Ho - Lee Model With Jump - Diffusion Process and Bond Markets

By

Wei Wang

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1/28/97
Date

S. Neftci (TDA)
Chair of Examining Committee

1/28/97
Date

Executive Officer

Professor Salih Neftci

Professor Thom Thurston

Professor Terence D. Agbeyegbe

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Wei Wang

Adviser: Professor Salih Neftci

This paper derives an arbitrage-free interest rate movements with jump diffusion process model (ARJ model) which is a general case of arbitrage-free Ho-Lee model (1986). After adding a jump term into, and adjusting the drift term of Ho-Lee model, the new model is still an arbitrage-free one. But it covers more information and characterizes much better interest rate movement than the Ho-Lee model does. It has features to fit the interest rate movements in addition to having all the properties of Ho-Lee model, which is of great significance in analyzing the bond markets. I further show that the ARJ model can be used to price interest rate contingent claims.

Contents

I. Introduction

Interest rate contingent claims are everywhere. Treasuries, corporate bonds, mortage-backed securities and collateralized mortgage obligations are but a few examples of them. The interest rate contingent claims price depends on the level and volatility of interest rates in the economy. Interest rate models simulate yield curve movements. With the dramatic growth in the interest rate contingent claims markets and the increase in interest rate risk, especially in recent decades, the knowledge of the whole interest rate term structure becomes necessary for the pricing of contingent claims. Usually, interest rate movements, just like the stock price movements, are assumed to have the property of normal stochastic distribution. A conclusion of this paper, however, is that the assumption is not sustained.

The value of these interest rate contingent claims is in large part determined by the yield curve and its stochastic movements. Once the interest rate behavior can be properly described, we will be able to predict more precisely the activities of interest rate contingent claims. Numerous approaches have been introduced over the last twenty years, many of them assume that interest rate movements are of the property of normal or lognormal distribution. These models can be classified into three categories: (1) One and two factor equilibrium models, such as the Vasicek model (1977), Cox, Ingersoll and Ross (1985), and Brennan & Schwartz (1979). (2) Arbitrage-free models, which are designed to be consistent with the term structure of a specific day, such as Ho and Lee (1986), Hull and White (1990), Black, Derman and Toy (1990) and Heath, Jarrow and Morton (1992). (3) State price kernel approaches, such as Turnbull and Milne (1991)

If the interest rate movements follow the pattern of a normal stochastic distribution, as these models assumed, they could be used satisfactorily to price the interest rate contingent claims. If it does not, the application of these models will result in the errors on the prices of contingent claims. Sanjiv Ranjan Das (1994) used a huge data set to show that interest rate movements, especially its short rate, is not normally or lognormally distributed. He showed that the skewness and kurtosis of the change in interest rate do not match those of the features of the normal distribution. Skewness is the function that characterizes the degree of symmetry of the distribution around its mean, where positive skewness indicates asymmetric tail extending toward positive values and negative skewness asymmetry toward negative values. Kurtosis is the function that characterizes relative peakedness or flatness of distribution compared to normal distribution where positive kurtosis indicates peak distribution and negative kurtosis flat distribution. Based on daily data of the U.S. Treasury bill of 3-month rate (which I collected from the Banker Trust, Co.), the kurtosis is above 27. (See table - 1). This can also be shown from the three graphs that I drew using the data. They are graphs respectively of the 3-month rate daily data, the 3-month rate weekly data, and the 3-month rate monthly data. The outlines of the graph of daily data are higher than that in the graph of the monthly data. This shows that the instantaneous shock of the movements of interest rate will become more obvious when the frequency is increased, which leads to the conclusion that the interest rate movements do not follow exactly the pattern of normal or lognormal distribution. This paper tries to prove that there is a better way to characterize the behavior of the interest rate movements, which assumes that interest rate movements follow a normal stochastic distribution with a jump term which is the result of multiply in the normal distribution with the Poisson process. The new assumption can better explain the phenomena that the instantaneous shock of the interest rate movements becomes more obvious when the frequency is increased.

Das has shown that when a jump term is added to the Vasicek model, the new model can better characterize the interest rate behavior than the Vasicek does. However, the model Das has developed is based on that of Vasicek, which is not an arbitrage free model. Arbitrage is the process of profiting from differences in price when the same security, currency, or commodity is traded on two or more markets. Arbitrage free or no-arbitrage is the reverse of arbitrage which means that there is no opportunities to get riskless profits. In an arbitrage-free interest rate model, the expectation of the discounted function of interest rate term structure movements should be consistent with the current market condition, which can be presented as,

$$E[\exp(\int_0^T r(t)\, dt)] = P(0, T)$$

P(0, T) is a discount function of zero-coupon bond price at present time with maturity T. E is the expectation under certain probability. r(t) is interest rate at time t.

Ho and Lee have developed a model which is known as the arbitrage-free model(AR model). By using this model, the pricing of contingent claims become much more reliable. However, like the Vasicek model, it is still based on the assumption that interest rate movements follow the pattern of normal stochastic distribution.

This paper develops a model base on the assumption that interest rate movements follow the pattern of normal stochastic distribution with a jump diffusion process which is determined by incoming information. The change of normal stochastic distribution is due to variation of capitalization rates, a temporary imbalance between supply and demand, or any other information which causes the fluctuation of the marginal interest rate. The change of the jump diffusion process which is usually modeled as a Poisson process, is due to any information which causes more than marginal change in interest rate movements. For example, see table 5. The 6-month bill jump 6 base point to 5.19 in April 8, 1996. One base point equal to one percent. This jump is because that the Laber Department announce a news that compnies added a hefty number of new jobs in March. (See New York Time, Business Day April 9, 1996 and April 10, 1996.)

The contribution of this paper is the combination of the arbitrage-free rate model with jump diffusion. The result of the new model is still an arbitrage-free one which means by using this model the prices of simple bonds will be consist with the market prices.

An outline of this paper is as follow: Section II reviews the arbitrage free models and notations. Section III presents the jump diffusion processes. Section IV presents the result of ARJ model from theoretically to empirically. Section V provides examples. Section VI summarizes the paper.

II. Arbitrage-Free Interest Rate Models

Since more than a decade ago, due to their sophistication in theory and reliability in practice, arbitrage-free interest rate models are widely used in financial markets. The basic idea of these interest rate models comes from the popular Black-Schole (1973) option model which is used to price options on an underlying stock or any risky asset. By applying Ito's laemma, Black and Schole obtained a closed form solution for the pricing of options, with the assumption that the underlying assets follow normal stochastic distribution. Merton (1974) first proposed the modified Black-Scholes model for bond option valuation. The same approach can be used to characterize the interest rate movements. Once we know the interest rate movements, we will be able to price bonds, bond options and other interest rate contingent claims. Therefore, to characterize the interest rate movements is the key for the pricing. As a result, many interest rate term structure models have emerged. The Vasicek model is one of them. Since it does not use today's term structure as an input, it is not guaranteed to be arbitrage free. On the contrary, Ho-Lee(1986) model, as the first in its kind, is designed to be consistent with today's term structures. In other words, it assumes that the interest rate movements fit today's term structure to ensure that they are consistent with an equilibrium framework. The assumption is made so that the movements permit no arbitrage profit opportunities, which is called as *arbitrage-free rate movements* (AR).

A. Ho-Lee Model: A Discrete Time Approach

The original Ho-Lee model, in which the discount price function was characterized for the first time, is in discrete time. Assuming that the discount function of bond price is $P_{i,n}(T)$, which means that at time n and state i, the bond present value of $1 with maturity T is $P_{i,n}(T)$. $P_{i,n}(.)$ must satisfy the following conditions:

1) it must be positive because it is the present value of the asset.

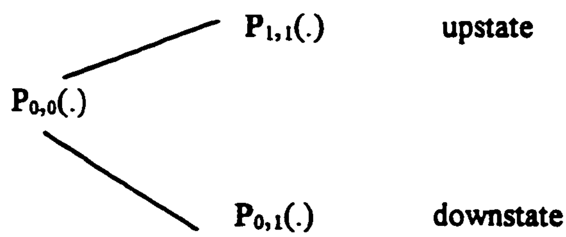
2) $P_{i,n}(0) = 1$ for all i, n

3) $\lim_{T\to\infty} P_{i,n}(T) = 0$ for all i,n

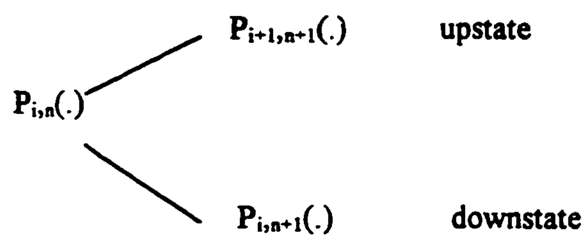
$P_{i,n}(0) = 1$ means that for a discount bond with today as its maturity date, its value worth $1.

$\lim_{T\to\infty} P_{i,n}(T) = 0$ means that a discount bond with a quite long maturity, its present value can be ignored.

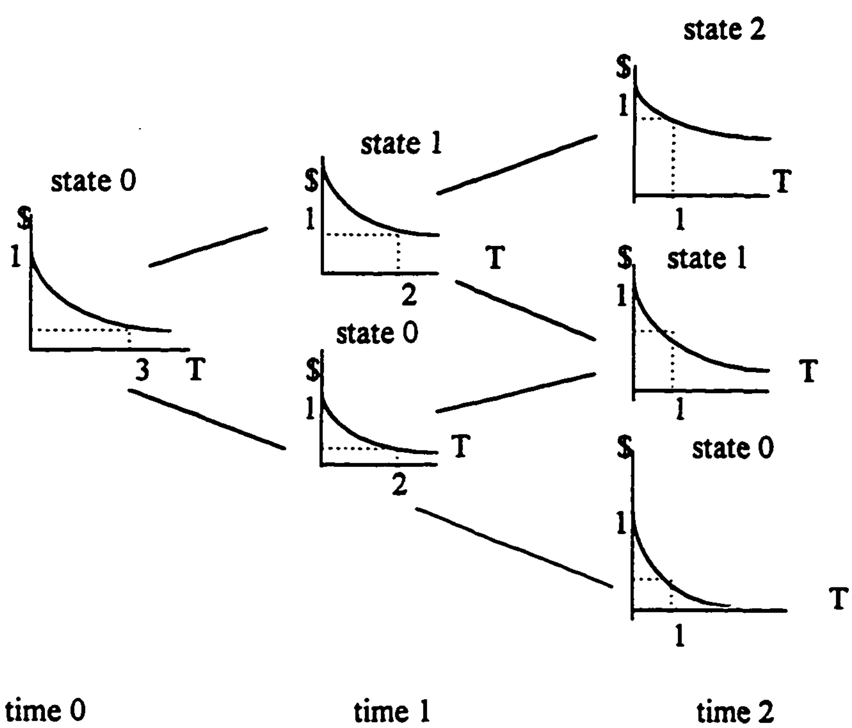
In the Ho-Lee model, $P_{0,0}(T)$ is given as a current discount function. T is the maturity (T = 1, 2,...). For example, assuming $P_{0,0}(3)$ =$ 0.90, the maturity is 3 months, the present value of $1 is 90 cents. At the initial time 0, we have 0 state. Therefore, the discount function is $P_{0,0}(T)$. At time 1, the discount function may have two possible movements--up or down. When moving up, the discount function will be $P_{1,1}(.)$. When moving down, the discount function will be $P_{0,1}(.)$. At time 2, the discount function may have three possible movements -----up, steady and down. The discount function are respectively $P_{2,2}(.)$, $P_{1,2}(.)$ and $P_{0,2}(.)$ and so on. At time n, after i upstate movements and (n-i) downstate movements, the discount function is $P_{i,n}(.)$. Thus, the binomial lattice at time 0 are:



the binomial lattice at time n are:



The three periods of the binomial lattice are:



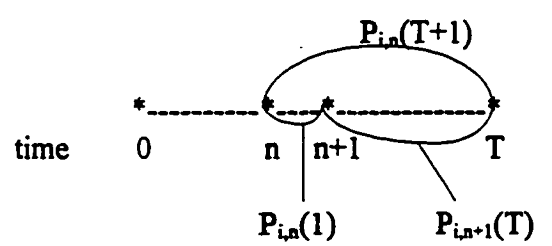
This figure is based on the paper of Ho &Lee (1986).

