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On the dynamics of lambda*tan(z)

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On the Dynamics of lambda*tan(z)

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0 Introduction

Iterating a complex analytic map gives rise to a discrete conformal dynamical system. The theory of these dynamical systems was first introduced by Fatou and Julia in the 1920's. Recently, interest in this subject has grown due to the graphics capabilities of computers and to the infusion of techniques from the theory of quasi-conformal mapping from the work of Sullivan, Douady and Hubbard. While much progress has been made, the full theory, even for quadratic polynomials of the form $z^2 + c$ is not yet fully understood.

In [DK], Devaney and Keen began the development of an iteration theory for meromorphic functions. They studied the class of meromorphic functions with polynomial Schwarzian derivative. This thesis is a study of a particular class of these meromorphic functions, the family $T_\lambda : z \mapsto \lambda \tan z$ where $\lambda \in \hat{\mathbb{C}} - \{0\}$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This family, like the quadratic family, depends on a single complex parameter, and so we expect to see phenomena that are generic for more general families of meromorphic functions.

As is usual in the study of dynamical systems, we look at the following two problems:

1. How does the dynamical system look for various values of the parameter λ; that is, how does the z-plane divide into stable and chaotic behavior?

2. How does the parameter plane divide into regions in which the dynamical systems are similar; that is, topologically or quasi-conformally conjugate?

For the first problem, we give a description of the stable behavior by studying the attracting periodic cycles. We see that T_λ has a single attracting fixed point if and only if $|\lambda| < 1$. There are at most two attracting cycles; if T_λ has only one attracting periodic cycle with period greater than one, then the period of the cycle is an even number. If there are two cycles, their multipliers are the same. We show that for appropriate values of λ, T_λ has attracting periodic cycles of arbitrary period.

The Julia sets are fractals in general. When $|\lambda| < 1$, $J(T_\lambda)$ is a Cantor set. When $\lambda \geq 1$ or $\lambda \leq -1$, $J(T_\lambda)$ is a smooth submanifold; that is, the real axis. When λi is a pre-periodic point or a pre-image of a pole, $J(T_\lambda)$ is the whole complex plane.

For the second problem, we see that the bifurcation picture is quite different from that for quadratic polynomials. There are infinitely many simply connected components of the λ-plane in which the functions T_λ are topologically conjugate; their attracting cycles therefore all have the same period. In analogy with the quadratic case, each component can be mapped to the unit disk by sending the point λ to the multiplier of an attracting cycle. Here the map is not a conformal isomorphism, but a universal covering of the *punctured* disk. We use this map to define *internal rays* inside the component as pre-images of the actual rays (measured in turns) in the unit disk.

Outside the unit disk, the components appear in pairs. The components themselves do not possess a physical center where the multiplier vanishes; instead, each pair of components share a *virtual center* at their unique common boundary point. At such a virtual center, the asymptotic value, λi is a pole or a preimage of a pole. These poles and pre-poles can be identified with the poles and pre-poles of the tangent function $\tan z$. This identification gives us a coding of the component-pairs.

We show that the boundary of any component is piecewise smooth. On the boundary of a component there are countably many saddle node bifurcation points. These bifurcation points are the end points of the rational internal rays. Just as for the Mandelbrot set, there is a *bud component* attached to this component at each of these rational bifurcation points. The period of the attracting cycle of T_λ for λ in a bud at the end-point of the rational ray with argument q/p is the product of p and the period of the attracting cycle of T_λ for λ in the root components. There is a one-one correspondence between the rational internal arguments and the Farey series which is generated by recursively inserting the rational number $\frac{a+c}{b+d}$ between any two rationals $\frac{a}{b}$ and $\frac{c}{d}$. Base on the Farey series, the diagram of circle packing on an interval reproduces the pattern of the bud components along a boundary piece with their periods and their sizes.

The organization of the paper is as follows :

In Section 1 we give the background and notation of the iteration theory and some important results about the stable and unstable sets. In Section 2 we study the stable domains of T_λ and give a classification of the attracting cycles of T_λ. In Section 3 we show that each component in the λ-plane corresponds to a quasi-conformal family of T_λ. In Section 4 we show how we generate the computer pictures of the λ-plane for T_λ. In Section 5 we discuss the saddle node bifurcation around the unit circle and the relation between the components attached to the unit disk and the internal angles of the points on the unit circle. Then in Section 6 we generalize the results to all components in the λ-plane. In Section 7 we discuss the virtual centers of component pairs and the period doubling bifurcation, and we give a way to code these virtual centers. In Section 8 we use the Farey series and corresponding circle packing diagram to describe the location of the bud components, their periods and sizes. Then in the appendix we have a note on generating computer pictures.

1 Background and Notation

Let $F(z) : \mathbf{C} \to \hat{\mathbf{C}}$ be a meromorphic function. Let F^n denote the n^{th} iterate of F. Given a point $z \in \mathbf{C}$, the sequence $\{z_n\}$ defined by $z_{n+1} = F(z_n) = F^{n+1}(z)$ is called the *forward orbit* of z and denoted by $O^+(z)$. For F non-rational, because infinity is an essential singularity of F, $F(z)$ is not defined if z is a pole. We therefore terminate the orbit $O^+(z)$, hence $O^+(z)$ is finite if z is a preimage of a pole.

If $F^n(z_0) = z_0$ for some n we say that z_0 is a periodic point, and $O^+(z_0)$ is a *periodic cycle*. If n is the smallest integer with that property, then n is the *period* of the cycle.

A number $c \in \mathbf{C}$ is called a *critical value* of F if $F(z) = c$ has a solution with multiplicity greater than one. Any such solution is called a *critical point*. Critical points are solutions of $F'(z) = 0$. A point $c \in \hat{\mathbf{C}}$ is an *asymptotic value* of F if there is a path $v : [0,1) \to \hat{\mathbf{C}}$ such that $\lim_{t \to 1} v(t) = \infty$ and $\lim_{t \to 1} F(v(t)) = c$. An asymptotic value c of F is *logarithmic* if there is a simply connected neighborhood U of c, and a simply connected unbounded open set $V \subset F^{-1}(U)$ such that $F|V : V \to U - \{c\}$ is a covering. Critical values and asymptotic values are also called *singular values*.

Let z_0 be a periodic point of F with period p. The complex number $\rho(z_0) = (F^p)'(z_0)$ is called the *multiplier* of z_0. By the chain rule all the multipliers $\rho(z_0), \rho(F(z_0)), \cdots, \rho(F^{p-1}(z_0))$ are the same. So ρ is also called the multiplier of the periodic cycle.
A periodic cycle and each periodic point in this cycle is:

1. attracting if $|\rho| < 1$

2. super-attracting if $|\rho| = 0$

3. neutral or indifferent if $|\rho| = 1$

4. repelling if $|\rho| > 1$.

Suppose z_0 is a periodic point of $F(z)$ and $\rho(z_0) = e^{2\pi i \alpha}$. Then z_0 is indifferent. When α is a rational number q/p, z_0 is called a *parabolic* or rational indifferent periodic point of F. A point z is called *stable* if there exists a neighborhood N of z such that $\{F^n|N\}$ is a normal family. The set of all stable points is called the *stable set* or *Fatou set* of F. Each connected component of the stable set is called a *stable domain* or *Fatou component*. The complement of the Fatou set is called the *Julia set* of F and is denoted by $J(F)$.

A stable domain D of F is:

1. an attracting domain if D contains an attracting periodic point of F.
2. a super-attracting domain if D contains a super-attracting periodic point of F.
3. a parabolic domain if ∂D contains a parabolic periodic point of F.
4. a Siegel disk if D is simply connected and $F|D$ is analytically conjugate to an irrational rotation of the unit disk. A Siegel disk has an irrational indifferent periodic point in its interior. Since this periodic point is mapped to the center of the unit disk under the conjugation, it is called the center of the Siegel disk.
5. an Arnold-Herman ring if D is conformally equivalent to an annulus $A = \{z | r_1 < |z| < r_2\}$ and $F|D$ is analytically conjugate to an irrational rotation of the annulus A.
6. a domain at ∞ if for any $z \in D$ there exists a neighborhood $N \subset D$ of z so that $\{F^n\}$ converges uniformly to ∞ on N.

Siegel disks and Arnold-Herman rings are also called rotation domains.

Attracting and super-attracting periodic points are stable and repelling periodic points are unstable. Near attracting or repelling periodic points the map has a local linearization. That is (See [BL]):

Theorem 1.1 (Kœnigs Linearization) *Let z_0 be a periodic point of a meromorphic map F of period p. If the multiplier ρ satisfies $|\rho| \neq 0, 1$, then there exists a local holomorphic function h with $h(z_0) = 0$ so that $h \circ F^p \circ h^{-1}(z) = \rho z$ in some neighborhood of the origin.*

The situation for neutral periodic points is more delicate. Let $F(0) = 0$, $F'(0) = \lambda$. Suppose $\lambda = e^{2\pi i \alpha}$. Then the origin is a neutral or indifferent fixed point of F. When α is a rational number q/p, the origin is a parabolic fixed point. The dynamics near parabolic periodic points are depicted in the Leau-Fatou Flower Theorem (See [BL]). Parabolic periodic points are in the Julia set. At a parabolic fixed point the map F cannot be locally linearized; instead we have the following theorem proved by Camacho (See [CA]):

Theorem 1.2 *Let $F(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$ be a holomorphic map in the neighborhood of the origin, and let λ be a p^{th} root of unity. Then either F^p is the identity or there is a local homeomorphism h, $h(0) = 0$, and an integer $k \geq 1$ such that $h \circ F \circ h^{-1}(z) = \lambda z(1 + z^{kp})$.*

When α is an irrational number, the irrational fixed point 0 is stable if and only if F is locally linearizable. There are essentially two different kinds of irrational fixed points:

Siegel fixed points are the centers of Siegel disks where the map F can be locally linearized; that is, F is locally conjugate to the irrational rotation $z \to e^{2\pi i\alpha}z$. Roughly speaking this means that the Siegel points are those with $\lambda = e^{2\pi i\alpha}$ where α cannot be closely approximated by rational numbers. Siegel points are in the Fatou set.

Cremer points are neutral irrational or periodic points that cannot be linearized. If $\lambda = e^{2\pi i\alpha}$ one can think of the α as one that may be very closely approximated by rational numbers.

Siegel proved that a Siegel disk exists if α is a *Diophantine number*; that is, for n sufficiently large and for ε sufficiently small, the distance between α and an arbitrary rational number p/q should satisfy $|\alpha - p/q| > \varepsilon/q^n$.

Recently Yoccoz gave the following complete characterization for Siegel domains (See [MI]):

Theorem 1.3 (Bryuno-Yoccoz) *For $\lambda = e^{2\pi i\alpha}$ with α irrational, the following three conditions are equivalent:*

1. *the quadratic map $\lambda z + z^2$ possesses a Siegel disk,*
2. *every holomorphic map of the form $z \to \lambda z + O(z^2)$ possesses a Siegel disk,*
3. *$\sum q_n^{-1} \log q_{n+1} < \infty$, where the q_n are the denominators in the continued fraction expansion of α.*

Let z_0 be an attracting fixed point of F. Then the *attracting basin* of z_0 is $\{z | F^n(z) \to z_0 \text{ as } n \to \infty\}$. The *immediate attracting basin* of z_0 is the component of the attracting basin containing z_0. When z_0 is an attracting periodic point of period p, the immediate attracting basin of z_0 is defined as the union of the immediate attracting basin of $z_0, F(z_0), \cdots, F^{p-1}(z_0)$ under $F^p(z)$.

A classic theorem due to Fatou states:

Theorem 1.4 *For F a rational map, the immediate attracting basin of F contains at least one singular value of F.*

This theorem is at the heart of the classification of eventually periodic stable domains. Sullivan proved the following "no wandering domain" theorem:

Theorem 1.5 *Let F be a rational map. Then all stable domains are eventually periodic.*

Thus, the work of Fatou and Julia, together with Sullivan's completes the classification of the stable domains of rational maps (See [SU]). To summarize, a periodic Fatou component for a rational map is one of the following:

1. attracting domain
2. super-attracting domain
3. parabolic domain
4. Siegel disk
5. Arnold-Herman ring.

Recall that the *Schwarzian derivative* $S(F)(z)$ of $F(z)$ is defined as

$$S(F)(z) = -2\frac{(F'(z)^{-\frac{1}{2}})''}{F'(z)^{-\frac{1}{2}}} = \frac{F'''(z)}{F'(z)} - \frac{3}{2}\left(\frac{F''(z)}{F'(z)}\right)^2 \tag{1}$$

Let $\mathbf{M}$ denote the class of meromorphic functions which have polynomial Schwarzian derivatives. Fatou's proof of Theorem 1.4 applies equally well to functions in $\mathbf{M}$ (See [FA]). Sullivan's "no wandering domain" theorem was extended to the critically finite entire transcendental functions; that is, those with finitely many critical and asymptotic values (See [GK]). It was also extended to the meromorphic functions in $\mathbf{M}$ (See [DK]). It was shown that there are no wandering domains and no domains at ∞ for $F \in \mathbf{M}$. The function we will study is $T_\lambda(z) = \lambda \tan z$, it is in $\mathbf{M}$ and its Schwarzian derivative $S(T_\lambda)(z) = 2$. The *Fatou-Sullivan classification* of the Fatou components holds in our case.

By a theorem of Nevanlinna in [NE] meromorphic functions with polynomial Schwarzian derivative have only finitely many asymptotic values and no critical values. Therefore super-attracting domains do not exist for $F \in \mathbf{M}$. So we have the following:

Theorem 1.6 *For $F \in \mathbf{M}$, the stable domains are all eventually periodic, the periodic domains are either*

1. *attracting domains*
2. *parabolic domains*

3. Siegel disks

4. Arnold-Herman rings.

Some important standard properties of Julia sets and Fatou sets of polynomials, rational functions and entire functions carry over to functions in M. That is, for $F \in \mathbf{M}$ we have (see [DK]):

1. $J(F)$ is the closure of the set of repelling periodic points of F.

2. Let $z \in J(F)$. Then $J(F) = \overline{\bigcup_{n=1}^{\infty}\{F^{-n}(z)\}}$, that is, $J(F)$ is the closure of the union of all pre-images of z.

   In addition these functions satisfy

3. $J(F)$ is the closure of the set consisting of the poles of F^n for $n > 0$; that is, the closure of all the pre-images of poles of F, or equivalently the closure of all points with finite orbit.

The first case holds for larger classes of meromorphic functions (See [BKL]). It was shown in [DT] that Julia sets of entire functions may contain forward invariant subsets that are homeomorphic to the product of a Cantor set and the line $[0, \infty)$. These are called *Cantor bouquets*. In [DK] it was shown that these Cantor bouquets also occur in Julia sets of $F \in \mathbf{M}$ when an asymptotic value is also a pole of F.

Now we are ready to discuss the dynamics of T_λ.

