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SYMMETRIC DISPERSION RELATIONS AND THEIR APPLICATIONS

by

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ABSTRACT

A study of the representation of the scattering amplitudes $G_k(s,t,u)$ having the symmetric properties in s,t,u is made and unsubtracted Dispersion Relations for $G_1(s,t,u)$ and $G_2(s,t,u)$ are derived. Then we use these Dispersion Relations to extract information about the $\rho\pi\pi$ Regge residue function, $\gamma_\rho(t)$. Our results are consistent with the result obtained by E.P. Tryon, who based his analysis on experimental data of Hyams et. al. and of Durusoy et. al., namely $\gamma_\rho(t)$ is found to vanish at $t_z = -0.41\ \mathrm{Gev}^2$.

This technique of using homogeneous variables is further applied in deriving new Dispersion Relations for $\pi\pi$ amplitudes which are symmetric in t and u only, and we use these Dispersion Relations in unsubtracted form to study the value of the amplitudes at the symmetry point.

CHAPTER I

INTRODUCTION TO DISPERSION RELATIONS

A. Dispersion Relations in High Energy Physics

Dispersion relations as applied to high energy physics was first suggested by Kronig (1). His suggestion was carried through and developed by Gell-Mann (2, 3), Goldberger (4-7), Mandelstam (8-12), among others (13-15).

Working from the causality postulate, Gell-Mann, Goldberger and Thirring (2) proved that the amplitude for the forward scattering of photons by Nucleons satisfies a dispersion relation

$$\mathrm{Re}\ A(\nu) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \quad \frac{\mathrm{Im}\ A(\nu')}{\nu' - \nu}$$

Goldberger (4) then generalized such dispersion relations to the scattering of particles with mass, such as pions, by Nucleons. Goldberger's proofs of his relations were mathematically unsound, as they involved integrations over divergent exponentials but rigorous proofs were late given by Symanzik (16-19) and by Bogoliubov (20).

Crossing, analyticity and unitarity are believed to be powerful constraints on the behaviour of the scattering amplitude and may alone define a unique nontrivial S-matrix (20). However, the unitarity condition involves a summation over an infinite number of states, and approximations will have to be made if we are to obtain a soluble system of equations. In the case of π-N scattering for example, until we are considerably above threshold, we know from experience that the amplitude for pion production is very small, so that it is a good approximation to restrict the summation to intermediate pion-nucleon states only . Once the energy becomes too high, above about 600 Mev, the approximation fails. In general, the approximation to the unitarity condition will consist in restricting the summation to states with small numbers of particles.

The difficulty in the proofs arises from the contribution of the unphysical region to the dispersion integral (21-25). For forward scattering of massless particles there is no unphysical region and the proof is straightforward. For forward scattering of pions by nucleons, the intermediate one-nucleon state is in the unphysical region (as it occurs at a value of s below $(M+\mu)^2$, (M = nucleon mass, μ = pion mass) and more work is involved. On extending the relations to non-forward scattering one also encounters a continuum in the unphysical region, as there is a range of values of s for which the cosine of the angle is less than -1. Lehman (24-25) has shown that the dispersion relations are valid provided that

$$0 < -t < \frac{32}{3} \frac{2M+\mu}{2M-\mu} \mu^2$$

The limitation on the value of t is rather due to the method of proof and not to a breakdown in the validity of the dispersion relations (26,27). However, when $|t|$ increases sufficiently, the range of cosine becomes large enough to cause the breakdown of the proofs we have at present.

Dispersion relations for nucleon-nucleon scattering have not yet been proved rigorously, even in the forward direction. The difficulty is due to a large unphysical region in the dispersion integral. As usual, one has to determine the imaginary part of the scattering amplitude at negative kinetic energies by means of the crossing relations. The crossed reaction is nucleon-anti-nucleon scattering, whose imaginary part will be non-zero above the energy of the lowest intermediate with the same quantum numbers. This is the two-pion state, even the one-pion state will contribute a pole term. However, the physical region only begins at an energy of two nucleon masses. There is thus a large intermediate unphysical region which contributes to the dispersion integral. In the hypothetical case where the mass of the nucleon is less than $(1 + \sqrt{2})$ times the mass of the pion, the dispersion relations for nucleon-nucleon scattering have been proved rigorously by Bremermann, Oehme and Taylor (14). Proofs of dispersion relations for scattering of strange particles on nucleons are subject to the same limitations as for nucleon-nucleon scattering.

B. Analytic Properties of Dispersion Relations

Lehman (24,25) has shown that the amplitude $A(s,t)$ for π-N scattering is an analytic function of s, the square of the energy, in the cut plane when the momentum transfer t is fixed, he also showed that the amplitude has analytic properties as a function of t when s is kept fixed.

The scattering amplitude is an analytic function of $Z = \cos\theta$, in an ellipse surrounding the physical region in the complex Z-plane (13). The foci are at ± 1, and the major axis Z_o is given by

$$Z_o = \left[1 + \frac{2\mu^2(2M+\mu)}{q^2\{s - (M-2\mu)^2\}}\right]^{\frac{1}{2}}$$

where

$$Z = \cos\theta = \frac{t}{2q^2} + 1$$

q: center of mass momentum

$0 \leq q^2 \leq -t/4$ corresponds to $-\infty \leq \cos\theta \leq -1$

Notice that, as s approaches infinity, the ellipse closes down on the physical region.

The absorptive part is an analytic function of Z in a larger ellipse and its major axis is $2Z_o^2 - 1$.

According to a well-known theorem in analysis, the Legendre expansion for a function may be used within its ellipse of analyticity, therefore we may use the Legendre expansion to calculate Im A(s, t), even in the unphysical region, provided

$$0 < -t < \frac{32}{3}\,\frac{2M+\mu}{2M-\mu}\,\mu^2$$

This is deduced from the condition $|1 + t/2q^2| \leq 2Z_o^2 - 1$

C. Function of Two Variables

Applications of analytic properties of scattering amplitudes to dynamical calculations require a knowledge of these properties as a function of two variables, s and t, both complex. In general such properties have not been proved rigorously. The one exception is pion-pion scattering, where certain rigorous properties have been obtained (Mandelstam 1960 b).

Mandelstam (1959) has given the general form of the dispersion relation (8,12):

$$A(s,t) = \frac{(s-s_o)(t-t_o)}{\pi}\int ds'dt' \frac{A_{13}(s',t')}{(s'-s_o)(s'-s)(t'-t_o)(t'-t)}$$

$$+ \frac{(t-t_o)(u-u_o)}{\pi}\int dt'du' \frac{A_{23}(t',u')}{(t'-t_o)(t'-t)(u'-u_o)(u'-u)}$$

$$+ \frac{(s-s_o)(u-u_o)}{\pi}\int ds'du' \frac{A_{12}(s',u')}{(s'-s_o)(s'-s)(u'-u_o)(u'-u)}$$

$$+ \frac{s-s_o}{\pi}\int ds' \frac{f_1(s')}{(s'-s_o)(s'-s)}$$

$$+ \frac{u-u_o}{\pi}\int du' \frac{f_2(u')}{(u'-u_o)(u'-u)}$$

$$+ \frac{t-t_o}{\pi}\int dt' \frac{f_3(t')}{(t'-t_o)(t'-t)} + \lambda$$

where $A_{13}(s',t')$, $A_{23}(s',t')$, $A_{12}(s',t')$ are the so-called "double spectral functions" and are connected with the absorptive parts A_1, A_2 and A_3 through the equations:

$$A_1 = f_1(s) + \frac{t-t_o}{\pi} \int dt' \frac{A_{13}(s',t')}{(t'-t_o)(t'-t)}$$

$$+ \frac{u-u_o}{\pi} \int du' \frac{A_{12}(s',u')}{(u'-u_o)(u'-u)}$$

D. Axiomatic Scattering Amplitude

Before going on with the discussion of $\pi\pi$ scattering, let us have a short review of some of the general properties of scattering amplitudes resulting from axiomatic field theory. Using the asymtotic condition, a scattering amplitude $A(p_1, p_2, p_3, p_4)$ is expressed as the Fourrier transform of the vacuum expectation value of retarded or advanced products of field operators. This expression allows an off-shell continuation ($p_i^2 \neq m_i^2$) of $A(p_1, p_2, p_3, p_4)$ and the locality of the field operators implies analyticity properties of the scattering amplitude (28-33). Assuming that the retarded or advanced products are tempered distributions implies that $A(p_1, p_2, p_3, p_4)$ is polynomially bounded as the p_i's tend to infinity. Now, if one uses the method developed by Bros, Epstein and Glaser (33,34), one first determines a domain of holomorphy Δ of A as a function of the vectors p_i. Then one determines the intersection D of Δ with the complex mass shell ($p_i^2 = m_i^2$). D can be described in terms of the usual invariants s, t and u.

Let $A^I(s,t,u)$ be the isospin I s-channel amplitude. The relevant properties of D are contained in the following statements:

a) Every physical point of the s-channel ($s \geq 4$, $t \leq 0$, $u \leq 0$) is on the boundary of D. $A^I(s,t,u)$ is the boundary value of an analytic function $F^I(s,t,u)$, holomorphic in D:

$$A^I(s,t,u) = \lim F^I(s+i\epsilon\ ,\ t,\ u-i\epsilon\ )$$

b) D ensures crossing symmetry. There is a path in D connecting any physical point of the s-channel to any physical point of the t- or u-channel. One has the crossing relation:

$$\lim\ F^I(s\ ,\ t+i\epsilon\ ,\ u-i\epsilon\ ) = \sum_{I'} C_{II'}\ A^{I'}(s,t,u)$$

$$\text{if } t \geq 4,\ s \leq 0,\ u \leq 0$$

where

$$C_{II'} = \begin{matrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{matrix}$$

c) $A^I(s,t,u)$ verifies fixed-t dispersion relations for $-t_o \leq t \leq 0$. This means:

$$\left\{ s,t \;\middle|\; t=t_1,\ -t_o \leq t_1 \leq 0,\ s \neq 4+\lambda\ ,\ s \neq -t_1-\lambda\ ,\ \lambda \geq 0 \right\} \subset D$$

d) for $s \geq 4$, $F^I(s,t,u)$ is holomorphic in the Lehmann ellipse $E(s)$; i.e.

$$\{s,t \mid s=s_1, \quad s_1 \geq 4, \quad t \in E(s_1)\} \subset D$$

$E(s)$ = ellipse of t-plane, foci at $t = o$ and $t = -(s-4)$, semi major axis $a(s) = \frac{1}{2}(s-4)\left[1 + \frac{256}{s(s-4)}\right]^{\frac{1}{2}}$

e) $F^I(s,t,u)$ is holomorphic in s and t in a neighbourhood of every point of the domain described in c):

$$\{s,t \mid |s-s_1| < \rho(s_1,t_1), \; |t-t_1| < \rho(s_1,t_1), \; -t_0 \leq t_1 \leq o, \; s_1 \neq 4+\lambda, \; s_1 \neq -t_1-\lambda, \; \lambda \geq o\} \subset D$$

The properties listed until now are consequences of locality and the mass spectrum of the pion-pion systems. As it was shown by Martin, the holomorphy of $F^I(s,t,u)$ as a function of the two complex variables s and t , in an arbitrarily small neighbourhood of the points (s_1,t_1) leads to a finite and large domain of holomorphy in s and t. This result is obtained from the positivity properties implied by unitarity. These positivity properties are readily seen if we write the s-channel partial wave expansion of $A^I(s,t,u)$:

$$F^I(s,t,u) = \sum_{\ell=0}^{\infty} (2l + 1)\, F^I_l(s)\, P_\ell\left(1 + \frac{2t}{s-4}\right)$$

As a consequence of unitarity:

$$\operatorname{Im} A^I_l(s + i\varepsilon) = \sqrt{\frac{s-4}{s}} \left|A^I_l(s)\right|^2$$

for $s \geq 4$, we have $\operatorname{Im} A^I_l(s + i\varepsilon) > 0$. Furthermore:

$$\frac{\partial^n}{\partial z^n} P_\ell(z) > 0 \qquad \text{for } Z \geq 1$$

Therefore $$\frac{\partial^n}{\partial t^n} \operatorname{Im} A^I(s + i\varepsilon, t) \geq 0$$

for $s>4$ and those positive values of t for which the partial wave expansion converges (i.e. at least for t in the Lehmann ellipse $E(s)$).

E. Pion-Pion Scattering

The fact that the two crossed reactions are the same as the direct reaction means that we do not get coupling of the equations with those for other processes, and the absence of spin also simplifies the problem greatly.

Let us start with a general equation. Assuming charge independence, we write the complete amplitude as

$$A(s,t,u)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} + B(s,t,u)\,\delta_{\alpha\gamma}\,\delta_{\beta\delta} + C(s,t,u)\,\delta_{\alpha\delta}\,\delta_{\beta\gamma}$$

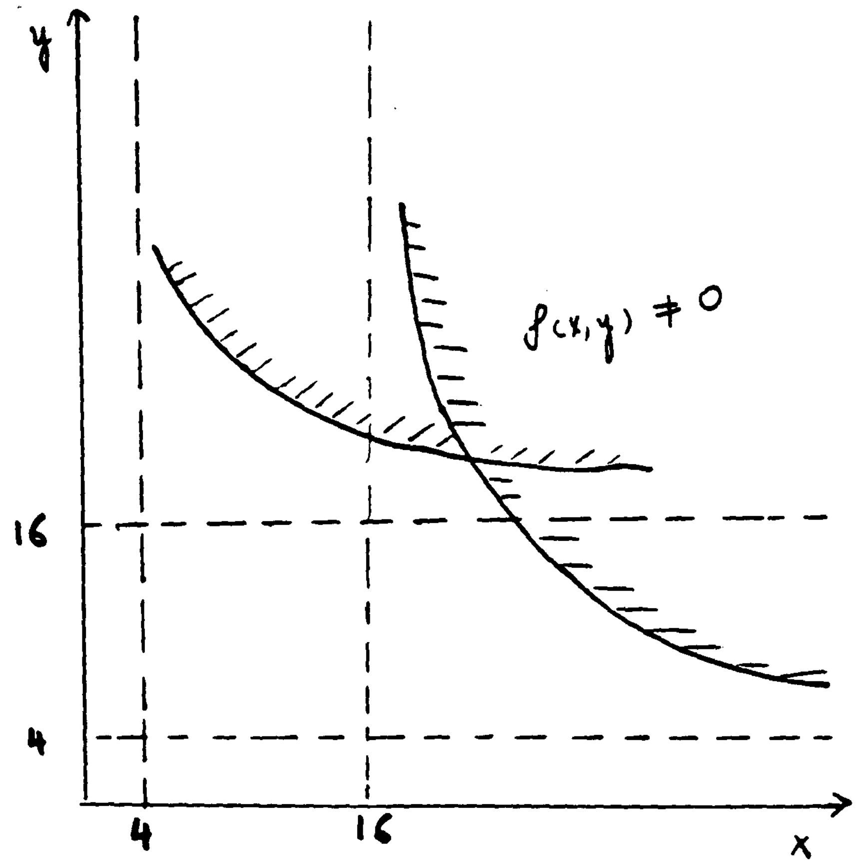
The connection between A, B, C and the amplitudes for well-defined total I-spin in the s-channel turns out to be:

$$A^0 = 3A + B + C$$
$$A^1 = B - C$$
$$A^2 = B + C$$

In $\pi\pi$ scattering, there are no poles and the continuous spectrum in the Mandelstam representation begins at $4m_\pi^2$ in each of the s,t,u. variables. The region in which any double spectral function $\rho(x,y)$ fails to vanish turns out to be bounded by the curves

$$x = \frac{16y}{y-4} \qquad x>y$$

$$y = \frac{16x}{x-4} \qquad y>x$$


y
f(x,y) ≠ 0
16
4
4
16
x

The large distance to the boundary from the corner y=x=4 is associated with the absence of three-pion vertex. If multiparticle singularities - in this case due to 4π and higher configurations - are consistently to be ignored, then in the $\pi\pi$ problem one need not worry at all about the double spectral functions, but can represent the entire amplitude by one dimensional dispersion integrals. Such would not be the case if a three-pion vertex existed, since the boundary then would be the dashed curve.

Introducing the CM variables v and $\cos\theta$, the crossing conditions become

$$A(v, \cos\theta) = A(v, -\cos\theta)$$

$$B(v, \cos\theta) = C(v, -\cos\theta)$$

$$A(v, \cos\theta) = B(v', \cos\theta')$$

where:

$$v = \frac{s-4m_\pi^2}{4}$$

$$v' = \frac{v}{2}(1 + \cos\quad) - (v + 1)$$

$$\cos\theta' = \frac{\frac{v}{2}(1 + \cos\theta) + (v + 1)}{\frac{v}{2}(1 + \cos\theta) - (v + 1)}$$

There is a point of maximum symmetry at s=t=u or

$v=v'=v_0=-2/3\ m_\pi^2$ $\cos\theta = \cos\theta' = 0$

At this point A, B, C are all real and equal to each other. It is therefore natural to introduce the arbitrary parameter by the definition:

$$A(-2/3,\ 0) = B(-2/3,\ 0) = C(-2/3,\ 0) = -\lambda$$

i.e.

$$A^0(-2/3,\ 0) = -5\lambda$$

$$A^1(-2/3,\ 0) = 0$$

$$A^2(-2/3,\ 0) = -2\lambda$$

Based on a study of the analytic properties of the general fourth-order Feynman diagram and a clever use of unitarity, Mandelstam formulated his famous representation of the scattering amplitude (8-12). This ambitiously describes the amplitude in the whole of the Mandelstam plane.