For each period, the discount function of bond price has a downward slope. This is because the present value of bond is lower as its maturity is longer. In the Ho-Lee model, in each state, we know the whole term structure but not the particular point of interest rate. Assuming at time 0, the bond life is 3 months. After one period, which is at time 1, the bond life is 2 months. The bond price is closer to $1 than that in time 0. Similarly, at time 2, the life of the bond is 1 month. Its value closer to $1 than that in time 1 and so on.

If there is no interest rate risk over the next period, then,

$$P_{i+1,n+1}(T) = P_{i,n+1}(T) = \frac{P_{i,n}(T+1)}{P_{i,n}(1)} \qquad (2.1)$$

In equation (2.1), $P_{i,n}(T+1) = P_{i,n+1}(T) * P_{i,n}(1)$, is that a T+1 period bond at time n can be decomposed into two discount bonds: one is T period bond at time n+1; the other is one period bond at time n. See the following figure



Now, assume there are interest rate risks over next period, and assume that h(T) and h*(T) are the perturbation function if it is up and down respectively.

$$h(T) = 1/\{\pi+(1-\pi)\delta^T\} \qquad (2.2)$$

$$h^*(T) = \delta^T/\{\pi+(1-\pi)\delta^T\} \qquad (2.3)$$

In equation (2.2) and (2.3), if T = 0, then

$$h(0) = h^*(0) = 1$$

π is the probability that the price will move up in the next period.

Therefore, the discount function of upward is

$$P_{i+1,n+1}(T) = \frac{P_{i,n}(T+1)h(T)}{P_{i,n}(1)} \qquad (2.4)$$

and the discount function of downward is

$$P_{i,n+1}(T) = \frac{P_{i,n}(T+1)h^*(T)}{P_{i,n}(1)} \qquad (2.5)$$

Assuming that $P_{i+1,n+1}(T) = P_{i,n+1}(T)$ if there is no risk and the equation equals to the implied forward discount function $F_{i,n}(T)$ which is defined that a discount price as seen at time n from next period, or n+1, to T. That is $P_{i+1,n+1}(T) = P_{i,n+1}(T) = F_{i,n}(T)$. The goal of this assumption is to make sure that there are no arbitrage opportunities. In a risk neuatral world, if the next period discount function not equal to the implied forward discount, investors can obtain arbitrage profits.

In a risky world, equation (2.4) and (2.5) are assumed.

Using equation (2.4), (2.5) and (2.1) continuously, we have

$$P_{i,n}(T) = \frac{P_{i,n-1}(T+1)h^*(T)}{P_{i,n-1}(1)} = \frac{P_{i,n-2}(T+2)h^*(T)\,h^*(T+1)}{P_{i,n-1}(1)\,P_{i,n-2}(1)} = \ldots.$$

the closed form solution of Ho-Lee model is:

$$P_{i,n}(T) = [P_{0,0}(T+n\,)/\,P_{0,0}(n)].\ \delta^{T(n-i)} \prod_{j=1}^{n} (h(T+n-j)/h(n-j)) \qquad (2.6)$$

$P_{0,0}(T+n)$ and $P_{0,0}(n)$ are the present discount function with maturity life T+n, and n respectively, which can be observed from present markets. So, the present value of a bond price having $1 value at maturity T is determined by two parameters δ and π. Once we know the value of them, the value of the bond is known. Since T can equal to 1, 2, 3, ...,360 months. The whole term structure is therefore, known.

The relationship between yield curve and discount price is

$$r_{i,n}(T) = - \ln P_{i,n}(T)/T.$$

So, from equation (2.6) we can get an interest rate function:

$$r_{i,n}(T) = - (\ln(P_{0,0}(T+n)) - \ln(P_{0,0}(n)))/T - \sum_{j=1}^{n} (\ln(h(T+n-j))-\ln(h(n-j)))/T - (n-i)\ln(\delta) \quad (2.7)$$

$r_{i,n}(T)$ is the interest rate at time n and state i,with maturity T.

For the best reference for this section, see [Ho-Lee 1986].

B. Ho-Lee Model: A Continuous-Time Form Approach

From the previous section, we see that continuous-time form of the Ho-Lee model should catch the two major features of Ho-Lee model: (1) it is an arbitrage-free rate model; (2) the change of the interest rate follows the normal stochastic process.

Let $r(t)$ be the term structure of instantaneous interest rate. i.e. short rate. Assume that during the short time period of dt, the change of instantaneous interest rate can be described as dr, and $F(t, T)$ be the instantaneous forward rate as seen at time t for a contract maturing at time T. Actually, we have $F(0,t) = -\partial \ln P(0,t)/\partial t$, and P(0, t) is the present discount zero-coupon bond price with maturity t. The following stochastic equation was proposed for the continuous-time form of the instantaneous interest rate change in Ho-Lee model by several authors, e.g. John Hull[H].

$$dr = \theta(t)\, dt + \sigma dz \tag{2.8}$$

where $\theta(t)$ is the drift or instantaneous interest rate return which is a function of t. It has been shown that $\theta(t) = F_t(0,t) + \sigma^2 t$ where $F_t(0,t)$ is the time derivative of the instantaneous forward rate. σ in σdz is the instantaneous standard deviation of the short rate and is constant (Ho and Lee suggest that the model can be extended to allow it to be time-dependent). And dz, is standard Brownian Motion. In other words, $\Delta z = \varepsilon\sqrt{\Delta t}$. ε is a random draw from standard normal distribution. According to what $\varepsilon \sim N(0,1)$.

We will further prove that the drift term $\theta(t) = F_t(0,t) + \sigma^2 t$ was chosen so that the model is arbitrage free. In other words, it ensures that the model fits the initial term structure.

Proposition 1. Assume that the interest rate movements follow the pattern of stochastic normal distribution, the change of interest rate has the form:

$$dr = \theta(t)\, dt + \sigma\, dz$$

where $\theta(t) = F_t(0,t) + \sigma^2 t$, and the function $F_t(0,t)$ is the time derivative of the instantaneous forward rate as seen at time 0 and contract maturing t. Then the interest rate model is arbitrage free.

Remark : Although Proposition 1 is well known. See John Hull[H], to make this paper self contained, and simplify the proof of Proposition 2 in the next section, a detailed proof of the proposition is given in the following.

Proof: If it is an arbitrage-free rate model, it must satisfy:

$$E\left[\exp\left(-\int_0^T r(t)\,dt\right)\right] = P(0,T)^{*}$$

or

$$\lim_{n \to \infty} E\left[\exp\left(-\sum_{i=0}^{n-1} r_i \Delta t\right)\right] = P(0,T)$$

where $P(0, T)$ is today's discount curve. This equation exhibits that the expectation of the future's discount zero-coupon bond price should equal to today's one. This condition ensures that there is no arbitrage opportunity.

let $\Delta t = T/n$. Assume that there are n periods from time 0 to maturity date T.

$0 = t_0 < t_1 < t_2 < \ldots < t_n = T$, r_i is the interest rate on the i-th period.

```
0                                                    T
*--------*--------*--------*..........*--------*
r0       r1                          rn-1
```

$$E\left[\exp\left(-\sum_{i=0}^{n-1} r_i \Delta t\right)\right]$$

$$= E\left[\exp\left(-\left(r_0 + (r_0 + \Delta r_0) + (r_0 + \Delta r_0 + \Delta r_1) + \ldots + (r_0 + \Delta r_0 + \ldots + \Delta r_{n-2})\right)\Delta t\right)\right]$$

$$= E\left[\exp\left(-\left((n-1)\, r_0 + (n-2)\,\Delta r_0 + (n-3)\,\Delta r_1 + \ldots + \Delta r_{n-2}\right)\Delta t\right)\right]$$

$$= E\left[\exp\left(-\left((n-1)\, r_0 + (n-2)(\theta(t_0)\,\Delta t + \sigma\Delta z_0) + (n-3)\,(\theta(t_1)\,\Delta t + \sigma\Delta z_1) + \ldots\right.\right.\right.$$

$$\left.\left.\left. + (\theta(t_{n-3})\,\Delta t + \sigma\Delta z_{n-3})\right)\Delta t\right)\right]$$

$$= E\left[\exp\left(-\left((n-1)\, r_0 + (n-2)\,(\theta(t_0)\,\Delta t) + (n-3)\,(\theta(t_1)\,\Delta t) + \ldots + (1)\,(\theta(t_{n-3})\,\Delta t)\right.\right.\right.$$

$$\left.\left.\left. + (n-2)\,(\sigma\Delta z_0) + (n-3)\,(\sigma\Delta z_1) + \ldots + (\sigma\Delta z_{n-3})\right)\Delta t\right)\right]$$

---

* $P(0, T) = \$1/(1+r_1)(1+r_2)\ldots(1+r_T) = \$1 * \exp(r_1\Delta t) * \exp(r_2\Delta t) \ldots \exp(r_T\Delta t) = \$1 * \exp(r_1 + r_2 + \ldots + r_T)\Delta t$

$$= E\left[\exp\left(-\left((n-1) r_0 \Delta t + \sum_{i=0}^{n-3} ((n-2-i)\, \theta(t_i)\, \Delta t\, \Delta t) + \sum_{i=0}^{n-3} ((n-2-i)\, \sigma\, \Delta t\, \Delta z_i)\right)\right]$$

$$= E\left[\exp\left(-\left((n-1) r_0 \Delta t + \sum_{i=0}^{n-3} ((n-2-i)\, \theta(t_i)\, \Delta t\, \Delta t)\right)\right)\right] * E\left[\exp\left(-\sum_{i=0}^{n-3} (n-2-i)\, \sigma\, \Delta t\, \Delta z_i\right)\right]$$

We will investigate the two expectations in the case $n \to \infty$. It's obvious that the first expectation is corresponding to the drift term $\theta(t)\,dt$. In the case, the deterministic interest rate function is denoted by $R(t)$. $dR(t) = \theta(t)\,dt$.

we have

$R(t) = \int \theta(t)\,dt = \int (F_t(0, t) + \sigma^2 t)\,dt = F(0, t) + \sigma^2 (t^2/2) + C$,

where C is a constant. On the other hand, we have the initial condition $R(0) = F(0, 0)$.

Therefore, $C = 0$, i.e. $R(t) = F(0, t) + \sigma^2 (t^2/2)$.

$$\lim_{n\to\infty} E\left[\exp\left(-\left((n-1) r_0 \Delta t + \sum_{i=0}^{n-3} ((n-2-i)\, \theta(t_i)\, \Delta t\, \Delta t)\right)\right)\right]$$

$$= \exp\left(-\int_0^T R(t)\,dt\right)$$

$$= \exp\left(-\int_0^T (F(0, t) + \sigma^2 (t^2/2))\,dt\right)$$

$$= \exp\left(-\int_0^T (F(0, t)\,dt\right) * \exp\left(-\int_0^T \sigma^2 (t^2/2)\,dt\right)$$

$$= P(0, T) * \exp(-\sigma^2 T^3/6)$$

where

$$\int_0^T F(0,t)\,dt = -\ln P(0,t)\Big|_0^T = -(\ln P(0,T) - \ln P(0,0)) = -\ln P(0,T)$$

so,

$$\exp\left(-\int_0^T F(0,t)\,dt\right) = P(0,T)^{*}$$

For the second expectation in which Δz_i is involved, the assumptions are as following:

$\Delta z_i = \varepsilon\sqrt{\Delta t}$. ε is a normally distributed random variable with zero mean and unit variance.