2 Stable Domains of T_λ

The map $T_\lambda(z)$ has exactly two asymptotic values $\pm\lambda i$, because

$$T_\lambda(z) = \lambda \tan z = -\lambda i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \tag{2}$$

and $e^{iz} \to 0$ (resp. ∞), $e^{-iz} \to \infty$ (resp. 0) as z moves to the point at infinity along a path in the upper (resp. lower) half plane. Since $T_\lambda(z)$ has no critical points, the orbits of these two asymptotic values of T_λ play a crucial role in the dynamics of T_λ. For iterations of T_λ, we have the following:

Lemma 2.1 *For $\lambda \in \mathbf{C}^* = \hat{\mathbf{C}} - \{0\}$ and $k = 1, 2, 3, \cdots$*

1. $T_\lambda{}^k(-z) = -T_\lambda{}^k(z)$
2. $T_{\overline{\lambda}}^k(\overline{z}) = \overline{T_\lambda{}^k(z)}$
3. $T_{-\lambda}{}^k(z) = (-1)^k T_\lambda{}^k(z)$

Proof. These properties are due to the odd symmetry of the tangent function. In Expression (2) we substitute the conjugates of λ and z, $\overline{\lambda}$ and $\overline{z}$, to get $T_{\overline{\lambda}}(\overline{z}) = \overline{T_\lambda(z)}$. The proof follows by induction on k. □

For the derivatives of iterations of T_λ, we have:

Lemma 2.2 *For $\lambda \in \mathbf{C}^*$ and $k = 1, 2, 3, \cdots$*

1. $[T_\lambda{}^k(-z)]' = [T_\lambda{}^k(z)]'$
2. $[T_{\overline{\lambda}}^k(\overline{z})]' = \overline{[T_\lambda{}^k(z)]'}$
3. $[T_{-\lambda}{}^k(z)]' = (-1)^k [T_\lambda{}^k(z)]'$

Proof. Since $\tan' z = 1 + \tan^2 z$, we have $T_\lambda'(-z) = T_\lambda'(z)$. Then by the chain rule and Lemma 2.1, we have

$$\begin{aligned} [T_\lambda^k(-z)]' &= \prod_{i=1}^{k} T_\lambda'[T_\lambda{}^{i-1}(-z)] = \prod_{i=1}^{k} T_\lambda'[-T_\lambda{}^{i-1}(z)] \\ &= \prod_{i=1}^{k} T_\lambda'[T_\lambda{}^{i-1}(z)] = [T_\lambda{}^k(z)]' \end{aligned}$$

$$
\begin{aligned}
[T_{\bar\lambda}^{k}(\bar z)]' &= \prod_{i=1}^{k} T_{\bar\lambda}'(T_{\bar\lambda}^{i-1}(\bar z)) = \prod_{i=1}^{k} T_{\bar\lambda}'(\overline{T_\lambda{}^{i-1}(z)}) \\
&= \prod_{i=1}^{k} \overline{T_\lambda{}'(T_\lambda{}^{i-1}(z))} = \overline{[T_\lambda{}^{k}(z)]'}
\end{aligned}
$$

$$
\begin{aligned}
[T_{-\lambda}{}^{k}(z)]' &= \prod_{i=1}^{k} T_{-\lambda}{}'[T_{-\lambda}{}^{i-1}(z)] = \prod_{i=1}^{k} T_{-\lambda}{}'[(-1)^{i-1}T_\lambda{}^{i-1}(z)] \\
&= \prod_{i=1}^{k} T_{-\lambda}{}'[T_\lambda{}^{i-1}(z)] = (-1)^k \prod_{i=1}^{k} T_\lambda{}'[T_\lambda{}^{i-1}(z)] \\
&= (-1)^k [T_\lambda{}^{k}(z)]'.
\end{aligned}
$$

Theorem 2.3 *Suppose T_λ has an attracting periodic cycle. Then we have one of the following cases:*

1. *T_λ has only one attracting fixed point. This can happen if and only if $0 < |\lambda| < 1$, and the fixed point is the origin. The attracting region is a domain about the origin. It is the full stable set.*

2. *T_λ has only one attracting periodic cycle of even period p and the periodic points appear in symmetric pairs. The stable set consists of p Fatou components each contains a periodic point and the pre-images these Fatou components.*

3. *T_λ has exactly two attracting periodic cycles of the same period p which are symmetric with respect to the origin. The multipliers of these two attracting cycles are the same. The stable set consists of p symmetric pairs of Fatou components containing the periodic points of the two cycles and their pre-images.*

Proof. We saw in the previous section that each attractor absorbs a singular value. So it is sufficient to follow the orbits of λi and $-\lambda i$ to determine all stable attracting periodic orbits of $T_\lambda(z)$. Therefore, there are at most two attracting cycles. From Identity (2.1.1) we know that the orbits of the two asymptotic values of $T_\lambda(z)$, λi and $-\lambda i$, are $\{T_\lambda{}^k(\lambda i)\}$ and $\{-T_\lambda{}^k(\lambda i)\}$.

1. Since $T_\lambda'(0) = \lambda$, the origin is an attracting fixed point if and only if $|\lambda| < 1$. On the other hand, the origin is an attracting fixed point if and only if $\{T_\lambda{}^k(\lambda i)\}$ and $\{-T_\lambda{}^k(\lambda i)\}$ converge to the origin. Therefore there is no other attracting cycle.

2. If $\{T_\lambda{}^k(\lambda i)\}$ and $\{-T_\lambda{}^k(\lambda i)\}$ converge to the same attracting cycle with period $p > 1$ (excluding case 1), then $\{T_\lambda{}^{np}(\lambda i)\}$ and $\{-T_\lambda{}^{np}(\lambda i)\}$ converge to two non-zero periodic points which are symmetric about the origin. Similarly, all the other periodic points in the cycle are grouped by pairs under the central symmetry. Therefore the attracting cycle itself is centrally symmetric and its period p is an even number.

3. The proof for two symmetric cycles is similar to that in case 2. Suppose z_0 is a periodic point of one of these two attracting cycles. Then $-z_0$ is in the other cycle. So the two multipliers are $\rho_1 = [T_\lambda{}^p(z_0)]'$ and $\rho_2 = [T_\lambda{}^p(-z_0)]'$, then by Lemma 2.2 $\rho_1 = \rho_2$. □

In Section 4 we will prove that the period in case 3 of Lemma 2.3 may be any positive integer.

Corollary 2.4 *For $|\lambda| > 1$, suppose the period of the attracting cycle of T_λ is an odd number. Then T_λ has two attracting periodic cycles which are symmetric with respect to the origin.*

Let us consider the negatives and the complex conjugates of λ for $|\lambda| > 1$.

Proposition 2.5 *For any λ outside the unit disk, the attracting cycles of T_λ and $T_{\overline{\lambda}}$ are complex conjugates of one another; so are their multipliers.*

Proof. From (2.1.2), we have

$$\overline{T_\lambda{}^k(\lambda i)} = T_{\overline{\lambda}}^k(\overline{\lambda i}) = T_{\overline{\lambda}}^k(-\overline{\lambda} i) \quad \text{for } k \in \mathbf{Z}^+.$$

Therefore the orbits of the asymptotic values of T_λ and $T_{\overline{\lambda}}$ are conjugate. Suppose z_0 is a periodic point of T_λ with period p. Then $\overline{z_0}$ is a periodic point of $T_{\overline{\lambda}}$. Their multipliers are

$$\rho(\lambda) = [T_\lambda{}^p(z_0)]' \text{ and } \rho(\overline{\lambda}) = [T_{\overline{\lambda}}^p(\overline{z_0}))]'$$

so by Lemma 2.2 $\overline{\rho(\lambda)} = \rho(\overline{\lambda})$. □

From Proposition 2.5 and (2.1.3), we have the following:

Corollary 2.6 *Suppose the period of an attracting cycle of T_λ is an even number p. Then the attracting periodic cycles of T_λ and $T_{-\lambda}$ are identical, and the periods of the attracting cycles of $T_{-\lambda}$, $T_{\bar\lambda}$ and $T_{-\bar\lambda}$ are all equal to p.*

Proposition 2.7 *For $|\lambda| > 1$, suppose the period of the attracting cycle of T_λ is an odd number $p > 0$. Then $T_{-\lambda}$ has an attracting cycle of period $2p$ that consists of the periodic points of the two attracting cycles of T_λ. Let $\rho(\lambda)$ and $\rho(-\lambda)$ be the multipliers of the attracting cycles of T_λ and $T_{-\lambda}$. Then $\rho(\lambda)^2 = \rho(-\lambda)$.*

Proof. Let z_0 be a periodic point of T_λ with an odd period p. Then by (2.1.3)

$$T_{-\lambda}{}^{2p}(z_0) = T_\lambda{}^{2p}(z_0) = T_\lambda{}^p(T_\lambda{}^p(z_0)) = z_0.$$

Suppose that there exists an integer $q < 2p$ such that $T_{-\lambda}{}^q(z_0) = z_0$, then $q|2p$ (meaning q is a factor of $2p$).

If q were even, then by (2.1.3),

$$T_{-\lambda}{}^q(z_0) = T_\lambda{}^q(z_0) = z_0,$$

but $T_\lambda{}^p(z_0) = z_0$ and p is odd, therefore $p < q < 2p$. This cannot happen since $q|2p$ and p is odd.

If q were odd, $q|2p$ implies $q|p$. Let d be the odd factor such that $p = dq$. By (2.1.4),

$$T_{-\lambda}{}^{2q}(z_0) = T_\lambda{}^{2q}(z_0) = z_0,$$

so $p \le 2q$ that implies $d < 2$. So $q = p$. But $T_\lambda{}^p(z_0) = z_0$ and $T_{-\lambda}{}^p(z_0) = -T_\lambda{}^p(z_0) = z_0$ imply $T_\lambda{}^p(z_0) = 0$. This cannot happen either, since $|\lambda| > 1$. Therefore the period of the attracting cycle of $T_{-\lambda}$ is $2p$.

Now we prove that each periodic point of T_λ is also a periodic point of $T_{-\lambda}$. Suppose $T_\lambda{}^p(z_0) = z_0$. We will show that

$$T_{-\lambda}{}^{2p}(T_\lambda{}^k(z_0)) = T_\lambda{}^k(z_0) \quad \text{for } k = 0, 1, \cdots, p-1.$$

When k is even, from (2.1.3) we have

$$\begin{aligned} T_{-\lambda}{}^{2p}(T_\lambda{}^k(z_0)) &= T_{-\lambda}{}^{2p}(T_{-\lambda}{}^k(z_0)) = T_{-\lambda}{}^k(T_{-\lambda}{}^{2p}(z_0)) \\ &= T_{-\lambda}{}^k(z_0) = T_\lambda{}^k(z_0). \end{aligned}$$

When k is odd, from (2.1.4) we have

$$\begin{aligned} T_{-\lambda}{}^{2p}(T_\lambda{}^k(z_0)) &= T_{-\lambda}{}^{2p}(-T_{-\lambda}{}^k(z_0)) = -T_{-\lambda}{}^k(T_{-\lambda}{}^{2p}(z_0)) \\ &= -T_{-\lambda}{}^k(z_0) = T_\lambda{}^k(z_0). \end{aligned}$$

Suppose $z_0, z_1, \cdots, z_{2p-1}$ are the $2p$ periodic points of T_λ and $T_{-\lambda}$ in such an order that $z_k = T_{-\lambda}{}^k(z_0)$, $k = 1, 2, \cdots, p-1$. By Lemma 2.1 and Theorem 2.3, for $k = 0, 1, 2, \cdots, p-1$

$$z_{p+i} = T_{-\lambda}{}^{p+i}(z_0) = T_{-\lambda}{}^i[T_{-\lambda}{}^p(z_0)] = T_{-\lambda}{}^i(-z_0) = -T_{-\lambda}{}^i(z_0) = -z_i.$$

Since the multipliers of the two attracting cycles of T_λ are the same, we have

$$\rho(\lambda) = \prod_{i=0}^{p-1} T_\lambda{}'(z_i) = \prod_{i=0}^{p-1} T_\lambda{}'(-z_i).$$

For the multiplier of the attracting cycle of $T_{-\lambda}$ we have

$$\rho(-\lambda) = \prod_{i=0}^{2p-1} T_{-\lambda}{}'(z_i) = \prod_{i=0}^{p-1} T_{-\lambda}{}'(z_i) \prod_{i=0}^{p-1} T_{-\lambda}{}'(-z_i).$$

So $\rho(\lambda)^2 = \rho(-\lambda)$. □

3 Quasi-Conformal Families

We are going to divide the parameter plane into regions in which the dynamical systems are quasi-conformally conjugate. First let us recall some facts about the quadratic map $P_c(z) = z^2 + c$. Any quadratic polynomial is conjugate to a unique $P_c(z)$ by an affine transformation. The map P_c has a unique critical point at the origin that plays a crucial role in the dynamics of P_c. The Julia set $J(P_c)$ is either connected or a Cantor set depending on whether or not the orbit $\{P_c^n(0)\}$ is bounded. The parameter plane is decomposed into the Mandelbrot set

$$M = \{c \in \mathbf{C} \mid J(P_c) \text{ is connected }\}$$

and its complement $\mathbf{C} - M$. We now know that the Mandelbrot set M is connected (See [DH]). It is conjectured that the interior of M is the set

$$M' = \{c \in \mathbf{C} \mid P_c \text{ has a finite attracting periodic cycle}\}.$$

For the tangent map T_λ, an analogous set in the λ-plane is defined as

$$\mathcal{H} = \{\lambda \in \mathbf{C} \mid T_\lambda \text{ has a finite attracting periodic cycle }\}.$$

It is shown in [DK] that

Theorem 3.1 *Suppose $F(z)$ is in the class $\mathbf{M}$ of meromorphic functions. Suppose all the asymptotic values of $F(z)$ lie in the immediate attracting basin of an attracting fixed point. Then the Julia set $J(F)$ is a Cantor set and $F|J(F)$ is conjugate to the shift map on infinitely many symbols.*

By Theorem 2.3 the two asymptotic values $\pm\lambda i$ of T_λ lie in the immediate attracting basin of a fixed point if and only if $\lambda \in D^* = \{\lambda : 0 < |\lambda| < 1\}$. Therefore $J(T_\lambda)$ is a Cantor set if $\lambda \in D^*$. The fixed point is at the origin. This contrasts with the situation for polynomial or entire maps in which finite attracting fixed points always have a simply connected immediate basin. However the component D^* of $\mathcal{H}$ is more like the complement of the Mandelbrot set $\mathbf{C} - M$. For $c \in \mathbf{C} - M$, the critical value c of P_c lies in the immediate basin of the super-attracting fixed point at infinity and the Julia set $J(P_c)$ is a Cantor set.

One difference between T_λ and P_c is that there are points $\lambda \in \mathbf{C} - \mathcal{H}$ such that the Julia set of T_λ is the whole sphere; by contrast, for all points $c \in \mathbf{C}$ $J(P_c)$ is never the whole sphere. We have

Proposition 3.2 *The Julia set of T_λ is the whole sphere, $J(T_\lambda) = \hat{\mathbf{C}}$, if λ satisfies one of the following:*

1. *The asymptotic values $\pm\lambda i$ of T_λ are pre-periodic.*

2. *The asymptotic values $\pm\lambda i$ are poles or pre-images of poles of T_λ.*

Proof. We use the classification of stable domains for T_λ in Theorem 1.6. Since T_λ does not have super-attracting domain, a stable component of T_λ requires at least one asymptotic value with infinite orbit. By symmetry both asymptotic values have infinite orbits or both have finite orbits. Therefore when both asymptotic values are pre-periodic, the stable components cannot exist.

If the asymptotic values are poles or pre-images of poles of T_λ, then the orbits of both asymptotic values are terminated at infinity, hence the stable components cannot exist either. In this case the Julia set contains forward invariant subsets called Cantor bouquets which are homeomorphic to the product of a Cantor set and the line $[0, \infty)$. □

To study the quasi-conformal families of T_λ, we need the following theorem:

Theorem 3.3 (Nevanlinna) *Maps with polynomial Schwarzian derivative of degree $p-2$ are exactly those functions which have p logarithmic singularities, $a_0, \cdots, a_{p-1}$. The a_i need not be distinct. There are exactly p disjoint sectors $W_0, \cdots, W_{p-1}$ at ∞, each with angle $\frac{2\pi}{p}$ in which F has the following behavior: there is a collection of disks B_i, one around each a_i, satisfying $F^{-1}(B_i - \{a_i\})$ contains a unique unbounded component U_i contained in W_i and $F : U_i \to B_i - \{a_i\}$ is a universal covering.*

We then use this to prove

Lemma 3.4 *If F has exactly two asymptotic values which are symmetric with respect to the origin and it has no critical points, and if F fixes zero, then $c^{-1}F(cz) = T_\lambda(z)$ where λ and c are uniquely determined by $S(F)(z)$ and the asymptotic value of F.*

Proof. The two asymptotic values are logarithmic. This follows from the proof of a lemma in [GK] which says that any asymptotic value of an entire transcendental function with finitely many singular values is logarithmic.