The Mandelstam representation by its global nature assumes the amplitude to be analytic in the whole of the cut-s and cut-t planes. The assumption of such a domain of analyticity has been postulated by Chew to be basic to S-matrix theory under the name of "maximal analyticity of the first kind" (21-23).

Unfortunately, we have no guarantee that the Mandelstam representation embodies the correct analytic structure of the amplitude. The technical difficulties of attempting such a global approach together with the vastness of the problem have led to a more limited scheme of study.

This is based on using only those analyticity properties deducible from axiomatic field theory (33). From such assumptions increasingly larger domains of analyticity in the complex s and t space have been proved for the $\pi\pi$ scattering amplitude (33).

It is well known that $\pi\pi$ dynamics is the simplest example of strong interaction amplitudes, and, therefore, the most interesting from the fundamental point of view; since all crossed channels are identical, it is the ideal place to play the "Analyticity-Crossing-Unitarity" game. Martin (36) has derived rigorous inequality constraints on the $\pi\pi$ partial wave amplitudes in the unphysical region $0 \leq s \leq 4m_\pi^2$. Roskies (39) has found sum rules involving integrals of the partial-wave amplitudes over $0 \leq s \leq 4m_\pi^2$, which follow from crossing symmetry. However, the main disadvantage of these methods lies in the fact that the crossing relations, or constraints, are written and used in unphysical regions. First crossing constraints in physical $\pi\pi$ partial waves have been obtained by Wanders (50) and by Roskies (61). However, their basic drawback is that s- and p-wave amplitudes are absent in those relations which only constrain higher waves. Recently, Roy (63) has written $\pi\pi$ equations which, in connection with the above remarks, can be seen to have the features of a) expressing each partial-wave amplitude in the physical-region over physical absorptive parts, and b) being well-defined up to energies around 1100 Mev, therefore providing direct consistency tests for experimental data. Roy uses crossing to express the t-dependent subtraction functions in twice-subtracted fixed-t dispersion relations, and then projects on partial waves. More recently, E.P. Tryon (43) has derived a new representation which expresses $\pi\pi$ amplitudes in terms of a single subtraction parameter and integrals over physical region absorptive parts. This representation is valid over the same portion of the physical region as the twice-subtracted representations of Roskies and Roy, and therefore constitutes a more powerful tool for studying the $\pi\pi$ interaction.

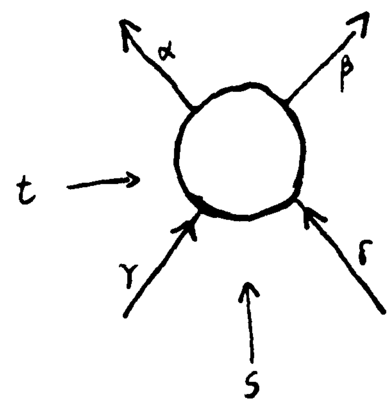
CHAPTER II

TOTALLY SYMMETRIC SCATTERING AMPLITUDE

A. Introduction

The amplitude for $\pi\pi$ scattering in the s-channel can be written as

$$A_{\alpha\beta,\gamma\delta}(s,t,u) = \delta_{\alpha\beta}\delta_{\gamma\delta}A(s,t,u) + \delta_{\alpha\gamma}\delta_{\beta\delta}B(s,t,u) + \delta_{\alpha\delta}\delta_{\beta\gamma}C(s,t,$$



where $\alpha\beta$ $(\gamma\delta)$ denote the isospin indices of the outgoing (incoming) pions.

By crossing symmetry

$$A(s,t,u) = A(s,u,t) \qquad \text{(II1)}$$

$$B(s,t,u) = A(t,s,u) \qquad \text{(II2)}$$

$$C(s,t,u) = A(u,t,s) \qquad \text{(II3)}$$

In the s-channel, the isospin amplitudes are given by

$$A^{0}(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,s,t) \tag{II4}$$

$$A^{1}(s,t,u) = A(t,s,u) - A(u,s,t) \tag{II5}$$

$$A^{2}(s,t,u) = A(t,s,u) + A(u,s,t) \tag{II6}$$

The amplitudes A^{0}, A^{1}, and A^{2} will be compatible with crossing symmetry and isospin invariance if and only if we can find a function $A(s,t,u)$ subject to (II1-II3) such that (II4-II6) are valid.

As a function of three variables, A can be written as a linear combination of functions which transform irreducibly under the permutation group operating on the variables s,t,u. R. Roskies (38) has shown that the most general A can be written as:

$$A(s,t,u) = f(s,t,u) + (2s-t-u)\, g(s,t,u) + (2s^2-t^2-u^2)\, h(s,t,u)$$

where f, g, h are totally symmetric in s, t, u. In terms of these new functions, we can write:

$$A^{0}(s,t,u) = 5f(s,t,u) + 2(2s-t-u)\, g(s,t,u) + 2(2s^2-t^2-u^2)\, h(s,t,u) \tag{II7}$$

$$A^{1}(s,t,u) = 3(t-u)\, g(s,t,u) + 3(t^2-u^2)\, h(s,t,u) \tag{II8}$$

$$A^{2}(s,t,u) = 2f(s,t,u) + (t+u-2s)\, g(s,t,u) + (t^2+u^2-2s)\, h(s,t,u) \tag{II9}$$

The amplitudes A^0, A^1, A^2 will be consistent with crossing and isospin invariance if and only if there exist three totally symmetric functions f, g, h such that (II7-II9) hold.

The three symmetric functions f, g, h now can be written in the vector form as:

$$G_0(s,t,u) = \frac{A^0(s,t,u)}{3} + \frac{2A^2(s,t,u)}{3}$$

$$G_1(s,t,u) = \frac{A^1(s,t,u)}{t-u} + \frac{A^1(t,u,s)}{u-s} + \frac{A^1(u,s,t)}{s-t}$$

$$G_2(s,t,u) = \left[\frac{A^1(s,t,u)}{t-u} - \frac{A^1(t,s,u)}{s-u}\right]\frac{1}{s-t}$$

$$+ \left[\frac{A^1(t,u,s)}{u-s} - \frac{A^1(u,t,s)}{t-s}\right]\frac{1}{t-u}$$

$$+ \left[\frac{A^1(u,s,t)}{s-t} - \frac{A^1(s,u,t)}{u-t}\right]\frac{1}{u-s}$$

Due to the antisymmetry of $A^1(s,t,u)$ with respect to t and u (Bose statistics), the denominators appearing in G_1 and G_2 do not induce new singularities at t = u, u = s and s = t. In fact, the functions $G_k(s,t,u)$ have the same analyticity properties as the amplitudes $A^I(s,t,u)$.

Using the crossing relations

$$A^I(s,t,u) = C_{st}^{II'} A^{I'}(t,s,u)$$

$$A^I(s,t,u) = C_{tu}^{II'} A^{I'}(s,u,t)$$

$$A^I(s,t,u) = C_{su}^{II'} A^{I'}(u,t,s)$$

where:

$$C_{st} = \begin{bmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{bmatrix}$$

$$C_{tu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_{su} = \begin{bmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}$$

We can write $A^I(s,t,u)$ in terms of $G_k(s,t,u)$:

$$A^0(s,t,u) = \frac{5}{3} G_0(s,t,u) + \frac{2}{9}(3s-4) G_1(s,t,u) - \frac{2}{27}(3s^2-16+6tu) G_2(s,t,u)$$

$$A^1(s,t,u) = \frac{(t-u)}{3} G_1(s,t,u) + \frac{(t-u)(3s-4)}{9} G_2(s,t,u)$$

$$A^2(s,t,u) = \frac{2}{3} G_0(s,t,u) - \frac{(3s-4)}{9} G_1(s,t,u) + \frac{(3s^2-16+6tu)}{27} G_2(s,t,u)$$

B .Homogeneous Variables

Because of the symmetry properties of $G_k(s,t,u)$, we can express G in terms of the symmetric variables:

$$x = -\frac{1}{16}(st + tu + us)$$

$$y = \frac{1}{64} stu$$

The singularities associated with this change of variables do not appear in $\tilde{G}_k(x,y) = G_k(s,t,u)$. The only singularities of $\tilde{G}_k(x,y)$ are the images of the singularities of $G_k(s,t,u)$ through the mapping $(s,t,u) \longrightarrow (x,y)$.

This change of variables maps the real (s,t,u)-plane onto a domain R of the real (x,y)-plane. This domain is bounded by two curves C_+ and C_- :

t: constant

$$
\begin{aligned}
16x &= -t(s + u) - us \\
&= t(t - 4) - us \\
&= t(t - 4) - u^2 \\
&= t(t - 4) - \left(\frac{4 - t}{2}\right)^2 \\
&= (t - 4)\left(\frac{3t}{4} + 1\right)
\end{aligned}
$$

therefore:

$$(3x + 1) = \left(\frac{3t}{8} - \frac{1}{2}\right)^2 \qquad x > -\frac{1}{3}$$

$$27y = 2\left(\frac{3t}{8} - \frac{1}{2}\right)^3 - 3\left(\frac{3t}{8} - \frac{1}{2}\right)^2 + 1$$

$C_\pm$:

$$y_\pm = \frac{1}{27}\left\{ 1 - 3(3x + 1) \pm 2(3x + 1)^{3/2}\right\}$$

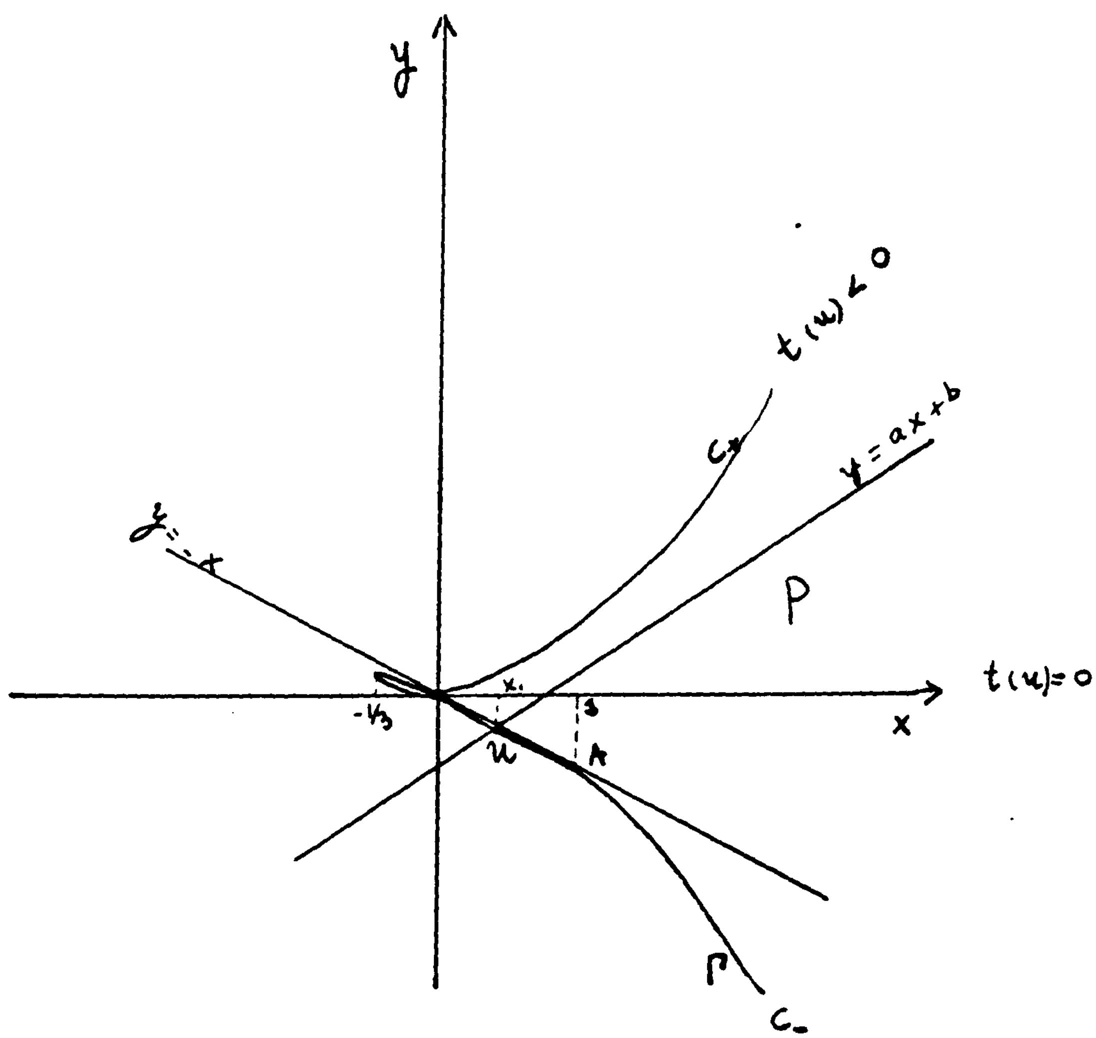
Real (x,y) points which are outside R correspond to complex values of s, t, and u.

Given x and y, the corresponding values of s, t and u are the roots of the third degree equation:

$$z^3 - 4z^2 - 16xz - 64y = 0$$

C. Analytic Properties of G_k

Through the mapping, the amplitude G becomes a function $\tilde{G}(x,y) = G(s,t,u)$. Due to the symmetry properties of G, $\tilde{G}$ is obviously a one-valued function wherever it is defined. In fact, the symmetry of G(s,t,u) entails a stronger result: the function $\tilde{G}(x,y)$ is analytic on the image $\tilde{D}$ of analiticity domain D of G(s,t,u).



The analytic properties of $\tilde{G}(x,y)$ (from now on we write $G(x,y)$) are not simple because its holomorphy domain is not a topological product of cut planes. Nevertheless, we get simple properties if we consider the values of $G(x,y)$ on the plane $y = a(x - x_0)$ or $y = ax + b$, where a and b are real. $G(x, ax + b)$ is holomorphic in the complex x-plane provided with a cut on the real axis $x_1(a,b) \leq x < \infty$. $x_1(a,b)$ is defined in the following way: consider C_- and its tangent at $A(x = 1, y = -1)$. The part of C_- below A and the part of tangent above A define a curve Γ. The real section of $y = ax + b$ cuts Γ at a point $\mathcal{U}$ whose abscissa is $x_1(a,b)$.

It is readily seen that if $(x,y) \to \infty$ in (x,y) plane

$$s \simeq 4\sqrt{x}$$
$$t \to -4a$$

$$x = -\frac{1}{16}\left[s(4 - s) + tu\right]$$
$$\simeq \frac{s^2}{16} \quad \therefore \quad s \simeq 4\sqrt{x}$$

$$y \simeq ax \simeq \frac{1}{64}\, 4\sqrt{x}\; t(4-4\sqrt{x}-t)$$

or $t \to -4a$ (or $t \simeq 4\sqrt{x}$, $s \to -4a$, etc.)

