in the direction of the force will add toward the work.[14]

When the forces are central forces, whose intensity depends only on the distance from a center, the three forces in the different directions may be written as the gradient (three-dimensional derivative) of the potential function $U(x,y,z)$. Leaving the multivariate case out for the moment, if we take just one of those forces for consideration, the expression for the work along x becomes:

$$\int_a^b U'(\mathbf{r}_x(t)) \; . \; \mathbf{r}_x'(t).$$

And something similar for y and z.

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[14] In general when the force and distance vary so will the work. We can however take a short stretch of time in which the work done may be practically constant, which will give us the increment dW.

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If we look closely at the expression under the integral sign we recognize the "chain-rule," i.e., d/dt: u(r(t)) = u'(r(t)). r'(t). In this case, the integrand can be written as the time-derivative of some function $\mathbf{U}(\mathbf{r}_x(t))$:

$$d/dt\ \mathbf{U}(\mathbf{r}_x(t)) = \mathbf{U}'(\mathbf{r}_x(t))\ .\ \mathbf{r}_x'(t).$$

And we can thus replace the right-hand expression with the left-hand expression under the integral sign. Now the d/dt sign and integration with respect to t are inverses and cancel out by the fundamental theorem, leaving:

$$\int_a^b \mathbf{U}'(\mathbf{r}_x(t)).\mathbf{r}_x'(t) = U(b) - U(a),$$[15]

or in three dimensions:

$$\mathbf{U}(x_1, y_1, z_1) - \mathbf{U}(x_0, y_0, z_0).$$[16]

This last expression $\mathbf{U}(x,y,z)$ is the analogue of the energy equation above E=CI. It gives the potential-value as a function of spatial position and it contains the information needed to calculate energy-values implicitly from it, assuming that it satisfies certain constraints.

The Importance of "Path Independence"

As we can verify by inspection, the potential energy converted to motion by falling depends only on the initial and final positions from a center-of-force and is completely

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[15] According to the fundamental theorem, we do not need to take account of the shape of $\mathbf{r}$, but only its values at (b) and (a).

[16] (M) pp. 439, 492, 574.

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independent of the path of $\mathbf{r}$ taken by a particle. Make no mistake, the path-independence of the energy is a direct consequence of the fundamental theorem, provided the function $\mathbf{U}$ has a gradient, or three partial derivatives that satisfy certain mathematical constraints.

And what are those constraints? It turns out that the existence of a gradient for $\mathbf{U}(x,y,z)$ depends solely on the geometric fact that the function is continuous in all three directions simultaneously, and in all mixed directions a particle might take. This is not assured a priori since $\mathbf{U}(x,y,z)$ might be continuous in each direction separately (in all paths through the (x,y), (y,z) and (z,y) planes) and still fail to be continuous in oblique cross sections through all three. Counterexamples can be found in most calculus texts. Mathematically the fact of continuity across directions is expressed by the "equality of mixed partials":

$$\partial^2\mathbf{U}/\partial x\partial y = \partial^2\mathbf{U}/\partial y\partial x;\ \partial^2\mathbf{U}/\partial x\partial z = \partial^2\mathbf{U}/\partial z\partial x;\ \partial^2\mathbf{U}/\partial y\partial z = \partial^2\mathbf{U}/\partial z\partial y.$$ [17]

This means, for example in the case of the first term, that the force along the x direction $\partial\mathbf{U}/\partial x$ changes with respect to y exactly as much as the force along the y direction $\partial\mathbf{U}/\partial y$ changes with respect to x. Another (perhaps deeper) way to look at it is that changes of step a particle takes with time

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[17] Arnold Sommerfeld. *Mechanics*, Martin O. Stern trans., (New York: Academic Press, 1964) p. 46.

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through the manifold in different directions are comparable across all mixed directional paths into the infinitely small.

Assuming the path independence of the potential energy function yields up the conservation of energy as an immediate consequence. If a point passes out of an initial position P_0 into a new position P_1 and then back to P_0 by any path, the potential function resets to the same initial value and the energy is conserved. The mathematical assumptions ensure that the mixture of directions in the path will not matter, nor will it matter whether the path is a straight line or a curve. Think of a stone rolling down a hill and rising to the same distance above the ground on the next hill, or a pendulum swinging forward and back to its initial position.

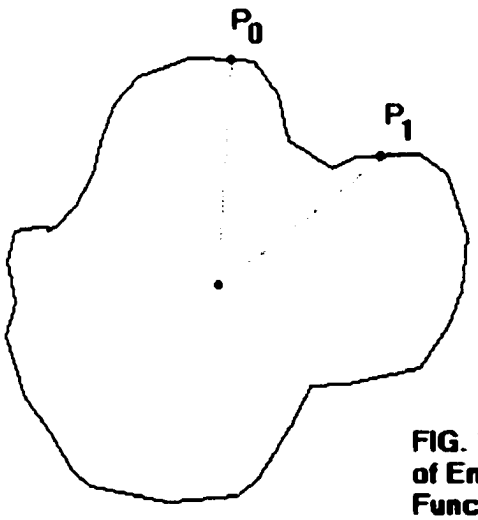


FIG. 12.1 Conservation of Energy for a Potential Function

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As Mach and Helmholtz showed, the assumption that a particle's energy is a function of its position rests on the principle of the excluded perpetuum mobile. For example if the stone were able to rise to a higher position using just the energy of falling and the shape of the path it could return to its original position with extra vis viva to spare and keep raising itself higher still by repeating the process. But that would generate energy from nothing. All of the real force of the above argument is thus contained in the excluded perpetuum mobile principle.

It is interesting that very simple geometric facts about continuity and the homogeneity of directions play a role in defining the potential energy function. We can add this to our list of odd properties of potential functions when they are conceived spatially. There is a peculiar vanishing of properties dependent on the path of a particle, and an increasing emergence of simpler properties of position. In 1867 Mach devoted some time to mathematical research on potential functions with Hermann Hankel,[18] and it seems to me that he probably had the well-known mathematical facts firmly in mind from then on.

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[18] In a letter from Hankel to Mach August 28, 1867, he mentions the job of finding a solution to the problem of finding potential functions, (WK), pp. 37-38.

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Relation of Potential to Distance and Spatial Extension

Now that the textbook treatment is over, we can present some specifically Machian ideas. I have used above the customary spatial description of potential functions and forces. However we must now try to see beyond the notion of points and properties at points to get at some of Mach's thinking about physics.

Mach's view, as we have seen in outline in previous chapters, was more or less that space is traced out by the same undulating processes that transfer energy. A quantity of stuff m moves between potential levels r and r' and traces out a swath of volume, as it converts potential energy to motion. As a consequence of this view, Mach could not accept that real physical properties like potential-values should be written as functions of spatial position (x,y,z). What would it mean to attribute efficacious physical properties to a bare point of space?[19] Wouldn't it make much more intuitive sense to derive spatial points in reverse from their physical properties?

In fact, and as we shall see further, Mach toyed with the idea of identifying the distance function of geometry with the potential energy function. The immediate problem with this mathematically-motivated analogy is: if the potential energy at points are somehow determinative of the spatial properties, why wouldn't all points with equal potential energy (such as

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[19] See his letter to Petzoldt (AU), p. 39: "A bare efficacious position has been observed by no one."

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all the points at a certain distance from the center of the earth) be in the same *place*? Equal values lie in a spherical shell, such that to every point of potential level, there corresponds a *sphere* of spatial points. This would tend to confirm that **U** should be defined over spaces, not spaces from **U**'s.

But the thing is not desperate. In his later writings about a "chemical manifold" Mach sought a way to resolve these difficulties:--if each value of **U** requires several spatial characteristics, perhaps that is because each point of space is really made up from a triple combination of several *different* potential energy functions U(x), V(y), W(z), each representing a different natural potential in a different direction.[20] This potential energy function instead of U(x,y,z) could be expressed in terms of the capacity value and the level value of each energy, U = W/V, abstracted from spatial properties. Thus although each U is a scalar-function, a collection of them arranged in a triple manifold would not be. The thought will be worked out in more detail in Chapters fourteen and sixteen.

Capacity, Intensity, Menge: The Vocabulary of Energetics

I intend to make use of Mach's vocabulary in what follows, so I will point out in advance that his terminology can be rather vexing, as there are not always well-established

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[20] (AS), p. 339.

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scientific names for the quantities he thought fundamental. The different terms of the general energy equation E = CI break down into the following. The "energy value" at a point is: **U** (x(t), y(t), z(t)), or else W/V, or W(V). The U's may be temperatures at points, or heights laid-off from a center of force. Mach sometimes calls these "physical characteristics" at points, other times he gave them values based only on the differences in potential levels between those points and neighboring points. Mach represented the capacity and intensity factors together in the more perspicuous quotient W/V:

> In the case of each type of energy we have to consider the energy value and the level value. The quantity values are given by the quotients W/V.[21]

It doesn't matter whether we write the potential W as a function of V as a quotient, or as W(V). What Mach calls *Menge* has the dimensions W/V. For example W/V translates into M/r for mass and distance, and into Q/T or Q(T) for quantity of heat (Q) and temperature (T). The Menge consists of the energy value Q or F as a function of the level value T or r. Mach called the potential energy function **U** the force-function, the potential or the potential value.[22]

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[21] (H), pp. 309-310.

[22] (M), pp. 439-440.

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Mach's Phenomenological Law of Energies

Even though the conservation of energy does not say anything about the energy transfer between initial and final states, Mach proposed a principle that holds for energies that we do know can equalize. He laid down a "law of differences" which runs:--the equalization of energy value (change of a potential level in time) is proportional to the excess of potential level above the average of neighboring potential levels (in space).[23] Mach called this a general "phenomenological" law of physics that holds over different domains and even has an analogue in nerve physiology.[24]

There are two special examples Mach used to illustrate the law. The first is found in his *Theory of Heat* during his discussion of Fourier.[25] Fourier started from the principle of conduction that the velocity of heat flowing through a surface is (all else being equal) proportional to the difference in temperatures going in and coming out. For

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[23] Space and time play a role in the statement of the law in this chapter, and even in Mach's own writings. However it is possible to imagine that "neighboring" points should be replaced by causally proximate points and that the underlying capacity and intensity factors be substituted for spatial and temporal changes, respectively. When I refer to the law of differences in general, it is this reformulation I mean.

[24] "Über die physiologische Wirkung räumlich vertheilter Lichtreize...Vierte Abhandlung" *SW* **57** (1868) pp. 11-19; (MB), pp. 302-305; "Über Herrn A. Guébhard's Darstellung der Äequipotentialcurven" *SW* **86** (1882):8-14; (M), pp. 288-289; (H), p. 115; (KE), p. 357.

[25] (H), pp. 83-85.

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example, we may consider an infinite solid surface lying between two parallel plates I (the heat source) and II (the cold sink). If the temperatures are held constant at both ends, then the temperatures of the individual points between will arrange themselves according to the law of a straight line. The temperature of any point of the material is always an average of its neighbors to the right and the left, and the heat will flow at a uniform velocity through the material; as much heat flows to any point as flows away from it in unit time.

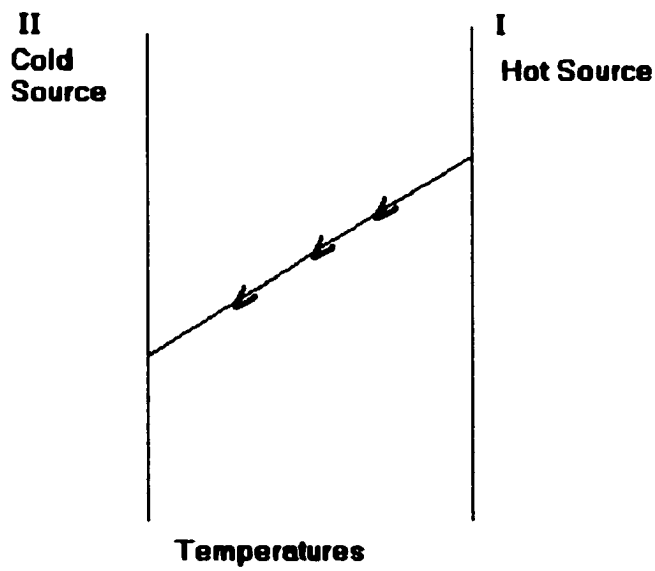


FIG. 12.2 Uniform Heat Flow

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By analogy when a mass changes its distances to large remote masses proportional to the time, the positions of the moving mass change at a constant velocity, and each position marked off to the distant masses is an average of neighboring positions (potential levels like temperatures).[26] This is both the natural equilibrium state of heat flow and of uniform motion in space.

If, on the other hand, the temperature at the source should suffer a rise the temperatures will pass into a curve, where the temperature at a midpoint A is suddenly below the average of its neighbors.

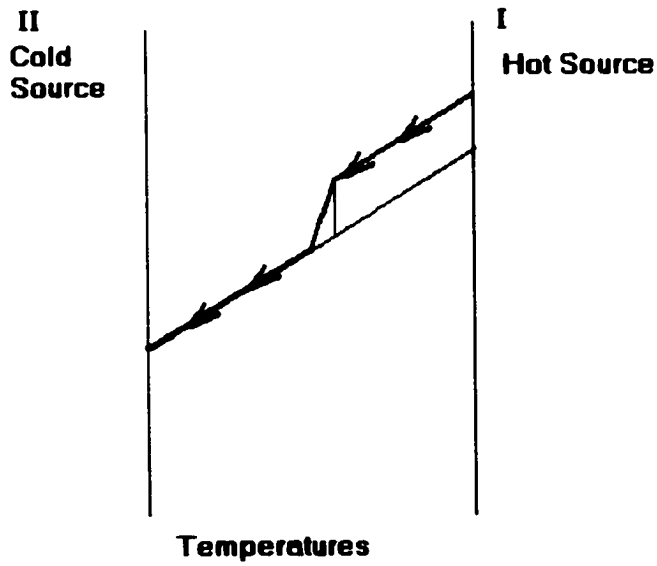


FIG. 12.3 Heat Flow with Equalization

[26] (H), p. 115; (M), p. 289.

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In this case more heat will flow to the point A than flows from it, and the temperature at A will rise with a velocity proportional to A's deviation from the mean of neighboring temperatures.[27] After this equalization, the temperatures will pass into a straight line again.

A second example of equilibration is in the relation of masses. Masses in no special force relation to one another change their distances proportional to the times. If a force relation holds, then the differences of distance between masses causes them to accelerate and close this difference at a rate proportional to the square of the time. This is unlike heat, where the equalization proceeds at a constant velocity in time that is proportional to the temperature difference.

In each case however the one condition calling for equalization is the difference of energy level (temperature, distance) above the average of neighboring energy levels, which Mach expresses by saying that if u represents some property of temperature, of potential function in short of energy level at a point, the conditions calling for equalization are:

$$\partial^2u/\partial x^2,\ \partial^2u/\partial y^2,\ \partial^2u/\partial z^2$$ [28]

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[27] (H), p. 86.

[28] (M), p. 288; (H), pp. 114-115.

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which represent the deviation of the energy level u from the mean energy level of neighboring points in the three directions. Mach wrote:

> The equation:
>
> $$\partial u/\partial t = k/c\rho \ (\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2)$$
>
> thus only represents that u tends to the mean temperature of the surroundings with a velocity that is proportional to the deviation from this mean. For the stationary state:
>
> $$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$$
>
> that is to say, this state occurs if the above deviation from the mean is zero or if every point has attained the mean temperature of the surroundings. The stationary (dynamical) state passes into a complete (statical) state of equilibrium if the flow of heat vanishes, so that:
>
> $$\partial u/\partial x = \partial u/\partial y = \partial u/\partial z = 0, \text{ or } u \text{ is constant.}$$[29]

The stationary state of dynamic equilibrium given by Laplace's equation ($\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$) describes the mean distribution of temperatures and the constant flow of heat, while the same equation also describes a body moving at a constant velocity in three directions, the partial derivatives are in this case spatial accelerations and not temperatures. Between these two cases only the rates of equalization of differences are different. Masses equalize by accelerations toward one another; temperatures equalize with velocities.

With these important disanalogies firmly fixed in thought--and there are far more than I have indicated--Mach still believed that differences and equalizations of physical

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[29] (H), p. 115.

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characteristics at points (given by $U(x,y,z)$) did point to a universal phenomenological principle for all of the energies in a physical continuum of points and properties at points:

> If we conceive u as the characteristic of a physical state of a material point (such as temperature, potential, concentration of a solution, velocity-potential, etc.) then every change of state, the continuance of a stationary process, equilibrium, is determined by the differences of values of u at the point (x,y,z) and the neighboring points. In a physical continuum, the behavior of every point is determined by the deviation of the value of its physical characteristic from a certain mean value of the characteristic of the neighboring points.[30]

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[30] (H), p. 115.

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Chapter Thirteen: Mach's Definition of Mass

In the scientific world, Mach's name has become synonymous with his definition of mass and his revision of Newton's bucket experiment. It is not widely known exactly how these two themes, along with his reflections on the excluded perpetual motion principle, formed part of Mach's elimination of spatial imagery from physics. The following chapters will trace out that motif.

Newtonian versus Machian Mass

Mass is a physical property we know of first and foremost through weight. Heavy objects attracted by the earth accelerate toward the center and press straight down on anything in their path. We also know of mass another way; if we attempt to accelerate a heavy object it offers resistance, on earth or even in empty space with no other bodies in the immediate neighborhood. Mass and weight are proportional because objects twice as heavy in weight also offer twice as great a resistance to acceleration. But the same mass may have many different weights on other planets, because of their different gravitational pulls.

Newton was the first to separate the two distinct properties and to determine by experiment that a body's mass was independent of its chemical constitution.[1] He defined

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[1] (M) pp. 235-238.

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mass as the "quantity of matter" contained in a body.

Now Mach's 1868 definition of mass was not a sharpening or refinement of the Newtonian quantity of matter but a completely new concept: *mass ratios* (m_α/m_β), and therein lies a tale. Individual mass-values did not exist for Mach; even the individual terms of the ratio, like m_α, are themselves ratios. Mass-values were the result of a system of comparisons between one mass-ratio with another and with a standard ratio.[2] To say that mass doesn't vary in a certain event means not that a certain lump of matter is the same as it was before, but that a systematic relationship between these ratios is upheld. Because mass-ratios are relations, there are properties of Newtonian quantity of matter that they lack: for example there is no mass-ratio for the entire universe. In addition because of the close relationship between Machian mass ratios and kinetic energy--manifested for example if the comparisons between masses are made by a system of momentum-conserving collisions--they can be included within the theory of relativity with only a few modifications.[3]

Mach's definition appealed to an experimental procedure. We let two bodies (K_α, K_β) act on one another by whatever forces

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[2] Arnold Koslow argues against Max Jammer that the non-existence of unique individual mass-values does not imply the same for mass-ratios "Mach's Concept of Mass: Program and Definition" *Synthése* **18** (1968) p. 226.

[3] See Wolfgang Yourgrau and Alwyn van der Merwe, "Did Ernst Mach 'Miss the Target'?" *Synthese* **18**(1968), p. 247-248.

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they mutually exert, electrical, magnetic and gravitational attractions, and forces of contact, such as the vis viva communicated in collisions. The forces produced in the one body (K_α) cause accelerations in the other (K_β), which must be accepted as a fact of experience, like the existence of forces themselves.

If we assign to the first body the mass-value m_α and the other a value m_β, then m_α acquires an acceleration $\phi_{\alpha\beta}$ (which we read as 'the acceleration of K_α due to K_β'). Meanwhile K_β acquires an acceleration $\phi_{\beta\alpha}$ (which we read as 'the acceleration of K_β due to K_α'). Because the forces both bodies exert on one another are equal by action and reaction, we write:

1. $$F_{\alpha\beta} = -F_{\beta\alpha}$$

2. $$m_\alpha \phi_{\alpha\beta} = - m_\beta \phi_{\beta\alpha}, \text{ or}$$

3. $$m_\alpha/m_\beta = - \phi_{\beta\alpha}/\phi_{\alpha\beta}.$$

The Definition of Mass as Elimination of Space

The original paper "Über die Definition der Masse" appeared in *Carls Reperatorium für physikalische Technik* along with two other notes in 1868. It had been submitted one year earlier and rejected by Poggendorf, the editor of the *Annalen der Physik*. Mach claimed that this early rejection prevented him from publishing his investigations of the law of inertia,[4]

[4] (CE), p. 80.

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which suggests that these ideas were connected.

Mach's 1867 appointment to a chair of experimental physics at Prague marked his return to physics. He had already been an Associate Professor of physics in his last year at Graz in 1866; yet up to 1868, apart from his Privatdozent years, he had published most of his work in sense physiology. However Mach's writings of the 1860s contained numerous intimations of his physical views and new theory of matter. He claimed to have developed his physical ideas much earlier in his lectures and private historical studies, but had avoided presenting them to the stiff-necked community of physicists. He said in 1872, reflecting back:

> ...these thoughts, which, as the notes and quotations from my earlier writings show, are not of very recent date, but which I have held since 1862, were not suited for discussion with my colleagues--I, at least, soon tired of such discussions. With the exception of some short notices written on the occasion of other works and in journals little read by physicists, but which may suffice to prove my independence, I have published nothing about these thoughts.[5]

As we have seen, one of these early articles "Bemerkungen über die Entwicklung der Raumvorstellungen" of 1866 was devoted to the elimination of space from physics and contained Mach's *naturphilosophische* construction of extended matter from attractive and repulsive pressures. It included a proposal to eliminate spatial distance from the law of force between attracting bodies. Instead of forces as functions of the

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[5] (CE), p. 16.

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distance between two bodies, he suggested, why not express that distance as a function of the forces?:

> Physics has long busied itself expressing the mutual action, mutual attraction (reciprocal acceleration, reciprocal pressures) of two material parts as a function of their distance (therefore of their spatial relation). The forces are functions of the distance. Now the spatial relations of material parts can, however, be known from the forces they exert on one another.[6]

We ordinarily write forces as a function of the distance, as in Newton's law of universal gravitation. Mach was proposing to tell the distances from the falling-off of pressures. This suggestion reveals that Mach thought spatial distance was not necessarily the independent variable, but that all we know is the reciprocal dependence of spatial distance and the strength of a force. Either one could be the independent variable, apparently. This of course gives no proof either way.