By Nevanlinna's theorem, $F(z)$ has a constant Schwarzian derivative, so we have $S(F)(z) = k$ where k is a constant. We are going to solve this

Schwarzian differential equation. Let $f(z) = (F'(z))^{-\frac{1}{2}}$, from Equation (1) we get a linear equation

$$f''(z) + \frac{k}{2} f(z) = 0$$

Its general solution is

$$f(z) = Ae^{\sqrt{-k/2}z} + Be^{-\sqrt{-k/2}z} \quad \text{where } A \text{ and } B \text{ are constants.}$$

If f_1 and f_2 are two linearly independent solutions, their Wronskian determinant $f_1' f_2 - f_1 f_2'$ is a non-zero constant c, so $(\frac{f_2}{f_1})' = \frac{c}{f_1^2}$. Therefore, $F(z) = \frac{f_2}{f_1}$ is a solution of $S(F)(z) = k$ and the general solution can be written as

$$F(z) = \frac{Ae^{\sqrt{-k/2}z} + Be^{-\sqrt{-k/2}z}}{Ce^{\sqrt{-k/2}z} + De^{-\sqrt{-k/2}z}} \quad \text{where } AD - BC \text{ is non-zero.}$$

Since $F(0) = 0$ and the asymptotic value $\frac{A}{C} = -\frac{B}{D}$, it can be rewritten as

$$F(z) = \frac{\hat{A}e^{\sqrt{-k/2}z} - \hat{A}e^{-\sqrt{-k/2}z}}{e^{\sqrt{-k/2}z} + e^{-\sqrt{-k/2}z}} \quad \text{where } \pm\hat{A} \text{ are the asymptotic values.}$$

So we now have

$$\frac{\sqrt{\frac{-k}{2}}}{i} F\left(\frac{iz}{\sqrt{\frac{-k}{2}}}\right) = \frac{\hat{A}\sqrt{\frac{-k}{2}}}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \hat{A}\sqrt{\frac{-k}{2}} \tan z\ .$$

QED.

Theorem 3.5 *Each component of $\mathcal{H}$ corresponds to a quasi-conformal family T_λ; that is, for any λ^* and λ in the same component of $\mathcal{H}$ there exists a quasi-conformal map g such that $T_{\lambda^*} \circ g = g \circ T_\lambda$.*

Proof. Suppose $\lambda_0 \in \mathcal{H}$ and T_{λ_0} has an attracting cycle of period p. Let $z_0, z_1, \cdots, z_{p-1}$ be the p periodic points in the attracting cycle of T_{λ_0}, and let ρ_0 be the multiplier of the attracting cycle. By Theorem 1.1, there exists a local holomorphic function h_{ρ_0} with $h_{\rho_0}(z_0) = 0$ such that

$$\psi_{\rho_0} = h_{\rho_0} \circ T^p_{\lambda_0} \circ h^{-1}_{\rho_0} \quad \text{has the form } \zeta \to \rho_0\zeta$$

in some neighborhood of the origin $D_{r_0} = \{z : |z| < r_0\}$. Let $N = h_{\rho_0}^{-1}(\overline{D_{r_0}})$.

We can find an annulus

$$A_{\rho_0} = \{\zeta : r_0|\rho_0| < |\zeta| < r_0\}$$

that is mapped one-one by ψ_{ρ_0} onto the annulus

$$A'_{\rho_0} = \{\zeta : r_0|\rho_0|^2 < |\zeta| < r_0|\rho_0|\}.$$

Now pick any ρ in $D_r = \{z : 0 < |z| < r\}$ where $|\rho_0| < r < 1$. Let A_ρ be the annulus $A_\rho = \{\zeta : r_0|\rho| < |\zeta| < r_0\}$ and set $\psi_\rho(\zeta) = \rho\zeta$.

We now define a measurable structure on A_{ρ_0} as follows:
Let $f : A_{\rho_0} \to A_\rho$ be a differentiable map such that

$$\hat{\mu}(\zeta) = \frac{f_{\overline{\zeta}}}{f_\zeta}$$

is a measurable function and $||\hat{\mu}(\zeta)||_\infty \leq k < 1$. Extend $\hat{\mu}$ to D_{r_0} by $\hat{\mu}(\psi_{\rho_0}^n(\zeta)) = \hat{\mu}(\zeta)$. We pull $\hat{\mu}$ back to N by $\mu(z) = \hat{\mu} \circ h_{\rho_0}(z)$. Extend $\mu(z)$ to the inverse orbit of $h_{\rho_0}^{-1}(A_{\rho_0})$ under $T_{\lambda_0}^p$ by

$$\mu(z) = \mu(T_{\lambda_0}^{np}(z))\frac{\overline{(T_{\lambda_0}^{np})'(z)}}{(T_{\lambda_0}^{np})'(z)} \quad \text{for } z \in T_{\lambda_0}^{-np}(h_{\rho_0}^{-1}(A_{\rho_0})).$$

This defines the measurable structure in the whole attracting basin of z_0 under $T_{\lambda_0}^p$. This structure is copied to the basin of z_i by $T_{\lambda_0}^i$. If T_{λ_0} has two attracting periodic cycles, by Proposition 2.3 the two cycles are symmetric with respect to the origin. So we can copy the structure to the other attracting cycle by central symmetry. Now we have the measurable structure defined everywhere in the stable set. Set $\mu(z) = 0$ everywhere else in $\mathbf{C}$. By the measurable Riemann mapping theorem (See [AL]), we can find a quasi-conformal map g on the plane that is a solution of the Beltrami equation $g_{\overline{z}}(z) = \mu(z)g_z(z)$. The map $g(z)$ is uniquely determined up to an affine transformation, therefore it can be determined by what it does to two points. We will assume g fixes the origin and maps the two asymptotic values to a pair of points symmetric with respect to the origin. The map $g \circ T_{\lambda_0} \circ g^{-1}$ is meromorphic by construction, it fixes the origin and has exactly two symmetric asymptotic values and no critical points. By Lemma 3.4, there exist constants λ and k such that $T_\lambda = (kg) \circ T_{\lambda_0} \circ (kg)^{-1}$. One can check that λ depends continuously on ρ with $\lambda_{\rho_0} = \lambda_0$. Therefore

λ and λ_0 are in the same component Ω of $\mathcal{H}$. Since it is possible for r to be arbitrarily close to 1, the multiplier ρ induces a covering map $\rho : \Omega \to D^*$. □

For example, we know that there is a component in the right half plane for λ in which T_λ has two fixed points. We denote this component by $\Omega_{\frac{1}{1}}$. When $\lambda_0 = 1.3669957433\ldots$ in $\Omega_{\frac{1}{1}}$, $\rho(\lambda_0) = \frac{1}{2}$, and the Julia set $J(T_{\lambda^*})$ is the real axis that is the equator on the Riemann sphere, and hence connected. For $\lambda \in \Omega_{\frac{1}{1}}$ not real, $J(T_\lambda)$ is a quasi-conformal image of $J(T_{\lambda^*}) = \hat{\mathbf{R}}$ and it is called a *quasi-circle*. In Fig. 1 we show the Julia sets $J(T_{1.5+0.5i})$ and $J(T_{1.5+1.5i})$ in one periodic section of period π. They are fractals; that is, quasi-self-similar and their Hausdorff dimensions are greater than one (See [MA]).

Figure 1: Julia sets for $T_{1.5+0.5i}$ and $T_{1.5+1.5i}$

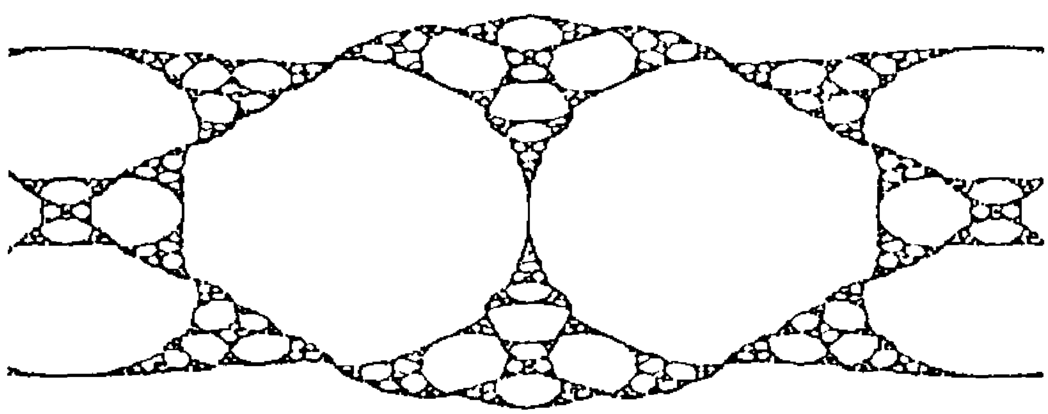


Figure 2: Julia set for $T_{2.17i}$

A picture of the Julia set is also shown for $T_{2.17i}$ that has two symmetric periodic cycles of period two.

Let us consider the λ on the boundary of the component D^*. Since the multiplier $T_\lambda'(0) = \lambda = e^{2\pi i\alpha}$, the origin is an indifferent fixed point.

When $\alpha = p/q$ with $p, q \in \mathbf{Q}/\mathbf{Z}$ the origin is a parabolic or rationally indifferent fixed point. The dynamics near the parabolic fixed points are depicted by the Leau-Fatou Flower Theorem and Theorem 1.2. We shall show in Section 5 that there are components budding from D^* at these λ. The parabolic fixed point, the origin in this case, is in the Julia set.

When α is irrational the origin is an irrational indifferent fixed point. The stability of an irrational indifferent fixed point is equivalent to linearizability in a neighborhood of that point, so there are two kinds of irrational indifferent fixed points: the Siegel points which are locally linearizable and the Cremer points which are not. A Siegel point is in the stable set and the maximal stable domain of T_λ containing this point is called a Siegel disk. A Siegel point is the center of the Siegel disk.

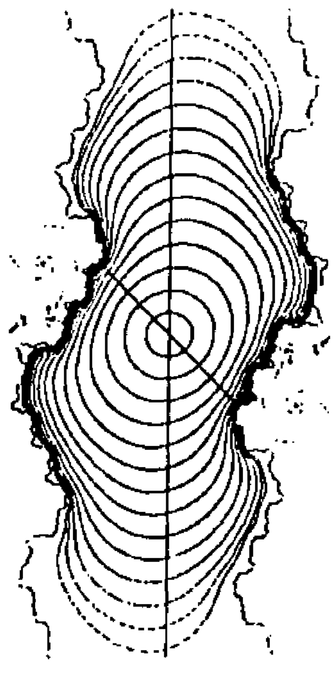


Figure 3: A Siegel disk for T_λ

For a holomorphic map $F(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$ in the neighborhood of the origin, Siegel gave a sufficient condition that α be a Diophantine number. A generalization of his theorem is the necessary and sufficient condition for linearizability given in Theorem 1.3. Therefore, for T_λ with $\lambda \in \partial D$, a Siegel disk exists if and only if the Bryuno condition holds. On the other hand, no Arnold-Herman rings exist for T_λ when $\lambda \in \partial D$. If an Arnold-Herman ring were to exist for some T_λ, then λ would have to be on the boundary of a component outside the unit disk.

An example of a Siegel disk is given in Fig. 3. We chose T_λ at $\lambda = e^{2\pi i \alpha}$ where $\alpha = (\sqrt{5}-1)/2$ is the golden mean. The picture shows the Siegel disk around the indifferent fixed point 0 of T_λ and part of the Julia set. Since the map $\tan z$ preserves the imaginary axis, T_λ maps the imaginary axis to a straight line of slope $2\pi\alpha - \frac{\pi}{2}$.

4 Computer Pictures of the λ-Plane

We know that an attracting periodic cycle of T_λ for $\lambda \in \mathcal{H}$ can be determined by following the orbit of one of the asymptotic values of T_λ. If there are two attracting cycles, then they are symmetric, and therefore of the same period.

We also know that each component $\Omega \in \mathcal{H}$ corresponds to a quasi-conformal family $\{T_\lambda\}_{\lambda \in \Omega}$, and that the periods of the attracting cycles of all T_λ in this family are the same. Therefore we can generate a computer picture of $\mathcal{H}$ by iterating $T_\lambda{}^n(\lambda i)$ to get the period of the attracting cycle of T_λ. A rudimentary numerical experiment can be carried out on the computer as follows:

First we set up a color lookup table $C(p)$ for periods p by assigning a color code to a period or several consecutive periods. Then we scan a rectangular area in λ-plane with a small grid. At each point λ, $T_\lambda(z)$ is iterated with initial condition $z = \lambda i$ and the orbit $T_\lambda{}^n(\lambda i)$ is kept in memory. At each step n we trace back by setting $p = 1, 2, \cdots$, and check if

$$|T_\lambda{}^n(\lambda i) - T_\lambda{}^{n-p}(\lambda i)| < \varepsilon?$$

If the inequality is true at some p, this p is returned as the period of $T_\lambda(z)$. If it is not true after a certain large number of steps, the number of iterations is returned. We then draw the point λ in color $C(p)$. For example, if the palette of a computer contains sixteen colors and we are going to iterate about one hundred steps at each point λ, then a color lookup table $C(p)$ can be set up as

1: White	2: LightRed	3: Red	4: Brown
5: Yellow	6: LightGreen	7: Green	8: LightCyan
14: Cyan	16: LightBlue	28: Blue	32: LightMagenta
56: Magenta	64: LightGray	112 Gray	$p > 112$: Black

At each λ, our program iterates at most 112 steps. Therefore all points λ in the parameter plane with period higher than 112 are drawn in black. For those points where the periods < 112 but not listed in the table, the nearest high order color is used. Starting from the number eight in the above table, the next number is twice the previous number. The advantage of this color table is that the period doubling can be easily seen in a 16-color picture.

The computer graph we obtained is essentially the picture of a subset $\mathcal{H}'$ of $\mathcal{H}$ where

$$\mathcal{H}' = \{\lambda \in \mathbf{C} : T_\lambda \text{ has an attracting periodic cycle with period} \leq 112\}.$$