D. Totally Symmetric Dispersion Relations

Using the known bounds on the asymtotic behavior of $A^I(s,t,u)$(28,35) as $s \to \infty$ at fixed t, we see that the definitions of $G(s,t,u)$ in terms of $A^I(s,t,u)$ imply that $G_k(x,\ a(x - x_o))$ verifies a once substracted dispersion relation. In fact we could write an unsubstracted dispersion relation for $G_2(x,\ a(x - x_o))$.

$$G_k(x,\ a(x - x_o)) = G_k(x_1,\ a(x_1 - x_o))$$

$$\frac{x-x_1}{\pi} \int_{C(a,x_o)} dx' \frac{\text{Disc. } G_k(x',\ a(x' - x_o))}{(x' - x_1)(x' - x)}$$

Disc. G_k is the discontinuity of G_k across the cut $C(a,\ x_o)$.

$$\text{Disc. } G_k(x,\ a(x - x_o)) = \frac{1}{2i} \Big(G_k(x + i\varepsilon,\ a(x + i\varepsilon - x_o))$$

$$- G_k(x - i\varepsilon,\ a(x - i\varepsilon - x_o)) \Big)$$

$$\text{for } (x,\ a(x - x_o)) \in C(a,x_o)$$

In terms of s,t,u the DR looks like :

$$G_k(s,t,u) = G_k(s_1,\ t_1,\ u_1) + \frac{1}{\pi} \int_4^\infty ds' \text{ Disc. } G_k(s',t')(s'-t')(2s'+t'-4)$$

$$\times \left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{(s'-s_1)(s'-t_1)(s'-u_1)} \right] \tag{II10}$$

Where:

$$t' = t'(s',\ a,\ x_o) = \frac{4-s'}{2} + \frac{1}{2}\sqrt{(s'-4)^2 - \frac{16a}{s'+4a} \left[s'(s'-4) - x_o \right]}$$

We verify that $\lim_{s' \to \infty} t'(s',\ a,\ x_o) = -4a$

The right-hand side integral is defined as long as a and x_o are such that $t'(s', a, x_o)$ is inside the analyticity domain of Disc $G_k(s',t')$ for all $s' \geqslant 4$. It is convenient to consider the special case where $x_1 = x_0$. Then, we may choose:

$$t_1 = 0$$
$$u_1 = -(s_o - 4)$$
$$s_1 = s_o = 2(1 + \sqrt{1 + 4x_o})$$

$$G_k(s,t) = G_k(s_o,0) + \frac{1}{\pi}\int_4^{\infty} ds' \, \text{Disc}.G_k(s',t')(s'-t')(2s'+t'-4)$$
$$\times \left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{s'(s'-s_o)(s'+s_o-4)}\right]$$

(II11)

Equations (II10) and (II11) can be expressed in the following form, which is very reminiscent of the Cini-Fubini representation:

$$G(s,t) = G(s_1, t_1) + \frac{1}{\pi}\int_4^{\infty} \frac{ds'}{s'^2} \, \text{Disc}.G(s',t')$$
$$\times \left[\frac{s^2}{s'-s} + \frac{t^2}{s'-t} + \frac{u^2}{s'-u} - \frac{s_1^2}{s'-s_1} - \frac{t_1^2}{s'-t_1} - \frac{u_1^2}{s'-u_1}\right]$$

And

$$G(s,t) = G(s_o, o) + \frac{1}{\pi}\int_4^{\infty} \frac{ds'}{s'^2} \, \text{Disc}.G(s',t')$$
$$\times \left[\frac{s^2}{s'-s} + \frac{t^2}{s'-t} + \frac{u^2}{s'-u} - \frac{s_o^2}{s'-s_o} - \frac{(4-s_o)^2}{s'+s_o-4}\right]$$

We are now ready to write DR's for the amplitudes A^I:

$$A^0(s,t) = \frac{1}{3}\Big[5G_0(s_0,0) + \frac{2}{3}(3s-4)\ G_1(s_0,0) - \frac{2}{9}(3s^2-16)\ G_2(s_0,0)$$

$$+ \frac{1}{3\pi}\int_4^\infty ds'\ (2s'+t'-4)(s'-t')\Big[\frac{1}{(s'-s)(s'-t)(s'-u)}$$

$$- \frac{1}{s'(s'-s_0)(s'+s_0-4)}\Big]$$

$$\times\Big[5\text{Disc.}G_0(s',t') + \frac{2}{3}(3s-u)\ \text{Disc.}G_1(s',t') - \frac{2}{29}(3s^2-16+6tu)\ \text{Disc.}G_2(s',t')\Big] \tag{II12}$$

$$A^1(s,t) = \frac{(t-u)}{9}\Big[3G_1(s_0,0) + (3s-4)\ G_2(s_0,0)\Big]$$

$$+ \frac{(t-u)}{9}\int_4^\infty ds'\ (s'-t')(2s'+t'-4)\Big[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{s'(s'-s_0)(s'+s_0-4)}\Big]$$

$$\times\Big[3\text{Disc.}G_1(s',t') + (3s-4)\ \text{Disc.}G_2(s',t')\Big]$$

$$\tag{II13}$$

$$A^2(s,t) = \frac{1}{3}\left[2G_o(s_o,o) - \frac{(3s-4)}{3} G_1(s_o,o) + \frac{(3s^2-16+6tu)}{9} G_2(s_o,o)\right]$$

$$+ \frac{1}{3\pi}\int_4^\infty ds'(s'-t')(2s'+t'-4)\left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{s'(s'-s_o)(s'+s_o-4)}\right]$$

$$\times\left[2\text{Disc.}G_o(s',t') - \frac{(3s-u)}{3}\text{Disc.}G_1(s',t') + \frac{(3s^2-16+6tu)}{9}\text{Disc.}G_2(s'.t')\right]$$

(II14)

where:

$$\text{Disc.}G_o(s',t') = \frac{1}{3}\,\text{Im}\,A^o(s',t') + 2\text{Im}\,A^2(s',t')$$

$$\text{Disc.}G_1(s',t') = \frac{(3s'-4)}{6(s'-t')(2s'+t'-4)}\left[2\text{Im}A^o(s',t') - 5\text{Im}A^2(s',t')\right]$$

$$+\left[\frac{1}{s'+2t'-4} - \frac{(2t'+s'-4)}{2(s'-t')(2s'+t'-4)}\right]\text{Im}A^1(s',t')$$

$$\text{Disc.}G_2(s',t') = -\frac{1}{2(s'-t')(2s'+t'-4)}\left[2\text{Im}A^o(s',t') - 5\text{Im}A^2(s',t')\right]$$

$$+ \frac{3(3s'-4)}{2(2t'+s'-4)(s'-t')(2s'+t'-4)}\ \text{Im}A^1(s',t')$$

From equations (II12) and (II14) we have:

$$a_o = \frac{5}{3} G_o(4,o) + \frac{16}{9} G_1(4,o) - \frac{64}{27} G_2(4,o)$$

$$a_2 = \frac{2}{3} G_o(4,o) - \frac{8}{9} G_1(4,o) + \frac{32}{27} G_2(4,o)$$

Note that equations (II12) and (II14) also give us the so-called Olsson's sum rule (52).

Equations (II12) and (II14) imply:

$$2a_o - 5a_2 = \frac{8(3G_1(4,0) - 4G_2(4,0))}{3}$$

If $G_1(s,t)$ and $G_2(s,t)$ obey unsubtracted DR's:

(It will be seen later that indeed G_1 and G_2 obey unsubtracted DR's)

$$G_1(4,0) = \frac{1}{\pi}\int_4^\infty ds' \quad \text{Disc.}G_1(s',o)\ \frac{2s'-4}{s'(s'-4)}$$

$$G_2(4,0) = \frac{1}{\pi}\int_4^\infty ds' \quad \text{Disc.}G_2(s',o)\ \frac{2s'-4}{s'(s'-4)}$$

Then:

$$2a_o - 5a_2 = \frac{8}{\pi}\int_4^\infty ds' \ (3\text{Disc.}G_1(s',o) - 4\text{Disc.}G_2(s',o))\ \frac{2s'-4}{3s'(s'-4)}$$

$$= \frac{24}{\pi}\int_4^\infty \frac{ds'}{s'(s'-4)} \quad \text{Im}T^1(s',o)$$

The practical consequence of this relation is that instead of the scattering lengths a_o and a_2 taking arbitrary values in the (a_o, a_2) plane, physical solutions to Roy-type equations (64-67), with two subtractions, have a_o and a_2 lying on a universal curve generated by that equation. This universal curve was first found by Morgan and Shaw (68) and it appears to be a general feature of physical $\pi\pi$ amplitudes that their scattering lengths lie on such a curve (69,70).

Furthermore, we have:

$$F_1(s,o) = \frac{s-4}{9}\left[3G_1(s,o) + (3s-4)\; G_2(s,o)\right]$$

As $s \longrightarrow 4$

$$a_1^1 = \frac{4}{27}\left[3G_1(4,o) - 4G_2(4,o)\right] + \frac{48}{27}\, G_2(4,o)$$

$$= \frac{(2a_o - 5a_2)}{18} + \frac{48}{27}\; G_2(4,o)$$

or:

$$2a_o - 5a_2 - 18a_1^1 = \frac{16}{\pi}\int_4^\infty ds' \; \frac{2\mathrm{Im}A^o(s',o) - 5\mathrm{Im}A^2(s',o) + 3\mathrm{Im}A^1(s',o)\frac{(4-3s')}{(s'-4)}}{s'^2(s'-4)} \tag{II15}$$

Equation (II15) is exactly the Wanders' sum rule (39). As a matter of fact eq. (II10) provides us with a means to write an infinite sets of sum rules; the only constraints are that a and x_o should be such that the lines on which the dispersion relations are written, do not enter the forbidden region, where the dispersion relations are not valid. We may for example remove $G(s_o,o)$ in favor of $G(4,o)$ which is related to the scattering lengths, by writing (II11) for the special case $a = o$, $s = 4$, $t = o = t'$. The result is:

$$G(4,o) = G(s_o,o) + \frac{1}{\pi}\int_4^\infty ds' \mathrm{Disc}.G(s',o)s'(2s'-4) \times \left[\frac{1}{(s'-4)s'^2} - \frac{1}{s'(s'-s_o)(s'+s_o-4)}\right]$$

or:

$$G(s,t) = G(s_o,o) + \frac{1}{\pi}\int_4^\infty ds'\ \text{Disc.}G(s',t')(s'-t')(2s'+t'-4)$$
$$\times\left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{s'(s'-s_o)(s'+s_o-4)}\right]$$
$$= G(4,o) + \frac{1}{\pi}\int_4^\infty ds'\ \text{Disc.}G(s',o)\ s'(2s'-4)$$
$$\left[\frac{1}{s'(s'-s_o)(s'+s_o-4)} - \frac{1}{s'^2(s'-4)}\right]$$
$$+ \frac{1}{\pi}\int_4^\infty ds'\ \text{Disc.}G(s',t')(s'-t')(2s'+t'-4)$$
$$\times\left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{s'(s'-s_o)(s'+s_o-4)}\right]$$

(II16)

Now, if equation (II16) is written for two different values of $s_1 = 2(1 + 1+4x_1)$ and $s_2 = 2(1 + 1+4x_2)$ and substracted from one another, we get this sum rule:

$$\frac{1}{\pi}\int_4^\infty ds'\ \text{Disc.}G(s',o)s'(2s'-4)\left[\frac{1}{s'(s'-s_1)(s'+s_1-4)} - \frac{1}{s'(s'-s_2)(s'+s_2-4)}\right]$$
$$= \frac{1}{\pi}\int_4^\infty ds'\ \text{Disc.}G(s',t'_\beta)(s'-t'_\beta)(2s'+t'_\beta-4)$$
$$\left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{s'(s'-s_2)(s'+s_2-4)}\right]$$
$$- \frac{1}{\pi}\int_4^\infty ds'\ \text{Disc.}G(s',t'_\alpha)(s'-t'_\alpha)(2s'+t'_\alpha-4)$$
$$\left[\frac{1}{(s'-s)(s'-t)(s'-4)} - \frac{1}{s'(s'-s_1)(s'+s_1-4)}\right]$$

where

$$t'_\alpha = t'(s', a_1, x_1)$$

$$t'_\beta = t'(s', a_2, x_2)$$

and

$$a_1 = \frac{y}{x-x_1}$$

$$a_2 = \frac{y}{x-x_2}$$

Each set of sum rule corresponds to a set of x_1 and x_2, such that the lines with slopes a_1 and a_2 are outside the forbidden region.

CHAPTER III

DETERMINATION OF THE SLOPE OF REGGE RESIDUE FUNCTION: A SURVEY

Before attempting to extract information about the $\rho\pi\pi$ Regge residue function from this new representation, we will prepare the groundwork for such endeavour by discussing in some detail some recent calculations of $\gamma_\rho(t)$. These will put our calculations in perspective.

Kaiser (40,41) has calculated $\pi\pi$ Regge residues from continuous moment sum rules (CMSR), using the data of Carroll et al (42). These data go up to $s^{1/2}$= 1.45 Gev, and beyond that energy he assumes Regge behaviour has set in.

Kaiser finds the ρ trajectory function, $\alpha_\rho(t)$, to be described rather well by linear form with intercept near 0.5. He calculates the full Regge residue $\beta_\rho(t)$ for $-1\ Gev^2 \leq t \leq 0$, and finds $\beta_\rho(t)$ has a zero close to $t = -0.4\ Gev^2$. Since $\gamma_\rho(t)$, the imaginary part of the residue is directly proportional to $\beta_\rho(t)$, it too has a zero at this value of t, while the real part of the residue ($\sim \beta_\rho(t) \tan \pi\alpha/2$) not only has this zero but also has another zero nearby at $t = -0.6\ Gev^2$, when $\alpha(t) = 0$.

Tryon (43) has also used the data of Carroll et al to determine $\gamma_\rho(t)$. To do this recall that the residue must be known at some value of t, since eq. (4) of Ref. (43) essentially only determines the derivative of γ_ρ. By taking differences between fixed-s and fixed-t DR's, he was able to eliminate the substraction terms and, thereby obtain sum rule equating certain integrals over absorptive parts to zero. Then this sum rule was used to extract the $\rho\pi\pi$ Regge residue function for a wide range of momentum transfer.

The useful feature of eq. (6) (Ref. 43) is that if γ_ρ is known for any single value of its argument, this value can be substituted for s on the right-hand side of the equation and then $\gamma_\rho(t)$ can be computed over a wide range of t from a knowledge of Im $T^1(s,t)$ between threshold and Λ, together with knowledge of the rapidly convergent integral $h(s,t)$. He notes that from the data of Carroll et al Im $T^1(s,t)$ has a zero around $t = -0.52$ Gev^2. This is for $s^{\frac{1}{2}} < 1.48$ Gev. He assumes that this zero continues to be at this fixed value of t above that energy. Then $\gamma_\rho(t)$ was given within 4% over the interval -1 Gev$^2 < t < 0.2$ Gev^2 by the simple curve:

$$\gamma_\rho(t) \simeq 0.82 + 2.04(t/\bar{s}) + 0.88(t/\bar{s})^2$$

with $\bar{s} = 1$ Gev^2, using the standard linear ρ trajectory with intercept of 1/2. He also checks that his results for $\gamma_\rho(t)$ do indeed make Im $T^1_{Regge}(s,t)$ the local average of the actual absorptive part Im $T^1(s,t)$ for $-1.$ Gev$^2 \leq t \leq 0.25$ Gev^2 and for energies where Carroll et al have data 0.60 Gev $\leq s^{\frac{1}{2}} \leq$ 1.48 Gev. He finds that it does. In a later work (44) the same sum rule is used together with the $\pi\pi$ data of Hyams et al (45) (for S, P, D, F waves, I=0,1) and Durusoy et al (46) (for S, D waves, I=2) which extend up to $M_{\pi\pi}$ = 1.9 Gev. The integrand of the sum rule receives no contribution whatever from S waves, so that we will be spared from the ambiguities which have plagued experimental studies of the I=0 S wave. Thus for $\Lambda \leq (1.1 \text{ Gev})^2$ the integral in eq. (4) (Ref. 44) is determined almost entirely by the ρ resonance, since D-wave absorptive parts are negligible below 1.1 Gev. Furthermore, the integral defining h(s,t) is rapidly convergent, and is thus determined primarily by the ρ resonance. As Λ increases above $(1.1 \text{ Gev})^2$, D waves begin to contribute to the integral but in such a way that $\gamma(t)$ is highly stable against variations of Λ. Here $\gamma_\rho(t)$ is found to vanish at t_z = -0.42 Gev^2, and the value of $\gamma(t)$ is given wthin $\pm$ 0.01 over the interval $-1.0 \leq t \leq 0.1$ Gev^2 by:

$$\gamma_\rho(t) = 0.67 + 1.78(t/\bar{s}) + 0.41(t/\bar{s})^2 - 0.17(t/\bar{s})^3$$

It is to be noted that the value $\gamma_\rho(0)$ = 0.82 (Ref. 43) and $\gamma_\rho(0)$ = 0.67 (Ref. 44) is only slightly different from the one required by Morgan and Shaw (47), namely, $\gamma_\rho(0) = 0.68$. But it is much larger than the one deduced by Olsson, $\gamma_\rho(0) = 0.17$ (48). This latter author has used πN charge exchange data together with universality to arrive at the result $\sum_I C_{1I}\, \sigma^I_{total} = 0.6\ v^{-\frac{1}{2}} \mu^{-2}$, where $v = \frac{1}{4}(s/\mu^2 - 4)$. It follows that Im $T^1_F = 0.048\ v^{\frac{1}{2}}$, where T^I denotes the forward amplitude in the t channel. In terms of the normalization in Ref. (43) and Ref. (44) $\gamma_\rho(0)$ would be 0.17.

The author of Ref. (44) has also attempted to calculate $\gamma_\rho(t)$, by interchanging s and t in eq.'s (1) and (4), thus a new equation is obtained, where $\gamma_\rho(t)$ is completely determined by integrals over absorptive parts, together with a known function f. However, $\gamma_\rho(t)$ obtained in this way proves to be extremely sensitive to the f_0 and g resonance parameters. Hence it is difficult to obtain a reliable result for $\gamma_\rho(t)$ in this way. The left-hand side of equation (13) (Ref. (44)) is independent of s, so the right-hand side should be independent of s also. However, using the experimental values for resonance parameters, the right-hand side is found to depend strongly on s, unless values for the f_0 and g parameters are very near the experimental ones.

Ukawa et al (49) has also used the data of Carroll et al. to calculate $\pi\pi$ Regge residues from continuous moment sum rules (CMSR). For $|t| < 32$ they find the same results as

Kaiser, namely $\gamma_\rho(t)$ has a zero close to $t = -0.4$ Gev^2. For t between 0 and -2.4 Gev^2 (which is -123 in pion mass units) they also calculate both the real and imaginary parts of the residue. It is doubtful whether such a computation is too reliable much beyond $t = -32$, since they are then integrating across the s-u double spectral region.