Mach's desire to eliminate space from basic laws of physics is especially interesting in light of his mass definition because it is often erroneously assumed in the literature that Mach intended his definition to reduce "unobservable" pressures or forces to grossly observable spatial bodies and accelerated motions.[7] That is an understandable, straightforward reading since it looks (in 3.) as if the mass-ratios (m_α/m_β) are defined in terms of the ratio

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[6] (R) p. 231.

[7] Russell *Analysis of Matter* (New York: Dover 1954) p. 19: "Kirchhoff and Mach developed a mechanics which dispensed with force..."

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of their accelerations ($-\phi_{\beta\alpha}/\phi_{\alpha\beta}$).

But Mach believed forces were the readily demonstrable data of experience; it was rather extended body and space that were metaphysical, physiological conceptions. He said in his criticism of Heinrich Hertz's *Mechanik* (which really *did* try to dispense with forces):

> To characterize forces as being frequently "empty running wheels" as being frequently not demonstrable to the senses, can scarcely be permissible. In any event, "forces" are decidedly in the advantage on this score, as compared with "hidden masses" and "hidden motions." In the case of a piece of iron lying at rest on a table, both the forces in equilibrium, the weight of the iron and the elasticity of the table are very easily demonstrable.[8]

In nature we can immediately demonstrate pushes and pulls by finding other standard pushes and pulls to oppose them. We can still feel the pressure in the tension each offers the other without producing motion. For example without motion, we can slip a platform beneath the iron to replace the pressure offered by the table, and we can switch the iron for an equivalent test weight. These motionless tensions are experimentally demonstrable facts.

Newton also recommended impressing test forces to determine and alter the tension in the cord connecting two rotating bodies, when the relative motions of two bodies did

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[8] (M) p. 319.

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not reveal their "true motions."[9] Mach stated in his notebook, that pressure should be measured with pressure [*Druck wie d. Druck messen*].[10] The experimental proposition that masses induce pressures in one another would be verified by impressing test pressures. Since *Druck* is experimentally fundamental, we need not even know yet that it can be further analyzed into the product of mass and acceleration.[11] We do need action and reaction, however, and Mach assumed Newton's third law ($F_{\alpha\beta} = -F_{\beta\alpha}$) as an unanalyzed fact of experience.[12]

While the mass-definition itself is not an "elimination" of spatial intuition, we must bear in mind that it formed a part of a larger program. In his "Raumvorstellungen" article, written just two years before the paper on mass, Mach had claimed that extended bodies of matter could be reduced to internal states of pressure. For instance, the presence of matter could reduce to pressures due to action by contact or collision; while attractions at a distance could be pressures due to fundamental forces, from which the spatial distances between the pieces of matter could be derived. Thus all of the relations between matter across spaces can be cashed out

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9 *Principia*, Motte and Cajori (trans.) (Berkeley: University of California Press, 1934) Scholium to Definitions, p. 12.

10 Notebook dated 4 May 1880 (NL 174/2/16).

11 Newton's second law or "definition" of force as F=ma.

12 "Über die Definition der Masse" p. 357

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in terms of pressures and nothing but pressures, which was probably the implicit goal of both articles together: the mass article to set up the philosophical programme of the "Raumvorstellungen" article.

Spatial Accelerations versus Intensive Pressures

In the "Definition of Mass" article we indeed find that pressures play the primary role. Newton's second law defined force as a product of mass and acceleration, but pressure was not a defined term for Mach. It was the empirical datum from which both the masses (m) and the spatial accelerations (a) were to be derived. There can be no logical objection to taking a seemingly more complex concept (F) as primitive, since, within bounds, anyone may choose his own defined and undefined terms for his own system of mechanics.

Mach began his article by attacking the logical circle in the concepts of mass, acceleration and *pressure*:

> m=p/g, p=mg This is either a very repugnant circle or we are required to conceive of force as "pressure."[13]

The reason there is a vicious circle, as he later observed in his *Mechanics*, was that mass and pressure are always conjoined: wherever we have masses they attract one another and induce mutually opposing pressures, and all pressures in

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[13] (UDM) p. 356.

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mechanics proper are due to the presence of masses.[14] Nowhere do we find a pressure or a mass alone. Mach thus proposed to put pressure at the head [*an der Spitze*] of mechanics along with action and reaction:

> In that principle of Newton, which is customarily placed at the head of mechanics and which runs "Actioni contrarium semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi," the "actio" is again a pressure or the principle is quite unintelligible unless we already possess the conceptions of force and mass. But pressure looks very strange at the head of the quite phoronomical mechanics of today. However this can be avoided.[15]

He "avoided the problem" by making the fundamental law of mechanics a *dynamical* principle: two qualitatively indistinguishable bodies always induce in one another equal and opposite accelerations (*Beschleunigungen*):

> *Proposition of Experience*: Bodies placed opposite one another communicate to each other accelerations in opposite senses in the direction of their line of junction (The law of inertia is already thereby included).[16]

This proposition is tantamount to assuming Newton's law of universal gravitation, because it says that all masses

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[14] (M), p. 239. The tension of a string, or normal force of a rigid body or fluid either depend indirectly on masses, or can be replaced in thought with them.

[15] (UDM), p. 357.

[16] (UDM) p. 359. The law of inertia is included, Mach thought, as the logically equivalent contrapositive of the experimental proposition: where bodies are in no position to exert forces on one another there are no accelerations, hence either rest or motion at a constant velocity, see M p. 172.

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attract and accelerate towards one another. In fact, in his historical treatment, Mach claimed that Newton had arrived at his definition of "quantity of matter" by conjoining masses and their concomitant pressures. According to Mach, Newton should have been surprised by his discovery that chemically different bodies possessed the same homogeneous properties of mass and gravitational attraction, as if their differences in composition didn't count at the deepest level.[17] If bodies were different all the way down they shouldn't attract one another equally, particle-by-particle; rather the attraction of the one particle should always exceed or fall short of the other particle if they are truly different in all respects. Only if all bodies were made of the same tiny homogeneous corpuscles should we expect them to exert similar forces on one another. Newton was thus led to regard every unit of attractive force produced by matter as due to the attractions of each homogeneous corpuscle of matter for one another. Hence the total attraction of the body is due to the sum total of all its corpuscles' attraction, i.e., the "quantity of matter." According to Mach, then, Newton thought of the quantity of matter as a pure number of homogeneous particles in the body. I assume that he was appealing to Newton's known preference for corpuscularian theories.[18]

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[17] (M) p. 239.

[18] Arnold Koslow presents a different case for Newtonian "quantity of matter" based on a controversy between Newton and the editor of the Second Edition of the

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Mach mentions not masses but *bodies* (K_α, K_β) and accelerations. The mental picture it brings up is of extended lumps of matter floating in space at distances which change according to the accelerations. It would seem, at least, that the existence of bodies and accelerations in space were the primary facts. But he said in a notebook entry that he intended to measure pressures with pressures as part of his mass definition:

> There exists something determinative of pressure independent of weight. But it increases and decreases with weight. Proportionality p=mg p'=m'g' only by equivalent bodies, or one must already posses the concept of mass, something determinative of pressure independent of material constitution. Measure pressure with pressure. What is pressure. Set it free of subjectivity. Through weight. Mass is: what communicates and receives the m-fold acceleration to an assumed unit body.[19]

Mach hid the distinction between acceleration [*Beschleunigung*]

Principia, R. Cotes. According to Koslow, Newton, urged on by Cotes, eventually settled on inertia as the measure of mass, and admitted at least the possibility of different kinds of matter. (See "Quantitative Nonnumerical Relations in Science: Eudoxus, Newton and Maxwell" in *Philosophical and Foundational Issues in Measurement Theory*, C.W. Savage and P. Ehrlich eds. (Hillsdale, NJ: Lawrence Erlbaum Publishers, 1992), pp. 157-163.)

[19] Es existiert ein Druck bestimmendes bei gegebener Beschleunigung unabhängig vom Gewicht. Wächst und nimmt aber mit d. Gewicht ab. Proportionalität p=mg p'=m'g' nur bei gleichen Körpern, oder man muss den Massbegriff schon haben, ein Druckbestimmendes unabhängig von d. materiellen Beschaffenheit. Druck wie d. Druck messen. Was ist Druck. Vom subjectiven frei machen. Durch Gewichte. Masse ist: welcher dem als Einheit angenommene Körper die m fache Beschleunigung ertheilt, welche er selbst erhält. Notebook dated 4 May 1880 (NL 174/2/16).

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and pressure [*Druck*] in his article, switching terms midstream. These terms are *not* synonymous: two equal spatial accelerations may represent two very different pressures depending on the masses involved. Mach wanted to measure pressures directly (the combination of mass and acceleration), with no independent means to measure acceleration unconjoined with mass. Nor is mass measurable except through pressure due to mass, which is why Mach complained of the logical circle in mass, acceleration and pressure. As he said, his proposal was to measure pressure with pressure [*Druck wie d. Druck messen*] which is much more in line with the philosophical programme of the 1866 "Raumvorstellungen" paper two years before.

A Gedankenexperiment

Mach was finally able to pose the question to himself directly "What is primary: pressure or spatial acceleration?" in a thought experiment. When two bodies face one another across a distance, or when a body is released above the ground, what happens in the very first instant before the body moves? This is assuming of course that there is an initial change involved from a state of rest to a state of motion. If force needs a distance to act over, the force will be zero until some space is covered, and naturally the acceleration is zero if the space covered is zero. But if both remain zero, bodies could never begin to move at all.

The problem had occurred to Galileo, who proposed a

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politic solution to it.[20] He said: start backwards with any finite velocity acquired by a falling body acted on by a constant force. This body has passed through every velocity between rest and the final velocity, infinitely many velocities. But in half the time the body always has half its final velocity, in a quarter of the time a quarter of that final velocity. So for any small time (not zero) we can always find *some* velocity for the body however small. Of course this is no answer as to whether forces can act *without* producing changes of velocity. Galileo simply conjoined force with change of velocity into the infinitely small so that there is no first moment, a procedure that becomes meaningless at zero where a body has not begun to move and the force has not yet begun to act.

It is true that forces doing work must cover distances, because they involve a fall from a higher to a lower potential level. But we know that mutually destroying forces can also act even when they leave a body fixed in position. In fact according to Mach's "Raumvorstellungen" article matter is fixed in position by internally combined pressures. This is reasonable, for indeed nothing holds its position simply by existing; positions must be physically fixed. Kant, too, thought of position as the result of pre-spatial pressures; he said in his *Metaphysical Foundations of Natural Science* that

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[20] *Dialogues Concerning the Two New Sciences* Third Day "Naturally Accelerated Motion (Chicago: Encyclopedia Britannica 1952) pp. 201-202.

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body holds a position in space by *forces* that keep it fixed there, not by its mere existence.[21]

The "*Beschleunigungsprincip*"

Mach said that we require a logically separate law of force prior to motion, in addition to the *Arbeitsprincip*, which describes the action of forces over spaces. He wrote in a letter to Popper:

> I believe that the concept of force and the concept of work have their uses. Why limit oneself to just one...When masses without velocity face one another the relation of work and vis viva cannot be of use. It does not tell us what will happen. It says only that when the pressure is so great, so is the change of velocity as great. The force however tells us what happens. Marcus Marci and Mariotte did not know what to say about the initial velocity of a freely falling body. They didn't know how a body should begin to move, since it has no velocity as yet, and finally assumed a finite velocity, which spoiled their investigations. This is a point that makes the force concept palpable.[22]

Mach wrote about this problem in various places in his notebooks of 1881 and 82. He remarked that it always takes work to make a body move, but for the case where two masses float opposite one another he asked why there is an *initial* change of velocity (or vis viva) if no work is done? He decided that an additional *Beschleunigungsprincip* must give the condition for a change from static position to spatial accelerations and vis viva, writing in 1890:

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[21] *Metaphysical Foundations of Natural Science*, Translated by James W. Ellington (Hackett: 1985), p 77. ff.

[22] Mach to Popper date uncertain 1883-84? in Blackmore and Hentschel #4.

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> The energy principle would allow that a body could hang in the air. The principle of acceleration gives the transformation its direction. It says more. Through a push, brings it out of position.[23]

For Mach the *Energieprincip* or *Arbeitsprincip* equated the "change" in potential energy (position) to the change in kinetic energy (vis viva):

$$\Sigma \int_a^b (Xdx + Ydy + Zdz) = 1/2\ \Sigma\ mv_1^2 - 1/2\ \Sigma\ mv_0^2$$

This equation relates the *Spannkraft* (literally the "tensing force") on the left due to static position to motional changes in the vis viva on the right. Now Helmholtz had claimed that we may regard the *Spannkräfte* as forces that "strive to move" a point out of its present position so long as they have not *yet* produced any motion.[24] As soon as the forces produce motion, the resident energy is no longer in the form of *Spannkraft* but has been converted over to vis viva. Since the one is destroyed with the appearance of the other no point can have its self-same *Spannkraft* and vis viva at once. Thus the

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[23] Das Energieprincip würde gestatten, dass ein Körper in der Luft hängen bliebe. Das Beschleunigungsprincip gibt der Umwandlung die Richtung. Es sagt mehr. Durch einen Stoss aus der Lage bringen. Notebook dated 6 December 1890 (NL 174/2/32).

[24] Hermann von Helmholtz *Über die Erhaltung der Kraft*, *Ostwalds Klassiker der exakten Wissenschaften* No. 1 (Leipzig: Wilhelm Engelmann, 1888) p. 12. The equation contains the same ambiguity: infinitesimal elements dx dy dz are considered infinitely *small* motions. They are both produced by forces that do work over infinitely small spaces and are regarded by Helmholtz as tendencies (spaces that forces *would* act over, but don't).

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problem of which side occasions the energy-transformation, which the energy equation cannot clarify.

Mach was interpreting motion as a special case of energy transfer from the left side of the equation to the form on the right; i.e., from the potential form based on static position to the kinetic form of vis viva. When he said that the force-principle gave the change its direction he meant that we must read the energy-principle from left to right, at least in cases where bodies are accelerated from rest.[25] He said it must be pushes and pulls that "bring the body out of its initial position," though Galileo was also correct that there is *no* moment of the process, not even the tiniest first element, when a freely acting force doing work does not produce a velocity.

The case is different for confined forces of the iron and the table that still act without producing motion, and are thus prior to spatial accelerations. But these forces only serve to fix bodies in position like an object pulled on by three cords in opposing directions.

Pieces of matter also exert attractive pressures on one another, and these do not depend on motion either but only on the relative positions of the two pieces of matter, their potential difference in Machian terms (supposing that this is

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[25] The equation can also be read the other way, to describe a body thrown straight up, whose vis viva is consumed into potential until it comes to rest at a height before falling again.

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not a spatial difference but more a pure difference in physical potential level, like a difference in temperature). This fact is expressed by Mach's *Beschleunigungsprincip*, which says that, in a freeze frame of time, before any accelerations have taken place, pressures will be generated originally by purely static differences in position (or potential level) between two bodies.[26] The *Beschleunigungsprincip* was Mach's fundamental law of motion of mechanics: in a more perspicuous form it says that accelerations, or rather the forces that produce them, depend on differences in the instantaneous potential level of bodies. The pressures due to accelerations, or as Mach would call them equalizations of level, are activated by instantaneous differences in *position* between the capacities. These differences are really the respective potential levels of the two capacities, just as temperatures are the potential levels for the flow of heat. Accelerations equalize the difference in position, as heat flow equalizes differences in temperatures. As Mach said in the *Mechanics*, we see in spatial accelerations an equalization of differences in potential level above the mean of neighboring levels.[27]

Simplification of the Problem

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[26] I leave out of account any velocities the bodies may already have acquired.

[27] (M) pp. 288-289.

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The experimental pressures we measure in Mach's definition thus reduce to two factors: the capacity factors resident in the bodies by assumption, and the intensity factor due to their instantaneous difference in position or potential level with respect to one another. Both are conceived as static and prior to motion and both always occur together.

The true problem of the mass definition now appears considerably simplified. We must get from the experimentally observed pressures to two specifications of pressure: those due to a capacity factor in the mass, or position, and those due to the difference of potential level between masses, which show up empirically as spatial accelerations equalizing differences in position. These two factors of capacity and intensity do not occur separately, but it is possible to separate them in thought by applying two different standards of comparison: a variety of masses at the same distance; the same mass at a variety of distances. For a definition of mass, we thus need a technique for fixing both factors independently.

Mass: a Standardized Capacity Factor

Mach proceeded as follows. There is no such thing as a body that does not accelerate according to universal gravitation. Thus all bodies will press one another according to their capacity and intensity for pressure. Wherever we find matter we also find pressure, whether at rest or

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accelerating. We can feel the pressures of contact resident in the body by trying to accelerate it, or collide with it, and we can feel the pressures due to fundamental forces by bringing sufficiently large bodies together.

The constant conjunction of force with matter leads us to believe that there is a capacity for acceleration resident in matter independent of its chemical composition or other resident properties we discover, such as magnetism and electricity. Mach claimed mass was but *one* of a number of such energetic capacities possessed by bodies.[28] We need to *isolate* this particular capacity to accelerate from other properties of a body due to its peculiar chemical composition.

Once the capacity for acceleration has been isolated from electrical and magnetic attractions, we discover the fact that if two bodies are equal in their capacity to accelerate one another and one of them is equal to a third, the other is also equal to this third. The separation of this property from the specific composition of bodies and the flux of experience is what Mach's definition aimed to prove. The mass concept thus represents a standardization and metricization of capacities that we already know they have. Mach's definition isolates the capacity factor and gives it a unique meaning, even though it does not appear alone in experience. This is why he insisted that mass be defined first for two bodies m,m of identical chemical composition first, and at a fixed distance

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[28] (M), p 240.

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from one another: that fixes both the capacity and intensity factors.[29] Then we can consider extending the concept to bodies of heterogeneous chemical compositions.[30] But for this extension we need an argument.

Mach argued further that if bodies did not press one another equally, either by attraction or by contact, then it would be possible to construct a perpetuum mobile machine. Take the pressures produced by contact. We set three bodies on a frictionless ring and start one in motion.[31] If the collision produces more vis viva in the second body than was in the first, and more in the third than was in the second, then when the machine goes through one complete cycle and the vis viva is communicated back to the first body, there will be more kinetic energy in the ring than there was to start with, a perpetuum mobile. Mach proof mimicked exactly (and is probably patterned on) Sadi Carnot's argument that the quantity of heat exchanged between two bodies depends only on their temperatures and not their chemical compositions.[32]

Mach assumed that two absolutely *equal* bodies at a distance from one another (a certain given difference in potential level), will produce equal and opposite pressures, which he will measure by means of motionless test-pressures:

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[29] (UDM), p. 357

[30] (UDM), p. 358.

[31] (UDM), p. 358.

[32] (H), pp. 203-204.

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$$F_{\alpha\beta} = -F_{\beta\alpha}.$$

Mach then assumed some separation between the experimentally fixed capacity and intensity factors so that he could analyze pressure into masses and accelerations:

$$m_\alpha \phi_{\alpha\beta} = - m_\beta \phi_{\beta\alpha}, \text{ or}$$

$$m_\alpha/m_\beta = - \phi_{\beta\alpha}/\phi_{\alpha\beta}.$$

This is the real problem. There is no way to separate the capacity and intensity factors completely, but it is possible to fix each of them independently of the other by carrying out a set of systematic comparisons with a unit test body at standardized distances. This is the real reason why there are mass-ratios but no absolute masses.

All of these conditions for the definition are to be specified by experimental physics.[33] It seems to me Mach could have relied on the conservation of energy here as well. Simply find an equivalent energy source, such as heat or electricity, for which one knows, and can vary, the capacity factor of the interaction, and compare this to mechanical mass. Since we are assured by Mach's perpetuum mobile argument that the relations of any two bodies to a standard body determines their relations to each another, we have a method for testing the capacities of all bodies in comparison with the capacity of mass. The conservation of energy in turn ensures that all of the capacities are invariant in their

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[33] (M), p. 303

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ability to press one another, no matter what chemical test body is employed.

If a mass A/a is twice as great as B/a, it will remain twice as great even when they are expressed in a different unit A/b and B/b. To transform to a new system of units, we would simply multiply by the ratio of the old test mass to the new test mass and divide through.

$$A/a.a/b = A/b;$$

$$B/a.a/b = B/b.$$

Capacity and Intensity as a Foundation for Physical Space

Mach was aware that mass-capacities based on position tended to be closely linked with electrical, thermal and other capacities. Much later Mach came to believe that mass-capacities were "volume elements" in a physical or chemical manifold involving all natural potentials, gravitation, electricity, heat and vis viva. But this is a subject for Chapter sixteen.

Although we cannot go into these claims at this juncture, we can consider some of Mach's own reflections on his mass definition in letters to his friends. Here he seems to be trying to convey the notion that the mass definition is similar to a metrical concept of space. His definition standardizes the capacity factor of mass through the

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interactions of bodies the same way that the metric definition of space standardizes a certain length through motions from place to place.

For example in a letter to Josef Popper-Lynkeus of December 30, 1883 (right after the first edition of the *Mechanik*) he wrote:

> Concerning mass I am convinced that I have hit the mark in not making this a fundamental concept. It is not a concept of experience like length, but rather a concept of space.[34]

Mach also corresponded extensively with Friedrich Adler about the latter's own mass-concept and how it differed from Mach's:

> For me the concept of mass is a dynamical concept of measure [*Massbegriff*], which as such assumes a relation to a comparison body and nothing else. When bodies dissolve, evaporate and so forth, this can cause difficulties in the application of the concept but does not affect the concept itself. Everywhere that the concept of mass or force presents itself in physics, only the mechanical side of the process is meant; only ponderable masses come into consideration, and reciprocal action against another ponderable body must take place. I have already said for 30 years that the validity of the same measure of the masses [*Massenmaasses*] for arbitrary physical circumstances is an hypothesis that has been valid until now...When we speak of the conservation of "mass" or the conservation of "matter" that can only mean one thing: in all changes of a closed system the sum of the mass-values of the bodies, in relation to one determinate comparison-body remains constant. In "matter" I see nothing but the proportionality of all physical capacities [*Capacitäten*] to one another.[35]

Adler wrote back on January 6, 1904 with an admirable précis of Mach's own mass concept:

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[34] (AU), Letter #3.