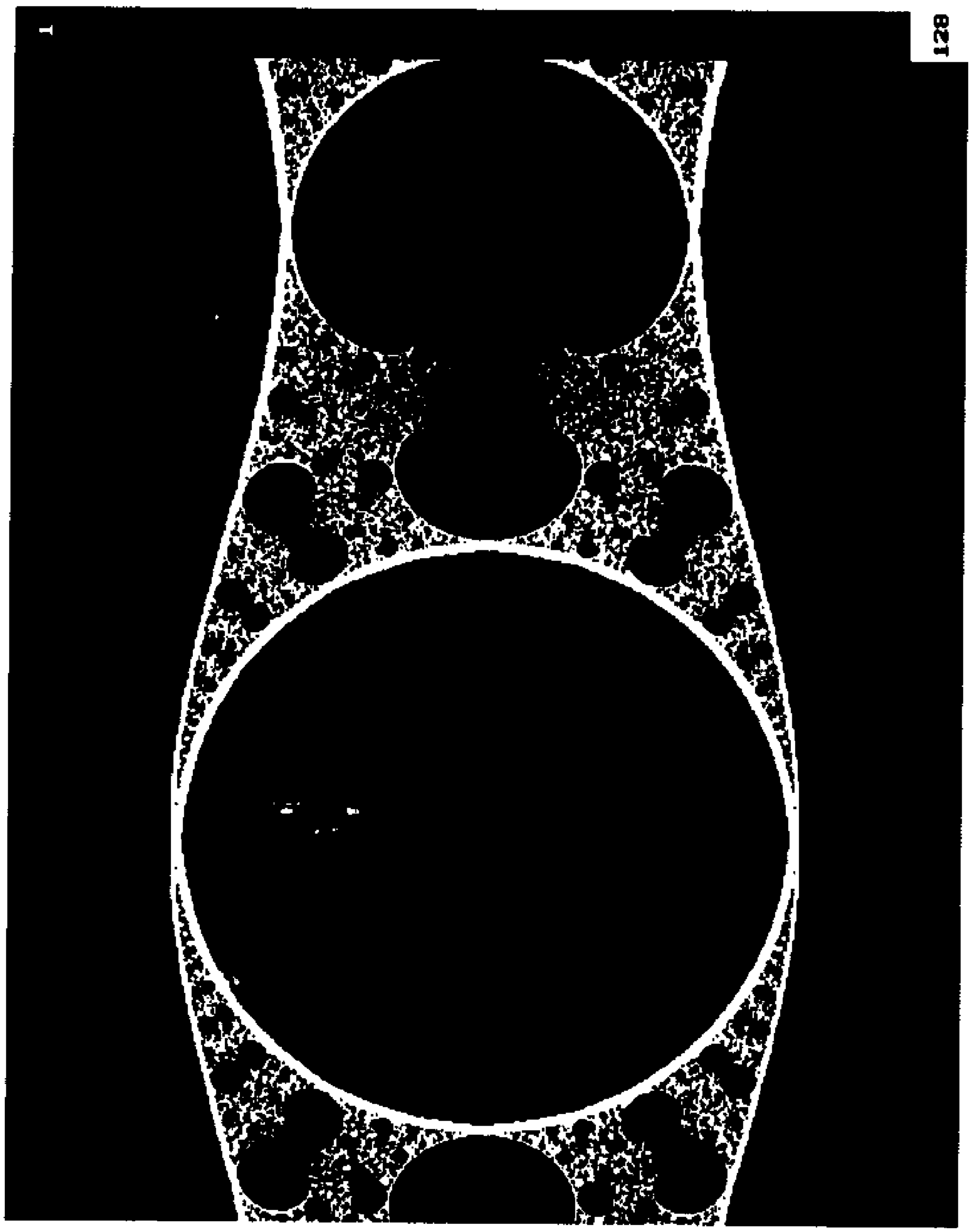


Figure 4: The unit disk and its surrounding components in $\mathcal{H}'$

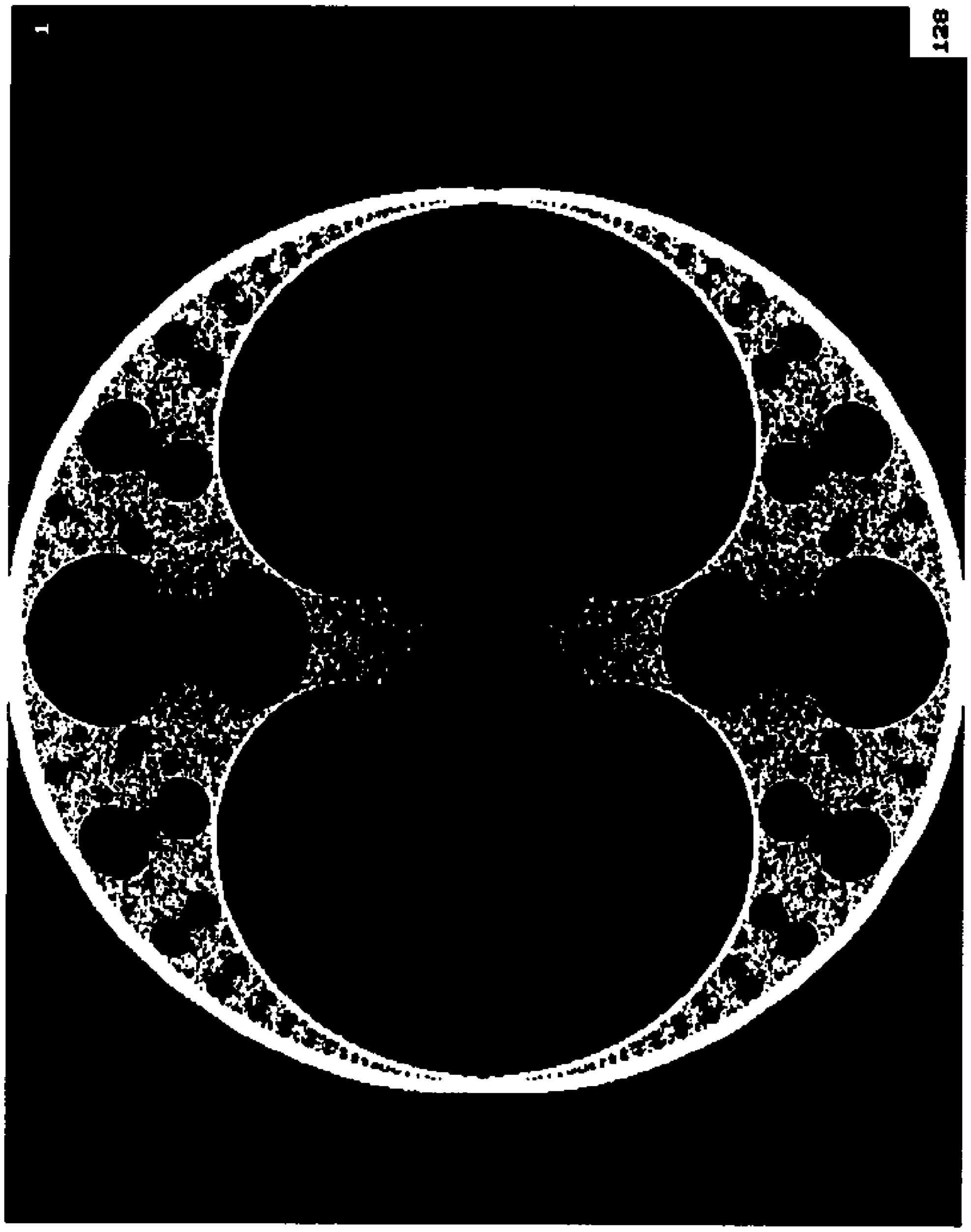


Figure 5: $\mathcal{H}'$ at ∞ (the center)

The computer picture of $\mathcal{H}$ in Fig. 4 is drawn around the origin and the one in Fig. 5 is drawn around the point at infinity.

In Fig. 4 the large disk centered at the origin consists of those λ such that T_λ has only one fixed point at the origin. By Theorem 2.3, it is the open punctured unit disk D^*.

In Fig. 5, the surrounding dark area is in the unit disk and the center is the point at infinity. Let (Ω_1, Ω_2) denote the component pair having a common boundary point at infinity. The period pair of T_λ for λ in Ω_1 and Ω_2 is $(1,2)$.

These two pictures together cover the whole sphere. We see that all the components except the unit disk appear in pairs and each pair has a common boundary point just like (Ω_1, Ω_2). We see that the colors of some component pairs are interchanged at the symmetric component pairs. Let (k,l) be the period pair of the attracting cycles of $(T_{\lambda_1}, T_{\lambda_2})$ for (λ_1, λ_2) in a component pair (Ω_k, Ω_l). Suppose k (or l) is an odd number. Then the period pair of the central symmetric component pair $(-\Omega_k, -\Omega_l)$ is (l,k). By Proposition 2.7, $l = 2k$ (or $k = 2l$ if l is odd). In fact, the period pair of any component pair (Ω_k, Ω_l) is either $(k, 2k)$ or $(2l, l)$ where k and l can be any integers. We will come back to this later.

The symmetry of $\mathcal{H}$ can be explained by the results in the previous section. From Proposition 2.5, we have:

Corollary 4.1 *The set $\mathcal{H}$ is symmetric with respect to the real axis.*

From Corollary 2.6, if for λ in a component T_λ has an attracting cycle of even period, then the four symmetric components located around $\lambda, -\lambda, \overline{\lambda}$ and $-\overline{\lambda}$ are in the same color. Namely, the periods of the attracting cycles of T_λ for λ in these four components are the same. The following is a result of Corollary 2.6 and Proposition 2.7 and shown in the computer pictures:

Corollary 4.2 *If a component Ω contains a section of the imaginary axis, then the period of T_λ for $\lambda \in \Omega$ is an even number. The component Ω itself is symmetric with respect to reflection in the imaginary axis.*

5 Bifurcations on the Unit Circle

Let $W_k = \{\lambda \in \mathbf{C} : T_\lambda$ has an attracting periodic cycle of period $k\}$

and let Ω be a connected component of W_k. We call $\Omega \in W_k$ a component of period k. We will prove in Theorem 6.2 that the multiplier ρ of the attracting cycle of T_λ for $\lambda \in \Omega$ induces a universal covering map $\rho : \Omega \to D^*$ which is lifted to a conformal isomorphism $\rho : \Omega \to \mathbf{H}$ where D^* is the open punctured unit disk $D - \{0\}$ and $\mathbf{H}$ is the upper half plane.

Definition. A boundary point $\lambda \in \partial\Omega$ is called a point of *internal angle* α if $\lambda = \rho^{-1}(e^{2\pi i\alpha})$.

The case $\Omega = D^* \subset W_1$ is special. Because the origin is the only attracting fixed point of T_λ for $\lambda \in D^*$ and $T'_\lambda(0) = \lambda$, the map $\rho : D^* \to D^*$ is the identity map. It extends continuously to the identity on the boundary. If a boundary point $\lambda = e^{2\pi i\alpha}$ has a rational internal angle $\alpha = q/p \in \mathbf{Q}/\mathbf{Z}$, the origin is called a *rational indifferent* fixed point or a *parabolic* fixed point. We want to study how the dynamics of T_λ change when λ moves inside D^* toward a boundary point of rational internal argument and then goes out. To this end, we first consider the boundary point $\lambda = e^{2\pi i\alpha}$ where $\alpha = 0$; that is, $\lambda = 1$. Since $T'_1(0) = 1$, the origin is a parabolic fixed point of $T_1(z) = \tan z$.

In a neighborhood of the origin, we have

$$T_\lambda(z) = \lambda z(1 + \frac{1}{3}z^2 + \frac{2}{15}z^4 + \frac{47}{315}z^6 + \cdots) \quad (3)$$

Therefore, for $\varepsilon > 0$ small,

$$T_1(z) = z + \frac{1}{3}z^3 + O(z^5)$$
$$T_{1\pm\varepsilon}(z) = z + z(\pm\varepsilon + \frac{1\pm\varepsilon}{3}z^2) + O(z^5).$$

When $0 < \lambda < 1$, $T_{1-\varepsilon}$ has an attracting fixed point at the origin and two real repelling fixed points $\pm p$ approximated by $\pm\sqrt{3\varepsilon/(1-\varepsilon)}$.

When $\lambda = 1$, the origin is a root of $T_1(z) = z$ with multiplicity three. The points near the origin on the real axis are moving away from it and those near the origin on the imaginary axis are moving toward it under iteration by T_1.

When $\lambda > 1$, $T_{1+\varepsilon}$ has a repelling fixed point at the origin and two imaginary attracting fixed points $\pm p$ approximated by $\pm i\sqrt{3\varepsilon/(1+\varepsilon)}$. This change, or splitting apart is called a ***saddle node bifurcation***.

Fig. 6 shows the orbits in these three cases.

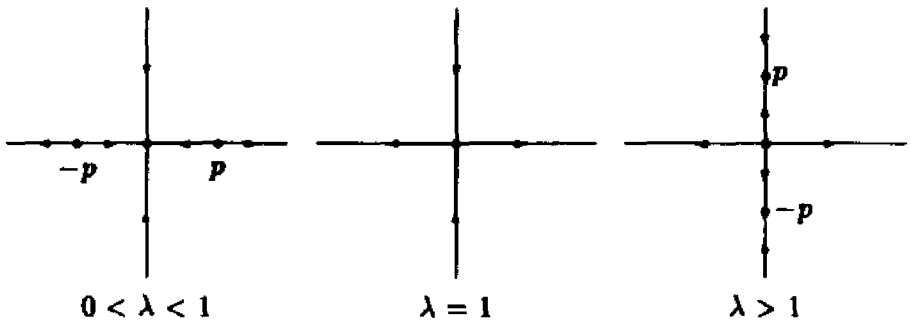


Figure 6: Phase portrait of T_λ for λ near 1

We next consider $\alpha = 1/2$; that is, $\lambda = -1$. We have

$$T_{-1}^2(z) = z + \frac{2}{3}z^3 + O(z^5).$$

For $\varepsilon > 0$ small, if we perturb $\lambda = -1$ by $-\varepsilon$, then $T_{-(1+\varepsilon)}^2(z)$ has a repelling fixed point at the origin and two attracting fixed points on the imaginary axis. These two points are the periodic points of $T_{-(1+\varepsilon)}$ with period 2. Again we have a saddle node bifurcation.

In fact, there are saddle node bifurcations at all points of rational internal angle around the unit disk. We have the following:

Theorem 5.1 *Let* $\lambda = e^{\frac{2\pi q i}{p}}$, $q/p \in \mathbf{Q}/\mathbf{Z}$, *and let* k *be 1 if* p *is even or 2 if* p *is odd. Then* $T_{(1+\varepsilon)^{1/p}\lambda}$ *has* k *attracting cycles of period* p *for* $|\varepsilon|$ *small and satisfying* $|\varepsilon - \frac{1}{kp}| < \frac{1}{kp}$.

As a corollary we see that cycles of arbitrary period exist.

Corollary 5.2 *For any integer* $p > 0$, *there exists* $\lambda \in \mathbf{C}$ *such that* $T_\lambda(z)$ *has one or two attracting cycles of period* p.

To prove Theorem 5.1, we first prove the following:

Theorem 5.3 *Let $f(z)$ be analytic in a neighborhood N of z_0 such that $f(z_0) = z_0$ and $f'(z_0) = \lambda$. Assume further that λ is a p-th root of unity. Then there is a map $\tilde{f}$, called a perturbation of f, such that $\tilde{f}$ has one repelling fixed point at z_0 and a finite number of attracting periodic cycles of period p in N.*

Proof. With a parallel translation $\zeta = z - z_0$, we may assume $f(0) = 0$. We can locally express $f(z)$ by a power series

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots.$$

Then by Theorem 1.2, there is a local homeomorphism h, $h(0) = 0$, and an integer $k \geq 1$ such that

$$h \circ f \circ h^{-1}(z) = \lambda(z + z^{kp+1}).$$

Hence the p-th iterate of $f(z)$ satisfies

$$h \circ f^p \circ h^{-1}(z) = \lambda^p(z + pz^{kp+1}) + O(z^{kp+2}) = z + pz^{kp+1} + O(z^{kp+2}),$$

and therefore zero is a root of $h \circ f^p \circ h^{-1}(z) - z = 0$ with multiplicity $kp + 1$.