Another set of sum rules was derived and the slope of the Regge residue function was calculated by Wanders (50). The idea of the method is to find quantities which can be expressed either by a fixed-t dispersion integral or by a fixed-u dispersion integral. The sum rules express that these two integrals are equal. The method here is similar to the one developed by Tryon (43,44), the only difference consists in that the sum rules here involve derivatives of absorptive parts with respect to the transfer t. The inputs are the data of Rarita et al. (51) obtained through factorization from their analysis of πN and NN scattering. It is assumed that:

$$\mathrm{Im}\, T'(\nu,t) = \beta_\rho(t)\, \nu^{\alpha_\rho(t)}$$

where $\nu = (s-4)/4$ and t are in pion mass square units, which define the scale of $\beta_\rho(t)$

with:

$$\beta_\rho(0) = 2.4$$

$$\alpha_\rho(0) = 0.5$$

$$\alpha'_\rho(0) = 0.017$$

And:

$$Im\, T^{\circ}(\nu,t) = \beta_P(t)\, \nu^{\alpha_P(t)} + \beta_{P'}(t)\, \nu^{\alpha_{P'}(t)}$$

with

$$\alpha_P(0) = 1 \qquad \alpha'_P(0) = 0.0057$$
$$\beta_P(0) = 7.3 \qquad \beta'_P(0) = 0.063$$
$$\alpha_{P'}(0) = 0.57 \qquad \alpha'_{P'}(0) = 0.043$$
$$\beta_{P'}(0) = 16.8 \qquad \beta'_{P'}(0) = -1.63$$

The value $\beta'_\rho(0)$ was calculated at different value of v_o, which separates low and intermediate energies from high energies.

$$\nu_0 = 41 \; (\nu_0^{1/2} \simeq 1.8\, Gev):$$
$$\beta'_\rho(0) = 0.77 - 0.23\,x - 3.2\, F_{BG}(41)$$
$$\nu_0 = 12 \simeq \tfrac{1}{2}(\nu_f + \nu_\rho) \; :$$
$$\beta'_\rho(0) = 0.81 - 1.7\, F_{BG}(12)$$
$$\nu_0 = 26 \simeq \tfrac{1}{2}(\nu_f + \nu_g) \; :$$
$$\beta'_\rho(0) = 0.66 - 2.5\, F_{BG}(26)$$

If we assume that $F_{BG}(v_o)$ is small and the inelasticity factor for ϱ-meson $x \simeq 0.3$, then

$$\nu_{41} \; : \; \beta'_\rho(0) \simeq 0.70$$
$$\nu_{12} \; : \; \beta'_\rho(0) \simeq 0.81$$
$$\nu_{26} \; : \; \beta'_\rho(0) \simeq 0.66$$

In the normalization of Ref. (43) and Ref. (44), we would have:

$$\nu_{12} \; : \; \gamma'_\rho(0) \simeq 0.33$$
$$\nu_{26} \; : \; \gamma'_\rho(0) \simeq 0.40$$
$$\nu_{41} \; : \; \gamma'_\rho(0) \simeq 0.35$$

Another estimation of $\beta'_\rho(0)$ was obtained from a crude interpolation of the ρ residue between t = o and t = t_ρ which gives $\beta'_\rho(0) \simeq 0.86$ or $\gamma'_\rho(0) \simeq 0.43$. All these results, obtained from a very rough approximation, seem to be consistent with those given in Ref. (44), at least in their order of magnitude.

In the Regge pole model, the high energy behavior of the charge exchange amplitude, which has pure isospin one in the t-channel, T^1, is controlled by the ρ Regge pole. Since $\alpha_\rho(t) < 1$ for $t < m_\rho^2$, we expect the amplitude T^1 to satisfy a once substracted dispersion relation for all $t \in (-28,4)$. From the s-u antisymmetry of this amplitude, the substraction function is determined and we have:

$$T^1(s,t) = \frac{1}{\pi}\int_4^\infty ds' \left(\frac{1}{s'-s} - \frac{1}{s'-u}\right) \operatorname{Im} T^1(s',t) \qquad \text{(III1)}$$

One immediate consequence of this equation is that the two substraction constants of the twice substracted dispersion relations and hence of Roy's equations, namely a_0^0 and a_0^2, become related to each other: if equation (III1) is evaluated at s-channel threshold, we get:

$$2a_0^0 - 5a_0^2 = \frac{24}{\pi}\int_4^\infty \frac{ds'}{s'(s'-4)} \operatorname{Im} T^1(s',0) \qquad \text{(III2)}$$

Equation (III2) is called the Osson sum rule (52). Souza (53) calculates the imaginary and the real part of the ρ Regge residue by using the dispersion relation of eq.(III1).

The low energy input for the s and p waves, are taken from the two solutions of Basdevant et al. (54) and Le Guillou et al.(55). For the higher partial waves, Souza uses Breit-Wigner forms for the f_o and g resonances and assumes Regge behaviour beyond $s = \frac{1}{2}(m_f^2 + m_g^2) = 115$. He finds that for $t \simeq -0.2$ Gev^2, both the real and imaginary parts of the ρ residue vanish. This corresponds to a large value of the scattering length a_o^o ($\simeq 0.6$).

Using a sum rule similar to the one in Ref. (43), Basdevant and Schomblond (56) try to calculate $\gamma_\rho(t)$ in this way: first with s = o they assume $\gamma_\rho(o)$ is given, they calculate $\gamma_\rho(t)$ as a function of t, thereby they find a direct correlation between the assumed $\gamma_\rho(o)$ and t_z for which $\gamma_\rho(t_z)=0$. Then Olsson sum rule is used to get a relationship between $\gamma_\rho(o)$ and a_o^o . This gives a correlation between t_z and a_o^o . They conclude that $t_z = -0.2$ Gev^2 is favoured by all values of $a_o^o > 0.16$. This is to be compared with Tryon's value of $t_z \simeq -0.52$ Gev^2 (43), Ukawa et al.'s $t_z \simeq -0.4$ Gev^2 and Eguchi et al.'s $t_z \simeq -0.3$ (57,58).

CHAPTER IV

DETERMINATION OF $\rho\pi\pi$ RESIDUE FUNCTION USING TOTALLY SYMMETRIC REPRESENTATION

A. Introduction

In this study, our purpose is to establish some sum rules for $\pi\pi$ scattering which are physically meaningful consequences of analycity, crossing symmetry and unitarity. Analyticity, crossing symmetry and unitarity are believed to be powerful constraints on the behaviour of the scattering amplitude, and may alone define a unique nontrivial S-matrix. The three conditions are of quite a different nature and this makes it hard to exploit them simultaneously and exhaustively. Crossing symmetry, by its very nature, depends on analytically continuing the scattering amplitude from one region to another. Therefore, its practical implementation depends strongly on the analytic properties we believe the amplitude to have and the tools we use to exploit them.

The Mandelstam representation assumes the $\pi\pi$ amplitude to be analytic in the whole of the cut-s and cut-t planes. However, we have no guarantee that the Mandelstam representation embodies the complete and correct analytic structure of the amplitude.

From the results of axiomatic field theory it has been proved (33) that the $\pi\pi$ scattering amplitude satisfies fixed momentum transfer dispersion relations for a limited region of t, namely, $-28 < t < 4$ (in pion mass units). Moreover Jin and Martin (59) have shown that such dispersion relations require at most two substractions. It is in terms of such fixed-t representations for the amplitude, rather than Mandelstam's two variable representation, that in practice we impose these rigorous analytic properties in constructing model $\pi\pi$ amplitudes. We can also write a dispersion relation at fixed-s. Thus we have represented the amplitude in two different ways. The s-channel partial wave dispersion relations involve contributions for the t- and u-channels through the left hand cut. However, these contributions cannot be evaluated in a closed form. We can write fixed-t and fixed-s dispersion relations, and by crossing symmetry these expressions must be equal in their common region of validity. Indeed, within such domain, namely $s,t \in (-28,4)$, using twice subtracted dispersion relations, we can write down necessary and sufficient conditions for crossing symmetry as has been done by Wanders (60), Roskies (61), Lyth (62) and Roy (63). In $\pi\pi$ scattering, because all three channels are identical, crossing symmetry is a particularly powerful constraint, that we want to exploit as fully as possible. As was discussed in Chapter II, a rigorous improvement on the one-dimensional fixed-t dispersion relations discussed above is provided by the representation written in the (x,y) plane, which has three channel symmetry. This corresponds to continuing the amplitude not at fixed s or t but along more complicated curves in the s-t plane thereby making fuller use of the rigorous analyticity domains available.

In this chapter we develop an approach which takes into account the full crossing symmetry. The scattering amplitudes G(s,t,u) we consider have simple symmetry properties in the variables s,t,u and they satisfy once subtracted dispersion relations. The difficulty mentioned at the beginning appears here too; the analyticity properties of G(x,y) are intricate and it is not easy to express unitarity in terms of this function. Nevertheless, it is possible to extract from analyticity of G(x,y) some new type of dispersion relation in one variable. This can be done if we consider only values taken by G(x,y) on the complex plane $y = a(x - x_0)$.

B. Unsubtracted Dispersion Relation for $G_1(x,y)$

We can write an unsubtracted dispersion relation for $G_1(x,y)$:

$$G_1(x,y) = \frac{1}{\pi} \int dx' \; \frac{1}{x'-x} \; \mathrm{Disc.}G_1(x', a(x' - x_0))$$

Convergence of the integral is ensured by familiar tenets of Regge theory.

In terms of s and t, after some calculation we get:

$$G_1(s,t) = \frac{1}{\pi}\int_4^{\infty} ds' \frac{(s'-t')(2s'-t'-4)}{(s'-s)(s'-t)(s'-u)} \text{Disc.} G_1(s',t') \quad \text{(IV1)}$$

Where:

$$t' = t'(s', a, x_o) = \frac{4-s'}{2} + \frac{1}{2}\sqrt{(s'-4)^2 - \frac{16a}{s'+4a}\left[s'(s'-4) - x_o\right]}$$

We shall denote the $\pi\pi$ amplitude with Isospin I in the s-channel by $A^I(s,t)$. We normalize the $A^I(s,t)$ such that:

$$A^I(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) A^{(\ell)I}(s)\, P_l\left(1+\frac{2t}{s-4}\right)$$

$$A^{(\ell)I}(s) = Q^{-1} \exp(i\delta_\ell^I) \sin\delta_\ell^I$$

$$Q = \left(\frac{s-4}{s}\right)^{\frac{1}{2}}$$

where pion mass square unit has been used. We note that for energies less than 1.9 Gev, only the S, P, D and F waves have non-negligible absorptive parts.

Furthermore, above 1.9 Gev, Regge theory provides good approximations for absorptive parts. Therefore, it is convenient to introduce a parameter Λ and to treat the region $s' < \Lambda$ and $s' > \Lambda$ in different ways, which exploit these facts. We also assume that for $s > \Lambda$ $\mathrm{Im}T^{2}(s,t)=0$.

In this spirit, we write $G_1(s,t)$ as:

$$G_1(s,t) = \frac{1}{\pi}\int_4^{\Lambda} \frac{ds'}{(s'-s)(s'-t)(s'-u)}\Big\{\left[2\mathrm{Im}A^{(0)0}(s') - 5\mathrm{Im}A^{(0)2}(s')\right] M_0(s')$$
$$+ \left[2\mathrm{Im}A^{(2)0}(s') - 5\mathrm{Im}A^{(2)2}(s')\right] 5M_0(s')P_2(z')$$
$$+ \left[3\mathrm{Im}A^{(1)1}(s') + 7\mathrm{Im}A^{(3)1}(s')\ ZD3(z')\right] M_1(s',z')\Big\}$$
$$+ \frac{1}{\pi}\int_{\Lambda}^{\infty} \frac{ds'}{(s'-s)(s'-t)(s'-u)}\Big\{\left[M_0(s') + M_1(s',z')/2z'\right] \mathrm{Im}T^1(s',t')$$
$$- \left[M_0(s') - M_1(s',z')/3z'\right] \mathrm{Im}T^0(s',t')\Big\} \qquad \text{(IV2)}$$

where:

$$M_0(s') = (3s' - 4)/6$$

$$M_1(s',z') = \frac{((3s' - 4)^2 - 3z'^2(s' - 4)^2}{4(s' - 4)}$$

$$z' = 1 + 2t'/(s' - 4)$$

$$P_2(z') = 0.5(3z'^2 - 1)$$

$$ZD3(z') = P_3(z')/z'$$

and

$$T^{I}(s,t) = \sum_{I'=0}^{2} C_{II'} A^{I'}(s,t)$$

The only approximation made in the above equation lies in the fact that $\mathrm{Im}A^{(\ell)I}(s')$ has been set equalto zero for $\ell \geqslant 4$, $s' < \Lambda$.

For $s' > \Lambda$, we have expressed $G(s,t)$ in terms of T^{I} because these are the amplitudes for which Regge theory prescribes simple asymptotic behavior. We assume that

$$\mathrm{Im}\, T^{1}(s,t) = \gamma_{\rho}(t)\ (^{\Lambda}/_{\bar{s}})^{\alpha_{\rho}(t)}$$

where γ_{ρ} is related by a well-known factor to the residue of the ρ pole in the J plane and α_{ρ} denotes the ρ trajectory. We use $\bar{s}=1\mathrm{Gev}^{2}$, which defines the scale of γ_{ρ}, Λ . In our calculations, we assume that

$$\alpha_{\rho}(t) = 0.5 + 0.9(t/\bar{s})$$

We shall assume that in the Regge region, T^{0} is dominated by Pomeron and f_{0} exchange:

$$\mathrm{Im}\, T^{0}(s,t) = \gamma_{P}(t)\ (^{\Lambda}/_{\bar{s}})^{\alpha_{P}(t)} + \gamma_{f}(t)\ (^{\Lambda}/_{\bar{s}})^{\alpha_{f}(t)}$$

Duality equates resonant absorptive parts with those resulting from exchange of normal Regge trajectories. Since

$$A^{2} = T^{0}/3 - T^{1}/2 + T^{2}/6$$

contains no resonances, it follows that

$$\gamma_{f}(t)/3 - \gamma_{\rho}(t)/2 = 0$$

$$\alpha_{f}(t) = \alpha_{\rho}(t)$$

where $\gamma_\rho^{(0)}$ characterizes the contribution of ρ exchange to Im $T^T(s,t)$. We incorporate an asymtotic total cross section of 20 mb in our assumption:

$$\gamma_\rho^{(0)} = 1.34$$
$$\alpha_\rho(t) = 1.+ .6(t/\bar{s})$$

For s'<A we use the experimental results of Hyams et al (for I=0 and 1) and of Durusoy et al (for I=2). These results are presented in the form of S,P,D and F waves phaseshifts and inelasticities over the energy range $.6 < s^{\frac{1}{2}} < 1.9$ Gev .

First we write $G_1(s,t)$ at s=4, t=27.165 (x=46.122, y=-46.122). This is the point where the lines of maximum and minimum slopes allowed meet in the x,y-plane, and therefore we expect to extract more informations by writing $G_1(s,t)$ at this point. $G_1(s,t)$ is written on the lines with slope ''a'' varying from -1 to 10.75, the only lines allowed, which are outside the forbidden region.

$\gamma_\rho(t)$ is assumed to have the form:

$$\gamma_\rho(t) = \gamma_\rho(0) + \sum_{k=1}^{N} \gamma_{\rho k} \left(t/\bar{s}\right)^k$$

N is the number of terms to be included in the approximation.

We do not expect $\gamma_\rho(t)$ to have many terms. As a matter of fact, fig. (3) of Ref. (43) and fig. (2a) of Ref. (44) show that $\gamma_\rho(t)$ is roughly linear in t.

In our calculations, however, the number of terms included are from 2 to 15, varying the values of a's, (from -1 to 10.75 in equal intervals) to have a set of

N equations with N unknowns. Since E.P. Tryon, using the same experimental data in his calculations, finds that $\gamma_\rho(0)=0.67$, we assume $\gamma_\rho(0)=0.67$ as input. The value of $\gamma_\rho(0)$ starts to stabilize with the second cycle of calculations (3 terms). Here are some of the values of $\gamma_\rho(t)$ we get:

$$\gamma_\rho(t)=0.67+2(t/\bar{s})+0.35(t/\bar{s})^2 \qquad \text{(IV4)}$$

$$\gamma_\rho(t)=0.67+1.81(t/\bar{s})+0.55(t/\bar{s})^2-0.75(t/\bar{s})^3 \qquad \text{(IV5)}$$

$$\gamma_\rho(t)=0.67+1.81(t/\bar{s})+0.54(t/\bar{s})^2-0.74(t/\bar{s})^3+0.01(t/\bar{s})^4 \qquad \text{(IV6)}$$

$$\gamma_\rho(t)=0.67+1.81(t/\bar{s})+0.53(t/\bar{s})^2-0.67(t/\bar{s})^3+0.12(t/\bar{s})^4+0.06(t/\bar{s})^5 \qquad \text{(IV7)}$$

If we approximate the absorptive parts in the low energy region by their resonances, we expect our results to depend strongly on Λ . The reason is simple: we have written

$$G_1 = G_{1L} + G_{1H}$$

where G_{1L} is the value of G_1 in the low energy region (the contribution from ρ , f, and g_0 resonances), G_{1H} is G_1 in the high energy region (the equation integrated from Λ to ∞), then

$$G_1 - G_{1L} = G_{1H} \qquad \text{(IV8)}$$

The left hand side of equation (IV8) is a constant, whereas the right hand side depends on the value we give to Λ .

In the low energy region the contribution from ρ resonance is the largest. G_ρ's vary from 0.109 to 0.156 for the values of a's, the slope of the lines on which DR's are written, from -1 to 10.75 (x_0 from 0 to 50.41). This confirms our suspicion that G_1 is dominated by ρ resonance. The contribution from f and g_0 resonances are small (G_f's take the value from 0.002 to 0.003, G_g's from 0.010 to 0.007).

It has been proved by Roskies (71) and others (72-75) that the size of the imaginary part of the d-wave controls the size of the imaginary parts of all the higher partial waves taken together. Therefore, the fact that we neglect all partial waves with $\ell > 3$ will not affect our results.