[35] Mach to Friedrich Adler December 27, 1903 (AU), #22.

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> Masses are, as constant relational values, a system of numerical magnitudes and nothing further. We take a new step when we affix [*zuordnen*] this system to a system of bodies. In every movement of one body we have to ascribe the mass value a new place. The states (temperature, color) that form the bodies may change but we will affix to the body the same mass-value, as long as the accelerations that occur in them point to it. Bodies change; all states occur discontinuously [*unstetig*] in space (changes of energy occur). The masses however move constantly in space, or, when we cannot establish this, the assumption of a constant motion appears to us the most convenient. The system of masses that is thought of spatially only out of economical considerations proves itself the most appropriate, in order to relate it to all other appearances. Before the conception of masses as a spatial system, whose arrangement constantly changes, there is no reason to designate this relational value as the capacity.[36]

The two seem to be debating how the metrical concept of mass is to be gleaned from the spatial relations of bodies, and in this sense the spatiality of the bodies is not reduced, as in the "Raumvorstellungen" article. However the only properties Mach's original bodies have are exhausted by their capacity and intensity factors, which show up empirically as the capacity to accelerate based on differences in spatial position. This is telling and gestures in the direction of a further reduction. I also find his admission to Popper that mass is a "concept of space" a hint in this direction, but no more than a hint.

Mach's Early Investigations of Inertia

Mach's theory of space at the time is greatly illuminated

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[36] Adler to Mach January 6, 1904 (AU), Letter #23.

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by his revisions to the Newtonian principle of inertia. It may be surprising that when Mach says "bodies always accelerate one another" he does not even speak of bodies that travel at constant velocities. That is because for Mach, rest and motion at a constant velocity were both special cases of accelerated motions. Ordinarily in a world of forces all bodies would be continually accelerated, but a body is fixed in position at rest when all of the accelerations on it cancel one another in magnitude and direction. Similarly a body moving at a constant velocity in space acquires an acceleration in one direction just as it is losing one in another, in a dynamic equilibrium. So long as we don't think of the agency of these accelerations this is a fairly tolerable picture. In any case it is a picture of mechanics in which statics and equilibrium are but special cases of the underlying dynamics of forces.

In many systems of mechanics the beginning notions are confined to mass, motion at a constant velocity and equilibrium problems with notions of work and vis viva introduced later. The reason is that perhaps static notions seem closer to an a priori ideal of geometry and to exemplary demonstrative reasoning--which Mach called in Archimedes the "Grecian mania for demonstration."[37] C.F. Gauss had shown earlier that the laws of statics could be deduced from his own

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[37] (M), p. 24.

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principle of least constraint.[38] Only the pedagogical necessity of giving a presentation clear to beginners demanded that statics precede dynamics:

> Proper as it is that in the gradual development of a science, and in the instruction of individuals, the easy should precede the difficult, the simple the complex, the special the general, yet the mind when once it has reached a higher point of view, demands the contrary course, in which all statics shall appear simply as a special case of mechanics.[39]

Mach's conception of the "natural" motions is more Aristotelian in nature than modern (17th century) conceptions of natural rectilinear motion. The historical irony would have delighted Mach, as he was also a "ptolemaic" astronomer of sorts. The inertial unforced motion of a body is for Mach an artificial set of conditions where all of the distances between the body and other distant bodies change proportional to the times. Mach shows in his notebook that in a system of such bodies, sufficiently removed from one another, travelling in all directions at different velocities, they all still change their relative positions proportional to the times:

$$x=\alpha+at \qquad x'=\alpha'+a't \qquad x'-x(\alpha'-\alpha)+(a'-a)+$$

$$y=\beta+bt \qquad y'=\beta'+b't \qquad y'-y(\beta'-\beta)+(b'-b)+$$

$$z=\gamma+ct \qquad z'=\gamma'+c't \qquad z'-z(\gamma'-\gamma)+(c'-c)+\ldots$$[40]

he adds to this: "Die Entfernung ändert sich sehr compliciert.

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[38] With great alacrity in fact, see (M), p. 445.

[39] quoted in (CE), p. 34.

[40] Notebook dated May 1879 (NL 174/2/14).

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Es ist kein Gesetz für zwei Körper. Die Richtung is deutlich darin." (The distance changes in a very complicated way. It is no law for two bodies. The direction is clearly within.)[41]

By writing down all of the distances from a body to the distant masses, Mach felt he had solved the problem of the direction and velocity mentioned in Newton's first law, which reads:

> Every body perseveres in its state of rest or uniform motion in a straight line, except insofar as it is compelled to change that state by impressed forces.[42]

To which Mach asked: perseveres in its velocity in a direction with respect to what? With a velocity measured with respect to what? Absolute space in itself or the fixed stars or to some local set of bodies like the solar system? On the other hand if the law of inertia really assumes a dynamical equilibrium and a changing distance proportional to the changes of *other* bodies, perhaps at great distances, then the coordinate system must be specified in the law itself, because these other bodies give actual causal conditions for the inertial motion of even one body. If their accelerations on the body did not all cancel out, the body would not preserve its velocity at all. On Mach's view velocity is a vector (a triple of numbers only because it implicitly contains a three-

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[41] See also Notebook of 26 January 1881 NL 174/2/17: "Durch die Änderung der Entfernung sind auch die Richtung[en] gegeben. Hiemit nicht nur die Geschwindigkeit sondern auch die Richtung gegeben."

[42] (M), p. 301.

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fold relation to all other bodies). He wrote more on this much later.

With the idea in mind that the law of inertia involves a dynamical reciprocity (*Wechselwirkung*) with all other bodies as its frame of reference, Mach posed the question:

> What would become of the principle of inertia if the whole of the heavens began to move and the stars swarmed in confusion? How would we apply it then? How would it have to be expressed then?[43]

Mach scarcely ever asked a question he didn't think he already possessed an answer to, and sure enough, the formula we would apply in that case is Mach's own:

> In so far as bodies are so distant from one another that they contribute no noticeable acceleration to one another, all distances vary proportionally to one another.[44]

This expression, which dated from Mach's notes to the first edition of the *Conservation of Energy* in 1872, is practically the same that reappeared prominently in the Mechanics ten years later:

> Instead of saying, the direction and velocity of a mass μ in space remain constant, we may also employ the expression, the mean acceleration of the mass μ with respect to the masses m, m', m''... at the distances r, r', r'' is $=0$ or $d^2/dt^2\ [\Sigma mr/\Sigma m]=0$.[45]

This extremely cumbersome (and ugly) expression is nothing

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[43] (CE), p. 78.

[44] (CE), p. 80.

[45] (M), p. 287.

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other than a list of each mass-weighted distance between the test potential μ and every other potential (m, m', m''), and the second derivative of that weighted-distance with respect to time. It is perhaps more Machian to say that in a system of bodies accelerating one another, the center of mass does not accelerate. In so far as all bodies moving at constant velocities are unaccelerated they are all candidates for the center of mass of the system, with respect to a system of an infinite number of surrounding masses. The expression is of course totally unusable for practical purposes and is a good object lesson not to regard all of Mach's physical theories as mere economical descriptions. Julian Barbour points out further that the expression cannot be a scalar equation in any case, as Mach writes it, because it describes the acceleration of a particle situated with respect to other bodies.[46] This however does not necessitate the adoption of any underlying three-fold system of spatial points. It indicates a three-foldness in the fundamental physical relationship between a test mass and the rest of the masses in the universe. The important issue of scalar versus vector equations will be revisited in chapter sixteen.

For the moment, Mach's dynamical view of inertia is bound to disquiet some readers' minds. One feels a certain prearticulate repulsion to Mach's proposal to make

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[46] Julian Barbour "General Relativity as a Perfectly Machian Theory" in Barbour and Pfister (eds.) *Mach's Principle* (Boston: Birkhäuser, 1995), p. 217.

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acceleration prior to motion at a constant velocity. This is because the concepts of time, position, distance, velocity and acceleration (at least in traditional mechanics based on statics) are neatly defined like Russian dolls each fitting into and depending on the concept before. A velocity is a change of position in time; an acceleration a change of velocity in time. It is indeed paradoxical to say that accelerations are somehow real while the other notions of space, time, velocity are dependent on it, when acceleration logically seems to depend on those other concepts. Kant himself felt the problem acutely when he wrote how odd it was acceleration should be absolute in a space and time that is merely relative, especially if accelerations are defined *in terms* of space and time.[47]

What Mach did say is that he regarded the law of force (viz. that equal masses in different positions induce equal and opposite accelerations or pressures) as a fact that must be accepted from experience. However accelerations (changes of changes of position) are equalizations of potential level, and are the conditions out of which spatial regions are defined from intensive elements, thus there is no circle. Kant's solution to the absolute acceleration puzzle was just the opposite: accept accelerations as facts, make absolute space the empirical precondition for such facts to be true,

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[47] *Metaphysische Anfangsgründe der Naturwissenschaft* Viertes Hauptstück Phänomenologie Lehrsatz 2. Anmerkung (Hamburg: Meiner 1997) p. 117.

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and yet make this presupposition transcendentally ideal, not true of nature an sich. For his part, Mach insisted that facts of experience do not "presuppose" the validity of anything, least of all absolute space or the principle of inertia, as Kant believed. If the principle of inertia is an empirical fact this means it could have been otherwise. When Mach was establishing his reputation, however, the followers of Kant were still strong and the law of inertia was often regarded as an a priori presupposition of experience. Mach argued in support of his own empirical conception of inertia that we could always imagine that masses did not induce accelerations in one another, but rather velocities as in equalizations between temperatures:

> If two masses opposite one another gave one another not accelerations, but rather perhaps velocities dependent on the distances, then there would be no law of inertia. Only experience teaches whether one or the other takes place. if we had only sensations of warmth, then there would only be equalizing velocities, which would be zero when the temperature differences were themselves =0.[48]

Accelerations of mere differences of position between two potentials induce changes of changes of those distances, whereas heat always flows at a constant velocity, faster or slower depending on the temperature differences. Mach repeated this argument again and again to contradict the aprioricity of the law of inertia. The facts of experience are the different natural potentials and the means by which

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[48] (UDM), p. 356.

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they will equalize any deviation of potentials above or below the mean value of neighboring potential levels. I think at this abstract level of consideration "neighboring" points should be understood as causally rather than spatially proximate. Clearly for Mach spatial differences covered by bodies in uniform motion are completely analogous to constant flows of heat from higher to lower temperatures, a case Mach called "dynamic equilibrium." Just as the position of a body with reference to surrounding bodies is the mean of its earlier and later positions when it travels at a constant velocity, so too a constant flow of heat passes between temperatures that are the mean of adjacent temperatures.

> That accelerations play a prominent role in the relations of the masses, must be accepted as a fact of experience; which does not however exclude attempts to elucidate this fact by a comparison of it with other facts, involving the discovery of new points of view. In all processes of nature the differences of certain quantities u play a determinative role. Differences of temperature of potential function and so forth, induce the natural processes which consist in the equalization of those differences. The familiar expressions ∂^2u/dx^2 ∂^2u/dy^2 ∂^2u/dz^2, which are determinative of the character of the equalization may be regarded as the measure of the departure of the condition of any point from the mean of the conditions of its environment--to which mean the point tends. The acceleration of masses may be analogously conceived.[49]

Mach's forcible expressions almost make up for the fact that his argument rests on an analogy between the natural processes and what he conceives to be the fundamental physical

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[49] (M), p. 288-289.

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characteristics. He was normally very sensitive to the disanalogies between the flow of heat and the position of a body. The very first such disanalogy that comes to mind (and it is a great one) is that heat flows by contact or radiation, whereas Mach never indicated any mechanism by which an equalization might flow from the infinitely distant masses to a body in the immediate neighborhood. This has always been the fatal flaw in Mach's theory of inertia and gravitation. How are the equalizations to travel such vast distances from those enormously distant masses to, let's say, a pen I push across the table? Even if we were to assume that the accelerations are causal and not spatial, we still would have the problem of the immense causal remoteness of the distant masses, and their ability to equalize differences over vast distances, whereas electrical and thermal equalizations cannot.

A body in a vast space isolated from the distant bodies still resists the efforts made to accelerate it. Where do these resisting forces come from? There is of course a brilliant literature on Mach's Principle that tries to deal with just this problem, e.g. by bringing in a gravitational medium, spread through space like a fluid, conditioned by the presence of all masses, which bodies pull and stretch as they accelerate. Mach himself considered such a gravitational medium as a possible way out, without explicitly endorsing

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it.[50] In fact according to Hugo Dingler, Ludwig Mach reported that the question had always been *besonders quälend* (especially tormenting) for his father. There is currently no consensus about where Mach's insights fit into contemporary science, or whether he remains the "schwerverständlicher Eigenbrötler" he appeared to his colleagues.

John Norton produced an exemplary article on Mach's own conception of "Mach's Principle" before Einstein coined the phrase.[51] He found no sign that Mach advocated a new physical mechanism to carry the gravitational forces. The only proposal Mach consistently makes, says Norton, is that space be eliminated from the discussion of inertia.

> The only unequivocal proposal Mach makes is for a simple redescription of the experiment and the law [of inertia] in a formulation that does not use the term 'space.' It remains unclear whether Mach intended to propose and endorse a new physical mechanism for the origin of inertial forces.[52]

This neat result tallies with everything I have found in Mach's writings, both published and unpublished. In fact in his lectures *Allgemeine Fragen der Naturwissenschaft* 1897 he even tried to dispense with media for matter and forces (as he did for electricity in the Mechanics)[53] because of the remnant

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[50] Mach to Petzoldt September 18, 1904.

[51] John Norton "Mach's Principle before Einstein" in Barbour and Pfister (eds.) *Mach's Principle* (Boston: Birkhäuser, 1995).

[52] Ibid., p. 23.

[53] (M), p. 548.

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of spatial extension in them. With the one exception of Adler, all of Mach's contemporaries, Petzoldt and Einstein included, interpreted him as proposing an exclusively spatial *Wechselwirkung* of ponderable bodies to account for inertial forces by propagating gravitational forces through space. The old philosophical springs which emphasized intensity above extension, and from which Mach's thought had drawn its source, had long dried out. The contemporary spatial reconstructions of his ideas were not what Mach envisaged, and may explain why he refused to associate himself with the developing general theory of relativity in his *Optik* preface of 1913. He said there that his "correspondence above all" led him to issue the denial. Sure enough, shortly before the date of the *Optik* Preface two letters, one from Einstein, the other from Petzoldt, contain explicitly spatial interpretations of his reflections in the *Mechanics*.[54]

Einstein cleverly anticipated Mach's objections in an

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[54] Einstein to Mach June 25, 1913: "Nächstes Jahr bei der Sonnenfinsternis soll sich zeigen, ob die Lichtstrahlen an der Sonne gekrümmt werden...Wenn ja, so erfahren Ihre genialen Untersuchungen über die Grundlagen der Mechanik...eine glänzende Bestätigung. Denn es ergibt sich mit Notwendigkeit, daß die *Trägkeit* in einer Art *Wechselwirkung* der Körper ihren Urprung hat, ganz im Sinne Ihrer Überlegungen zum Newton'schen Eimer-Versuch." Petzoldt to Mach June 15, 1913: "Vorgestern sprach ich den Relativitätstheoretiker Prof. [Max von] Laue der von Zurich zu einem Vortrag in der physikalischen Gesellschaft gekommen war. Einstein will mit der Gravitationserklärung im wesentlich fertig sein und diese Arbeit für das Beste erklärt haben, war ihm gelungen sei. Dabei macht er, nach Laues Äusserung, die Zentrifugalvorgänge von Relativdrehungen der Massen gegeneinander abhängig, also wie Sie es tun."

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undated letter,[55] where he emphasized that the coordinates in his tensor expressions were arbitrarily assigned and could be replaced just as easily by others. Perhaps this was not enough for Mach who wanted not only arbitrary but eliminable coordinates and an intensive treatment of physical concepts in terms of world elements. Mach even said in an earlier letter to Petzoldt that a "bare efficacious position has been observed by nobody."[56] This is another sign that Mach would have regarded position as fixed by the pressures composing matter. Presumably Mach's objections to Petzoldt included bare efficacious spatial coordinates, like Einstein's, however arbitrarily assigned and varied. The question was whether Einstein actually explained space away by non-spatial energies, or whether he had left a residue framework of semi-absolute space-time coordinates, for example in the flat

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[55] (AU), Letter #82a: "Für mich ist es absurd, dem "Raum" physikalische Eigenschaften zuzuschreiben. Die Gesamtheit der Massen erzeugt ein $G_{\mu\nu}$-Feld (Gravitationsfeld), das seinerseits den Ablauf aller Vorgänge, auch die Ausbreitung der Lichtstrahlen und das Verhalten der Maßstabe und Uhren regiert. Das Geschehen wird zunächst auf vier ganz willkührliche raum-zeitliche Variable bezogen. Diese müssen dann, wenn den Erhaltungssätzen des Impulses und der Energie genüge geleistet werden soll, derart spezialisiert werden, dass nur, (ganz) lineare Substitutionen von einem berechtigen Bezugssystem zu einem anderen führen. Das Bezugssystem ist der bestehenden Welt mit Hilfe der Energiestoffe sozusagen angemessen und verliert seine nebulöse apriorische Existenz."

[56] Mach to Petzoldt September 18, 1904 (AU), p. 39.

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Minkowskian manifold.[57] If so then Einstein's treatment still depended logically on a framework of space-time points--absolute space albeit in a watered down sense. The one contemporary writer who noticed this circle in relativity was Russell, who wrote of Einstein's theory:

> It is natural to ask: could we not dispense with coordinates altogether, since they have become little more than conventional names systematically assigned? Perhaps this will become possible in time but at present the necessary mathematics is lacking. We wish, for example, to differentiate, and we cannot differentiate a function unless its arguments and values are numbers...To define such a process is a problem in mathematical logic, probably soluble but unsolved hitherto. If it were solved, it might become possible to avoid the elaborate and round-about process of assigning coordinates and then treating almost all their properties as irrelevant, which is what is done when the method of tensors is employed.[58]

Probably Mach would have rejected most of the discussion that has since swirled around inertial (gravitational) forces propagating instantaneously through space at a distance or through a medium. This conception, whatever its merits, is

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[57] Barbour suggests--analogously to Mach's revision of the Newtonian principle of inertia--that the flat absolute Minkowskian 4-space of special relativity is only that rare and special case in which the dynamical forces in a region exactly cancel one another, as described by general relativity: "What we need is a change of perspective--away from the view that flat Minkowski space is the 'natural ground state of the universe'(or, alternatively a fixed external framework) and to the recognition that it is a highly atypical solution of a sophisticated dynamical theory." See "General Relativity as a Perfectly Machian Theory" in Barbour and Pfister eds. *Mach's Principle* (Boston: Birkhäuser, 1995), pp. 226-227.

[58] Bertrand Russell *The Analysis of Matter* (London: George, Allen and Unwin, 1927) pp. 67-69.

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completely alien to Mach's intent to eliminate physiological space-saturated thinking from physics. His silence on the question of spatial propagation speaks loudest of all.

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Chapter Fourteen: The Perpetuum Mobile of the First Kind

Mach's lecture *Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit* was held on November 15, 1871 at the Royal Bohemian Society of the Sciences in Prague and published the following year. He had finally emerged from his long sojourn in the wilderness, studying the physiology of the senses and investigating the history of physics with original sources, especially during a period of illness in the summer of 1870.[1] *The Conservation of Energy* deserves to be considered the most straightforward exposition of his ideas on the "elements," the fruits of his painful intellectual struggle of 1863-1871. If there is any book of Mach's to start with, this is the one.

Many of his views did undergo changes later, in particular his ideas on thermodynamics, which caused him to give up his beloved principle of conservation of energy for heat. However even in 1909, Mach continued to refer this short, fierce exposition in an unaltered edition:

> Here are to be found the fundamental ideas of the *Mechanik* of 1883, of the *Analyse der Empfindungen* of 1886, which was primarily addressed to biologists, and of the *Wärmelehre* of 1896, and the *Erkenntnis und Irrtum* of 1905.[2]

Yet the Conservation of Energy was a roman à clef. Both

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[1] K.D. Heller *Ernst Mach Wegbereiter der modernen Physik* (Wien: Springer 1966), p.23.

[2] (CE), p. 10. Preface of 1909.

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in title and content, the pamphlet was an attack on Helmholtz's famous treatise *Über die Erhaltung der Kraft* (1847)[3] The narrow thesis of Mach's booklet is that the law of the conservation of energy is *not* an exclusively mechanical law to be understood in terms of particles and central forces, as Helmholtz had. Rather Mach hoped to show in the opening historical section that mechanics had always assumed the conservation of energy in the form of the excluded perpetuum mobile principle, and had never succeeded in proving it from more fundamental ideas. In the logical section he hoped to show that the law could not be demonstrated in mechanics without begging the question. Other goals of the book were to oppose the mechanical theory of heat (the idea that heat is nothing but the motion of particles)[4] oppose atomism in chemistry in favor of pure energy transformations[5] and to argue against substance, space and time and in favor of Mach's conception of energies in terms of potential and potential function.[6] It was a *Streitschrift* and like all such polemics its weakness was that Mach did not show in a positive way how he intended to do science differently, how to lay down the fundamental dimensions of capacity and intensity factors and

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[3] *Ostwalds Klassiker* No. 1 (Leipzig: Wilhelm Engelmann 1889).

[4] (CE), p. 37.

[5] (CE), p. 54.

[6] (CE), p. 47, 85.

