Geometrical aspects of linear differential equations over compact Riemann surfaces with reductive differential Galois group

by

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Abstract

GEOMETRICAL ASPECTS OF LINEAR DIFFERENTIAL EQUATIONS OVER COMPACT RIEMANN SURFACES WITH REDUCTIVE DIFFERENTIAL GALOIS GROUP

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Suppose $L(y) = 0$ is a linear differential equation with reductive Galois group over the function field of a compact Riemann surface. We prove that any solution to the equation can be written as a product of a solution to a first order equation and a solution to the pullback of an equation of a special form (a "standard equation"). We classify standard equations using ruled surfaces. We relate the symmetries of $L(y) = 0$ to the outer-automorphisms of the differential Galois group.

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Chapter 1

Introduction

Let k denote the field of meromorphic functions on a compact Riemann surface X, and let v denote a derivation on k. Equivalently, one can regard v as a meromorphic tangent vector field on X. By a linear differential equation over X we mean an equation of the form

$$L(y) = a_n v^n(y) + a_{n-1} v^{n-1}(y) + \ldots + a_1 v(y) + a_0 y = 0,$$

where $a_i \in k$ for $i \in \{0, 1, \ldots, n\}$.

In this thesis we study equations having reductive differential Galois groups.

For the necessary background on differential Galois theory we refer to [21], [25]. For the basic properties of equations with reductive differential Galois group see [8]. For background on Riemann surfaces see [10], [15] or [11].

The thesis is divided into two parts. The first concerns pullbacks, which can defined as follows. Consider a subfield $k_0 \subseteq k$, with $k_0 \neq \mathbb{C}$. If X_0 is a compact Riemann surface with field of meromorphic functions k_0, the inclusion $k_0 \subseteq k$ corresponds to a ramified covering $\pi : X \to X_0$. Let v_0 be a derivation on k_0, and consider a differential equation $L_0(y) = 0$ over k_0, say

$$L_0(y) = \alpha_n v_0^n(y) + \alpha_{n-1} v_0^{n-1}(y) + \ldots + \alpha_1 v_0(y) + \alpha_0 y = 0,$$

where $\alpha_i \in k_0$ for $i \in \{0, 1, \ldots, n\}$, and let $F_1, \ldots, F_n$ a full system of (locally defined) solutions of $L_0(y) = 0$. We say that $L(y) = 0$ is the *pullback* of $L_0(y) = 0$ via π if $F_1 \circ \pi, \ldots, F_n \circ \pi$ is a full system of solutions of $L(y) = 0$. In Chapter 2 we will deal with the question of determining whether a given equation is a pullback. More precisely, given $L(y) = 0$ we ask the question: do such a k_0 and $L_0(y) = 0$ exist? If so, can they be determined? We prove that under the reductive hypothesis the answer to both questions is yes.

> $L(y) = 0$ *is the pullback of an equation defined over its projective Fano curve.*

Closely linked to the study of pullbacks is the study of *projective equivalence*. Assume we are given second equation $L_1(y) = 0$ over k with full system of solutions $f_1, \ldots, f_n$. We say that $L(y) = 0$ and $L_1(y) = 0$ are projectively equivalent if the functions $f \cdot f_1, \ldots, f \cdot f_n$ form a full system of solutions of $L(y) = 0$, where f is the solution of some first order ordinary linear differential equation over k. An early result by Klein in this subject asserts that if $k = \mathbb{C}(z)$, $v = \frac{d}{dz}$ and $L(y) = 0$ is of second-order with finite projective Galois group, then $L(y) = 0$ is projectively equivalent to the pullback of a hypergeometric equation [2] (the projective Galois group is the quotient of the Galois group by its center). This result by Klein has been extended to arbitrary compact Riemann surfaces in [1] and to third order equations in [3]. The collection of hypergeometric equations arising in Klein's theorem can be reduced to a finite collection of "standard equations". Chapter 2 offers an extension of this result to arbitrary order for equations with reductive Galois group.

> *If $L(y) = 0$ has only algebraic solutions then it is projectively equivalent to the pullback of a standard equation over its projective Fano curve.*

Using this extension of Klein's result we can say that the solutions of $L(y) = 0$ are of the form $f \cdot (F \circ \pi)$ for some solution F of a standard equation. This leads to the following question: can one classify the standard equations? We conclude Chapter 2 by proving that

> *Standard equations are classified by ruled surfaces.*

The second part of the thesis examines symmetries of equations, which can be defined as follows. Assume $\sigma : X \to X$ is an automorphism, or equivalently that $\sigma^* : k \to k$ is a $\mathbb{C}$-automorphism. We say that σ is a symmetry of $L(y) = 0$ if the equation $L(y) = 0$ is the pullback of itself via σ. In other words, if $f_1, \ldots, f_n$ is a full system of solutions of $L(y) = 0$ then so is $f_1 \circ \sigma, \ldots, f_n \circ \sigma$. Note that if k is Galois over k_0 and $L(y) = 0$ is, as above, the pullback of $L_0(y) = 0$ via π; then any automorphism σ^* of k over k_0 induces a covering transformation $\sigma : X \to X$ of π which is a symmetry of $L(y) = 0$.

It follows that if σ is a symmetry of $L(y) = 0$ then $L(y) = 0$ is the pullback of an equation over the fixed field k^σ. Symmetries, also called reversibles, will be studied in Chapter 3. In particular, we will link symmetries of $L(y) = 0$ to the outer-automorphisms of the Galois group using the Fano group.

> *The group of symmetries of a standard equation corresponds to the inner-automorphisms of the Fano group that fix the Galois group.*

The Fano group is a generalization of a primitive version of the Galois group used by G. Fano [9], [27].

The main tool in this work is a result of E. Compoint which gives us the algebraic relations among the solutions of $L(y)$ and their derivatives [7]. We recall and generalize Compoint's theorem in Appendix A.

Because the results of the thesis allow us to describe the solutions to $L(y) = 0$ in terms of covering maps, solutions of first order equations, and solutions of standard equations, all these entities can be applied to construct closed form solutions to our given equation. The second appendix, Appendix B, gives some computational applications. For details on the computations involved we refer to [18].

Although the results of the thesis have thus far been given in the language of ordinary linear differential equations, our restriction to equations over compact Riemann surfaces allows for a formultation in terms of connections. In general there are three different, but equivalent, approaches to the study of linear differential equations: differential operators; differential modules; and connections. The equivalence between the three approaches is meticulously covered in [25]. I will assume readers are comfortable jumping from one viewpoint to another. On occasion I will recall specific aspects of the differential Galois theory of linear differential equations, but in general I assume this topic is familiar.

In addition to Riemann surfaces, we will also make use of vector bundles, sheaves, connections, and pullbacks. References for the first two concepts is [11] and [12]; references for the latter two are [23] and [20].

The exposition in Chapter 2 is algebraic-geometric in nature, whereas the exposition in Chapter 3 is analytic-geometric. In particular, local coordinates will play a key role during our analytic approach. Most of the concepts and tools that will be used are introduced in Chapter 2. Some of these concepts will be revisited in Chapter 3. Because of the difference between the two viewpoints it seems appropriate to give equivalent formulations in that chapter using the analytic language. Compoint's theorem will be stated once in each of these two chapters, in accordance with the two viewpoints.

The concept of a meromorphic vector bundle is ubiquitous in this thesis. One may choose to work with holomorphic vector bundles with meromorphic connections, but the formalism would become quite cumbersome in the context of projective equivalences. A short exposition of the concept of a meromorphic vector bundle and its link to that of a holomorphic vector bundle is found in Appendix C.

Chapter 2

Pullbacks and Standard Equations

In [1], [2] Baldassari and Dwork give a contemporary formulation of a result, known to Klein, on second order ordinary linear differential equations with algebraic solutions. The result is most easily stated in terms of projective equivalence.

Theorem 2.0.1. *If an ordinary second order linear differential equation has finite projective Galois group, it is projectively equivalent to a pullback, by a rational map, of a hypergeometric equation.*

An extension of this result for the third order case was obtained by M. Berkenbosch [3], who simultaneously gave an algorithmic implementation of Klein's result which simplifies Kovacic's algorithm. Berkenbosch introduces the concept of a "standard equation" in order to state his generalization. A standard equation is an ordinary linear differential equation which is minimal in the sense that any other ordinary linear differential equation must be a pullback thereof. The purpose of this chapter is to formulate an extension of Klein's theorem that also covers many non-algebraic cases. We treat the problem in terms of differential modules and connections. The author believes that a geometric, rather than algebraic, approach to the problem is much more elucidating.

Our main tool in achieving this extension is a theorem of E. Compoint That result gives a very concrete description of the maximal differential ideal involved in the construction of a Picard-Vessiot extension for the connection. In the first part of the chapter we introduce the geometric concepts involved. In the second we give an algebraic interpretation of these concepts. In the third and final part, we prove the generalization, study some of the consequences for the algebraic case, and introduce the classifying ruled surfaces.

2.1 Geometric considerations

We first establish notation. X denotes a compact Riemann surface, $k := \mathbb{C}(X)$ is the associated field of meromorphic functions, and $\Pi : E \to X$ is a rank n holomorphic vector bundle induced from a meromorphic vector bundle over X (Appendix C). The sheaf of meromorphic sections associated to Π will be denoted by $\mathscr{E}$, and the sheaf of meromorphic functions on X by $\mathscr{M}$. The sheaf of meromorphic 1-forms on X, and the sheaf of meromorphic tangent fields on X, will be denoted by $\Omega^1_{\mathscr{M}}$ and $\mathscr{T}X$ respectively. The sheaf of differential forms $\Omega^1_{\mathscr{M}}$ is the meromorphic dual of $\mathscr{T}X$. Given an $f \in k$ there is a global meromorphic differential form $df \in \Omega^1_{\mathscr{M}}(X)$ defined as follows:

$$\begin{aligned} df : \mathscr{T}X(X) &\longrightarrow k \\ v &\longmapsto df(v) : p \mapsto v_p(f). \end{aligned}$$

Any global tangent field $v \in \mathscr{T}X(X)$ induces a derivation in k, i.e. the map

$$\begin{aligned} v : k &\longrightarrow k \\ f &\longmapsto v(f) : p \mapsto v_p(f) \end{aligned}$$

is additive and satisfies the Leibniz rule:

$$\begin{aligned} v(f+g) &= v(f) + v(g) \\ v(fg) &= v(f)g + fv(g), \quad \forall (f,g) \in k^2. \end{aligned}$$

Once we fix v, the field k together with the derivation defined by v is a differential field. The field of complex numbers $\mathbb{C}$ can be identified with a subfield by regarding the complex numbers as constant functions. With this identification in mind we see that the kernel of v, known as the *constants* of the differential field, is $\mathbb{C}$ provided $v \neq 0$.

REMARK 2.1.1. In broad terms what we do in this chapter is to study the following geometric construction. Consider a linear differential equation

$$v(f^i) = a^i_j f^j, \quad i \in \{1, \ldots, n\}, \quad a^i_j \in k$$

and an open $U \subset X$ over which we have a full-system of solutions (y^i_j). The analytic map

$$\begin{array}{rcl} U & \longrightarrow & GL_n(\mathbb{C}) \\ p & \longmapsto & (y^i_j(p)) \end{array}$$

induces an algebraic map

$$\begin{array}{rcl} U & \longrightarrow & GL_n(\mathbb{C})/G \\ p & \longmapsto & (y^i_j(p)) \cdot G \end{array}$$

where G is the Galois group of our linear differential equation. Because the last map is algebraic, it can be extended to a meromorphic map defined globally over X. The idea is to see to which extend this last map characterizes our differential equation.

2.1.1 Differential Modules

Fix a non-trivial derivation $v \in \mathscr{T}X(X)$ of k.

Definition 2.1.2. A *differential k-module* (rigorously a (k, v)-module) is a finite dimensional k-vector space M together with an additive map

$$\partial : M \to M$$

satisfying the Leibnitz rule:

$$\begin{array}{rcll} \partial(m_1 + m_2) & = & \partial m_1 + \partial m_2 & \forall (m_1, m_2) \in M^2 \\ \partial f m & = & v(f)m + f\partial m & \forall (f, m) \in k \times M \end{array}$$

An $m \in M$ such that $\partial m = 0$ is called a *horizontal element.*

REMARK 2.1.3. If f is a constant, i.e. if $v(f) = 0$, the Leibnitz rule implies that $\partial fm = f\partial m$. The collection of horizontal elements thus forms a vector space over the field of constants $\mathbb{C}$.

REMARK 2.1.4. Fix a basis $e_1, \ldots, e_n$ of M and set

$$\partial e_j = -a^i_j e_i \quad \forall j \in \{1, \ldots, n\}.$$

If $m = f^i e_i$, then

$$\begin{aligned} \partial m &= v(f^i)e_i + f^i \partial e_i \\ &= v(f^i)e_i - f^i a^j_i e_j \\ &= (v(f^i) - a^i_j f^j)e_i. \end{aligned}$$

Solving the equation $\partial m = 0$ therefore amounts to solve the matrix differential equation

$$v(f^i) = a^i_j f^j, \quad i \in \{1, \ldots, n\}.$$

REMARK 2.1.5. Consider two differential k-modules (M_1, ∂_1) and (M_2, ∂_2). Given $(f, m_1, m_2) \in k \times M_1 \times M_2$, consider the two following equalities in $M_1 \otimes_k M_2$:

$$\begin{aligned} \partial_1 fm_1 \otimes m_2 + fm_1 \otimes \partial_2 m_2 &= v(f)(m_1 \otimes m_2) \\ &\quad + f(\partial_1 m_1 \otimes m_2) + f(m_1 \otimes \partial_2 m_2) \\ \partial_1 m_1 \otimes fm_2 + m_1 \otimes \partial_2 fm_2 &= f(\partial_1 m_1 \otimes m_2) \\ &\quad + v(f)(m_1 \otimes m_2) + f(m_1 \otimes \partial_2 m_2) \end{aligned}$$

So we conclude that under the map

$$\partial_1 \otimes \partial_2 : m_1 \otimes m_2 \longrightarrow \partial_1 m_1 \otimes m_2 + m_1 \otimes \partial_2 m_2$$

the k-vector space $M_1 \otimes M_2$ inherits a differential k-module structure. The same holds for symmetric and exterior powers: given $(m, m_1, m_2) \in M^3$, in $M \otimes_k M$ we have

$$\begin{aligned} \partial \otimes \partial(m_1 \otimes m_2 - m_2 \otimes m_1) &= (\partial m_1 \otimes m_2 - m_2 \otimes \partial m_1) \\ &\quad -(\partial m_2 \otimes m_1 - m_1 \otimes \partial m_2) \\ \partial \otimes \partial(m \otimes m) &= (\partial m + m) \otimes (\partial m + m) \\ &\quad -(\partial m \otimes \partial m + m \otimes m). \end{aligned}$$

Inductively we see that symmetric products and exterior powers of differential modules inherit differential structures. The same holds for the dual $M^* := \mathrm{Hom}_k(M, k)$ of M. By defining

$$\partial^* : \mu \longmapsto [m \mapsto v(\mu(m)) - \mu(\partial m)]$$

for any $\mu \in M^*$ one sees that

$$\partial^*(f\mu) : m \mapsto (v(f)\mu + f\partial^*\mu)(m).$$

In particular, if m is horizontal then $\partial^*\mu(m) = v(\mu(m))$. Summarizing, a differential k-module endows a canonical differential structure on any tensorial construction (duals, tensor products, symmetric powers, exterior powers, sums, ...) over k.

REMARK 2.1.6. If we denote $\mu(m)$ by $< \mu, m >$ the formula for the dual structure becomes the classical Lagrange identity

$$v < \mu, m > = < \partial^*\mu, m > + < \mu, \partial m > .$$

REMARK 2.1.7. In most of the cases, given a differential k-module M, we will not be able to find a basis composed of horizontal elements.

The definition of a differential k-module depends on our choice of a derivation on k, in our case it was v. Since we are in the realm of Riemann surfaces, which are actually one dimensional manifolds over the complex numbers, we can circumvent this restriction using connections.

2.1.2 Connections and Pullbacks

A meromorphic connection is a $\mathbb{C}$-linear map (linear over the constants)

$$\nabla : \mathscr{E} \longrightarrow \mathscr{E} \bigotimes_{\mathscr{M}} \Omega^1_{\mathscr{M}}$$

satisfying the Leibnitz rule

$$\nabla(fV) = V \otimes df + f\nabla V \qquad \forall (f, V) \in k \times \mathscr{E}(X).$$

Given a meromorphic tangent field v, we define the $\mathscr{M}$-linear contraction map by

$$\begin{aligned} \imath_v : \mathscr{E} \bigotimes_{\mathscr{M}} \Omega^1_{\mathscr{M}} &\longrightarrow \mathscr{E} \\ V \otimes \omega &\longmapsto \omega(v)V. \end{aligned}$$

Not every element in $\mathscr{E} \otimes_k \Omega^1_{\mathscr{M}}$ can be written in the form $V \otimes \omega$, but as the map is $\mathscr{M}$-linear, it suffices to define the map on such elements. The composition of ∇ followed by this contraction map is denoted by ∇_v. This is commonly called the *covariant derivative* along v.

REMARK 2.1.8. The k-vector space of global sections $\mathscr{E}(X)$ of Π is isomorphic to k^n (see Appendix C), so $(\mathscr{E}(X), \nabla_v)$ is a differential k-module.

REMARK 2.1.9. Derivations on k are in a natural bijective correspondence with global sections of $\mathscr{T}X$ and $\mathscr{T}X(X) \simeq k$. Suppose $\nu \in \Omega^1_{\mathscr{M}}$ is the dual of a derivation $v \in \mathscr{T}X$, i.e. $\nu(v) = 1$. Because $\Omega^1_{\mathscr{M}}$ is one-dimensional, for every $\omega \in \Omega^1_{\mathscr{M}}$ we have $\omega = \omega(v)\nu$, and we conclude that tensoring with ν is the inverse to $\imath_v$. Define the following map ∇':

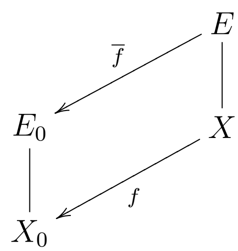
$$\begin{aligned} \nabla' : \mathscr{E} &\longrightarrow \mathscr{E} \bigotimes_{\mathscr{M}} \Omega^1_{\mathscr{M}} \\ V &\longmapsto \nabla_v V \otimes \nu. \end{aligned}$$

Then

$$\begin{aligned} \nabla' V &= \nabla_v V \otimes \nu \\ &= \imath_v(\nabla V) \otimes \nu \\ &= \nabla V. \end{aligned}$$

Endowing $\mathscr{E}(X)$ with a differential k-module structure is thereby seen to be the same as defining a meromorphic connection over $\mathscr{E}$. In particular, all tensorial constructions over meromorphic vector bundles with connection inherit a connection. This remark is independent of the fact that X is one-dimensional.

We now define connection pullbacks. Take a meromorphic vector bundle $\Pi_0 : E_0 \to X_0$ over a compact Riemann surface together with a meromorphic connection ∇_0 and a morphism $f : X \to X_0$. If $\Pi : E \to X$ is the pullback of $\Pi_0 : E_0 \to X_0$, and $(\overline{f}, f) : (E, X) \to (E_0, X_0)$ stands for the canonical vector bundle morphism (i.e. $\Pi_0 \circ \overline{f} = f \circ \Pi$):



The *pullback* (*connection*) $f^*\nabla_0$ at $p \in X$ is given by

$$[(f^*\nabla_0)_v V](p) = [(\nabla_0)_{f_* v} \overline{f} V](f(p))$$

(the vector field $f_* v$ is well defined only locally, but since $(\nabla_0)_{\bullet}\bullet$ is tensorial on $\cdot$ the value of $[(\nabla_0)_{f_* v} \overline{f} V](f(p))$ is uniquely determined by $f_* v(p)$).

2.1.3 Projective equivalence

Given a second vector bundle $P : L \to X$ with a meromorphic connection ∇' and associated sheaf of sections $\mathscr{L}$, the tensor product induces a canonical meromorphic connection $\nabla' \otimes \nabla$ on $P \otimes \Pi : L \otimes_X E \to X$:

$$\nabla' \otimes \nabla : W \otimes V \longmapsto \nabla' W \otimes V + W \otimes \nabla V.$$

REMARK 2.1.10. If W happens to be horizontal, i.e. $\nabla' W = 0$, then

$$(\nabla' \otimes \nabla) W \otimes V = W \otimes \nabla V.$$

Assume now that $P : L \to X$ is 1-dimensional (i.e. a line bundle), and fix a global non-zero meromorphic section $s \in \mathscr{L}(X)$. We can then identify

$\mathscr{E}$ with $\mathscr{L} \otimes \mathscr{E}$ through the morphism $V \mapsto s \otimes V$. It must be noted that this identification is not unique, since it depends on the choice of s.

Definition 2.1.11. Given another meromorphic connection ∇_1 on $\Pi : E \to X$, we say that ∇ and ∇_1 are *projectively equivalent* if there exist a 1-dimensional meromorphic vector bundle $P : L \to X$ with a connection ∇' such that

$$\nabla_1 \simeq \nabla' \otimes \nabla.$$

REMARK 2.1.12. In view of the non-uniqueness of the identification of $\mathscr{E}$ with $\mathscr{L} \otimes \mathscr{E}$, we need to clarify this definition. As every choice of $s \in \mathscr{L}(X)$ gives a different identification, the notation $\nabla_1 \simeq \nabla' \otimes \nabla$ indicates only that for some such s we have

$$s \otimes [\nabla_1]_v(V) = \nabla'_v s \otimes V + s \otimes \nabla_v V.$$

We will see below that this means that the map $V \mapsto s \otimes V$ is a horizontal isomorphism $(E, \Pi, \nabla_1) \to (L \otimes_X E, P \otimes \Pi, \nabla' \otimes \nabla)$. Under this identification, if $\nabla' s = 0$ then $\nabla_1 V = s \otimes \nabla V$ when ∇_1 and ∇ are projectively equivalent.

REMARK 2.1.13. Note that since $(P, \nabla') : L \to X$ is 1-dimensional, the same holds for the dual (L^*, P^*, ∇'^*). Under the canonical isomorphisms we have

$$\mathscr{L}^* \otimes \mathscr{L} \simeq \mathrm{Hom}_{\mathscr{M}}(\mathscr{L}, \mathscr{L}) \simeq \mathscr{M}.$$

Moreover in terms of the basis $s = s_1$ of $\mathscr{L}(X)$, if $s^1 \in \mathscr{L}^*$ is such that

$s^1(s_1) = 1$, then

$$\begin{aligned}
[\nabla'^* \otimes \nabla']_v(s^1 \otimes s_1)s_1 &= [\nabla'^*_v s^1](s_1)s_1 + s^1(s_1)\nabla' s_1 \\
&= (v(s^1(s_1)) - s^1(\nabla'_v s_1))s_1 + \nabla' s_1 \\
&= \nabla' s_1 - s^1(\nabla'_v s_1)s_1 = 0.
\end{aligned}$$

So the connection on $\mathscr{L}^* \otimes \mathscr{L}$ is trivial, and if $\nabla_1 \simeq \nabla' \otimes \nabla$ then

$$\nabla'^* \otimes \nabla_1 \simeq \nabla'^* \otimes \nabla' \otimes \nabla \simeq \nabla.$$

We conclude that projective equivalence is an equivalence relation.

2.1.4 Linear first integrals and Fano curve

We now study symmetric products and duals of vector bundles with connections in more detail. For the remainder of this section we set $M = \mathscr{E}(X)$ and we fix a non-zero derivation $v \in \mathscr{T}X(X)$.

Definition 2.1.14. Let $\phi \in M^* = \mathrm{Hom}_k(\mathscr{E}(X), k)$. We say that ϕ is a *linear first integral* (of (E, ∇)) if $\phi(V) \in k$ is constant whenever V is horizontal.

REMARK 2.1.15. Let ϕ be a linear first integral and let $V \in M$ be horizontal. Then

$$\begin{aligned}
0 &= v(\phi(V)) \\
&= [\nabla^*_v \phi](V).
\end{aligned}$$

In particular, linear first integrals form a vector space over the constants.

Definition 2.1.16. Denote by $\mathrm{S}_k^d(M)$ the $\mathbb{C}$-vector space of linear first integrals of the d-th symmetric product of (E, ∇). As a convention we set $\mathrm{S}_k^0(M) = \mathbb{C}$. We define the graded algebra of linear first integrals of M as

$$\mathrm{S}_k(M) = \bigoplus_{d \geq 0} \mathrm{S}_k^d(M).$$

REMARK 2.1.17. If $V \in M$ is horizontal, then so is V^d in the d-th symmetric power. Thus, given $\phi \in \mathrm{S}_k^d(M)$, $\phi(V^d)$ is constant. The elements in $\mathrm{S}_k^d(M)$ are called *first integral* of M. Once we fix a basis of M, an element in $\mathrm{S}_k^d(M)$ is given by a homogeneous polynomial of order d in the coordinates on M. With this in mind $\mathrm{S}_k(M)$ corresponds to the collection of rational $\mathbb{C}$-valued functions over E containing horizontal sections of $\mathscr{E}$ within their level sets.

REMARK 2.1.18. If we pick a $V \in M$ we obtain a homomorphism

$$V : \mathrm{S}_k(M) \longmapsto k$$

of $\mathbb{C}$-algebras by evaluating each first integral in $\mathrm{S}_k^d(M)$ at V^d.

REMARK 2.1.19. Let $U \subseteq X$ be an open set and pick $V \in \mathscr{E}(U)$, then we can evaluate first integrals in $\mathrm{S}_k^d(M)$ at V to obtain an element of $\mathscr{M}(U)$. Indeed, since the first integrals in $\mathrm{S}_k^d(M)$ are globally defined meromorphic functions we can restrict them to $\Pi^{-1}U$ and evaluate them at V^d. So in this case V can be identified with a homomorphism

$$V : \mathrm{S}_k(M) \longmapsto \mathscr{M}(U)$$

of $\mathbb{C}$-algebras. Note that the latter remark is the particular case $U = X$.

REMARK 2.1.20. The fundamental theorem of ordinary differential equations guarantees that if p is not a singular point of ∇, then for a sufficiently small open set $U \subseteq X$ containing p we can find a frame $(V_1, \ldots, V_n)$ of $\mathscr{E}(U)$ composed of horizontal elements.

Definition 2.1.21. Let H be a (local) horizontal invertible element of the differential k-module $\mathrm{Hom}_k(\mathscr{E}(U), \mathscr{E}(U)) \simeq [\mathscr{E}^* \otimes_{\mathscr{M}} \mathscr{E}](U)$, i.e. a frame of $\mathscr{E}(U)$ composed of horizontal sections. The *Fano curve* of (E, ∇) is defined as the $\mathbb{C}$-valued point

$$H : \mathrm{S}_k(\mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X))) \longrightarrow \mathbb{C} \subseteq \mathscr{M}(U)$$

(see Remark 2.1.19). If $\mathrm{S}_k(\mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X)))$ is finitely generated we define the *projective Fano curve* X_0 to be the non-singular model of the projective variety defined by the maximal homogeneous ideal contained in the kernel of the Fano curve (i.e. in the kernel of H).