$\gamma_\rho(t)$ is valid over a range of t from -43 to 4 in pion mass units, or from -0.82 to 0.08 Gev^2. This coincides with the line of maximum slope a = 10.75, and a = -1 in the (x,y) plane. The integration in the high energies region is made from $\Lambda^{\frac{1}{2}}$ = 1.8 Gev, to over 72 x 10^6 m_π^2 and the error from this approximation is only $(\Lambda/s')^{\frac{1}{2}} \simeq 0.0014$ or 0.14%.

Equation (IV1)'s advantage over other usual fixed-t and fixed-s DR's lies in the fact that $\gamma_\rho(t)$ extracted from it has a larger domain of validity (t varies from -43 to 4).

From equation (IV1) we can derive an equation the integrand of which is independent of S waves, so that we will be spared from the ambiguities which have plagued experimental studies of the I=0 S waves, by doing in this way: first, we write G_1 on a line with a slope a_1, then we write G_1 at the same point (s, t, u) but dispersed on a line with a slope a_2. If we take these two equations and subtract from each other, all S waves will disappear. The equation looks like: (see next page).

We define:

$$t_1' = t_1'(s', a_1, x_{o1})$$

$$t_2' = t_2'(s', a_2, x_{o2})$$

$$z_1' = 1 + \frac{2t_1'}{s'-4}$$

$$z_2' = 1 + \frac{2t_2'}{s'-4}$$

$$ZD3(z_1') = P_3(z_1')/z_1'$$

$$ZD3(z_2') = P_3(z_2')/z_2'$$

$$\frac{1}{\pi}\int_{4}^{\Lambda}\frac{ds'}{(s'-s)(s'-t)(s'-u)}\Bigg\{\Big[2\mathrm{Im}A^{(2)0}(s')-5\mathrm{Im}A^{(2)2}(s')\Big]\frac{15Mo(s')}{2}(z_1'^2-z_2'^2)$$

$$+\ 3\mathrm{Im}A^{(1)1}(s')\ (s'-4)(z_2'^2-z_1'^2)/4$$

$$+\ 7\mathrm{Im}A^{(3)1}(s')\Big[(M_1(s',z_1')ZD3(z_1')\ -$$

$$M_1(s',z_2')ZD3(z_2'))\Big]\Bigg\}$$

$$+\ \frac{1}{\pi}\int_{\Lambda}^{\infty}\frac{ds'}{(s'-s)(s'-t)(s'-u)}\Bigg\{9\mathrm{Im}T^{1}(s',t_1')/2-\mathrm{Im}T^{0}(s',t_1')-$$

$$\Big[9\mathrm{Im}T^{1}(s',t_2')/2+\mathrm{Im}T^{0}(s',t_2')\Big]\ Mo(s')$$

$$+\Big[\mathrm{Im}T^{1}(s',t_1')/2+\mathrm{Im}T^{0}(s',t_1')/3\Big]\ M_1(s',z_1')$$

$$-\Big[\mathrm{Im}T^{1}(s',t_2')/2+\mathrm{Im}T^{0}(s',t_2')/3\Big]\ M_1(s',z_2')\Bigg\}=0$$

(IV10)

Equation (IV10) involves only P, D, F waves. We can now evaluate this equation at different values of a's and thereby get a set of N equations with N unknowns. The solution of this equation gives us:

$$\gamma_{\rho}(t) = 0.69 + 1.84(t/\bar{s}) + 0.50(t/\bar{s})^2 - 0.75(t/\bar{s})^3$$

C. Unsubtracted DR for $G_2(x,y)$

Now we write unsubtracted dispersion relation for G_2 and try to extract information about the $\rho_{\pi\pi}$ Regge residue function from this equation. The unsubtracted $G_2(s,t)$ has the form:

$$G_2(x,a(x-x_o)) = \frac{1}{\pi}\int dx' \frac{1}{x'-x} \text{Disc.} G_2(x',a(x'-x_o))$$

or in terms of the variable s,t,u:

$$G_2(s,t) = \frac{1}{\pi}\int_4^{\infty} ds' \frac{(s'-t')(2s'+t'-4)}{(s'-s)(s'-t)(s'-u)} \text{Disc.} G_2(s',t')$$

If all partial waves with $l \geq 4$ are neglected for $s' < \Lambda$ we can write $G_2(s,t)$ as:

$$G_2(s,t) = \frac{1}{2\pi}\int_4^{\Lambda}\frac{ds'}{(s'-s)(s'-t)(s'-u)}\Big\{ 5\mathrm{Im}A^{(0)2}(s') - 2\mathrm{Im}A^{(0)0}(s') + \Big[5(5\mathrm{Im}A^{(2)2}(s') - 2\mathrm{Im}A^{(2)0}(s')\Big]P_2(z') + 9\mathrm{Im}A^{(1)1}(s')\ (3s'-4)/(s'-4) + \frac{21\mathrm{Im}A^{(3)1}(s')\ (3s'-4)ZD3(z')}{(s'-4)}\Big\}$$

$$+ \frac{1}{2\pi}\int_{\Lambda}^{\infty}\frac{ds'}{(s'-s)(s'-t)(s'-u)}\Big\{(\mathrm{Im}T^{0}(s',t') - 9\mathrm{Im}T^{1}(s',t')) + \frac{(\mathrm{Im}T^{0}(s',t') + 1.5\mathrm{Im}T^{1}(s',t'))\ (3s'-4)}{(s'-4)z'}\Big\}$$

where:

$$z' = 1 + 2t'/(s'-4)$$

$$P_2(z') = 0.5(3z'^2 - 1)$$

$$P_3(z') = 0.5(5z'^3 - 3z')$$

$$ZD3(z') = P_3(z')/z'$$

For the same reason stated previously in the case of $G_1(s,t)$, we write $G_2(s,t)$ at s=4, t=27.165 (x=46.122, y=-46.122). We assume that the contribution to G_2 of waves higher than F is negligible. This is inferred from the fact that, within all reasonable estimates, the contribution of F and G waves is indeed negligible. Here again ρ resonance dominates G_2 (G_ρ = 0.14). $G_2(s,t)$ involves not only P, D and F waves, but also S-waves, however we can use the same method as in the case of G_1 to eliminate S-waves.

More information will be gained if we solve the equations for $G_1(s,t)$ and $G_2(s,t)$ simultaneously. The results will be displayed in figure (4) and figure (5).

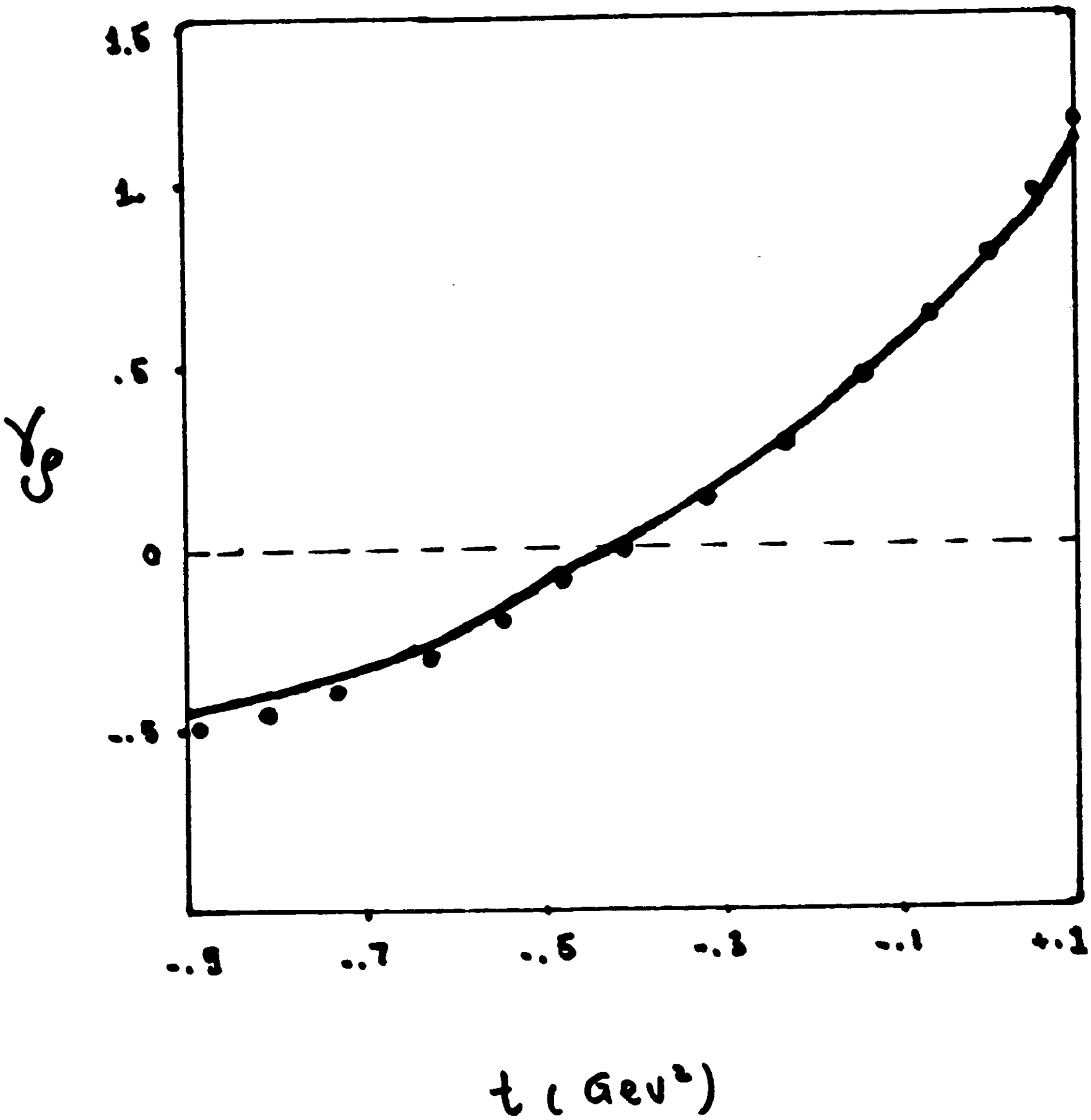


FIGURE 1

Figure 1:

Solid line displays $\gamma(t)$ defined by equation (7) of Reference (44).

Closed circles depict $\gamma(t)$ defined by (IV4), (IV5),(IV6), and (IV7). Number of terms included are from 3 to 6. Note that the values of $\gamma(t)$ are the same everywhere for t from -0.9 to 0.1 . From this graph $\gamma(t_z)=0$, with $t_z = -.40\ \mathrm{Gev}^2$.

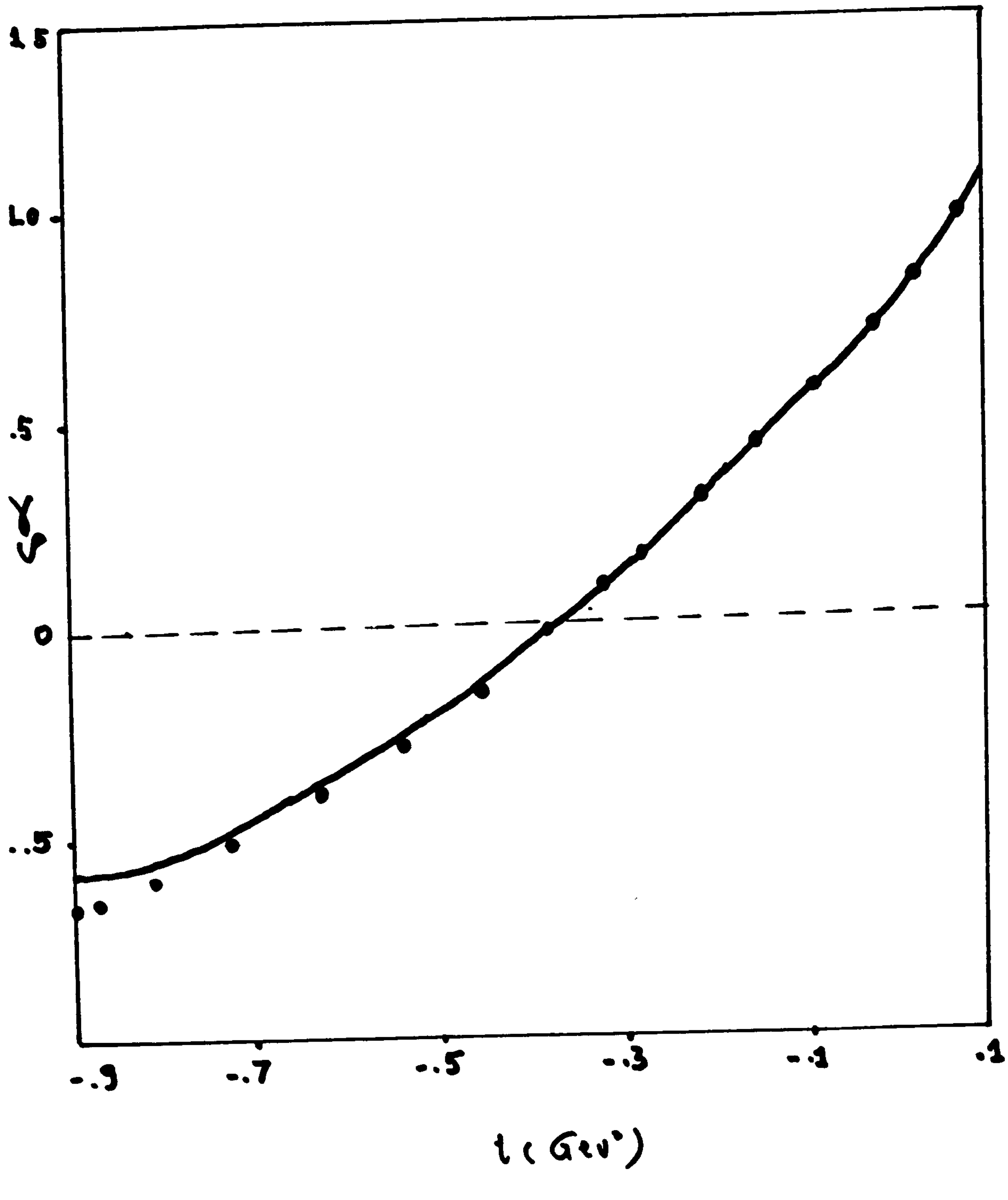


FIGURE 2

Figure 2:

Solid line displays values of $\gamma_{\rho}(t)$, assuming $\mathrm{Im}T^{0}(s,t)=0$.

Closed circles depicts values of $\gamma_{\rho}(t)$, assuming $\gamma_{P}(t)=1.34\ (s/\bar{s})^{\alpha_P(t)}$ and $\gamma_{f}(t)=3/2\ \gamma_{\rho}(t)$, $\alpha_{f}(t)=\alpha_{\rho}(t)$.

The closeness of the two curves shows that the contribution of $\mathrm{Im}T^{0}(s,t)$ to $\partial\gamma_{\rho}(t)/\partial t$ is negligible.

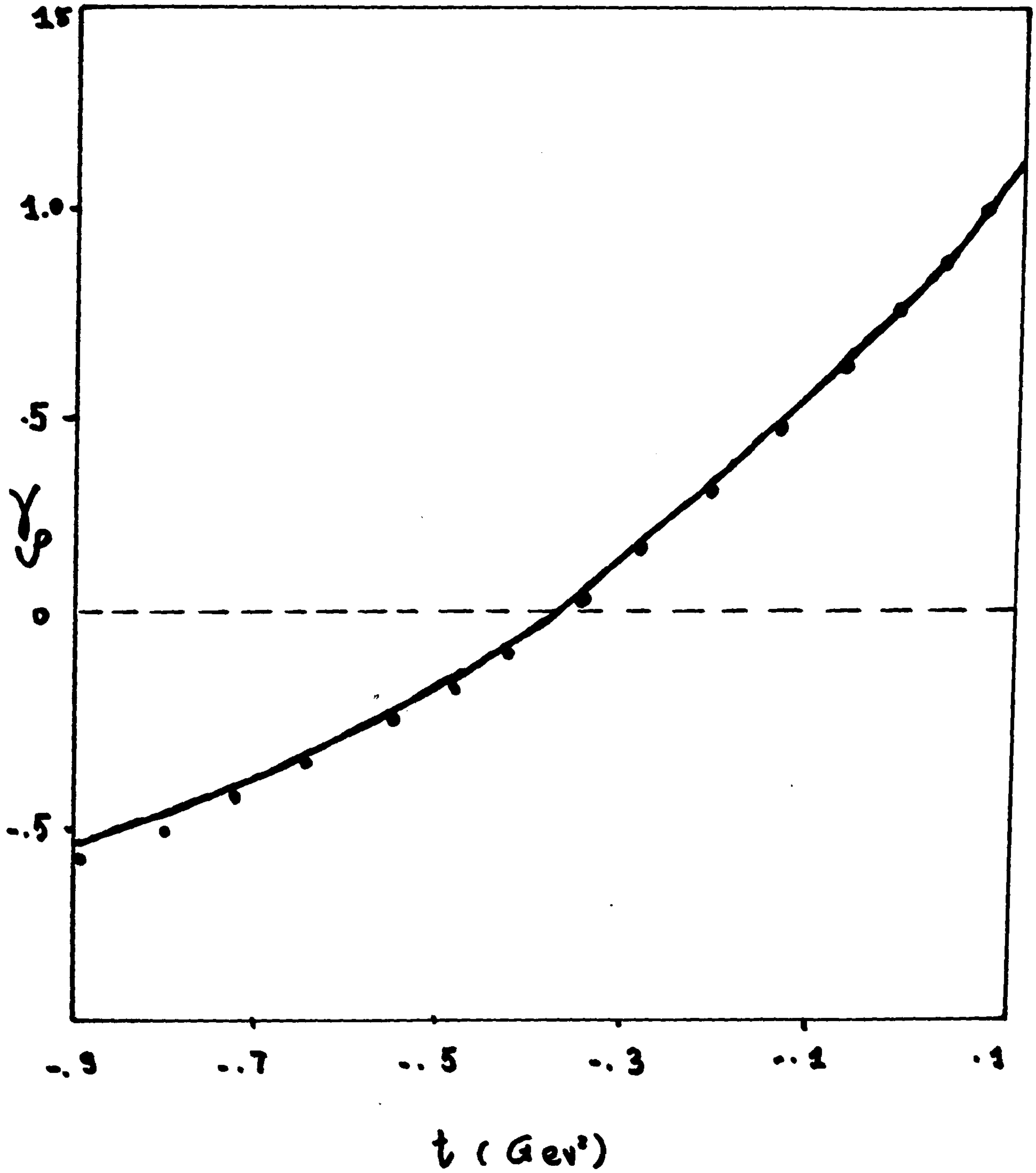


FIGURE 3

Figure 3:

Values of $\gamma_\rho(t)$ for t from −0.9 to 0.1 Gev^2, obtained by solving unsubtracted Dispersion Relation for $G_1(s,t)$.