Let $\tilde{f} = h^{-1} \circ (1+\varepsilon)^{1/p} h \circ f$. Suppose $|1+\varepsilon| > 1$ and $|\varepsilon|$ is small. Then we have that $h \circ \tilde{f}^p \circ h^{-1}(z) =$

$$= (1+\varepsilon)\{z + [1 + (1+\varepsilon)^k + \cdots + (1+\varepsilon)^{(p-1)k}]z^{kp+1}\} + O(z^{kp+2}) =$$

$$= z + z\{\varepsilon + (1+\varepsilon)[1 + (1+\varepsilon)^k + \cdots + (1+\varepsilon)^{(p-1)k}]z^{kp}\} + O(z^{kp+2}).$$

Therefore $h \circ \tilde{f}^p \circ h^{-1}(z) - z = 0$ has $kp + 1$ roots in the neighborhood of zero. They are zero and the kp non-zero roots ξ_i approximated by the kp roots of

$$z^{kp} = \frac{-\varepsilon}{(1+\varepsilon)[1 + (1+\varepsilon)^k + (1+\varepsilon)^{2k} + \cdots + (1+\varepsilon)^{(p-1)k}]}.$$

Compute the derivative:

$$[h \circ \tilde{f}^p \circ h^{-1}(z)]' =$$

$$1+\varepsilon+(1+\varepsilon)(kp+1)[1+(1+\varepsilon)^k+(1+\varepsilon)^{2k}+\cdots+(1+\varepsilon)^{(p-1)k}]z^{kp}+O(z^{kp+1}).$$

Evaluating it at $z = 0$ we have $[h \circ \tilde{f}^p \circ h^{-1}(z)]'_{z=0} = 1 + \varepsilon$. Therefore when $|1+\varepsilon| > 1$, zero is a repelling fixed point of $h \circ \tilde{f}^p \circ h^{-1}(z)$. For the non-zero fixed points we evaluate at ξ_i and get

$$[h \circ \tilde{f}^p \circ h^{-1}(z)]'_{z=\xi_i} = 1 - kp\varepsilon + O(\varepsilon^{1+\frac{1}{kp}}).$$

We see that if $|\varepsilon - \frac{1}{kp}| < \frac{1}{kp}$ then all the ξ_i, $1 \le i \le kp$ are attracting fixed points of $h \circ \tilde{f}^p \circ h^{-1}(z)$. Here the inequality $|\varepsilon - \frac{1}{kp}| < \frac{1}{kp}$ implies $|1 + \varepsilon| > 1$. Therefore if ε is chosen so that $|\varepsilon - \frac{1}{kp}| < \frac{1}{kp}$ the topological conjugacy h gives a one to one correspondence between the set of periodic points of $\tilde{f}^p(z)$ and those of $h \circ \tilde{f}^p \circ h^{-1}(z)$ and $\tilde{f}$ is the perturbation of f we are looking for. □

Proof of Theorem 5.1. From Expression (3), in a neighborhood of the origin we may assume that

$$T_\lambda{}^p(z) = \lambda^p z(1 + \sum_{i=1}^{\infty} a_i z^{2i}).$$

One can check by direct calculation that:
when p is even, a_i has a factor $1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{p-2}$ for $i < p/2$,
when p is odd, a_i has a factor $1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1}$ for $i < p$.

$$\lambda^p - 1 = \begin{cases} (\lambda^2 - 1)(1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{p-2}) & \text{if p is even} \\ (\lambda - 1)(1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1}) & \text{if p is odd} \end{cases}$$

Suppose $\lambda \neq \pm 1$, then $a_i = 0$ for $i < kp$. So we have

$$T_\lambda{}^p(z) = z + az^{kp+1} + O(z^{kp+3})$$

where $a \neq 1$ and $k = 1$ if p is even or $k = 2$ if p is odd. Let $g(z) = (\frac{a}{p})^{\frac{1}{kp}} z$. Then

$$g \circ T_\lambda{}^p \circ g^{-1}(z) = z + pz^{kp+1} + O(z^{kp+3}).$$

On the other hand, following the proof of Theorem 5.3, we know that for T_λ there exists a local homeomorphism h, $h(0) = 0$, and an integer $\tilde{k} \ge 1$ such that

$$h \circ T_\lambda{}^p \circ h^{-1}(z) = z + pz^{\tilde{k}p+1} + O(z^{\tilde{k}p+3}).$$

So $\tilde{k} = k$ and $h(z) = \text{constant} * z$. The perturbation $\tilde{T}_\lambda$ of T_λ is

$$h^{-1} \circ (1 + \varepsilon)^{\frac{1}{p}} h \circ T_\lambda(z) = (1 + \varepsilon)^{\frac{1}{p}} T_\lambda(z) = T_{(1+\varepsilon)^{\frac{1}{p}}\lambda}(z)$$

Therefore, an estimate of the diameter of the component attached to D^* at $e^{\frac{2\pi s}{p}}$ is

$$\{(1 + \varepsilon)^{\frac{1}{p}} e^{\frac{2\pi s i}{p}} \ : \ |\varepsilon - \frac{1}{kp}| < \frac{1}{kp}\}.$$

QED.

Definition. Given a component Ω the component attached to Ω is called a *bud component of* Ω.

Corollary 5.4 *Let α be any rational number q/p with $gcd(q,p) = 1$. Then there is a bud component attached to D^* at the point $e^{2\pi i\alpha}$. For λ in this component, the period of the attracting cycle of T_λ is p.*

We denote this component by $\Omega_{\frac{q}{p}}$ and call it the bud component of D^* of internal argument $\frac{q}{p}$. For instance, $\Omega_{\frac{1}{1}}$ is the component of period one in the right half plane attached to D^* at 1, and $\Omega_{\frac{1}{2}}$ is the component of period two in the left half plane attached to D^* at -1. $\Omega_{\frac{1}{4}}$ and $\Omega_{\frac{3}{4}}$ are components of period four attached to D^* at i and $-i$ (See Fig. 7).

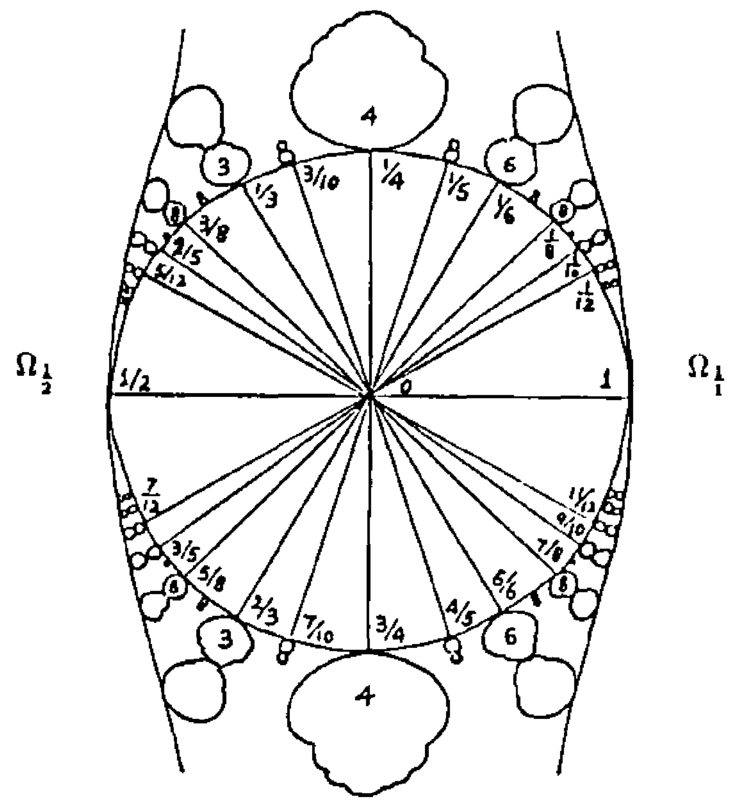


Figure 7: Internal rays in D^* of angle $\frac{q}{p}$ and components of period p

Note that the larger the period p is, the smaller the size of the bud component. However, when p is odd, the bud component of period p and the bud component of period $2p$ are about the same size. Let p be an odd number and let q_1 and q_2 be such that $gcd(p, q_1) = 1$ and $gcd(2p, q_2) = 1$. Let $\Omega_{\frac{q_1}{p}}$ be a bud component of period p attached to D^* at the point of internal argument $\frac{q_1}{p}$, and let $\Omega_{\frac{q_2}{2p}}$ be a bud component of period $2p$ at the point of internal argument $\frac{q_2}{2p}$. By Theorem 5.1 an estimate of the period p bud component $\Omega_{\frac{q_1}{p}}$ is given by

$$\{(1+\varepsilon)^{\frac{1}{p}} e^{\frac{2\pi q_1 i}{p}} : |\varepsilon - \frac{1}{2p}| < \frac{1}{2p}\},$$

while the period $2p$ bud component $\Omega_{\frac{q_2}{2p}}$ is estimated by

$$\{(1+\varepsilon)^{\frac{1}{2p}} e^{\frac{2\pi q_2 i}{2p}} : |\varepsilon - \frac{1}{2p}| < \frac{1}{2p}\}.$$

The size of the bud component of period $2p$ is slightly smaller than the one of period p; they differ at most by a factor $(1 + \frac{1}{2p})^{\frac{1}{2p}}$. Therefore when p is large they are about the same size.

We can characterize the periods of the bud components of D^* at each bifurcation point of internal angle q/p.

Proposition 5.5 *Let Ω be the component of period p attached to D^* at $e^{2\pi i\alpha}$, then the period p^* of the component Ω^* attached to D^* at $-e^{2\pi i\alpha}$ is given by*

$$p^* = \begin{cases} p/2 & \text{if } p = 2(2k+1) \\ p & \text{if } p = 4k \\ 2p & \text{if } p = 2k+1. \end{cases}$$

Proof. The period p^* of the symmetric component Ω^* to Ω is determined by the point $-e^{2\pi i\alpha}$. This point can be written as $e^{\pi i} e^{\frac{2\pi q i}{p}} = e^{\frac{2\pi i(2q+p)}{2p}}$. Let us check the internal argument $\frac{2q+p}{2p}$:
If $p = 2a$ and a is odd, then q is odd and $a+q$ is even; but $\frac{2q+p}{2p} = \frac{2q+2a}{4a} = \frac{(a+q)/2}{a}$, so $gcd(\frac{a+q}{2}, a) = 1$ and $p^* = a = p/2$.
If $p = 2a$ and a is an even number, then $\frac{2q+p}{2p} = \frac{2q+2a}{4a} = \frac{q+a}{2a}$; but $q+a$ is odd, so $gcd(q+a, 2a) = 1$ and $p^* = 2a = p$.
If p is odd, then $2q+p$ is odd and $gcd(2q+p, 2p) = 1$. So $p^* = 2p$. □
Check in Fig. 7 for some examples.

6 Internal Rays and Buds

We saw in Section 3 that the component D^* of $\mathcal{H}$ for T_λ is like the complement of the Mandelbrot set $\mathbf{C} - M$ for the quadratic maps. To study all the components in $\mathcal{H}$ we will pursue the analogy with the quadratic maps $P_c(z) = z^2 + c$. The set M' is an open set with infinitely many connected components. In [DO] Douady and Hubbard proved the following:

Theorem 6.1 *For each component W of M', the multiplier ρ induces a conformal isomorphism $\rho_W : W \to D$ that extends to a homeomorphism of $\overline{W}$ onto $\overline{D}$.*

This implies that each component W has a unique c-value such that $\rho(c) = 0$. That point is called the *center* of W. Let $\mathcal{R}(\alpha)$ denote the preimage of the radius with argument α of D under $\rho(\lambda)$; that is,

$$\mathcal{R}(\alpha) = \{\lambda \in W \mid \rho(\lambda) = re^{2\pi i\alpha},\ 0 < r < 1\}.$$

Then $\mathcal{R}(\alpha)$ is called the *internal ray of W with argument α.*

The component $D^* = D - \{0\} \in \mathcal{H}$ is special. The map $\rho : D^* \to D^*$ is the identity and the internal ray with argument α is the radius $\{re^{2\pi i\alpha} | 0 < r < 1\}$. For T_λ the set $\mathcal{H} - D^*$ is analogous to M'. We have the following result in analogy to the quadratic case:

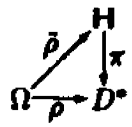
Theorem 6.2 *For each component Ω of $\mathcal{H} - D^*$, the multiplier ρ induces a conformal isomorphism $\tilde{\rho} : \Omega \to \mathbf{H}$.*

Proof. Let ρ be the multiplier of the attracting cycle of T_λ for $\lambda \in \Omega$. Following [DO] we showed in Theorem 3.5 that ρ induces a covering map $\rho : \Omega \to D^*, \lambda \mapsto re^{2\pi i\alpha}$. Let Ω be a component of period p. For any λ in Ω there exists a periodic point z_λ of period p and $\rho = (T_\lambda{}^p)'(z_\lambda)$. We solve for z_λ to get a local analytic solution, and then we substitute z_λ in $z_\lambda = T_\lambda{}^p(z_\lambda)$ to get a function $\phi(\lambda, \rho) = 0$. For example, for any $\lambda \in \Omega_1$ we can solve for z_λ from $\rho = T_\lambda{}'(z_\lambda)$ to get $z_\lambda = \pm \arctan \sqrt{(\rho - \lambda)/\lambda}$, and then substitute z_λ in $T_\lambda(z_\lambda) = z_\lambda$ to get

$$\lambda = \rho \cos^2 \sqrt{\lambda(\rho - \lambda)}.$$

Let $w = 2\pi\alpha - i \log r$. Then the exponential map $\pi : \mathbf{H} \to D^*, w \mapsto e^{iw}$ is a universal covering map, where the covering space $\mathbf{H}$ is the upper half plane.

So we can lift $\rho : \Omega \to D^*$ to get $\tilde{\rho} : \Omega \to \mathbf{H}$ which gives a commutative diagram



The universal covering map π is invariant under translation. To determine the lift $\tilde{\rho}$, we choose a point from the set $\rho^{-1}(\frac{1}{2})$ and denote it by $\lambda^0_{1/2}$. We then choose the internal ray containing $\lambda^0_{1/2}$ as the internal ray with argument zero and denote it by $\mathcal{R}(0)$. Let $\tilde{\rho}$ map $\lambda^0_{1/2}$ to $-i \log \frac{1}{2}$. Then the lift $\tilde{\rho}$ is unique. For any real number α the internal ray $\mathcal{R}(\alpha)$ of Ω is mapped to a vertical line $w = 2\pi\alpha$ in $\mathbf{H}$. One can check that $\tilde{\rho} : \Omega \to \mathbf{H}$ is a homeomorphism and so is its inverse, hence the two covering maps ρ and $\tilde{\rho}$ are equivalent. Therefore $\rho : \Omega \to D^*$ is a universal covering too. Since ρ and π are holomorphic, $\tilde{\rho}$ is a conformal isomorphism. □

Under the map $\tilde{\rho} : \Omega \to \mathbf{H}$, the boundary of Ω corresponds to the real axis. Another difference between W for quadratic maps and our Ω is that the component Ω does not have a physical center. The common end-point of all the internal rays of Ω is a boundary point of Ω that corresponds to the point at infinity under $\tilde{\rho}$. We call this point the *virtual center* of Ω. Let $S_k = \{w \in \mathbf{H} \mid 2k\pi < \text{real part of } w < 2(k+1)\pi\}$ where $k = 0, \pm 1, \pm 2, \cdots$. Every open set S_k is a vertical strip of $\mathbf{H}$. Let $V_k = \tilde{\rho}^{-1}(S_k)$. The countably many pairwise disjoint open subsets $\{V_k\}$ of Ω are obtained by cutting Ω along $\mathcal{R}(k)$ for all integers k. Then ∂V_k consists of three curves $\mathcal{R}(k), \mathcal{R}(k+1)$ and $\{\tilde{\rho}^{-1}(2\pi\alpha) \mid k < \alpha < k+1\}$ together with their end-points. These three curves are all regular simple arcs. Hence ∂V_k is a Jordan curve. By the uniformization theorem and Carathéodory theorem the conformal isomorphism $\tilde{\rho}|_{V_k}$ extends to a homeomorphism of $\overline{V}_k$ onto $\overline{S}_k$.

Corollary 6.3 *For any two components Ω_a and Ω_b of $\mathcal{H}$, there exists a conformal isomorphism $\mu : \Omega_a \to \Omega_b$.*

The boundary piece $\{\tilde{\rho}^{-1}(2\pi\alpha) \mid k < \alpha < k+1\}$ of Ω is a regular arc. It may not be regular at the end-points $\tilde{\rho}^{-1}(2k\pi)$ and $\tilde{\rho}^{-1}(2(k+1)\pi)$. For $\Omega_{\frac{1}{1}}$, one can check that $\frac{\partial \tilde{\rho}^{-1}(\alpha+i\beta)}{\partial \alpha}|_{\alpha=2k\pi,\beta=0} = 0$ for $k = \pm 1, \pm 2, \cdots$. So we have

Proposition 6.4 *The boundary of Ω is a union of piecewise smooth curves.*

The points where the boundary of Ω fails to be smooth are called the *cusps* of Ω. Each cusp is mapped under $\tilde{\rho}$ to a point $2k\pi$ for some integer k.

Fig. 8 shows three cusps of the component $\Omega_{\frac{1}{1}}$ in the right half plane. It shows the smooth boundaries of V_0, V_1, V_2 and part of V_3. The components $\Omega_{\frac{1}{1}}$ in the right half plane and $\Omega_{\frac{1}{2}}$ in the left half plane are symmetric with respect to the imaginary axis.

The computer pictures show that there are saddle node bifurcation points along the boundary of any component $\Omega \in \mathcal{H}$ and there are components attached to Ω at those points. To give proofs that these pictures are really what happens we need to generalize Theorem 5.1 and Theorem 5.3.