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how to express the customary physical quantities and equations in these terms. Indeed we are not even given a clear picture of the view under attack, for which we must turn to Helmholtz himself.

Helmholtz "Über die Erhaltung der Kraft"

Helmholtz, too, had started from the principle of excluded perpetual motion, but he saw it as identical with the assumption that all natural processes could be understood as motions of particulate matter pushed and pulled by forces emanating from their centers, their intensities depending on spatial distances:

> The derivation of the proposed laws can be attacked from two points of departure, either from the law that it is impossible to gain an unlimited amount of work from the actions of any combination of natural bodies, or from the assumption that all activities in nature are to be traced back to attractive and repulsive forces, whose intensity depends only on the distance of the two points acting on one another. That both laws are identical is shown in the beginning of the treatise itself.[7]

By this "identity" of the two laws Helmholtz also meant that wherever we can introduce the phenomenological principle of excluded perpetual motion, we can also say that the underlying mechanisms (i.e. of heat, electricity) are particulate matter and central forces. This was the thesis Mach sought to attack.

Helmholtz expressed the excluded perpetuum mobile

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[7] Ibid. p. 1, 6.

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principle with his customary clarity:

> Let us think of a system of natural bodies, which stand in certain spatial relations to one another and which, under the influence of their reciprocal forces, are set in motion until they arrive at certain other positions: so can we consider their acquired velocities as a certain mechanical work, and change them into such work. If we want now to make the same work over again, we must reset the bodies in their original conditions, by applying other forces at our disposal. For this we will need therefore a certain amount of work from the latter. In this case our principle demands that the amount of work that is obtained when the bodies go over from the first position to the second and is lost in going from the second back to the first is always the same, whatever the manner, path or velocity of this transition may be. For if it were greater along any one path than another, we could use the first to obtain the work and use the second to reset the positions, for which we would only need a part of the work we had just obtained. Thus we would obtain an indefinite amount of mechanical force and would have constructed a perpetuum mobile, which could not only keep itself in motion, but would also be able to render up force to the outside.[8]

The principle is easily illustrated by a stone rolling down a hill and up another of equal height on the surface of the earth. If the velocity acquired by the stone depended on the path of descent, we could roll it down one path to acquire a velocity and raise it to the same height along a different path with some extra velocity still left in it. Then of course we could repeat the process over and over again and use the excess vis viva to power a generator. The difficulty is not in grasping the principle itself but in trying to find out the reasons why it is true, for there do not seem to be any simpler facts than these.

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[8] Ibid, p. 8.

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The excluded perpetuum mobile principle is not well named, since, as Helmholtz indicated, it is not a perpetual change of position that is ruled out but the creation of a perpetual source of *work*. The orbit of a planet, or the uniform motion of a body forever in space are perpetual changes of place, but not perpetual sources of work. Mach also saw fit to make the distinction in his amendment to Stevinus' endlessly moving chain for example,[9] and in his further examples of excluded perpetual motion.[10]

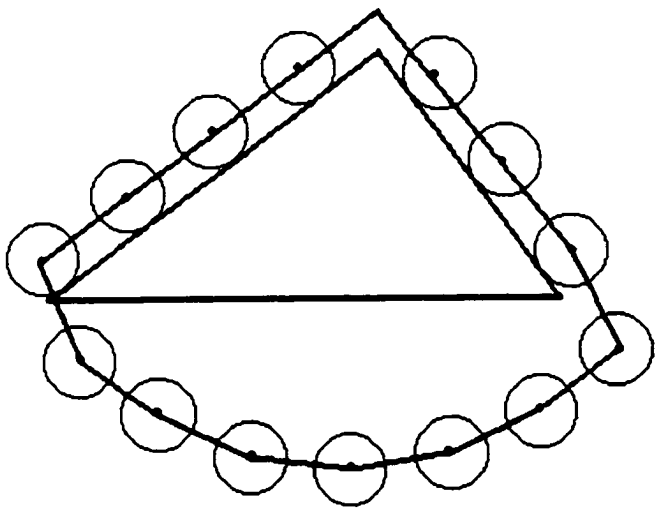


FIG. 14.1
Stevinus' Endless Chain

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[9] (CE), p. 23.

[10] (CE), pp. 71-72.

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Stevinus argued that the four weights on the flatter incline must balance the three on the steeper incline, such that the weight supported is inversely proportional to the steepness of the slope on which it is situated. If not the chain would be set in motion and once again four weights would ascend to a new position and so on ad infinitum. As Mach remarks, we may well conceive of the chain in frictionless perpetual motion as long as it is uniform. What we object to is an initial self-acceleration of the chain from rest into motion and all of the additional accelerations that it gives itself by virtue of its position.[11] These are uncompensated creations of vis viva from nothing and could be used as a source of perpetual work.

Really any uncompensated element of work-energy α appearing isolated and alone, once or repeatedly, is what constitutes a violation of the principle. This would also apply to a sudden appearance of a quality like pressure in a body, or even the creation of a sensation-energy in the nerves from nothing. Hence Mach's positive version of the excluded perpetual motion: for every element of work α (a sensation, a pressure, an acceleration from rest) a corresponding element of work β is needed to compensate it. We have a new metaphysical enemy, the *Ding bei sich* instead of the *Ding an sich*.

Instead of the principle that no element occurs alone as a *Ding bei sich*, Mach adopts the principle that all elements

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[11] (CE), p. 23.

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occur together, and as mutually compensating functions of one another. This principle Mach saw as more fundamental, and it relates strongly to his idea that nature forms a whole.

> "It is not possible to create work out of nothing." If a group of phenomena is to become the source of continual work, it means it shall become the source of continual variation of another group of phenomena. For by means of the general connexion of nature, all phenomena are also connected with mechanical phenomena and therefore with the performance of work. Every source of continual variation of phenomena is a source of work and vice versa.[12]

This is the rationale for Mach's fundamental equation for the work-energy elements:

$$f(\alpha, \beta, \gamma, \delta, \ldots, \omega) = 0.$$

He insisted that the excluded perpetuum mobile principle is just a "special case" of the fact that all work-elements are functionally dependent on one another and that none goes uncompensated. Such are the facts from which he wished to deduce the seemingly irreducible principle.

As Mach pointed out carefully in this and all subsequent expositions, he was not saying that every element depends directly on everything else. There may very well be sets of elements dependent on one another but independent of the rest.[13] One suspects that these "sub-groups shut up in

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[12] (CE), p. 69.

[13] (CE), pp. 70-71.

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themselves" are the equivalent of bodies or egos, within a universally closed system. We are extremely lucky however that nature provides us with examples of relatively closed systems to work with, the earth, the solar system, the fixed stars, for they give us a chance to observe the conditions of a miniature-universe and make very broad judgments about the universe itself.

A second limitation, perhaps logically connected with the first, is that there are fewer equations rigidly relating the changes of elements than there are elements; there is a certain variability to nature not predictable from energy data alone.[14] This is Mach's way of saying that he does not expect a complete description of nature from energy data, and in particular the specifics of energy transfer, the paths, the mechanisms, and so forth. These are matters of positive natural investigation.

Helmholtz's use of the *perpetuum mobile* principle was humbler. He said that one should imagine a mass m and a system A exerting forces on it dependent on the position of m against (gegen) A.[15] As m moves to various positions against A the intensity of the forces the system A exerts against it change, and m either picks up velocity (vis viva) or loses it. According to Helmholtz the excluded perpetuum mobile principle demands that when m comes back to its initial position against

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[14] (CE), pp. 61-62.

[15] *Über die Erhaltung der Kraft* p. 10.

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A where the intensity of forces is the same, m's velocity (or vis viva) must be the same too. That is m's vis viva is strictly a function of its position against A and thus the intensity of forces as a function of m's position against A.[16] The assumption of central forces thus manages the whole deduction. Mach criticized it directly in his lecture "Mechanik" of 1870-71:

> If a force function exists, from which the forces may be derived, then is the work a function of the coordinates. This is a circle, a tautology. The knowledge is important that in many natural forces the work depends solely on the coordinates.[17]

Mach also said in the Mechanics that if we are to know that only the positions m takes up against the system A are decisive and not the paths it takes, we must already possess the concept of work that is determinative of the vis viva of m, that only the forces acting along the line of junction between m and A produce vis viva in m and not motions in other directions.[18]

Of course Helmholtz's hope in the "Erhaltung der Kraft" was not to prove the law of vis viva a priori but to make plausible the assumption of central forces based on further

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[16] Ibid., p. 10.

[17] Wenn eine Kraftfunction existiert, von der die Kräfte ableiten [läßt?], so ist die Arbeit Function d. Coordinaten. Dies ist ein Cirkel, eine Tautologie. Wichtig ist die Erkenntnis, dass bei vielen Naturkräften die Arbeit bloss von der Coordinaten abhängt (NL 174/1/001).

[18] (M), p. 67.

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metaphysical assumptions. Those assumptions are what Mach's book is directed against.

For instance, Helmholtz laid down the hypothesis that nature consists of quality-less extended matter in space which can only suffer changes of position. Lacking any other manner of influence, the only way one piece of matter can affect another is through the space between them. The intensity of forces, having nothing else to depend on, depends on the distance. Hence central forces are the only ones metaphysically possible.[19] One gets the sense that Mach must have read the opening pages of Helmholtz's treatise over and over again, until he could maintain the very opposite of each of the theses of Helmholtz.

First matter, for Mach, is not dead quality-less stuff; it has a multiplicity of physical properties underneath its mere extension. Mach would even go so far as to call these properties sensations. Second the extension of matter and its action across space are secondary, derived from qualitatively explainable pushes and pulls of varying intensity--intensities are not necessarily functions of spatial properties of matter like position and distance; it might just as well be the reverse. Thirdly, the law of excluded perpetual motion could be interpreted as the deeper functional dependence of those intensive qualities on one another. Causes are not necessarily to be interpreted as motional changes. For Mach,

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[19] pp. 4-6.

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the law of the functional dependence of elements embraces mechanics and is more fundamental than the mechanical metaphysics of Helmholtz[20] or even the perpetuum mobile principle used by Stevinus, Huygens and other investigators.

To reverse someone's metaphysical conception of the world is not, of itself, to cease doing metaphysics, and Mach's views are just as loaded as Helmholtz's--a point he was not always cognizant of. Mach tended to identify his views with the facts in a way unbecoming of his critical intelligence. As if Helmholtz did not think his views were the facts?

Still, Mach does deserve the credit for working out an alternative to Helmholtzian metaphysics. During the 1870s Mach often wrote of his confidence that behind the phenomena of matter and motion, there were processes that could not be thought of spatially. For example in two notebook passages contemporary with the *Conservation of Energy*:

> The healthy fecund kernel, which is thought to derive from mechanics, can be carried within each of the sciences on its own. There are few general fundamental laws. Behind the motional processes that appear. Wherewith the old fear of ghosts is reintroduced. History.[21]

For example, Mach looked to the theory of heat as an example

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[20] (CE), p. 50.

[21] Den gesundten kräftigen Kern, den man meint der Mechanik entlehnen zu müssen, kann jede Naturwissenschaft in sich selber tragen. Es sind wenige allgemeine Grundsätze. Hinter den erscheinenden Bewegungs-vorgänge. Womit die alte Gespenster-furcht wieder eingeführt würde. Geschichte. Notebook dated 12 April 1873 (NL 174/2/3)

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of a non-spatial process:

> Temperatures are arrived at spatially without themselves being spatial. [22]

Mach saw the motion in place of particulate matter (locomotion) as a superficial metaphor for change in physical science, but not as the actual nature of change, even if the motional metaphor were pursued into infinitely small regions. Qualitative changes ran deeper and could even explain spatial changes.[23]

Mach's demand for a purely qualitative science and a construction of extended matter and motion only intensified as time went on. In his notebooks of the 1890s Mach compared the Hemholtzian-Wundtian metaphysics to the otiose *Naturphilosophie* of Oken:

> Motion is an analogy like any other, which can be quite fruitful in special cases,...and can even pass further than what has been assumed. The whole atomic theory. The general phrase that everything is motion is no more fruitful than the phrases of Oken's *Naturphilosophie*, which for example has it that the world arises with the greatest of ease through the division of zero into a+ and a-.
>
> A Greek philosopher. Pressure and impulse. Much happens as if it rested upon pressure and impulse. But that now we should remain with them, given our richer intuitions, whereby precisely pressure and impulse are no longer so simple! Motion is in most cases a fruitless *naturphilosophische* phrase. (As with Oken 0=+a, -a.)[24]

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[22] Die Temperaturen sind räumlich gewonnen aber selbst nicht räumlich. Lectures, Summer 1872 (NL 174/1/003).

[23] (CE), pp. 49-50.

[24] Die Bewegung so eine Analogie wie andere, die ja in Specialfällen recht fruchtbar sein kann...kann auch...weitere passen als man selbst angenommen hat. Die

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Concerning his own counterproposal to spatial physics Mach was quite reticent, especially in print. He explained that the belief in the mechanical theory of all natural processes was so ensconced in his colleagues' minds that he hesitated to oppose it directly.[25] In his notebooks and lectures he spoke of the *Gespensterfurcht* that often greeted expositions of his own theories. Thus, he worked his way through the whole body of mechanics first before making his own point.

The Principle of Virtual Work

Mach wanted to show in the *Conservation of Energy* that the functional dependence of energies was the logical basis for all principles of mechanics, and had been assumed even when the concept of energy was unknown. The endless chain of Stevinus,[26] Galileo's principle that a body must rise to the

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ganze Atomtheorie. Die allgemeine Phrase dass alles Bewegung sei ist nicht fruchtbarer als die Phrasen der ehemaligen Oken'schen Naturphilosophie, welche z.B. die Welterschöpfung mit grösster Leichtigkeit durch Spaltung der Nulle in +a in -a entstehen liess.

Ein griecher Philosoph. Druck und Stoss. Manches geht so vor, als ob es auf Druck und Stoss beruhren würde. Dass wir aber jetzt, bei so viel riecheren Anschaunngen, wobei gerade Druck und Stoss nicht mehr so einfach sind, dabei bleiben sollen!...Die Bewegung ist in d. meisten Fällen eine unfruchtbare naturphilosophische Phrase. So wie Oken (0=+a, -a). Notebook dated 6 December 1890 (NL 174/2/32).

25 (CE), p. 39.

26 (CE), p. 21.

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same height from which it fell, from which he deduced the principle of inertia,[27] and Huygen's principle of the center of oscillation for systems of bodies and rigid connections[28], and the principle of virtual velocities--all of these refer back to the excluded perpetual motion and are deduced by contradiction from that postulate.

The most important application of the postulate is in proving the principle of virtual velocities for a system of bodies and rigid connections. The principle was enunciated by Stevinus[29] for all machines (the inclined plane, the lever, and the pulley) and formally perfected by Johann Bernoulli and Lagrange. It states that if a machine is in a state of equilibrium--such as a system of weights held in balance with one another through any mixture of pullies, levers, inclined planes and connecting cords--the instantaneous distances that *would* be covered if the weights moved in the direction of the forces that move them all add to zero. As Mach pointed out, the tiny elements of motion through spaces by forces ($F.\delta s$)[30] are really the elements of work. Every virtual displacement is a possible motion consistent with the rigid connections of bodies in the system:-- if some body were moved by twice the

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[27] (CE), pp. 23-28.

[28] (CE), p. 28.

[29] (CE), p. 31.

[30] In a more elaborate treatment of mechanics the quantities δq are called variations but as Mach says, not as a rule in the principle of virtual velocities, (M), p. 540.

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force through half the distance in one part of the machine, it would be matched somewhere else by a body moved by half the force through twice the distance, so that the elements of work done in the machine all cancel out.

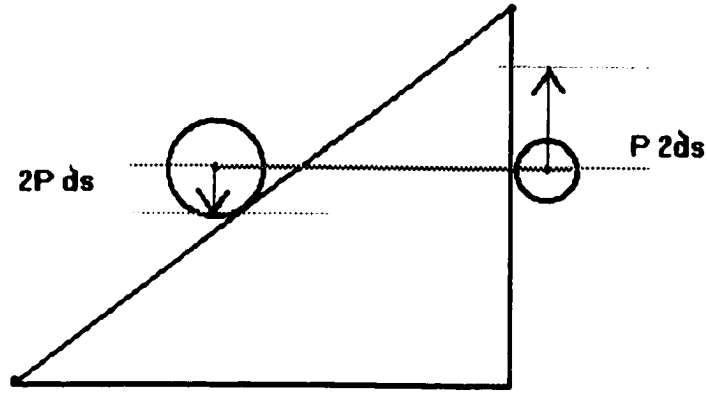


FIG. 14.2
Principle of Virtual Velocities

As Mach pointed out against Helmholtz, since we know that only the spaces covered in the direction of the forces count in the tally, we must already possess the notion of work and thus what we are really asserting is that the work elements add to zero. We cannot prove this within mechanics without somehow

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assuming the concept of work already and begging the question--thus in any case we must begin with work-elements.

Now the principle of virtual velocities is strangely formulated. A system in equilibrium is not supposed to be able to move at all and indeed it does not. Why then do we consider infinitesimal and, on top of that, virtual but not actual motions? As Mach explained in the *Mechanics*, the reason why the displacements must be infinitesimal is because we want to be able to consider systems with moving constraints and non-constant forces, in which equilibrium may only be due to the "momentary possibility of displacement in a momentary configuration of the system."[31] Over an extremely short span a system of changing forces may be regarded as a system of mutually offsetting constant forces in each moment of time.

The second requirement is that the displacements produced by the forces be *virtual but not actual* motions. Arnold Sommerfeld for example calls the virtual displacements "test quantities":

> The δq have nothing to do with the actual motion. They are introduced so to speak as test quantities, whose function it is to make the system reveal something about its internal connections and about the forces acting on it.[32]

The paradox is explained by the difference between potential energies (Helmholtz's "tensing-force") and kinetic energies of

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[31] (M), p. 70.

[32] *Mechanics*, Martin O. Stern trans., (New York: Academic Press), p. 51.

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motion. Before a body has converted its potential energy to motion it possesses a *tendency* to move in the way that the principle of virtual velocities describes. But as Helmholtz said, we must regard the potential energy or *Spannkraft* as this *tendency* to move before actual motion is produced. If not a body will have both its resident potential energy producing motion *and* the kinetic energy due to this same motion all at the same time, which is impossible since the one disappears with the appearance of the other. There is no getting around the fact that the potential or virtual work is a non-motional cause of motion, woven into the principle of virtual velocities and thus into the foundation of mechanics.

In the *Conservation of Energy* Mach pays special attention to Lagrange's attempted proof of the principle of virtual velocities in the *Méchanique Analytique*. To do so he had to assume the principle of excluded perpetual motion. Lagrange started from the idea of Torricelli's that in a system in equilibrium if one weight falls, another rises, in the proportion mgh=m'g'h' no matter how they are connected, thus the center of mass of the connected system cannot sink.[33]

> Lagrange, as is known, conceived simple pulleys arranged in the directions of the forces of the system, passed a cord through these pullies and appended to its free extremity a weight which is a common measure of all the forces of the system. With no difficulty, now, the number of elements of each pulley may be so chosen that the forces in question shall be replaced by them. It is then clear that if the weight at the extremity cannot sink equilibrium subsists, because heavy bodies cannot of

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[33] (CE), p. 31.

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> themselves move upwards. If we do not go so far, but wish to abide by Torricelli's idea, we may conceive every individual force of the system replaced by a special weight suspended from a cord passing over a pulley in the direction of the force and attached at its point of application. Equilibrium subsists then when the common center of gravity of all the weights together cannot sink. The fundamental supposition of this demonstration is plainly the impossibility of a perpetual motion.[34]

Mach remarks, concluding his historical review, that everywhere the principle of the excluded perpetual motion is the foundation for mechanics, but is never proved, or provable, within mechanics or even from mechanical assumptions such as the particulate extended matter and central forces of Helmholtz--since the assumption of central forces already contains the concept of work.

> The whole of mechanics is thus based upon an idea, which, though unequivocal, is yet unwonted and not coequal with the other principles and axioms of mechanics. Every student of mechanics, at some stage of his progress, feels the uncomfortableness of this state of affairs; everyone wishes it removed; but seldom is the difficulty stated in words.[35]

The excluded perpetual motion principle is a dynamical principle, which involves the transfer of energy from virtual potential to vis viva in the motion of bodies. Mach encouraged us to see all cases of statics and equilibrium as special cases of dynamics, where the forces are trying to do work by accelerating the masses of the system, but cannot because other forces are coupled to them through the system's

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[34] (CE), p. 32, Cf. (M), pp. 77-79.

[35] (CE), p. 32.

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connections and are set working against them. The only real condition for static equilibrium is that the possible work of the forces cancels out. Mach thus indicated that statics, like the principle of inertia, really rests on an unarticulated dynamical foundation in the modes of transfer and conservation of energies.[36] Employing concepts of motion as it does, mechanics can only express its fundamental law as a virtual motion from one configuration to another that need not actually take place. Mach goes further and sees the dynamical elements at work opposing one another even under states of rest when they produce only virtual motions. Far from being mere "virtual" existences however, these elements of the *Spannkräfte* are in fact unextended in either space or time and prior to motions.

The Conservation of Energy and the Law of Sufficient Reason

Mach actually used the principle of excluded perpetual motion in a variety of senses, before he discovered that it had two logically independent meanings. In the restricted sense it forbids the creation of work from nothing to which Mach added that it forbids creations of work not compensated by opposing elements. A universal functional connection of work elements is upheld.

But there is also a second sense in which Mach claims to use the principle in his *Mechanics*. Put roughly: any work

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[36] (CE), pp. 34-35.

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that possibly can be done by a system within constraints, will be done. Anything that can happen will happen and anything that can't happen, won't.[37] As an example, heavy weights fall whenever possible and *never* rise of themselves. We can view the spontaneous falling of weights as the unidirectional conversion of potential to vis viva whenever possible. According to this principle a machine is best viewed as a kind of energy-conversion device for transferring potential energy of weights into kinetic energy, by accelerating, or back again by decelerating.