Closed circles are for values of $\gamma_\rho(t)$ obtained from equation $G_1(s,t) = G_L(s,t) + G_H(s,t)$, where $G_L(s,t)$ is the value of $G_1(s,t)$ in the low energies region, G_H value of $G_1(s,t)$ in the high energies region, eq.(IV2).

Line is for values of $\gamma_\rho(t)$ obtained from equation $G_1(s,t) - G_1(s,t) = 0$, where G_1's are written on the lines with slopes a_1 and a_2. This equation does not include S-waves, eq.(IV10).

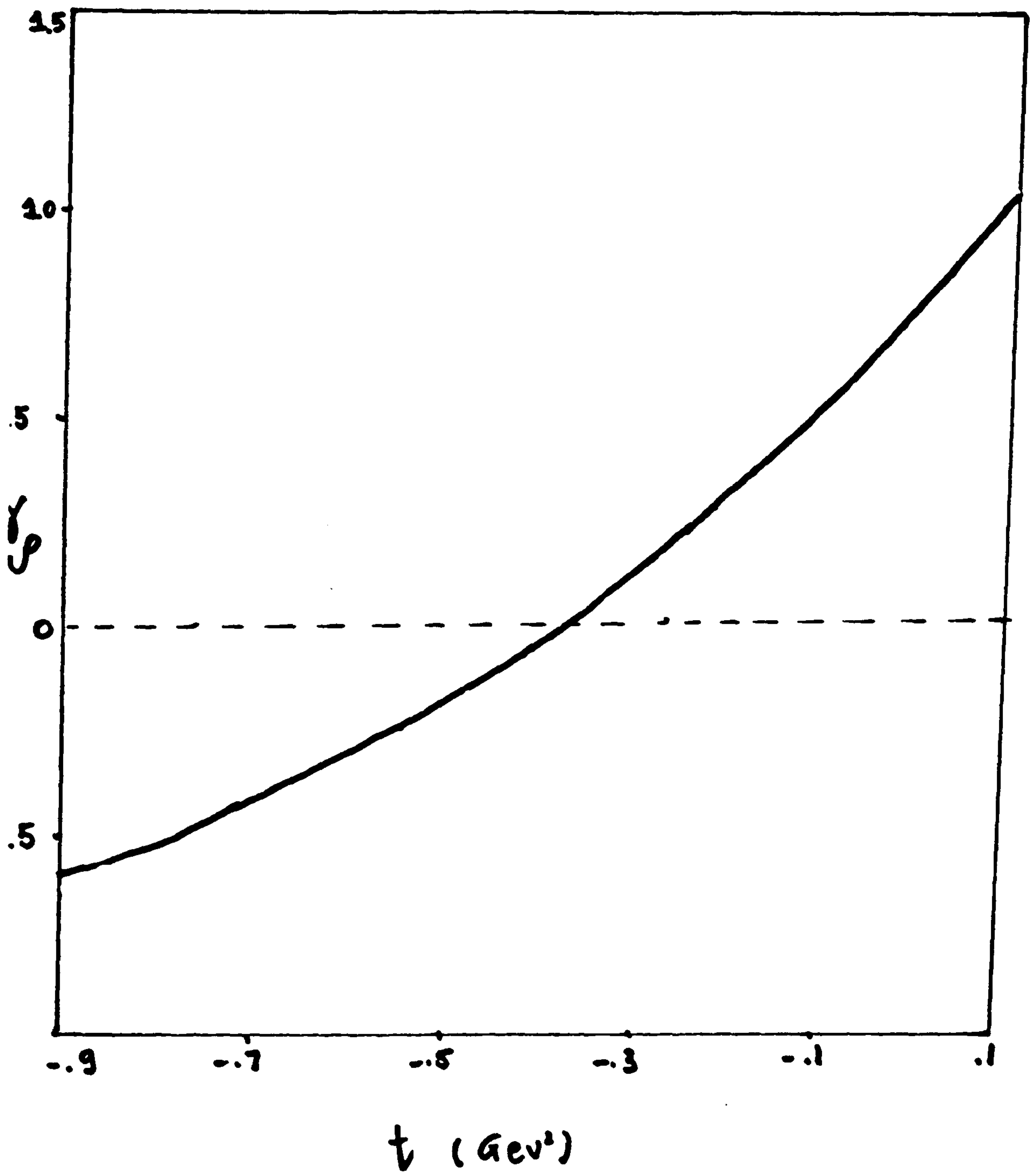


FIGURE 4

Figure 4:

Values of $\gamma(t)$ obtained by solving unsubtracted Dispersion Relations for $G_1(s,t)$ and $G_2(s,t)$ simultaneously.

Solid line displays values of $\gamma(t)$ with 3 terms, including $\gamma(0) = 0.74$.

Values of $\gamma(0)$ are found to fluctuate from 0.71 to 0.82 if the number of terms in the approximation varies from 4 to 10.

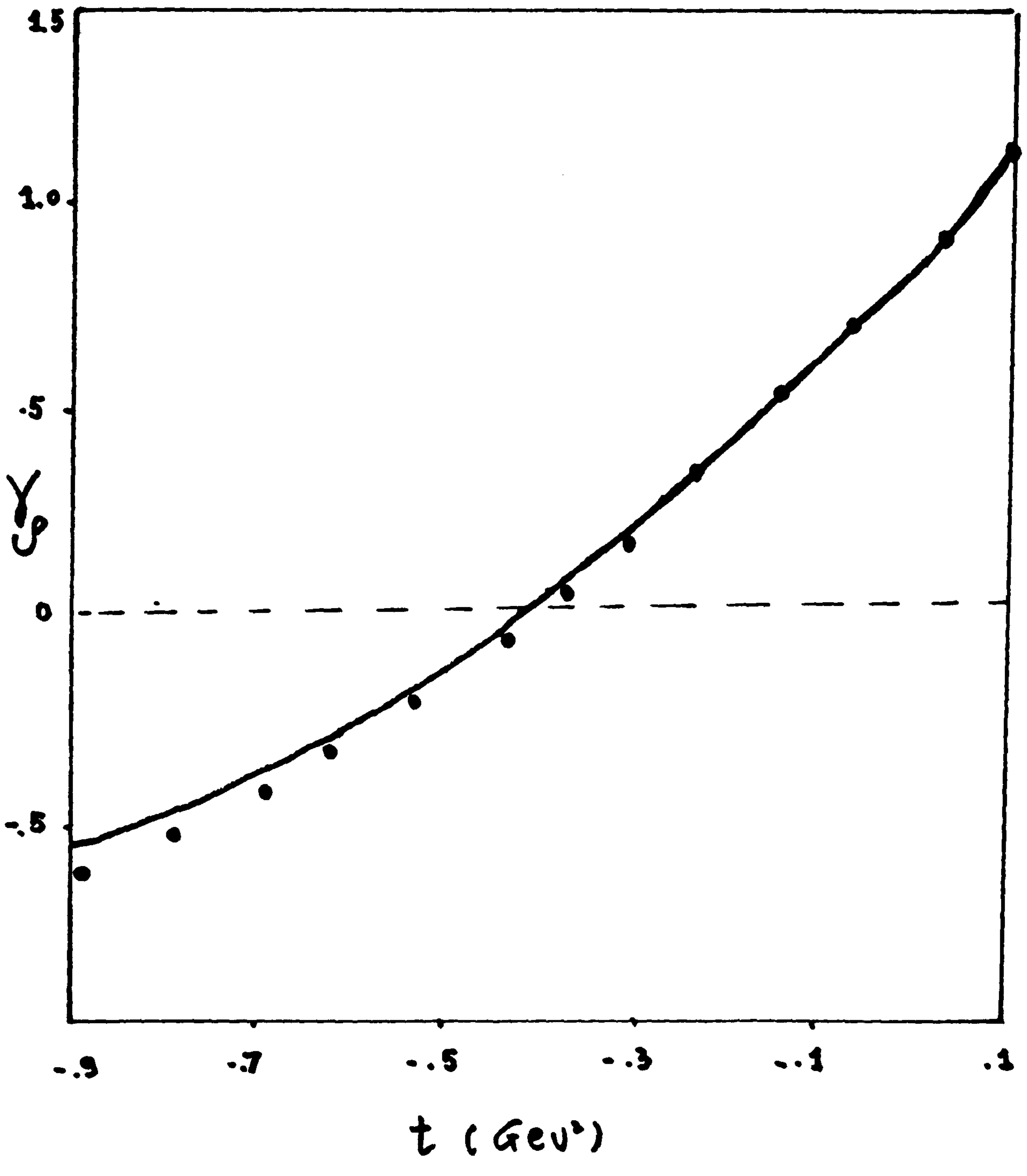


FIGURE 5

Figure 5:

Values of $\gamma_\rho(t)$ obtained by solving $G_1(s,t)$, and by solving $G_1(s,t)$ and $G_2(s,t)$ are compared.

Line represents values of $\gamma_\rho(t)$ from $G_1(s,t)$.

Closed circles display values of $\gamma_\rho(t)$ when $G_1(s,t)$ and $G_2(s,t)$ are solved simultaneously.

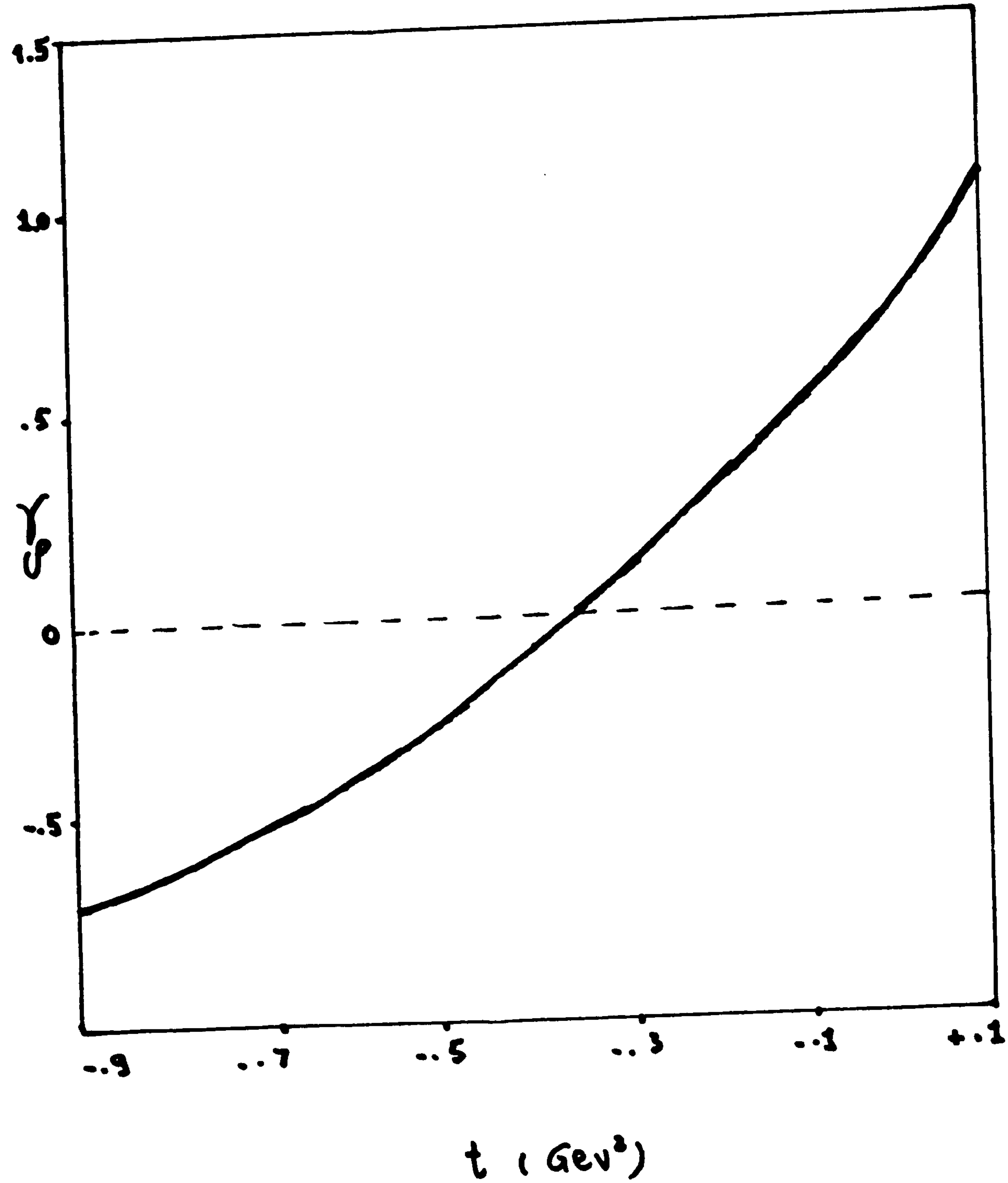


FIGURE 6

Figure 6:

Values of $\gamma_p(t)$ obtained by solving unsubtracted Dispersion Relation for $G_2(s,t)$. We assume $\gamma_p(0)=0.67$, $\text{ImT}^0(s,t)= 1.34(s/\bar{s})^{\alpha(t)}$ and the number of terms included in the approximation are from 3 to 6.

CHAPTER V

PARTIALLY SYMMETRIC AMPLITUDES

A. Introduction

In this part we show how the use of homogeneous variables can be applied to the case of a scattering amplitude which is not totally symmetric. We consider the scattering of identical spin zero mesons described by the s-channel amplitude A(s,t,u). This amplitude is symmetric under the exchange $t \leftrightarrow u$. We assume now that the symmetric t- and u-channels are distinct from the s-channel. Therefore, crossing symmetry implies no new symmetry for A(s,t,u) and we have:

$$A(s,t,u) = A(s,u,t) \qquad \text{(V1)}$$

B. Homogeneous variables

As a consequence of eq. (V1), we can write A(s,t,u) as a function F(x,y) of the variables

$$x = \frac{1}{16}(t+u) = \frac{1}{16}(4-s)$$

$$y = -\frac{1}{64}tu$$

without introducing kinematical singularities.

This change of variables maps the real (s,t,u) plane onto a domain R of the real (x,y)-plane.

$$y = -\frac{1}{64} tu = -\frac{1}{64} \left(\frac{4-s}{2}\right)^2 = -\frac{1}{4} x^2$$

$$R: \quad y > -\frac{x^2}{4}$$

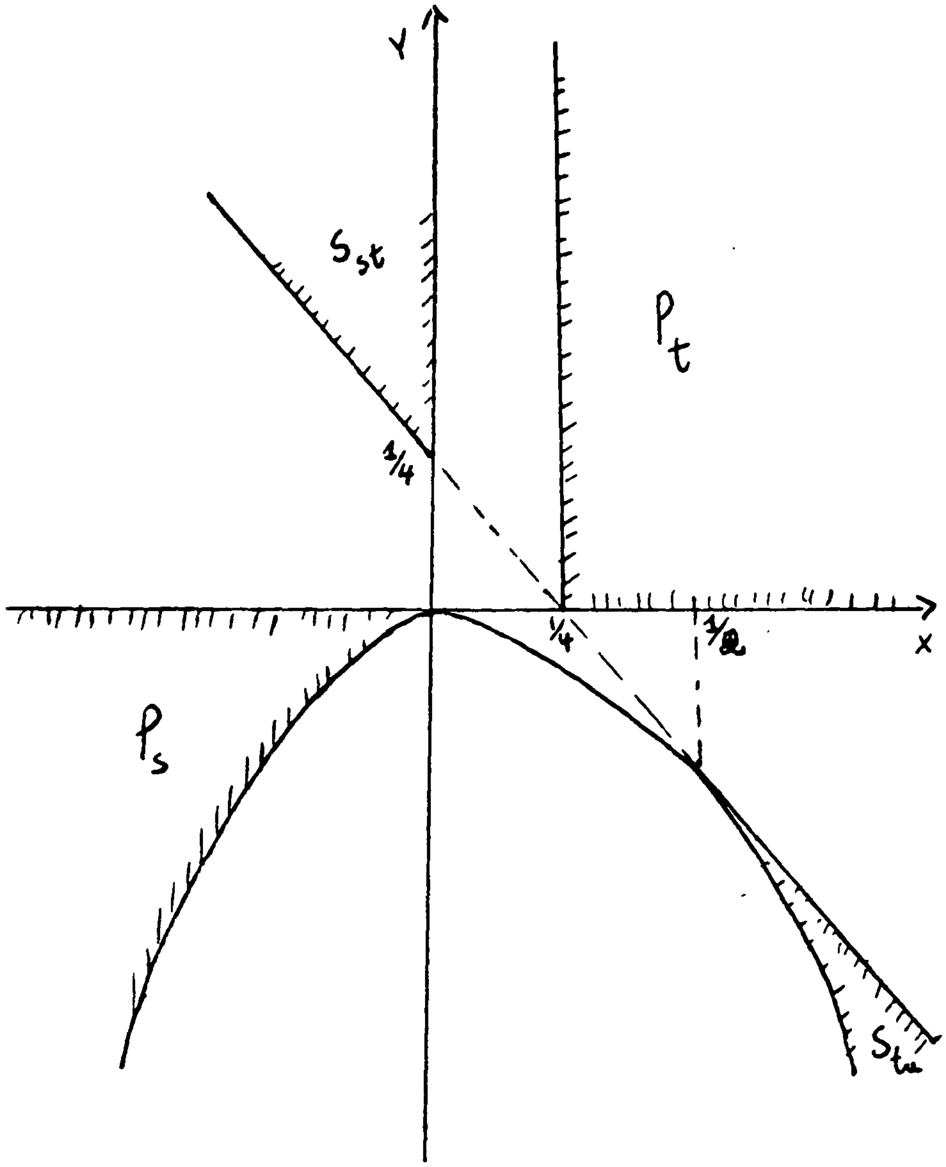
The image of a straight line t = real constant (or u = real constant) is a tangent to the boundary C of R.