REMARK 2.1.22. There are many aspects of this definition that require elaboration.

- The Fano curve is $\mathbb{C}$-valued: by definition, a first integral is constant when evaluated at a horizontal element.

- The Fano curve is independent of the choice of H up to isomorphism: if $\tilde{H}$ is another (local) horizontal frame defined over U, then it differs from H by an element of $GL_n(\mathbb{C})$ which one can see as acting on $\mathrm{S}_k(\mathrm{Hom}_k(\mathscr{E}(U), \mathscr{E}(U)))$ sending one Fano curve to another.

- The Fano curve is also independent of the choice of U up to isomorphism: if $\tilde{U}$ is another open set where one can define a (local) horizontal frame, then by taking a path from U to $\tilde{U}$ we can prolong H holomorphically to a frame over $\tilde{U}$. Because H is horizontal and the first integrals are constant over horizontal sections, as well as being globally defined, the prolongation does not change the homomorphism of $\mathbb{C}$-algebras.

REMARK 2.1.23. Let $\phi \otimes V \in [\mathscr{E}^* \otimes_{\mathscr{M}} \mathscr{E}](X)$, so that:

$$[\nabla^* \otimes \nabla](\phi \otimes V) \;=\; \nabla^* \phi \otimes V + \phi \otimes \nabla V.$$

Then under the canonical isomorphism $\mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X)) \simeq [\mathscr{E}^* \otimes_{\mathscr{M}} \mathscr{E}](X)$ we obtain:

$$\begin{aligned}
[\nabla^* \otimes \nabla]_v(\phi \otimes V)\ (W) &= [\nabla^*_v \phi](W)V + \phi(W)\nabla_v V \\
&= (v(\phi(W)) - \phi(\nabla_v W))V + \phi(W)\nabla_v V \\
&= \nabla_v[\phi(W)V] - \phi(\nabla_v W)V \\
&= \nabla_v[(\phi \otimes V)\ (W)] - (\phi \otimes V)\ (\nabla_v W).
\end{aligned}$$

This implies that $H \in \mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X))$ is horizontal if and only if it is a connection preserving map, i.e.

$$\nabla[H(W)] = H(\nabla W).$$

(In terms of differential k-modules, H is horizontal if and only if H is a morphism of differential modules).

REMARK 2.1.24. Given a (local) horizontal endomorphism of vector bundles $H : E \to E$ and a frame F for $\mathscr{E}(X)$, we associate to H an $n \times n$ matrix. The Fano curve describes the rational level sets of H in $E^* \otimes_X E$ in terms of the associated matrix. The projective Fano curve will play a similar role with classes of projectively equivalent connections, but the explanation requires some algebraic preliminaries.

2.1.5 The geometric Galois group

As above, we fix a (local) horizontal frame H for $\mathscr{E}(U)$. The collection of (local) horizontal frames $\tilde{H}$ for $\mathscr{E}(U)$ defining the same map

$$\tilde{H} : \mathrm{S}_k(\mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X))) \longmapsto \mathbb{C}$$

as H will be denoted by Ó. In particular, if we see the frames as invertibles in $\mathrm{Hom}_{\mathscr{M}}(\mathscr{E}, \mathscr{E})(U)$, the collection of elements in H^{-1}Ó forms a group. The group $G = H^{-1}$Ó is called the *geometric Galois group* of ∇.

REMARK 2.1.25. As in the case of the Fano curve, it follows from this definition that the geometric Galois group is independent of H and U, up to isomorphism. Given another frame $\overline{H}$ over U, there is a $g \in \mathrm{GL}_n(\mathbb{C})$ such that $\overline{H} = Hg$, and the geometric Galois group determined by $\overline{H}$ is the conjugate of G by g.

REMARK 2.1.26. The geometric Galois group measures the rational symmetries of the connection. More explicitly, given a $g \in G$ and the (local) horizontal frames H and Hg cannot be distinguished using rational (global) first integrals.

2.2 Algebraic Interpretation

To give an algebraic interpretation of the geometric objects introduced above, we require a global frame $F = e^i \otimes e_i \in \mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X)) \simeq [\mathscr{E}^* \otimes_{\mathscr{M}} \mathscr{E}](X)$ of $E \to X$ so that we can do some computations using coordinates.

As above, we fix a derivation $v \in \mathscr{T}X(X)$. As discussed earlier, the solutions to the equation $\nabla_v V = 0$ are given by the solutions to the matrix differential equation

$$v(f^i) = a^i_j f^j \quad i \in \{1, \dots, n\}, \tag{2.1}$$

where $f^i = e^i(V)$ and $\nabla_v e_i = -a^j_i e_j$.

2.2.1 Picard-Vessiot Extensions and Galois groups

Let $H = (V_1, \dots, V_n)$ be a frame of $\mathscr{E}(U)$ composed of horizontal elements. If we denote by $y^i_j = e^i(V_j) \in \mathscr{M}(U)$ the coordinates of these horizontal elements in our original frame F, then

$$v(y^i_j) = a^i_k y^k_j$$

and the Picard-Vessiot extension is given by the subfield

$$K := k(y^i_j)$$

of $\mathscr{M}(U)$. The inclusion map into this extension is given by the restriction map:

$$\begin{aligned} k \simeq \mathscr{M}(X) &\longrightarrow \mathscr{M}(U) \\ f &\longmapsto f\restriction_U . \end{aligned}$$

Formally, the Picard-Vessiot extensions can be obtained as follows (a rigorous exposition may be found in [25]). Consider the ring

$$k[X^i_j, \frac{1}{\det}]_{i,j\in\{1,\ldots,n\}},$$

which results from the ring of polynomials $k[X^i_j]$ in $n \times n$ variables by inverting the determinant polynomial $\det := \det(X^i_j)$. We turn this ring into a differential ring extension of (k, v) by setting

$$v(X^i_j) = a^i_l X^l_j$$

and using the Leibnitz rule and the quotient rule

$$v(ab^{-1}) = [v(a)b - av(b)]b^{-2}$$

we extend the derivation to the whole ring. An ideal $I \subseteq k[X^i_j, \frac{1}{\det}]$ is differential if it is closed under derivation, i.e. $v(I) \subset I$. Maximal differentials ideal are prime. We obtain a Picard-Vessiot extension for ∇ by taking the fraction field of the quotient of $k[X^i_j, \frac{1}{\det}]$ by a maximal differential ideal I.

Note that we can make $\mathrm{GL}_n(\mathbb{C})$ act on $k[X^i_j, \frac{1}{det}]$ by differential automorphisms over k by setting for $(g^i_j) \in \mathrm{GL}_n(\mathbb{C})$

$$\begin{array}{rcl} (g^i_j) : k[X^i_j, \frac{1}{\det}] & \longrightarrow & k[X^i_j, \frac{1}{\det}] \\ X^i_j & \longmapsto & X^i_l g^l_j \end{array}$$

We can identify the Galois group G with the elements of $\mathrm{GL}_n(\mathbb{C})$ sending I to itself. In particular, we may take I as the kernel of the evaluation map of

k-algebras:

$$\begin{array}{rcl} k[X^i_j, \dfrac{1}{\det}] & \longrightarrow & \mathscr{M}(U) \\ X^i_j & \longmapsto & y^i_j \end{array}$$

REMARK 2.2.1. Probably the most striking feature of the Galois group is that it is an algebraic group.

Let us consider with more care the relationship between the geometric Galois group and the (algebraic) Galois group. As above, H will denote the (local) frame of $\mathscr{E}(U)$ composed of the horizontal sections $(V_1, \ldots, V_n)$ and we set $y^i_j = e^i(V_j)$. An element of $S^d_k(\mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X)))$ corresponds on the frame F to a homogeneous polynomial $P(X^i_j)$ of degree d such that $P(y^i_j) = c$, where $c \in \mathbb{C}$. So $P(X^i_j) - c$ is in the kernel I of the evaluation map. Moreover, if $g = (g^i_j) \in \mathrm{GL}_n(\mathbb{C})$ is in our algebraic definition of the Galois group, then $P(X^i_l g^l_j) - c$ is again in I, meaning that $P(y^i_l g^l_j) = c$, so the geometric Galois group contains the (algebraic) Galois group.

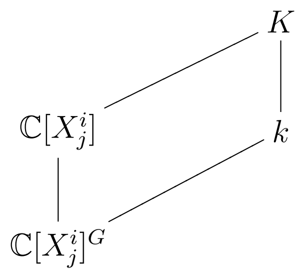
REMARK 2.2.2. In [7, Theorem 4.2] Compoint proves that when the (algebraic) Galois group is reductive and unimodular, the geometric and the (algebraic) Galois group coincide. In view of Compoint's result we will restrict ourselves to that case.

2.2.2 Compoint's Theorem and the projective Fano Curve

In analogy with classical Galois theory, differential Galois theory also admits a Galois correspondence. In particular, the fixed field K^G is the ground field k. Furthermore using the action in the previous section, if $P(X^i_j) \in k[X^i_j]$ is invariant under the action of G, then the Galois correspondence implies $P(y^i_j) \in k$. From now on we will assume that G is unimodular and reductive. In particular this will imply that $\mathrm{S}_k(\mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X)))$ is finitely generated, and so we will be able to associate to ∇ a projective Fano curve X_0. We now introduce a theorem that allows us to effectively compute X_0.

Theorem 2.2.3 (Compoint)**.** *If G is reductive and unimodular, then I is generated by the G-invariants it contains. Moreover, if $P_0, \ldots, P_r$ is a set of generators for the $\mathbb{C}$-algebra of G-invariants in $\mathbb{C}[X^i_j]$, and if $f_0, \ldots, f_r \in k$ are such that $P_i - f_i \in I$, then I is generated over $k[X^i_j, \frac{1}{det}]$ by $P_i - f_i$, $i \in \{0, \ldots, r\}$.*

REMARK 2.2.4. Compoint's theorem says that I is uniquely determined by the restriction of $k[X^i_j, \frac{1}{det}] \to K$ to $\mathbb{C}[X^i_j]^G \to k$. We will denote by J the maximal homogeneous ideal contained in the kernel of the last map.



Proposition 2.2.5. *Keeping the same notation and hypotheses as in the theorem and remark, the projective variety defined by the homogeneous ideal J is isomorphic to the projective Fano curve of ∇.*

Proof: Let $P(X_j^i) \in J$ of degree d, then $P(X_j^i)$ corresponds, in our frame F, to a first integral of degree d, i.e. an element $\lambda \in \mathrm{S}_k(\mathrm{Hom}_k(\mathscr{E}(X), \mathscr{E}(X)))$ vanishing at $H = (V_1, \ldots, V_n)$ (being pedantic, it vanishes at H^d). This means that λ is in the homogeneous ideal defining the projective Fano curve. Conversely [7, Theorem 4.2] says that such a λ, i.e. a first integral of degree d vanishing at H, corresponds to a homogeneous polynomial invariant under the G-action of degree d, $P(X_j^i) \in \mathbb{C}[X_j^i]$, such that $P(y_j^i) = 0$, i.e. $P(X_j^i) \in J$. ★

REMARK 2.2.6. We keep the notation from Compoint's theorem, and we denote by n_i the degree of P_i. Then the function field of the projective Fano curve is

$$\mathbb{C}(X_0) = \mathbb{C}(\frac{f_i^{m_i}}{f_j^{m_j}} | m_i n_i = m_j n_j,\ f_j \neq 0).$$

Consider the canonical projection $\mathrm{pr} : SL_n(\mathbb{C}) \to PSL_n(\mathbb{C})$ and set

$$\widetilde{G} := \mathrm{pr}^{-1}(PG),$$

where $PG = G/Z(G)$ (and $Z(G)$ stands for the center of G). Then $Z(\widetilde{G})$ is the cyclic group of n elements and the elements fixed in $\mathbb{C}(y^i_j)$ by this center are $\mathbb{C}(y^i_j)^{Z(\widetilde{G})} = \mathbb{C}(\frac{y^i_j}{y^\iota_\kappa}, (y^i_j)^n |\ y^\iota_\kappa \neq 0)$. Also, because

$$\mathbb{C}(y^i_j)^{\widetilde{G}} = [\mathbb{C}(y^i_j)^{Z(\widetilde{G})}]^{PG} \quad (PG = P\widetilde{G})$$

we then have

$$\mathbb{C}(y^i_j)^{\widetilde{G}} = \mathbb{C}(X_0)$$

provided that $P(y^i_j) = 1$ for some homogeneous $P(X^i_j) \in \mathbb{C}[X^i_j]^{\widetilde{G}} \setminus \mathbb{C}$ of order n. Let us make this last assertion explicit.

Lemma 2.2.7. *If $P(y^i_j) = 1$ for some homogeneous $P(X^i_j) \in \mathbb{C}[X^i_j]^{\widetilde{G}} \setminus \mathbb{C}$ of order n, then $\mathbb{C}(y^i_j)^{\widetilde{G}} = \mathbb{C}(X_0)$.*

Proof: From the equality $\mathbb{C}(y^i_j)^{Z(\widetilde{G})} = \mathbb{C}(\frac{y^i_j}{y^\iota_\kappa}, (y^i_j)^n |\ y^\iota_\kappa \neq 0)$ we see that $\mathbb{C}(y^i_j)^{Z(\widetilde{G})}$ is generated as an algebra over $\mathbb{C}$ by the ratios $\frac{y^i_j}{y^\iota_\kappa}$ together with the functions parameterizing a horizontal frame of the n-th symmetric power of ∇. So $\mathbb{C}(y^i_j)^{\widetilde{G}} = [\mathbb{C}(y^i_j)^{Z(\widetilde{G})}]^{PG}$ is generated as an algebra over $\mathbb{C}(X_0)$ by the rational first integrals of the n-th symmetric product of ∇. Dividing each of this rational first integral by powers of $P(y^i_j) = 1$ we obtain them as elements in $\mathbb{C}(X_0)$, so we have $\mathbb{C}(y^i_j)^{\widetilde{G}} = \mathbb{C}(X_0)$. ★

REMARK 2.2.8. A particular case of the previous lemma occurs when

$$\det(y^i_j) = 1.$$

Lemma 2.2.9. *Under the hypotheses of Compoint's theorem, if* $\det(y^i_j) = 1$ *there is a choice of $v \in \mathscr{T}X$ such that in (2.1) one has*

$$(a^i_j) \in M_{n\times n}(\mathbb{C}(X_0)) + \mathfrak{g}(k),$$

where $\mathfrak{g}$ is the Lie algebra of G.

Proof: Let $P_l = P_l(X^i_j) \in \mathbb{C}[X^i_j]^{\widetilde{G}}$, for $l \in \{1, \dots, r\}$ be generators of $\widetilde{G}$-invariant subalgebra $\mathbb{C}[X^i_j]^{\widetilde{G}}$. Let $f_l \in k$ be such that $P_l - f_l \in I$, $l \in \{1, \dots, r\}$. We denote by $\frac{\partial P_l}{\partial X^\iota_\kappa}(X^i_j)$ the partial derivative of P_l with respect to X^ι_κ.

We introduce a differential field $\mathbb{C}(y^i_j)(b^\iota_\kappa)_{\iota,\kappa \in \{1,\dots,n\}}$, where the b^ι_κ's are variables, with derivation $\tilde{v}_0$ defined by:

$$\begin{aligned} \tilde{v}_0(y^i_j) &= b^i_k y^k_j \\ \tilde{v}_0(b^i_j) &= 0. \end{aligned}$$

Using the chain rule we obtain

$$\begin{aligned} \tilde{v}_0(P_l(y^i_j)) &= \frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_j)\tilde{v}_0(y^\iota_\kappa) \\ &= \frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_j) b^\iota_\lambda y^\lambda_\kappa. \end{aligned}$$

We extend the action of $\widetilde{G}$ on $\mathbb{C}(y^i_j)$ to $\mathbb{C}(y^i_j)(b^\iota_\kappa)_{\iota,\kappa \in \{1,\dots,n\}}$ by letting each b^ι_κ be fixed by $\widetilde{G}$. The chain rule then implies that for $(g^i_j) \in \widetilde{G}$ one has

$$\begin{aligned} \tilde{v}_0(P_l(y^i_l g^l_j)) &= \frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_l g^l_j)\tilde{v}_0(y^\iota_\mu g^\mu_\kappa) \\ &= \frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_l g^l_j) b^\iota_\lambda y^\lambda_\mu g^\mu_\kappa. \end{aligned}$$

The equality $P_l(y^i_j) = P_l(y^i_l g^l_j)$ in turn implies

$$\frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_j) y^\lambda_\kappa = \frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_l g^l_j) y^\lambda_\mu g^\mu_\kappa \quad \forall\ \iota, \lambda,$$

for all $(g^i_j) \in \widetilde{G}$, whence $\frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_j) y^\lambda_\kappa \in \mathbb{C}(y^i_j)^{\widetilde{G}}$ for each ι, λ.

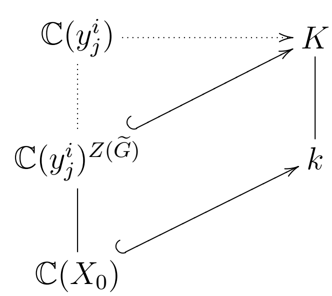
Let v be a non-trivial derivation of $\mathbb{C}(X_0) = \mathbb{C}(y^i_j)^{\widetilde{G}}$ (unless $\mathbb{C}(X_0) = \mathbb{C}$, in which case we let v be any element in $\mathscr{T}X$). Note that since P_l is $\widetilde{G}$-invariant we have $P_l(y^i_j) = f_l \in \mathbb{C}(X_0) \subseteq \mathbb{C}(y^i_j)$. Consider the following system of linear equations in the variables b^ι_λ with coefficients in $\mathbb{C}(X_0)$:

$$\frac{\partial P_l}{\partial X^\iota_\kappa}(y^i_j) y^\lambda_\kappa b^\iota_\lambda = v(f_l) \quad l \in \{0, \dots, r\}. \tag{2.2}$$

This system has solutions in k (apply v on both sides of the equalities $P_l(y^i_j) = f_l$ in K). The system of equations is therefore consistent, and the system can thus be solved in the field of coefficients $\mathbb{C}(X_0)$. Specialize b^ι_λ to such solutions, so that $(b^i_j) \in M_{n\times n}(\mathbb{C}(X_0))$. When we apply v on $P_l(y^i_j) = f_l$ in K, we obtain the solutions a^ι_λ to (2.2). Hence the $(a^i_j) - (b^i_j)$ is a solution to the homogeneous system associated to (2.2); but the left hand side of the equations in the system are the polynomials defining $\mathfrak{g}$, so $(a^i_j) - (b^i_j) \in \mathfrak{g}(k)$.

★

REMARK 2.2.10. We have



Definition 2.2.11. We say that the projective Fano curve X_0 is *degenerate* if C is not 1-dimensional, i.e. if $\mathbb{C}(X_0) = \mathbb{C}$.

Proposition 2.2.12. *The projective Fano curve is degenerate if and only if G is connected and $\mathbb{C}(y^i_j)$ corresponds to the field of rational functions over a coset of G in $GL_n(\mathbb{C})$.*

Proof: Let $I_{\mathbb{C}}$ be the kernel of the evaluation map $\mathbb{C}[X^i_j, \frac{1}{\det}] \to K$ sending $X^i_j \mapsto y^i_j$. Then $I_{\mathbb{C}}$ is a prime ideal. $I_{\mathbb{C}}$ is invariant under the G-action, so passing to the quotient we see that $(I_{\mathbb{C}})^G = I_G$ is the kernel of the restriction of the evaluation map to $\mathbb{C}[X^i_j, \frac{1}{\det}]^G \to K^G = k$. Again, I_G is a prime ideal. Because the Fano curve is degenerate, the maximal homogeneous ideal J contained in I_G corresponds at the level of varieties to a line, and I_G to a point in that line. So we conclude that G acts transitively by left multiplication over the subvariety of $\mathrm{GL}_n(\mathbb{C})$ defined by $I_{\mathbb{C}}$. In other words, $I_{\mathbb{C}}$ defines a coset of G. Finally, because $I_{\mathbb{C}}$ is prime, this coset is connected, whence G is connected. This proves the necessity in the statement of the proposition. The sufficiency follows immediately by noting that G acts transitively on its cosets. ★

REMARK 2.2.13. When G is connected the field of rational functions over a coset of G is isomorphic to $\mathbb{C}(G)$ because the coset and G are isomorphic varieties.

REMARK 2.2.14. When $\mathbb{C}(X_0) = \mathbb{C}$, the system (2.2) is homogeneous so $(a^i_j) \in \mathfrak{g}(k)$.

2.2.3 Projective Equivalence and Pullbacks

Let us consider the algebraic properties of projective equivalence and of pullbacks.

We put another meromorphic connection ∇_1 on $\Pi : E \to X$ and we assume that ∇ and ∇_1 are projectively equivalent. This means that there is a 1-dimensional meromophic bundle $P : L \to X$ with connection ∇' such that ∇_1 can be identified with $\nabla' \otimes \nabla$. We make this identification explicit by fixing (as before) a (global) frame F for $\mathscr{E}(X)$ and a non-zero (global) section $s^1 \in \mathscr{L}^*(X)$. The mapping $V \mapsto s_1 \otimes V$ is a horizontal isomorphism $(E, \Pi, \nabla_1) \to (L \otimes_X E, P \otimes \Pi, \nabla' \otimes \nabla)$, where $s^1(s_1) = 1$.

Let $U \subseteq X$ be an open set avoiding the singularities of ∇ and of ∇', and let $h \in \mathscr{L}(U)$ be such that $\nabla' h = 0$. We set $f := s^1(h)$. As usual, we choose $L = (V_1, \dots, V_n)$ a (local) horizontal frame of $\mathscr{E}(U)$ with respect to ∇, and $y^i_j = e^i(V_j)$. In particular the coordinates of $h \otimes V_j$ on the frame $(s_1 \otimes e_1, \dots, s_1 \otimes e_n)$ are $s^1 \otimes e^i(h \otimes V_j) = f y^i_j$.

Let $W_j \in \mathscr{E}(U)$ be defined for $j \in \{1, \dots, n\}$ by

$$W_j = f y^i_j e_i,$$

so that $s^1 \otimes e^i(s_1 \otimes W_j) = f y^i_j$ and $s_1 \otimes W_j = h \otimes V_j$. Thus

$$\nabla_1 W_j = \nabla' \otimes \nabla(s_1 \otimes W_j) = \nabla' \otimes \nabla(h \otimes V_j) = 0.$$

In particular, a Picard-Vessiot extension for ∇_1 is generated by $(f y^i_j)$.

Proposition 2.2.15. *Under the hypotheses of Compoint's theorem, if ∇ and ∇_1 are projectively equivalent then their projective Fano curves coincide.*

Proof: Let $P(X^i_j) \in \mathbb{C}[X^i_j]^G$ be homogeneous of degree d such that $P(y^i_j) = 0$. Then $P(fy^i_j) = f^d P(y^i_j) = 0$. ★

To ease notation we will forget the fact that (b^i_j) defines the connection giving us the Picard-Vesiot extension $\mathbb{C}(y^i_j)/\mathbb{C}(y^i_j)^G$. This will simplify our algebraic explanation of the pullback. Let $\Pi_0 : E_0 \to X_0$ be an n-dimensional meromorphic vector bundle with connection ∇_0 given by

$$\nabla_{0v_0} e_i = -b^j_i e_j$$

for a given (global) frame $F_0 = e^i \otimes e_i$ of $\mathscr{E}_0(X_0)$. The pullback to X of (E_0, Π_0, ∇_0) is (algebraically) defined by taking the tensor product

$$\mathscr{E}_0 \otimes_{\mathscr{M}_0} \mathscr{M}$$

and regarding it as a sheaf of differential $\mathscr{M}$-modules. In particular

$$\mathscr{E}_0 \otimes_{\mathscr{M}_0} \mathscr{M}(X) = \mathscr{E}_0(X_0) \otimes_{\mathbb{C}(X_0)} k.$$

Definition 2.2.16. Let k_0 be a subfield of k. We say that ∇ is *defined over* k_0 if in (2.1)

$$(a^i_j) \in M_{n\times n}(k_0) + \mathfrak{g}(k)$$

REMARK 2.2.17. It follows from the definition and Lemma 2.2.9 that when ∇ is such that $\det(y^i_j) = 1$ the connection is defined over X_0.

Theorem 2.2.18. *If ∇ has reductive Galois group the connection is projectively equivalent to a connection defined over its projective Fano curve.*

Proof: We fix the notation as above; in particular the horizontal sections of ∇ satisfy the linear differential equation

$$v(f^i) = a^i_j f^j.$$

It is a well-known fact that if we tensor ∇ with the connection given by the equation

$$v(f) = a^i_i f$$

then the resulting connection has unimodular Galois group [25, Exercises 1.35.5]. In particular we may take (y^i_j) such that $\det(y^i_j) = 1$. Furthermore, [8, Proposition 2.2] guarantees that the Galois group remains reductive after tensoring. We may therefore assume that ∇ satisfies the hypotheses of Compoint's theorem. The result now follows from Lemma 2.2.9. ★

REMARK 2.2.19. In view of Proposition 2.2.12, the classification of connections with degenerate Fano curve is very simple: they correspond to extensions isomorphic to $G(k)/k$. Indeed, they are just the pullbacks of extension $\mathbb{C}(G)/\mathbb{C}$, where G is a connected algebraic group.

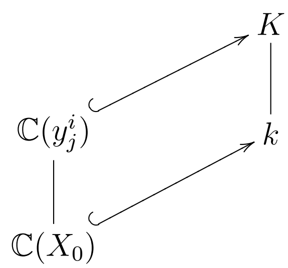
Definition 2.2.20. We say that ∇ is *standard* if $K = \mathbb{C}(y^i_j)$ and k is Galois over $\mathbb{C}(X_0)$.

Corollary 2.2.21. *Assume ∇ has reductive Galois group and its projective Fano curve is not degenerate. The connection ∇ is projectively equivalent to the pullback by a rational map of a standard connection over X_0 if and only if in Lemma 2.2.9 one has*

$$(a^i_j) \in M_{n\times n}(\mathbb{C}(X_0)).$$

In particular, if G is finite then ∇ is projectively equivalent to the pullback of a standard connection over X_0.

Proof: After tensoring like in the proof of the theorem we obtain the following diagram:



If ∇ is the pullback of a standard connection over X_0 then in Lemma 2.2.9, a^{ι}_{λ} is a solution to the system 2.2 in $\mathbb{C}(X_0)$ and so $(a^i_j) - (b^i_j) = 0$. Conversely $(a^i_j) \in M_{n\times n}(\mathbb{C}(X_0))$ defines a connection in a rank n bundle over X_0 and since $K = k(y^i_j)$, we conclude that ∇ is the pullback through the map $\mathbb{C}(X_0) \to k$.