Formally: $\varepsilon \sim N(0,1)$.

Also Δz_i and Δz_j are independent with each other when $i \neq j$.

Let $Y = -\sum_{i=0}^{n-3} ((n-2-i)\,\sigma\,\Delta t\,\Delta z_i)$,

then Y is a variable which follows the normal distribution with 0 as its mean, and its variance is calculated as following: **

$$s^2 = \sum_{i=0}^{n-3} (n-2-i)^2 \sigma^2 \Delta t^3$$
$$= \sigma^2 \Delta t^3 (1^2 + \ldots + (n-3)^2 + (n-2)^2)$$
$$= \sigma^2 \Delta t^3 (n-2)(n-1)(2(n-2)+1)/6$$

In other words, $Y \sim N(0, s^2)$. Hence, E [exp(Y)] is the moment generating function with $\mu = 0$, $\sigma = s$ and $t = 1$, (see [Ross] Page 306). We have $E[\exp(Y)] = \exp((1/2)\,s^2)$.

So, the second expectation term

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* See John Hull " Options, Futures, and Other Derivative Securities" Second Edition. PP399

** See Sheldon Ross " A First Course in Probability", Third Edition, 1988. PP 217

$$\lim_{n \to \infty} E\left[\exp\left(- \sum_{i=0}^{n-3} (n-2-i)\, \sigma\, \Delta t\, \Delta z_i \right)\right]$$

$$= \lim_{n\to \infty} E[\exp(Y)]$$

$$= \lim_{n \to \infty} \exp((1/2)\, s^2)$$

$$= \lim_{n \to \infty} \exp((1/2)\, \sigma^2 \Delta t^3 (n-2)(n-1)(2(n-2)+1)/6)$$

substitute Δt with T/n, and rewrite the term $(n-2)(n-1)(2(n-2)+1)$ as $2n^3 + O(n^2)$,

the above equation will be

$$\lim_{n \to \infty} \exp((1/12)\, \sigma^2 (T/n)^3 (2n^3 + O(n^2)))$$

$$= \lim_{n \to \infty} \exp[(\sigma^2 T^3/6)(1 + O(1/n))]$$

$$= \exp(\sigma^2 T^3/6)$$

Now, we have the following equation:

$$E\left[\exp\left(- \int_0^T r(t)\, dt\right)\right]$$

$$= \lim_{n \to \infty} E\left[\exp\left(- \sum_{i=0}^{n-1} r_i \Delta t\right)\right]$$

$$= E\left[\exp\left(-\left((n-1) r_0 \Delta t + \sum_{i=0}^{n-3} ((n-2-i)\, \theta(t_i)\, \Delta t\, \Delta t)\right)\right)\right] * E\left[\exp\left(- \sum_{i=0}^{n-3} (n-2-i)\, \sigma\, \Delta t\, \Delta z_i\right)\right]$$

$= P(0, T) * \exp(-\sigma^2 T^3 /6) * \exp(\sigma^2 T^3 /6)$

$= P(0, T)$

The proof is completed. #

According to Ho-Lee model, the discount zero-coupon bond price at time t with maturity T in terms of the short rate is

$$P(t,T) = A(t,T)\exp(-r(T-t))$$

where

$$\ln A(t,T) = \ln P(0,T) - \ln P(0,t) - (T-t)(\partial \ln P(0, t)/ \partial t) - 1/2\, \sigma^2 t(T-t)^2$$ *

An other advantage of Ho-Lee model is that it enables us to utilize the full information of the term structure to price contingent claims with the possibility to reach closed form solutions for some path-dependent options. See recent results by Wang (1995).

* See John Hull " Options, Futures, and Other derivative Securities", Second Edition.

III. Jump-Diffusion Processes

Many term structure models assume that the short rate follows the pattern of normal stochastic process. However, interest rate jump behavior has been observed in the economy of the bond markets. In financial markets, there are generally two kinds of information, one is kind of endogenous, the other is kind of exogenous. All endogenous effects are like diffusion process which have been approached in the normally distributive models. Like those mentioned in section II. All exogenous effects are like jump events. This kind of information can be the intervention by Federal Reserve; supply and demand shocks; and the economic news. For the purpose to catch the jump feature, some of this kind models have been developed. For example, Ball and Torous (1983), Beckers (1981), and Oldfield, Rogalski and Jarrow (1977) developed models which cover the jump behavior on stock price movements. Das has shown (1994) that the instantaneous interest rate movements have jump events. This phenomenon also can be observed from the historical interest rate data. Empirically, according to the historical data, it can be found that the skewness of the change of interest rate was far from zero and kurtosis was far from 3, while a normal distribution variable has its skewness equal to zero and its kurtosis 3. By figuring out the skewness and kurtosis of the data, we can tell whether the data is of the nature of symmetry or flat tail. I used the US treasury 91 day bill daily data to calculate the skewness and kurtosis. See the table - 1. In the table the skewness of the change of interest rate is -1.062 and kurtosis is 27.932 which show that the short rate did not follow the pattern of normal distribution. Also, from the table, we can see that the kurtosis will increase when the frequency of data reading increased. To analyze the interest rate movements properly, its movements can be described by two vibrations. One is the normal stochastic distribution which is modeled by a standard geometric Brownian motion. The other is modeled by jump process reflecting the non-marginal impact of the information. The jump diffusion process is usually described by a Poisson process which assumes that the occurring of the events is independent with each other and identical. The process has stationary and independent increments. During the time Δt and with rate h, $h > 0$, the probability of an event occurring is:

Prob{the event does not occur in the time interval $(t, t+\Delta t)$} = $1 - h\Delta t + o(\Delta t)$,

Prob{the event occurs once in the time interval $(t, t+\Delta t)$}= $h\Delta t + o(\Delta t)$,

Prob{the event occurs more than once in the time interval $(t, t+\Delta t)$}= $o(\Delta t)$[*] ,

h is the mean number of event occurring per unit time. $o(\Delta t)$ is the asymptotic order symbol. Event occurring means the coming of an 'abnormal' information about the short rate.

The number of events in interval of length Δt is Poisson distributed with mean $h\Delta t$. That is, for all $t \geq 0$,

$$\text{Prob}\{x(t + \Delta t) - x(t) = i\} = \exp(-h\Delta t)*(h\Delta t)^i / (i!)$$

i is the number of events happen (i = 1,2,3,....). The mean and variance of the variable x(t) is $h\Delta t$.

Table 6 shows one example of observation of jump in interest rate in 1996. This jump event happened is because the Labor Department announced that the payrolls increased in April. This news leads to 30-year yield jump 5 base point which is that one base point is one percent.

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[*] For this part, see Sheldon M. Ross " Introduction to Probability Models", Fifth Editon, 1993, pp 211.

IV. Ho-Lee Model with Jump-Diffusion Process

This section shows in theory that the interest rate movement follows the pattern of the normal distribution with a jump diffusion process (ARJ), which is a generalization of Ho-Lee model. The ARJ model has the same assumptions as Ho-Lee model does except that it further assumes that the short rate follows the pattern of a normal distribution with a jump diffusion process. In ARJ, the stochastic variable dz and the Poisson process variable $d\pi$ are used to define respectively the normal distribution and the jump diffusion process.

The problem is how to retain the arbitrage - free feature of the Ho-Lee model when the jump diffusion process is added into it.

In the modern arbitrage theory, there are two ways to measure the interest rate changes. One is the applying PDE's implicit in arbitrage free portfolios. The other is the martingales. The later is done through transforming the underlying probability distributions. The two methods are connected in the sense that whenever that the movements of the bond price become martingale, the model is arbitrage free [Harrison & Krep]. A stochastic process X is a martingale if, for any times t and s > t, we have $E_t (X_s) = X_t$.

There are two ways to martingalize the interest rate movements. The first approach is to find a new probability measure which converts the bond prices into martingales [Neftci]. The other is to use the standard method for changing the mean of a random variable which simply adds a term to the random variable. We adopt the second method to ARJ model, which means adding a correction to the drift term.

A. Interest Rate Movements

In this section, I propose a model (ARJ) which is designed to be arbitrage free by adjusting the drift rate term in the model. The jump term is assumed to be a variable with Poisson process and has a normal distribution jump scale. The interest rate movements of the ARJ model are assumed to be:

$$dr = \omega(t)\, dt + \sigma\, dz + J(\mu,\gamma^2)\, d\pi(h),$$

where $\omega(t)$ is chosen to be $\omega(t) = F_t(0,t) + \sigma^2 t + G(t, h, \mu, \gamma)$ so that the model is arbitrage free. The function $G(t, h, \mu, \gamma) = -\mu h + (3/2)\gamma^2 h^2 t^2$ is the correction of the drift term caused by interest rate jump. More specifically,

$$dr = \omega(t)\, dt + \sigma\, dz + \begin{cases} J(\mu,\gamma^2) & \underline{h\Delta t} \quad \text{if the Poisson event occurs,} \\ 0 & \underline{1 - h\Delta t} \quad \text{if the Poisson event does not occur,} \end{cases}$$

This equation defines the probability $h\Delta t$ of a jump, in which:

σ^2 is the instantaneous diffusion variance of the interest rate;

$F_t(0,t)$ is derivative of instantaneous forward rate;

π is Poisson arrival process with parameter h.

J is the jump magnitude which follows the normal distribution with mean μ and variance γ^2.

J and π are independent with each other, z and π are independent with each other.

The ARJ is arbitrage free. The result is as following:

Proposition 2: The ARJ model is described as

$$dr = \omega(t)\, dt + \sigma\, dz + J(\mu,\gamma^2)\, d\pi(h)$$

where $\omega(t) = F_t(0,t) + \sigma^2 t - \mu h + (3/2)\gamma^2 h^2 t^2$. Then the model is arbitrage free.

Proof: If it is an arbitrage-free model, it must satisfy:

$$E\left[\exp\left(-\int_0^T r(t)\, dt\right)\right] = P(0,T)$$

or

$$\lim_{n \to \infty} E\left[\exp\left(-\sum_{i=0}^{n-1} r_i \Delta t\right)\right] = P(0,T)$$

where $P(0, T)$ is today's discount curve.

let $\Delta t = T/n$. Assume that there are n periods from time 0 to maturity date T.