P_s: $s > 4 \longrightarrow x < o \qquad -\frac{x^2}{4} < y < o$

P_t: $t > 4 \longrightarrow x > \frac{1}{4} \qquad y > o$

S_{st}: $s > 4,\ t > 4$ and $s > 4,\ u > 4 \qquad x < o \qquad y > \frac{1}{4} - x$

S_{tu}: $u > 4,\ t > 4 \longrightarrow x > \frac{1}{2} \qquad -\frac{x^2}{4} < y < \frac{1}{4} - x$



P_s, P_t: physical regions of s- and t-channels

S_{st}, S_{tu}: double-spectral regions.

C. Partially Symmetric Dispersion Relations

The analytic properties of $F(x,y)$ are not simple, nevertheless, we get simple properties if we consider the values taken by $F(x,y)$ on the complex plane $y = a(x - x_o)$ (a, x_o real). $F(x, a(x-x_o))$ is holomorphic in the cut x-plane. There are two cuts:

a left-hand cut $-\infty < x \leq 0$

and a right-hand cut $x_1 \leq x < \infty$

The lower limit is the solution of the equation:

$$a(x - x_o) = y_1(x)$$

$$y_1 = \begin{cases} \frac{1}{4} - x & \text{for } x < \frac{1}{2} \\ -\frac{x^2}{4} & \text{for } x > \frac{1}{2} \end{cases}$$

For $x \to -\infty$, $y = a(x - x_o)$

$$s = -16x$$

$$y \simeq ax \simeq -\frac{1}{64} tu = -\frac{1}{64} t(4-s-t) \simeq -\frac{1}{64} t(4+16x-t)$$

$$ax \simeq -\frac{xt}{4} \longrightarrow t \to -4a$$

For $x \to \infty$ we have

$$t \simeq 16x, \qquad u \to -4a$$

We know that unitarity and analyticity imply the following, fixed t, asymtotic bound (35-37):

$$|\mathrm{Im}\ A(s,t,u)| < C\ s^{1+\epsilon}, \qquad \epsilon < 1 \quad \text{for } s \to \infty, \qquad t < 4$$

Therefore, we have

$$|\mathrm{Im}\ F(x,\ a(x - x_0))| < C'\ x^{(1+\epsilon)} \qquad \text{for} \quad x \to \infty \quad, a > -1$$

Thus, even without assuming Regge theory, we can write Dispersion Relations which require at most two subtractions (59).

For instance, we may write:

$$F(x,\ a(x - x_0)) = \frac{1}{x_1}\Big\{F(x_1,\ a(x_1 - x_0))\ x - F(o,\ -ax_0)(x - x_1)\Big\}$$

$$+ \frac{1}{\pi} x(x - x_1) \left\{ \int_{-\infty}^{0} dx' \frac{\mathrm{Im}\ F(x' + i\epsilon,\ \ a(x' + i\epsilon - x_0))}{x'(x' - x_1)(x' - x)} \right. \qquad \text{(V2)}$$

$$\left. + \int_{x_1}^{\infty} dx' \frac{\mathrm{Im}\ F(x' + i\epsilon,\ a(x' + i\epsilon - x_0))}{x'(x' - x_1)(x' - x)} \right\}$$

Evidently we cannot use equation (V2) to make any meaningful calculation, simply because the subtraction constants are unknown.

It will be seen later that even Regge theory is assumed, the amplitudes in this representation do not obey unsubtracted Dispersion Relations. However, by combining amplitudes with different isospins an unsubtracted Dispersion Relation can be obtained. Tnerefore, first, we write unsubtracted Dispersion Relations for single isospin amplitudes (the integrals of which diverge):

$$F^I(x,ax+b) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx'\frac{\text{Disc.}F^I(x',ax'+b)}{x'-x}$$

For I = 0,2 we have:

$$A^I(s,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} ds'\frac{\text{Im}A^I(s',t')}{s'-s}$$

$$= \frac{1}{\pi}\int_{4}^{\infty} ds'\frac{\text{Im}A^I(s',t')}{s'-s} + \frac{1}{\pi}\, C_{st}^{II'}(-1)^{I'}\int_{4}^{\infty} du'\frac{(u'-t')}{(u'-t)(u'-u)}\,\text{Im}T^{I'}(u',t_u) \quad \text{(V3)}$$

Where:

$$t' = \frac{1}{2}\left\{4-s' + \sqrt{(s'-4)^2-16as'-16b}\right\}$$

$$u' = \frac{1}{2}\left\{4-s' - \sqrt{(s'-4)^2-16as'-16b}\right\}$$

$$t_u = \frac{-4(au'-4a-b)}{u'+4a}$$

$$C_{st} = \frac{1}{6}\begin{bmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^I(s,t) = T^I(t,s)$$

When $s' \to \infty \qquad t' \to -4a$

$u' \to \infty \qquad t_u \to -4a$

$A^1(s,t)$ is not t,u symmetric, therefore, we write Dispersion Relation for $A^1(s,t)/(t-u)$. Then for I=1, we have:

$$A^1(s,t) = \frac{1}{\pi}\int_4^\infty ds' \frac{(t-u)}{(s'-s)(t'-u')} \mathrm{Im}A^1(s',t') - \frac{1}{\pi}\sum C_{st}^{1I'}(-1)^{I'}\int_4^\infty du' \frac{(t-u)}{(u'-t)(u'-u)} \mathrm{Im}T^{I'}(u',t_u) \quad (V4)$$

D. New Sum Rules and their Applications

Equations (V3) and (V4) have the advantages of being simple and having explicit contributions from the s-channel and the t,u-channel. However, they do not converge and therefore, do not satisfy unsubtracted Dispersion Relations. But, we get a convergent sum rule by writing Dispersion Relation for $A^0(s,t)-A^2(s,t)$.

$$\begin{aligned} A^0(s,t)-A^2(s,t) = {} & \frac{1}{\pi}\int_4^\Lambda ds' \frac{\mathrm{Im}A^0(s',t')-\mathrm{Im}A^2(s',t')}{s'-s} \\ & - \frac{3}{2\pi}\int_4^\Lambda du' \frac{(u'-t')}{(u'-t)(u'-u)}\left[\mathrm{Im}A^1(u',t_u)-\mathrm{Im}A^2(u',t_u)\right] \\ & + \frac{3}{2\pi}\int_\Lambda^\infty \frac{ds'}{s'-s}\left[\mathrm{Im}T^1(s',t')+\mathrm{Im}T^2(s't')\right] \\ & - \frac{3}{2\pi}\int_\Lambda^\infty du' \frac{(u'-t')}{(u'-t)(u'-u)}\left[\mathrm{Im}T^1(u',t_u)-\mathrm{Im}T^2(u',t_u)\right] \quad (V5) \end{aligned}$$

Where $\Lambda = (1.9\ \mathrm{Gev})^2$

We choose $s=t=u=C_0=4/3$, then $A^0(C_0,C_0)-A^2(C_0,C_0) = -3\lambda$, from this choice we have (assuming $\mathrm{Im}T^2(s',t')=0$ for $s'>\Lambda$):

$$\lambda = -\frac{1}{3\pi}\int_4^{\Lambda}\frac{ds'}{s'-C_0}\left[\mathrm{Im}A^0(s',t')-\mathrm{Im}A^2(s',t')\right]$$

$$+\frac{1}{2\pi}\int_4^{\Lambda} du'\frac{u'-t'}{(u'-C_0)^2}\left[\mathrm{Im}A^1(u',t_u)-\mathrm{Im}A^2(u',t_u)\right]$$

$$-\frac{1}{2\pi}\int_{\Lambda}^{\infty}\left[ds'\frac{\mathrm{Im}T^1(s',t')}{s'-C_0} - du'\frac{(u'-t')}{(u'-C_0)^2}\mathrm{Im}T^1(u',t_u)\right] \quad \text{(V6)}$$

We can also write Dispersion Relation for

$$\frac{A^0(s,t)}{3} - \frac{(s-c)A^1(s,t)}{2(t-u)} + \frac{A^2(s,t)}{6} \quad \text{(V7)}$$

The value for λ can be calculated from the combination (V7) by choosing $s=t=u=4/3$:

$$\lambda = \frac{1}{\pi}\int_4^\Lambda ds' \frac{2\mathrm{Im}A^0(s',t')+\mathrm{Im}A^2(s',t')}{4(4-3s')} + \frac{1}{\pi}\int_4^\Lambda ds' \frac{\mathrm{Im}A^1(s',t')}{4(t'-u')}$$

$$+ \frac{1}{\pi}\int_4^\Lambda du' \left[\frac{(3u'+24a+4)}{2(4-3u')(u'+4a)} \left(\frac{\mathrm{Im}A^0(u',t_u)}{6} - \frac{\mathrm{Im}A^1(u',t_u)}{4} + \frac{7\mathrm{Im}A^2(u',t_u)}{12} \right) \right.$$

$$\left. - \frac{(-9u'^2+24u'-16)}{9u'^2+72au'-96a-16} \left(\frac{\mathrm{Im}A^0(u',t_u)}{6} - \frac{\mathrm{Im}A^1(u',t_u)}{4} - \frac{5\mathrm{Im}A^2(u',t_u)}{12} \right) \right]$$

$$+ \frac{1}{\pi}\int_\Lambda^\infty ds' \frac{3t'-4}{(4-3s')(t'-u')} \left(\frac{\mathrm{Im}T^0(s',t')}{6} + \frac{\mathrm{Im}T^1(s',t')}{4} \right)$$

$$+ \frac{1}{\pi}\int_\Lambda^\infty du' \frac{(3u'+24a+4)}{(4-3u')(u'+4a)} \frac{(36au'+12u'-48a-16)}{(9u'^2+72au'-96a-16)} \left(\frac{\mathrm{Im}T^0(u',t_u)}{6} - \frac{\mathrm{Im}T^1(u',t_u)}{4} \right) \tag{V8}$$

Equations (V6) and (V8) are valid only if the line $y=ax+b$, on which the Dispersion Relations are written, does not cross the forbidden regions (the double spectral region and the regions outside the large Lehman ellipses).This puts restrictions on the maximum value that s and a can have.Investigations of the boundary curves of the Mandelstam double spectral functions and of the large Lehmann ellipses show that $s_{max} = 120.31\ m_\pi^2$ and $a_{max} = 9.57$.

The line which passes through the point s=t=u=4/3 and does not cross the forbidden regions has the maximum slope a = 3.87. Theoretically, we can use eq.'s V6 and V8 to calculate λ . To do so, between threshold and $M_{\pi\pi}$ = 0.9 Gev, we use the parametrization derived in reference (44) Between 0.9 and 1.9 Gev, the experimental data of Hyams <u>et</u>. <u>al</u>. and of Durusoy <u>et</u>. <u>al</u>. are used. Above 1.9 Gev, we assume Regge theory to evaluate $ImT^0(s,t)$ and $ImT^1(s,t)$.

However, results from these calculations are not reliable, because the exact value of the cut-off Λ is unknown. More accurate informations will be obtained if we write unsubtracted DR's on a line with a maximum slope a for which $P_3(z)=0$, at $s=s_g=2.84$ Gev^2; because this line does <u>not</u> pass through the point s=t=4/3, we also write once-subtracted DR's on a line with slope a=0 (equations V9 and V10).

The results are displayed in figures 7 and 8.The uncertainties of our results come from the uncertainties of the experimental data for S^0, S^2 waves as well as for ρ , f_0 and g resonances. We found that $\Delta\lambda$ =0.015 (equation V9) and $\Delta\lambda$ =0.012 (equation V10).

The results obtained from these equations are consistent with the one obtained by E.P. Tryon, namely λ =-0.013±0.010.

$$A^0(s_0,t_0) - A^2(s_0,t_0) = A^0(s,t) - A^2(s,t)$$

$$+ \frac{1}{\pi}\int_4^{\Lambda} ds'\left[\frac{1}{s'-s_0} - \frac{1}{s'-s}\right]\left[\mathrm{Im}A^0(s',t_0') - \mathrm{Im}A^2(s',t_0')\right]$$

$$- \frac{3}{2\pi}\int_4^{\Lambda} ds'\left[\frac{1}{s'-t_0} + \frac{1}{s'-u_0} - \frac{1}{s'-t} - \frac{1}{s'-u}\right]\left[\mathrm{Im}A^1(s',\tilde{t}_0) - \mathrm{Im}A^2(s',\tilde{t}_0)\right]$$

$$+ \frac{3}{2\pi}\int_{\Lambda}^{\infty} ds'\left[\frac{1}{s'-s_0} - \frac{1}{s'-s}\right]\mathrm{Im}T^1(s',t_0')$$

$$- \frac{3}{2\pi}\int_{\Lambda}^{\infty} ds'\left[\frac{1}{s'-t_0} + \frac{1}{s'-u_0} - \frac{1}{s'-t} - \frac{1}{s'-u}\right]\mathrm{Im}T^1(s',\tilde{t}_0)$$

$$= \frac{1}{\pi}\int_4^{\Lambda}\frac{ds'}{s'-s}\left[\mathrm{Im}A^0(s',t') - \mathrm{Im}A^2(s',t')\right]$$

$$+ \frac{1}{\pi}\int_4^{\Lambda} ds'\left[\frac{1}{s'-s_0} - \frac{1}{s'-s}\right]\left[\mathrm{Im}A^0(s',t_0') - \mathrm{Im}A^2(s',t_0')\right]$$

$$- \frac{3}{2\pi}\int_4^{\Lambda} ds'\left[\frac{1}{s'-t} + \frac{1}{s'-u} - \frac{1}{s'+4a}\right]\left[\mathrm{Im}A^1(s',t_u) - \mathrm{Im}A^2(s',t_u)\right]$$

$$- \frac{3}{2\pi}\int_4^{\Lambda} ds'\left[\frac{1}{s'-t_0} + \frac{1}{s'-u_0} - \frac{1}{s'-t} - \frac{1}{s'-u}\right]\left[\mathrm{Im}A^1(s',\tilde{t}_0) - \mathrm{Im}A^2(s',\tilde{t}_0)\right]$$

$$+ \frac{3}{2\pi}\int_{\Lambda}^{\infty}\frac{ds'}{s'-s}\mathrm{Im}T^1(s',t') + \frac{3}{2\pi}\int_4 ds'\left[\frac{1}{s'-s_0} - \frac{1}{s'-s}\right]\mathrm{Im}T^1(s',\tilde{t}_0)$$

$$- \frac{3}{2\pi}\int_{\Lambda}^{\infty} ds'\left[\frac{1}{s'-t} + \frac{1}{s'-u} - \frac{1}{s'+4a}\right]\mathrm{Im}T^1(s',t_u)$$

$$- \frac{3}{2\pi}\int_{\Lambda}^{\infty} ds'\left[\frac{1}{s'-t_0} + \frac{1}{s'-u_0} - \frac{1}{s'-t} - \frac{1}{s'-u}\right]\mathrm{Im}T^1(s',\tilde{t}_0)$$

(V9)