The algebraic case follows from the fact that if G is finite then $\mathfrak{g} = 0$. ★

2.3 The algebraic case

REMARK 2.3.1. The content of this section is a generalization of the presentation of [1, Lemma 1.5] to higher order.

Given a n-th order linear differential equation $L(y) = 0$ over X with algebraic solutions, and a point $p \in X$, we denote by

$$E(L,p) = \{\alpha_{1,p}, \alpha_{2,p}, \dots, \alpha_{n,p}\}$$

the collection of generalized exponents of $L(y) = 0$ at p, ordered so that $\alpha_{i,p} < \alpha_{j,p}$ if $i < j$. We set

$$\Delta(L, p) = \alpha_{n,p} - \alpha_{1,p} - (n-1)$$

and we let $e(L, p)$ be the least common denominator of $E(L, p) - E(L, p) = \{a - b |\ a, b \in E(L, p)\}$. If $S \subseteq X$ we set

$$\Delta(L, S) := \sum_{p \in S} \Delta(L, p).$$

As $E(L, p) = \{0, 1, \ldots, n-1\}$ for almost all $p \in X$, we have $\Delta(L, p) = 0$ for all but finitely many $p \in X$, and for a sufficiently large S, the number $\Delta(L, S)$ attains limiting value $\Delta(L)$.

Lemma 2.3.2. *Let $f : X \to X_0$ be a morphism of compact Riemann surfaces of degree M, and assume that $L(y) = 0$ is the pullback of $L_0(y) = 0$ through f. Furthermore, assume all solutions to $L_0(y) = 0$ are algebraic. Then, if g (resp. g_0) denotes the genus of X (resp. of X_0), we have:*

$$M\left(\frac{\Delta(L_0)}{n-1} - 2(g_0 - 1)\right) = \frac{\Delta(L)}{n-1} - 2(g-1).$$

Proof: Let $S_0 \in X_0$ be a finite collection of points containing all ramifications of f and all singularities of $L_0(y) = 0$, and set $S := f^{-1}(S_0)$. So, if e_{p_0} denotes the ramification index of p_0 in f, then

$$\begin{aligned}
\Delta(L, f^{-1}(p_0)) &= \sum_{p|p_0} \Delta(L,p) \\
&= \sum_{p|p_0} \alpha_{n,p} - \alpha_{1,p} - (n-1) \\
&= \sum_{p|p_0} e_{p_0}(\alpha_{n,p_0} - \alpha_{1,p_0}) - (n-1)) \\
&= \frac{M}{e_{p_0}}(e_{p_0}(\alpha_{n,p_0} - \alpha_{1,p_0}) - (n-1)) \\
&= M(\Delta(L_0,p_0) + (n-1)) - \mathrm{Card}(f^{-1}(p_0))(n-1).
\end{aligned}$$

Thus $\Delta(L, f^{-1}(p_0)) + (n-1)\mathrm{Card}(f^{-1}(p_0)) = M(\Delta(L_0,p_0) + (n-1))$ and

$$\Delta(L,S) + (n-1)\mathrm{Card}(S) = M(\Delta(L_0,S_0) + (n-1)\mathrm{Card}(S_0))$$

As S_0 contains all ramifications of f, the Hurwitz genus formula implies

$$2(g-1) - 2M(g_0-1) = M\ \mathrm{Card}(S_0) - \mathrm{Card}(S).$$

Combining the last two equalities we obtain the desired conclusion by noticing that S_0 contains all singularities of $L_0(y) = 0$. ★

Corollary 2.3.3. *Under the hypotheses of the lemma we have:*

$$\sum_{p_0 \in S_0} \left(\frac{1}{e(L_0,p_0)} - 1 \right) = 2(g_0 - 1) - \frac{2(g-1)}{M}.$$

Proof: Suppose f corresponds to the field extension

$$\mathbb{C}(X_0) \subseteq \mathbb{C}(X_0)[\frac{y_1}{y_n}, \ldots, \frac{y_{n-1}}{y_n}],$$

where $y_1, \ldots, y_n$ denote a full system of solutions of $L_0(y) = 0$. Then $e_{p_0} = e(L_0, p_0)$, and it follows that

$$\begin{aligned}
\frac{1}{n-1}(M\Delta(L_0) - \Delta(L)) &= \frac{1}{n-1}\Big\{M \sum_{p_0 \in S_0} (\alpha_{n,p_0} - \alpha_{1,p_0} - (n-1)) \\
&\qquad - M \sum_{p_0 \in S_0} (\alpha_{n,p_0} - \alpha_{1,p_0} - \frac{n-1}{e(L_0,p_0)})\Big\} \\
&= \frac{M}{n-1} \sum_{p_0 \in S_0} \left(\frac{n-1}{e(L_0,p_0)} - (n-1) \right) \\
&= M \sum_{p_0 \in S_0} \left(\frac{1}{e(L_0,p_0)} - 1 \right).
\end{aligned}$$

The statement now follows from the previous lemma. ★

EXAMPLE 2.3.4. Consider the three following equations: Ulmer's G_{54} equation [28]

$$y''' + \frac{3(3x^2-1)}{x(x-1)(x+1)}y'' + \frac{221x^4 - 206x^2 + 5}{12x^2(x-1)^2(x+1)^2}y' + \frac{374x^6 - 673x^4 + 254x^2 + 5}{54x^3(x-1)^3(x+1)^3}y = 0$$

with singular points at 0, 1, -1 and ∞, with respective exponents

$$\{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\}, \quad \{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\} \quad \{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\}, \quad \{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\};$$

the Geiselmann-Ulmer $F_{36}^{SL_3}$ equation [14]

$$y''' + \frac{5(9x^2 + 14x + 9)}{48x^2(x+1)^2}y' - \frac{5(81x^3 + 185x^2 + 229x + 81)}{432x^3(x+1)^3}y = 0$$

with singular points at 0, 1 and ∞, with respective exponents

$$\{1, \frac{3}{4}, \frac{5}{4}\}, \quad \{\frac{5}{6}, \frac{11}{6}, \frac{1}{3}\}, \quad \{-1, \frac{-3}{4}, \frac{-5}{4}\};$$

and the equation

$$y''' + \frac{1}{48}\frac{41z^2 - 50z + 45}{(z-1)^2 z^2}y' - \frac{1}{432}\frac{364z^3 - 665z^2 + 1030z - 405}{(z-1)^3 z^3}y = 0$$

with singularities at 0, 1 and ∞ with respective exponents

$$\{\frac{3}{4}, 1, \frac{5}{4}\}, \quad \{\frac{1}{2}, 1, \frac{3}{2}\}, \quad \{-\frac{4}{3}, -\frac{13}{12}, -\frac{7}{12}\}.$$

For the first equation we have $g_0 = 0$, $M = |PG_{54}| = 18$ and $e_0 = e_1 = e_{-1} = e_\infty = 2$, so that

$$\sum_{i \in \{0,1,-1,\infty\}} \left(1 - \frac{1}{e_i}\right) = 2;$$

for the second equation we have $g_0 = 0$, $M = |F_{36}| = 36$ and $e_0 = e_\infty = 4$, $e_1 = 2$ so that

$$\sum_{i \in \{0,1,\infty\}} \left(1 - \frac{1}{e_i}\right) = 2;$$

and, for the third equation we again have $g_0 = 0$, $M = |F_{36}| = 36$ and $e_0 = e_\infty = 4$, $e_1 = 2$. This tells us that the algebraic extension given by the ratio of solutions is a curve of genus 1. Indeed, the three equations are related to the generalized hypergeometric equation defining ${}_3F_2(-\frac{1}{12}, \frac{1}{6}, \frac{2}{3}, \frac{1}{2}, \frac{3}{4}; z)$, i.e.

$$y''' + \frac{3}{4}\frac{5z-3}{z(z-1)}y'' + \frac{1}{24}\frac{43z-9}{z^2(z-1)}y' - \frac{1}{108z^2(z-1)}y = 0.$$

The first of our equations is projectively equivalent to the pullback of this one by the map $z(x) = \frac{1}{16}\frac{(x^2+1)^4}{x^2(x+1)^2(x-1)^2}$; the second one is projectively equivalent to the pullback by the map $z(x) = \frac{4(x-1)}{x^2}$; and the third one is the normalized form which is standard. If one takes three linearly independent solutions to each of these equations, we can see that they satisfy a homogeneous equation of degree 3 in three variables defining an elliptic curve.

EXAMPLE 2.3.5. Consider the following two equations. The first is van

Hoeij's $H_{72}^{SL_3}$ equation [28], i.e.

$$\begin{aligned} 0 &= y''' + \frac{21x^2 - 24x - 1}{(3x^2+1)(x-1)} y'' + \frac{1}{48} \frac{4437x^3 - 5973x^2 + 171x - 683}{(3x^2+1)^2(x-1)} y' \\ &\quad + \frac{1}{216} \frac{13338x^4 - 22647x^3 + 1983x^2 - 7297x - 737}{(3x^2+1)^3(x-1)} y. \end{aligned}$$

The singular points are 1 (which actually is an apparent singularity), $\frac{i\sqrt{3}}{3}$, $-\frac{i\sqrt{3}}{3}$ and ∞, with respective exponents

$$\{0,1,3\}, \quad \{-\frac{7}{12}, -\frac{1}{3}, -\frac{1}{12}\}, \quad \{-\frac{7}{12}, -\frac{1}{3}, -\frac{1}{12}\}, \quad \{\frac{13}{12}, \frac{4}{3}, \frac{19}{12}\}.$$

The second is

$$y''' + \frac{1}{432} \frac{405z^2 - 469z + 384}{(z-1)^2 z^2} - \frac{1}{11664} \frac{10935z^3 - 18803z^2 + 27196z - 10368}{(z-1)^3 z^3} = 0.$$

Here the singular points are 0, 1 and ∞, with respective exponents

$$\{\frac{2}{3}, 1, \frac{4}{3}\}, \quad \{\frac{5}{9}, \frac{8}{9}, \frac{14}{9}\}, \quad \{-\frac{5}{4}, -1, -\frac{3}{4}\}.$$

For the first equation we have $g_0 = 0$, $M = |H_{72}| = 72$ and $e_{\frac{i\sqrt{3}}{3}} = e_{-\frac{i\sqrt{3}}{3}} = e_\infty = 4$, so that

$$\sum_{j \in \{\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}, \infty\}} \left(1 - \frac{1}{e_j}\right) = \frac{9}{4} = \frac{2(10-1)}{M} + 2.$$

For the second equation we have $g_0 = 0$, $M = |H_{216}| = 216$ and $e_0 = e_1 = 3$, $e_\infty = 4$, so that

$$\sum_{i \in \{0,1,\infty\}} \left(1 - \frac{1}{e_i}\right) = \frac{25}{12} = \frac{2(10-1)}{M} + 2.$$

This tells us that the algebraic extension given by the ratio of solutions is a curve of genus 10. Indeed, the two equations are related to the generalized hypergeometric equation ${}_3F_2(-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}, \frac{1}{3}, \frac{2}{3}; z)$, i.e.

$$y''' + \frac{1}{3}\frac{11z-6}{z(z-1)}y'' + \frac{1}{432}\frac{-96+757z}{z^2(z-1)}y' - \frac{17}{5832}\frac{1}{z^2(z-1)} = 0.$$

The first of our equations is projectively equivalent to the pullback of this one by the map $z(x) = \frac{1}{2}\frac{(x+1)^3}{(1+3x^2)}$; whereas the second one is the normalized form and is standard. If one takes three linearly independent solutions to each of these equations, we can see that they satisfy a homogeneous equation of degree 6 in three variables defining a curve of genus 10.

2.4 The classifying ruled surfaces.

For the remainder of the chapter we will assume that the projective Fano curve X_0 is non-degenerate. Recall that this curve is defined by the homogeneous elements in $\mathbb{C}[X^i_j]^G$ vanishing when we evaluate them at (y^i_j). The aim of this section is to explain how we can associate a ruled surface to each class of projectively equivalent standard connections over X_0.

Definition 2.4.1. A *ruled surface* is a surface Σ (a two dimensional $\mathbb{C}$-variety), together with a surjective map $\pi : \Sigma \to X_0$, where X_0 is a curve (a one dimensional $\mathbb{C}$-variety), such that the fibers $\pi^{-1}y$ are isomorphic to $\mathbb{P}^1(\mathbb{C})$, for every $y \in X_0$.

Consider an irreducible standard connection ∇_0 over X_0 with unimodular reductive Galois group G and fundamental system of solutions (y^i_j); and we fix $P(X^i_j) \in \mathbb{C}[X^i_j]^G$ such that $P(y^i_j) = 1$. Let $P_1, \ldots, P_r \in \mathbb{C}[X^i_j]^G$ be homogeneous generators of the subalgebra of G-invariants in $\mathbb{C}[X^i_j]$. We denote by I^G the kernel of the evaluation $\mathbb{C}$-morphism

$$\begin{array}{rcl} \mathbb{C}[X^i_j]^G & \longrightarrow & \mathbb{C}(X_0) \\ P_l(X^i_j) & \longmapsto & P_l(y^i_j) \quad \forall l \in \{1, \ldots, r\}, \end{array}$$

which includes the homogeneous ideal J that defines the Fano curve. Geometrically we have a curve $V(I^G)$ in $(\mathbb{C}^{n\times n})^G$ generating the cone $V(J)$.

We embed $\mathbb{C}^{n\times n}$ into $\mathbb{P}^{n\times n}(\mathbb{C})$ by introducing the homogeneous coordinates $(\mathfrak{z} : \mathfrak{x}^i_j)$ for $\mathbb{P}^{n\times n}(\mathbb{C})$ and identifying $\mathbb{C}^{n\times n}$ with $\mathfrak{z} = 1$. In particular we put $X^i_j = \frac{\mathfrak{x}^i_j}{\mathfrak{z}}$. We extend the action of G on $\mathbb{C}^{n\times n}$ to $\mathbb{P}^{n\times n}(\mathbb{C})$ by declaring

$$(g^i_j) : \mathfrak{z} \mapsto \mathfrak{z},\ \mathfrak{x}^i_j \mapsto \mathfrak{x}^i_l g^l_j.$$

Now consider $\mathbb{P}^{n\times n}(\mathbb{C}) \times \mathbb{P}^{n\times n-1}(\mathbb{C})$, where the second factor has homogeneous coordinates $(\mathfrak{y}^i_j)$. Again, we extend the action of G by declaring that on the second factor we have

$$(g^i_j) : \mathfrak{y}^i_j \mapsto \mathfrak{y}^i_l g^l_j.$$

So that the variety Y defined by the homogeneous equations $\mathfrak{x}^i_j \mathfrak{y}^\iota_\kappa - \mathfrak{x}^\iota_\kappa \mathfrak{y}^i_j$ is invariant under the G-action. Indeed,

$$(g^i_j)(\mathfrak{x}^i_j \mathfrak{y}^\iota_\kappa - \mathfrak{x}^\iota_\kappa \mathfrak{y}^i_j) = g^l_j g^\lambda_\kappa (\mathfrak{x}^i_l \mathfrak{y}^\iota_\lambda - \mathfrak{x}^\iota_\lambda \mathfrak{y}^i_l).$$

So we have a commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{P}^{n\times n}(\mathbb{C}) \times \mathbb{P}^{n\times n-1}(\mathbb{C}) \\ & \searrow & \downarrow \\ & & \mathbb{P}^{n\times n}(\mathbb{C}) \end{array}$$

of G-morphisms which yields

$$\begin{array}{ccc} Y^G & \hookrightarrow & (\mathbb{P}^{n\times n}(\mathbb{C}) \times \mathbb{P}^{n\times n-1}(\mathbb{C}))^G \\ & \searrow^{\varpi} & \downarrow \\ & & (\mathbb{P}^{n\times n}(\mathbb{C}))^G \end{array}$$

We set $\Sigma := \varpi^{-1}\overline{V(J)}$.

Lemma 2.4.2. *The map:*

$$\begin{aligned} \pi : \Sigma &\longrightarrow X_0 \\ (\mathfrak{z} : \mathfrak{x}^i_j, \mathfrak{y}^\iota_\kappa) \cdot G &\longmapsto (\mathfrak{y}^\iota_\kappa) \cdot G \end{aligned}$$

defines a ruled surface.

Proof: Let $(q^\iota_\kappa) \cdot G \in X_0$. Let $R(X^i_j) \in \mathbb{C}[X^i_j]^G$ be homogeneous and such that $R(q^\iota_\kappa) \neq 0$. If $Q(X^i_j) \in \mathbb{C}[X^i_j]^G$ is such that $Q(q^\iota_\kappa) = 0$, and if $(p^i_j) \cdot G \in \mathbb{P}^{n\times n-1}(\mathbb{C}))^G$ is such that $(\mathfrak{z} : p^i_j, q^\iota_\kappa) \in \Sigma$, then

$$Q(p^i_j)R(q^\iota_\kappa) - R(p^i_j)Q(q^\iota_\kappa) = 0,$$

implying that $Q(p^i_j) = 0$. Therefore $(p^i_j) = (q^\iota_\kappa)$. We have then $\pi^{-1}(q^\iota_\kappa) = \overline{\{(\mathfrak{z} : q^i_j, q^\iota_\kappa) \cdot G |\ \mathfrak{z} \in \mathbb{C}\}}$, so that $\pi^{-1}(q^\iota_\kappa)$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$. ★

REMARK 2.4.3. If follows from the proof that the ruled surface Σ is determined by the homogeneous polynomials

$$Q(\mathfrak{x}^i_j)R(\mathfrak{y}^\iota_\kappa) - R(\mathfrak{x}^i_j)Q(\mathfrak{y}^\iota_\kappa), \quad Q(X^i_j) \in J,\ R(X^i_j) \in \mathbb{C}[X^i_j]^G.$$

In particular two projectively equivalent standard connections have the same ruled surface because their ideals J coincide. Now we will prove that the connection is uniquely determined then by $\pi : \Sigma \to X_0$ together with the curve $\varpi^{-1}(\overline{V(I_G)})$.

Lemma 2.4.4. *To a standard connection ∇_0 over X_0 with unimodular reductive Galois group we associate a ruled surface $\pi : \Sigma \to X_0$. If two standard connections are projectively equivalent then their associated ruled surfaces coincide. Among the connections associated to $\pi : \Sigma \to X_0$, ∇_0 is characterized by the curve $\varpi^{-1}(\overline{V(I_G)})$.*

Proof: The first two statements in the lemma follow from the previous one and the remark just above. Now if we restrict ourselves to the open set $U = \{\mathfrak{z} \neq 0\}$ on $\mathbb{P}^{n\times n}(\mathbb{C})$ and to the open set $V = \{P(q^\iota_\kappa) \neq 0\}$ on X_0, then the map (using the notation from Compoint's Theorem 2.2.3)

$$\begin{array}{rcl} \mathbb{C}[X^i_j]^G & \longrightarrow & \mathbb{C}(X_0) \\ P_i & \longmapsto & f_i \end{array}$$

sends V into $V(I_G)$. This map is the one defining the maximal differential ideal I in Compoint's Theorem. ★

REMARK 2.4.5. The ruled surfaces obtained by blowing-up the vertex of a cone can also be obtained by taking the projective bundle defined by a rank-two holomorphic vector bundle over the base curve X_0. In order to obtain Σ through a vector bundle, we are going to construct a two-dimensional vector bundle over $\mathbb{P}^{n\times n-1}(\mathbb{C})^G$ which we will then pull back to X_0. We adapt the exposition in [15, Example V.2.11.4] to our specific setting.

Given a positive integer N we denote by $\mathbb{C}[X^i_j]^G_{\geq N}$ the $\mathbb{C}[X^i_j]^G$-algebra of polynomials of degree greater than or equal to N. Note that

$$\mathbb{C}[X^i_j]^G = \bigoplus_{m\in\mathbb{N}} \mathbb{C}[X^i_j]^G_m$$

where $\mathbb{C}[X^i_j]^G_m$ denotes the polynomial of degree m; and

$$\mathbb{C}[X^i_j]^G_{[N]} = \bigoplus_{m\in\mathbb{N}} \mathbb{C}[X^i_j]^G_{mN}$$

can both be used as homogeneous coordinates of $\mathbb{P}^{n\times n-1}(\mathbb{C})^G$.

Now assume $P_1, \ldots, P_r \in \mathbb{C}[X^i_j]^G$ is a minimal set of generators of $\mathbb{C}[X^i_j]^G$. Denote by n_i the degree of P_i, $i \in \{1, \ldots, r\}$, and by $N = [n_1, \ldots, n_r]$ the least common multiple of these degrees. Set $M_0 = \mathbb{C}[X^i_j]^G$ and $M_N = \mathbb{C}[X^i_j]^G_{\geq N}$. ($M_N$ is graded by $\deg - N$.) Each of these two free graded $\mathbb{C}[X^i_j]^G$-modules define two rank one holomorphic vector bundles $L_0 \to \mathbb{P}^{n\times n-1}(\mathbb{C})^G$ and $L_N \to \mathbb{P}^{n\times n-1}(\mathbb{C})^G$ respectively. Let $V \to \mathbb{P}^{n\times n-1}(\mathbb{C})^G$ be the fibred sum of L_0 and L_N. Denote by $\mathscr{O}$ the sheaf of holomorphic sections of $L_0 \to \mathbb{P}^{n\times n-1}(\mathbb{C})^G$ and by $\mathscr{O}(\mathrm{N})$ the sheaf of holomorphic sections of $L_N \to \mathbb{P}^{n\times n-1}(\mathbb{C})^G$. In particular we have that the sheaf of sections of $V \to \mathbb{P}^{n\times n}(\mathbb{C})^G$ is $\mathscr{V} = \mathscr{O} \oplus \mathscr{O}(N)$.

We want to construct the graded $\mathbb{C}[X^i_j]^G$-module giving V. Set

$$\mathbb{C}[X^i_j]^G[Z] \otimes_{\mathbb{C}[X^i_j]^G} \mathbb{C}[X^i_j]^G_{\geq N} =: \mathbb{C}[X^i_j]^G[Z][Y^i_j]^G_{\geq N},$$

where Z will have degree 0 and $\mathbb{C}[Y^i_j]^G_{\geq N}$ will be graded, as above, by $\deg -N$. To obtain the desired module we take the quotient algebra defined by the

$\mathbb{C}[X^i_j]^G$-morphism

$$\begin{array}{rcl} \mathbb{C}[X^i_j]^G[Z][Y^i_j]^G_{\geq N} & \longrightarrow & \mathbb{C}[X^i_j]^G \\ Z & \longmapsto & 1 \\ Q(Y^i_j) & \longmapsto & Q(X^i_j), \quad \text{for } Q(Y^i_j) \in \mathbb{C}[Y^i_j]^G_{\geq N}. \end{array}$$

The kernel of this morphism is generated by the elements

$$Q(X^i_j)R(Y^i_j) - R(X^i_j)Q(Y^i_j) \quad Q(X^i_j), R(X^i_j) \in \mathbb{C}[X^i_j]^G_{\geq N},$$

which would yield the same ideal defining Y^G if instead of using $\mathbb{C}[X^i_j]^G$ as homogeneous coordinate ring of $\mathbb{P}^{n\times n-1}(\mathbb{C})^G$ we were to use $\mathbb{C}[X^i_j]^G_{[N]}$. So we conclude that Y^G corresponds to the projective bundle defined by $V \to \mathbb{P}^{n\times n}$.

REMARK 2.4.6. Now we denote the pullback to X_0 of $\mathscr{V}$ (resp. of $\mathscr{O}$, of $\mathscr{O}(N)$) by $\mathscr{V}_{X_0}$ (resp. by $\mathscr{O}_{X_0}$, by $\mathscr{O}_{X_0}(N)$). Then as Y^G is obtained as the projective bundle from $\mathscr{V}$, the portion Σ over X_0 is obtained by taking the projective bundle defined by $\mathscr{V}_{X_0}$. In particular, since $\mathscr{V}_{X_0} = \mathscr{O}_{X_0} \oplus \mathscr{O}_{X_0}(N)$ and $\mathscr{O}_{X_0}$ is the sheaf of holomorphic functions over X_0, we conclude that Σ is uniquely determined by $\mathscr{O}_{X_0}(N)$.

It is costumary to normalize $\mathscr{V}$ by tensoring with $\mathscr{O}(-N) = \mathrm{Hom}_{\mathscr{O}}(\mathscr{O}(N), \mathscr{O})$. This does not change the induced projective bundle, so that the rank two vector bundle inducing Σ is $\mathscr{O}_{X_0} \oplus \mathscr{O}_{X_0}(-N)$.

Theorem 2.4.7. *Let ∇_0 and ∇ be two standard connections, with reductive Galois groups, defined on the same meromorphic vector bundle $E \to X_0$. Then ∇_0 and ∇ are projectively equivalent if and only if their associated ruled surface coincide.*

Proof: The necessity has been already established in Lemma 2.4.4. We retain the same notation as above for the connection ∇_0. In particular $\pi : \Sigma \to X_0$ is the associated ruled surface, and $P(y^i_j) = 1$ for some fixed $P(X^i_j) \in \mathbb{C}[X^i_j]^G$. Because $\overline{V(J)}$ is a projective variety of dimension two, we have that the irreducible variety $\overline{V(I_G)}$ is in the closure of the curve $V(J) \cap V(P(X^i_j) - 1)$, and they coincide if $V(J) \cap V(P(X^i_j) - 1)$ is irreducible. Without loss of generality we may assume that ∇_0 and ∇ are normalized so that their n-th exterior product has rational sections. In such a case $P(X^i_j) = \det(X^i_j)$, $P(X^i_j) - 1$ is prime and $\overline{V(I_G)} = \overline{V(J) \cap V(P(X^i_j) - 1)}$. It follows that the curve in Σ defining ∇_0 and ∇ coincide (cf. Lemma 2.4.4), whence $\nabla = \nabla_0$.

★

Chapter 3

Symmetries of linear differential equations

In 1900, G. Fano addressed the following problem [9]: what are the consequences of algebraic relations between the solutions of a linear differential equation? The problem was apparently proposed to him by F. Klein. A particular concern was whether or not a linear differential equation with solutions satisfying a homogeneous polynomial can be “solved in terms of linear equations of lower order”. This has been successfully studied by M. Singer, cf. [27], and more recently by K.A. Nguyen, cf. [24].

Fano considered the group of projective automorphisms of the projective variety having the solutions of the differential equation as coordinate functions. This could be viewed as a primitive version of the differential Galois Group. Here we replace the Fano group with one slightly smaller: the group of projective automorphisms of the projective variety having as coordinate ring the $\mathbb{C}$-algebra generated by the solutions of the differential equation together with the i'th derivatives, $i \in \{1, \ldots, n-1\}$ (where n is the order of the equation).