Originally Mach viewed the second form of the principle as equivalent to the principle of sufficient reason, or as he sometimes says, as the contrapositive of the law of causality.[38] Of course, it is impossible to apply these abstract principles a priori since we must find out empirically which properties in fact are the causes of processes and which are irrelevant.[39] By the law of sufficient reason Mach simply meant: any energetic process that has just as much reason to happen as not to happen, will not occur.[40] Energetic processes that *do* occur must have some underlying asymmetry in their initial conditions which gives a univocal path the process is compelled to take, such as the

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37 (M), p. 470.

38 (CE), p. 65.

39 (CE), p. 65.

40 (CE), p. 65-66, (UDM), pp. 355, 357. (M), 1-89.

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falling of bodies, the flow of heat or the equalization of a difference in energy levels and not the spontaneous generation of a greater difference. Giving up sufficient reason, Mach thought, would mean embracing indeterminism or the idea that changes could occur without any natural cause.

Nothing in the principle of the conservation of energy, or the original excluded perpetuum mobile principle, really demands that energy transfers *must* happen or what paths from asymmetries in initial conditions they may take. For all we know energies could transfer forwards and backwards, bodies could slow and rise in the air of their own accord, and the first law would be upheld. All that the first form of the excluded perpetual motion principle demands is equality between the energies of the initial and the final states. With the first principle alone bodies might not fall and transfer their gravitational potential energy to vis viva, but might heat up or electrify instead, producing no contradiction with the postulate.

At times Mach expressed a difference between the two principles, calling the first form the *Arbeitsprincip* and the principle of sufficient reason governing the transfers or happenings the *Beschleunigungsprincip*. Indeed while the *Arbeitsprincip* goes at the head of all of the sciences, the *Beschleunigungsprincip* goes at the head of mechanics since that branch of science describes those energy transfers that run from gravitational potential energy to vis viva and back

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again. Mach sometimes wrote that in mechanics instead of systems obeying the law of least action, we should really say that the path the system takes obeys the law of *most* action, since the masses try to fall as far as they can, and produce more vis viva than any other path would in the same set of connections. In this sense for Mach mechanics is simply the description of all possible energy transfers between potential and kinetic energies based on the idea that all other things being equal bodies will accelerate and take the path through the connections of the system where as much mass will sink as can sink, as quickly as it can. Thus mechanics falls into place as the study of *one* set of energy transfers. Mach said in remarks on Gauss's principle of least constraint:

> Once we have recognized work as the factor determinative of motion, once we have grasped the meaning of the principle of virtual displacements to be, that motion can never take place except where work can be performed, the following converse truth will also involve no difficulty, namely, that *all* the work that can be performed in an element of time actually is performed. Consequently the total diminution of work due in an element of time to the connections of the system's parts is restricted to the portion annulled by the counter-work of those parts.[41]

Thus mechanical systems don't really minimize their paths or work cancelled out in their mutual constraints, but rather *maximize* the work they are able to do in free motion and in acting against one another.

> It will be seen that the principle of least action, like all other minimum principles in mechanics, is a simple expression of the fact that in the instances in question

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[41] (M), p. 453.

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> precisely so much happens as possibly can happen under the circumstances, or is determined, viz. *uniquely* determined by them.[42]

In fact in Mach's earliest preserved notebooks of the 1870s, he was struggling toward a dynamical conception of least action, as resulting from the overcoming of forces by one another in a *Wechselwirkung* of masses:

> In what relationship does action and reaction stand toward work?...Follow it back to the reaction and demonstrate the deeper grounds of the conservation of energy. At bottom this is the intent of Jac. Bernoulli, which leads to D'Alembert's principle. The Principle of the Maximum of Work: if they oppose one another so far. So far do they work against one another.[43]

> Pressure and counterpressure. Work and counter-work.
> Velocity by the Wechselwirkung of masses
> Work by the Wechselwirkung of masses...
> Reaction. Different masses between one another...
> General law of reaction is D'Alembert's principle.
> Look over [or even overlook, ignore E.B.] the center of oscillation. Equality of action and reaction with consideration of D'Alembert...
> Principle of virtual velocities is actually the principle of work.
> Principles of application. Lagrange.
> D'Alembert's principle is actually the principle of work.[44]

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[42] (M), p. 470.

[43] In welcher Beziehung steht Wirkung und Gegenwirkung zur Arbeit?...Auf die Gegenwirkung zurückzuführen und der Tiefere Grund der Arbeitserhaltung nachzuweisen. Im Grunde ist diese Bestrebung die Jac. Bernoulli'sche welche zum d'Alembert'schen Princips führt Princips des Maximums der Arbeit: so weit sie sich entgegengesetzt. So weit wirken sie sich entgegen. Notebook dated "70er Jahre" (NL 174/2/1).

[44] Druck und gegendruck. Arbeit und Gegenarbeit.
Geschwindigkeit bei d. Wechselwirkung d. Massen
Arbeit bei der Wechselwirkung der Massen...
Gegenwirkung. Verschiedene Massen untereinander...

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D'Alembert's principle--one of the supreme achievements in mechanics, logically equivalent to Gauss's principle of least constraint and Hamilton's principle of least action--is a dynamical principle that embraces statics implicitly as a special case of dynamics. The principle treats a system that is not in equilibrium such as a weight and an unequal overweight on a pulley. It reduces the forces to the impressed forces, the forces each body produces as if it were free to move, and the equilibrated forces, those that each weight impresses on the other through their connection. The system of equilibrated forces must be in equilibrium and can be reduced to the static principle of virtual velocities. But the system of impressed forces and the system of equilibrated forces are also in equilibrium, so that they too can be reduced to the principle of virtual velocities and to the conservation of work in machines.

Mach conceived of the world as consisting of various energetic potentials straining against one another, sometimes equalizing one another out and sometimes doing work by overcoming one another when one force is stronger, more

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Allgemeineres Gesetz d Gegenwirkung ist das D'Alembertsche Princip.
Schwingungsmittelpunkt nachsehen. Gleichheit von Wirkung und Gegenwirkung mit Rücksicht auf D'Alembert...
Princip d. virtuellen Geschwindigkeit ist eigentlich das Arbeitsprincip.
Anwendungsprincipien. Lagrange.
D'Alembert'sches Princip ist eigentlich das Arbeitsprincip. Notebook dated 12 April 1873 (NL 174/2/3).

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intense than the forces constraining it. Even free motions of accelerating bodies in space are not unconstrained motions, but rather the overcoming of the less intense inertial forces holding the bodies in position, whether due to the distant masses or perhaps even confined pressures making-up the body.

If all the forces are either cancelled or constrained by others, the spatial concept of work as "force through a distance" could be replaced by the concept of one force overcoming a weaker constraining force in time. The latter conception could be a means to reduce the spatial elements to pre-spatial pressures, intensities and time, quite in line with the programme Mach announced in the *Raumvorstellungen* paper.

With his own oppositional view of elements, this seems to be what Mach was getting at in saying that D'Alembert's principle, which describes forces overcoming one another, is a general law of force acting against force in time, equivalent to the principle of virtual work, which describes forces acting through infinitely small and sometimes non-existent distances. Thus taking this thought to its conclusion, those forces that act over distances seem to be reducible to forces that act over and above constraining forces, with nothing intrinsically spatial about the action. The intensity factor of distance is more precisely expressed as the ratio between the intensities of the overcoming force and the constraining forces.

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The reduction of statics to a dynamics of energy, which Mach pursued at length in his *Conservation of Energy* and *Mechanics*, was I believe scheduled to end with a dynamical construction of matter, space and time in elements, as in the "Raumvorstellungen" paper. With this the last vestiges of the solemn statics of the Greeks were to be swept away.[45]

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[45] (M), pp. 60-61.

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Chapter Fifteen: The Perpetuum Mobile of the Second Kind and the Un-Conservation of Energy

Introduction: Two Senses of the Perpetuum Mobile Principle

According to Mach's historical researches on the excluded perpetual motion principle in the *Conservation of Energy*, the *Mechanics* and the *Theory of Heat*, the classical principle used by Huygens, Lagrange and others really contained two distinct ideas: energy cannot appear from nothing and spontaneous transformations of energy from one form to another follow a principle of sufficient reason touched on in the last chapter. Mach, too, distinguished in mechanics between the *Energieprincip*, which set the initial and final states of a transformation equal to one another, and the *Beschleunigungsprincip*, which gave a sense to the transformation, by ascribing to masses the spontaneous tendency to accelerate.[1]

As Mach noted, Torricelli, Huygens and Lagrange all used the idea that the center of mass of a connected system of bodies could not ascend of itself without giving rise to a perpetual motion. Suppose that the center of gravity of a system could rise spontaneously. That would give us a new initial state with a higher potential energy and then, if the

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[1] This is assuming that the masses have no initial vis viva to be destroyed by the accelerations, which is often an important consideration.

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other half of the principle held equating final states to initial states, we could let the center of mass fall and reap the gains in the excess energy of motion created.

Mach thought both ideas were firmly anchored in the same classical perpetuum mobile principle. This brought him into conflict with the founders of thermodynamics who postulated two excluded perpetual motion principles, one for the conservation of energy and one for the second law of thermodynamics, which we shall discuss at length in this chapter.

Mach's Prague "Unified Science" Programme

The Prague years were the most productive of Mach's career, and have received considerable attention from scholars because of the ethno-political conflicts of the period. During this time Mach presided over the division of the University of Prague into German and Czech-speaking halves. He also constructed and directed his own Institute, trained students and awarded degrees.

Although it would take a detailed investigation to determine all of the different projects pursued at the Institute, Mach did have one overarching scientific program, which he announced in the *Conservation of Energy*: to discover all of the energetic equivalences and their laws of transformation. This master plan certainly stamped much of the work done at Mach's behest. Indeed his notebooks contain

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assignments for his students to investigate the equivalences of mechanical work in optics, electricity, and so on. Mach's notebooks of the period are replete with analogies between the different energies, as well as the important disanalogies which he took quite seriously as problems to be solved.

To discover equivalences, Mach compared the dimensional formulas of physical quantities in extraordinary detail. In addition to its role in an equation or set of equations, every quantity in physics must have dimensions, or units that serve for comparison with quantities from other theories. As dimensional formulas in his notebooks testify, this was Mach's technique for pursuing his programme of unified science. The dimensions of potential, potential level and *Menge* (their quotient) can be found in each domain, although they had often not been identified in words.

The first entry in Mach's notebooks that indicates a "Master Plan" in the Prague years occurred in 1874, when he drew up a table of the natural potentials known to him:

M=Mechanical Energy (Gravitational Potential Energy)

W=Heat (Wärme)

E=Electrical potential (Electricität)

Mg=Magnetism (Magnetismus)

L=Vis Viva (Lebendige Kraft)[2]

He then said that there were twenty combinations to consider

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[2] (NL 174/2/5).

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as the laws of transfer between these types. Mechanics in the abstract deals with transfers from potential to vis viva and back again, so there are two "laws" here (M,L) and (L,M). Because direction matters, these are not equivalent.

> Twenty combinations are to be systematically tried. Write down the fundamental laws and concepts on individual sheets of paper and then try to combine them. Electricity and heat are as much material as masses.[3]

In notebook NL 174/2/14 he drew up the table in checkerboard fashion, crossing each form with each. Clearly laws of transformation like the *Beschleuningungsprincip*, had to be discovered empirically. But for all such transformations that did happen, Mach held to a general phenomenological principle of *Ausgleich* (equalization) of differences of potential above the mean. He conceptualized energy transformations when they occurred as waves of action, a moving *Ausgleich* of differences.[4]

The Special Nature of Heat

There was one great disanalogy between energies that Mach had to grapple with: the special nature of heat. All of the other forms transform into one another and back again reversibly: electricity can become vis viva can become

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[3] 20 Combinationen sind systematisch zu versuchen. Die Hauptsätze und Hauptbegriffe auf besondere Blätter schreiben und dann versuchen zu combinieren. Electricität und Wärme so gut Stoff wie die Massen. Ibid.

[4] Notebook dated 12 April 1873 (NL 174/2/3)

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gravitational potential in any order. But if the energy takes the form of heat and flows spontaneously from high to low temperatures, it cannot all be transformed reversibly back again to mechanical work or its equivalents. Here we have an equalization that goes only one way. The second main asymmetry is that temperature (the potential level for heat) has an absolute level-value and an absolute zero while other potential levels are relative to their surroundings.[5] In his *Wärmelehre*, Mach approached this important phenomenon of thermal irreversibility carefully by means of historical and conceptual analysis.

The Carnot Engine

Mach found in Carnot's reversible cyclic process what he thought was the first application of the excluded perpetual motion principle *outside* mechanics.[6] Yet thermodynamics has been interpreted in two non-equivalent ways: the mechanical-statistical way based on the hypothesis that heat is a molecular vibration, and the phenomenological way in which energy-transfers between heat and work are considered as facts of experience. Although the physics community has long embraced the mechanical view, the irreversibility in the second law has never been adequately explained, and there are

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[5] Velocity-squared, the level-value for vis viva is also an absolute due to the absolute speed of light.

[6] (CE), p. 36

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still reasonable arguments against the identity between the heat of entropy and disordered motion.

The important question at issue is whether the special properties of heat in irreversible processes are to be admitted as fundamental to the description of the world, like a fundamental energy-kind on par with electromagnetism and gravitation, or whether heat does not play a role in the deep structure of events. Following the phenomenological view, Mach held that thermal properties were more fundamental even than the motion of particulate matter. How one deals with this problem will have a great influence on the version of reality one endorses, especially since a physically irreversible process at the most basic level would be a way to account for the direction of time.

Let us examine the equivalence between heat and mechanical work in more detail. The most obvious way heat can do work is by the expansion of heated bodies. If we think of a gas enclosed in a tube with a moveable piston head, we can heat the gas through a plate in the bottom of the tube and let it press against the piston-head to do work. The analogy between the thermal qualities and the mechanical ones is very exact. It turns out that the quantity of heat (Q) that disappears when the gas pushes up against the piston head corresponds to the greatness of the force (F) that acts to

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raise the weight.[7] The temperature difference between the initial and final temperatures of the gas corresponds to the height traversed by the weight, for a constant area.

If we consider only the possibility that heat produces work by expansion, then using the example of an ideal gas in a piston the facts of thermodynamics can be elegantly described by the Carnot engine. The engine is a single piston capable of moving without friction from a high temperature source of heat to a low temperature sink in such a way that the theoretical maximum of work done by heat is realized in a reversible way. Although heat is forced to flow from one place to another by means of pressure, no uncompensated energy transfers occur. Bodies at unequal temperatures never come into thermal contact, and all the variations of temperature that occur are consequences of variations of volume (the piston head is always pressed down on the gas with a force equal to the force of expansion).

We consider a high temperature source T_1 and a low temperature sink T_0. For these two bodies with very great (practically infinite) capacities for heat are used. Between them is an absolutely non-conducting insulator. The Carnot engine is obviously an ideal process or limiting case, rather

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[7] The force in this (reversible) case will not accelerate the mass because it is offset by a force pressing down on the piston head that increases with it. Were the heat to accelerate the mass on the piston head, only some of the heat would be converted into work, the rest would be lost.

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than a real experiment.[8] Mach was quite aware of the need for idealization in physics[9] and encouraged it for clearly picturing and isolating facts in thought.

We start out the piston on the high temperature source T_1. The gas in the piston is already at temperature T_1 and raises the piston head uniformly by absorbing heat at that temperature from the heat source. The pressure on the piston head is modulated such that the gas can naturally expand without accelerating the weight. The gas would ordinarily cool as it expands, but because it takes a quantity of heat Q_1 (an energy-value *Menge* Q_1/T_1) from the high temperature source, it remains at the same temperature as heat flows through it. This is called an *isothermic expansion*.

Now we move the cylinder over the insulator. The gas is allowed to expand further as the pressure on the piston head is relaxed, but this time it is allowed to fall in temperature to T_0 without a change in volume (this is called *adiabatic expansion*). When the piston has reached the temperature of the low temperature sink T_0, we move it over to the sink and press down the piston head until the original volume is restored. Ordinarily this would cause an increase in temperature but since the heat is allowed to flow out through the bottom plate to the low temperature sink, the temperature

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[8] (H), p. 203.

[9] (UDM), p. 355.

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remains constant and a second quantity of heat Q_0 is expelled there (a *Menge* Q_0/T_0). This is an *isothermic compression*. Finally we move the cylinder back over the insulator and keep the pressure on without decreasing the volume, this time allowing the temperature to increase. This is called an *adiabatic compression*. According to Carnot, the final temperature and volume reached are the same as the initial temperature and volume with which the cycle began, thus the process can be repeated.

That the end state of the cycle is the same as the initial state involves an extra assumption. But if all the stages of the cycle are states of equilibrium, thermal or mechanical, it can run backwards as well as forwards, using work done on the piston to transfer heat in reverse from the low temperature sink to the high temperature source.

With the help of the principle of excluded perpetual motion, Carnot proved that the heat's capacity to produce work in the cycle did not depend on the chemical nature of the bodies the heat flows through, but only on the temperatures. The bodies are homogeneous abstract capacities for heat independent of their chemical differences. If this were not true and, for example, mercury vapor had a higher yield of work from heat than air, then we could use the air piston to generate work by letting down a quantity of heat and then use the work produced to run the mercury vapor piston in the opposite sense, replacing all of the heat let down by the air

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piston back to its original source, and still have additional work to spare to power a generator. Thus the work done cannot depend on the bodies used, but only on the temperatures.[10]

Now Carnot originally thought that the quantity of heat Q_1 that flowed into the piston at the high temperature source was the same as the quantity of heat Q_0 expelled at the low temperature sink, the way that 100 gallons of water does work by falling over the paddles of a water mill while remaining the same 100 gallons. There is a logical gap in the argument just here. Later J.P. Joule and J.R. Mayer demonstrated that a quantity of heat actually disappeared in becoming work. For example when a steam engine was driving a locomotive, it was noticed that less heat was passed to the condenser (the low temperature source) than when the engine was idling.

Clausius finally cleared up this problem. He said that of the heat Q_1 absorbed by the piston at the high temperature source T_1, some of it is transformed into work and *disappears* while pushing up the piston-head. The quantity that remains, Q_0, is expelled at the low temperature sink T_0. There are thus two parts of the absorbed heat to consider: the heat transformed to work, and the heat that is let down from the high to the low temperature level, which Clausius called the entropy. As it turned out, the amount of heat converted to work is always proportional to the amount that is let down.

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[10] (H), pp. 202-204.

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And for the reversible Carnot cycle consisting of a sequence of equilibrium states, they are equal. This fact is written:

$$-(Q_1-Q_0)/T_1 + Q_0\ (1/T_0 + 1/T_1) = 0$$

Where Q_1, Q_0, T_1, and T_0 have the significance given above).[11]

In a carefully controlled reversible cyclic process consisting only of equilibrium states, the heat of entropy, $Q_0(1/T_0-1/T_1)$, is equal to the amount of heat that was converted to work in the piston. By the reverse application of that work, the heat of entropy can be made to flow backwards from the low to the high temperature--which heat would never do spontaneously. In cases where the entropy increase is greater than the work performed, however, the reverse application of the work will not be enough to reset the process to its initial state.

Mach pointed out that when heat is confined to reversible Carnot cycles it behaves similar to the other energies, with a potential and potential level. He announced this in a journal note of 1870, which he reprinted in the *Conservation of Energy*:

> If a certain potential value P of an agent at the potential level V passes over into another form--for example, if the potential of an electrical discharge is transferred to heat--then another potential value P′ of this same agent sinks simultaneously from the higher potential level V to the lower one V′. And the said

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[11] (H), p. 284.

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values are connected with one another by the equation:

$$-P/V + P'(1/V'-1/V) = 0.$$[12]

Mach considers only those thermal processes where the entropy analogue $(P'(1/V'-1/V))$ is equal to the energy transformed $(-P/V)$. Here it is true that the entropy $Q(1/T_1-1/T_0)$ is analogous to the potential energy function of mechanics (W/V). It has the same dimensions and can be differentiated like $U(x,y,z)$, to give quantities similar to mechanical forces $\partial U/\partial x$, $\partial U/\partial y$, $\partial U/\partial z$ that change a thermodynamic system--Pierre Duhem was the first to show this.

But as we mentioned, whenever heat flows spontaneously between unequal temperatures--or otherwise falls in its potential to do work for example by the spontaneous expansion of a gas out of the piston--the process cannot be reversed without expending additional work from outside the system. Other forms of energy do not behave this way. Even when equilibrium is not maintained, the other forms can be returned to their initial states with the same amount of work that was liberated by displacing them, like the pendulum bob, which, without resistance, would attain the same height from which it fell.

Mach's Critique of the Postulates of Clausius and Lord Kelvin

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[12] "Eine Bemerkung über den zweiten Hauptsatz der mechanistischen Wärmetheorie" *Lotos* 21 pp. 17-18 (trans CE p. 86).

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Phenomenological thermodynamics rests on the postulates of Clausius and Lord Kelvin. But Mach claimed that in fact these postulates add nothing to what we already know by analogy from the behavior of other energies and the ideas of conservation and sufficient reason contained in the classical excluded perpetual motion principle:

> The mere exact knowledge of the conservation of energy is sufficient to obtain the theorem of Carnot and Clausius; and because this theorem holds for the different forms of energy a special position for heat is not conditioned by this theorem.[13]

For Mach, the theorem of Carnot and Clausius, which includes reversible processes, does not entail any special irreversibility in the spontaneous flow of heat from hotter to colder bodies, which he of course acknowledged. For Mach even the univocal flow of heat from hotter to colder bodies is not yet a new phenomenon from the univocal acceleration of a body or the spontaneous attractions and repulsions of electric charges and magnets, if we are simply talking about the possibility the heat could spontaneously start a perpetual motion. These ideas are also contained in the sense of the original postulate of excluded perpetual motion, where all spontaneous and uncompensated rises in potential are ruled out. Even the entropy has an analogue that holds for the reversible energy forms: there is gravitational entropy, electrical entropy and so on.