Proposition 6.5 *Let the multiplier ρ of a periodic cycle of period n of $f(z)$ be a p-th root of unity and let $f(z)$ be analytic in a neighborhood of the periodic cycle. Then there is a perturbation $\tilde{f}$ of f such that $\tilde{f}$ has a finite number of attracting periodic cycles of period np.*

Proof. Let z_i, $i = 1, \cdots, n$ be n periodic points of f in an attracting cycle of period n. Let N_i, $i = 1, \cdots, n$ be n disjoint neighborhoods of z_i. Then f^n is analytic in N_i, $f^n(z_i) = z_i$, $f^{n\prime}(z_i) = \rho$ and $\rho^p = 1$. By applying Theorem 5.3 to f^n in N_i, there is a perturbation $\tilde{f}^n$ of f^n that has a finite number of attracting periodic cycles of period p in N_i. But the image of a periodic point of period p under $\tilde{f}^n$ in N_i is actually mapped under $\tilde{f}$ through all n disjoint neighborhoods

$$N_i \to N_{i+1} \to N_{i+2} \to \cdots \to N_n \to N_1 \to N_2 \to \cdots \to N_i.$$

So a periodic point of $\tilde{f}^n$ with period p is a periodic point of $\tilde{f}$ with period np. $\square$

Now we can apply this to our situation.

Theorem 6.6 *For a given component Ω_n of period n, there are components Ω_{np}, called buds attached to Ω_n at the points of internal argument $\frac{q}{p}$ where $p > 0$ and $gcd(q,p) = 1$. The period of Ω_{np} is np.*

Proof. Let $\mathcal{R}(\frac{q}{p}) = \{\lambda \in \Omega_n | \rho(\lambda) = re^{\frac{2\pi qi}{p}}, 0 < r < 1\}$ be the internal ray with argument $\frac{q}{p}$. We can locate the point $\lambda^* \in \partial\Omega_n$ of rational internal argument $\frac{q}{p}$. So $\lim_{\lambda \xrightarrow{\mathcal{R}} \lambda^*} \rho(\lambda) = \lim_{\lambda \xrightarrow{\mathcal{R}} \lambda^*} \prod_{i=1}^{n} T_\lambda'(z_i) = e^{\frac{2\pi qi}{p}}$. Then by Proposition 6.5 there is a perturbation $\tilde{T}_\lambda$ of T_{λ^*}; namely the perturbation λ of λ^* such that the period of the attracting cycle of T_λ is np. All these λ's

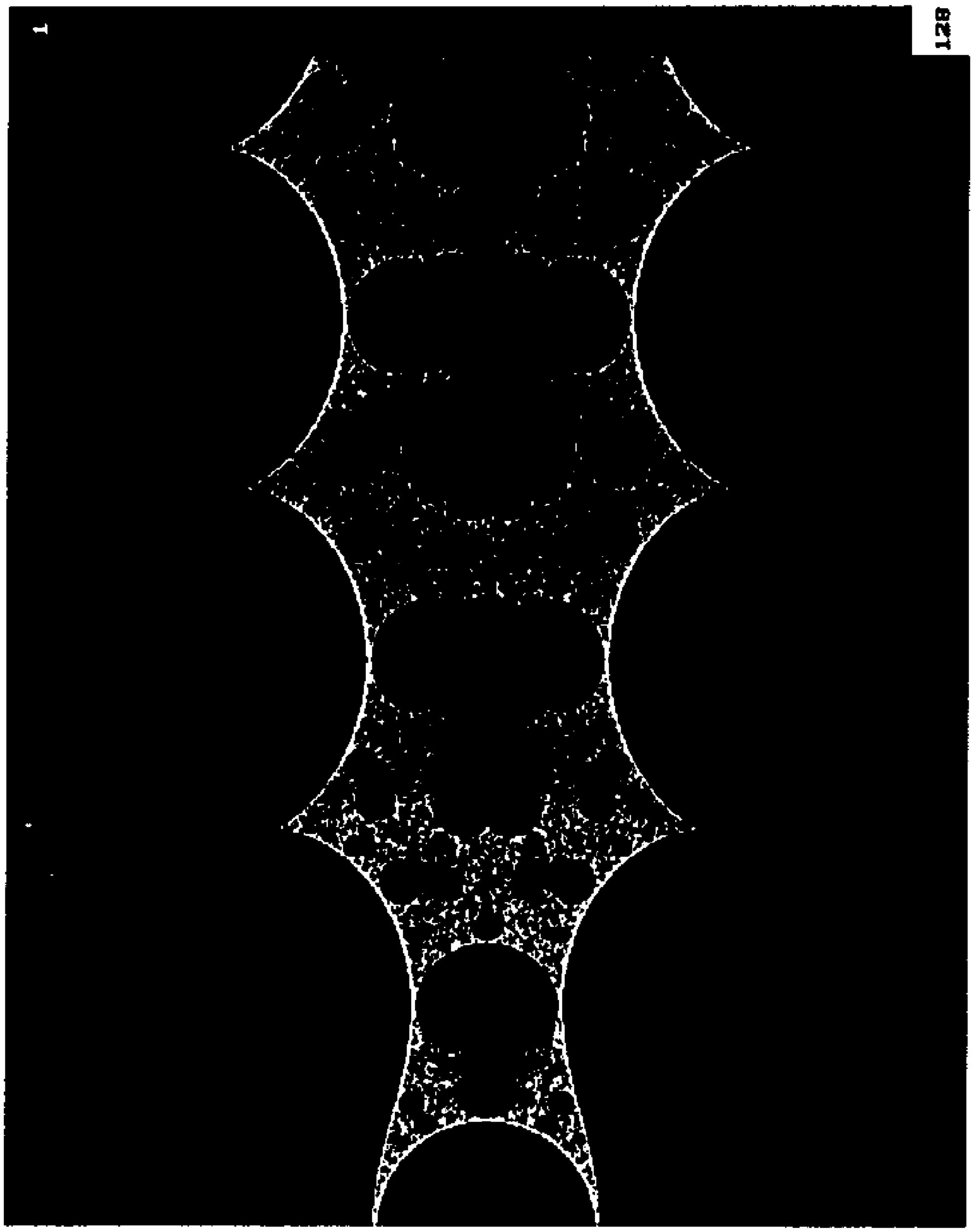


Figure 8: The boundary pieces with cusps of $\Omega_{\frac{1}{1}}$ and $\Omega_{\frac{1}{2}}$

make up a component of period np budding from Ω_n at λ^*. This component is denoted by Ω_{np}. The point λ^* where the bud Ω_{np} attached to Ω_n is called the *root* of Ω_{np}. □

Let Ω_{np} be the bud component attached to Ω_n at the boundary point λ^* of Ω_n of internal argument $\frac{q}{p}$. The point λ^* is the root of Ω_{np}. Let $\rho : \Omega_{np} \to D^*$ be the conformal projection induced by the multiplier. Following the proof of Proposition 6.5, there are n periodic points $z_i, i = 1, 2, \cdots, n$ of T_{λ^*} of period n with $\prod_{i=1}^{n} T'_{\lambda^*}(z_i) = e^{\frac{2\pi q i}{p}}$. For $\lambda \in \Omega_{np}$, in each of the n disjoint neighborhoods N_i of z_i, there are p periodic points ξ_{ij} of T_λ of period np and $\xi_{ij} \to z_i$ as $\lambda \to \lambda^*$ for $j = 1, 2, \cdots, p$. Therefore in the bud Ω_{np} the multiplier of the attracting cycle of period np satisfies

$$\rho(\lambda) = \prod_{i=1}^{n} \prod_{j=1}^{p} T'_\lambda(\xi_{ij}) \to \prod_{j=1}^{p} \prod_{i=1}^{n} T'_{\lambda^*}(z_i) = \prod_{j=1}^{p} e^{\frac{2\pi q i}{p}} = e^{2\pi q i} \text{ as } \lambda \to \lambda^*.$$

Therefore, like the cusps, the root λ^* of a component Ω is mapped under $\bar{\rho}$ to a point $2k\pi$ for some integer k. The computer pictures show that the boundary of Ω is smooth at its root λ^*. For $\Omega_{\frac{1}{1}}$ we can show that $\partial\Omega_{\frac{1}{1}}$ is smooth at its root 1.

Some components may have two roots. If a component intersects the imaginary axis, by Corollary 4.2 this component is symmetric with respect to the imaginary axis and of even period. If the root of this component is not on the imaginary axis, then the reflection of the root with respect to the imaginary axis is also a root of this component.

By the same arguments in the proof of the theorem, the buds in turn have buds. For any component Ω we can locate the bud components attached to Ω by following the internal rays of rational argument. We denoted the bud component of D^* attached to it at its boundary point of internal argument $\frac{q_1}{p_1}$ by $\Omega_{\frac{q_1}{p_1}}$. The component has period p_1. We can then locate the bud components of $\Omega_{\frac{q_1}{p_1}}$. The one attached to it at its boundary point with internal argument $\frac{q_2}{p_2}$ is denoted by $\Omega_{\frac{q_1}{p_1}\frac{q_2}{p_2}}$. This bud component has period p_1p_2. Suppose we are at a component $\Omega_{\frac{q_1}{p_1}\frac{q_2}{p_2}\cdots\frac{q_k}{p_k}}$ of period $p_1p_2\cdots p_k$ where $\frac{q_1}{p_1}, \frac{q_2}{p_2}, \cdots \frac{q_k}{p_k}$ are all in $\mathbf{Q}/\mathbf{Z}$. Following the internal ray of $\Omega_{\frac{q_1}{p_1}\frac{q_2}{p_2}\cdots\frac{q_k}{p_k}}$ of argument $\frac{q_{k+1}}{p_{k+1}} \in \mathbf{Q}/\mathbf{Z}$, we can locate a bud component attached to it at the point of internal argument $\frac{q_{k+1}}{p_{k+1}}$. The period of this bud is $p_1p_2\cdots p_kp_{k+1}$. This gives us a way to code the components.

However, we may need to locate a component attached to the current component at the virtual center. In this case, we will see in Proposition 7.1 that the period of that component is either twice or half the period of the current component. Since all components except the component D^* appear in pairs and each pair has a unique virtual center, such that the two components are attached at their shared virtual center, coding the virtual centers gives a coding of the component pairs.

7 Virtual Centers and Coding

We have seen in the computer pictures that all components except the unit disk appear in pairs and each component pair has a unique common boundary point. We know that the point at infinity is the unique common boundary point of the two components $\Omega_{\frac{1}{1}}$ and $\Omega_{\frac{1}{2}}$ and it is the common end-point of all internal rays in both components where the multipliers vanish. It is called the *virtual center of the component pair.* This component pair serves as a model of all component pairs. We have the following:

Proposition 7.1 *Let $B_n = \{\lambda \in \mathbf{C} | T_\lambda{}^{n-1}(\lambda i) = \infty\}$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then B is the set of the virtual centers of all component pairs. For any component pair (Ω_a, Ω_b), suppose their periods are (p, q) and $p < q$. Then for $\lambda \in \Omega_a$, T_λ has two attracting cycles of period p, and for $\lambda \in \Omega_b$, T_λ has one attracting cycle of period $q = 2p$.*

The set B consists of the point at infinity and all λ such that the asymptotic values $\pm\lambda i$ are poles or pre-images of poles of T_λ. For $\lambda \in B$, we know that $J(T_\lambda) = \hat{\mathbf{C}}$ by Corollary 3.2.

Proof. If $n = 1$, B_1 contains only one point, the point at infinity; it is the unique common boundary point of the component pair $(\Omega_{\frac{1}{1}}, \Omega_{\frac{1}{2}})$. To see the period doubling bifurcation near the point at infinity we follow the angle zero internal ray in the component pair that is on the real axis. For $\lambda > 1$, T_λ has two symmetric attracting fixed points $\pm yi$ on the imaginary axis. We rewrite $\lambda \tan(yi) = yi$ as $\tanh y = \frac{y}{\lambda}$. When λ moves along the positive real axis toward the point at infinity, the two fixed points $\pm yi$ move along the imaginary axis to infinity (See Fig. 9. Left), and the two multipliers $\lambda \tan'(\pm yi)$ go to zero. For $\lambda < -1$ on the real axis, T_λ has two symmetric attracting periodic points $\pm yi$ on the imaginary axis. When λ moves along the negative real axis toward the point at infinity, the two period two points $\pm yi$ move along the imaginary axis to infinity (See Fig. 9. Right), and the multiplier $\lambda^2 \tan'(yi) \tan'(-yi)$ goes to zero. The point at infinity is the virtual center of $(\Omega_{\frac{1}{1}}, \Omega_{\frac{1}{2}})$. Since the multiplier is zero there, we can think of the limit point of $\pm yi$ at infinity as a "super-attracting" fixed point with multiplicity two (See Fig. 9. Center). Of course, there is no basin of attraction for this fixed point. It splits up into two attracting fixed points as λ moves to the right half plane, or into two attracting periodic points of period two as λ moves to the left half plane.

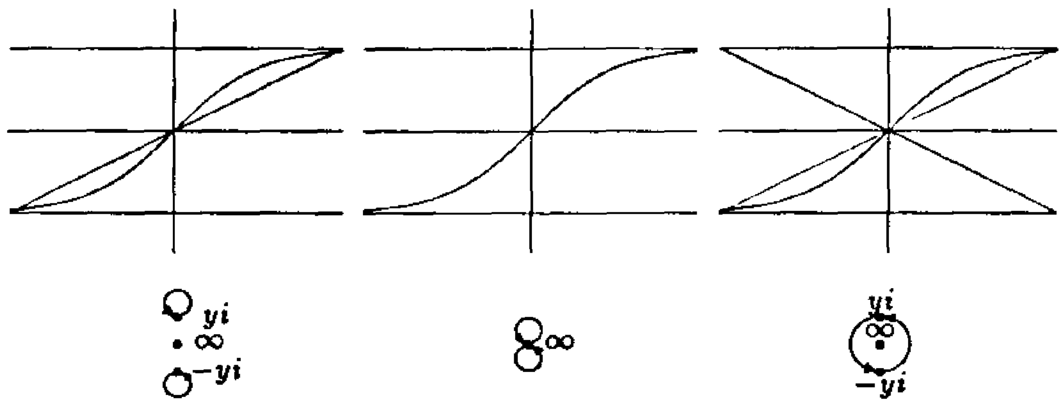


Figure 9: The solutions of $\tanh y = \frac{y}{\lambda}$

When $n = 2$, the set B_2 consists of those λ such that $T_\lambda(\lambda i) = \infty$; that is,

$$B_2 = \{\lambda_m = \frac{(2m+1)\pi i}{2} | m = 0, \pm 1, \pm 2, \cdots\}.$$

The points in B_2 are evenly distributed on the imaginary axis at intervals of π. They accumulate at infinity; $\lim_{m\to\infty} \lambda_m = \infty$. Each $\lambda_m \in B_2$ is a unique common boundary point of a component pair with period (2,4). For example, $\lambda_0 = \frac{\pi i}{2}$ is the unique common boundary of the component pair $(\Omega_{\frac{1}{1}\frac{1}{2}}, \Omega_{\frac{1}{4}})$ with period (2,4). Again we follow the angle zero internal ray in the component pair that is now on the imaginary axis. Let $\lambda = li$. For $l > \frac{\pi}{2}$ so that $\lambda \in \Omega_{\frac{1}{1}\frac{1}{2}}$, T_λ has two symmetric attracting cycles of period two. Each cycle consists of a periodic point x on the real axis and a periodic point yi on the imaginary axis. We rewrite $\lambda \tan(\lambda \tan yi) = yi$ as $\tan(l \tanh y) = \frac{y}{-l}$. As l goes to $\frac{\pi}{2}$ from above, the two periodic points $\pm yi$ of period two go to infinity along the imaginary axis (See Fig. 10. Left), and the two period two points $\pm x$ go to the two poles $\pm\frac{\pi}{2}$ along the real axis. Meanwhile the two multipliers $\lambda^2 \tan'(\pm x) \tan'(\pm yi)$ go to zero. When $1 < \lambda < \frac{\pi}{2}$, $\lambda \in \Omega_{\frac{1}{4}}$, T_λ has an attracting periodic cycle of period four that consists of two real periodic points $\pm x$ and two pure imaginary periodic points $\pm yi$. As l goes to $\frac{\pi}{2}$ from below, the two periodic points $\pm yi$ of period four go to infinity along the imaginary axis (See Fig. 10. Right), and the two period four points $\pm x$ go to the two poles $\pm\frac{\pi}{2}$ along the real axis. The multiplier $\lambda^4 \tan' x \tan'(yi) \tan'(-x) \tan'(-yi)$ goes to zero. The point $\lambda_0 = \frac{\pi i}{2}$ is the virtual center of $(\Omega_{\frac{1}{1}\frac{1}{2}}, \Omega_{\frac{1}{4}})$. At the virtual center λ_0, we can