We treat the problem in terms of connections. Suppose we are given a ramified covering map $\phi : X' \longrightarrow X$ of compact Riemann surfaces, together with a meromorphic vector bundle E with a connection ∇ over X. We can use ϕ to pull back the bundle and the connection, obtaining $E' = \phi^* E$ together with $\nabla' = \phi^* \nabla$. In this context one obtains [23, Appendix B] a natural injection

$$\mathrm{Gal}(\nabla') \longrightarrow \mathrm{Gal}(\nabla),$$

where the induced map on the Lie algebras is an isomorphism (i.e. the connected components of 1 are isomorphic). Note that, a covering transformation $\sigma \in \mathrm{Aut}_{\mathbb{C}}(X')$ of $\phi : X' \longrightarrow X$ lifts to a horizontal automorphism of the vector bundle (E', ∇'). The automorphisms lifting to the connection are called symmetries (of the connection). A concise exposition on symmetries is given in [6].

Conversely, suppose we begin with a connection on a meromorphic vector bundle over a Riemann surface which admits a symmetry (such equations are often called reversible). We can then consider the quotient Riemann surface with the canonically induced vector bundle and connection. This new connection has Galois group with bigger monodromy subgroup, but with identity component isomorphic to that of the original Galois group.

Checking for symmetries of a given connection over the Riemann sphere is quite easy (just consider permutations of singular points). Methods for revealing them in arbitrary contexts are far from simple. For example, one can consider the work by Dwork and Baldassarri [1], [2]. The study of symmetries is intimately linked to the study of descent conditions and to the identification of pullbacks. These pullbacks, on their turn, are important in the classification of equations and in algorithmic implementations.

The purpose of this chapter is to suggest how the outer-automorphisms in the Fano group of the Galois group correspond to symmetries of the connection, and to give a proof in the case when the Galois group is reductive and unimodular and the connection is standard (Theorem 3.2.12). For ease of reference we will recall some definitions from the previous chapter.

3.1 Algebraic justification

REMARK 3.1.1. The argument behind the proof of Theorem 3.2.12 is of a geometric nature. Nevertheless, it is possible to describe the phenomena studied in this chapter algebraically. We do so informally in this section. For instance, the examples in the last section correspond more to the algebraic point of view than to the geometric one.

We consider a field k over $\mathbb{C}$ of transcendence degree one, together with a non-trivial derivation v. Recall that a non-trivial derivation is a non-zero additive map $v : k \to k$ satisfying the Leibnitz rule:

$$v(fg) = v(f)g + fv(g).$$

The collection of linear differential operators over (k, v) forms a noncommutative ring extension $k[v]$ of k with multiplication given by

$$v \cdot f = v(f) + f \cdot v.$$

As a consequence we see that every linear differential operator $L \in k[v]$ can be written in the form

$$L = a_n v^n + a_{n-1} v^{n-1} + \ldots, +a_1 v + a_0$$

with $a_i \in k$ for $i \in \{0, 1, \ldots, n\}$.

Let $\sigma^* \in \mathrm{Aut}_{\mathbb{C}}(k)$ be an automorphism k over $\mathbb{C}$. We can naturally lift σ^* to an automorphism of $k[v]$ by setting

$$\sigma^* v : f \mapsto v(\sigma^* f)$$

Note that $\sigma^* v$ is a non-trivial derivation. In particular, since the k-module of derivations of k has rank one over k, there is a non-zero element $v(\sigma) \in k$ such that:

$$\sigma^* v = \frac{1}{v(\sigma)} v.$$

One can define σ^* to be a symmetry of L if $\sigma^* L = f_\sigma \cdot L$ for some $f_\sigma \in k$. This means that the solutions to the equation $L(y) = 0$ and to $\sigma^* L(y) = 0$ coincide; classically we say that σ^* is a period of L.

Assume now that v is invariant under σ^* (i.e. that $v(\sigma) = 1$). Then v determines a derivation on the subfield $k^{\sigma^*} \subseteq k$ fixed by σ^*. In this case, if σ^* is a symmetry of L, then L has coefficients in k^{σ^*} and L restricts to a differential operator on (k^{σ^*}, v). Theorem 3.2.12 says that the symmetries of

L manifest themselves as outer-automorphisms of the Galois group of L over (k, v).

REMARK 3.1.2. In broad terms the geometric idea behind proof of the main result is the following. Let X be the projective algebraic curve with $\mathbb{C}(X) = k$. We fix a point $p \in X$. The differential module $k[v]/L$ defines a vector bundle with connection over X [25, Chapter 2]. The Galois group G of L over k acts on a fiber over p of this vector bundle. Given $\sigma^* \in \mathrm{Aut}_{\mathbb{C}}(X)$, it defines an automorphism σ of X. If σ^* is a symmetry of L we have two ways of identifying the fiber over p and the fiber over $\sigma(p)$: by analytic extension through a path from $\sigma(p)$ to p, or via the symmetry σ. Jumping from one identification to the another amounts to acting on G by an outer-automorphisms provided L is standard (cf. Definition 3.2.8). The geometric interpretation of L having coefficients in k^{σ^*} when v is σ^*-invariant is: the connection defined by $k[v]/L$ descents to a connection over X^σ. Explicitly it descends to the connection defined by $k^{\sigma^*}[v]/L$.

REMARK 3.1.3. Let us illustrate the standard hypothesis with an example from number fields. We take as base field $\mathbb{Q}(i)$. Consider the Galois extension of the polynomial

$$X^4 - i = 0;$$

namely $\mathbb{Q}(i)(e^{\frac{\pi}{8}i})$. The Galois group is $\mathbb{Z}/4\mathbb{Z}$, and is generated by

$$e^{\frac{\pi}{8}i} \mapsto ie^{\frac{\pi}{8}i}.$$

The arithmetic interpretation of the standard hypothesis is that

$$\mathbb{Q}(i)(e^{\frac{\pi}{8}i}) = \mathbb{Q}(e^{\frac{\pi}{8}i})$$

which implies $\mathbb{Q}(i) \subseteq \mathbb{Q}(e^{\frac{\pi}{8}i})$. We can act on the extension $\mathbb{Q}(i) \subset \mathbb{Q}(i)(e^{\frac{\pi}{8}i})$ by complex conjugation. Complex conjugation transforms the polynomial $X^4 - i$ into $X^4 + i$. Notice that the Galois extension over $\mathbb{Q}(i)$ of these two polynomials is the same. As above the Galois group of $X^4 + i$ is generated by $e^{\frac{-\pi}{8}i} \mapsto ie^{\frac{-\pi}{8}i}$, i.e. by

$$e^{\frac{\pi}{8}i} \mapsto -ie^{\frac{\pi}{8}i} = i^3 e^{\frac{\pi}{8}i}.$$

We put everything together in the tower

$$\mathbb{Q} \subset \mathbb{Q}(i) \subset \mathbb{Q}(i)(e^{\frac{\pi}{8}i})$$

corresponding to the polynomial $X^8 + 1 = (X^4 - i)(X^4 + i)$. The Galois group of the tower is the dihedral group of order 8, and complex conjugation corresponds to the outer-automorphism of the cyclic subgroup inside the dihedral group. The discussion is summarized by the exact sequence

$$1 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_{2\cdot 4} \longrightarrow \langle i \mapsto -i \rangle \longrightarrow 1.$$

3.2 Setting and definitions

REMARK 3.2.1. We remind that we use the Einstein summation convention for indices.

Let X be a (connected) compact Riemann surface with field of meromorphic functions k. Let

$$\Pi : E \longrightarrow X$$

be an n-dimensional meromorphic vector bundle, with a meromorphic connection

$$\nabla : \mathscr{E} \longrightarrow \Omega^1_{\mathscr{M}} \otimes_k \mathscr{E}$$

where $\Omega^1_{\mathscr{M}}$ denotes the meromorphic differential forms over X and $\mathscr{E}$ the meromorphic sections of Π. We also denote by $\mathscr{T}X$ the vector fields of meromorphic tangent vectors to X. There is a natural map

$$\begin{aligned} \mathscr{T}X \otimes_k \Omega^1_{\mathscr{M}} &\longrightarrow k \\ v \otimes \eta &\longmapsto \eta(v) \end{aligned}$$

which canonically extends to

$$\begin{aligned} \mathscr{T}X \otimes_k \Omega^1_{\mathscr{M}} \otimes_k \mathscr{E} &\longrightarrow \mathscr{E} \\ v \otimes \eta \otimes X &\longmapsto \langle v, \eta \otimes X \rangle := \eta(v)X. \end{aligned}$$

Given a meromorphic tangent vector field $v \in \mathscr{T}X$ we denote by ∇_v the derivation on $\mathscr{E}$

$$\nabla_v(X) = \langle v, \nabla X \rangle.$$

Definition 3.2.2. Let $\sigma \in \mathrm{Aut}_{\mathbb{C}}(X)$ (or equivalently $\sigma^* \in \mathrm{Aut}_{\mathbb{C}}(k)$). We say that σ is a *symmetry* of ∇ if there is a horizontal vector bundle morphism

$$\tilde{\sigma} : (E, \nabla) \to (E, \nabla)$$

lifting σ, i.e. $\tilde{\sigma}(\nabla_v X) = \nabla_{\sigma_* v} \tilde{\sigma}(X)$ and $\sigma \circ \Pi = \Pi \circ \tilde{\sigma}$.

$$\begin{array}{ccc} E & \overset{\tilde{\sigma}}{\dashrightarrow} & E \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ X & \underset{\sigma}{\longrightarrow} & X. \end{array}$$

The group of symmetries of ∇ will be denoted by $\mathrm{Aut}_{\nabla}(X)$.

REMARK 3.2.3. A symmetry of ∇ permutes the singular points.

Let (U, z) be a holomorphic chart of X centered at $p \in U$, where $U \subseteq X$ is an open ball avoiding the singularities of ∇. E is holomorphic and trivial above U and ∇ induces a holomorphic connection ∇' on $\Pi^{-1}U \to U$. Let $(\Pi^{-1}U, z, y^1, \dots, y^n)$ be a trivializing chart of E. There exists a holomorphic horizontal frame $V_1, \dots, V_n$ over U, i.e.

$$v_j^i(z) = y^i(V_j(z))$$

with $v_j^i(z)$ holomorphic in U such that $\nabla'_{\frac{\partial}{\partial z}} V_j = 0$ and $\det(v_i^j)(z)$ does not vanish in U (see Remark 2.1.20 or [19]).

Definition 3.2.4. The Fano group G_F of ∇ is the subgroup of $GL_n(\mathbb{C})$ fixing the homogeneous ideal in $\mathbb{C}[X_j^i]$ generated by the G-invariant homogeneous polynomials $P[X_j^i] \in \mathbb{C}[X_j^i]$ vanishing at v_j^i.

REMARK 3.2.5. The Fano group defined here differs from that considered in [24]. In fact our group G_F contains the group G^+ used in [24]. For instance, if the connection is standard (cf. Definition 3.2.8) and the Galois group is finite, G^+ coincides with G but G_F may be larger (cf. Example 3.6.1). In [9], G. Fano worked with automorphisms of projective varieties, so our approach is in the same spirit.

REMARK 3.2.6. It follows directly from the definition that there is a canonical inclusion of the representation in $GL_n(\mathbb{C})$ of the Galois group G given by v_j^i into the Fano group G_F. Let us make this remark more explicit.

Fix $v \in \mathscr{T}X$, $v \neq 0$, and a global meromorphic frame $(e_1, \ldots, e_n)$ of $\mathscr{E}$, i.e. an n-tuple of meromorphic sections such that on some Zariski open subset $X' \subseteq X$, the n-tuple $(e_1(q), \ldots, e_n(q))$ is a basis of $\Pi^{-1}(q)$ for each $q \in X'$. Let $a^i_j \in k$ be such that

$$\nabla_v e_j = -a^i_j e_i$$

(recall Remark 3.2.1). Thus, in this frame, the equation $\nabla_v X = 0$ is equivalent to $X' = AX$, where $A = (a^i_j)$ and $X' = v(x^i)e_i$ if $X = x^i e_i$, $x^i \in k$.

We review the construction of a Picard-Vessiot extension from the previous chapter. We define the differential ring extension $(k[X^i_j, \frac{1}{\det}], \tilde{v})$ of (k, v), where (X^i_j) is an $n \times n$ matrix of indeterminates, $\det := \det(X^i_j)$ and

$$\tilde{v}(X^i_j) = a^i_k X^k_j.$$

Note that we can make $GL_n(\mathbb{C})$ act on $k[X^i_j, \frac{1}{\det}]$ through differential automorphisms over (k, v) by setting for $(g^i_j) \in GL_n(\mathbb{C})$

$$\begin{array}{rcl} (g^i_j) : k[X^i_j, \dfrac{1}{\det}] & \longrightarrow & k[X^i_j, \dfrac{1}{\det}] \\ X^i_j & \longmapsto & X^i_l g^l_j \end{array}$$

A Picard-Vessiot extension of k for the matrix differential equation $X' = AX$ is given by the quotient field of

$$k[X^i_j, \frac{1}{\det}]/I,$$

where I is a maximal differential ideal. Since $GL_n(\mathbb{C})$ acts through differential automorphisms, the action permutes the maximal differential ideals. A representation of the differential Galois group of ∇ is given by [21, Corollary 4.10]

$$G = \{(g^i_j) \in GL_n(\mathbb{C}) |\ (g^i_j) : I \mapsto I\}$$

the stabilizer of I under this action.

Let us assume that the holomorphic chart (U, z) and v are such that v restricted to U coincides with $\frac{\partial}{\partial z}$. Fix an injection $\iota : k \to \mathbb{C}[\frac{1}{z}][[z]]$. Identifying $v^i_j(z)$ with their power series expansion we have a differential homomorphism

$$\begin{array}{rcl} \Phi : (k[X^i_j, \dfrac{1}{\det}], \tilde{v}) & \longrightarrow & (\mathbb{C}[\dfrac{1}{z}][[z]], \dfrac{\partial}{\partial z}) \\ X^i_j & \longmapsto & v^i_j(z) \end{array}$$

such that $\Phi(f) = \iota(f)$ if $f \in k$, i.e.

$$\begin{array}{ccc} k[X^i_j, \frac{1}{\det}] & \xrightarrow{\Phi} & \mathbb{C}[\frac{1}{z}][[z]] \\ \uparrow & \nearrow_{\iota} & \\ k & & \end{array}$$

If we set $I = \ker(\Phi)$, then I is a maximal differential ideal and the choice of $v^i_j(z)$ induces the representation of the Galois group of ∇ by G.

In order to state our theorem we need to introduce the following concepts [17] [3]:

Definition 3.2.7. Let $P[X^i_j] \in \mathbb{C}[X^i_j]$ be a homogeneous non-constant polynomial. If $P[v^i_j](z) = \iota(f)$ for some $f \in k$ we say that f is a *dual first integral* of ∇ with degree defined to be the degree of P. We denote by k_∇ the field generated over $\mathbb{C}$ by the quotients of dual first integrals of the same degree.

Definition 3.2.8. The connection ∇ is called:

- *standard* if $k(v_j^i) = \mathbb{C}(v_j^i)$ and k is Galois over k_∇.
- *basic* if $k = k_\nabla$.

REMARK 3.2.9. We already defined standard connection in the previous chapter. Both definitions coincide as $k_\nabla = \mathbb{C}(X_0)$, where X_0 is the projective Fano curve of ∇.

REMARK 3.2.10. Since $k_\nabla \subseteq \mathbb{C}(v_j^i)$ then if ∇ is basic we have $\mathbb{C}(v_j^i) = k_\nabla(v_j^i) = k(v_j^i)$. In particular basic connections are standard.

REMARK 3.2.11. As a corollary of Lemma 3.4.2, we will see that, under the unimodular and reductive hypothesis, the automorphisms of k over k_∇ can be identified as a subgroup of the symmetries of ∇ provided k is standard. In particular, this means that ∇ would descend to a connection over k_∇. Note that k and k_∇ are $\mathbb{C}$-algebras. Another approach to the descent problem is seen in [16], where the treatment is in terms of Galois cohomology and deals with descent on the field of constants.

Theorem 3.2.12. *Let ∇ be a standard connection. If the Galois group of ∇ is unimodular and reductive the sequence*

$$1 \longrightarrow Z(G) \longrightarrow G \longrightarrow Aut_{G_F}(G) \longrightarrow Aut_\nabla(X) \longrightarrow 1$$

is exact.

REMARK 3.2.13. In the statement above $Aut_{G_F}(G)$ denotes the image, in the group of automorphisms of G, of the normalizer $N_{G_F}(G)$ of G in G_F. $Z(G)$ denotes the center of G.

The remainer of the chapter is devoted to a proof of this theorem. As in the previous chapter, the hypothesis on the differential Galois group allows us to use the following result [7]:

Theorem 3.2.14 (Compoint)**.** *When G is reductive and unimodular the ideal I is generated by the G-invariants contained therein. Moreover, if $P_1, \dots, P_r$ is a set of homogeneous generators for the $\mathbb{C}$-algebra of G-invariants, with respective degrees $n_1, \dots, n_r$, in $\mathbb{C}[X_j^i]$, and if $f_1, \dots, f_r \in k$ are such that $P_i - f_i \in I$, then I is generated over $k[X_j^i, \frac{1}{\det}]$ by $P_i - f_i$, where $i \in \{1, \dots, r\}$.*

REMARK 3.2.15. In [7] and in [4] the statement of this theorem is reserved for $k = \mathbb{C}(z)$ and $v = \frac{\partial}{\partial z}$. But the proof by F. Beukers in [4] carries through *mutatis mutandis* when $\mathbb{C}(z)$ is replaced by k. The careful reader would note that F. Beukers does not make the unimodular hypothesis explicit in his paper, but the assumption is needed to guarantee the invariance of det.

REMARK 3.2.16. In the notation of the theorem we obtain

$$k_\nabla = \mathbb{C}(\frac{f_i^{m_i}}{f_j^{m_j}})_{\{(i,j) \in \{1,\dots,r\}^2 \mid \ m_i n_i = m_j n_j, \ f_j \neq 0\}}.$$

Proposition 3.2.17. *If ∇ is such that $k(v_j^i) = \mathbb{C}(v_j^i)$ and 1 is a dual first integral of ∇ (i.e. if there is a homogeneous $P(X_j^i) \in \mathbb{C}[X_j^i]^G \setminus \mathbb{C}$ such that $P(v_j^i) = 1$), then k/k_∇ is Galois with abelian Galois group. In particular, ∇ is standard.*

Proof: The assumption on ∇ implies that $\mathbb{C}(v_j^i)^G = k(v_j^i)^G = k$. In the notation of Theorem 3.2.14 we therefore have $\mathbb{C}(v_j^i)^G = \mathbb{C}(f_1, \dots, f_r)$. By hypothesis there is a homogeneous $P(X_j^i) \in \mathbb{C}[X_j^i]^G$ such that $P(v_j^i) = 1$.

Denote by d the degree of $P(X_j^i)$, and by $[d, n_l] := dm_l = n_l d_l$ the least common multiple of d and of the degree of $P_l(X_j^i)$, n_l. Now

$$f_l^{d_l} = \frac{f_l^{d_l}}{1^{m_l}} \in k_\nabla$$

Thus k is the splitting field over the $\mathbb{C}$-algebra k_∇ of the polynomial

$$\prod_{l=1}^{r} (Z^{d_l} - f_l^{d_l}).$$

★

REMARK 3.2.18. It is easy to produce examples to justify the unimodular hypothesis of the theorem. Indeed, consider

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2z & 0 \\ 0 & \frac{1}{z} + 2z \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

A fundamental system of solutions is given by

$$\begin{pmatrix} e^{z^2} & 0 \\ 0 & ze^{z^2} \end{pmatrix},$$

and the connection is therefore basic. Indeed $Y_2^2/Y_1^1 = z$. The Galois group is $\mathbb{G}_{m,\mathbb{C}}$, which is represented by

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}^* \right\}.$$

The ideal of homogenous polynomials vanishing at the solutions is generated by $zX_1^1 - X_2^2$, X_2^1 and X_1^2. There is no vanishing G-invariant homogeneous polynomial of order greater than 0. It follows that G_F is $GL_2(\mathbb{C})$, $N_{G_F}(G) =$

G_F, $G_F/G = PGL_2(\mathbb{C})$, but the connection is symmetric only with respect to $z \mapsto -z$. Indeed, one can lift this symmetry by mapping (y_1, y_2) into $(y_1, -y_2)$.

REMARK 3.2.19. An example justifying the requirement on ∇ to be standard is a little more delicate; when one lifts a standard equation with symmetries by means of a Galois covering, the symmetries are lifted together with the equation. So the way to obtain the example is by lifting an equation with symmetries, say the standard $D_{2\cdot 4}$ equation [3] defined on the x-sphere, through a non-Galois covering, say through $x \mapsto z(z-1)^2$, to the z-sphere.

3.3 Geometric construction: a covering space with covering group $Aut_{G_F}(G)$

REMARK 3.3.1. We keep the same notation from the previous section, we will assume, from now on, that ∇ has reductive unimodular Galois group and that ∇ is standard or basic.

Following J.A. Weil [29] we will work with first integrals (i.e. with the solutions of the adjoint system) rather than with horizontal sections. Recall, from Definition 2.1.14, that a linear first integral of $\nabla'_{\frac{\partial}{\partial z}}$ over $\Pi^{-1}U$ refers to a function $\Phi : \Pi^{-1}U \to \mathbb{C}$ of the form

$$\Phi(z, y^1, \ldots, y^n) = \frac{\partial \Phi}{\partial y^j}(z) y^j,$$

where $\frac{\partial\Phi}{\partial y^j}(z)$ is holomorphic in U and constant on the horizontal sections of $\nabla'_{\frac{\partial}{\partial z}}$, i.e. if $\nabla'_{\frac{\partial}{\partial z}} X = 0$ then $z \mapsto \Phi(X(z))$ is a constant function.

Let Φ^i, $i \in \{1, \ldots, n\}$, be a system of linear first integrals of $\nabla'_{\frac{\partial}{\partial z}}$ over $\Pi^{-1}U$ such that $\frac{\partial\Phi^i}{\partial y^j}$ is an invertible matrix. We call such a system a *fundamental system* of linear first integrals.

Let F be the homogeneous ideal in $\mathbb{C}[X^i_j, \frac{1}{\det}]^G$ generated by the homogeneous polynomials of $\mathbb{C}[X^i_j]^G$ vanishing at $\frac{\partial\Phi^i}{\partial y^j}$.

Lemma 3.3.2. *The ideal F is prime.*

Proof: Without loss of generality we may assume that $\frac{\partial\Phi^1}{\partial y^1} \neq 0$, so that X^1_1 is not in the kernel of the evaluation map $\mathbb{C}[X^i_j, \frac{1}{\det}] \to k(v^i_j) : X^i_j \mapsto \frac{\partial\Phi^i}{\partial y^j}$. So we define the map

$$\begin{array}{rcl} \mathbb{C}[\dfrac{X^i_j}{X^1_1}, \dfrac{(X^1_1)^n}{\det}] & \longrightarrow & k(\dfrac{\partial\Phi^i}{\partial y^j}) \\ \dfrac{X^i_j}{X^1_1} & \longmapsto & \dfrac{\frac{\partial\Phi^i}{\partial y^j}}{\frac{\partial\Phi^1}{\partial y^1}}. \end{array}$$

Every element in the kernel of this map defines, after homogenizing with X^1_1, a homogeneous element in the kernel of the evaluation map. Conversely, every homogeneous element of degree d in the kernel of the evaluation map, after dividing by $(X^1_1)^d$, defines an element in the kernel of this map. Thus, since $k(\frac{\partial\Phi^i}{\partial y^j})$ is an integral domain and the homogeneous elements in the kernel of the evaluation map form a prime ideal. The lemma now follows by letting G act and taking invariants. ★

Note that if $AX = X'$ is the matrix equation form of $\nabla_v X = 0$, the adjoint system[1] for the linear first integrals is given by $X(-A) = X'$. Indeed, if S is a fundamental system of solutions for $AX = X'$ then from

$$S^{-1}S = I_n$$

and we see that S^{-1} is a fundamental system of linear first integrals. Moreover

$$0 = I_n' = (S^{-1}S)' = (S^{-1})'S + S^{-1}S' = (S^{-1})'S + S^{-1}AS$$

and therefore $(S^{-1})' = -S^{-1}A$. When considering first integrals we use the differential ring extension $(k[X_j^i, \frac{1}{\det}], \overline{v})$ of (k, v), where $\overline{v}X_j^i = -X_k^i a_j^k$, and we let $GL_n(\mathbb{C})$ act to the left.

A maximal differential ideal I determines a representation of the Galois group $G \subseteq GL_n(\mathbb{C})$ given by the stabilizer of I. On the other hand I also determines the homogeneous ideal $F \subseteq \mathbb{C}[X_j^i]^G$ generated by the homogeneous elements in $I \cap \mathbb{C}[X_j^i]^G$. According to Compoint's Theorem, any maximal differential ideal of the differential ring $(k[X_j^i, \frac{1}{\det}], \overline{v})$, with stabilizer G and defining the ideal F, is uniquely determined by a $\mathbb{C}$-homomorphism $\phi : R \to k$. Indeed, in the notation of the statement of Compoint's theorem we have a map

$$\begin{array}{rcl} \mathbb{C}[X_j^i, \dfrac{1}{\det}]^G & \longrightarrow & k \\ P_i & \longmapsto & f_i. \end{array}$$

The ideal F, by definition, is contained in the kernel, so this map factors through a unique map $\phi : R \to k$.

[1]The common notation is $X' = -A^t X$ but our notation is aimed to make more transparent the interplay between both systems.

Lemma 3.3.3. *The group $H := Aut_{G_F}(G)$ acts on k_∇.*

Proof: The group $H = N_{G_F}(G)/Z(N_{G_F}(G))$ acts on the Zariski open subset of $\mathrm{Proj}(\mathbb{C}[X^i_j])$ defined by $\det \neq 0$, and by the definition of G_F this action fixes the Zariski closed subset defined by the homogeneous ideal generated by F in $\mathbb{C}[X^i_j]$. So passing to the quotient by the action of $G/Z(G)$, we have that H acts on $\mathrm{Proj}(\mathbb{C}[X^i_j]^G)$, leaving the variety defined by the homogeneous prime ideal F invariant. In particular, it defines an automorphism of the field of meromorphic functions over this variety, which is actually k_∇. ★

In particular, if ∇ is basic the group also describes automorphisms of k (cf. Definition 3.2.8).

$$\begin{array}{c} k[X^i_j, \frac{1}{\det}]/I \\ | \\ k \\ | \\ k_\nabla \\ | \\ (k_\nabla)^H \end{array}$$

We will now give a geometric construction that will make clear the interplay between all the groups and fields involved.

Fix a meromorphic tangent vector field v over X. Let $p \in X$ over which ∇_v is not singular and fix a frame of holomorphic horizontal sections $V_1, \ldots, V_n$ of ∇_v around p. We lift the connection to the frame bundle $GL(\Pi)$.

We extend $U \subseteq X$ to the maximal open set X' over which the frame

$(V_1, \ldots, V_n)$ can be extended holomorphically (as a multi-valued frame).