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[13] (H), p. 310.

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Postulates of Lord Kelvin and Clausius

Now Lord Kelvin's postulate stated:

> It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of surrounding objects. If this axiom be denied for all temperature it would have to be admitted that a self-acting machine might be set to work and produce mechanical effect by cooling the sea or earth with no limit but the total loss of heat from the earth or sea, or, in reality, from the whole material world.[14]

Clausius' postulate stated that heat cannot be made to flow from a colder to a hotter body without adding work from the environment.[15]

Mach remarked that as soon as we regard the flow of heat connected with the difference of temperatures in Clausius' postulate as the stated source of work in Lord Kelvin's postulate, the two are equivalent. Kelvin's postulate says that we cannot extract work from the heat of a body that is cooled *below* the temperature of surrounding bodies since there is then no way heat can do work by flowing away from this body to a colder body. Clausius' postulate says in effect the same thing, that heat will only flow spontaneously from hotter to colder bodies, and we must do work to make it flow backwards against its natural tendency.

Both postulates try to cover artificially reversible Carnot cycles and irreversible spontaneous flows of heat, and

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[14] (H), p. 264.

[15] (H), p. 261.

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for that very reason they do not distinguish between these cases. According to Mach, the use of an excluded perpetual motion "of the second kind" does not in fact mark the distinction between reversible and irreversible processes.[16]

The excluded perpetual motion principle is indeed mentioned, but in its conventional sense. Heat is somehow allowed to flow, or be extracted at no cost, from colder to warmer bodies, whereupon the heat can then be made to do work by letting it flow spontaneously back to the cold source. Mach claimed that this was exactly like imagining that a mass should spontaneously rise without absorbing any other work from the environment. The mass could then fall and we would be that much better off for the work gained. That is exactly what Huygens and Lagrange were trying to show when they proved the principle of virtual velocities: that masses always spontaneously sink and never rise.

But as Mach points out, *all* natural potentials produce work by transforming a part of their energy to other forms and by letting down what remains from higher to lower potential levels. As a mass falls for example, it transforms some of its potential energy to vis viva, as the mass itself is let

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[16] To be completely fair, Lord Kelvin's postulate does seem to me to suggest a new "second" kind of excluded perpetual motion argument, which some have called the third law of thermodynamics. Suppose that there was no minimum temperature. Then we could always extract heat by cooling the surrounding bodies and obtaining energy from them. All of the heat seemingly lost for work could then be made to flow to a colder source and generate again that lost work.

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down to a lower potential level, analogous to the entropy of heat. And once it has been let down to the lower potential level, quite naturally the agent has less capacity to perform work than it had before.

But seeing it Mach's way for the moment, which of these postulates, Clausius' or Lord Kelvin's, would be violated by the supposition that the spontaneous flow of heat was a reversible process. Supposing heat should generate some work by spontaneously flowing from warm to cold bodies, letting down a heat of entropy, and then that the reverse application of the work should then be able to make all of the heat of entropy flow backwards from the cold to the warm body? Since replacing all of the heat exactly cancels the work done, there is no danger of constructing a perpetual motion machine.

Thus from Mach's point of view, the two postulates as they stand do not illuminate any special disanalogy between heat and other energy forms. One would have to add that once heat has flowed spontaneously from hotter to colder temperatures, and done thereby a certain work, it is impossible by inanimate material agency to replace the heat back to the hotter source without an *additional* expenditure of work from the environment.

Still, the excluded perpetual motion of the second kind does call attention to an idea contained in the classical principle: the impossibility of getting access to energy in one form without the spontaneous fall in potential level that

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accompanies the transfer of energy from one form to another, the so-called sufficient reason. For Mach the classical principle of excluded perpetual motion included this idea.

The Disanalogy between Heat and other Energies

In reversible processes, Mach noted that energies can fall in their potential level to produce work only when they transform and raise others up. Thus the abstract amount of "potential to do work" is always preserved in a different form. If electrical discharge lifts a weight, the electricity falls in potential level, but the weight is higher and can fall by transforming to vis viva. Mach said that heat was different because it could fall in its potential to do work without raising up *any other* potential level in nature, and therein lies the rub.

> Every transformation of a kind of energy A is connected, in the case of heat as well, with a fall of potential of the kind of energy considered. But while, for other kinds of energy, with the fall of potential a transformation is inversely connected--and consequently a loss in energy of the kind which falls in potential level--heat behaves in another way. Heat may suffer a fall in potential without experiencing a loss of energy--at least according to the usual way of measuring it. If a weight sinks, it must necessarily generate kinetic energy or heat or some other energy. An electric charge,too, cannot undergo a fall of potential without a loss of energy, that is to say without a transformation. Heat, on the other hand, may be transformed with a fall of temperature to a body of greater capacity though the energy of heat remains the same.[17]

Mach added that the entropy mentioned in the second law is

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[17] (H), p. 311.

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only capable of an increase in the case of heat-energy--entropy increases (increases of Q/T) denote the heat's lowered potential to do work.

> We have for the simplest reversible Carnot's process, whatever the form of energy $-W_1/V_1 + W_2/V_2 = 0$...While the energy value of a heavy mass, of an electric charge, and so on, sinks as the height of level demands in proportion to this height, this is not the case with heat. Indeed to take an extreme example, the level may sink on mere conduction and without variation of the energy value so that $W_1=W_2$. Since $V_1>V_2$, the entropy increases in this process. Thus although an analogue of entropy can be set up for every kind of energy, this quantity is only capable of increase in the case of heat.[18]

Mach had a very surprising reaction to these special properties, stemming from the economy of thought. Through an increase in entropy that cannot be reset, Mach held that we must consider not only the potential to do work, but the energy itself irrevocably lost, in effect destroying the analogy between the entropy of heat and other quantities of energy, except for reversible processes:

> The measure of energy rests on the fact that we can make any physical reaction vanish and replace it by mechanical work, and conversely. But is there any meaning in attributing a work value to a quantity of heat which cannot be transformed into work? Accordingly it seems that the principle of energy, just like any other substance view, only holds for a limited domain of facts, and over this limit we are only too prone to stray. I am certain that a doubt as to the unlimited validity of the principle of energy will seem as surprising at the present time as a doubt of the constancy of the quantity of heat would have seemed to the followers of Black. But we must reflect that every dominating theory tends to extend its dominion beyond what is proper...The principle of energy is a special form of viewing facts, but its

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[18] (H), p. 312.

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domain of application is not unlimited.[19]

For Mach to turn against the conservation of energy, and appear as an apostate of "energetics" and its view of energy as the ultimate *Urstoff*, must have shocked Ostwald and Helm who were struggling to establish energetics in the scientific community of the 1890s when Mach's *Wärmelehre* appeared. It was not a popular book even among Mach's acquaintances.

It's a curious thing to say that heat should lose its energy. The heat of entropy is considered an energy at a lowered potential level that cannot be restored without absorbing additional energy. However it is still a heat and can flow to a still lower temperature doing some work that way; but this will run out when absolute zero is reached. Mach's objections have more to do with a lack of complete intertranslatability between heat and all other energy sources.

Mach versus Planck on Irreversible Processes

Max Planck--with Ostwald and Boltzmann one of the foremost authorities on thermodynamics in the late nineteenth century--conducted a polemic with Mach over the *Wärmelehre* and other issues. According to Planck, Mach regarded all entropy increases as the letting down of a quantity of heat from a higher to a lower temperature. Planck emphasized that there

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[19] (H), pp. 318-319.

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were irreversible processes in which a working substance lost its potential for work but did *not* fall in temperature, for example the expansion of a gas out of a tube or the diffusion of two gasses. As Mach wrote in the *Wärmelehre*, however these processes are indirectly comparable with heats of entropy because they, too, cannot be reset without an extra expenditure of energy. The criterion is very vague however. What work is lost if I let cream spontaneously mix with coffee?

For his part, Planck thought entropy increases more fundamental than changes of potential level for work, although that was one way to understand them. These processes indicated for Planck a preference in nature for certain final states of processes over their initial states, and thus implicitly played the role of a principle of sufficient reason, which Mach had seen in the excluded perpetual motion principle. According to Erwin Hiebert:

> Planck stood firm in the view that there are irreversible processes in which there is no change in temperature. He insisted that the essential meaning of the second law simply rests on the fact that many different kinds of irreversible processes really do exist: the conduction of heat; the free expansion of gasses; the freezing of supercooled water; the condensation of superheated steam; explosive processes; chemical processes...in fact every change of a system moving toward a more stable condition of equilibrium.[20]

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[20] Hiebert p. 45., Cf p. 41 and the following quote from Planck:

> Es gibt irreversible Processe, deren Endzustand genau dieselben einzelnen Energieformen aufweist wie der Anfangszustand, wie z.B. die Diffusion zweier

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It seems that for Planck, entropy increases for heat were a new phenomenon--only superficially related to potential level--which belonged within a larger class of irreversible processes, despite the fact that they sometimes involved temperatures. The question concerned the existence of irreversible processes in nature, of which the flow of heat from warmer to colder bodies was one, and the diffusion of gasses was another.

Questions of interpretation often clouded the meaning of Mach's and Planck's exchanges. I do not see why Mach could not have agreed with Planck, as he said he did,[21] that there was a primary distinction between reversible and irreversible processes.

The issue is actually quite complicated and requires more thorough study than I can give. Let me merely illustrate a few points. Mach argued in the *Wärmelehre* that when the heat of entropy increased beyond complete recovery from the final states, it was no longer even related to energies, potential

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vollkommener Gase, die weitere Verdünnung einer sehr verdünnten Lösung. Ein solcher Process erzeugt keine merkliche Wärmetönung, überhaupt keinen endlichen Umsatz von Energie (eine "Disgregationsenergie" gibt es nicht), er geht nur desshalb vor sich, weil ihm eine Entropievermehrung entspricht. Man könnte hier also in voller Analogie von einer "Zerstreuung der Materie" reden, und hätte damit nur eine andere Seite der Sache bezeichnet. Allgemein erschöpfend aber kann der Inhalt des zweiten Hauptsatzes nur durch den Begriff der Entropie ausgedrückt werden, deren Wachstum das allgemeinste Maass der Irreversibilität bildet.

[21] "Leitgedanken..." (DL), p. 136.

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levels and so forth, as it is in the reversible case. Mach closed the *Wärmelehre* with what many probably took at the time as a cynical remark, but which perhaps reveals his historically guided wisdom:

> The following are the principal results of the foregoing investigation. The various energies show in their behavior a likeness which has its historical grounds in the fact that the heights of level were measured in units of mechanical work. With respect to the energy of heat, however, this likeness is owing to a historical chance. Besides this agreement the energy of heat deviates from other energies in the fact that it can undergo a fall of potential without a decrease of energy, and that the zero point of the level cannot be chosen arbitrarily. The principle of energy consists in a special form of viewing facts, but its domain of application is not unlimited.[22]

Mach appears to be suggesting that heat represents a different phenomenon that shows its differences by its irreversibility and inequivalence to other energies.

Mach often remarked that heat was a process that ran deeper than the superficial historical comparisons between energies that had guided inquiry. The lesson Mach wished to draw from thermodynamics was that the science had been pursued from only one side--the interface with mechanical energy--but might be pursued from another side, the hidden side that energetic conceptions did not reveal, perhaps because of their irrepeatable nature and thus unsuitability for economical picturing in thought like reversible processes.

The conceptions "quantity of heat" "weight of heat"

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[22] (H) p. 319.

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> "equivalence value" and "entropy" must be carefully held apart just as the domains of fact for which they were established.[23]

He had clearly chosen his own theory of economy and irrepeatability of natural events over the continuities that bound them together in thought.

For Mach as for Planck, heat's irreversible spontaneous flow from higher to lower potential levels seems to have marked it out as belonging to a different class of phenomena, perhaps more closely related to irreversible processes in nature such as diffusion than to the reversible transformations of energies. The entropy of heat just happens to overlap the two classes as a matter of historical circumstance, since mechanical energies were investigated first. Entropy-increases half belong to the class of thermal phenomena, with drops of temperature, and half belong to the general class of irreversible phenomena, which cannot be reset without energy expenditures and thus are metaphorically--but only it seems metaphorically--at a "lower" potential level for work.

Mach on Irreversibility

Mach had one more surprise up his sleeve strongly hinted at in the *Wärmelehre*: to interpret *all* natural processes as belonging to the irreversible class--which even Planck would

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[23] (H), p. 318.

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not do--and to interpret reversibility, and indeed energy itself, as an economical abstraction from irreversible components.

> It is to be remarked that even processes which can be reversed all contain an irreversible element in the velocity, acceleration, and so on--, namely time.[24]

For Mach, in the isothermic expansion and diffusion of gasses the velocities and accelerations of molecules point at a deeper irreversibility in natural processes in the direction of the velocity and the time. Mach even believed that the ultimate processes in nature could be thermal.[25]

The increase in entropy was seen by Mach as one of the foundations for the representation of time in all irreversible natural processes:

> For me Time-Space questions are essentially physical questions. The riddle of time will be solved, I believe, through the correct conception of the second law.[26]

Now I believe we have arrived at Mach's finalized view: that the time-variable in all natural processes conceals an irreversible energy consumption, as opposed to the superficially reversible processes.

This view has rather far ranging consequences. For example, even energy transformations described by the first law with equal initial and final states are an economical

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[24] (H), p. 489 Note #25 to chapter 19.

[25] (CE), pp. 94-95.

[26] Mach to Adler Letter (AU) #37 p. 56.

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abstraction and not the true face of the natural processes. Even seemingly reversible energy transfers which appear to run backwards and forwards indifferently are derivative from the irreversible processes, as on Mach's view static phenomena were derivative from dynamic phenomena. Just as statics turned out to be a special case of dynamics, reversible dynamics of energy turns out to be a special case of irreversible dynamics.

The Second Law as a Principle of Sufficient Reason

Reversible energy transfers do present something of a paradox: since their initial states always equal their final states, why is there any reason for say masses to accelerate instead of electrifying, why one form of energy rather than another? And if there is as much reason for one as another why doesn't nothing at all happen instead? This kind of thinking often leads a philosopher or scientist to seek for a principle of sufficient reason. It led Mach to seek for the principles of statics in dynamics, for example. He regarded statical cases as dynamic forces in equilibrium, such as two weights accelerating equally on equal lever arms.

Mach included the idea of unequivocal determination in the classical perpetuum mobile principle, although as we know he made an explicit distinction, for example between the *Beschleuningungsprincip* and the *Arbeitsprincip*. Unlike the classical principle that work cannot be created from nothing

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or without compensating work elements, the second idea is that a process of transfer cannot be determined without some asymmetry in the initial conditions. If all energy transfers were in fact undetermined as to their initial and final states not only could they all be reversed, there would be no unique reason for them to happen in the first place. The *Beschleunigungsprincip* is of this kind as well as the principle of least action derived from it. The spontaneous flow of heat from hotter to colder sources was also supposed to be of this kind. Mach even remarked on Boltzmann's "beautiful discovery"[27] that the second law actually corresponded to the mechanical law of least action.

Mach identified the perpetuum mobile principle and the principle of sufficient reason in his *Conservation of Energy*, without mentioning the distinction between reversible and irreversible processes. His argument there, as in the *Mechanics*, was confined to statics. A level can only be seen as a case of symmetry with identical weights on identical lever-arms, if we already know that these two factors are the causes of motion. But what about dynamical processes where we discuss not just equilibrium but actual transformations of energy, accelerating weights, flowing heats? Here again it is possible to imagine setting the processes against one another as in a Carnot engine, so that we have equilibrium again of a reversible kind.

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[27] (H), p. 334.

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Let us extrapolate this view of things where the processes of nature are ruled by sufficient reason as primary. It is possible to imagine cases of energy transfer in which there is no increase of entropy but no equilibrium either. These are all of the energetic processes that do not involve heat. This is a sort of second-stage of reversible dynamic processes. Finally there come all of the irreversible processes, determined as to their initial and final states. It could indeed be that Mach would have identified these processes as the genuine cases of motion, even of reversible motions, just as the dynamical causes of motion are recognized as the factors determining static equilibrium. A reversible dynamic process is thus one in which the irreversible factors that cause motion in a determinate direction offset one another. That a basic "chemical" asymmetry remains in nature is evidenced by the fact that any process is able to happen at all.

This view of all of the processes of the world perhaps signals to Mach that in nature the irreversible factors compensate one another just enough to make possible an economical, symmetrical picture of reversible energy transfers and the conservation of energy, without there actually being any such. The use of the excluded perpetuum mobile as a principle of sufficient reason is thus the *more* fundamental sense of the original postulate and even overrides the conservation of energy itself.

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The principle of excluded perpetual motion would finally reduce to an assertion of a supreme underlying asymmetry in the initial states of *all* processes in nature that gives them a sufficient reason to transform to their final states. The final states never contain more potential than the initial states but may contain less. The classical excluded perpetual motion principle should then be interpreted to mean that two irreversible energy transformations that each have a sufficient reason may offset one another and give the appearance of a reversible process.

The Mach-Planck Polemics: Some Clarifications

In phenomenological thermodynamics the canonical view, under the influence of Planck and Ostwald, became to assume two perpetuum mobile postulates. These correspond roughly to Mach's own division between the *Energieprincip* and the *Beschleunigungsprincip* in mechanics. The first is the classical principle that work and its equivalents cannot be created from nothing. This principle says nothing about the direction of energetic processes from A to B to C or from C to B to A. Therefore, energy transfers may run in any order and there is no reason for a process to proceed in one direction as opposed to another. Planck wrote:

> The second law does not hatch out (*auskommen*) of the concept of energy, in no way is it exhausted by the first law, in that one analyzes every natural process into a series of energy changes and now asks after the direction of each individual change. Granted, one can mention by name the different types of energy in each individual

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> case which reciprocally replace one another; for indeed the principle of energy must always be fulfilled. But there always remains a certain arbitrariness, how the causal conditions of the transformations are expressed, and this arbitrariness cannot be avoided through any general rule.[28]

The excluded perpetual motion of the second kind is, according to Planck, a logically independent principle which forbids the uncompensated production of work from heat, except in reversible cyclic processes, but not the uncompensated production of heat from work. Thus it picks a univocal direction for energy transformations between heat and other forms of work.

Now of course Mach and (the early) Planck agreed in making irreversible processes the most fundamental.[29] We can see why, for Mach, Planck's critique would not have forced him to change his views. Planck complained about this in his reply "Mach's Theory of Physical Knowledge--A Reply":[30]

> He speaks very often in his book about perpetual motion, but he does not attach any clear cut physical meaning to this expression. He continually confuses perpetual motion of the first kind (production of work from nothing) with perpetual motion of the second kind (uncompensated production of work from heat)...Mach does not even devote a syllable to the fact that the two basic principles about the impossibility of perpetual motion

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[28] Max Planck *Vorlesungen über Thermodynamik* (1897) pp. 73-75, quoted in Erwin N. Hiebert The Conception of Thermodynamics in the Scientific Work of Mach and Planck (Freiburg: Ernst-Mach-Institut, 1966).

[29] Planck was to change his position on the second law in a few years under the influence of Boltzmann's statistical conception of the second law.

[30] *Physikalische Zeitschrift* **11** (1910) p. 136.

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> are completely different from each other, that the first is reversible (the impossibility of abolishing work) but the second not (uncompensated production of heat from work being by no means impossible), that the energy principle (the first law) rests on the first and that the Carnot-Clausius principle (the second law) rests on the second, that the second is completely equivalent to the known Clausius principle of the transfer of heat from a lower to higher temperature, and that this principle has as a presupposition that without the assumption of a non-reversible process proof of the Carnot-Clausius principle is not to be managed.[31]

As we have seen, Mach only considered the Carnot-Clausius law to apply to a system in which a reversible cyclic transfer of heat to work occurs, thus only the classical excluded perpetual motion principle is needed. The Kelvin and Clausius postulates can be seen to back this version only, because they include the Carnot cycle and thus cannot be predicated on irreversibility. I think Mach understood the received view that irreversibility was supposed to be guaranteed by these arguments; he just didn't believe it. He called the application of the Carnot-Clausius law to irreversible processes an "extension" of the law beyond its initial scope. And then he restricted this scope by refusing to maintain that the energy of heat was still conserved mechanical energy after an increase in entropy.

The misunderstandings rest on certain verbal distinctions and exactly what is meant by "reversible cyclic process" and the "Carnot-Clausius Principle." The differences in style are interesting: Mach trying to keep the *Spielraum* as wide as

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[31] (DL), pp. 143-144.

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possible, Planck trying to slam the door hard. Planck had a good point that Mach did not clarify especially well the perpetual motion principle in such a way that makes sense of the difference between reversible and irreversible processes--at least not in his published work. Mach should not have assumed that everyone would take the time to re-read his early articles and sift every aphoristic remark.

Mach on the other hand did not understand that Planck was interpreting the postulates of Lord Kelvin and Clausius to apply to uncompensated, irreversible productions of work from heat. For Planck the reversible Carnot cycle is merely a special limiting case of irreversible spontaneous heat flows.

Conclusion

It seems to me likely that Mach's view of time was also connected with irreversible processes, and that he regarded repeatable processes as economical abstractions, for example the equality of initial and final states. The destruction of energy is a special process that runs one way in nature like a fixed difference between right and left, or a molecule whose mirror image never occurs. This is not a disaster for science, it just means, as Mach said, that owing to its special economico-historical circumstances of discovery, we could not have expected the energy law to have an unbounded validity.

It could still be the case perhaps that reversible

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processes exist when irreversible processes offset one another, as the dynamic causes of motion offset one another in a case of statics. Time is thus a real chemical direction for Mach, inherent in the fundamental processes, and as we shall see in the next and final chapter, so are the directions of space. After years as a skeptic, he had in fact found his way to a special brand of realism about space and time.

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Chapter Sixteen: From Intensity to Extension: Mach's Late Realism and the Geissler Argument

Introduction

In this chapter on Mach's late writings about space, our task is to examine the lingering question of a physical multiple manifold made up of qualities. The chapter will include previously unpublished material from the notebooks to uncover some very original observations. After a review of the evidence, I will close this study with my own reconstruction of the Machian ideas considered so far.