$0 = t_0 < t_1 < t_2 < \ldots < t_n = T$, r_i is the interest rate after the $i\Delta t$ period.

```
0                                            T
*--------*--------*--------*.........*--------*
r0       r1                          rn-1
```

$$E\left[\exp\left(-\sum_{i=0}^{n-1} r_i \Delta t\right)\right]$$

$= E[\exp(-(r_0 + (r_0 + \Delta r_0) + (r_0 + \Delta r_0 + \Delta r_1) + \ldots + (r_0 + \Delta r_0 + \ldots + \Delta r_{n-2}))\Delta t)]$

$= E[\exp(-((n-1) r_0 + (n-2)\Delta r_0 + (n-3)\Delta r_1 + \ldots + \Delta r_{n-2})\Delta t)]$

$= E[\exp(-((n-1) r_0 + (n-2)(\omega(t_0)\Delta t + \sigma\Delta z_0 + J\Delta\pi_0) + (n-3)(\omega(t_1)\Delta t + \sigma\Delta z_1 + J\Delta\pi_1) +$

$$\ldots + (\omega(t_{n-3})\,\Delta t + \sigma\Delta z_{n-3} + J\Delta\pi_{n-3}))\,\Delta t\,)]$$

$$= E\,[\exp(-(\,(n-1)\,r_0 + (n-2)\,(\omega(t_0)\,\Delta t\,) + (n-3)\,(\omega(t_1)\,\Delta t\,) + \ldots + (1)\,(\omega(t_{n-3})\,\Delta t\,)$$

$$+ (n-2)\,(\,\sigma\Delta z_0\,) + (n-3)\,(\,\sigma\Delta z_1\,) + \ldots + (\,\sigma\Delta z_{n-3}\,)$$

$$+ (n-2)\,(J\Delta\pi_0) + (n-3)\,(J\Delta\pi_1) + \ldots + (\,J\Delta\pi_{n-3}\,)\,)\,\Delta t\,)]$$

$$= E\,[\exp(-((n-1)\,r_0\,\Delta t + \sum_{i=0}^{n-3}((n-2-i)\,\omega(t_i)\,\Delta t\,\Delta t) + \sum_{i=0}^{n-3}((n-2-i)\,\sigma\,\Delta t\,\Delta z_i) + \sum_{i=0}^{n-3}((n-2-i)\,\Delta t\,J\Delta\pi_i\,)\,)]$$

$$= E\,[\exp(-((n-1)\,r_0\,\Delta t + \sum_{i=0}^{n-3}((n-2-i)\,\omega(t_i)\,\Delta t\,\Delta t\,))] * E\,[\,\exp(-\sum_{i=0}^{n-3}(n-2-i)\,\sigma\,\Delta t\,\Delta z_i\,)]$$

$$* E\,[\,\exp(-\sum_{i=0}^{n-3}((n-2-i)\,\Delta t\,J\,\Delta\pi_i\,)\,)\,] \qquad (4.1)$$

The first two expectations have been calculated in proposition 1 except the correction term $G(t, \mu, \gamma, h)$. It's obvious that the first expectation is corresponding to the drift term $\omega(t)dt$. In the case, the deterministic interest rate function is denoted by $R(t)$. $dR(t) = \omega(t)dt$, $R(0) = F(0, 0)$. we have

$$R(t) = \int \omega(t)\,dt = F(0, t) + \sigma^2 (t^2/2) - \mu h t + (1/2) h^2 \gamma^2 t^3$$

By a similar argument, we have following equation:

$$\lim_{n\to\infty} E\,[\exp(-((n-1)\,r_0\,\Delta t + \sum_{i=0}^{n-3}((n-2-i)\,\omega(t_i)\,\Delta t\,\Delta t\,))]$$

$$= \exp\left(- \int_0^T R(t)\, dt\right)$$

$$= \exp\left(- \int_0^T (F(0, t) + \sigma^2 (t^2/2) - \mu h t + (1/2) h^2 \gamma^2 t^3) dt\right.$$

$$= \exp\left(- \int_0^T (F(0, t)\, dt\right) * \exp\left(- \int_0^T ((1/2) \sigma^2 t^2 - \mu h t + (1/2) h^2 \gamma^2 t^2) dt \right)$$

$$= P(0, T) * \exp(- \sigma^2 T^3/6 + (1/2) \mu h T^2 - (1/8) h^2 T^4 \gamma^2) \qquad (4.2)$$

Also, similar to Proposition 1, we have

$$\lim_{n \to \infty} E\left[\exp\left(- \sum_{i=0}^{n-3} (n-2-i) \sigma \Delta t \Delta z_i \right)\right] = \exp(\sigma^2 T^3/6) \qquad (4.3)$$

We now focus on the last expectation term:

$$\lim_{n \to \infty} E\left[\exp\left(- \sum_{i=0}^{n-3} ((n-2-i) \Delta t J \Delta \pi_i) \right) \right]$$

$$= \lim_{n \to \infty} E\left[\exp\left(- \sum_{i=1}^{n-2} ((n-2-i) \Delta t J \Delta \pi_i) \right) \right]$$

$$= \lim_{n \to \infty} E_J\left[\prod_{i=1}^{n-2} E(\exp(-i \Delta t J \Delta \pi_i)) \right]$$

$$= \lim_{n \to \infty} E_J [\prod_{i=1}^{n-2} (\exp (-i \Delta t J * 1) * h \Delta t + \exp (-i \Delta t J * 0) * (1 - h \Delta t))]$$

$$= \lim_{n \to \infty} E_J [\prod_{i=1}^{n-2} (1 + (\exp (-i \Delta t J) - 1) \; h \Delta t)]$$

$$= E_J [\lim_{n \to \infty} \prod_{i=1}^{n-2} (1 + (\exp (-i \Delta t J) - 1) h \Delta t)]$$

$$= E_J [\lim_{n \to \infty} \exp (\ln \prod_{i=1}^{n-2} (1 + (\exp (-i \Delta t J) - 1) h \Delta t))]$$

$$= E_J [\exp (\lim_{n \to \infty} \sum_{i=1}^{n-2} \ln (1 + (\exp (-i \Delta t J) - 1) \; h \Delta t))]$$

take Taylor expansion of $\ln (1 + (\exp (-i \Delta t J) - 1) \; h \Delta t)$

$$= E_J [\exp (\lim_{n \to \infty} \sum_{i=1}^{n-2} (-i h J \Delta t^2 + O (\Delta t^3)))]$$

$$= E_J [\exp (\lim_{n \to \infty} ((- h J \Delta t^2) (n-2)(n-3) / 2 + (n-2) * O (\Delta t^3)))]$$

$$= E_J [\exp (\lim_{n \to \infty} ((- h J T^2 / n^2)(n^2/2 + O(n)) + O((n-2)* (T/n)^3)))]$$

$$= E_J [\exp (-(1/2) T^2 h J)]$$

$$= \exp (-(1/2) T^2 h \mu + (1/2) (-T^2 h /2)^2 \gamma^2) \qquad (4.4)$$

the last equation is the calculation of the moment generating function of the normal distribution. See Ross page 306.

Combine the results (4.2), (4.3) and (4.4), go back to the main road (4.1),

$$\lim_{n \to \infty} E\left[\exp\left(-\sum_{i=0}^{n-1} r_i \Delta t\right)\right]$$

$$= P(0, T) * \exp(-\sigma^2 T^3/6 + (1/2)\mu h T^2 - (1/8) h^2 T^4 \gamma^2) * \exp(\sigma^2 T^3/6)$$

$$* \exp(-(1/2) T^2 h \mu + (1/2)(-T^2 h/2)^2 \gamma^2)$$

$$= P(0, T)$$

This show that the Ho-Lee model with jump diffusion is still an arbitrage - free model.

The proof of proposition 2 is completed.

B. Contingent Claim Valuation

In the remaining part of this section, I will give a close-form solution of zero-coupon bond price assuming the interest rate changes follow the pattern of normal distribution with a jump term. According to the extended version of Ito's Lemma (see Kushner & Dupuis (1992), Merton (1971)) the partial differential equation for $P(r, \tau)$ is:

$$dP = (\omega(\tau) P_r + P_\tau + (\sigma^2/2)P_{rr})\, d\tau + \sigma P_r\, dz + ([P(r+J, \tau) - P(r, \tau)] | J(\mu,\gamma^2))\, d\pi \qquad (4.5)$$

where the subscripts denote the relevant derivatives. $P(r, \tau)$ is the discount zero-coupon bond price. τ is time period which equal to T - t.

According to the CIR (1985a) Theorem 2, the expected return per unit time on the bond must satisfy:

$$E(dP/P) = r + \lambda\sigma(P_r/P) + \lambda_J \mathrm{Var}\{[P(r+J,\tau) - P(r,\tau)] | J(\mu,\gamma^2))d\pi\}/P \qquad (4.6)$$

In equation (4.6), λ is the market price per unit of diffusion risk; λ_J is the price of jump risk. The assumption of the price of jump risk is proportional to the variance of the jump,

or

λ_J . Var $\{[P(r+J,\tau) - P(r,\tau)] | J(\mu,\gamma^2))d\pi\}$

Base on the equation (4.5) and (4.6), a closed form solution of bond price has been developed.

Proposition 3. Base on the assumption of ARJ model, the bond pricing equation will be:

$P(r, \tau) = A(\tau) \exp(-r\tau)$

where

$A(\tau) = \exp(D(\tau))$

$D(\tau) = N\tau^3 + M\tau^2 - T(\partial \ln P(0,T)/\partial T) + t(\partial \ln P(0,t)/\partial t) + \ln[P(0,T)/P(0,t)]$

and $\tau = T-t$, $P(r, \tau)$ is the discount zero-coupon bond price with short rate r and time period to mature τ. The parameters

$M = (\lambda\sigma - \sigma^2 T)/2$

$N = (\sigma^2/2) + (h/3)[(\gamma^2 + \mu^2)/2 - \lambda_J \gamma^2] + (1/2)h^2\gamma^2$

$G(\tau) = -\mu h + (3/2)\gamma^2 h^2 \tau^2$

See appendix A for the proof of Proposition 3.

The discount zero coupon bond pricing equation which was given in Proposition 3 also satisfies the following equations:

$$\lim_{\tau\to\infty} P(r,\tau) = 0, \forall r$$

$$\lim_{r\to\infty} P(r, \tau)=0, \forall \tau$$

$$\lim_{\tau\to 0} P(r, \tau) = 1, \forall r$$

The ARJ model is the extension of Ho-Lee model, so the solution of part A in this section must reduce to the original Ho-Lee model when jump never happens, i.e. h = 0, and the probability of a jump event happened is zero. A verification is given as following:

When there is no jump during the period of Δt, which means that the jump rate equal to 0, or h = 0, and no market price of risk. Proposition 3 will become:

$$M = -(\sigma^2 T)/2$$

$$N = \sigma^2/2$$

Since $G(\tau) = 0$ when h = 0.

$$\begin{aligned} D(\tau) &= N\tau^3 + M\tau^2 + \tau F(0, \tau) - \int F(0, \tau)\, d\tau \\ &= (\tau^3 \sigma^2)/2 - (\sigma^2 T\tau^2)/2 + \tau F(0, \tau) - \int F(0, \tau)\, d\tau \\ &= (\tau^3 \sigma^2)/2 - (\sigma^2 (\tau+t)\tau^2)/2 + \tau F(0, \tau) - \int F(0, \tau)\, d\tau \\ &= -(\sigma^2 t\tau^2)/2 + \tau F(0, \tau) - \int F(0, \tau)\, d\tau \\ &= -(\sigma^2 t(T - t)^2)/2 - (T-t) [\partial \ln P(0,\tau)/ \partial\tau] + \ln[P(0,T)/P(0,t)] \end{aligned}$$

where

$$-\tau F(0, \tau) = \tau [\partial \ln P(0,\tau)/ \partial\tau]$$

and

$$\int_t^T F(0, \tau)\, d\tau = -\ln[P(0,T)/P(0,t)] \qquad (4.7)$$

while in the Ho-Lee model

$\ln A(t,T) = \ln[P(0,T)/P(0,t)] - (T-t)[\partial \ln P(0,t)/\partial t] - (\sigma^2 t(T-t)^2)/2$

and $\ln A(t,T) = D(\tau)$

The above have show

(4.7) can be explained as the following.

The forward rate, $f(t,T_1,T_2)$ as relates to discount zero-coupon bond prices is:*

$$f(t,T_1,T_2) = [\ln P(t, T_1) - \ln P(t, T_2)]/(T_2 - T_1) \qquad (4.8)$$

$f(t,T_1,T_2)$ is forward rate as seen at time t for the period between time T_1 and T_2.