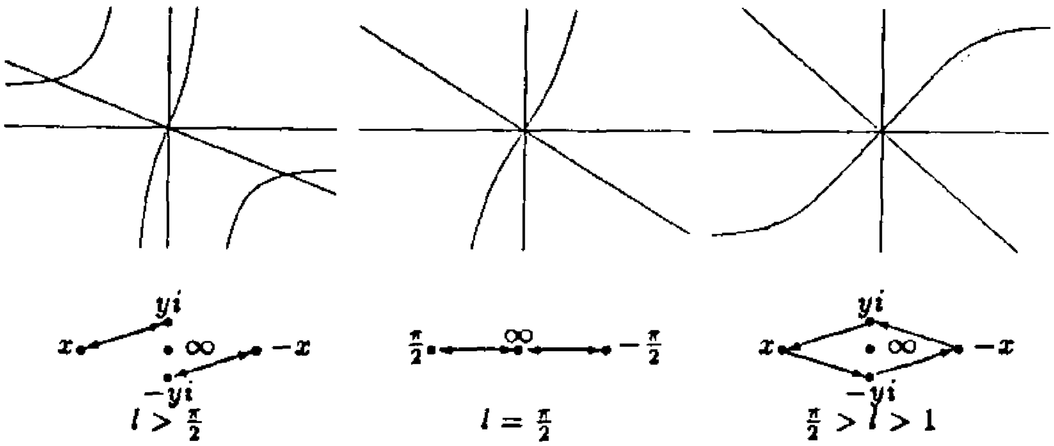


Figure 10: The solutions of $\tan(l \tanh y) = \frac{y}{-l}$

think of the point at infinity as an attracting periodic point of period two with multiplicity two (See Fig. 10. Center). It splits up into two attracting periodic points of period two as l moves up, or into two attracting periodic points of period four as l moves down.

In general, when $n > 2$, the set B_n consists of the common boundary points of all component pairs with period $(n, 2n)$. For any given component pair (Ω_a, Ω_b) with period $(n, 2n)$. The virtual center can be located by tracking along an internal ray as the modulus of the multiplier goes to zero. For $\lambda \in \Omega_a$, suppose $T_\lambda{}^p(z_\lambda) = z_\lambda$. Since $\tan' z = 1 + \tan^2 z$, $T_\lambda'(z) = \lambda\{1 + [\frac{1}{\lambda}T_\lambda(z)]^2\}$. Then the multiplier of the attracting cycle can be written as

$$[T_\lambda{}^n(z_\lambda)]' = \prod_{k=1}^{n} T_\lambda'[T_\lambda{}^{k-1}(z_\lambda)] = \lambda^n \prod_{k=1}^{n} \{1 + [\frac{1}{\lambda}T_\lambda{}^k(z_\lambda)]^2\}.$$

In the polar coordinate system $[T_\lambda{}^n(z_\lambda)]' = re^{2\pi i\alpha}$. Suppose λ moves along the internal ray $\mathcal{R}$ of angle α to a limit point λ^* as $r \to 0$; that is, $\lim_{\lambda \xrightarrow{\mathcal{R}} \lambda^*}[T_\lambda{}^n(z_\lambda)]' = 0$. Therefore the point λ^* is the virtual center of Ω_a. For some $1 \leq k \leq n$ we have

$$\lim_{\lambda \xrightarrow{\mathcal{R}} \lambda^*} 1 + [\frac{1}{\lambda}T_\lambda{}^k(z_\lambda)]^2 = 0,$$

so

$$\lim_{\lambda \xrightarrow{\mathcal{R}} \lambda^*} T_\lambda{}^k(z_\lambda) = \pm\lambda^* i \quad \text{and} \quad \lim_{\lambda \xrightarrow{\mathcal{R}} \lambda^*} T_\lambda{}^{k-1}(z_\lambda) = \infty.$$

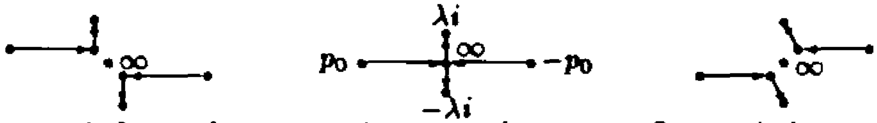
Therefore,

$$\lim_{\lambda \xrightarrow{\mathbb{R}} \lambda^*} T_\lambda{}^{n-k}(\pm\lambda i) = \lim_{\lambda \xrightarrow{\mathbb{R}} \lambda^*} T_\lambda{}^{n-k}\,[T_\lambda{}^k(z_\lambda)] = \lim_{\lambda \xrightarrow{\mathbb{R}} \lambda^*} T_\lambda{}^n(z_\lambda) = \lim_{\lambda \xrightarrow{\mathbb{R}} \lambda^*} z_\lambda = z_{\lambda^*}$$

and

$$\lim_{\lambda \xrightarrow{\mathbb{R}} \lambda^*} T_\lambda{}^{n-1}(\pm\lambda i) = \lim_{\lambda \xrightarrow{\mathbb{R}} \lambda^*} T_\lambda{}^{k-1}\,[T_\lambda{}^{n-k}(\pm\lambda i) = \lim_{\lambda \xrightarrow{\mathbb{R}} \lambda^*} T_\lambda{}^{k-1}(z_\lambda) = \infty.$$

So $T_{\lambda^*}{}^{n-2}(\pm\lambda^* i)$ is a pole. The period doubling bifurcation near the common boundary point λ^* of Ω_a and Ω_b is shown in Fig. 11. When λ reaches



Two period n cycles $\lambda \in \Omega_a$ | $\pm p_0$ are poles $\lambda = \lambda^*$ | One period $2n$ cycle $\lambda \in \Omega_b$

Figure 11: The attracting cycles of T_λ

the virtual center λ^* from inside of both Ω_a and Ω_b, the multipliers vanish. In this sense the point at infinity plays the role of a super-attracting periodic point of period n with multiplicity two. It has the two asymptotic values as its images and the two poles $\pm p_0$ as its pre-images, so it looks like a saddle node. We know that the Julia set of T_{λ^*} is the whole sphere. However the two cycles of T_{λ^*}, $(\infty, \pm\lambda^*, T_{\lambda^*}(\pm\lambda^*), \cdots, T_{\lambda^*}{}^{n-2}(\pm\lambda^*))$ have a property of super-attracting cycles; that is, each contains a singular value, the asymptotic value.

We continue our development of the coding for the virtual centers with $n = 3$. The set B_3 consists of λ such that $T_\lambda{}^2(\lambda i) = \infty$; that is, $T_\lambda(\lambda i)$ is a pole. Let $p_m = \frac{(2m+1)\pi}{2}$ where $m = 0, \pm1, \pm2, \cdots$. Then λ can be determined by solving the equation $T_\lambda(\lambda i) = p_m$. The numerical solution $\lambda = (x, y)$ can be obtained by iterating $\lambda_{k+1} = \varphi(\lambda_k) = (\varphi_1(\lambda_k), \varphi_2(\lambda_k))$. From $\lambda_{k+1} = -i \arctan \frac{p_m}{\lambda_k}$ we get the following formula

$$\begin{cases} \varphi_1(x,y) &= \frac{1}{2}\ln \frac{\sqrt{4x^2y^2+(p_m^2+x^2-y^2)^2}}{x^2+(p_m+y)^2} \\ \varphi_2(x,y) &= \frac{1}{2}\arctan \frac{-2xp_m}{x^2+y^2-p_m^2} + n\pi \qquad n \in \mathbb{Z} \end{cases}$$

An index n has been introduced to indicate the branch of the solution of the arctangent. So we have

$$B_3 = \{\lambda_{m,n} \in \mathbb{C} \,|\, \lambda \text{ is a solution of } T_\lambda(\lambda i) = p_m, m = 0, \pm1, \pm2, \cdots\}.$$

When m goes to infinity, δ_m goes to infinity and $T_\lambda(\lambda i) = \delta_m$ becomes the formula used to define B_2. So we have

$$\lim_{m\to\infty} \lambda_{m,n} = \lambda_n \in B_2.$$

The $\lambda_{m,n}$'s are the centers of the component pairs of period (3,6). These pairs are located in the wedges of the component pairs of period (2,4), and they accumulate at the centers λ_n of the component pairs (2,4). For example, we used the iteration to get

$$\begin{array}{ll}
\lambda_{0,0} = (0.600791, 1.035870) & \lambda_{0,1} = (0.510192, 3.237279) \\
\lambda_{1,0} = (0.326474, 1.493972) & \lambda_{1,1} = (1.014660, 4.130291) \\
\lambda_{2,0} = (0.199102, 1.544433) & \lambda_{2,1} = (0.659176, 4.586390) \\
\lambda_{3,0} = (0.142588, 1.557564) & \lambda_{3,1} = (0.451528, 4.662387) \\
\lambda_{4,0} = (0.110996, 1.562848) & \lambda_{4,1} = (0.344152, 4.685051) \\
\cdots & \cdots
\end{array}$$

We can see that $\lim_{m\to\infty} \lambda_{m,0} = \frac{\pi}{2} i$ and $\lim_{m\to\infty} \lambda_{m,1} = \frac{3\pi}{2} i$.

B_4 consists of all the solutions of $T_\lambda{}^2(\lambda i) = \delta_{n_1}$. One more index has been introduced and λ_{n_1,n_2,n_3} is the center of the component pair (4,8). Similarly we have

$$\lim_{n_1\to\infty} \lambda_{n_1,n_2,n_3} = \lambda_{n_2,n_3} \in B_3 \qquad \lim_{n_1,n_2\to\infty} \lambda_{n_1,n_2,n_3} = \lambda_{n_3} \in B_2.$$

In general, any point in B_p can be coded as $\lambda_{n_1,n_2,\cdots,n_{p-1}}$ where the indices are determined by the branches of the solutions in all intermediate steps. The point $\lambda_{n_1,n_2,\cdots,n_{p-1}}$ is in the neighborhood of $\lambda_{n_{p-1}}$. When $n_{p-1} \geq 0$ (resp. < 0), the B_p point is in the upper half (resp. lower half) plane. Each B_p point is the virtual center of a component pair with period $(p, 2p)$. This gives us a way to code all the component pairs in the λ-plane.

8 Farey Series and the Ordering of Buds

For any component $\Omega \in \mathcal{H}$ of period n, all the bud components of Ω along $\partial\Omega$ are arranged in a certain order. This order is determined by the periods np of the bud components which are in turn determined by the internal arguments q/p of the root points and the period n of Ω. The computer picture shows that there is a certain pattern of the locations, the periods and the sizes of all the bud components along the boundary of a component Ω. In this section we are going to give a description of this pattern as an ordering of the buds.

Let $\tilde{\rho} : \Omega \to \mathsf{H}$ be the conformal isomorphism induced by the multiplier map ρ. We define the boundary pieces G_k of Ω as

$$G_k = \{\tilde{\rho}^{-1}(2\pi\alpha) \mid k < \alpha < k+1\} \quad \text{for all integers } k.$$

Since the covering map $\pi : \Omega \to \mathsf{H}$ commutes with translation, and $\tilde{\rho} : \Omega \to \mathsf{H}$ is the lift of the multiplier map $\rho : \Omega \to D^*$. We have the following.

Corollary 8.1 *For each component Ω, the orderings of the periods of the bud components along G_k for all integers k are the same. Moreover for all components with the same period, the orderings of the periods of the bud components along the boundaries of these components are the same.*

For example, there are only two components of period one in $\mathcal{H}$ which are D^* and $\Omega_{\frac{1}{1}}$. The ordering of the periods of bud components along the boundary of D^*, $\{e^{2\pi i\alpha} \mid 0 < \alpha < 1\}$ and the boundary pieces G_k of $\Omega_{\frac{1}{1}}$ for all integers k are all the same. These can be seen in Fig. 8. However, in terms of sizes of the buds, the patterns of the arrangement of the buds are different along D^* and G_k.

By observation, we found an equivalent relation between the *Farey series* and the periods and sizes of the bud components along each boundary piece G_k. We then drew a diagram by packing circles on the interval $[0,1]$ based on the Farey series. The pattern of the circle packing matches that of the arrangement of the buds, their periods and their sizes along G_k.

Definition. The Farey addition of the fractions is defined as

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$$

Suppose $\frac{a}{b} < \frac{c}{d}$. Then $\frac{a}{b} \oplus \frac{c}{d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$; that is, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

We will construct the Farey series for the components of period one, $\Omega_{\frac{1}{1}}$ and D^*. To depict the ordering along the boundary piece G_k of $\Omega_{\frac{1}{1}}$, we construct the Farey series between 0 and 1 in the following way:
Given two points $\frac{a}{b} < \frac{c}{d}$, we mark the point $\frac{a}{b} \oplus \frac{c}{d}$. Starting with 0 and 1 which are written as $\frac{0}{1}$ and $\frac{1}{1}$, we mark the point $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$, then we mark $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$ and $\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$, and then we mark $\frac{0}{1} \oplus \frac{1}{3} = \frac{1}{4}$ and $\frac{2}{3} \oplus \frac{1}{1} = \frac{3}{4}$; repeat this procedure to get our points of the first stage:

$$\frac{0}{1} \cdots \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6} \cdots \frac{1}{1}$$

At the second stage new points are obtained by applying the Farey addition to all marked points in the first stage. We get

$$\cdots \frac{2}{11}\ \frac{2}{9}\ \frac{2}{7}\ \frac{2}{5}\ \frac{3}{5}\ \frac{5}{7}\ \frac{7}{9}\ \frac{9}{11} \cdots$$

The process of adding new points by the Farey addition is then repeated on all the marked points from the previous stages to get the points of Stage 3:

$$\cdots \frac{3}{17}\ \frac{3}{16}\ \frac{3}{14}\ \frac{3}{13}\ \frac{3}{10}\ \frac{3}{11}\ \frac{3}{8}\ \frac{3}{7}\ \frac{4}{7}\ \frac{5}{8}\ \frac{7}{10}\ \frac{8}{11}\ \frac{10}{13}\ \frac{11}{14}\ \frac{13}{16}\ \frac{14}{17} \cdots$$

Proceed inductively to get the Farey series between 0 and 1. The points of the Farey series are exactly those in the Farey diagram defined in [SE]. We will show that any given rational number has a position in the Farey series.