Mach deepened his thinking on the problem of space in writings that included his lecture "Allgemeine Fragen der Naturwissenschaft" (Summer 1897), Notebook entries, and his article "Eine Betrachtung über Raum und Zeit" (1910). In this period he returned in some respects to the Herbartianism of his youth, while cobbling together meditations from the physical investigations of a lifetime. To borrow an analogy from painting, these writings are enigmatic late works by the master who had composed the *Conservation of Energy*, and the *Mechanics*.

The "Chemical" Manifold

The new theory of matter Mach had announced in his earliest *Privatdozent* writings finally took the form of a chemical manifold. Mach understood "chemical" reactions as

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those from all natural potentials to all others (from vis viva to heat or from gravitational potential to electricity). Mach was impressed by phenomena that criss-crossed the boundaries of physics,[1] for example, the fact that light could be polarized by passing it through a solution, that various compounds involved different heats of combination, and of course that electricity and magnetism were extremely important in chemistry.

Mach discussed his chemical manifold at length in Chapter twenty-one of his *Theory of Heat*. He started by saying that he believed chemistry more general than physics. While each physical science deals with the change of one property of matter at a time (its electrical charge, its gravity, its temperature), chemical changes involve alterations of all of them. Through a so-called "chemical" alteration a mass of material may change its specific heat, charge, or temperature:

> In physical process one property of the complex changes alone-or at least preeminently; in chemical processes, the entire complex changes.[2]

Because the mass-factors mark capacities that remain constant through chemical changes--especially through the law that elements combine by weight in compounds--Mach remarked there was a nearly irresistible tendency to regard these processes as atomic at bottom, involving the combination of small massy particles.

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[1] (R), p. 232.

[2] (H), p. 328.

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> If, now, chemical processes are conceived as combinations and separations of radically different bodies in themselves unchangeable the tendency to purely mechanical explanation breaks out and spreads further and further until its supporters endeavor even to build up the "qualitatively" different elements out of one kind of fundamental matter. To this attempt the fact that there is in mechanical respects only one kind of mass (or matter) appears to furnish support.[3]

Mach however thought chemical changes were alterations of a homogeneous multidimensional capacity-factor between different chemical potential levels. The chemical potential level for the capacity under consideration is also a multiple manifold of all individual levels.

> May we not imagine that in the domain of chemistry... nothing further takes place than that masses--homogeneous throughout--of one chemical level go over to another chemical level?[4]

The laws of general energy transformation stand above the specific energy transformation explored in each particular science--for example, mechanics as the science of all transformations between potential energy and vis viva.

> From all the foregoing, it appears that chemical processes lie much deeper than physical ones. This also appears in the following way. Physical processes are subject to certain equations which present permanencies of combination or relation of the elements entering into the equation. If a chemical change takes place, those equations are replaced by entirely new ones. The rules which would determine completely the transformation from one system of equations to the other would be the complete chemical laws, and compared with physics, would represent permanencies of a higher order.[5]

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[3] (H), p. 328.

[4] (H), p. 329.

[5] (H), p. 332.

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This multiple manifold of the different natural potentials and level values was Mach's answer to a difficulty that had troubled him since the 1870s, and which we have touched on in previous chapters. In the *Conservation of Energy* he had proposed that velocity was a potential level for bodies exchanging vis viva through collision. However dimensional analysis revealed that velocity, a vector, could not give a scalar potential value. We recall that potential functions give scalars as their values, while vectors like velocity and acceleration are triples and require three independent numbers to represent their properties. Originally Mach had proposed to break "kinetic potential" into M as capacity factor and $V^2/2$ as the intensity factor.

Mach finally sought to evade the problem of vector quantities altogether by giving each different direction in space a qualitatively different physical energy.[6]

> Physical space also has three essentially different directions, which are most clearly manifested in a triclinal medium, in the behavior of an electromagnetic element.[7]

Consequently, he believed the independent components of any vector quantity in space represented three different *chemical* directions or processes. The vector nature of certain physical quantities is mathematically fundamental to their representation in space, but this is not the fundamental

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[6] (H), pp. 331-332.

[7] (AS), p. 339.

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physical description. It is of course possible to imagine destroying the velocity of a moving body with three independent energies (heat, electrical charge, gravitational potential) to annihilate its three components separately. And it is possible to show that it also takes energy to change the components' relation to one another, i.e., changing the direction of the vector. In this sense, the three components are indeed three separate but interdependent energies. But this only confirms the mathematical representation of velocity as a triple-something. Hcwever, it is still not clear why the three components of a moving body cannot *all* be mechanical energies (*vires vivae*). Where do the other potentials appear in a purely mechanical problem?

Mach often stressed that there were no purely mechanical problems. He viewed mechanics as a superficial science that ignored the fact that the natural processes always involve heat, electricity, magnetism and gravitation at once. For example the rigidity of bodies in mechanics is partly due to inner electrical attractions. That mechanical problems do not exist is a rather amusing claim to discover in one of the most distinguished treatises on the science of mechanics. Further, Mach thought multiple energies were in play in any example we might choose, even if we are forced to look to the very smallest elements of rigid bodies to find them. So perhaps he would have said that three mutually independent spatial directions in purely mechanical problems are a sign of other

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deeper energies ignored in the problem, or too small to influence it. Triples would then be basic for the restricted study of mechanical problems but not for an expanded physics. This was why Mach proposed to expand his consideration to a science of all energy transformations instead. For example on page 332 of the *Theory of Heat*, he says:

> Just as at present in stereochemistry we endeavor to explain chemical relations by spatial relations, it is quite possible that we may some day attain to the understanding of space, its dimension number and so on, by chemical paths.

Another interesting feature of the chemical manifold is the type of transformations between energy levels that occur. While the equilibration of a hot and a cold body embraced one property of temperature represented on a continuum, chemical equilibration for Mach was discrete and discontinuous embracing sudden changes between several properties at once. That suddenness and discontinuity, as well as the special conditions for energy transformations, was, for Mach, the mark of the chemical. Na and Cl must combine in an exact proportion, and the amount left over will not combine at all.

> This fact obtrudes itself, so it seems to me, as such a striking difference from what takes place in physical processes that to close our eyes to it and to deceive ourselves by ingenious assumption about its significance can hardly commend itself to us.[7]

Sometimes the level values will depend on each other. In other cases, level values of gravitational potential may be more closely related to electrical level and may not go over

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[7] (H), p. 330.

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into vis viva at all. In those cases the gravitational potential and vis viva of a body will be independent. All natural potentials can thus be dependent or independent of one another depending on the particular circumstances.

It also turned out that for Mach the potentials themselves were not the ultimate realities but rather the existing differences and equalizations always active in the phenomena. Probably these raw differences in level were the world elements, with capacities as certain derived invariances that are functions of these differences. This is likely since the quantity factor, or magnitude of the potential W is always a function of the level value V: W/V, or W(V):

> The real world is not the potentials but the differences.[9]

In *Knowledge and Error*, Mach summed up his general physical views in the following way:

> Physical differences determine everything that happens, and the decrease of differences is preponderant in the segment of reality that we are contemplating. Where many differences of the same kind similarly determine events at a point, the determining factor is the mean of those differences. The equations of Laplace, and Poisson which are applicable to so many areas of statics, dynamics, heat, electricity and so on respectively state the value of this mean as zero or whatever else it happens to be. Symmetrical differences with regard to a point determine symmetrical events there; or in special cases of multiple symmetry, an absence of events.[10]

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[9] Die realen Welt sind nicht die Potentiale, sondern die Differenzen. Notebook dated 27 November 1908 (NL 174/2/52).

[10] (KE) p. 357.

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Mach's 1897 Lecture: General Questions of Natural Science

Further details appeared in Mach's lecture "Allgemeine Fragen der Naturwissenschaft" (Summer 1897).[11] There he treated three views of space: the mechanical-atomic theory, the ether theory of W.K. Clifford and his own *chemische Auffassung*.

Mach saw the strongest argument for the atomic theory in the idea that the laws of nature held all the way down to the tiniest particles. Unless the natural processes were uniform down to the smallest parts of matter, like the smallest capacities exchanging heat, or the smallest masses attracting one another by gravitation, the laws would take no regular form. Some small parts would act on one another differently and set up different rates that would never equalize.[12]

Mach also remarked (as he had many years earlier in the *Compendium*) that certain phenomena, like transverse waves, polarization in optics and the radiation of heat in thermodynamics only seemed to make sense when explained as the motion of discrete particles in empty space. "Atomistic mächtiger stütze durch die Thermodynamik," he said[13] (Mighty support for atomism from thermodynamics). Oscillations, heat currents and radiation for example, seem to require space in

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[11] NL (174/1/019).

[12] See Mach's letter to Popper of March 14, 1895, (AU), p. 21.

[13] Ibid.

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which the parts of the body can move against one another. Thirdly, he remarked again on the theory of proportions in chemistry, i.e., that the different compounds are combinations in fixed proportion by weight, and thus the same quantity of matter could be seen moving around under the differing chemical combinations:

> Wonderful proportions. Unexplained. Hence the outward appearance of the atomic theory.[14]

Next, Mach moved to W.K. Clifford's Riemann-inspired theory of matter as little "hills" [*Rünzel*] or the curvature of space, where space is conceived as an Aether-continuum, or stuff. He criticized this theory to some extent.

> The continuous aether gives a very obscure Shearing-elasticity...But what is the *material* of the aether? Doesn't it raise the same problem over again?[15]

He added that the idea of a medium in which the different forces are currents is merely a mental picture, or analogy with sense perception.

> Magnetic currents. Gravitational currents. Just analogies. Pictures. Just as much as heat-stuff. Electricity as fluid, in which nobody seriously believes.[16]

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[14] Proportionen Wunderbar. Unerklärt. Hieraus d. Aussehen d. Atomtheorie. (NL 174/1/19).

[15] Der continüerliche Aether gibt eine schwer verständliche Scheerungs-elasticität....Was aber ist die Aethers*materie*? Fangen die nicht wieder diesselben Probleme an? (NL 174/1/19).

[16] Magnetische Ströme. Gravitationsströme
Nur Analogien. Bilder.
Gerade so gut wie d. Wärmestoff. Elektricität als

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On the whole, Mach's pronouncements on ether theories like Clifford's were extremely ambivalent. He could be admiring of a paper he cited in a late edition of the Mechanics by Paul Gerber "Ueber die räumliche und zeitliche Ausbreitung der Gravitation" *Zeitschrift für Mathematik und Physik* II, 1898, in which gravity was given the velocity of light to explain the advance of the perihelion of mercury. Mach said "This speaks for the Aether as the medium of Gravity." He also contemplated, but never embraced, the idea that the inertial forces on a body are due to a gravitational medium. A body will move at a constant velocity without resistance in a frictionless fluid, but suffers resistance from the fluid on accelerating. Mach apparently thought the idea of an ether acting by contact on an accelerating body was at least preferable to instantaneous forces acting from infinitely distant masses on a body in our neighborhood--if these were the only choices. As we saw in the chapter on the definition of mass, Mach was silent on the question of how inertial forces were propagated from the distant masses to objects.

Mach also opposed ether theories; he pointed out in the *Mechanics* that the medium of electromagnetic interactions was an idealization for him. He praised Hertz's deduction of electromagnetic waves from Maxwell's equations and not from any medium. He claimed physical constants of electricity and

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Flüssigkeit, an die niemand in Ernst glaubt.

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magnetism sufficed for any propagation of light waves in empty space and that the theory of electricity could be developed from the concept of electrical potential and the dielectric constant (potential function), without "the hypothesis of a fluid or medium." Clearly, media and the ether were mental images for Mach, deriving from human sensibility, picture theories based on physiological sense perception.

Mach believed his chemical manifold escaped all of the metaphysical difficulties in the concept of space. When he spoke, for example, of a manifold of "substances, sensations or whatever" in "*Mischungen*," he said he meant the different natural potentials (gravitational, electric, thermodynamic) as they would be found combined in a chemical substance:

> The chemical is perceived through different mixtures, combinations. Consider a row of materials a,b,c,d,e,...n, arbitrarily many. The completed continuous system in all possible relations would be an n-fold manifold. Could be treated analytically. We cannot visualize the n dimensions all at once, but can one after another.[17]

The connection with Riemann is immediate, suggesting Mach's research on the color manifold:

> Gauss. Stimulus. Riemann. Manifolds in general. Analytical geometry. Abstraction. a a^2 a^3 a^4? a^5? $f(x,y,z)=0$, $f(x,y,z,u)=0$, $f(x,y,z,u,v)=0$. Physical manifold so many-fold as the spatial. Given but not

---

[17] Chemie durch in diff. Mischungen vernehmen...
Denken Sie sich um eine Reihe von Stoffen a,b,c,d,e...n, beliebig viele. Das vollständige continuierliche System von Mischungen in allen möglichen Verhältnissen wäre eine n fache Mannigfaltigkeit. Könnte analytisch behandelt werden. Wir könnten die n dimensionen nicht auf einmal anschaulich haben, aber doch nacheinander. (NL 174/1/19)

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necessary for thought...Herbart intelligible space. Riemann. Helmholtz.[18]

Mach's Notebooks

In his notebooks Mach had been pursuing his course for some forty or fifty years. There are periodic references to the chemical manifold and what Mach sought to accomplish by it:

> Encapsulated [shut-in] manifold. Potential differentiation. As in analytical chemistry. Results.[19]

And again:

> Space is composed of three kinds of chemical changes.[20]
>
> For three dimensions. Four elements necessary. Four qualities. Chemical. When the aether manages everything in a moment of action three specific directions. Measure everything with the wave-lengths of light. Quality with magnitudes.[21]

Like Herbart before him, Mach made space the production

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[18] Gauss. Anregung. Riemann. Allgemeine Mannigfaltigkeiten. Analytische Geometrie. Abstraction. a a^2 a^3 a^4? a^5? $f(x,y,z)=0$, $f(x,y,z,u)=0$, $f(x,y,z,u,v)=0$. Physikalische Mannigfaltigkeit so vielfach als die räumliche. Gegeben aber nicht denknotwendig...Herbart intelligibler Raum. Riemann. Helmholtz.

[19] Eingeschachtelte Mannigfaltigkeit.
Potentielle Differenzierung
Wie bei d. analytischen Chemie.
Ergebnisse. Notebook dated 19 August 1884 (NL 174/2/22).

[20] Dreierlei chemische Änderungen sind der Raum. Notebook dated 2 January 1886 (NL 174/2/23).

[21] Für 3 Dimensionen. 4 Elemente nöthig. 4 Qualitäten. Chemisch. Wenn d. Aether alles vermittelt im Moment d. Wirkens drei spec. Richtungen. Alles mit Lichtwellenlängen messen. Qualität auf Grössen. Notebook dated 28 November 1904 (NL 174/2/48).

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of line-elements in time by travelling waves or oscillations like light.

> Explain space by way of time. The lengths of lines by way of lengths of time.[22]

As we have seen, the significance of light as a chemical agent had occupied him since his 1866 "Raumvorstellungen" paper. He believed that light was just one of a class of oscillations "Potentialfunctionströme" propagated from place to place by travelling over differences among the various potential levels, all according to the tendency to approach the mean of neighboring levels:

> Space is a three-fold quality-characteristic just like any other physical quality. Perhaps chemical, perhaps electrical...the spatially closer relationship is also temporally closer. Temporal propagation of the equalization to the mean.[23]

> First of all eliminate space. To get time pure. The smallness of the difference is the measure of time. The diminishment of difference is a determinate goal. Middle...Space reduced to the temporal approach to the mean.[24]

> Analogy heat current. Temperature current. Potential

---

[22] Durch die Zeit den Raum erklären. Die Linienlänge durch die Zeitlänge.

[23] Der Raum ist eine dreifache Qualitätscharacteristik wie jede andere physikalische Qualität. Vielleicht chemisch, vielleicht elektrisch...Die räumlich nähere Beziehung ist auch zeitliche näher. Zeitliche Fortpflanzung des Mittelausgleichs. Notebook dated 20 September 1903 (NL 174/2/46).

[24] Zuerst den Raum eliminieren. Um die Zeit rein zu bekommen. Die Kleinheit der Differenz das Mass der Zeit. Differenzverkleinerung ist ein bestimmtes Ziel. Mittel...Raum auf zeitliches Mittelsnahen reducirt. 20 September 1903 (NL 174/2/46).

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function current.[25]

As in the last quote, Mach saw light and heat as "streams of potential function"--in the case of light, a periodic streaming of the electric potential of the light back and forth over the dielectric constant of empty space.[26] As in Hertz's deduction, no ether is mentioned. Mach wrote earlier that the speed of light had a special constant value because it depended on two fundamental physical constants of electricity and magnetism, in other words, the exchange of two chemical qualities:

> Light as chemical oscillation...Light. The difference will diminish and approach a constant limiting value. Difference-drop tubes. When the velocity falls the difference diminishes. When it rises, the difference enlarges. Otherwise it remains constant.[27]

By the 1890s Mach was holding tight to his idea that the Herbartian means of drawing out lines, surfaces and even delineating bodies in space were given by streams of light and

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[25] Analogie Wärmestrom. Temperaturstrom. Potentialfunctionstrom. 23 November 1878 (NL 174/2/13).

[26] Boltzmann had shown that Snell's law of refraction could be deduced from the values of the dielectric constants in the two substances, i.e., air and glass, alone. Thus, the speed of light really depended on the potential levels being equalized. A mention of Maxwell and Boltzmann's use of dielectricity appeared in Mach's notebook for 1879 (NL 174/2/14).

[27] Das Licht als chemische Schwingung...
Licht. Die Differenz wird sich verkleinern und sich einem constanten Grenzwerth nähern. Differenzfallröhren. Wenn die Geschwindigkeit abnimmt verkleinert sich die Differenz. Wenn sie zunimmt vergrössert sie sich. Sonst bleib sie constant. Notebook dated 17 May 1870 (NL 174/2/9).

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heat radiation. He even wrote in *Knowledge and Error* that outwardly propagating light wave-lengths would make a serviceable objective measure of both time and space.

The "Spatiality" of Space: Physical Capacity and Geometric Volume

Returning to the "Allgemeine Fragen" lecture of 1897 NL 174/1/014 Mach wrote that the invariant abstract capacities of the chemical manifold could be used to give it metrical properties. Mach noticed that both in Wechselwirkungen of temperatures and accelerations, the capacity-factors are constant:

> Because there are fixed capacities, the product $c\tau$ represents something constant in the Wechselwirkung. As does $m\phi$.[28]

He also remarked that energy is a concept of measure most likely to replace spatial congruence in a physical manifold:

> The concept of work is determinative in questions of motion. Energy a universal measure.[29]

This means the equivalences Q/T, Mass/distance, W/V, will support the proportionality of the Q, M, W through all three changes. Mach's notion of a capacity as a fixed equation

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[28] Weil es feste Capacitäten gibt stellt das Product $c\tau$ etwas constantes in d. Wechselwirkung vor. So wie $m\phi$. Notebook dated 26 January 1881 (NL 174/2/17).

[29] Begriff Arbeit in Bewegungsfragen maassgebend. Energie allgemeines Maass. Notebook dated 5 Juli 1888 (NL 174/2/28).

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between elements of a manifold was almost certainly an analogue to Helmholtz's definition of a geometric body as a set of invariant equations between every two points belonging to this body.

> Substance.
> Body consists of a certain connection of circumstances. In white light it is red, for example...At certain distances it casts certain optical images, is felt by touch,...etc. The perspective images in complete dependence, representable by an equation from the position of three points in space. This equation is the only thing that endures. The equation that represents the dependence on one another. The equation subsists between the properties of certain reaction-activities. The equation subsists between the properties, which themselves consist of certain reactions. For example mass is constant.[30]

At long last Mach had selected the capacity factor of transformations as the foundation of the spatial waves passing through the qualities of his chemical manifold:

> Space-conservation plays the same role in geometry as the continuity and conservation of mass in physics.[31]

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[30] Substanz.
Der Körper besteht in einem gewissen Zusammenhang d. Umstände. Er ist bei weissem Licht z.B. roth...Gibt bei bestimmten Entfernungen und Lagen bestimmte optische Bilder, fühlt sich betastend...u.s.w. Die perspectivischen Bilder in voller Abhängigkeit, durch eine Gleichung darstellbar von den Raumlagen dreier Punkte. Diese Gleichung ist das einzige Beständige. Die Gleichung welche die Abhängigkeit von einander darstellt. Die Gleichung besteht zwischen den Merkmalen gewisser Reactions-thätigkeiten. Die Gleichung besteht zwischen den Merkmalen welche sich aus gewissen Reactions-thätigkeiten bestehen. z.B. Masse constant. Notebook dated 6 December 1890 (NL 174/2/32).

[31] Die Raumerhaltung dieselbe Rolle in der Geometrie, wie die Continuität u. Massenerhaltung in d. Physik. NB dated 9 November 1900 (NL 174/2/43).

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Transformations. n Dimensions. Nebengleichung. Homogeneous volume= Capacity
$dx^2+dy^2+dz^2$ dxdydz.[32]

Velocity is a three-fold level-value. Collision just like temperature by conduction. Three-fold manifold. The squared velocity some other characteristic...How does it happen that...always the same mass-relationships are the same in equivalents for every combination, so must universal rules subsist for the potential levels, too. Think this through. All capacities go along proportional to the capacities for acceleration, all other things being equal.[33]

Mach could also write to Friedrich Adler December 27, 1903:

When we speak of the conservation of "mass" or the conservation of "matter" this can only mean one thing: in all changes of a closed system the sum of the mass values of bodies remains constant, relative to a certain unchanging test-body. In matter I see nothing but the proportionality of all physical capacities to one another.[34]

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[32] Transformationen. n Dimensionen. Nebengleichung. Homogenes Volumen = Capacität
$dx^2+dy^2+dz^2$ dxdydz. NB dated 14 July 1907 (NL 174/2/51).