$P(t, T_i)$ is the discount zero-coupon bond price at time t with maturity T_i, i=1, 2.

substitute $t = 0$, $T_1 = t$ and $T_2 = T$, the equation (4.8) will be:

$$f(0,t,T) = (\ln P(0,t) - \ln P(0,T))/(T - t)$$

Since the variable F(t, T) is the limit of $f(t, T, T + \Delta t)$ as Δt tends to zero, the instantaneous forward rate as seen at time 0 for a contract maturing at time t will be:

$$F(0, t) = \lim_{\Delta t \to 0} f(0, t, t + \Delta t) = \lim_{\Delta t \to 0} \frac{\ln P(0,t) - \ln P(0, t + \Delta t)}{\Delta t} = -\frac{\partial \ln P(0,t)}{\partial t} \qquad (4.9)$$

integrate the equation (4.9) with respect to τ, and τ is in the range of t to T.

$$\int_t^T F(0, \tau)\, d\tau = -\int_t^T \partial \ln P(0, \tau)/\partial \tau\, d\tau = -\ln P(0,\tau)\Big|_t^T = -(\ln P(0,T) - \ln P(0,t))$$

* See John C. Hull " Options, Futures, and other Derivative Securities" Second Edition, page 399.

So, base on the proof of the above, we can reach the conclusion that the ARJ model is a generalizel form of the Ho-Lee model. This tells us that the ARJ model has more properties than that of the Ho-Lee model by adding a jump term, but it does not lose the properties which the Ho-Lee model has. Theoretically, this prove that ARJ model have improved the Ho-Lee model. It overcomes the short hand of Ho-Lee model which can not approach the interest rate jump diffusion behavior.

C. Estimating The ARJ Model

This section describes the methods of estimating the parameters. Generally, there are two ways to estimate parameters for the interest rate movements models. One is by using historical data to estimate parameters, such as the method of maximum likelihood estimation. For a series of random variable, if their joint density function is simple and known to us, maximum likelihood estimation is the best choice in estimating the parameters. See [Vasicek],[CIR] etc.. But models using maximum likelihood estimation may not adequeting mesure today's parameters, because they are based on historical data. Therefore, in the financial markets, another parameter estimating method merged and become much more popular. This method is called calibration, which uses the current market asset prices to estimate the parameters. In apply this calibration method, it's not necessary to know the value and the complexity of the joint density function. See [Ho-Lee], [BDT], [HJM]. The other advantage is that the prices of bonds, reached by using the calibration method, are guaranteed to match the current market prices.

Since the joint density function of the ARJ model, is quite complicate and in order to match the ARJ model with today's market prices, I choose the calibration method to estimate the four parameters in the ARJ model which are: μ, h, σ and γ.

The following is the estimation procedure by using calibration method.

Let fi , i=1, 2, ... n, be a bond option price at the ith period based on the market price and fi(μ, h, σ, γ) , i= 1,2, ... n, be a bond option price based on the ARJ model. The parameters are obtained based on

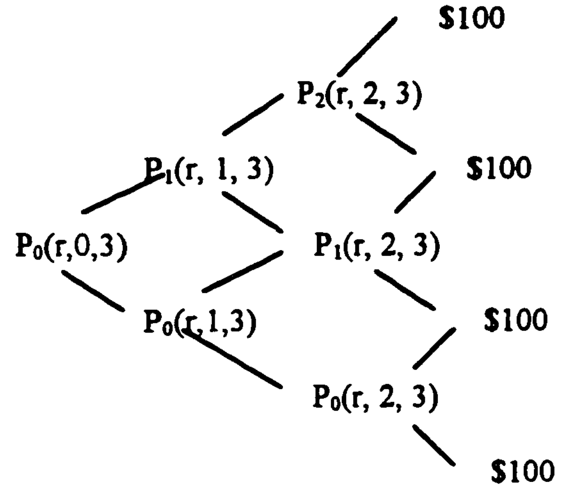
$$\min_{(\mu,\ h,\ \sigma,\ \gamma)} \sum_{i=1}^{n} (fi(\mu,\ h, \sigma, \gamma) - fi)^2$$

The procedure of the parameters estimation is:

1. Estimating parameters
2. Calculating values of bonds or European bond options base on the parameters
3. Comparing calculated values with market values
4. Adjusting values of parameters
5. repeating step 2 to step 4, until no further improvement possible

If we use binominal trees method to estimated the parameters, the following is the example:

The bond price base on the ARJ model is $P_i(r, t, T)$ or $P_i(r, \tau)$,τ= T-t. i,is the state of the bond price. For a three period bond, which has face value is $100, the lattics are:



Assume the strike price is K, then the bond option price on each state is

Ci=max[Pi(r, t,T)-K, 0]

The following is the estimation result. I used Monte Carlo simulation method to generate interest rate lattices. Base on those possible interest rate lattices to get bond prices. Taking the average of the bond prices, the result is what we want. Comparing the price with the bond price in the market by adjust the parameters of the ARJ pricing model, I get the parameters. On July 30, 1996, the bond price is $98.288 for the yield to mature 120 days. Based on the ARJ model, at time May 29,1996, to evaluate the bond price after tow months which maturity day is Nov. 29, 1996, the bond price is $98.28 based on the parameters in table 2. In table 2, it exhibit the yield curve is base on May 29, 1996. This estimation method is similar to the above which suggesting to use option price.

Table 3 exhibit the estimated parameters which base on the April 30, 1996 yield curve. The bond price base on ARJ model and the table 3 parameters is $98.708. The market price on July 30, 1996 is $98.704.

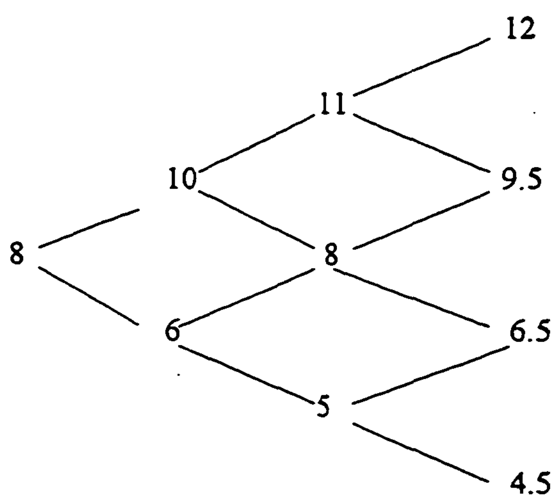
Comparing table 2 with table 3, we find out the parameters are same except for volatility. This is because that the volatility dimisivesreduce as the bond gots cluson to muturity..

V. Empirical Result

While still waiting the result to come out. I'll explain here the methods and procedures I used in applying the ARJ model to price bonds and bond options.

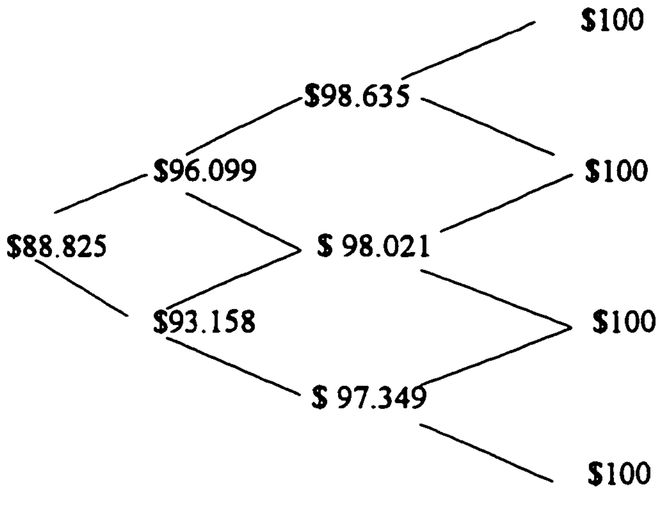
The methodology of pricing interest rate derivatives normally is by using the binomial tree lattices. Since it is important to understand the pricing of bond or bonds options, I will give an example in the following.

Suppose the binomial lattice of one-year rate



Assuming the face value of zero-coupon bond price is \$100. We choose time period as 3 months. Then, from the up rate 12%, the present value of the bond price is \$100*exp(-0.12*(3/12)) which equal to \$97.045. If the rate reach 9.5%, the present value of the bond price is \$100*exp(-0.095*(3/12)) = \$97.653. Then the average price of the bond after one period is \$97.349. And so on.

We have the binomial lattice of one-year bond price:



So, the present value of zero-coupon bond with face value $100 and maturity 1 year is $88.825. This pricing method is widely used in the financial market. But for some interest rate models, it is not convenient to use it. For this reason, a simulation method like the Monte Carlo method is introduced.

Since the ARJ model has three factors. The binomial tree method, which fits better in one or two factor models is not good to be used in the ARJ model.

The Monte Carlo simulation method is a numerical procedure for estimating the value of a European style derivative security when exact formulas are not available. The method is more effective in pricing the interest rate contingent claims of derivative securities with their payoff being dependent on the history of the underlying variable or variables. There are three random variables in ARJ model: (1) the standard normal random variable with zero mean and unit variance; (2) the normal random variable with mean μ, and variance γ^2 and (3); the Poisson process variable. The procedure for the simulation of interest rate can be described as the following:

(1) Generating the three random variables; X_1, X_2 X_3.

Assuming:

a) X_1 is standard normal random variable with mean zero and variance 1, formally,

$$X_1 \sim N(0, 1)$$

b) X_2 is normal random variable with mean μ and variance γ^2, formally,

$$X_2 \sim N(\mu, \gamma^2)$$

c) X_3 is random variable from Poisson process.

(2) Calculating the change of interest rate by using the three random variables.

$$\Delta r = \omega(t)\, \Delta t + \sigma \Delta X_1 + \Delta X_2 \Delta X_3$$

(3) In next period of time, the interest rate changes from r_0 to r_1, where

$$r_1 = r_0 + \Delta r$$

(4) Repeating the step (2) and (3), a complete probability distribution of the interest rate is obtained. The details of the procedure is described in Table 4 which shows only one possible pattern of interest rate movements. Different random samples would lead to different interest rate movements. In column 2 of table 4 the number is generated from standard normal distribution. In column 3 of table 4, the number is generated from a normal distribution with mean 0.002 and variance 0.03^2. In column 4 of table 4, the number is the random drawing from the Poisson distribution when the rate is 0.5. Column 5 shows the change in interest rate during periods of length 0.01 year or 3.65 days base on the random number of column 2 to column 4. The column 1 is the interest rate at the start of each period base on the three random samples.

Once the whole term structure is obtained, the interest rate contingent claims can be priced based on the term structure.

For example, the discount zero-coupon bond price can be calculated base on the table 4. Suppose the bond will mature at t = 0.01*10 = 0.1 years, so, at time 0.1 year or after 36.5 days, the bond have value of $1. The present value of the bond base on the path in table 4 is

$P = \$1 * e^{-0.01*10*0.06025} * e^{-0.01*9*0.06048} * \ldots * e^{-0.01*0.06}$

$= \$0.967424$.

Normally, we will generate thousands interest rate lattics, then calculate the average price which is much more robust and reliable.

This method for value the zero coupon bonds is the Monte Carlo Simulation. It is quite popular in financial markets when the pricing models do not have a closed form solution.

VI. Summary

In this paper, I propose an approach to track interest rate behavior. This approach has been done through the way by combining the arbitrage-free Ho-Lee model with a jump term. The new model (ARJ) has remained the same properties which the important one is the arbitrage free as the Ho-Lee does. Besides, the new model can catch the jump feature which is one of interest rate movements behavior. Through the methodology of Monte Carlo simulation, we will be able to price simple bonds and bond options and other derivatives.

This paper has implications important to financial research. It provides the procedure for pricing many interest rate contingent claims with the arbitrage-free model.

Appendix A

The following is the proof of proposition 3 in section IV.

Proof:

According to Ito's Lamma and here all t substitute to τ.