Proposition 8.2 *Let p and q be two integers such that $0 < q < p$. Then the rational number q/p is in the Farey series between 0 and 1.*

Proof. We will start with the first stage points $\frac{1}{n+1}$ and $\frac{1}{n}$. For a fixed integer $q > 0$, then $\frac{1}{n+1} < \frac{q}{p} < \frac{1}{n}$ if and only if $nq < p < (n+1)q$. We want to show that all possible rational numbers with numerator q; that is, all rationals $\frac{q}{nq+1}, \frac{q}{nq+2}, \cdots, \frac{q}{nq+q-1}$ are in the Farey series. We now write down the points at each stage:

$$\frac{1}{n+1} \qquad\qquad\qquad\qquad \frac{1}{n}$$

$$\frac{2}{2n+1}$$

$$\frac{3}{3n+2} \qquad\qquad \frac{3}{3n+1}$$

$$\frac{4}{4n+3} \qquad \frac{5}{5n+3} \qquad \frac{5}{5n+2} \qquad \frac{4}{4n+1}$$

$$\frac{5}{5n+4}\ \ \frac{7}{7n+5}\ \ \frac{8}{8n+5}\ \ \frac{7}{7n+4}\ \ \frac{7}{7n+3}\ \ \frac{8}{8n+3}\ \ \frac{7}{7n+2}\ \ \frac{5}{5n+1}$$

$$\cdots\ \ \cdots\ \ \cdots\ \ \cdots\ \ \cdots\ \ \cdots\ \ \cdots\ \ \cdots\ \ \cdots$$

The Farey series between $\frac{1}{n+1}$ and $\frac{1}{n}$ forms an infinite binary tree. For each fixed integer q, all rationals $\frac{q}{nq+1}, \frac{q}{nq+2}, \cdots, \frac{q}{nq+q-1}$ can be located in this binary tree. The proof follows by induction on q. □

There is a one-one correspondence between the marked points in Farey series and the points on the boundary piece G_0 of $\Omega_{\frac{1}{1}}$ with rational internal argument. In fact, the periods of the bud components along this boundary piece are arranged in the same order as the denominators of the Farey series points between 0 and 1. Therefore the ordering itself is symmetric with respect to the middle point $\frac{1}{2}$. These can be seen in the computer pictures. Each stage of the marked points in the Farey series corresponds to a level of bud components attached to $\Omega_{\frac{1}{1}}$.

The relation between the Farey series and the periods and sizes of the bud components becomes clearer in the diagram of circle packing of the interval $[0,1]$ in Fig. 12. First we draw an interval $[0,1]$, and then at each rational point $q/p \in [0,1]$ we draw a circle with radius $\frac{r}{p^2}$. The interval $[0,1]$ is the common tangent line of the infinitely many circles. The radius is selected from $2 < r < 3$ and $r = 2.2$ in Fig. 12.

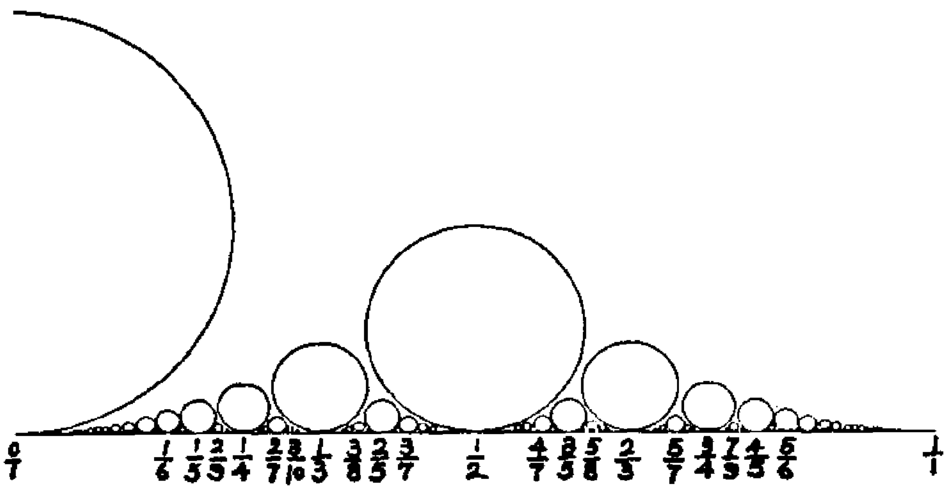


Figure 12: The circle packing on $[0,1]$

The punctured open disk D^* is the other component of period one. We will generate a circle packing diagram to match the pattern of the locations, periods and sizes of the bud components of D^* in the first quadrant. The diagram for the ordering in the other three quadrants can be obtained from this one. Let q/p be a point in the Farey series between 0 and $\frac{1}{4}$ that corresponds to a point on the unit circle in Quadrant I. Then $\frac{1}{2} - \frac{q}{p}$ corresponds to the reflection of that point in Quadrant II, $\frac{1}{2}+\frac{q}{p}$ corresponds to the rotation of that point in Quadrant III, and $1-\frac{q}{p}$ corresponds to the complex conjugate of that point in Quadrant IV. See Fig. 7 for some examples.

We generate the Farey series between $\frac{0}{1}$ and $\frac{1}{4}$ by choosing the first stage points as

$$\frac{0}{1} \cdots \frac{1}{14}\ \frac{1}{12}\ \frac{1}{10}\ \frac{1}{8}\ \frac{1}{6}\ \frac{1}{4}$$

and then use the Farey addition to get the second stage points

$$\cdots \frac{2}{30}\ \frac{2}{26}\ \frac{2}{22}\ \frac{2}{18}\ \frac{2}{14}\ \frac{2}{10}$$

cancelling by 2 we get

$$\cdots \frac{1}{15}\ \frac{1}{13}\ \frac{1}{11}\ \frac{1}{9}\ \frac{1}{7}\ \frac{1}{5}$$

The union of the first stage and second stage points is exactly the set of the first stage points smaller than $\frac{1}{4}$ in the previous construction of the Farey series for $\Omega_{\frac{1}{1}}$. It is more convenient however, not to reduce the fractions but to proceed with these points to get the third stage points:

$$\cdots \frac{3}{34}\ \frac{3}{32}\ \frac{3}{28}\ \frac{3}{26}\ \frac{3}{22}\ \frac{3}{20}\ \frac{3}{16}\ \frac{3}{14}$$

Continue inductively without reduction of fractions to get the Farey series. Using the same method we can show that any given rational number has a position in the Farey series. One can check that this is the same as the one constructed with different first stage points. The circle packing diagram is shown in Fig. 13. The interval $[0,\frac{1}{4}]$ is the common tangent line of the sequence of circles. At each rational point $\frac{q}{p}\in[0,\frac{1}{4}]$, we draw a circle with radius $\frac{r}{p^2}$ where $\frac{q}{p}$ is a point in the Farey series (without reduction of the fractions). This diagram corresponds to the first quadrant in Fig. 7.

Given a component Ω_n of period n, by Theorem 6.6, if a bud component is attached to Ω_n at the point of internal argument $\frac{q}{p}$ where $p>0$ and $gcd(q,p)=1$, then the period of that bud is np. Therefore the Farey series and the corresponding circle packing diagram can be adapted to the study of the periods and sizes of buds of a component Ω_n of any period n.

For quadratic maps P_c, Yoccoz gave an estimate of the size of the buds of a component of the Mandelbrot M. Let Ω be a component of the Mandelbrot set M and suppose $b\in\partial\Omega$ is a point of internal argument $q/p\in\mathbb{Q}/\mathbb{Z}$. The connected component of M attached to Ω at b is called the limb of M relative to Ω of internal argument q/p and denoted by $M_{\Omega,q/p}$. Yoccoz proved the following.

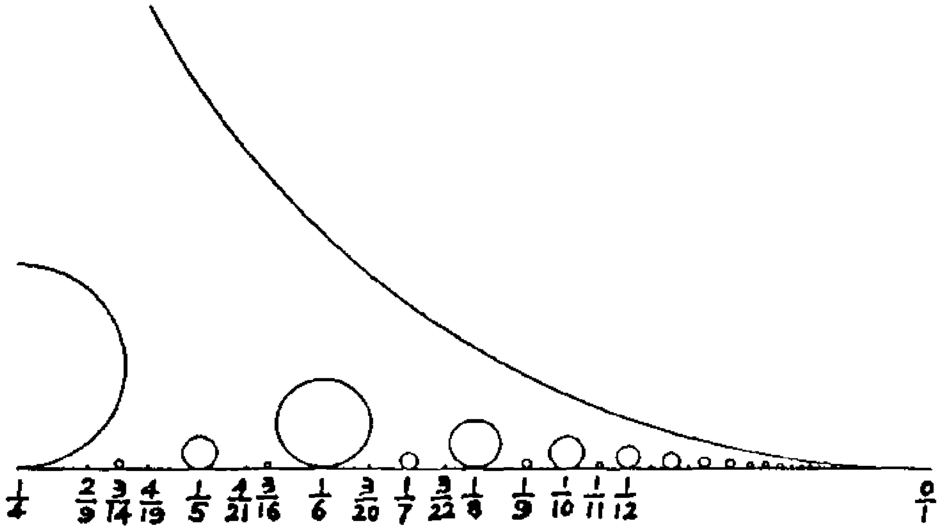


Figure 13: The circle packing on $[0, \frac{1}{4}]$

Theorem 8.3 *For any component Ω of M, there exists a constant C_Ω such that the Euclidean diameter of $M_{\Omega, q/p}$ is $\leq \frac{C_\Omega}{p}$ for all $q/p \in \mathbf{Q}/\mathbf{Z}$.*

For T_λ we obtained an under-estimate in Theorem 5.1 for the bud components attached to D^*. Our computer pictures indicate that there should be an estimate on the size of the components of $\mathcal{H}$ analogous to Yoccoz' result.

Conjecture. For any component Ω in $\mathcal{H}$, there exists a constant C_Ω such that the Euclidean diameter of the bud component attached to Ω at the point of internal argument q/p is $\leq \frac{C_\Omega}{p}$ for all $q/p \in \mathbf{Q}/\mathbf{Z}$.

Along each boundary piece G_k of Ω_1, bud components of any period exist. These boundary pieces G_k are disjoint open subsets and accumulate at infinity. So we have

Proposition 8.4 *Let N be a neighborhood of infinity. Then for any integer $p > 0$ there is $\lambda \in N$ such that T_λ has an attracting cycle of period p.*

This reminds us of the classical theorem of Weierstrass. By that theorem the holomorphic function T_λ comes arbitrarily close to any complex value in every neighborhood of infinity for infinity is an essential singularity of T_λ. This is analogous to a situation for the quadratic map $P_c(z)$ described in [TL]; that is, there exists a similarity between the Julia set in the z-plane and the Mandelbrot set in the c-plane around every Misiurewicz point.

Appendix: A Note on Generating Computer Pictures

In Section 4 we described a procedure to generate computer pictures of T_λ in the λ-plane. The inequality used in that procedure to get the period of an attracting cycle may sometimes give us wrong information.

Recall that at the n^{th} iterate we checked if the inequality

$$|T_\lambda{}^n(\lambda i) - T_\lambda{}^{n-p}(\lambda i)| < \varepsilon?$$

is true for some p, and if so this p is returned as the period of the attracting cycle of T_λ.

A problem occurs near all saddle node bifurcation points. Since these bifurcation points are dense on boundaries of all components, the problem is spread over the boundary area of all components. We explain this problem by an example. For λ in component $\Omega_{\frac{1}{1}}$, we know that T_λ has two attracting fixed points, and there is a bud component attached to $\Omega_{\frac{1}{1}}$ at the boundary point of internal angle $\frac{1}{3}$. For λ in this bud component T_λ has two attracting periodic cycles with period three. Let ξ be an attracting fixed point of T_λ for $\lambda \in \Omega_{\frac{1}{1}}$ near the boundary point of internal angle $\frac{1}{3}$. Then the absolute value of the multiplier is close to one, and the orbit of the asymptotic value $O^+(\lambda i)$ approaches ξ very slowly. The orbit looks more like a period three cycle. For some given ε we will have

$$|T_\lambda^n(\lambda i) - T_\lambda^{n-3}(\lambda i)| < \varepsilon \quad \text{but} \quad |T_\lambda^n(\lambda i) - T_\lambda^{n-1}(\lambda i)| > \varepsilon.$$

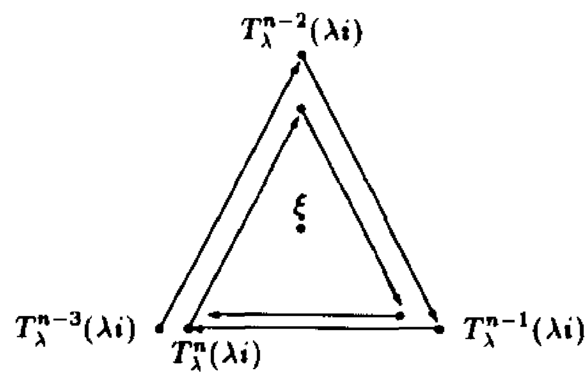


Figure 14: A false period 3 cycle obtained at a fixed point

This situation is shown in Fig. 14. There are many ways to fix the problem. We used a very simple way to do it. When the inequality holds for some p, instead of using the returned p, we replace the ε with a much larger ε^* and then trace back by setting $p = 1, 2, \cdots, n-1$ to check the inequality again. Suppose in our example shown in Fig. 14 the orbit of the asymptotic value $\{T_\lambda^k(\lambda i)\}$ is in the neighborhood $|z - \xi| < \varepsilon^*/2$ of ξ, then $|T_\lambda^n(\lambda i) - T_\lambda^{n-1}(\lambda i)| < \varepsilon^*$. Hence we get the correct period of the attracting cycle. Our computer experiment shows that this method works in all components in the λ-plane.

Our computer pictures were generated on an IBM PC/AT compatible computer with a VGA monitor and a math coprocessor, and these pictures were printed out on an HP LaserJet II printer. Both are currently the most widely available equipment around. Since there are a lot of graphics tools libraries for the C language, we chose it as our programming language. However the Fortran language is much more convenient to compute the complex functions. So the programs we wrote are in mixed languages. Most of our programs are large and take dozens of hours of computing time on a personal computer. As an example we will list the source code of the program which computes the Siegel disk at the golden mean. It will take several minutes to draw a picture of the Siegel disk and part of its boundary near the point at infinity on a VGA monitor.

The complex function subroutine is kept in file LTAN.FOR and the C main program is kept in file SIEGEL.C. The compilers we used here are Microsoft C V5.0 and Microsoft Fortran V5.0. To compile the programs and link them with large model 80X87 floating point library, type in:

```
FL /AL /c  LTAN.FOR
CL /AL /c  SIEGEL.C
LINK /NOE  SIEGEL+LTAN,,,LLIBC7+LLIBFOR7
```

Then type in SIEGEL to run the program.

The fortran subroutine in the file LTAN.FOR:

```
C       Fortran subroutine: T=L*tan(Z)
        SUBROUTINE LTAN (L,Z,T)
        COMPLEX*16 L [FAR, REFERENCE]
        COMPLEX*16 Z [FAR, REFERENCE]
        COMPLEX*16 T [FAR, REFERENCE]
        T = L*CDSIN(Z)/CDCOS(Z)
        END
```

The C main program in the file SIEGEL.C:

```
#include < stdio.h >
#include < math.h >
#include < graph.h >

extern void fortran ltan (double far *, double far *, double far *);

double far L[2], U[2], V[2];  /* Declare the complex numbers */

main()
{
int i, j, k=1;
double x, y, v0, v1, d;

x=4.*atan(1.)*(sqrt(5.)-1.);  /* Let α be the golden mean */
L[0]=cos(x); L[1]=sin(x);  /* Let lambda=exp(2πiα) */
x=639./8.8; y=479./6.6;  /* window area= [-4.4 : 4.4, -3.3 : 3.3] */
_setvideomode(_VRES16COLOR);  /* Set VGA 16-color mode */
for(i=1; i<=15; i++){  /* The foliation and the boundary */
   U[1]=U[0]=.25+i/18.;  /* Pick a start point for a circle */
   if(i==15){  /* Pick the start point for boundary */
      U[0]=-L[1]; U[1]=L[0];  /* Get the inverse of the asymptotic value */
      k=10;
      }
   _setcolor(i);  /* Select a color for a circle */
   for(j=0; j<200+i*k*100; j++){  /* Iteration # depends on start point */
      ltan(L,U,V);  /* Iterate lambda*tan(z) */
      d=V[0]*V[0]+V[1]*V[1];
      v0=V[0]/d; v1=-V[1]/d;  /* Get the inverse of the point */
      _setpixel((short) ((4.4+v0)*x), (short) ((3.3-v1)*y));  /* Draw a point */
      _setpixel((short) ((4.4-v0)*x), (short) ((3.3+v1)*y)); /* & its reflection */
      U[0]=V[0]; U[1]=V[1];
      }
   }
printf("%c",7);  /* Beep when it is done */
while(getch()!='q'){};  /* Wait for pressing Q-key to quit */
_setvideomode(_TEXTC80);  /* Set text mode */
}
```

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