[33] Geschwindigkeit ein Niveau von dreifacher Mannigfaltigkeit. Beim Stoss wie Temperatur bei Leitung. 3 fach Mannigfaltigkeit. Geschwindigkeitsquadrat andere characteristik...Wie kommt es das es...immer dieselben Massenverhältnisse weil in Aequivalente dieselben sind für all Verbindungen, so müssen auch für die Potentialstufen allgemeine Regeln bestehen. Das ausdenken. Alle Capacitäten gehen bei sonst gleichen Zuständen den Beschleunigungscapacitäten proportional. NB 4 December 1891 (NL 174/2/33).

[34] Wenn wir von Erhaltung der "Masse" oder von Erhaltung der "Materie" sprechen, kann das doch nur den Sinn haben: Bei allen Veränderungen eines abgeschlossenen Systems bleibt die Summe der Massenwerthe der Körper, bezogen auf einen bestimmten unveränderten Vergleichkörper, constant. In der Materie sehe ich nichts als die Proportionalität aller physikalischen Capacitäten zueinander. (AU), p. 31.

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Mach's use of capacity as a standard measure of space in the chemical manifold seems finally to have supplanted the geometric congruence of rigid bodies in his mind, as indeed he wrote: "homogeneous volume = capacity." In *Space and Geometry*, Mach argued that physical geometry was founded, in good Helmholzian fashion, on a constant volume-element that moved to and fro over the potential differences of the different coordinates. Physical capacity and geometric volume had become one and the same.

"The Spatiality of Space": Four Machian Theses

By saying that what is real is not the potentials but the differences, I believe Mach would have regarded point-potentials as somewhat ideal compared with the abstract intensive qualities and capacities. For example, he could have conceived points of matter in position as "junctions" between natural potentials transferring level, fixed in place by forces. Where they couldn't transfer due to natural circumstances, of independence or mediate causal interference, an independent gap or a difference would appear.

But what about empty space? It is still hard to explain the brute extendedness of it. Here I believe that Mach was influenced by Herbart's idea that spatial gaps were nothing but the physical independence of certain energies from one another--the greater the physical independence the greater the apparent spatial gap. For example gravity and magnetism can

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transfer potential energy if a body is both a capacity for gravity (a mass) and a capacity for magnetism (such as a lodestone). Otherwise each potential acts as if the other is not there, and a causal independence opens between the two. Another source of gaps are mediate causal relations between the same capacities. Sometimes A can only influence C through intermediate contact with B. Mach dwelled on the idea of causally mediate transfers of energy as the foundation for spatial and temporal extensions in *Knowledge and Error*.[35]

Starting with Mach's lectures of 1897, a pattern of ideas began to emerge from his writings, as to how physics would form the foundation of geometry. I will list these in my own formulations:

1. energy transfers between levels compose points or locations; position in the manifold results from forces confined at a point there, and not by mere existence.

2. independent level-values incapable of transfer, and mediate causal connections between homogeneous energies, are the basis for spatial gaps.

3. line and volume-tracing processes such as light and heat radiation are waves of potential transfer or equalization according to the law of differences;

3. capacities are the constant volume-elements that move from level to level in the chemical manifold, while the level values are the intensities at different coordinates.

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[35] (KE), Chapter 24.

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Intensity differences determine differing coordinate positions as in Mach's color space.

As we shall see, all of these theses suggest the direction of Mach's thought back towards a Herbartian construction of space from multiply dependent-and-independent energies.

Mach's Late Realism

Just because Mach thought extension was a set of causal relations between unextended elements, did not mean he was a relativist about space; nor did he regard it as unreal. It was simply less fundamental than the chemical phenomena he was considering.[36] In Mach's final phase of "chemical space" he adopted a robust realism about properties like direction and extension, even while reducing them to simpler physical factors. Space is real because it represents a real combination of things, without being fundamental to the description of the unextended things that make it up.

Mach's article "Eine Betrachtung über Raum und Zeit,"[37] one of his best, was the culmination of many years of thought on these questions. Here he attempted to bring his ideas on

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[36] A reductionist who regards only atoms as real cannot understand how the macroscopic arrangement of atoms in a body (such as a neutron star) could come to affect the internal structure of the atoms, any more than he could understand how one person's failure to find employment could be affected by the interest rates in the economy.

[37] *Das Wissen für Alle* (Wien: 1910) (PWV), pp. 492-508.

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physiological space and time together with his physical views. Physiological time of memory, for example, he explained by the feeling of energy consumption by the body--the time sensation. Physical time was explained by a fundamental chemical direction of processes. But since physiological time rested on organic processes most aptly compared with combustion and the production of heat, physical, physiological, and psychological time all ran in the same direction. Mach described a steam engine running backwards in time, with the passengers and their memories running backwards. A passenger remembers things that later happen and then forgets the memory altogether. Mach wrote that here we have both a physical and a physiological impossibility. Likewise, the asymmetries of physiological space had always impressed him, for example, the asymmetry of the right and left sides of the body. He remarked that this asymmetry can be used as a convention in physical problems such as the right-hand rule in Ampéres law.

Mach believed the physiological asymmetry of directions and time rested on an actual *physical* asymmetry to be found in the chemical manifold, as well as in the human body. In an interesting way, Mach thus brought physical and physiological space and time into connection. Physiological space and time were no longer completely metaphysical illusions, but contained some extramental reality in their chemical directions. In particular memory and time sensation seemed to be both physically and physiologically real. Mach closed the

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article by suggesting a physiological revival of certain of Kant's a priori judgments about space and time.[38]

Mach relayed his late realistic views of space and time to Friedrich Adler on March 28, 1909:

> For me Time-Space questions are essentially physical questions. The question of time will be solved, I believe, through the correct conception of the second law. As for space, it is perhaps most intimately connected with the fact that one can orient oneself electrodynamically with respect to one's own body; that right and left cannot be exchanged without exchanging either up and down or front and back, in short with the inter-relation of the three dimensions.[39]

The Geissler Argument

Between 1905 and 1910 Mach had discovered an article by the mathematician Kurt Geissler.[40] Geissler held that in a three dimensional system, if we reverse the directions on the x axis transforming $x \rightarrow -x$, in order to keep the relationships between the other coordinates the same (i.e., for a rigid form such as a hand to keep its shape) we will have to reverse the directions of one of the other axes, either hold the y constant and reverse the z, or hold z constant and reverse the y value.

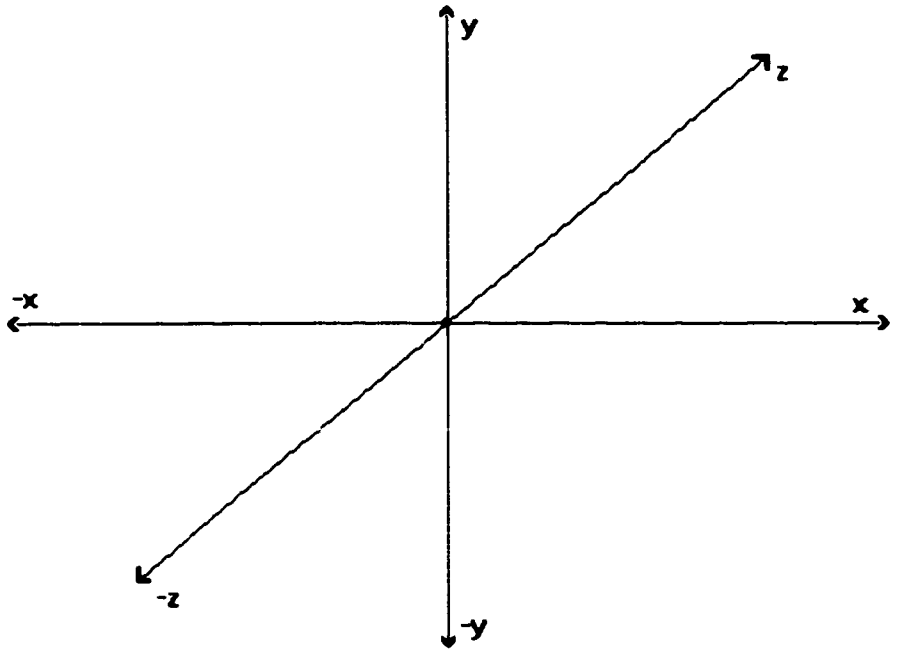
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38 (PWV), p. 508.

39 (AU), p. 56.

40 "Grundgedanken einer übereuklid. Geometrie" *Jahresbericht d. deutschen math. Vereinigung* **13** (1904).

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16.1 The Geissler Argument

Specifically, if we want to hold y constant we will have to rotate the x, z plane around y, taking z ➡ -z. If we want to hold z constant, we will have to rotate the x, y plane about z taking y ➡ -y. This is the reason why a right hand looks like a left hand in a mirror. If there is a giant mirror in the (y,z) plane for example and we stand at x, parts of our bodies furthest from us are closest to the mirror and appear to us at x to be closer. Thus the x directions, reckoned from our standpoint on x, are reversed, but we see this as a hand reversed from right to left in order to keep the other relations the same between the points of the mirror hand and our bodies.

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This fact does not seem to be mathematical or logical in nature, since it is logically conceivable that a set of relations R(x,y,z) between two points could be set up such that a property A (rigidity*)[41] would be unaffected by changes in the values of only one relation or of any of them separately. However that may be, Mach did not appeal to mathematical or a priori truth, but rather concluded that Geissler's geometric dependence of directions really rested on a physical three-fold dependence of chemical directions. I think that Mach was saying implicitly that the three dimensions--the system of underlying chemical qualities--formed the equivalent of a physically closed system for him. The strange simultaneous dependence *and* independence of geometric directions signals a deeper chemical dependence and independence of the qualities that make up the space.

The ideas of Geissler fell on fertile soil in Mach's mind, since he was already gravitating back towards the Herbartianism of his youth. In *Knowledge and Error* Mach even referred to Herbart's intelligible space as the model of a physical space constructed by sheer "quality and magnitude," which he indicated was the right intuition to build on.[42] Of course, he added morosely that such intuitions would take at least a century to develop.

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[41] Rigidity* here being so defined that if a thing passes into its mirror image, it remains the same object.

[42] (KE), p. 349

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From Intensity to Extension: The Space Construction

It seems to me the Geissler argument recalls the homogeneity of directions demanded by both potential functions and distance functions, an identity Mach himself sought to establish in various places. I believe a Machian physical construction of extension would have pursued this course with Herbart as his philosophical guide. What follows will be my reconstruction, which I will call "philosophical space" in order to distinguish it from the actual processes of physics and the ready-made space of geometry.

The different qualities are represented by six potentials: call them {D, V}, {E, M}, {H, U}. Each represents an independent direction in quality that takes on numbers as its intensities. I assume that qualities exhibit opposing directions when they are capable of influencing one another's magnitudes reciprocally, and independent directions when the magnitude of one has no effect on the magnitude of another.

Now any potential of the six may conceivably act on any other, depending on the circumstances at a point considered, not just the ones they happened to be paired with above. D may come to be independent of V and dependent on E, for example. These are what we call the various types of physical problems in mechanics, electricity, optics, etc or cross-boundary problems. Depending on the contingent circumstances, the qualities may pair off and separate in quite complex alternating patterns.

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These alternations must be conceived of as discontinuous and immediate, switching on and off between opposition and independence in a series of freeze-frame instants. Incidentally this is in keeping with Mach's discontinuity theory of the chemical, and his earlier proposal of discontinuous qualitative changes in the *Conservation of Energy*. Thus D and V for example do not slowly pass out of their causal dependence and into dependence on the E and M coming to meet them. The qualities switch partners and take on new ones suddenly. Their individual changes of intensity are likewise sudden and immediate just so long as their end states always agree once the switches are made. We mentioned above that one quality may change while another remains fixed in magnitude and direction, for example (D-V) means that D is fixed and V changes away from it. This means that in the next instant we will find D with the same magnitude it had before, but opposed by some different quality such as M. We will however find V discontinuously altered in magnitude and dependent on another quality that complements its new magnitude, such as E.

I think it is appropriate to mention that because heat is one of these natural potentials, these alternating processes exchanging energy from one form to another really always have a certain chemical direction. Reversible processes will be understood as offsetting irreversible processes. This Machian view sketched in Chapter fifteen is by no means the standard

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view since most physicists do not include heat among the fundamental properties of nature. For our purposes however heat is a fundamental quality like the others, with the difference that heat has no true backwards direction but is produced in one chemical direction only, siphoning off magnitudes from the other qualities.

It looks to me as if a total of six "reversible" potential-directions are required and one irreversible direction, which, since we do not know exactly what it is we will call (T). These will be all we need for the space construction: that is, seven fundamental chemical directions in all. If two paired directions are needed to construct a one-dimensional line, one should say that the "chemical" manifold has three and one-half dimensions and not four.

I. We may now consider the production of progressively more extended magnitudes by qualities in their chemical dependence and independence from each other. Let us start with any two qualities such as D and V. The Not-Together of D and V occurs when their values depend on other potentials in the set. This allows D and V to dissociate, and climb in either the intensity of the D property or the V property.

Suppose that E and M are the properties that D and V depend on in gaps. In this combination of four properties {D,V,E,M} we could think of E and M as dependent on each other when not dependent on D and V, which, collecting up both

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processes, gives us two interdependent lines.[43] These are the lines Herbart discussed in his *Metaphysik*, traced out in two directions:

Line 1:

"Right": (D+V) (V-D) (D+V) . . .

"Left": (V+D) (D-V) (V+D) . . .

Line 2:

"Right: (E+M) (M-E) (E+M) . . .

"Left": (M+E) (E-M) (M+E) . . .

For two lines generated together against each other like this, I don't see any way to call them truly independent processes sitting perpendicular to one another; rather the production of the one is rigidly connected with the production of the other, both in the directions of the generating processes and in the sequences of points and gaps.

So let us just say that the two processes tracing out lines DV, VD, EM and ME are interdependent-lines. Every time D and V are together, E and M must be too. When D and V are apart, so are E and M. Each is a set of points set apart from one another, but neither one is truly extended. Rather we have a case mentioned in Chapter five, where we look head-on down the length of a wire and are unable to perceive its

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[43] There are other logically possible cases I leave out of account here such as when D and V switch partners with E and V.

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extension, because we lack a dissociating direction.

II. To dissociate these lines from one another, we bring in the third process in its two directions (H+U)(U-H)(H+U)... and (U+H)(H-U)(U+H).... Now D and V can be independent of one another by passing into dependence on E and M, and also by passing into dependence on H and U. E and M can be independent of one another by passing into dependence on D and V but indeed also by passing into dependence on H and U. To put things more illustratively, if we were sitting on one line, we would be capable of perceiving the extensions of the other two through their dissociations from us.

III. Now we bring in Geissler's observation, which requires that if one of the processes reverses direction from right to left, the second can continue to progress in the same direction only if the third reverses to offset the change in the first. This means for example that if the process in the (D+V)(V-D)... direction is to change to the (V+D)(D-V)... direction, one of the other processes {E,M} or {H,U} must reverse with it. If the process (D+V)(V-D)(D+V) is to change direction this means that instead of (V-D), (D-V) will occur, a totally different physical situation since it involves a fixed D and a changing V, not vice versa. This means the qualities paired with D and V, must also reverse. Either (E-M) must become (M-E) or (H-U) must become (U-H).

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D+V	E+M	H+U
		U+H
	M+E	H+U
		U+H
	E-M	H-U
		U-H
	M-E	H-U
		U-H
V+D	E+M	H+U
		U+H
	M+E	H+U
		U+H
	E-M	H-U
		U-H
	M-E	H-U
		U-H
D-V	E+M	H-U
		U-H
	M+E	H-U
		U-H
	E-M	H+U
		U+H
	M-E	H+U
		U+H
V-D	E+M	H-U
		U-H
	M+E	H-U
		U-H
	E-M	H+U
		U+H
	M-E	H+U
		U+H

FIGURE 16.2

Assuming we have only six potentials to work with it is necessary that when D and V are independent, and do not depend on E and M, they must depend on H and U. Similarly, when E and M are independent, and do not depend on D and V they must depend on H and U.

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IV. Each point is signified by a joining of qualities such as V+D, which indicates that the point is located at a point of transfer between V and D. The gaps are signified by D-V which says that V has departed from the intensity of D, without indicating what its new value is. It is these agreements and differences in the intensity of the qualities that determine everything that happens in the manifold. The existence of the independent gaps means that regular waves of changing local intensity may be propagated from place to place. The quantities carried by these waves are represented by the capacity factors of the transfers that occur at every point of space, and which travel harmlessly through one another in gaps. At points of space, the potential levels at which the transfers of quantity occur are represented by the coordinates at those points, the local intensity of the qualities that meet and fix location there. Indeed all of the processes within the manifold come down to the exchanging of intensities between one quality and another through waves of varying capacity and intensity. These waves will overlap and combine values at points where transfers of energy occur and otherwise they will form spatial independencies between one point and another. For example, seen from the inside if we are riding on a single wave, we will see others where we interact with them, assuming values based on the value of our own wave, and the rest of them will appear as if in blank empty space. The various capacities that move about wave-like

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through the manifold indeed give it an extension, but the processes have not yet been standardized into metric distances.

V. Now for the metric considerations. The manifold possesses a coordinate system already in the sense that there exist processes for getting from any fixed point α_l to any other α_m along a directed path. The value of the coordinate will depend on the process generating the path and the fixed point chosen as the starting point. Because the fixed point is determined qualitatively it is immediately recognizable on inspection. The extension of the manifold depends on how many fixed points and processes are required to get from the fixed point to another point in the manifold.

Collecting up all of the directed paths from every point to every other, we have infinitely many conceivable paths. The problem of the metric is to define an equivalence relation (partition, group) on the paths that are equal in the property of distance, which we now introduce for that purpose. The three requirements that directed paths must fulfill are:

1. **AB** = **BA**
2. **AB** + **BC** = **AC**
3. **AA** = **BB** = **CC** = 0

The endpoints represent paths traced in the direction indicated from the one coordinate to the other. In this type of fundamental construction it will also be necessary to show

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that there are equal paths traversed by processes over different times, and equal paths that involve all possible mixtures of the homogeneous underlying qualities.

For the purpose of equating paths with one another through the above transformations, we will need to find a certain constant stuff that remains the same as it travels through the qualities of the manifold. The basic quantity of the stuff will not be affected whether the process moves forwards or backwards, at different times, or whether it involves combinations of several paths in various orders. This constant stuff, fixed through the interaction of qualities in the coordinate-tracing process, resembles the moveable metre stick that serves as a constant length on translation.

In the case of these energetic paths the constant something is given by the universal capacity factor in energy changes, whether masses, thermal capacity, electrical or magnetic capacity and so forth. Already in tracing out the extended pathways, capacities move between the different potential levels of the qualities, and at different intensities of action. When we metricize this property we are asserting that the path taken by a capacity across a potential difference is universally comparable with all others, no matter what the differences in the individual type of quality and capacity may be. Thus the identity of the capacity factor across all of the energetic transformations serves us as a

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definition of distance, just as Mach standardized the capacity of bodies to accelerate in his definition of mass.

This means in practical terms that wherever a coincidence occurs at a point, a capacity of one kind at one potential level is set equivalent to another capacity of other capacity at an equivalent level according to the principle $W/V=W'/V'$. They are not identical, as they involve different qualities, but they may be seen as equivalent in value, as if the capacity were a certain "quantity" of mechanical energy that retains its potential as it crosses over to electrical energy.

Metric comparisons between different processes are possible wherever they coincide and resist one another through their qualities. But what about qualities that are independent? There we have to rely on the principle that for certain energy values W/V, if $W/V=W'/V'$ and $W'/V'=W''/V''$, then $W''/V''=W/V$, in other words the principle of the conservation of energy becomes our metric principle for defining distances, just as Mach believed.

The manifold of abstract capacities and potential functions is thus replete with waves of equivalent and commensurable capacity-volume, passing over level-values, composing points in matter where they add or coincide, and gaps where they pass through one another. The propagating waves of quality-volume gives sense to the perpetual tracing out of extended *spheres* in the manifold, while the empty space comes to represent a very generalized physical independence or

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inability of two properties to influence one another.

Basic Propositions

The plausibility of the Herbart-Riemann-Mach construction I have considered hangs on four experiential propositions, psycho-physical axioms which I have found articulated in various places in Mach's writings:

1. The existence of world-elements that are homogeneous psychical and physical energies: $\{\alpha, \beta, \gamma, \ldots\}$.

2. The existence of a functional dependence of elements: $f(\alpha, \beta, \gamma, \ldots)=0$, which states that every element must depend on some other and cannot occur alone. This ensures a kind of instantaneous physical co-presence or whole of nature.

3. Independencies exist among individual elements, such that not every particular element always depend on a particular other. This opens the possibility for spatial gaps and extension traced out by travelling oscillations in the elements.

4. Regularly alternating processes (equalizations to the mean according to the law of differences) exist produced by the dependence and independence of elements.

The truth of these propositions rests on experience.

Final Summation

Now at the end of this study we may draw some conclusions about Mach's goals for a future science and see if this footpath merits widening into a high-road.

Among Mach's goals was the demand for a science of "direct experience" in which human beings could enjoy immediate knowledge of other minds, animal sensations and

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perhaps, at a very distant time, direct experience of the first natural world elements themselves. This new science would eschew the schematic, mnemonic forms of repeating experiences in space and time. There would be an end of reversibility and symmetry, except as economical measures, as the schematic forms of physics are entirely replaced by dynamical constructions. In its most extreme sense, the doctrine would imply that each separate experience was absolutely unique in nature. Were it to be granted, this Heraclitean claim lurking behind Mach's writings on economy and symmetry, would open a breach between the most deeply entrenched practices in the sciences, dating to the Pre-Socratics, and a truly uncanny frontier.

At the end of his economy lectures, Mach remarked that in the science of the future human beings would confront the so-called lifeless world directly and with greater sympathy. To this effect he quoted a certain Chinese philosopher Licius, who we may imagine speaks for him, and who, when passing a heap of human bones told his students: "These and I alone have the knowledge that we neither live, nor are dead."

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