The extension of $(V_1, \ldots, V_n)$, together with its orbit under the Galois group G, defines a subsheaf $\mathfrak{F}$ of $GL(\Pi) \restriction_{X'}$ whose sections are horizontal holomorphic frames under ∇_v. The sheaf $\mathfrak{F}$ gives rise to a regular covering of X' with covering group G [22] [5, Théorème 5.3.1]. Denote by $\widetilde{X'}$ the covering space corresponding to the center $Z(G)$ of G.

The diagram above implies the following tower of covering spaces:

$$\begin{array}{c} \widetilde{X'} \\ \Big| \\ X' \\ \Big| \\ X'_{\nabla} \\ \Big| \\ (X'_{\nabla})^H \end{array}$$

where X'_{∇} corresponds to the projection of X' to the Riemann surface with meromorphic functions k_{∇}. In the case that ∇ is basic we have $X' = X'_{\nabla}$. So we obtain $\widetilde{X'}$ covering X' with covering group PG, and $\widetilde{X'}$ covering $(X'_{\nabla})^H$ with covering transformations $H = Aut_{G_F}(G)$. By the Galois correspondence, if ∇ is basic, the covering group of X' over $(X'_{\nabla})^H$ is given by the outer-automorphisms of G in G_F. If ∇ is standard then the outer-automorphisms of G in G_F define covering transformations of X' over $(X'_{\nabla})^H$ since G fixes X'_{∇} and X' is Galois over X'_{∇}.

3.4 The map $Aut_{G_F}(G) \longrightarrow Aut_\nabla(X)$

Given a maximal differential ideal $I \subseteq k[X_j^i, \frac{1}{\det}]$, we obtain a representation of the Galois group, namely, the stabilizer $G \subseteq GL_n(\mathbb{C})$ of I. Any $g \in GL_n(\mathbb{C})$ sends I into another maximal differential ideal $I^g := g(I)$. The stabilizer of I^g is now the conjugate gGg^{-1} of G. So the collection of maximal differential ideals whose stabilizer is G, or equivalently, whose representation of the Galois group is G, will be given by the orbit of I under $N_{GL_n(\mathbb{C})}(G)$. We set

$$\mathcal{I} := \{I^g |\ g \in N_{GL_n(\mathbb{C})}(G)\}.$$

On the other hand, each maximal differential ideal $I^g \in \mathcal{I}$ defines the homogeneous ideal $F^g := g(F) \subseteq \mathbb{C}[X_j^i]^G$ given by the homogeneous elements in $I^g \cap \mathbb{C}[X_j^i]^G$. So we consider the sub-collection

$$\mathcal{F} := \{I^g \in \mathcal{I} |\ F = F^g\}$$

of $\mathcal{I}$ which is precisely the orbit of I under the elements of $G_F \cap N_{GL_n(\mathbb{C})}(G)$. So we have

$$\mathcal{F} := \{I^g |\ g \in N_{G_F}(G)\}.$$

Compoint's Theorem implies that for every $I^g \in \mathcal{F}$ we have a unique map

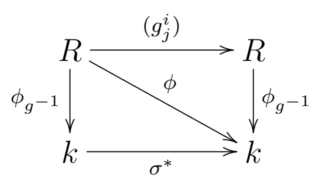
$$\phi_g : R \to k.$$

Note that since $g \in N_{G_F}(G)$ fixes $F \subseteq R$, it defines an automorphism of R. In particular,

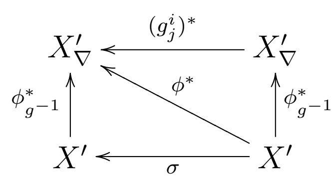
$$\phi \circ g = \phi_g.$$

REMARK 3.4.1. The open set X' from our previous section is the variety given by $X \setminus S$, where S is the collection of points where $\phi(\det)$ vanishes, together with the singular points of ∇.

Consider a $g = (g^i_j) \in N_{G_F}(G)$, so that now we have two k-valued points ϕ and $\phi_{g^{-1}}$. The k-valued point $\phi = \phi_e$ corresponds to the ideal I and $\phi_{g^{-1}}$ to $I^{g^{-1}}$. Under the action notation we write $\phi = \phi_{g^{-1}} \circ (g^i_j)$. Similarly, one can see (g^i_j) acting on k_∇ (Lemma 3.3.3). Indeed, instead of seeing (g^i_j) as acting linearly on G-invariant polynomials, one can see it as acting linearly on quotients of rational first integrals of same degree. On the other hand, k is Galois over k_∇, so the automorphism (g^i_j) on k_∇ lifts to an automorphism of k. We denote by σ^* such a lifting (g^i_j). In this fashion we obtain a commutative diagram:



Reversing the arrows:



Lemma 3.4.2. *The automorphism σ (or equivalently a lifting of $(g^i_j)^*$) is a symmetry of ∇.*

Proof: If $P[X^i_j] \in \mathbb{C}[X^i_j]^G$ and $f \in k$ are such that $\phi_{g^{-1}}(P[X^i_j]) = f$, then

$$\begin{aligned} \phi(P[X^i_j]) &= \phi_{g^{-1}}(P[g^i_k X^k_j]) \\ &= \sigma^* \phi_{g^{-1}}(P[X^i_j]) \end{aligned}$$

i.e.,

$$\phi(P[X^i_j]) \;=\; \sigma^* f.$$

We again take holomorphic charts (U, z) and $(\pi^{-1}U, z, y^1, \ldots, y^n)$ as before in Section 3.2. Set $V = \sigma(U)$ and consider a chart of E giving a vector bundle trivialization $(\Pi^{-1}V, w, x^1, \ldots, x^n)$ with $w \circ \sigma = z$. Note that

$$\sigma_* \frac{\partial}{\partial z}(w) = \frac{\partial}{\partial z}(w \circ \sigma) = \frac{\partial}{\partial z}(z) = 1,$$

hence $\sigma_* \frac{\partial}{\partial z} = \frac{\partial}{\partial w}$.

Let $p \in U$ ($\sigma(p) \in V$) and consider some linear first integrals $\frac{\partial \Psi^i}{\partial x^j}$ defined by $\phi_{g^{-1}}$ in $\Pi^{-1}V$. Since the k-valued point given by Compoint's Theorem is nothing other than the restriction of the evaluation homomorphism, the equalities above implies

$$\begin{aligned}
\frac{\partial \Phi^i}{\partial y^j}(p) \;&=\; g^i_k \frac{\partial \Psi^k}{\partial x^j}(p) \\
&=\; \frac{\partial \Psi^i}{\partial x^j}(\sigma(p)),
\end{aligned}$$

hence that

$$\frac{\partial \Phi^i}{\partial y^j}(p) \;=\; \frac{\partial \Psi^i}{\partial x^j}(\sigma(p)) = \frac{\partial \Psi^i}{\partial y^l}(\sigma(p)) \frac{\partial y^l}{\partial x^j}(\sigma(p)). \tag{3.1}$$

But $\frac{\partial \Phi^i}{\partial y^j}$ is invertible in U, whence

$$\begin{aligned}
\frac{\partial x^j}{\partial y^l}(\sigma(p)) \;&=\; \frac{\partial y^j}{\partial \Phi^i}(p) \frac{\partial \Psi^i}{\partial y^l}(\sigma(p)) \\
&=\; f^j_l(p).
\end{aligned}$$

Letting G act on both sides of (3.1), it follows that f_l^j describes (by analytic extension) a meromorphic function over X (Galois correspondence). So the map $U \to V$ described by

$$\begin{aligned} w(z, y^1, \dots, y^n) &= \sigma(z) \\ x^j(z, y^1, \dots, y^n) &= f_i^j(z) y^i \end{aligned}$$

gives the transform on the fiber coordinates that lifts the covering transformation σ to a horizontal automorphism. ★

If there is another σ' with the property $\phi^* = \phi^*_{g^{-1}} \circ \sigma'$ then

$$\phi^* \circ \sigma^{-1} \circ \sigma' = \phi^*_{g^{-1}} \circ \sigma \circ \sigma^{-1} \circ \sigma' = \phi^*_{g^{-1}} \circ \sigma' = \phi^*.$$

This says that $\sigma^{-1} \circ \sigma'$ is a covering transformation of ϕ^*, and so σ and σ' are two liftings of the automorphism of k_∇ defined by (g_j^i). The previous lemma asserts that σ' is a symmetry. Conversely, given a symmetry of the form in the previous lemma (e.g. the identity or σ), then for any covering transformation of ϕ^* (or equivalently any Galois automorphism of k over k_∇), the symmetry composed with the covering transformation gives us another symmetry. So we conclude that ∇ descends to a connection over the compact Riemann surface with field of meromorphic functions k_∇. This last connection with Picard-Vessiot extension $k_\nabla(v_j^i) = \mathbb{C}(v_j^i)$ is by construction a basic connection. We obtain the following proposition:

Proposition 3.4.3. *Any standard connection ∇ is the pullback of a basic connection over the Riemann surface with field of meromorphic functions k_∇.*

Finally, as $H = Aut_{G_F}(G)$ acts on k_∇ as symmetries of this last basic connection (Lemma 3.4.2), we can in turn descend the vector bundle and connection all the way down to the Riemann surface X_0 with field of meromorphic functions $(k_\nabla)^H =: k_0$. Denote by E_0 this quotient vector bundle over X_0 and by ∇_0 the resulting quotient connection. Then by definition (E, ∇) is the pullback of (E_0, ∇_0), and we have the following tower of Galois extensions:

$$\begin{array}{c} k(v^i_j)^{Z(G)} \\ | \\ k \\ | \\ k_0 \end{array}$$

Lemma 3.4.4. *$k_0(v^i_j) = k(v^i_j)$ is a Picard-Vessiot extension for ∇_0. The Galois group of ∇_0 is represented by a subgroup of $N_{G_F}(G)$ with projective Galois group $H = Aut_{G_F}(G)$.*

Proof: The first statement on the lemma follows from the fact that ∇ is the pullback of ∇_0. The establish the second assertion it suffices to notice $k_0 = (k_\nabla)^H = (\mathbb{C}(v^i_j)^{Z(G)})^H$. ★

To obtain the map $Aut_{G_F}(G) \to Aut_\nabla(X)$ identify the group of symmetries with the Galois group $Aut_{k_0}(k)$, where in the map is seen to arise from the Galois action on the tower immediately before the statement of the lemma.

3.5 Right exactness

Take $\sigma \in Aut_{\nabla}(X)$. Put $V = \sigma(U)$ and consider a chart of E giving a vector bundle trivialization $(\Pi^{-1}V, w, x^1, \ldots, x^n)$ with $w \circ \sigma = z$. Denote by ∇' the holomorphic connection on V induced by ∇. Since σ is a symmetry there is a horizontal vector bundle isomorphism

$$\tilde{\sigma} : (\Pi^{-1}U, \nabla') \quad \longrightarrow \quad (\Pi^{-1}V, \nabla')$$

lifting σ. Note that $\tilde{\sigma}^* w = w \circ \tilde{\sigma} = (w \circ \Pi)\tilde{\sigma} = w \circ \sigma\Pi = z \circ \Pi = z$ and $\sigma_* \frac{\partial}{\partial z} = \frac{\partial}{\partial w}$.

The hypothesis $\sigma \in Aut_{\nabla}(X)$ implies that each pullback $\tilde{\sigma}^*\Psi^i$ along $\tilde{\sigma}$ of a fundamental system of linear first integrals Ψ^i of $\nabla'_{\frac{\partial}{\partial w}}$ over $\Pi^{-1}V$ is a linear first integral of $\nabla'_{\frac{\partial}{\partial z}}$ over $\Pi^{-1}U$. In other words, there exist c^i_k such that

$$\begin{aligned} c^i_k \Phi^k(z, y^1, \ldots, y^n) &= \tilde{\sigma}^*\Psi^i(z, y^1, \ldots, y^n) \\ c^i_k \frac{\partial \Phi^k}{\partial y^j}(z) y^j &= \frac{\partial \Psi^i}{\partial x^j}(\sigma(z)) \tilde{\sigma}^* x^j(y^1, \ldots, y^n). \end{aligned}$$

Taking x^i such that $\tilde{\sigma}^* x^j(y^1, \ldots, y^n) = y^j$ we have

$$c^i_k \frac{\partial \Phi^k}{\partial y^j}(z) = \frac{\partial \Psi^i}{\partial x^j}(\sigma(z)). \tag{3.2}$$

Choose $f \in k$ such that $f(\sigma(q))v_{\sigma(q)} = (\sigma_* v)_{\sigma(q)}$, for every $q \in X$. If $AX = X'$ is the matrix equation form of $\nabla_v X = 0$, then on U we have

$$\nabla' X = (AX - X') \otimes dz$$

Let γ be a path from $p \in U$ to $\sigma(p) \in V$ avoiding the singularities of ∇. If

$\nabla'_{\frac{\partial}{\partial z}} X = 0$, and if X is analytically extended along γ, we have

$$\begin{aligned}
\nabla'_{\sigma_* \frac{\partial}{\partial z}} X &= \langle (AX - X') \otimes dz, \sigma_* \frac{\partial}{\partial z} \rangle \\
&= \langle (AX - X') \otimes dz, f(z) \frac{\partial}{\partial z} \rangle \\
&= f(z) \nabla'_{\frac{\partial}{\partial z}} X \\
&= 0.
\end{aligned}$$

So we may take Ψ^i as the analytic extension of Φ^i along γ, and (3.2) becomes:

$$c^i_k \frac{\partial \Phi^k}{\partial y^j}(z) = \frac{\partial \Phi^i}{\partial x^j}(\sigma(z)) \tag{3.3}$$

Lemma 3.5.1. *The matrix* (c^i_k) *defined on (3.3) is in the normalizer of* G *in* G_F*.*

Proof: As in the previous section, let X_0 be the Riemann surface obtained as the quotient space of X by the group of symmetries $Aut_\nabla(X)$. By the definition of symmetry there exists a vector bundle E_0 and a connection ∇_0 such that (E, ∇) is the pullback of (E_0, ∇_0) under the covering map induced by the action.

The projection γ_0 of γ to X_0 is a closed curve, and (c^i_k) therefore defines an element of the monodromy of ∇_0.

Let $P[X^i_j] \in \mathbb{C}[X^i_j]^G$, i.e. a G-invariant polynomial such that

$$P[\frac{\partial \Phi^i}{\partial y^j}](z) = \iota(f)(z)$$

for some $f \in k$.

If $f = 0$, then under analytic continuation along γ_0 it remains the case that $P[c^i_k \frac{\partial \Phi^k}{\partial y^j}(z)] = 0$. This implies $(c^i_k) \in G_F$.

On the other hand, k is an extension of the field of meromorphic functions over X_0. Under analytic continuation along γ_0, an arbitrary non-zero f is mapped into f_σ by a covering (i.e. Galois) automorphism of X (of k) over X_0 (over $\mathbb{C}(X_0)$). Whence $P[c^i_k \frac{\partial \Phi^k}{\partial y^j}(z)] = \iota(f_\sigma)(z)$, and so by the Galois correspondence $P[X^i_j]$ is invariant under $G^{(c^i_k)}$ (the conjugate of G by (c^i_k)).

A symmetric argument allows us to conclude that the invariant polynomials under G and under $G^{(c^i_k)}$ coincide. So Compoint's Theorem implies $G^{(c^i_k)} = G$. This completes the proof. ★

The Lemma implies the map defined in Section 3.4 is surjective, and the theorem follows.

3.6 Examples

EXAMPLE 3.6.1. Consider the elliptic curve defined by the equation $z^3 - z = w^2$ with field of meromorphic functions $\mathbb{C}(z, w)$. We take its invariant differential form $\frac{dz}{2w}$ and its dual tangent vector field $v = 2w\frac{\partial}{\partial z}$. We denote by σ the fourth order automorphism $\sigma : w \mapsto iw, z \mapsto -z$; so that $\sigma_* v = -iv$.

The differential equation

$$v^3(y) - \frac{3(5z^2-3)}{w}v^2(y) + \frac{1}{3}\frac{194z^4 - 230z^2 + 108}{w^2}v(y) - \frac{4}{27}\frac{364z^6 - 665z^4 + 1030z^2 - 405}{w^3}y = 0$$

has differential Galois group G_{27}. Indeed, written in terms of z and $\frac{\partial}{\partial z}$, the equation corresponds to:

$$y''' - \frac{3}{z}y'' + \frac{1}{12}\frac{77z^4 - 122z^2 + 81}{(z^3-z)^2}y' - \frac{1}{54}\frac{364z^6 - 665z^4 + 1030z^2 - 405}{(z^3-z)^3}y = 0.$$

This equation is irreducible and has unimodular Galois group. Furthermore to get the Galois group using methods of [28] we notice that it has a two-dimensional space of third degree invariants, which corresponds to the dual first integrals 0 and z^3w^3. The wronskian of the equation in terms of v, z, w is zw, the ratio of two sixth degree invariants is z and the ratio of two ninth degree invariants is w, so the induced connection is basic.

If X, Y and Z are the solution based at point $p : (z(p), w(p)) = (0, 0)$, with respectively z-exponents 2, 3/2 and 5/2, and leading coefficient 1, then

$$Y^2Z + X^2Y - \frac{1}{81}Z^3 = 0.$$

This vanishing homogenous polynomial is a G_{27} invariant which describes an elliptic curve, and the subgroup of $GL_3(\mathbb{C})$ leaving this elliptic curve invariant is the lifting of $F_{36} \subset PSL_3(\mathbb{C})$. So the group $G_F/Z(G_F)$ is F_{36}. The group PG_{27} (the projective version of G_{27}) is a normal subgroup of index four in F_{36}; the quotient is a cyclic group of order four. On the other hand the equation is invariant under σ and we obtain

$$1 \longrightarrow Z(G_{27}) \longrightarrow G_{27} \longrightarrow F_{36} \longrightarrow \langle\sigma\rangle \longrightarrow 1.$$

The equation descends into the Riemann sphere parameterized by $x = z^2$ to an equation with Galois group $F_{36}^{SL_3}$, i.e.

$$y''' + \frac{1}{48}\frac{41x^2 - 50x + 45}{x^2(x-1)^2}y' - \frac{1}{432}\frac{364x^3 - 665x^2 + 1030x - 405}{x^3(x-1)^3}y = 0,$$

which is given in terms of x and $\frac{\partial}{\partial x}$.

Algebraically we have $v = 2w\frac{\partial}{\partial z} = 4wz\frac{\partial}{\partial x}$ and $\mathbb{C}(z,w) \supseteq \mathbb{C}(z) \supseteq \mathbb{C}(x)$, with $\mathbb{C}(z) = \mathbb{C}(z,w)^{\sigma^2}$ and $\mathbb{C}(x) = \mathbb{C}(z,w)^{\sigma}$. We also have that $\frac{\partial}{\partial z}$ is invariant under σ^2 and that $\frac{\partial}{\partial x}$ is invariant under σ. We thus have the following tower of linear operators:

$$\begin{array}{lcl} \mathbb{C}(z,w)[v] & & \\ \quad\Big| & & \\ \mathbb{C}(z)[\frac{\partial}{\partial z}] & & = (\mathbb{C}(z,w)[v])^{\sigma^2} \\ \quad\Big| & & \\ \mathbb{C}(x)[\frac{\partial}{\partial x}] & & = (\mathbb{C}(z,w)[v])^{\sigma} \end{array}$$

The first of the equations corresponds to an operator in $\mathbb{C}(w,z)[v]$, which when written in terms of $\frac{\partial}{\partial z}$ defines the operator in the second equation, which is in $\mathbb{C}(z)[\frac{\partial}{\partial z}]$. Finally writing it in terms of $\frac{\partial}{\partial x}$, we obtain the operator in $\mathbb{C}(x)[\frac{\partial}{\partial x}]$ corresponding to the third equation. So the operator in the first equation is actually an operator in the bottom of the tower.

EXAMPLE 3.6.2. Consider the differential equation $L(y) = 0$ given by:

$$y'' - \frac{z^4 - 3z^2 - 1}{1 + z^4} y = 0.$$

Two linearly independent solutions are given by:

$$Y_1 = \sqrt[4]{z^4 - 1}e^{\int \frac{1}{\sqrt{z^4-1}}}, \quad Y_2 = \sqrt[4]{z^4 - 1}e^{-\int \frac{1}{\sqrt{z^4-1}}}.$$

So according to the Kovacic algorithm, the Differential Galois group is the infinite dihedral group D_∞. A basis of invariants of this group on $\mathbb{C}[X_j^i]$ is given by

$$\begin{aligned}
X_{2,1} := X_1^1 X_2^2 - X_1^2 X_2^1 &\longmapsto -2 \\
X_{4,1} := (X_1^1 X_2^1)^2 &\longmapsto z^4 - 1 \\
X_{4,2} := (X_1^2 X_2^2)^2 &\longmapsto \frac{(z^6 - z^4 + 1)^2}{(z^4 - 1)^3} \\
X_{4,3} := (X_1^1 X_2^2 + X_1^2 X_2^1)^2 &\longmapsto \frac{4z^6}{z^4 - 1} \\
X_{4,4} := (X_1^1 X_2^1)(X_1^1 X_2^2 + X_1^2 X_2^1) &\longmapsto 2z^3 \\
X_{4,5} := (X_1^2 X_2^2)(X_1^1 X_2^2 + X_1^2 X_2^1) &\longmapsto \frac{2z^3(z^6 - z^4 + 1)}{(z^4 - 1)^2} \\
X_{4,6} := X_1^1 X_2^1 X_1^2 X_2^2 &\longmapsto \frac{z^6 - z^4 + 1}{z^4 - 1},
\end{aligned}$$

where the arrow refers to the image on the Picard-Vessiot extension under the map

$$X_{ij} \longmapsto Y_j^{(i-1)}.$$

We have:

$$\frac{4X_{4,1} + X_{2,1}^2}{2X_{4,4}} = z.$$

So the equation is basic. The homogeneous relations that vanishe under this evaluation homomorphism are given by the homogeneous ideal generated by

$$\begin{aligned}
X_{4,4}X_{4,5} - X_{4,6}X_{4,3} \\
X_{4,6}^2 - X_{4,1}X_{4,2} \\
X_{4,1}X_{4,2} - \frac{1}{16}(X_{4,3} - X_{2,1}^2)^2 \\
X_{4,3}X_{2,1}^2 - (X_{4,3} - 2X_{4,6})^2 \\
X_{4,4}^2 - X_{4,1}X_{4,3} \\
X_{4,5}^2 - X_{4,2}X_{4,3}.
\end{aligned}$$

All of these are invariant under the group G_F generated by D_∞ together with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and so correspond to automorphisms of the space

$$\mathrm{Proj}(\mathbb{C}[X_{2,1}, X_{4,1}, X_{4,2}, X_{4,3}, X_{4,4}, X_{4,5}, X_{4,6}]).$$

Again we have an exact sequence:

$$1 \longrightarrow G \longrightarrow G_F \longrightarrow \langle z \mapsto -z \rangle \longrightarrow 1.$$

EXAMPLE 3.6.3. We study in more detail the comments in Remark 3.2.19. Consider the differential equation [28] $L(y) = 0$ given by

$$y''' + \frac{3(3x^2-1)}{x(x-1)(x+1)}y'' + \frac{221x^4 - 206x^2 + 5}{12x^2(x-1)^2(x+1)^2}y' + \frac{374x^6 - 673x^4 + 254x^2 + 5}{54x^3(x-1)^3(x+1)^3}y = 0.$$

Its Picard-Vessiot extension has differential Galois group G_{54} of order 54. The singular points of $L(y) = 0$ are 0, 1, -1 and ∞, with respective exponents

$$\{-\frac{1}{6}, \frac{5}{6}, \frac{-2}{3}\}, \quad \{-\frac{1}{6}, \frac{5}{6}, \frac{-2}{3}\}, \quad \{-\frac{1}{6}, \frac{5}{6}, \frac{-2}{3}\}, \quad \{\frac{11}{6}, \frac{17}{6}, \frac{4}{3}\}.$$

The ramification data in 0, 1 and -1 is the same, so one can expect some kind of symmetry in between these three points. A quick glance at the equation reveals that all the coefficients of the numerator have even power of x, and the denominator present the same exponents for $x-1$ and for $x+1$. This equation admits one symmetry $x \mapsto -x$.

With some computation we can see that if X denotes the solution based at 0 with exponent $-\frac{1}{6}$, Y the one with $-\frac{5}{6}$ and Z the final one, then

$$YZ^2 + X^3 - \frac{16}{81}XY^2 = 0.$$

This corresponds to an elliptic curve. The other third degree G_{54}-semi-invariant is given by:

$$XZ^2 + \frac{32}{162}X^2Y + \frac{256}{19683}Y^3 = \left(\frac{1}{x^3(x^2-1)^3}\right)^{\frac{1}{2}}.$$

The vanishing G_{54}-invariant polynomials are then spanned by

$$(YZ^2 + X^3 - \tfrac{16}{81}XY^2)(XZ^2 + \tfrac{32}{162}X^2Y + \tfrac{256}{19683}Y^3)$$
$$(YZ^2 + X^3 - \tfrac{16}{81}XY^2)^2.$$

Now, as in the case of the equation in the first example over the elliptic curve, we obtain the projection version of G_{54} as a normal subgroup of index two in F_{36}. Although it may seem like the theorem applies to this case, the problem here is that the equation is neither basic nor standard. In fact, if we take the symmetric product of this equation with

$$y' - \frac{2}{3}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1} - \frac{x}{x^2+1}\right)y = 0$$

we obtain the equation

$$y''' + \frac{5x^4-1}{x(x^2-1)(x^2+1)}y'' + \frac{1}{12}\frac{45x^8+20x^6-130x^4+20x^2-3}{x^2(x^2-1)^2(x^2+1)^2}y' - \frac{20}{27}\frac{x^4-6x^2+1}{x(x^2+1)^3(x^2-1)}y = 0.$$

The solutions to this last equation are given by the solutions to the former multiplied by $\sqrt[3]{\frac{(x^3-x)^2}{x^2+1}}$. The Galois group is unmodified, as are the vanishing homogeneous G_{54}-invariants. This equation has group of symmetries isomorphic to the dihedral group of order eight:

$$\{x \mapsto x, x \mapsto \frac{1}{x}, x \mapsto -x, x \mapsto \frac{-1}{x}, x \mapsto \frac{-x+1}{x+1}, x \mapsto \frac{x+1}{x-1}, x \mapsto \frac{x+1}{-x+1}, x \mapsto \frac{x-1}{x+1}\},$$

and it descends to the sphere parameterized by $z = \frac{1}{16}\frac{(x^2+1)^4}{x^2(x+1)^2(x-1)^2}$ to the basic equation

$$y''' + \frac{1}{2}\frac{8z-5}{z(z-1)}y'' + \frac{5}{48}\frac{21z-5}{z^2(z-1)}y' - \frac{5}{864}\frac{1}{(z-1)z^3}y = 0,$$

which is expressed in terms of z and $\frac{\partial}{\partial z}$. The quotient group of F_{36} by PG_{54} corresponds to the symmetry coming from lifting this equation to the sphere parameterized by $\sqrt{z} = \frac{1}{4}\frac{(x^2+1)^2}{x(x+1)(x-1)}$, and the remaining symmetries arise from the lifting to the original sphere parameterized by x.