$P_t = - P_\tau$

The partial differential equation for $P(r, \tau)$is:

$$dP = (\omega(\tau)\, P_r - P_\tau + (\sigma^2/2)P_{rr})\, d\tau + \sigma P_r\, dz + ([P(r+J, \tau) - P(r, \tau)] | J(\mu,\gamma^2))\, d\pi \qquad (4.5)$$

where the subscripts denote the relevant derivatives.

Take expectation of (4.3), we get

$$E(dP) = E\,[(\omega(\tau)\, P_r - P_\tau + (\sigma^2/2)P_{rr})d\tau] + E[\sigma\, P_r\, dz] + E[([P(r+J, \tau) - P(r, \tau)] | J(\mu,\gamma^2))d\pi]$$

$$= \omega(\tau)\, P_r - P_\tau + (\sigma^2/2)P_{rr} + 0 + h\, E[[P(r+J, \tau) - P(r, \tau)] | J(\mu,\gamma^2)]$$

Substitute the above equation into the equation (4.4), we get

$$\omega(\tau)\, P_r - P_\tau + (\sigma^2/2)P_{rr} + h\, E[[P(r+J, \tau) - P(r, \tau)] | J(\mu,\gamma^2)] =$$

$$rP + \lambda\sigma P_r + \lambda_J\, Var\, \{[P(r+J, \tau) - P(r, \tau)] | J(\mu,\gamma^2)d\pi\}$$

$$\Rightarrow 0 = [\omega(\tau)-\lambda\sigma]P_r+(\sigma^2/2)P_{rr} - rP - P_\tau + hE[([P(r+J,\tau)-P(r, \tau)] | J(\mu,\gamma^2))]$$

$$- \lambda_J Var\{[P(r+J,\tau)- P(r,\tau)] | J(\mu,\gamma^2)d\pi\} \qquad (4.10)$$

According to the Ho -Lee model assumption, the bond price is given by:

$$P(r,\tau) = A(\tau)\exp(-rB(\tau)) \qquad (4.11)$$

take derivatives of the price equation (4.11).

The first order derivatives are:

$$P_r = A(\tau)\exp[-rB(\tau)][-B(\tau)] = - A(\tau)B(\tau)\exp[-rB(\tau)]$$

$$P_\tau = A_\tau \exp[-rB(\tau)] + A(\tau)\exp[-rB(\tau)](-r)B_\tau$$

$$= (A_\tau - r\, A(\tau)B_\tau) \exp[-rB(\tau)]$$

The second order derivative is:

$$P_{rr} = A(\tau)B(\tau)^2 \exp[-rB(\tau)]$$

then substitute P_r, P_τ, P_{rr} in the function (4.10)

we get

$$0 = [\omega(\tau) - \lambda\sigma][- A(\tau)B(\tau)] + (\sigma^2/2)\, A(\tau)B(\tau)^2 - r\, A(\tau) - (A_\tau - r\, A(\tau)B_\tau) +$$

$$hE\{A(\tau)\exp[-JB(\tau)] - A(\tau)\} - \lambda_J \mathrm{Var}\{ A(\tau)\exp[-JB(\tau)] - A(\tau)]d\pi\}$$

$\Rightarrow$

$$0 = [\omega(\tau) - \lambda\sigma][- A(\tau)B(\tau)] + (\sigma^2/2)\, A(\tau)B(\tau)^2 - r\, A(\tau) - A_\tau + r\, A(\tau)B_\tau +$$

$$A(\tau)[\, hE\{ \exp[-JB(\tau)] - 1\} - \lambda_J \mathrm{Var}\{ \exp[-JB(\tau)] - 1]\}h] \qquad (4.12)$$

Using a Taylor series expansion for the jump component in (4.12)

we know that

$$\exp[-x] = 1 - x + (x^2)/(2!) - (x^3)/(3!) + \ldots..$$

So, the Taylor series expansion for the jump component is

$$\exp[-(JB(\tau))] - 1 = -J\, B(\tau) + [J\, B(\tau)]^2/(2!) - \ldots... \qquad (4.13)$$

Take expectation of the above equation

we have

$$E\{\exp[-(JB(\tau))] - 1\} = -B(\tau)E[J] + [B(\tau)]^2/(2!)E[J^2]$$

$$\cong -B(\tau)\mu + [B(\tau)]^2/(2!)(\gamma^2 + \mu^2)$$

The high order of (J B(τ)) was dropped, because they are too small to affect the result.

Take the variance of the equation (4.13):

$$Var\{\exp[-(J B(\tau))] - 1\} \cong [B(\tau)]^2 Var[J] = \gamma^2 [B(\tau)]^2$$

Substitute the mean and variance into the equation (4.12)

So, the equation (4.12) becomes

$$0 = [\omega(\tau) - \lambda\sigma][-A(\tau)B(\tau)] + (\sigma^2/2) A(\tau)B(\tau)^2 - r A(\tau) - A_\tau + r A(\tau)B_\tau +$$

$$A(\tau)[h[-B(\tau)\mu + B(\tau)]^2/(2!)(\gamma^2 + \mu^2)] - \lambda_J h\gamma^2 [B(\tau)]^2$$

$$= [\omega(\tau) - \lambda\sigma][-A(\tau)B(\tau)] + (\sigma^2/2) A(\tau)B(\tau)^2 - r A(\tau) - A_\tau + r A(\tau)B_\tau +$$

$$A(\tau) h[B(\tau)]^2[(\gamma^2 + \mu^2)/2 - \lambda_J\gamma^2] - A(\tau)B(\tau)]h\mu$$

Let $w = h[B(\tau)]^2[(\gamma^2 + \mu^2)/2 - \lambda_J\gamma^2] - B(\tau)h\mu$

So, $0 = [\omega(\tau) - \lambda\sigma][-A(\tau)B(\tau)] + (\sigma^2/2) A(\tau)B(\tau)^2 - r A(\tau) - A_\tau + r A(\tau)B_\tau + A(\tau)w$

or $0 = (-r A(\tau) + r A(\tau)B_\tau) + \lambda\sigma[A(\tau)B(\tau)] - \omega(t) A(\tau)B(\tau) + (\sigma^2/2) A(\tau)B(\tau)^2 + A(\tau)w - A_\tau$

(4.14)

from the deferential equation (4.14) we can get

$$0 = -r A(\tau) + r A(\tau)B_\tau \qquad (4.15)$$

and

$$0 = \lambda\sigma[A(\tau)B(\tau)] - \omega(t) A(\tau)B(\tau) + (\sigma^2/2) A(\tau)B(\tau)^2 + A(\tau)q - A_\tau \qquad (4.16)$$

from the equation (4.15)

$B_\tau = 1$

$B(\tau) = \tau = T - t$

from the equation (4.14), we have

$A_\tau / A(\tau) = \lambda\sigma B(\tau) - \omega(\tau) B(\tau) + (\sigma^2/2) B(\tau)^2 + w$

take expectation of above equation, we have

$$\int(d A(\tau)/A(\tau)) = \lambda\sigma \int B(\tau)\, d\tau - \int \omega(\tau) B(\tau) d\tau + (\sigma^2/2) \int B(\tau)^2 d\tau + \int w d\tau$$

$$= (\lambda\sigma\tau^2)/2 - \int \omega(\tau)\, \tau d\tau + (\sigma^2/6)\, \tau^3 + \int w d\tau \qquad (4.17)$$

the last term in the equation (4.17) will be:

$$\int w d\tau = \int \{h\, [\, B(\tau)]^2 [(\gamma^2 + \mu^2)/2 - \lambda_J \gamma^2] - B(\tau) h\mu\}\, d\tau$$

$$= h[(\gamma^2 + \mu^2)/2 - \lambda_J \gamma^2] \int [\, B(\tau)]^2\, d\tau - h\mu \int B(\tau) d\tau$$

$$= (h\tau^3/3)[(\gamma^2 + \mu^2)/2 - \lambda_J \gamma^2] - (h\mu\tau^2)/2$$

Since

$\omega(t) = F_t(0,t) + \sigma^2 t + G(t, h, \mu, \gamma)$

and $\tau = T - t$

substitute t to τ,

$$\omega(\tau) = - F_\tau(0,\tau) + \sigma^2 (T - \tau) + G(\tau, h, \mu, \gamma)$$

where

$G(\tau, h, \mu, \gamma) = - h\mu + (3/2) h^2 \gamma^2 \tau^2$

The second term in equation (4.17) will be:

$$\int \omega(\tau)\, \tau d\tau = \int (- F_\tau(0,\tau) + \sigma^2 (T - \tau) + G(\tau))\tau d\tau$$

$$= - \int F_\tau(0,\tau)\, \tau d\tau + \sigma^2 T \int \tau\, d\tau - \int \sigma^2 \tau^2 d\tau + \int G(\tau)\tau d\tau$$

$$= - \tau F(0, \tau) + \int F(0, \tau)\, d\tau + (\sigma^2\tau^2 T/2) - (\sigma^2\tau^3)/3 + \int \tau(- h\mu + (3/2)h^2\gamma^2\tau^2)d\tau$$

$$= - \tau F(0, \tau) + \int F(0, \tau)\, d\tau + (\sigma^2\tau^2 T/2) - (\sigma^2\tau^3)/3 - \tau^2 h\mu/2 + (1/2)h^2\gamma^2\tau^3$$

The equation (4.17) becomes

$$\int(d A(\tau)/A(\tau)) = (\lambda\sigma\tau^2)/2 + \tau F(0, \tau) - \int F(0, \tau)\, d\tau - (\sigma^2\tau^2 T/2) + (\sigma^2\tau^3)/3 + \tau^2 h\mu/2 - (1/2)h^2\gamma^2\tau^3$$

$$+ (\sigma^2/6)\, \tau^3 + (h\tau^3/3)[(\gamma^2 + \mu^2)/2 - \lambda_J\gamma^2] - (h\mu\tau^2)/2$$

$$= \{(\sigma^2/2) + (1/2)h^2\gamma^2 + (h/3)[(\gamma^2 + \mu^2)/2 - \lambda_J\gamma^2]\}\, \tau^3 + ((\lambda\sigma - \sigma^2 T)\tau^2)/2 + \tau F(0, \tau) - \int F(0, \tau)d\tau$$

Let

$M = (\lambda\sigma - \sigma^2 T)/2$

$N = (\sigma^2/2) + (h/3)[(\gamma^2 + \mu^2)/2 - \lambda_J\, \gamma^2] - (1/2)h^2\gamma^2$

So, $\ln A(\tau) = N\tau^3 + M\tau^2 + \tau F(0, \tau) - \int F(0, \tau)\, d\tau$

let $D(\tau) = N\tau^3 + M\tau^2 + \tau F(0, \tau) - \int F(0, \tau)\, d\tau$

So, $A(\tau) = \exp(D(\tau))$

The proof of proposition 3 is completed. #

Table 1

The Statistics of Interest Rate For Three Kinds of Reading

Data Frequency	Mean	Variance	Skewness	Kurtosis	No. of Obs
Daily					
r	5.601	2.631	-0.067	-0.903	2,526
dr	-0.010	0.005	-1.062	27.932	
Weekly					
r	4.957	2.693	0.343	-1.103	326
dr	-0.009	0.011	-0.298	2.240	
Monthly					
r	5.560	2.765	-0.040	-0.921	199
dr	-0.014	0.075	-1.124	4.021	

Notes:

(1) The daily data is from Jan, 17, 1986 to Jan. 17, 1996.
(2) The weekly data is from Sep. 22, 1989 to Dec. 22, 1995.
(3) The monthly data is from Dec. 31, 1985 to Nov. 30, 1995.
(4) USTBAUS TREAS 91 - DAY BILL YIELD
(5) CURRENCY; U.S. DOLLAR
(6) The Data is From Banker Trust Inc.