$\lambda =$

$$- \frac{1}{12\pi}\int_4^\infty ds' \, \frac{2\mathrm{Im}A^0(s',t') + \mathrm{Im}A^2(s',t')}{s'-s} + \frac{1}{4\pi}\int_4^\infty ds' \, \frac{s'-s_0}{(t'-u')(s'-s)} \, \mathrm{Im}A^1(s',t')$$

$$- \frac{1}{12\pi}\int_4^\infty du' \left(\frac{1}{u'-t} + \frac{1}{u'-u} - \frac{1}{u'+4a}\right)\left(\mathrm{Im}T^0(u',t_u) - \tfrac{3}{2}\mathrm{Im}T^1(u',t_u) + \tfrac{7}{2}\mathrm{Im}T^2(u',t_u)\right)$$

$$- \frac{1}{4\pi}\int_4^\infty du' \, \frac{(s'_t-s_0)}{(u'-t)(u'-u)} \left(\frac{\mathrm{Im}T^0(u',t_u)}{3} - \frac{\mathrm{Im}T^1(u',t_u)}{2} - \frac{5\mathrm{Im}T^2(u',t_u)}{6}\right)$$

$$- \frac{1}{12\pi}\int_4^\infty ds' \left(2\mathrm{Im}A^0(s',t'_0) + \mathrm{Im}A^2(s',t'_0)\right)\left(\frac{1}{s'-s_0} - \frac{1}{s'-s}\right)$$

$$+ \frac{1}{4\pi}\int_4^\infty ds' \, \frac{\mathrm{Im}A^1(s',t'_0)}{t'-u'_0} - \frac{1}{4\pi}\int_4^\infty ds' \, \frac{\mathrm{Im}A^1(s',t'_0)(s'_0-s_0)}{(t'_0-u'_0)(s'-s)}$$

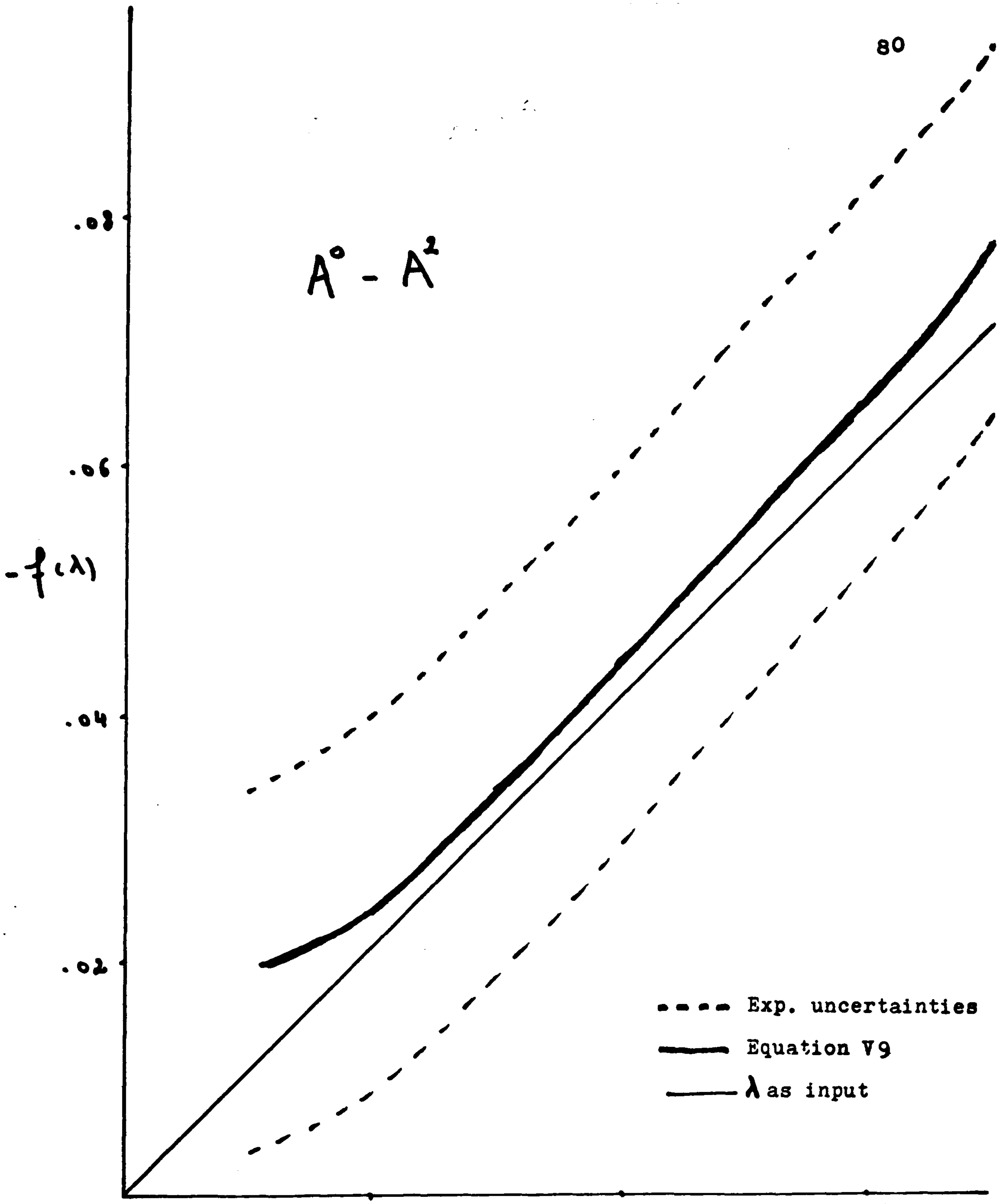
$$- \frac{1}{12\pi}\int_4^\infty du' \left(\frac{1}{u'-t_0} + \frac{1}{u'-u_0} - \frac{1}{u'-u} - \frac{1}{u'-t}\right)\Big(\mathrm{Im}T^0(u',\tilde{t}_0) - \tfrac{3}{2}\mathrm{Im}T^1(u',\tilde{t}_0) + \tfrac{7}{2}\mathrm{Im}T^2(u',\tilde{t}_0)\Big)$$

$$- \frac{1}{4\pi}\int_4^\infty du' \left(\frac{s'_0-s_0}{(u'-t_0)(u'-u_0)} - \frac{s'_0-s_0}{(u'-t)(u'-u)}\right)\left(\frac{\mathrm{Im}T^0(u',\tilde{t}_0)}{3} - \frac{\mathrm{Im}T^1(u',\tilde{t}_0)}{2} - \frac{5\mathrm{Im}T^2(u',\tilde{t}_0)}{6}\right)$$

where

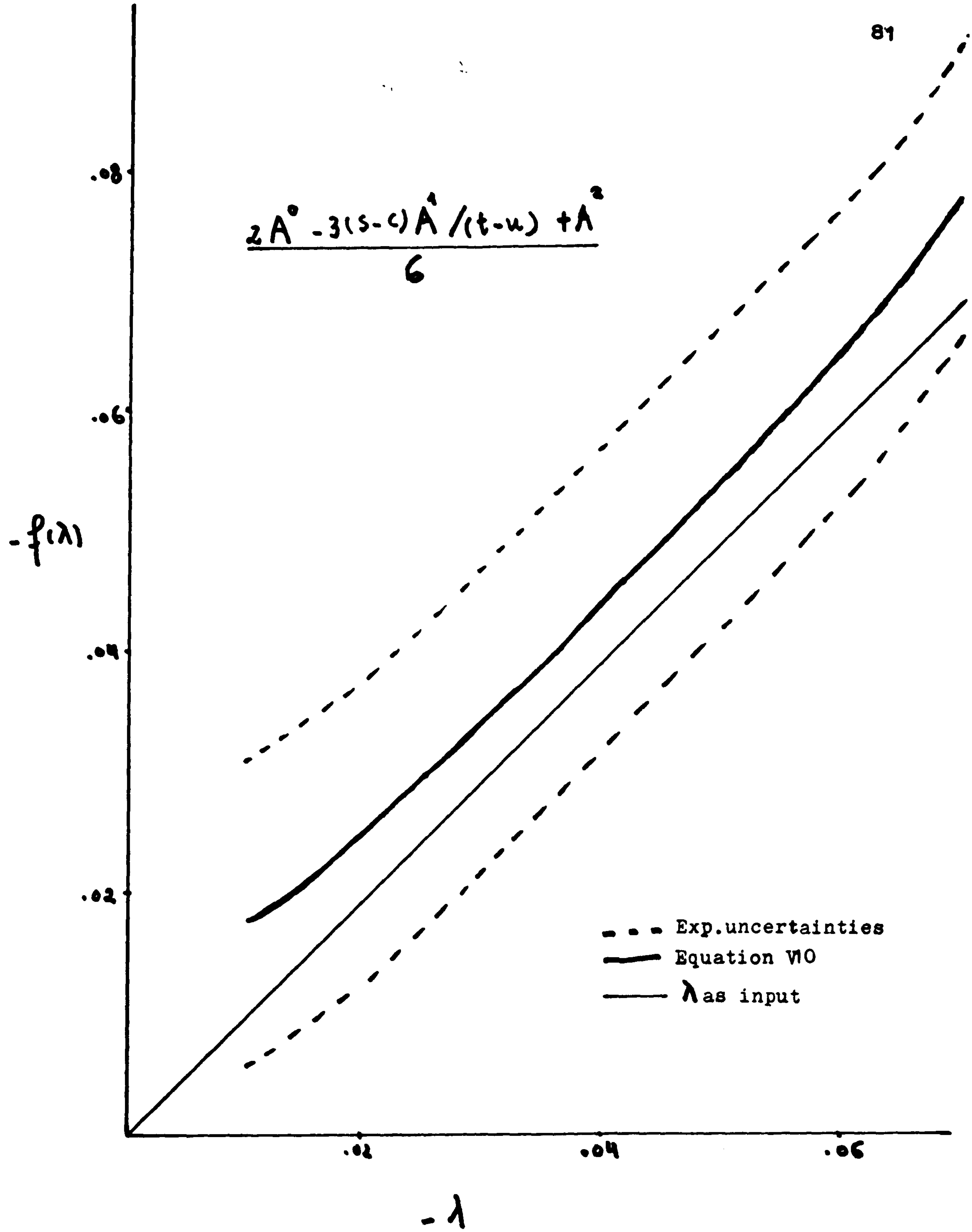
$$t'_0 = .5\left\{4-s' + \sqrt{(s'-4)^2-16b}\right\} \qquad \text{(V10)}$$

$$\tilde{t}_0 = \frac{4b}{u'}$$



A° - A²
-f(λ)
.08
.06
.04
.02
.02
.04
.06
-λ
Exp. uncertainties
Equation V9
λ as input

FIGURE 7



$\frac{2A^0 - 3(s-c)A^1/(t-u) + A^2}{6}$
.08
.06
.04
.02
$-f(\lambda)$
.02
.04
.06
$-\lambda$
Exp.uncertainties
Equation VI0
λ as input

FIGURE 8

CONCLUSION

We have studied an approach which takes into account the full crossing symmetry. The scattering amplitudes $G_k(s,t,u)$ we considered have simple properties in the variables s,t, and u.

We have written unsubtracted dispersions in this new representation and tried to extract information about $\rho\pi\pi$ Regge residue function. It was found that $\left.\frac{\partial \gamma_\rho(t)}{\partial t}\right|_{t=0} = \gamma'_\rho$ =1.81. This is to be compared with γ'_ρ = 1.78 by E.P.Tryon, who used the same inputs in his calculations. Although, in our equations, the contributions of $\text{Im}T^0(s,t)$ are small, the lack of precise knowledge of the Pomeron parameters make it difficult to have a more accurate knowledge of the $\gamma_\rho(t)$ parameters. We found that $\gamma_\rho(0)$ fluctuates between 0.71 and 0.82.

We have also written Dispersion Relations which are symmetric in t and u only, and we used these Dispersion Relations to study the value of the amplitudes at the symmetry point . Here again the uncertainties of our results come from the uncertainties of the experimental data we used. We found that $\Delta\lambda$=0.015 (equation V9) and $\Delta\lambda$ =0.012 (equation V10).

However, we did not exhaust all the advantages that the new representation might offer: the restriction $y = a(x - x_0)$ limits the applications to only a family of curves in the (s,t,u) space. Other relationships between x and y will open the way to the applications to other families of curves. The problem here is to find these relationships, which might enlarge further the domain of validity of our equations. This possibility may merit further study.

REFERENCES

1. R. Kronig, Physica 12, 543(1946).
2. M. Gell-Mann, M.L. Goldberger, and W. Thirring, Phys. Rev. 95, 1612(1954).
3. M. Gell-Mann, Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, Interscience, N.Y. (1956).
4. M.L. Goldberger, Phys. Rev. 99, 979(1955).
5. M.L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys., Lpz., 2, 226(1957).
6. M.L. Goldberger and S.B. Treiman, Phys. Rev. 110, 1178(1958a), Phys. Rev. 111, 354(1958b).
7. M.L.Goldberger, A.S. Wightman, R. Omnes, G Kallen, G.F. Chew, S.B. Treiman, and Y. Yamaguchi, Relations de Dispersion et Particules elementaires, Ed. C. Dewitt and R. Omnes (Hermann, Paris, 1960).
8. S. Mandelstam, Phys. Rev. 112, 1344(1958).
9. Phys. Rev. 115, 1741(1958a).
10. Phys. Rev. 115, 1752(1958b).
11. Phys. Rev. Letters 4, 84(1960a).
12. Nuovo Cimento 15, 658(1960b).
13. O.W. Greenberg and F.E. Low, Phys. Rev. 124, 2047(1961).
14. H.J. Bremermann, R. Oehme, and J.G. Taylor, Phys. Rev. 109, 2178(1958).
15. B.R. Desai, Phys. Rev. Letters 6, 497(1961).

16. K. Symanzik, Phys. Rev. 105, 743(1957).

17. Progr. Theor. Phys., Japan, 20, 690(1958).

18. Dispersion Relations, Lecture Note, University of Rome (1959).

19. J. Math. Phys. 1, 249(1960).

20. N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, Interscience, N.Y. (1957).

21. G.F. Chew and S. Mandelstam, Phys. Rev. 119, 467(1960).

22. Nuovo Cimento 19, 752(1961).

23. G.F. Chew, S. Mandelstam, and H.P. Noyes, Phys. Rev. 119, 478(1960).

24. H. Lehmann, Nuovo Cimento 10, 579(1958).

25. Nuovo Cimento 14, 153(1959).

26. F.J. Yndurain, Rev. Mod. Phys. 44, 645(1972).

27. B.R. Webber, Phys. Rev. 3, 1971(1971).

28. A. Martin, Phys. Rev. 129, 1432(1963).

29. Nuovo Cimento 42A, 930(1966).

30. Nuovo Cimento 43A, 1220(1966).

31. Nuovo Cimento 53A, 946(1968).

32. Nuovo Cimento 58A, 303(1969).

33. Scattering Theory, Lectures Notes in Physics, vol. 3, Springer.

34. R.J. Eden, Rev. Mod. Phys. 43, 15(1971).

35. M. Froissart, Phys. Rev. 123, 1054(1961).

36. A. Martin, Phys. Rev. 129, 432(1963).

37. Y.S. Jin and A. Martin, Phys. Rev. 35B, 1375(1964).

38. R. Roskies, Nuovo Cimento 65A, 467(1970).

39. Helv. Phys. Acta 39, 228(1966).

40. G. Kaiser, Nuovo Cimento Letters 2, 1373(1971).

41. Nucl. Phys. B43, 345(1972).

42. J.T. Carroll et al., Phys. Rev. Letters 28, 318(1972).

43. E.P. Tryon, Phys. Rev. D8, 1586(1973).

44. Phys. Rev. D11, 698(1975).

45. B. Hyans et al., Nucl. Phys. B64, 134(1973).

46. N.B. DuRusoy et al., Phys. Lett. 45B, 517(1973).

47. D. Morgan and G. Shaw, Nucl. Phys. B10, 261(1967).

48. M.G. Olsson, Phys. Rev. 162, 1338(1967).

49. A. Ukawa et al., Nucl. Phys. B72, 474(1974).

50. G. Wanders, Nuovo Cimento 63A, 108(1969).

51 W. Rarita et al., Phys. Rev. 165, 1615(1968).

52. M.G. Olsson, Phys. Rev. 162, 1338(1967).

53. M. Souza, Saclay preprint D. Ph-T/73-54, July, 1973.

54. J.L. Basdevant et al., Phys. Letters 41B, 173(1972).

55. J.C. Le Guillou et al., Nuovo Cimento 5A, 659(1971).

56. J.L. Basdevant and C. Schomblond, Phys. Letters 45B, 48(1973).

57. T. Eguchi et al., Phys. Letters 48B, 56(1973).

58. Nucl. Phys. 74B, 102(1974).

59. Y.S. Jin and A. Martin, Nuovo Cimento B135, 1369 and 1375(1964).

60. G. Wanders, Nuovo Cimento 63A, 108(1969).

61. R. Roskies, Phys. Rev. D2, 247 and 1649(1970).

62. D.H. Lyth, Nuovo Cimento Letters 2, 724(1969).

63. S.M. Roy, Phys. Letters 36B, 353(1971).

64. J.L. Basdevant et al., Phys. Letters 41B, 178(1972).

65. B. Bonnier and P. Gauron, Nucl. Phys. B52, 506(1973).

66. B. Bonnier and J. Procureur, Nucl. Phys. 1361, 155(1973).

67. M.R. Pennington and S.D. Protopopescu, Phys. Rev. D7, 1429 and 2591(1973).

68. D. Morgan and G. Shaw, Phys. Rev. D2, 520(1970).

69. J.L. Petersen, Phys. Reports 2, 155(1971).

70. M.R. Pennington, Nuovo Cimento 25A, 149(1975).

71. R. Roskies. Phys. Rev. D2, 247 and 1649(1970).

72. P. Grassberger, Nucl. Phys. 1370, 141(1974).

73. A.K. Common, Nuovo Cimento 65A, 581(1970).

74. F.J. Yndurain, Nuovo Cimento 64A, 225(1969).

75. P. Grassberger and H. Kuhnelt, Nucl. Phys. B57, 332(1973).