EXAMPLE 3.6.4. Consider the differential equation [26] $L(y) = 0$ given by

$$y''' + \frac{21(x^2 - x + 1)}{25x^2(x-1)^2}y' + \frac{21(-2x^3 + 3x^2 - 5x + 2)}{50x^3(x-1)^3}y = 0.$$

Its Picard-Vessiot extension has differential Galois group A_5, the rotational icosahedral group of order 60. This equation admits one symmetry: $z \mapsto -z+1$. The equation is a symmetric power of a second order linear differential equation and so its solutions satisfy the equation

$$XY - Z^2 = 0.$$

The group of automorphisms obtained by conjugating A_5 within the subgroup of $GL_3(\mathbb{C})$ fixing the homogeneous ideal generated by this conic is S_5 (conjugation with a diagonal matrix with determinant -1 together with the inner automorphisms). As in the previous example, because the equation is not standard, the quotient group corresponds to the symmetries of another equation which can be pulled back to obtain this one. Indeed we can take the symmetric product with the equation

$$y' - \frac{2}{3}\left(\frac{1}{x} + \frac{1}{x-1}\right)y = 0$$

to obtain

$$y''' + \frac{2(2x-1)}{x(x-1)}y'' + \frac{1}{75}\frac{163x^2 - 163x + 13}{x^2(x-1)^2}y' - \frac{11}{1350}\frac{2x^3 - 3x^2 - 3x + 2}{x^3(x-1)^3}y = 0,$$

which is an equation with the same Galois group, but with symmetries forming the group S_3:

$$\{x, 1-x, \frac{1}{x}, \frac{x-1}{x}, \frac{1}{1-x}, \frac{x}{x-1}\}.$$

The equation then descends to the sphere parameterized by $z = \frac{4}{27}\frac{(x^2-x+1)^3}{x^2(x-1)^2}$ as the basic equation

$$y''' + \frac{1}{2}\frac{7z-4}{z(z-1)}y'' + \frac{1}{900}\frac{1389z-200}{z^2(z-1)}y' - \frac{11}{5400}\frac{1}{z^2(z-1)}y = 0,$$

which is expressed in terms of z and $\frac{\partial}{\partial z}$. The quotient group of S_5 by A_5 corresponds to the symmetry arising from lifting this equation to the sphere parameterized by $\frac{(x-c)^3}{x(x-1)}$, where $c = \frac{1+i\sqrt{3}}{2}$; the remaining symmetries come from the lifting to the original sphere parameterized by x.

3.7 Comments

The computations involved in obtaining the Fano group, as well as the normalizer of the differential Galois group, are quite complicated. Among other things, they require extensive use of Van Hoeij and Weil's algorithm [17]. In the case where the Galois group is finite, things may be simplified. Indeed, if the charts $(U, z, y^1, \ldots, y^n)$ are taken so that the section $(z, 1, 0, \ldots, 0)$ is cyclic under ∇_v, then the Picard-Vessiot extension is given by $\iota(k)[\frac{\partial \Phi^i}{\partial y^1}]$. So we should be able to replace R with $\mathbb{C}[X_1^i]^G/F_0$, where F_0 is the contraction of F. This ideal is what Fano originally considered, cf. [27].

Appendix A

Compoint's Theorem

Let $K = \mathbb{C}(X)$ be the field of meromorphic functions over a connected compact Riemann surface X. By fixing a non-zero meromorphic tangent vector field v over X, we see K as a differential field with field of constants $\mathbb{C}$. Given $f \in K$, we denote its derivative $v(f)$ by f'.

We consider a matrix linear differential equation

$$(y^i)' = a^i_j y^j, \quad (a^i_j)_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{C}(X)),$$

and a full system of (locally defined) solutions f^i_j. We pose the following question: *What are the algebraic relations among the* $\{f^i_j\}_{1\leq i,j\leq n}$*? In other words, what are the polynomials* $P(X^i_j) \in K[X^i_j]_{1\leq i,j\leq n}$ *such that* $P(f^i_j) = 0$*?*

Under appropriate hypothesis on the Galois group of the equation above, the problem has been completely solved by Compoint [7]. Compoint's solution asserts that these algebraic relations can be written in terms of invariant polynomials (Theorem A.0.5 below). Now, to fully obtain a basis of invariants one can rely on the algorithm by M. van Hoeij and J.A. Weil [17] together with the bounds obtained in [8]. In his paper Compoint also asserts that his solution fails if we relax his hypothesis. However by modifying his argument, we are able to generalize his result to a larger class of equations (Theorem A.0.6). Our proof will follow F. Beukers' [4]. We treat the problem in terms of differential modules.

Definition A.0.1. A *differential module* M over K is a K-vector space together with an additive map ∂ such that

$$\partial f m = f' m + f \partial m$$

for every $f \in K$ and $m \in M$. We refer to ∂ as the *differential* of the module. If $\partial m = 0$ we say that m is *horizontal*.

REMARK A.0.2. It is quite evident that the collection of horizontal elements forms a vector space over the field of constants. Now, given a differential module M and a basis $e_1, \ldots, e_n$, if $\partial e_j = \sum_i -a^i_j e_i$, then to solve the equation $\partial m = 0$ in this basis amounts to solving the system of equations

$$(x^i)' = a^i_j x^j.$$

Indeed if $m = x^j e_j$, then

$$\begin{aligned} \partial m &= (x^j)' e_j - x^j \sum_i a^i_j e_i \\ &= \left((x^i)' - a^i_j x^j\right) e_i \end{aligned}$$

Let M_1 and M_2 be two differential modules over K with differentials ∂_1 and ∂_2 respectively. We turn $M_1 \otimes_K M_2$ into a differential module by differentiating as follows: $m_1 \otimes m_2 \mapsto \partial_1 m_1 \otimes m_2 + m_1 \otimes \partial_2 m_2$. Note that this map is well-defined since

$$\begin{aligned}\partial_1 f m_1 \otimes m_2 + f m_1 \otimes \partial_2 m_2 &= f' m_1 \otimes m_2 + f \partial_1 m_1 \otimes m_2 + f m_1 \otimes \partial_2 m_2 \\ &= m_1 \otimes f' m_2 + \partial_1 m_1 \otimes f m_2 + m_1 \otimes f \partial_2 m_2 \\ &= m_1 \otimes \partial_2 f m_2 + m_1 \otimes \partial_2 f m_2.\end{aligned}$$

More generally, starting with a differential module M, any tensorial construction on M inherits a unique differential module structure.

Definition A.0.3. Given a finite dimensional differential module M over K, a *Picard-Vessiot* extension for M is a differential field extension E of K such that:

a) the constants in E coincide with the constants in K;

b) the differential module $E \otimes_K M$ has a basis over E composed of horizontal elements; and

c) If $\{f_j = f_j^i \otimes e_i\}_{1 \leq j \leq n}$ is a basis of horizontal elements over E of $E \otimes_K M$, then $E = K(f_j^i)_{1 \leq i,j \leq n}$.

REMARK A.0.4. In the context of the definition, it is common to refer to the matrix $(f_j^i)_{1 \leq i,j \leq n}$ as a *fundamental matrix of solutions*. One can easily see that condition c) does not depend on the choice of the K-basis $e_1, \ldots, e_n$ of M.

A Picard-Vessiot extension for M can be obtained as follows. Take the ring of polynomials with coefficients in K and $n \times n$ variables, $K[X_j^i]_{1 \leq i,j \leq n}$, where n is the dimension of M. We turn it into a differential ring extension of K by setting

$$(X^i_j)' = a^i_k X^k_j$$

and we extend the differential to $K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n}$ where $W = \det(X^i_j)_{1\leq i,j\leq n}$, by putting $\left(\frac{a}{b}\right)' = \frac{a'b-ab'}{b^2}$ whenever $a, b \in K[X^i_j]_{1\leq i,j\leq n}$. Finally, to obtain a Picard-Vessiot extension it suffices to form the quotient of $K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n}$ by a maximal differential ideal I, which turns out to be prime, and to take the field of fractions of the resulting quotient domain. For a rigorous exposition of the details involved one could, for example, read [25, Section 1.3].

Each Picard-Vessiot extension has an associated Galois group G. This group is an algebraic group, but for this exposition the full structure is not needed. For our purposes it can be described as follows: the group $GL_n(\mathbb{C})$ acts from the right on $K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n}$ by differential K-automorphisms by setting

$$(g^a_b)_{1\leq a,b\leq n} : X^i_j \longmapsto X^i_k g^k_j.$$

The Galois group G is then the stabilizer of the maximal differential ideal I. Again, full details are in [25, Section 1.4]. Now we state Compoint's theorem:

Theorem A.0.5 (Compoint)**.** *In the context above, if G is reductive and unimodular then the ideal I is generated by its G-invariants.*

The remainder of this appendix is devoted to proving the following generalization:

Theorem A.0.6. *In the context above, if G is reductive and the image of* $\det : G \to \mathbb{C}^*$ *is finite then I is the radical of the ideal generated by the G-invariants it contains.*

In the hypotheses of the theorem it may seem that the image of the determinant map depends on the representation of the Galois group. This is not the case as we will see in the third part of the appendix.

The statement of the generalization of the theorem should come as no surprise to a reader having some familiarity with the topic. As a matter of fact, it should be enough to convince him/her to consider an N-th symmetric product of M with $N \gg 0$, and notice that for such a module the Galois group is unimodular. One can then apply Compoint's theorem to this symmetric power and recover the original ideal I. The proof below is just a rigorous exposition of that heuristic.

The most important aspect of Compoint's theorem, Corollary A.1.5 below, is that we obtain an explicit, finite, simple and useful set of generators of I.

A.1 The proof

We start by fixing some notation. Given a $g = (g^i_j)_{1\leq i,j\leq n} \in GL_n(\mathbb{C})$, the image by the above-mentioned right action on $P \in K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n}$ will be denoted by P^g. Similarly, there is left action:

$$(g^a_b)_{1\leq a,b\leq n} : X^i_j \longmapsto \sum_k g^i_k X^k_j$$

and the image of P will be denoted by ${}^gP$. We also extend both actions and their corresponding notation to the field $K(X^i_j)_{1\leq i,j\leq n}$

Theorem A.1.1 (First main theorem of Invariant Theory)**.** *Let*

$$P \in \mathbb{C}[X^i_j]_{1\leq i,j\leq n}$$

and assume that there is a character $\chi : GL_n(\mathbb{C}) \mapsto \mathbb{C}^*$ *such that for every* $g \in GL_n(\mathbb{C})$*,* $P^g = \chi(g)P$*. Then* $P = \lambda \det(X^i_j)^N_{1\leq i,j\leq n}$ *for some* $N \in \mathbb{N}$ *and some* $\lambda \in \mathbb{C}$*. Furthermore,* $\chi(g) = \det(g)^N$*.*

Proof: We may start by assuming that P is homogeneous. As χ is a character of $GL_n(\mathbb{C})$, $\chi(g) = \det(g)^{N_0}$ for some $N_0 \in \mathbb{Z}$. Hence P is homogeneous and invariant under $SL_n(\mathbb{C})$, and therefore $P = \lambda \det(X^i_j)^N_{1\leq i,j\leq n}$. The proof of these statements may be found in [13, Paragraph 15]. We now explain why the hypothesis forces the polynomial to be homogeneous. Indeed, we just proved that the theorem can be applied to each homogeneous component of P, so that under the action of $g \in GL_n(\mathbb{C})$, each component is multiplied by $\det(g)^{N_0}$ for some N_0 depending on the degree. However, because every component is multiplied by the same coefficient $\chi(g)$ we conclude that P has a unique component, and is therefore homogeneous. ★

Corollary A.1.2. *Let* $r \in K(X^i_j)_{1\leq i,j\leq n}$ *and assume that there is a character* $\chi : GL_n(\mathbb{C}) \mapsto K^*$ *such that for every* $g \in GL_n(\mathbb{C})$*,* $r^g = \chi(g)r$*. Then* $r = \lambda \det(X^i_j)^N_{1\leq i,j\leq n}$ *for some* $N \in \mathbb{Z}$ *and some* $\lambda \in K$*. Furthermore,* $\chi(g) = \det(g)^N$*.*

Proof: Let $P, Q \in K[X^i_j]_{1\le i,j\le n}$ be such that $r = P/Q$, where P and Q have no common factors. From the equality $r^g = \chi(g)r$ we obtain the relation $PQ^g = \chi(g)P^gQ$, which implies that P divides P^gQ. As $K[X^i_j]_{1\le i,j\le n}$ is a unique factorization domain and P and Q are co-prime, we conclude that P divides P^g. Symmetrically, P^g divides P, hence $P^g = \chi_P(g)P$ for some $\chi_P(g) \in K$. Similarly $Q^g = \chi_Q(g)Q$ for some $\chi_Q(g) \in K$. So in order to complete the proof, it suffices to prove the result for $r \in K[X^i_j]_{1\le i,j\le n}$.

Given $a \in X$, denote by r_a, r^g_a and $\chi_a(g)$ the specialization of r, r^g and $\chi(g)$ to a. We now have $r_a \in \mathbb{C}[X^i_j]_{1\le i,j\le n}$, and for every $g \in GL_n(\mathbb{C})$ there is a $\chi_a(g) \in \mathbb{C}^*$ such that $r^g_a = \chi_a(g)r_a$. The previous theorem implies that $r_a = \lambda_a \det(X^i_j)^N_{1\le i,j\le n}$ for some $N \in \mathbb{N}$ and some $\lambda_a \in \mathbb{C}$. The function $a \mapsto N$ takes integer values, so it is locally constant, and since X is connected, it must be constant. Finally, specializing at $X^i_j = \delta^i_j$, we see that $a \mapsto \lambda_a = r_a(\delta^i_j)$ is meromorphic over X, hence $r = \lambda \det(X^i_j)^N_{1\le i,j\le n}$ for some $\lambda \in K$. ★

Proposition A.1.3. *Let $P_1, \ldots, P_r \in K[X^i_j, \frac{1}{W}]_{1\le i,j\le n}$ be linearly independent over K. Moreover, suppose that they generate an r-dimensional K-vector space stable under the right $GL_n(\mathbb{C})$-action. Then there exist elements $g_1, \ldots, g_r \in GL_n(\mathbb{C})$ such that*

$$\det({}^{g_i}P_j)_{1\le i,j\le r} = \lambda \det(X^i_j)^N_{1\le i,j\le n}$$

for some $\lambda \in K^*$ and $N \in \mathbb{Z}$. Furthermore, one can choose $g_1 = e$, the identity element.

Proof: Consider the vector space $\big(K(X^i_j)_{1\le i,j\le n}\big)^r$. Denote by s the rank over $K(X_j^i)_{1\leq i,j\leq n}$ of the set of vectors

$$\left\{({}^gP_1,\ldots,{}^gP_r)\in\left(K(X_j^i)_{1\leq i,j\leq n}\right)^r \mid g\in GL_n(\mathbb{C})\right\},$$

and choose $g_1,\ldots,g_s$ such that the vectors $({}^{g_i}P_1,\ldots,{}^{g_i}P_r)$, $1\leq i\leq s$, are linearly independent. Replacing g_i by $g_1^{-1}g_i$ we may assume $g_1=e$.

We claim that $s=r$. Assume to the contrary that $s<r$. As $g_1=e$, the determinant of the $(s+1)\times(s+1)$-matrix

$$\begin{pmatrix} P_1 & \cdots & P_{s+1} \\ {}^{g_1}P_1 & \cdots & {}^{g_1}P_{s+1} \\ \vdots & & \vdots \\ {}^{g_s}P_1 & \cdots & {}^{g_s}P_{s+1} \end{pmatrix}$$

is zero. Expanding this determinant along the top row we have

$$P_1\Delta_1-P_2\Delta_2+\ldots+(-1)^sP_{s+1}\Delta_{s+1}=0.$$

By re-indexing the $P_i's$ if necessary, we may assume that the last minor $\Delta_{s+1}=\det({}^{g_i}P_j)_{1\leq i,j\leq s}$ does not vanish, and we obtain

$$\frac{\Delta_1}{\Delta_{s+1}}P_1-\frac{\Delta_2}{\Delta_{s+1}}P_2+\ldots+(-1)^{s-1}\frac{\Delta_s}{\Delta_{s+1}}P_s=-(-1)^sP_{s+1}$$

Since the left $GL_n(\mathbb{C})$-action acts on the rows of our $(s+1)\times(s+1)$-matrix, each minor Δ_i is multiplied by the same quantity under a $g\in G$, and $\frac{\Delta_i}{\Delta_{s+1}}$ is invariant, implying that these minor quotients are actually in K. This contradicts our hypothesis on the linear independence of $P_1,\ldots,P_r$. So $s=r$, and $\det({}^{g_i}P_j)_{1\leq i,j\leq r}\neq 0$.

The right and left $GL_n(\mathbb{C})$-actions commute, and since the K-vector space spanned by $P_1, \ldots, P_r$ is invariant under the right $GL_n(\mathbb{C})$-action, we conclude that the right $GL_n(\mathbb{C})$-action acts on the columns of $\det({}^{g_i}P_j)_{1\leq i,j\leq r}$, so that for every $g \in GL_n(\mathbb{C})$ there is a $\chi(g) \in K^*$ such that

$$(\det({}^{g_i}P_j)_{1\leq i,j\leq r})^g = \chi(g) \det({}^{g_i}P_j)_{1\leq i,j\leq r}.$$

The corollary above now implies this proposition. ★

Before we prove our theorem we need the following consequence of the Galois Correspondence.

Lemma A.1.4. *If $P \in K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n}$ is G-invariant, then there is an $f \in K$ such that $P - f \in I$.*

Proof: Under the canonical map $K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n} \to K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n}/I$, the image of P is invariant under G, so by Galois correspondence P maps to an f in the ground field K. Thus, the image of $P - f$ under the canonical map is zero. ★

Proof of Theorem A.0.6: Let $Q \in K[X^i_j]_{1\leq i,j\leq n}$. As the $GL_n(\mathbb{C})$-action does not modify the degree, the degree of the elements in the orbit of Q is bounded, and these elements therefore span a finite dimensional K-vector space. The same holds if $Q \in K[X^i_j, \frac{1}{W}]_{1\leq i,j\leq n}$, for $W^g = \det(g)W$.

Now take $Q \in I$. We will prove that a power of Q lies in the ideal generated by the G-invariants in I. Denote by $Q_0^{GL_n(\mathbb{C})}$ the K-vector space spanned by $\{Q^g\}_{g\in GL_n(\mathbb{C})}$, and by r_0 its dimension. Similarly, denote by Q_0^G

the space spanned by $\{Q^g\}_{g\in G}$ and by t_0 its dimension. Note that $Q_0^G \subseteq I$ as I is G-stable. Let $Q_1, \ldots, Q_{r_0}$ be a basis of $Q_0^{GL_n(\mathbb{C})}$ such that $Q_1 = Q$ and $Q_1, \ldots, Q_{t_0}$ is a basis of Q_0^G. Additionally, as G is reductive we can assume that $Q_{t_0+1}, \ldots, Q_{r_0}$ span a G-stable space. Indeed, this last condition follows since every linear representation of a reductive group over $\mathbb{C}$ is completely reducible.

Let $N_0 \gg 0$ be such that if $g \in G$ then $\det(g^{N_0}) = 1$. Denote by $Q^{GL_n(\mathbb{C})}$ the space spanned by the N_0-fold products of the $Q_1, \ldots, Q_{r_0}$, and by r its dimension. Similarly denote by Q^G the sub-space spanned by the N_0-fold products of the $Q_1, \ldots, Q_{t_0}$, and by t its dimension. Note that $Q^G \subseteq I$. Let $P_1, \ldots, P_r$ be a basis of $Q^{GL_n(\mathbb{C})}$ such that $P_1 = Q^{N_0}$, $P_1, \ldots, P_r$ is a basis of Q^G and the space spanned by $P_{t+1}, \ldots, P_r$ is G-stable. The previous proposition implies that there exist $g_1 = e, g_2, \ldots, g_r \in GL_n(\mathbb{C})$ such that

$$\det({}^{g_i}P_j)_{1\leq i,j\leq r} = \lambda \det(X_j^i)_{1\leq i,j\leq n}^N$$

for some $\lambda \in K^*$ and $N \in \mathbb{Z}$. Notice that the action of $GL_n(\mathbb{C})$ on $Q^{GL_n(\mathbb{C})}$ is the N_0-th symmetric power of the action on $Q_0^{GL_n(\mathbb{C})}$, thus N is a multiple of N_0 and $\det(g)^N = 1$.

As $\det(X_j^i)_{1\leq i,j\leq n} = W$ is a unit in $K[X_j^i, \frac{1}{W}]_{1\leq i,j\leq n}$, we must have

$$\det(X_j^i)_{1\leq i,j\leq n} \notin I.$$

Expanding the determinant $\det({}^{g_i}P_j)_{1\leq i,j\leq r}$ through the first row, we see from $P_1, \ldots, P_t \in I$ that $t < r$ (or else $t = r$ and we would have $W^N \in I$).

We compute $\det({}^{g_i}P_j)_{1\leq i,j\leq r}$ by expanding through the first t columns: given a collection of t indices $J = \{j_1, \ldots, j_t\}$, with $1 \leq j_1 < j_2 < \ldots < j_t \leq$

r, we denote by P_J the determinant

$$P_J = \det({}^{g_{j_k}}P_j)_{1\le j,k\le t},$$

and by Q_J its co-determinant so that

$$\det({}^{g_i}P_j)_{1\le i,j\le r} = \sum_J P_J Q_J.$$

Note that $P_J \in I$ if $1 \in J$. But, as $\det({}^{g_i}P_j)_{1\le i,j\le r} \notin I$, there is a collection of indices $\underline{k} = \{k_1, \ldots, k_t\}$ such that $1 \notin \underline{k}$. Because P^G is G-stable and the left and right actions commute, for every $g \in G$ there is a $\chi(g) \in K$ such that $P_J^g = \chi(g)P_J$ for every J. Similarly, because the space spanned by $P_{t+1}, \ldots, P_r$ is also G-stable, for every $g \in G$ there is a $\chi'(g) \in K$ such that $Q_J^g = \chi'(g)Q_J$ for every J. Thus, from the expansion of $\det({}^{g_i}P_j)_{1\le i,j\le r}$ we read that $\chi(g)\chi'(g) = \det(g)^N = 1$. In particular, for any two collection of t indices J and L, $P_J Q_L$ is G-invariant.

The previous lemma implies that there is an $f \in K$ such that $P_{\underline{k}}Q_{\underline{k}} - f \in I$. Since $P_{\underline{k}}Q_{\underline{k}} \notin I$ we know that $f \neq 0$.

Set $k_0 = 1$, $P_0 = P_1 = Q^N$. Denote by $\underline{k}(s)$ the sequence

$$\{k_0, k_1, \ldots, \hat{k_s}, \ldots, k_t\}$$

where the hat indicates omission. In particular $\underline{k}(0) = \underline{k}$. Also, the determinant of the $(t+1)\times(t+1)$ matrix

$$\begin{pmatrix} {}^{g_{k_0}}P_0 & {}^{g_{k_0}}P_1 & \cdots & {}^{g_{k_0}}P_t \\ {}^{g_{k_1}}P_0 & {}^{g_{k_1}}P_1 & \cdots & {}^{g_{k_1}}P_t \\ \vdots & \vdots & & \vdots \\ {}^{g_{k_t}}P_0 & {}^{g_{k_t}}P_1 & \cdots & {}^{g_{k_t}}P_t \end{pmatrix}$$

is zero since the two first columns are the same. Expanding along the first column we have

$$0 = Q^{N_0} P_{\underline{k}} - {}^{g_{k_1}} Q^{N_0} P_{k(1)} + \ldots + (-1)^t \; {}^{g_{k_t}} Q^{N_0} P_{k(t)}.$$

Multiplying by $Q_{\underline{k}}$ on both sides and then subtracting fQ^{N_0} we obtain

$$-fQ^{N_0} = Q^{N_0}(P_{\underline{k}} Q_{\underline{k}} - f) - {}^{g_{k_1}} Q^{N_0} P_{k(1)} Q_{\underline{k}} + \ldots + (-1)^t \; {}^{g_{k_t}} Q^{N_0} P_{k(t)} Q_{\underline{k}}.$$

Finally, note that if $s > 0$ then $1 \in k(s)$ and so $P_{k(s)} \in I$. Whence we have an expression for Q^{N_0} as a linear combination of G-invariants in I. ★

Corollary A.1.5. *Assuming the context of Theorem A.0.6, suppose the polynomials $P_1, \ldots, P_r$ generate the $\mathbb{C}$-algebra of G-invariants in $\mathbb{C}[X^i_j]_{1 \le i,j \le n}$, and $f_1, \ldots, f_r \in K$ are such that $P_1 - f_1, \ldots, P_r - f_r \in I$. Then I is the radical of the ideal generated by $\{P_i - f_i\}_{i \in \{1,\ldots,r\}}$.*

Proof: Because G is reductive, the $\mathbb{C}$-algebra of G-invariants in $\mathbb{C}[X^i_j]_{1 \le i,j \le n}$ is finitely generated. If $P \in \mathbb{C}[X^i_j, \frac{1}{W}]_{1 \le i,j \le n}$ then $W^N P \in K[X^i_j]_{1 \le i,j \le n}$ for some $N \gg 0$. Additionally, if P is G-invariant, and if N is such that W^N is G-invariant, then $W^N P$ can be written as a K-linear combination of G-invariants in $\mathbb{C}[X^i_j]_{1 \le i,j \le n}$. ★

A.2 Comment

If our original differential module M happens to be one-dimensional, then G is either $\mathbb{C}^*$ or is a finite cyclic group of roots of unity. In the last case the

Picard-Vessiot extension is given by the adjunction of the radical of an element of K. In general, withdrawing the unidimensionality of M, if $(f^i_j)_{1\leq i,j\leq n}$ is a fundamental matrix of solutions and $w = \det(f^i_j)_{1\leq i,j\leq n}$ denotes the wronskian, then one can prove that

$$w' = \mathrm{tr}(a^i_j)_{1\leq i,j\leq n} \cdot w,$$

where tr denotes the trace. Indeed, consider the n-th exterior power of M, and, by abuse of notation, denote its additive map by ∂:

$$\begin{aligned}
\partial(e_1 \wedge e_2 \wedge \ldots \wedge e_n) &= (a^j_1)e_j \wedge e_2 \ldots \wedge e_n + e_1 \wedge (a^j_2)e_j \wedge \ldots \wedge e_n + \ldots \\
&\quad + e_1 \wedge e_2 \wedge \ldots \wedge (a^j_n)e_j \\
&= a^1_1 e_1 \wedge e_2 \ldots \wedge e_n + e_1 \wedge a^2_2 e_2 \wedge \ldots \wedge e_n + \ldots \\
&\quad + e_1 \wedge e_2 \wedge \ldots \wedge a^n_n e_n \\
&= \mathrm{tr}(a^i_j) e_1 \wedge e_2 \wedge \ldots \wedge e_n.
\end{aligned}$$

If $(g^a_b) \in G$, then it sends f^i_j into $f^i_k g^k_j$ and $\det(f^i_j)$ into $\det(f^i_j) \cdot \det(g^a_b)$. In particular, if the image of $\det : G \to \mathbb{C}^*$ is finite, then $w = \sqrt[N]{f}$ for some $N \in \mathbb{N}$ and some $f \in K$.