Table 2

Parameters Estimation

Parameters	Estimation	Yield Curve(May 29,1996)	
h	0.5	3m	5.16%
		6m	5.35%
μ	0.02	1yr	5.69%
		2yr	6.17%
γ	0.03	3yr	6.33%
		4yr	5.14%
σ	0.18	5yr	6.55%
		7yr	6.71%
		10yr	6.78%
		20yr	7.02%
		30yr	6.94%

Notes:

The parameters estimated is based on the July 30, 1996 bond market price \$98.288, yield to mature in 120 days. Based on the ARJ model, the bond price P(r, t, T)=P(5.35%, 2m, 6m)=\$98.28 under the above parameters.

Table 3

Parameters Estimation

Parameters	Estimation	Yield Curve(April 30,1996)	
h	0.5	3m	5.12%
		6m	5.30%
μ	0.02	1yr	5.62%
		2yr	6.04%
γ	0.03	3yr	6.17%
		4yr	5.66%
σ	0.15	5yr	5.14%
		7yr	6.41%
		10yr	6.58%
		20yr	7.02%
		30yr	6.90%

Notes:

The parameters estimated is based on the July 30, 1996 bond market price $98.704, yield to mature in 91 days.(Wall Street Journal Wedensday, July 31, 1996.) Based on the ARJ model, the bond price P(r, t, T)=P(5.12%, 3m, 6m)=$98.708 under the above parameters.

Table 4

Simulation of Interest Rate When Derivative of Forward Rate and G(t) are Constant and Equal to 0.12 and Sigma = 0.20 During Periods of Length 0.01 Year

Change of r = (0.12+0.20*t) * 0.01 +0.20*0.1*N(0,1) + 0.1*Poisson Random Number * N(0.002,0.03)

Interest Rate at Start Period (In Percentage)	Random Number N(0,1)	Random Number N(0.002,0.03) 0.03 is S.D	Random Number Poisson h=0.5	Change in Interst Rate During Period
6.000	-0.026	-0.010	0	0.001
6.001	0.255	0.025	1	0.009
6.010	-0.024	-0.024	0	0.001
6.010	-0.940	-0.028	0	-0.018
5.993	0.549	0.029	1	0.015
6.008	0.699	-0.023	1	0.013
6.021	0.564	0.031	0	0.013
6.034	0.730	-0.019	1	0.014
6.048	-1.174	-0.010	0	-0.022
6.025	1.170	0.018	0	0.025

Notes:

Here we assume that the function G is constant and the derivative of instantaneous forward rate is known and constant

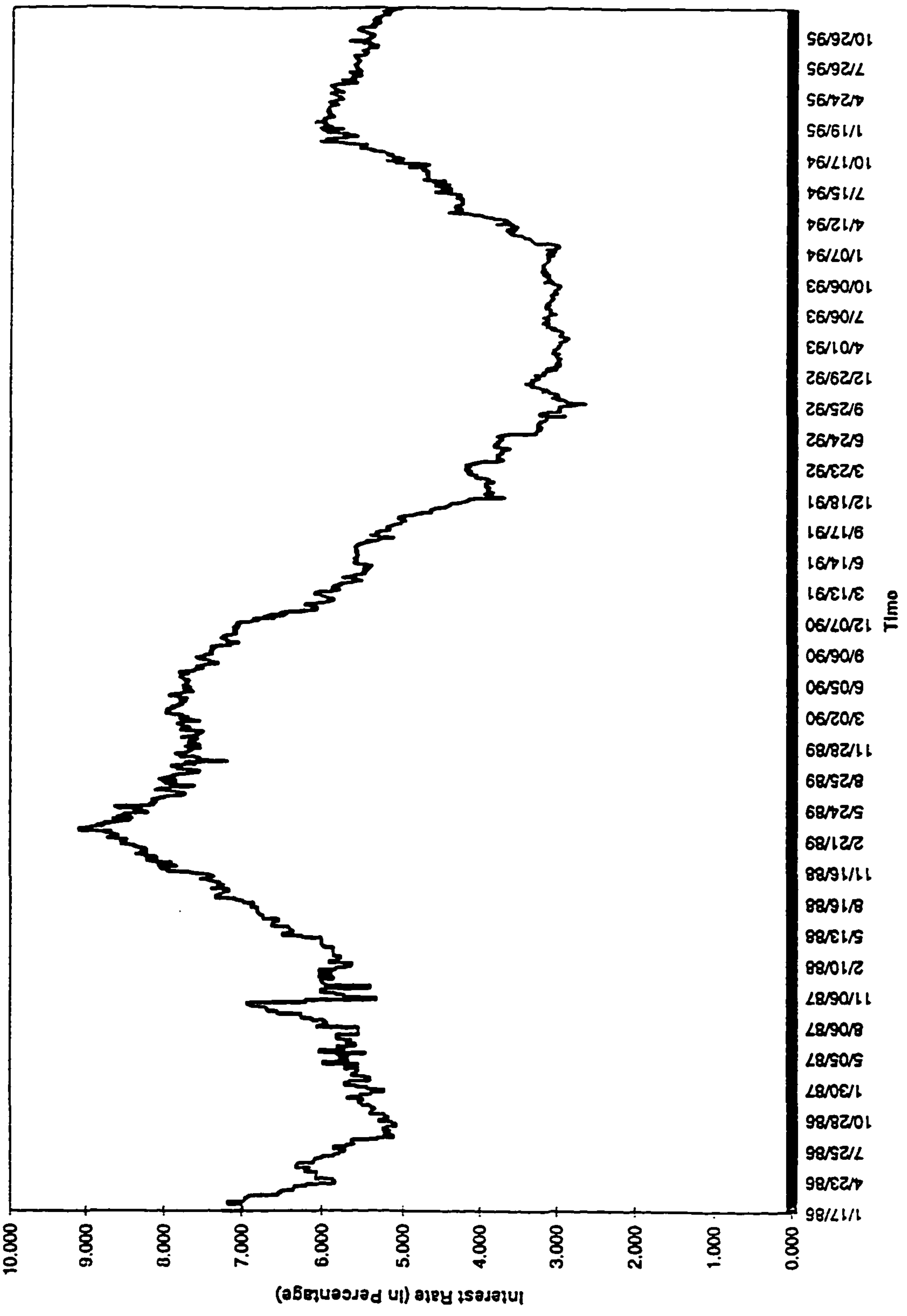
Table 5

Jump observations

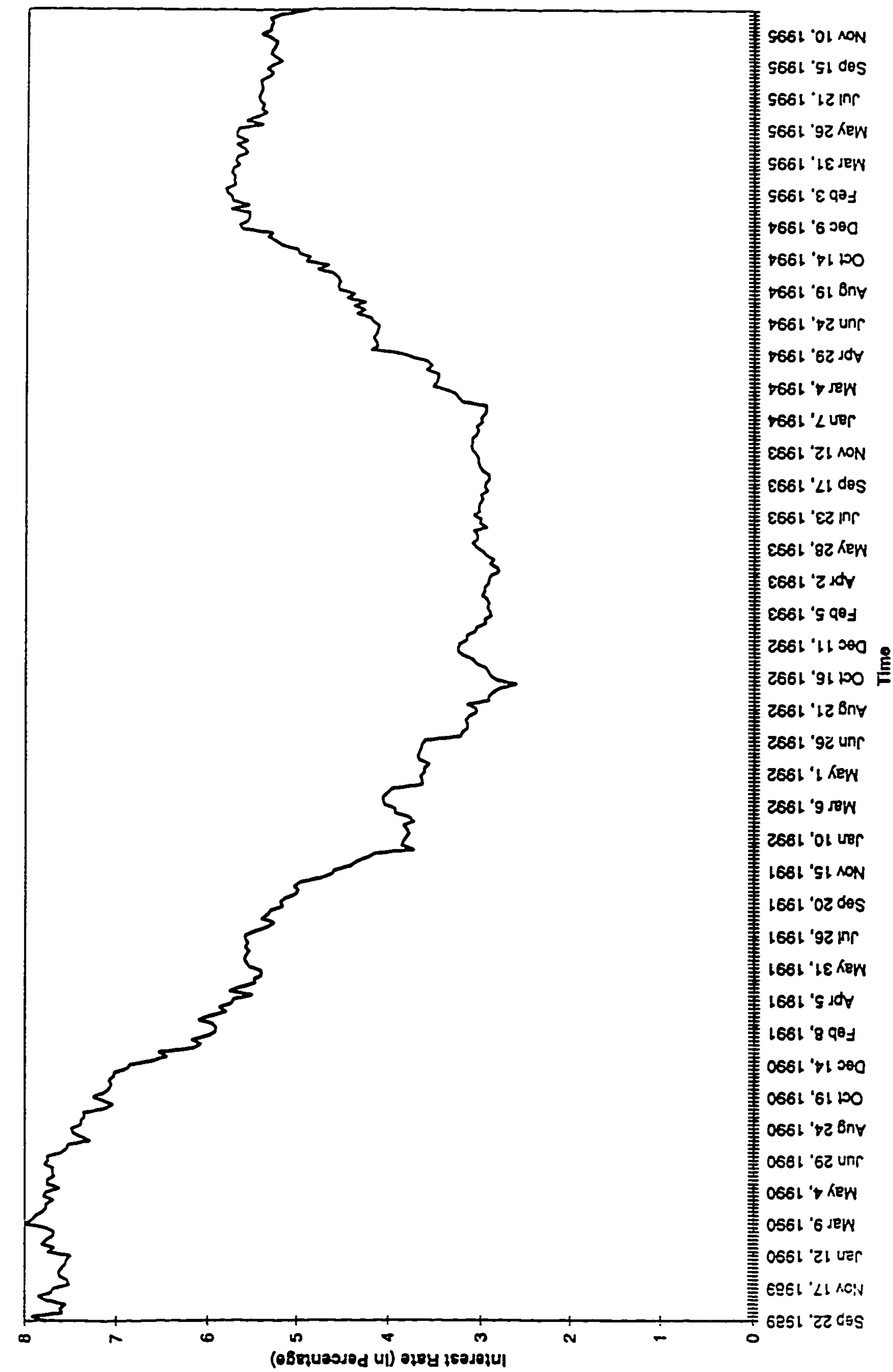
Key Rates (in percent)	April 5, 1996	April 8, 1996	April 9, 1996
3-Month Bill	5.00	5.03	4.94
6-Month Bill	5.13	5.19	5.08
10-yr.	6.55	6.59	6.54
30-yr.	6.82	6.87	6.83

The 30-yr. bill jump 5 base point to 6.87 in April 8, 1996. That happened "since Friday's report by the Labor Department that companies added a hefty number of new jobs to their payrolls in March." (New York Times, Business Day, April 9, 1996 and April 10, 1996.)