Using these facts we can make a coordinate-free characterization of the hypotheses of Theorem A.0.6. We define the determinant group of G as the Galois group of the differential module given by the n-th exterior power of M. So Compoint's Theorem A.0.5 holds whenever the Galois group is reductive and its determinant group is trivial, and its extension Theorem A.0.6 holds whenever the Galois group is reductive and the determinant group is finite.

Definition A.2.1. We say that M_1 and M_2 are *finitely projectively equivalent* if there is a one-dimensional differential module M_0 over K such that:

a) M_2 can be identified with $M_0 \otimes M_1$ as differential modules over K; and

b) the Picard-Vessiot extension of M_0 is algebraic over K.

REMARK A.2.2. This is a special case of the concept of projective equivalence introduced in [16]. It follows immediately from the definition that, in the same notation, we have:

a) if (f^i_j) is a fundamental matrix solution of M_1 with respect to a basis, and if g is a horizontal section of M_0 then (gf^i_j) is a fundamental matrix solution of M_2;

b) as M_0 is one-dimensional and g is algebraic over K, $g^N \in K$ for some $N \in \mathbb{N}^*$; and,

c) if in M_0 the differential is given by $\partial e = ae$, with $a \in K$, then the differential in M_2 in the basis $\{e \otimes e_1, \ldots, e \otimes e_n\}$ is

$$\partial e \otimes e_j = (a\delta^i_j + a^i_j)e_i.$$

Proposition A.2.3. *The determinant group of G is finite if and only if M is finitely projectively equivalent to a differential module whose Galois group is unimodular.*

Proof: If the determinant group is finite then the wronskian w, as above, is algebraic. Tensoring with the module defining the n-th root of $\frac{1}{w}$, we obtain a module with unimodular Galois group. Conversely, the hypothesis on M implies that the wronskian is an element of K^* multiplied by a radical of an element of K, say an N-th root, so the determinant group of G is cyclic of order dividing N. ★

Corollary A.2.4. *G is reductive and the determinant group of G is finite if and only if M is finitely projectively equivalent to a differential module whose Galois group is unimodular and reductive.*

Proof: It suffices to use the previous proposition and to notice that the property of being reductive is not lost by finite projective equivalences. This follows from [8, Proposition 2.2] and Remark A.2.2 above. ★

Appendix B

Computational examples

There is a well-known correspondence between n-th order ordinary linear differential equations with coefficients in the field of meromorphic functions of a compact Riemann surface and n-dimensional meromorphic connections on meromorphic vector bundles over that surface (cf. [25]).

In a similar fashion, there is a correspondence between the approach to projective equivalence considered in [1] and the approach in [16]. In [1], two second order linear differential equations are projectively equivalent if there basis of solutions with identical ratios; and in [16], two meromorphic connections are projectively equivalent if one can obtain one from the other by tensoring with a 1-dimensional vector bundle with connection.

The advantages of considering equivalent classes of projectively equivalent equations, rather that working with a single equation, are heavily exploited in the literature. The best-known advantage is the reduction to the case of equations with constant wronskian. Indeed, to achieve this reduction it suffices to multiply each solution by the inverse of an n-th root of the wronskian. Equations with constant wronskian have the powerful characteristic of having a unimodular Galois group.

In this appendix I will use several examples to illustrate one more application of this idea of jumping from one equation to another through projective equivalence. This new application will allow us to uncover symmetries of the connection, and as a result to descent to a quotient Riemann surface. In particular we will obtain the pullback explained in [1] in a more explicit way. Essentially, this new approach does not differ from that of J.A. Weil in [29], since the second order case is a geometric interpretation of his algorithm. The algorithm in [29], [18] represents joint work of M. van Hoeij, M. Berkenbosch and J.A. Weil. For the definition of symmetry I refer to Chapter 3. In colloquial language a symmetry of a connection is a change of coordinates which leave the equation invariant.

We will start by showing three examples of second order. Among the three examples, the second is of special interest since it explains, among other simplifications, how an equation corresponding to the Schwarz triple $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ is actually a pullback of an equation corresponding to the triple $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$. In the third example we deduce an expression in terms of radicals for the solution to the standard equations with projective Galois groups A_4 and S_4. The remaining examples deal with third order cases. In contrast to the former examples, where the classification explained in [1] [2] relies completely on Galois coverings of the sphere by the sphere, the third order cases may involve coverings by surfaces of higher genus. Remarkable identities will arise in the cases where the solutions describe curves having prescribed genus. This last assertion will be made more explicit in due time.

In must be noted that a generalization of the above mentioned second order classification to third order has been carried out by M. Berkenbosch [3]. Our approach differs in that we rely solely on symmetries in order to achieve standard equations.

To obtain the Galois group of the new equations, as well as the first integrals, we use the algorithms in [17] and in [28].

B.1 Pépin's equation, 1881

The equation

$$y'' + \frac{21}{100}\frac{x^2 - x + 1}{x^2(x-1)^2}y = 0$$

has Galois group $A_5^{SL_2}$ [26], and the singularities are 0, 1 and ∞ with respective exponents

$$\{\frac{3}{10}, \frac{7}{10}\}, \quad \{\frac{3}{10}, \frac{7}{10}\}, \quad \{\frac{-7}{10}, \frac{-3}{10}\}.$$

After tensoring with

$$y' - \frac{1}{3}(\frac{1}{x} + \frac{1}{x-1})y = 0$$

the equation becomes

$$y'' + \frac{2}{3}\frac{2x-1}{x(x-1)}y' - \frac{11}{900}\frac{x^2 - x + 1}{x^2(x-1)^2}y = 0,$$

and the exponents at 0, 1 and ∞ are now

$$\{\frac{-1}{30}, \frac{11}{30}\}, \quad \{\frac{-1}{30}, \frac{11}{30}\}, \quad \{\frac{-1}{30}, \frac{11}{30}\}.$$

The tensored equation is invariant under the group of symmetries

$$S_3 = \{x, 1-x, \frac{1}{x}, \frac{x-1}{x}, \frac{1}{1-x}, \frac{x}{x-1}\},$$

and descends to the sphere parameterized by $z = \frac{1-3x-3x^5+5x^3+x^6}{(x^2-x)^2}$ as

$$y'' + \frac{2}{3}\frac{7z-15}{(z-6)(4z+3)}y' - \frac{11}{900}\frac{1}{(z-6)(4z+3)}y = 0.$$

The singularities at 6, $\frac{3}{4}$ and ∞ have respective exponents

$$\{0, \frac{1}{3}\}, \quad \{0, \frac{1}{2}\}, \quad \{\frac{-1}{60}, \frac{11}{60}\}.$$

This equation, after the Möbius transformation

$$z \mapsto -(4/27)z + 8/9,$$

is the same as J.A. Weil's S_{A_5} equation [29].

B.2 The triples $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ are pull-backs of the triple $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$

Consider Weil's S_{A_4} equation [29]

$$y'' + \frac{1}{6}\frac{7x-4}{x(x-1)}y' - \frac{1}{48}\frac{1}{x(x-1)}y = 0.$$

The singularities are 0, 1 and ∞, with respective exponents

$$\{0, \frac{1}{3}\}, \quad \{0, \frac{1}{2}\}, \quad \{\frac{-1}{12}, \frac{1}{4}\}.$$

Tensoring with

$$y' - \frac{1}{12}\frac{1}{x-1}y = 0$$

one obtains

$$y'' + \frac{2}{3}\frac{2x-1}{x(x-1)}y' - \frac{5}{144}\frac{1}{(x-1)^2 x} = 0,$$

with exponents at 0, 1 and ∞ given respectively by

$$\{0, \frac{1}{3}\}, \quad \{\frac{-1}{12}, \frac{5}{12}\}, \quad \{0, \frac{1}{3}\}.$$

The equation is invariant under the change of variable $x \mapsto \frac{1}{x}$ and descends to the sphere parameterized by $z = \frac{x^2+1}{x}$ as

$$y'' + \frac{2}{3}\frac{2z+1}{(z-2)(z+2)}y' - \frac{5}{144}\frac{1}{(z-2)^2(z+2)}y = 0.$$

The singularities are now at 2, -2 and ∞ and the respective exponents are

$$\{0, \frac{1}{2}\}, \quad \{\frac{-1}{24}, \frac{5}{24}\}, \quad \{0, \frac{1}{3}\}.$$

This equation, under the change of variable $z \mapsto \frac{-4}{z-2}$, is the same as J.A. Weil's S_{S_4} equation [29].

Comparing solutions of the original and tensored equations we obtain the identities

$$\begin{aligned} {}_2F_1(\frac{-1}{12}, \frac{1}{4}, \frac{2}{3}; x) &= (x-1)^{1/12} {}_2F_1(\frac{-1}{24}, \frac{5}{24}, \frac{2}{3}; -\frac{4x}{(x-1)^2}), \\ \sqrt[3]{x}\ {}_2F_1(\frac{1}{4}, \frac{7}{12}, \frac{2}{3}; x) &= (x-1)^{1/12} \sqrt[3]{\frac{-4x}{(x-1)^2}}\ {}_2F_1(\frac{7}{24}, \frac{13}{24}, \frac{2}{3}; -\frac{4x}{(x-1)^2}). \end{aligned}$$

Now we focus our attention on Weil's S_{D_4} equation [29]

$$y'' + \frac{1}{2}\frac{2x-1}{(x-1)x}y' - \frac{1}{16}\frac{1}{(x-1)x}y = 0.$$

The singularities are 0, 1 and ∞, with respective exponents

$$\{0, \frac{1}{2}\}, \quad \{0, \frac{1}{2}\}, \quad \{\frac{-1}{4}, \frac{1}{4}\}.$$

After tensoring with

$$y' - \frac{1}{12}(\frac{1}{x} + \frac{1}{x-1})y = 0$$

we obtain

$$y'' + \frac{2}{3}\frac{2x-1}{(x-1)x}y' - \frac{5}{144}\frac{x^2-x+1}{(x-1)^2x^2}y = 0.$$

The exponents at 0, 1 and ∞ are now

$$\{\frac{-1}{12}, \frac{5}{12}\}, \quad \{\frac{-1}{12}, \frac{5}{12}\}, \quad \{\frac{-1}{12}, \frac{5}{12}\}.$$

The tensored equation is invariant under the group of symmetries

$$S_3 = \{x, 1-x, \frac{1}{x}, \frac{x-1}{x}, \frac{1}{1-x}, \frac{x}{x-1}\}$$

and descends to the sphere parameterized by $z = \frac{1-3x-3x^5+5x^3+x^6}{(x^2-x)^2}$ as

$$y'' + \frac{2}{3}\frac{7z-15}{(z-6)(4z+3)}y' - \frac{5}{144}\frac{1}{(z-6)(4z+3)}y = 0.$$

The singularities are now 6, $\frac{3}{4}$ and ∞, and the respective exponents are

$$\{0, \frac{1}{3}\}, \quad \{0, \frac{1}{2}\}, \quad \{\frac{-1}{24}, \frac{5}{24}\}.$$

This equation, after the Möbius transformation

$$z \mapsto -(4/27)z + 8/9,$$

is the same as J.A. Weil's S_{S_4} equation.

We obtain the identities:

$$\sqrt[4]{4x-2+4\sqrt{(x-1)x}} = \sqrt[12]{x(x-1)}\ {}_2F_1\left(\frac{-1}{24},\frac{5}{24},\frac{2}{3};\frac{4}{27}\frac{(x^2-x+1)^3}{x^2(x-1)^2}\right)$$

$$\frac{1}{\sqrt[4]{4x-2+4\sqrt{(x-1)x}}} = \sqrt[12]{x(x-1)}\sqrt[3]{\frac{4}{27}\frac{(x^2-x+1)^3}{x^2(x-1)^2}}\ {}_2F_1\left(\frac{7}{24},\frac{13}{24},\frac{2}{3};\frac{4}{27}\frac{(x^2-x+1)^3}{x^2(x-1)^2}\right)$$

If $z = \frac{4}{27}\frac{(x^2-x+1)^3}{x^2(x-1)^2}$, then

$$x = \frac{1}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{A^{2/3}+(3z-1)A^{1/3}+z(9z-8)}}{\sqrt[6]{A}},$$
$$\text{where } A = 27z^3-36z^2+8z+8\sqrt{z^2-z^3}$$

Combining these identities we obtain radical expressions for the solutions to S_{S_4} and S_{A_4}.

B.3 The triple $(\frac{1}{2},\frac{1}{2},\frac{1}{2n})$ descends to the triple $(\frac{1}{2},\frac{1}{2},\frac{1}{2(2n)})$

Consider Weil's $S_{D_{2n}}$ equation [29]

$$y''+\frac{1}{2}\frac{2x-1}{(x-1)x}y'-\frac{1}{4n^2}\frac{1}{(x-1)x}y=0.$$

The singularities are 0, 1 and ∞, with respective exponents

$$\{0,\frac{1}{2}\},\quad \{0,\frac{1}{2}\},\quad \{\frac{-1}{2n},\frac{1}{2n}\}$$

The equation is invariant under the change of variable $x \mapsto 1-x$ and descends to the sphere parameterized by $z = 4(x-x^2)$, resulting in

$$y''+\frac{1}{2}\frac{2z-1}{(z-1)z}y'-\frac{1}{4(2n)^2}\frac{1}{(z-1)z}y=0.$$

This is Weil's $S_{D_{4n}}$ equation.

B.4 A standard equation with projective Galois group F_{36}

B.4.1 Ulmer's G_{54} equation

Consider the equation [28]

$$y''' + \frac{3(3x^2-1)}{x(x-1)(x+1)}y'' + \frac{221x^4-206x^2+5}{12x^2(x-1)^2(x+1)^2}y' + \frac{374x^6-673x^4+254x^2+5}{54x^3(x-1)^3(x+1)^3}y = 0.$$

It has Galois group G_{54} of order 54. The singular points are 0, 1, -1 and ∞ with respective exponents

$$\{-\frac{1}{6}, \frac{5}{6}, -\frac{2}{3}\}, \quad \{-\frac{1}{6}, \frac{5}{6}, -\frac{2}{3}\}, \quad \{-\frac{1}{6}, \frac{5}{6}, -\frac{2}{3}\}, \quad \{\frac{11}{6}, \frac{17}{6}, \frac{4}{3}\}.$$

We tensor it with

$$y' + \frac{1}{2}(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1})y = 0,$$

obtaining

$$y''' + \frac{3}{2}\frac{3x^2-1}{x(x-1)(x+1)}y'' + \frac{1}{3}\frac{8x^4-11x^2-1}{x^2(x-1)^2(x+1)^2}y' - \frac{2}{27}\frac{x^6-5x^4-5x^2+1}{x^3(x-1)^3(x+1)^3}y = 0.$$

The space of third degree invariants of this equation is of dimension 2, and its Galois group is not unimodular. The singularities 0, 1, -1 and ∞, have exponents

$$\{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\}, \quad \{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\}, \quad \{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\}, \quad \{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\}.$$

So the equation is invariant under the group of symmetries

$$D_8 = \{x, \frac{1}{x}, -x, \frac{-1}{x}, \frac{-x+1}{x+1}, \frac{x+1}{x-1}, \frac{x+1}{-x+1}, \frac{x-1}{x+1}\},$$

and it descends to the sphere parameterized by

$$z = \frac{1}{32}\left(\frac{2(1+7x^2+7x^6+x^8)}{x^2(x+1)^2(x-1)^2} - 6\right) = \frac{1}{16}\frac{(x^2+1)^4}{x^2(x+1)^2(x-1)^2}$$

as

$$y''' + \frac{3}{4}\frac{5z-3}{z(z-1)}y'' + \frac{1}{24}\frac{43z-9}{z^2(z-1)}y' - \frac{1}{108z^2(z-1)}y = 0.$$

This differential equation corresponds to the generalized hypergeometric differential equation defining ${}_3F_2(-\frac{1}{12}, \frac{1}{6}, \frac{2}{3}; \frac{1}{2}, \frac{3}{4}; z)$. It has singularities at 0, 1 and ∞ with exponents

$$\{0, \frac{1}{4}, \frac{1}{2}\}, \quad \{0, \frac{1}{2}, 1\}, \quad \{\frac{-1}{12}, \frac{1}{6}, \frac{2}{3}\}.$$

A solution to our original equation is therefore

$$\frac{1}{\sqrt{x(x-1)(x+1)}}\ {}_3F_2\left(-\frac{1}{12}, \frac{1}{6}, \frac{2}{3}, \frac{1}{2}, \frac{3}{4}; \frac{1}{16}\frac{(x^2+1)^4}{x^2(x+1)^2(x-1)^2}\right)$$

B.4.2 The Geiselmann-Ulmer $F_{36}^{SL_3}$ equation

The equation

$$y''' + \frac{5(9x^2+14x+9)}{48x^2(x+1)^2}y' - \frac{5(81x^3+185x^2+229x+81)}{432x^3(x+1)^3}y = 0$$

has Galois group $F_{36}^{SL_3}$ of order 108 [14]. The singular points are 0, 1 and ∞ with respective exponents

$$\{1, \frac{3}{4}, \frac{5}{4}\}, \quad \{\frac{5}{6}, \frac{11}{6}, \frac{1}{3}\}, \quad \{-1, \frac{-3}{4}, \frac{-5}{4}\}.$$

We tensor it with

$$y' - \frac{1}{2}\left(\frac{1}{x} + \frac{3}{2}\frac{1}{x-1}\right)y = 0,$$

obtaining

$$y''' + \frac{3}{4}\frac{5x-2}{(x-1)x}y'' + \frac{1}{24}\frac{45x^2-28x-8}{(x-1)^2x^2}y' - \frac{1}{27}\frac{x-2}{(x-1)^2x^3}y = 0.$$

The singularities 0, 1 and ∞ now have exponents

$$\{-\frac{1}{6}, \frac{1}{3}, \frac{4}{3}\}, \quad \{0, \frac{1}{4}, \frac{1}{2}\}, \quad \{0, \frac{1}{4}, \frac{1}{2}\}.$$

The equation is invariant under the change of variable $x \mapsto \frac{x}{x-1}$, and descends to the sphere parameterized by

$$z = \frac{4(x-1)}{x^2}$$

where the equation becomes

$$y''' + \frac{3}{4}\frac{5z-3}{z(z-1)}y'' + \frac{1}{24}\frac{43z-9}{z^2(z-1)}y' - \frac{1}{108z^2(z-1)}y = 0.$$

This is the same as the generalized hypergeometric equation of the previous example, and a solution to our original equation is therefore given by

$$x^{\frac{1}{2}}(x-1)^{\frac{3}{4}}\ {}_3F_2(-\frac{1}{12}, \frac{1}{6}, \frac{2}{3}, \frac{1}{2}, \frac{3}{4}; \frac{4(x-1)}{x^2}).$$

B.4.3 The ${}_3F_2(-\frac{1}{12}, \frac{1}{6}, \frac{2}{3}, \frac{1}{2}, \frac{3}{4}; z)$ equation

We now center our attention on

$$y''' + \frac{3}{4}\frac{5z-3}{z(z-1)}y'' + \frac{1}{24}\frac{43z-9}{z^2(z-1)}y' - \frac{1}{108z^2(z-1)}y = 0.$$

Since two projectively equivalent linear differential equation have the same projective Galois group, instead of computing the group of this equation we can work with that of its normalization, i.e.

$$y''' + \frac{1}{48}\frac{41z^2-50z+45}{(z-1)^2z^2}y' - \frac{1}{432}\frac{364z^3-665z^2+1030z-405}{(z-1)^3z^3}y = 0.$$

The singularities are 0, 1 and ∞ with respective exponents

$$\{\frac{3}{4}, 1, \frac{5}{4}\}, \quad \{\frac{1}{2}, 1, \frac{3}{2}\}, \quad \{-\frac{4}{3}, -\frac{13}{12}, -\frac{7}{12}\}.$$

The equation has two semi-invariants of order two and degree three, so the Galois group is $F_{36}^{SL_3}$. The sixth degree invariants have value 0 and $z^5(z-1)^3$, the ninth degree 0 and $z^7(z-1)^5$, so in particular the ratio of two eighteenth degree invariants is $\frac{z}{z-1}$.

Probably the most important feature of this example is the fact that the solutions describe an elliptic curve. By this we mean that if X denotes the solution with exponent $\frac{3}{4}$, Y the solution with exponent 1 and Z the solution with exponent $\frac{5}{4}$ around 0 (normalized to have first coefficient 1 in their power series expansions), then X, Y and Z satisfy the homogeneous relation

$$XZ^2 - Y^2Z - \frac{1}{81}X^3 = 0$$

corresponding to a curve of genus 1 in the projective plane. If we set $g = 1$ $M = |F_{36}| = 36$ and $e_0 = e_\infty = 4$ and $e_1 = 2$ (the values coming from the exponents of the ratio of solutions to the equation) then we have (recall Corollary 2.3.3):

$$2 - \sum_{i \in \{0,1,\infty\}} \left(1 - \frac{1}{e_i}\right) = \frac{-2(g-1)}{M}$$

B.5 A standard equation with projective Galois group H_{216}

B.5.1 van Hoeij's $H_{72}^{SL_3}$ equation

Consider van Hoeij's $H_{72}^{SL_3}$ equation [28]:

$$\begin{aligned} 0 &= y''' + \frac{21x^2-24x-1}{(3x^2+1)(x-1)}y'' + \frac{1}{48}\frac{4437x^3-5973x^2+171x-683}{(3x^2+1)^2(x-1)}y' \\ &\quad + \frac{1}{216}\frac{13338x^4-22647x^3+1983x^2-7297x-737}{(3x^2+1)^3(x-1)}y. \end{aligned}$$

The singular points are 1 (which in fact is an apparent singularity), $\frac{i\sqrt{3}}{3}$, $-\frac{i\sqrt{3}}{3}$ and ∞, with respective exponents

$$\{0,1,3\}, \quad \{-\frac{7}{12}, -\frac{1}{3}, -\frac{1}{12}\}, \quad \{-\frac{7}{12}, -\frac{1}{3}, -\frac{1}{12}\}, \quad \{\frac{13}{12}, \frac{4}{3}, \frac{19}{12}\}.$$

Tensoring with

$$y' - \frac{5}{9}(\frac{1}{3x^2+1})y = 0$$

we obtain

$$y''' + \frac{11x^2-14x-1}{(3x^2+1)(x-1)}y'' + \frac{1}{48}\frac{757x^3-1333x^2+11x-203}{(3x^2+1)^2(x-1)}y' - \frac{17}{216}\frac{x^3-3x^2+3x-1}{(3x^2+1)^3} = 0,$$

and the exponents become

$$\{0,1,3\}, \quad \{-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}\}, \quad \{-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}\}, \quad \{-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}\}.$$

So the equation is invariant under a cyclic group of order three that rotates the points $\frac{i\sqrt{3}}{3}$, $-\frac{i\sqrt{3}}{3}$ and ∞ and leaves 1 invariant. It therefore descends to the sphere parameterized by

$$z = \frac{1}{2}\frac{(x+1)^3}{(1+3x^2)}$$

as

$$y''' + \frac{1}{3}\frac{11z-6}{z(z-1)}y'' + \frac{1}{432}\frac{757z-96}{z^2(z-1)}y' - \frac{17}{5832}\frac{1}{z^2(z-1)}y = 0.$$

This equation has singularities at 0, 1 and ∞ and with respective exponents

$$\{0, \frac{1}{3}, \frac{2}{3}\}, \quad \{0, \frac{1}{3}, 1\}, \quad \{-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}\},$$

This is the generalized hypergeometric differential equation defining the function ${}_3F_2(-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}, \frac{1}{3}, \frac{2}{3}; z)$. A solution to van Hoeij's equation is therefore given by

$$(3x^2+1)^{\frac{5}{9}}{}_3F_2(-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}, \frac{1}{3}, \frac{2}{3}; \frac{1}{2}\frac{(x+1)^3}{(1+3x^2)}).$$

B.5.2 The ${}_3F_2(-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}, \frac{1}{3}, \frac{2}{3}; z)$ equation

We will follow the same reasoning (and phrasing) from the previous example, now centering our attention on

$$y''' + \frac{1}{3}\frac{11z-6}{z(z-1)}y'' + \frac{1}{432}\frac{-96+757z}{z^2(z-1)}y' - \frac{17}{5832}\frac{1}{z^2(z-1)} = 0.$$

As two projectively equivalent linear differential equation have the same projective Galois group, instead of computing the Galois group of this equation we consider its normalization

$$y'''+\frac{1}{432}\frac{405z^2-469z+384}{(z-1)^2z^2}y'-\frac{1}{11664}\frac{10935z^3-18803z^2+27196z-10368}{(z-1)^3z^3}y = 0.$$

The singularities are 0, 1 and ∞ with respective exponents

$$\{\frac{2}{3}, 1, \frac{4}{3}\}, \quad \{\frac{5}{9}, \frac{8}{9}, \frac{14}{9}\}, \quad \{-\frac{5}{4}, -1, -\frac{3}{4}\}.$$

The third symmetric power of the associated operator has a second order right factor

$$y'' - \frac{2}{3}\frac{9z-5}{z(z-1)}y' + \frac{1}{48}\frac{585z^2-649z+224}{z^2(z-1)^2}y'.$$

Furthermore, the space of fourth degree invariants of this second order factor is one-dimensional, spanned by $z^8(z-1)^7$. We conclude that the Galois group is $H_{216}^{SL_3}$ [28]. The ninth degree invariant of the normalized operator has value $z^6(z-1)^5$, so the ratio of two twenty-fourth degree invariants is $z-1$. It is remarkable that the solutions to the equation describe a curve of genus 10. By this we mean that if X denotes the solution with exponent $\frac{5}{9}$, Y the solution with exponent $\frac{8}{9}$, and Z the solution with exponent $\frac{14}{9}$ around 1 (normalized with first coefficient 1 in their power series expansions), then X, Y and Z satisfy the (horrible) homogeneous relation:

$$\frac{174142586880000}{5821656268296953}Y^6 - \frac{26741631626956800}{5821656268296953}Y^3Z^3 + Z^6 + \frac{169126071054336000}{5821656268296953}XY^3Z^2$$

$$-\frac{61786141208337600}{5821656268296953}XZ^5 - \frac{356644017930240000}{5821656268296953}X^2Y^3Z + \frac{262289575363968000}{5821656268296953}X^2Z^4$$

$$+\frac{250765325107200000}{5821656268296953}X^3Y^3 - \frac{79529414492160000}{831665181185279}X^3Z^3 + \frac{590778725990400000}{5821656268296953}X^4Z^2$$

$$-\frac{250765325107200000}{5821656268296953}X^5Z^1 = 0$$

corresponding to a curve of genus 10 in the projective plane. If we set $g = 10$ $M = |H_{216}| = 216$ and $e_0 = e_1 = 3$ and $e_\infty = 4$ (the values coming from the exponents of the ratio of solutions to the equation) then we have:

$$2 \; - \sum_{i\in\{0,1,\infty\}} \left(1 - \frac{1}{e_i}\right) = \frac{-2(g-1)}{M}.$$

Appendix C

Meromorphic vector bundles over compact Riemann surfaces

Although we assume familiarity with vector bundles, reviewing the definition will prove useful for introducing the meromorphic case. A more detailed exposition of the concept of holomorphic vector bundle may be found in [12].