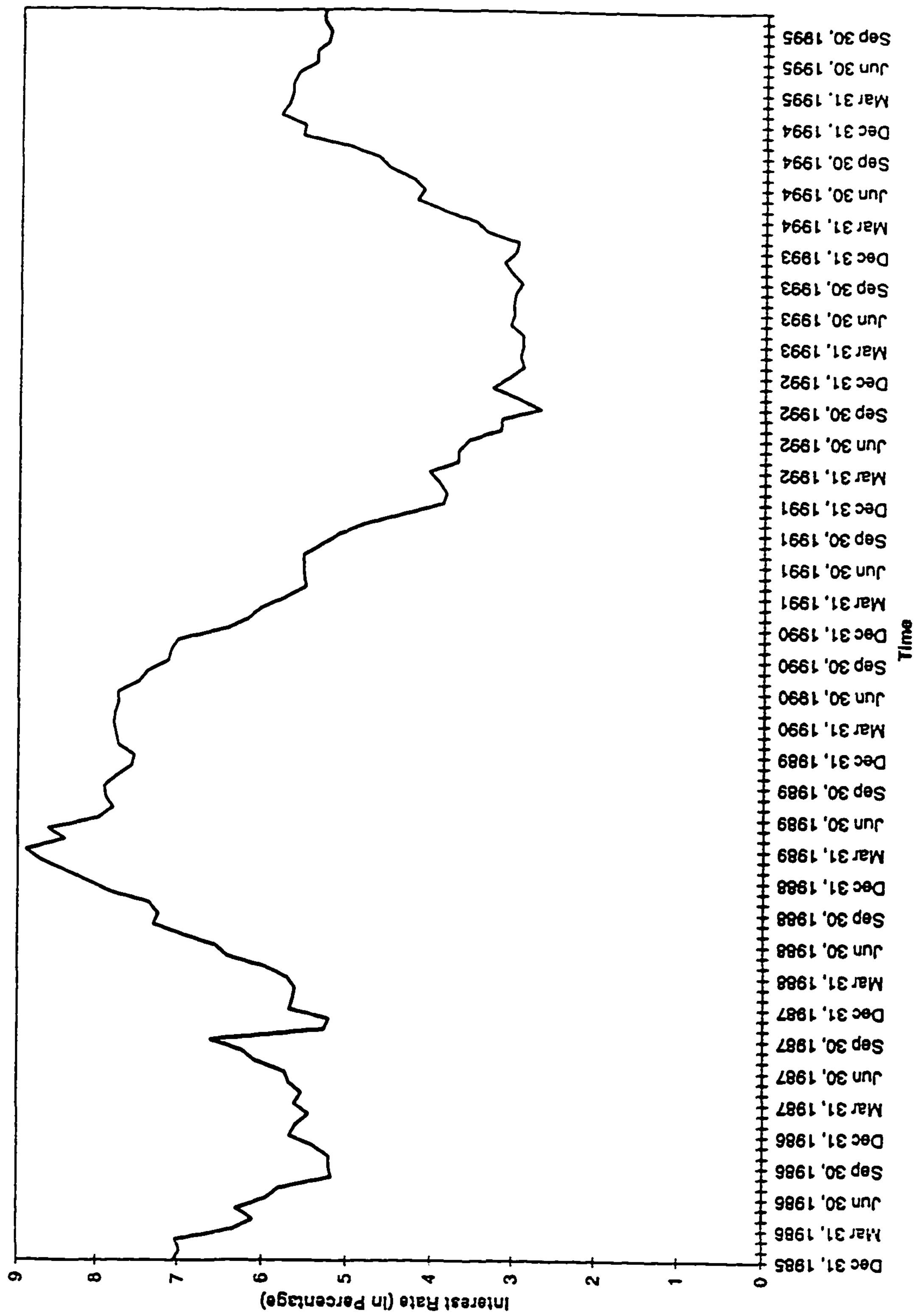
3-Month Bill Daily Data



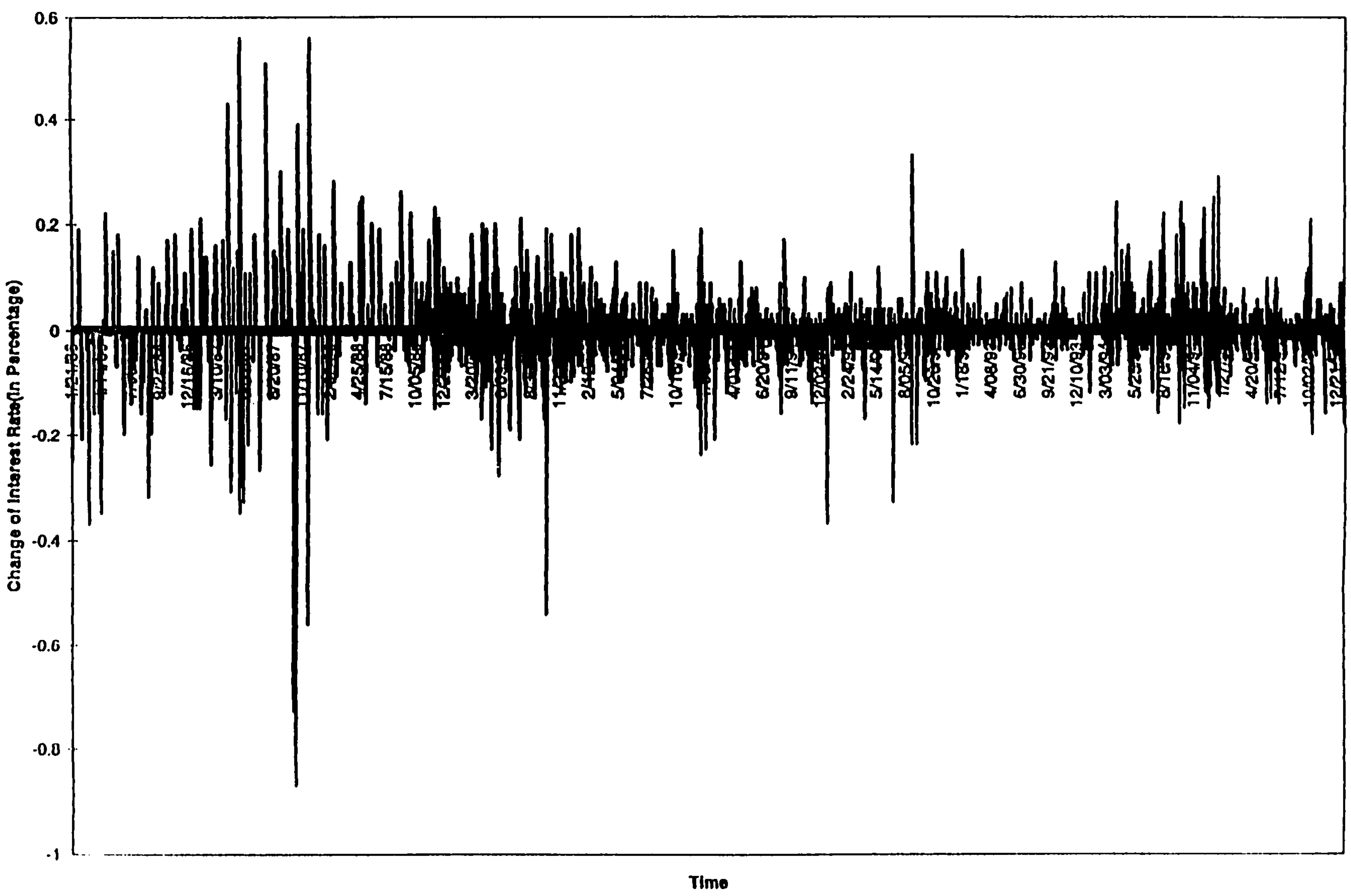
3-Month Bill Weekly Data

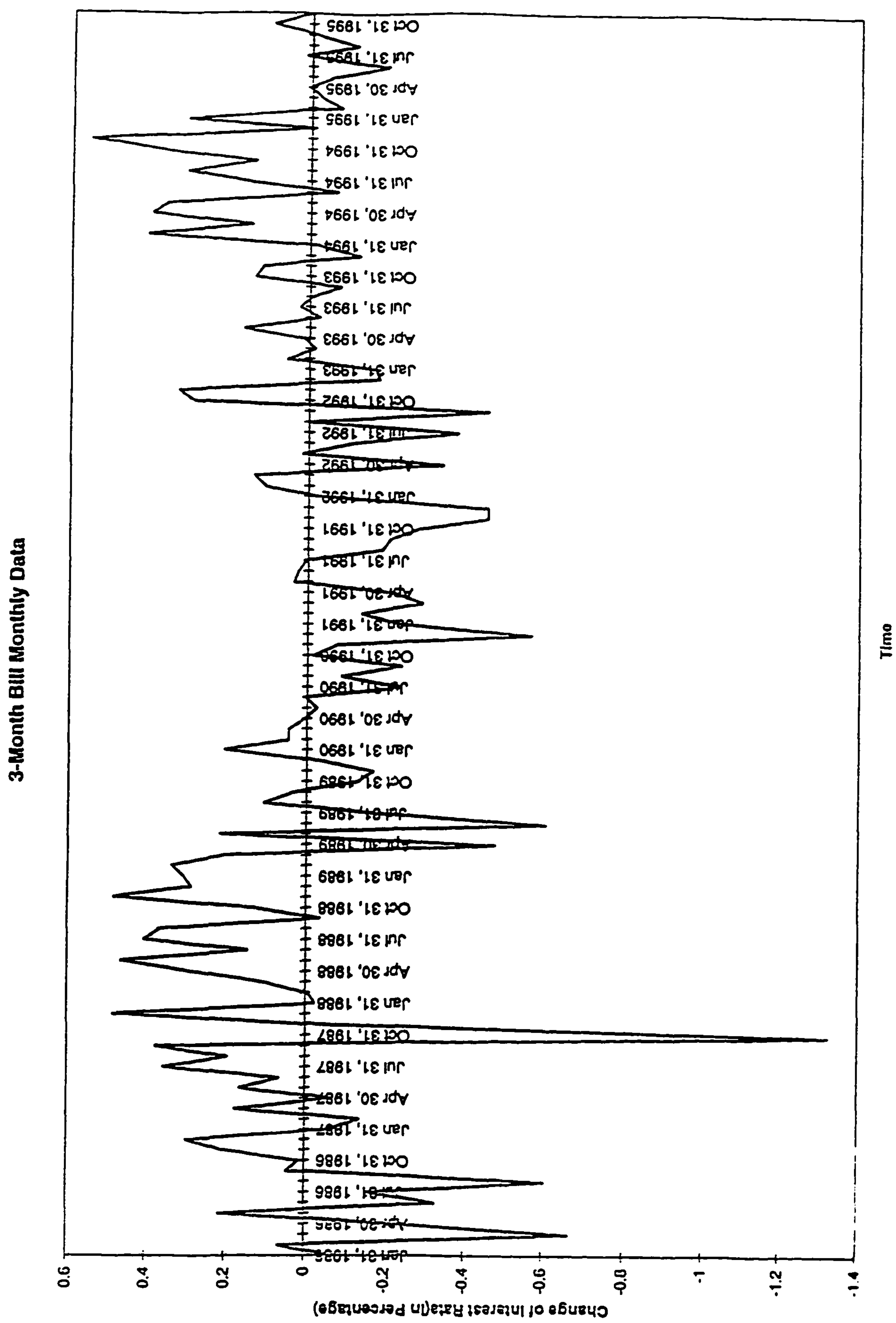


3-Month Bill Monthly Data



3-Month Bill Daily Data




3-Month Bill Monthly Data
Change of Interest Rate(In Percentage)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
Time
Jan 31, 1985
Apr 30, 1985
Oct 31, 1986
Jan 31, 1987
Apr 30, 1987
Jul 31, 1987
Oct 31, 1987
Jan 31, 1988
Apr 30, 1988
Jul 31, 1988
Oct 31, 1988
Jan 31, 1989
Apr 30, 1989
Jul 31, 1989
Oct 31, 1989
Jan 31, 1990
Apr 30, 1990
Jul 31, 1990
Oct 31, 1990
Jan 31, 1991
Apr 30, 1991
Jul 31, 1991
Oct 31, 1991
Jan 31, 1992
Apr 30, 1992
Jul 31, 1992
Oct 31, 1992
Jan 31, 1993
Apr 30, 1993
Jul 31, 1993
Oct 31, 1993
Jan 31, 1994
Apr 30, 1994
Jul 31, 1994
Oct 31, 1994
Jan 31, 1995
Apr 30, 1995
Jul 31, 1995
Oct 31, 1995

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