C.1 Holomorphic vector bundles

We fix a compact Riemann surface X and denote the field of meromorphic functions over X by k. Given an open set $U \subseteq X$ the collection of holomorphic functions $\phi : U \to \mathbb{C}$ will be denoted by $\mathcal{H}(U)$. Given any product $A \times B$ we will denote respectively by $\Pi_1 : A \times B \to A$ and by $\Pi_2 : A \times B \to B$ the first and second canonical projections.

Definition C.1.1. A *rank* n (*holomorphic*) *vector bundle* over X is a complex manifold E together with a holomorphic map $\Pi : E \to X$ and a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$, where $\{U_\alpha\}_{\alpha \in A}$ is a covering of X and for each $\alpha \in A$,

$$\phi_\alpha : \Pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^n$$

is a bi-holomorphic map with $\Pi_1 \circ \phi_\alpha = \Pi$, satisfying the following properties:

1. for any two $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$ there is a holomorphic map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{C})$ such that

$$\phi_\beta \circ \phi_\alpha^{-1} \restriction_{U_\alpha \cap U_\beta} = id_{U_\alpha \cap U_\beta} \times g_{\beta\alpha};$$

2. for any two $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$

$$g_{\beta\alpha} = g_{\alpha\beta}^{-1}$$

that is $g_{\beta\alpha}(p) = (g_{\alpha\beta}(p))^{-1}$ for every $p \in U_\alpha \cap U_\beta$;

3. for any three $\alpha, \beta, \gamma \in A$ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ we have

$$g_{\gamma\alpha} = g_{\gamma\beta} g_{\beta\alpha},$$

i.e. $g_{\gamma\alpha}(p) = g_{\gamma\beta}(p) g_{\beta\alpha}(p)$ for every $p \in U_\alpha \cap U_\beta$.

If $U \subseteq X$ is an open set such that $\Pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{C}^n$, we say that U *trivializes* the vector bundle.

REMARK C.1.2. We can define vector bundles in terms of cocycles in Čech cohomology. A holomorphic map $g : U \to GL_n(\mathbb{C})$ can be seen as an element in $GL_n(\mathcal{H}(U))$. Indeed given a standard coordinate system $(x^i_j)_{i,j\in\{1,\dots,n\}}$ of $GL_n(\mathbb{C})$, if we denote by g^i_j the element in $\mathcal{H}(U)$ given by the composition

$$g^i_j = x^i_j \circ g,$$

then we can identify g with the matrix having ij-th entry g^i_j, for each $i, j \in \{1, \dots, n\}$. Since X is compact, we can cover this Riemann surface by a finite collection $\{U_\alpha\}_{\alpha\in A}$ of trivializing open sets. The vector bundle Π gives us, for each non-empty intersection of two trivializing open sets U_α and U_β, a transition element $g_{\beta\alpha} \in GL_n(\mathcal{H}(U_\alpha \cap U_\beta))$. So we obtain an element

$$s \in \prod_{(\alpha,\beta)\in A\times A} GL_n(\mathcal{H}(U_\alpha \cap U_\beta))$$

whose (α, β) entry is precisely $g_{\beta\alpha}$, that is $(s)_{(\alpha,\beta)} = g_{\beta\alpha}$. We can map the product $\prod_{A\times A} GL_n(\mathcal{H}(U_\alpha \cap U_\beta))$ into $\prod_{A\times A\times A} GL_n(\mathcal{H}(U_\alpha \cap U_\beta \cap U_\gamma))$ via

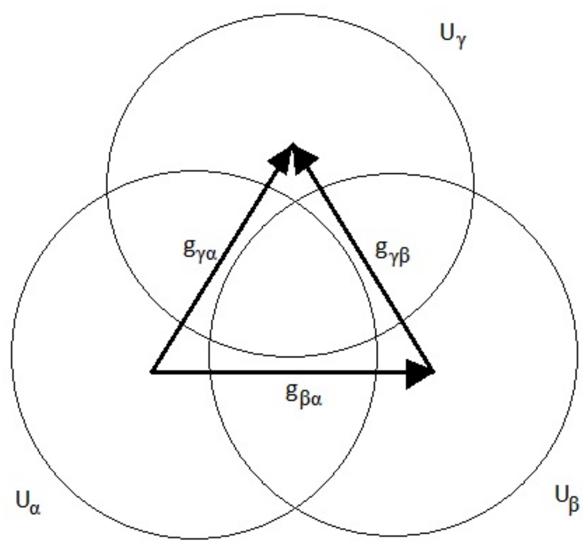


Figure C.1: vector bundle as a cocycle

the coboundary map δ defined by

$$(\delta\xi)_{(\alpha,\beta,\gamma)} = (\xi)^{-1}_{(\gamma,\alpha)}(\xi)_{(\gamma,\beta)}(\xi)_{(\beta,\alpha)} \restriction_{U_\alpha \cap U_\beta \cap U_\gamma} .$$

So a combination of Items 2. and 3. in the definition of vector bundle tells us that

$$\begin{aligned}(\delta s)_{(\alpha,\beta,\gamma)} &= g^{-1}_{\gamma\alpha} g_{\gamma\beta} g_{\beta\alpha} \restriction_{U_\alpha \cap U_\beta \cap U_\gamma} \\ &= Id,\end{aligned}$$

where Id stands for the identity operator in $GL_n(\mathcal{H}(U_\alpha \cap U_\beta \cap U_\gamma))$. This means that s defines a 1-cocycle in the Čech cohomology. Conversely, a 1-cocycle in Čech cohomology gives the transition elements defining a vector bundle. To get the total space E we start with the disjoint union

$$\bigsqcup_{\alpha \in A} U_\alpha \times \mathbb{C}^n,$$

and we identify elements in $(U_\alpha \cap U_\beta) \times \mathbb{C}^n \subseteq U_\alpha \times \mathbb{C}^n$ with elements in $(U_\alpha \cap U_\beta) \times \mathbb{C}^n \subseteq U_\beta \times \mathbb{C}^n$ via the transition map $g_{\beta\alpha}$. The projection Π is given by the projections on the first factor.

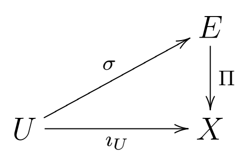
REMARK C.1.3. In analogy with the definition of cocycles in terms of the intersection of two trivializing sets and a co-boundary map, we can also define cocycles using the trivializing sets and a co-boundary map into the intersection of two trivializing sets to get the zeroth Čech cohomology. Indeed we define the *coboundary* map from $\prod_A GL_n(\mathcal{H}(U_\alpha))$ into $\prod_{A\times A} GL_n(\mathcal{H}(U_\alpha \cap U_\beta))$ by:

$$(\delta\xi)_{(\alpha,\beta)} = (\xi)^{-1}_\beta (\xi)_\alpha.$$

We say that $\xi \in \prod_A GL_n(\mathcal{H}(U_\alpha))$ is a *cocycle* if $(\delta\xi)_{(\alpha,\beta)} = Id$ for all $(\alpha, \beta) \in A \times A$.

REMARK C.1.4. Although we defined vector bundle using local holomorphic functions, the bundles that we will use will be defined using functions that are meromorphic over X.

Definition C.1.5. Given an open set $U \subseteq X$, a (*holomorphic*) *section* of Π over U is a holomorphic map $\sigma : U \to E$ such that the diagram



commutes (i.e. $\Pi \circ \sigma = \imath_U$), where $\imath_U : U \to X$ is the inclusion.

REMARK C.1.6. Consider a trivializing set U_α and a bi-holomorphic map $\phi_\alpha : \Pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^n$. Given a section σ over U_α we have $\Pi_1 \circ \phi_\alpha \circ \sigma = \Pi \circ \sigma = \imath_U$ and

$$\Pi_2 \circ \phi_\alpha \circ \sigma : U \longrightarrow \mathbb{C}^n.$$

Thus, if $e^1, \dots, e^n$ is a system of linear coordinates for $\mathbb{C}^n$, we have n holomorphic functions

$$e^i \circ \Pi_2 \circ \phi_\alpha \circ \sigma =: f^i \in \mathcal{H}(U_\alpha), \quad i \in \{1, \dots, n\}.$$

Conversely, n holomorphic functions $f^1, \dots, f^n \in \mathcal{H}(U_\alpha)$ define a unique section σ over U_α, i.e.

$$\sigma = \phi_\alpha^{-1} \circ (id_{U_\alpha} \times f^1 \times \dots \times f^n)$$

REMARK C.1.7. In general holomorphic vector bundles may not have global holomorphic sections (holomorphic sections over X).

Definition C.1.8. A *meromorphic section* of Π over U is a holomorphic section σ of Π over $U \setminus S$, where $S \subseteq X$ is a discrete set, such that for every trivializing open set $U_\alpha \subseteq U$ (with bi-holomorphic map $\phi_\alpha : \Pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$) the holomorphic functions

$$f^i := e^i \circ \Pi_2 \circ \phi_\alpha \circ \sigma \restriction_{U_\alpha \setminus S}, \quad i \in \{1, \dots, n\},$$

define meromorphic functions over U_α.

REMARK C.1.9. Just as a holomorphic section over U_α is given by n holomorphic functions, a meromorphic section over U_α is given by n meromorphic functions.

Lemma C.1.10. *The collection of global meromorphic sections of* Π *form an* n*-dimensional* k *vector space.*

Proof: From [11, §29] we obtain that the collection of global meromorphic sections of Π is a non-zero k vector space. Let s be non-zero global meromorphic section of Π. Give a $p \in X$ set

$$s_p := \{f(p) \cdot s(p) \subseteq \Pi^{-1}(p) | \; f \text{ meromorphic at } p, f \cdot s \text{ holomorphic at } p\}.$$

The subspace $\cup_{p \in X} s_p \subseteq E$ describes a rank 1 subbundle of Π. The lemma follows by considering the quotient bundle with fibers $\Pi^{-1}(p)/s_p$ and applying induction on n. ★

C.2 Meromorphic vector bundles

A more detailed exposition of the concepts introduced here may be found in [23].

The definition of a holomorphic vector bundle makes reference to a total space, but it is completely determined by the transition maps. With this observation as a motivation, we can define meromorphic vector bundles, not by a total space satisfying some prescribed properties, but by a 1-cocycle in the Čech cohomology. We denote by $\mathcal{M}$ the sheaf over X of meromorphic functions. In particular $\mathcal{M}(X) = k$.

Definition C.2.1. A *rank n meromorphic vector bundle* is a 1-cocycle in the Čech cohomology of the sheaf over X given by $U \mapsto GL_n(\mathcal{M}(U))$. That is, such a bundle consists of:

1. a finite covering $\{U_\alpha\}_{\alpha \in A}$; and

2. for every pair $(\alpha, \beta) \in A \times A$ a $g_{\beta\alpha} \in GL_n(\mathcal{M}(U_\alpha \cap U_\beta))$ such that

$$g_{\beta\alpha} = g_{\alpha\beta}^{-1};$$

   and such that

3. for every $(\alpha, \beta, \gamma) \in A \times A \times A$ one has

$$g_{\gamma\alpha} = g_{\gamma\beta} g_{\beta\alpha}$$

   on $U_\alpha \cap U_\beta \cap U_\gamma$.

We say that a meromorphic vector bundle is *trivial* if it is given by the trivial cocycle, that is every $g_{\beta\alpha}$ is the identity map.

REMARK C.2.2. Let us consider the relation in between holomorphic and meromorphic vector bundles. As every holomorphic function is meromorphic, every holomorphic vector bundle defines a meromorphic vector bundle. Assume, conversely, that we have a meromorphic vector bundle as defined above. Let $S = \{p_1, \ldots, p_r\}$ be the collection of poles of all the transition maps g^{α}_{β}, $(\alpha, \beta) \in A \times A$. We refine our covering as follows. Let $B = A \cup S$, for every $\alpha \in A$ we set

$$V_\alpha := U_\alpha \setminus S;$$

for every $p \in S$, we pick $\alpha_p \in A$ such that $p \in U_{\alpha_p}$ and we set

$$V_p := V_{\alpha_p} \cup \{p\}.$$

Thus $\cup_{\alpha \in A} V_\alpha = X \setminus S$ and $S \subset \cup_{p \in S} V_p$, so that $\cup_{b \in B} V_b = X$. Now for any $a, b \in B$ such that $a \neq b$, $S \cap (V_a \cap V_b) = \emptyset$, so the transition functions defined by

$$\begin{aligned} h_{\beta\alpha} &:= g_{\beta\alpha} \restriction_{V_\alpha \cap V_\beta} \\ h_{p\beta} &:= g_{\alpha_p\beta} \restriction_{V_{\alpha_p} \cap V_\beta} \\ h_{\beta p} &:= g_{\beta\alpha_p} \restriction_{V_{\alpha_p} \cap V_\beta} \\ h_{pq} &:= g_{\alpha_p\alpha_q} \restriction_{V_{\alpha_p} \cap V_{\alpha_q}} \end{aligned}$$

are holomorphic. This way we obtain a holomorphic 1-cocycle in the Čech cohomology, thereby defining a holomorphic vector bundle whose induced meromorphic vector bundle coincides in the cocycle sense with our original bundle. It must be noted that this holomorphic vector bundle is not unique, different representatives of the same meromorphic 1-cocycle may lead to different holomorphic vector bundles.

Proposition C.2.3. *Every meromorphic vector bundle over X is trivial.*

Proof: Given a meromorphic vector bundle there is an induced holomorphic vector bundle as in the remark above. It follows from Lemma C.1.10 that the global meromorphic sections of this holomorphic vector bundle are isomorphic to k^n. A k-basis of the space of global meromorphic sections, by restriction to the V_a's, defines a 0-cocycle in the Čech cohomology that through the coboundary renders our meromorphic bundle trivial. ★

Definition C.2.4. Given a meromorphic vector bundle over X, let $\Pi : E \to X$ be an induced holomorphic vector bundle. If $U \subseteq X$ is an open set, a (*meromorphic*) *section* of the meromorphic vector bundle over U will mean a meromorphic section of Π over U.

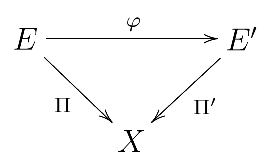
REMARK C.2.5. In view of the above-mentioned relation in between holomorphic vector bundles and meromorphic vector bundles, we refer to the meromorphic vector bundle by the map $\Pi : E \to X$ corresponding to an induced holomorphic vector bundle and refer to the *rank* of Π as the rank of the meromorphic bundle. In this thesis every vector bundle is considered with its meromorphic structure. In particular, we only deal with meromorphic sections unless otherwise stated.

Definition C.2.6. Let $\Pi : E \to X$ and $\Pi' : E' \to X$ be two meromorphic vector bundles of respective ranks n and n'. By refining the coverings of X trivializing Π and Π' we may assume that both bundles are trivialized by the same covering $\{U_\alpha\}_{\alpha \in A}$. A *bundle morphism* φ of Π to Π' is a collection of meromorphic linear maps $\varphi_\alpha \in M_{n' \times n}(\mathcal{M}(U_\alpha))$ such that:

$$\varphi_\beta \restriction_{U_\alpha \cap U_\beta} \circ \, g_{\beta\alpha} = g'_{\beta\alpha} \circ \ \varphi_\alpha \restriction_{U_\alpha \cap U_\beta}$$

for every $(\alpha, \beta) \in A \times A$.

REMARK C.2.7. Although a morphism of vector bundles does not corresponds to a holomorphic map of the total spaces, it proves convenient to denote the bundle morphism ϕ by



Indeed, the morphism corresponds to a holomorphic map of total spaces once we remove the poles of the φ_α's from X, for we are then dealing with the meromorphic maps

$$id_{U_\alpha} \times \varphi : U_\alpha \times \mathbb{C}^n \to U_\alpha \times \mathbb{C}^{n'}.$$

C.3 Meromorphic connections

Consider a meromorphic vector bundle $\Pi : E \to X$ and denote by $\mathscr{E}$ the sheaf of sections of Π (i.e. given an open set $U \subseteq X$, $\mathscr{E}(U)$ is the collection of meromorphic sections of Π). Similarly, denote by $\mathscr{T}X$ the sheaf of meromorphic tangent vector fields over X and by $\Omega^1_{\mathscr{M}}$ the sheaf of meromorphic differentials. Note that the sheaves $\mathscr{T}X$ and $\Omega^1_{\mathscr{M}}$ correspond to the meromorphic sections of the tangent bundle and of the bundle of differential forms respectively, so that $\mathscr{E}(X) \simeq k^n$, $\mathscr{T}X(X) \simeq k$ and $\Omega^1_{\mathscr{M}}(X) \simeq k$.

Definition C.3.1. A *meromorphic connection* over Π is a $\mathbb{C}$-linear map

$$\nabla : \mathscr{E} \longrightarrow \mathscr{E} \otimes_{\mathcal{M}} \Omega^1_{\mathscr{M}},$$

i.e. for every open $U \subseteq X$ a map $\mathscr{E}(U) \to \mathscr{E}(U) \otimes_{\mathcal{M}(U)} \Omega^1_{\mathscr{M}}(U)$ satisfying the Leibniz rule

$$\nabla f\sigma = \sigma \otimes df + f\nabla\sigma$$

for every section σ of Π. A section σ is called *horizontal* if $\nabla\sigma = 0$.

REMARK C.3.2. Since $\Omega^1_{\mathscr{M}}$ is the meromorphic dual of $\mathscr{T}X$ we have a canonical $\mathcal{M}$-linear map

$$\Omega^1_{\mathscr{M}} \otimes_{\mathcal{M}} \mathscr{T}X \longrightarrow \mathscr{M}.$$

In particular, if we fix a $v \in \mathscr{T}X(X)$ we have an evaluation map $\Omega^1_{\mathscr{M}} \to \mathscr{M}$ given by $\omega \mapsto \omega(v)$. The composition of ∇ followed by this evaluation map on the $\Omega^1_{\mathscr{M}}$-factor will be denoted by ∇_v. We then have

$$\nabla_v\sigma = v(f)\sigma + f\nabla_v\sigma.$$

REMARK C.3.3. Because $\mathscr{T}X(X)$ is of dimension 1 over k, $\nabla\sigma = 0$ if and only if $\nabla_v\sigma = 0$ for some non-zero $v \in \mathscr{T}X(X)$.

REMARK C.3.4. Because every meromorphic vector bundle is trivial, a connection ∇ on Π is completely determined by its action on the global sections. So to obtain a connection it suffices to define a map

$$\mathscr{E}(X) \longrightarrow \mathscr{E}(X) \otimes_k \Omega^1_{\mathscr{M}}(X)$$

satisfying the Leibnitz rule. Thus if we have a basis of global sections $\sigma_1, \ldots, \sigma_n$ in the n-dimensional k-vector space $\mathscr{E}(X)$, the connection ∇ is uniquely determined by the global sections $\nabla\sigma_1, \ldots, \nabla\sigma_n$ in $\mathscr{E}(X) \otimes_k \Omega^1_{\mathscr{M}}(X)$.

Definition C.3.5. $\mathscr{E}(X)$ is an n-dimensional k-vector space. We say that ∇ is *decomposable* if there are two non-zero proper subspaces $V, W \subset \mathscr{E}(X)$ such that $\mathscr{E}(X) = V \oplus W$ and V and W are respectively mapped to $V \otimes_k \Omega^1_{\mathscr{M}}(X)$ and to $W \otimes_k \Omega^1_{\mathscr{M}}(X)$ via ∇. If no such subspaces V and W exist we say that ∇ is indecomposable.

C.4 Pulling back bundles and connections

We recall the following standard result.

Proposition C.4.1. *Let $\pi : \widetilde{X} \to X$ be a holomorphic morphism of compact Riemann surfaces. Then π is either constant or surjective.*

Proof: Non-constant holomorphic maps $U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$ are open. Thus if π is not constant, then π is an open map so $\pi(\widetilde{X}) \subseteq X$ is open. On the other hand, since $\widetilde{X}$ is compact and X is Hausdorff, $\pi(\widetilde{X}) \subseteq X$ is closed. Because X is connected $\pi(\widetilde{X}) = X$. ★

We consider a (ramified) covering $\pi : \widetilde{X} \to X$ of X by another compact Riemann surface $\widetilde{X}$. Algebraically, this corresponds to an algebraic extension $\widetilde{k}$, the field of meromorphic functions over $\widetilde{X}$, of k. Indeed k can be mapped into $\widetilde{k}$ by a pulling back the meromorphic functions over X to $\widetilde{X}$ via π:

$$\begin{array}{rcl} k & \longrightarrow & \widetilde{k} \\ f & \longmapsto & f \circ \pi. \end{array}$$

More generally, we have an injective map for every open $U \subseteq X$, i.e.

$$\begin{array}{rcl} \mathcal{M}(U) & \longrightarrow & \mathcal{M}_{\widetilde{X}}(\pi^{-1}U) \\ f & \longmapsto & f \circ \pi, \end{array}$$

where $\mathcal{M}_{\widetilde{X}}(V)$ denotes the field meromorphic functions over an open set $V \subseteq \widetilde{X}$. In this way any element of $GL_n(\mathcal{M}(U))$ can be seen as an element of $GL_n(\mathcal{M}(\pi^{-1}U))$. In analogy with the holomorphic case, an element $g \in GL_n(\mathcal{M}(U))$ correponds to a meromorphic map $g : U \to GL_n(\mathbb{C})$ which we can pull back to $\widetilde{X}$ via π.

Definition C.4.2. The *pullback of* Π *through* π is the meromorphic vector bundle over $\widetilde{X}$ defined by the cocycle given by the pullback $g_{\beta\alpha} \circ \pi$ of the transition maps $g_{\beta\alpha}$ to $GL_n(\mathcal{M}(\pi^{-1}(U_\alpha \cap U_\beta)))$.

REMARK C.4.3. Topologically, the total space of the pullback bundle is obtained by patching the complex manifolds $\pi^{-1}U_\alpha \times \mathbb{C}^n$, $\alpha \in A$, via the transition maps $g_{\beta\alpha} \circ \pi$. (To be precise: we patch and use the transition map of the refinement giving the induced holomorphic structure on Π.) The pullback of $\Pi : E \to X$ via π is denoted by $\pi^*\Pi : \pi^*E \to \widetilde{X}$. On the level of the complex manifolds E and π^*E we have a commuting diagram of holomorphic morphism of complex manifolds

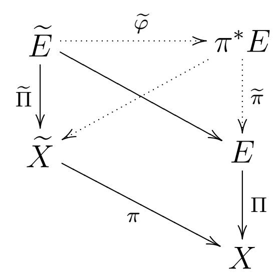
$$\begin{array}{ccc} \pi^*E & \xrightarrow{\widetilde{\pi}} & E \\ {\scriptstyle \pi^*\Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \widetilde{X} & \xrightarrow[\pi]{} & X \end{array}$$

$(\Pi \circ \widetilde{\pi} = \pi \circ \pi^*\Pi)$ which is locally given by:

$$\begin{aligned} \pi^{-1}U_\alpha \times \mathbb{C}^n &\longrightarrow U_\alpha \times \mathbb{C}^n \\ (p, x) &\longmapsto (\pi(p), x). \end{aligned}$$

REMARK C.4.4. Since a meromorphic section σ of Π over a trivializing open set U_α is given by a collection $f^1, \dots, f^n \in \mathscr{M}(U_\alpha)$ by pulling back these functions via π to $\pi^*f^1, \dots, \pi^*f^n \in \mathscr{M}_{\widetilde{X}}(\pi^{-1}U_\alpha)$, we see that the section σ lifts to a meromorphic section $\pi^*\sigma$ of $\pi^*\Pi$ over $\pi^{-1}(U_\alpha)$. In particular, global meromophic sections lift to global meromorphic sections. The sheaf of meromorphic section of $\pi^*\Pi : \pi^*E \to \widetilde{X}$ will be denoted $\pi^*\mathscr{E}$.

Definition C.4.5. Given two meromorphic vector bundles $\Pi : E \to X$ and $\widetilde{\Pi} : \widetilde{E} \to \widetilde{X}$, a *morphism* of $\widetilde{\Pi}$ to Π is a pair $(\pi, \widetilde{\varphi})$, where $\pi : \widetilde{X} \to X$ is a holomorphic morphism of Riemann surfaces and $\widetilde{\varphi} : \widetilde{E} \to \pi^*E$ is a morphism of meromorphic vector bundles making the following diagram commute



REMARK C.4.6. Consider a morphism of compact Riemann surfaces

$$\pi : \widetilde{X} \to X.$$

Because the elements of $\mathscr{T}X(X)$ correspond to derivations of $\mathcal{M}(X)$, the pullback via π of the tangent bundle of X lifts to a subbundle of the tangent bundle of $\widetilde{X}$. Similarly, the bundle of differential forms over X is pulled-back to a subbundle of the bundle of differential forms over $\widetilde{X}$.

REMARK C.4.7. A basis of global sections $\sigma_1, \dots, \sigma_n$ in the n-dimensional k-vector space $\mathscr{E}(X)$, lifts via π to a basis of global sections $\pi^*\sigma_1, \dots, \pi^*\sigma_n$ in the n-dimensional $\widetilde{k}$-vector space $\pi^*\mathscr{E}(\widetilde{X})$. Similarly, the global sections $\nabla\sigma_1, \dots, \nabla\sigma_n$ in $\mathscr{E}(X) \otimes_k \Omega^1_{\mathscr{M}}(X)$ lift to global sections $\pi^*\nabla\sigma_1, \dots, \pi^*\nabla\sigma_n$ in $\pi^*\mathscr{E}(\widetilde{X}) \otimes_{\widetilde{k}} \Omega^1_{\mathscr{M}}(\widetilde{X})$.

Definition C.4.8. Let ∇ be a meromorphic connection on $\Pi : E \to X$ and let $\pi : \widetilde{X} \to X$ be a morphism of compact Riemann surfaces. The *pullback connection* $\pi^*\nabla$ on $\pi^*\Pi$ is the connection such that if $\sigma_1, \dots, \sigma_n$ is a basis of $\mathscr{E}(X)$ then

$$\pi^*\nabla(\pi^*\sigma_i) = \pi^*(\nabla\sigma_i), \quad i \in \{1, \dots, n\}$$

REMARK C.4.9. A more geometric, but equivalent, definition of pullback connection is given in Chapter 2.

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