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ERGODIC PROPERTIES OF GENERALIZED AFFINITIES OF COMPACT NILMANIFOLDS

by

EVELYN MAYER

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Faculty in Mathematics in partial ful-
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INTRODUCTION

Let N be a connected, simply-connected nilpotent Lie group and let Γ be a discrete subgroup of N such that N/Γ is compact. Let A', an automorphism of N, be such that $A'(\Gamma) \subset \Gamma$. We will show that $\Gamma/A'(\Gamma)$ is finite and so A' induces a mapping $A\colon N/\Gamma \to N/\Gamma$ which is a finite covering. We will call A a generalized automorphism of N/Γ. Now, since N is a unimodular group, Haar measure on N induces a unique probability measure μ on N/Γ. Since $A\colon N/\Gamma \to N/\Gamma$ is a finite covering, it is well-known that A is a measure preserving transformation. We note that A^m, m in the set Z^+ of positive integers, is well defined but unless $A'(\Gamma) = \Gamma$, A^{-m}, m in Z^+, is not.

Let $n^* \in N$. We define the mapping $n\Gamma \to n^*A(n)\Gamma$, $n \in N$, as a generalized affinity and denote it by F. It is clear that F is a measure preserving transformation of N/Γ. Our problem in this paper is to give effective necessary and sufficient conditions for F to act ergodically on N/Γ. This problem has been studied by W. Parry in[7]. However in this paper we will follow the approach for determining ergodicity of flows on solvmanifolds as presented by L. Auslander in[a].

We will now define some concepts that will enable us to state our two main results. Let L(N) denote the Lie algebra of N and let A also denote the automorphism of L(N) that A induces. Let U_1 be the subalgebra of L(N) generated by the subspace of L(N) on which A has eigenvalues with non-zero real parts. Let U be the subgroup of N with Lie algebra U_1. We will call U the unstable subgroup of N relative to A.

Let us adopt the notational convention that a subscript zero denotes the identity component of a group and a bar denotes the closure operation. We can now state our first main result.

__Theorem A.__ Let A be a generalized automorphism of the nilmanifold N/Γ. Then A acts ergodically on N/Γ if and only if $N = (U\Gamma)^-$, where $U\Gamma$ is the subgroup of N generated by U and Γ, where U is the unstable subgroup of N relative to A.

In order to state our second main result we will have to establish a few more ideas. Let $F = n_0 A$ be a generalized affinity on N/Γ. By an inner automorphism of N we can arrange it so that, if A_s is the semisimple part of A, then $A_s(n_0) = n_0$. If the previous condition is satisfied we will say that F is in standard position. We will call the unstable subgroup of A the unstable subgroup of F.

__Theorem B.__ Let F be a generalized affinity of N/Γ in standard position and let U denote the unstable subgroup of N relative to F. Then F acts ergodically on N/Γ if and only if 1. The action A^* of A induced on $N/(\Gamma U)^-_0 = N^*$ is unipotent.

2. Let $A^*(t)$ be the unipotent group of automorphisms of N^* such that $A^*(1) = A^*$. Let M be the nilpotent Lie group $A^*(t) \ltimes N^*$. Let $\Gamma^* = \Gamma/(\Gamma U)^-_0 \cap \Gamma$ and let

$\Delta = A^*(m) \ltimes \Gamma^*$, m in Z. Further let p(t) be the one parameter group through $n_0^* A^*(1)$ in M. Then p(t) acts ergodically on M/Δ.

The basic new ingredient in this work that enables us to go beyond the results in [2] is the generalized Mautner Lemma.

PART I

1. SOME PRELIMINARIES

In this section we shall review how, in the past, ergodic properties of affinities of the torus and of compact nilmanifolds were analyzed using abelian harmonic analysis. This is in contrast to the approach we shall take later on of using nilpotent theory. We shall discuss, in the following order, the results in [10] and [11] on ergodicity of automorphisms and generalized automorphisms of the torus, the results in [8] on ergodicity of generalized affinities of the torus, and the results in [7] on ergodicity of generalized affinities of compact nilmanifolds. We will then mention the work on skew product flows in [1], showing how we can relate the ergodic properties of an automorphism to the ergodic properties of a group-induced flow.

Definitions. Let V^n/L denote the n-dimensional torus group, where V^n is the n-dimensional real vector space and L the integer lattice subgroup of V^n. Let A' be a linear transformation of V^n which maps L onto itself. Then A' can be represented as a matrix with integer entries and determinant ± 1, which induces a measure-preserving automorphism, A, of V^n/L onto itself. For $f \in L^2(V^n/L)$ and $x \in V^n/L$, define the unitary operator $U_A: f(x) \rightarrow f(A(x))$.

Let $\ell \in L$, let $x \in V^n$, and let $\ell \cdot x$ denote the dot product in V^n. Then the characters $e^{2\pi i \ell \cdot x}$, denoted by $\phi_\ell(x)$, can be viewed first as functions on V^n/L and second as an orthonormal basis for $L^2(V^n/L)$.

<u>Theorem 1. ([illegible])</u> An automorphism A of the torus V^n/L is ergodic if and only if A has no eigenvalues which are roots of unity.

Proof. It suffices to show that A is ergodic if and only if U_A restricted to the group $\{\phi_\ell\}$ of characters of V^n has no finite orbits, i.e. $U_A^n \phi_\ell = \phi_\ell$ implies $\phi_\ell = 1$, [See [9] p.55]. Let n be the least positive integer such that $U_A^n \phi_\ell = \phi_\ell, \phi_\ell \neq 1$. Then for $f = \phi_\ell + U_A(\phi_\ell) + (U_A)^2(\phi_\ell) + \ldots + (U_A^{n-1})(\phi_\ell)$, $U_A(f) = f$. Furthermore, since the characters are orthogonal, the inner product $(f,1) = 0$, and since the characters are linearly independent, $f \neq 0$. Hence f isn't constant and so A is not ergodic.

Conversely, assume there are no finite orbits of U_A. Let p_ℓ denote the projection of $f \in L^2(V^n/L)$ onto ϕ_ℓ. Then we may expand f in a Fourier series, $f = \sum_{\ell \in L} p_\ell e^{2\pi i \ell \cdot x}$. We have $U_A f(x) = \sum_{\ell \in L} p_\ell(f) e^{2\pi i \ell \cdot A(x)} = \sum_{\ell \in L} p_\ell(f) e^{2\pi i (A'\ell) \cdot x}$,

where A' is the transpose of A acting on L. This, in turn, equals $\sum_{\ell \in L} p_{(U_{A'}-1)(\ell)} e^{2\pi i \ell \cdot x}$. Therefore, taking the inner product of both sides with each character, and using the orthogonality property of the characters,

$U_A f = f$ if and only if $p_\ell(f) = p_{(U_{A'}-1)(\ell)}(f)$. Hence we have

an infinite number of coefficients in the Fourier series of f which are equal in modulus, since U_A has no finite orbits implies , $U_{A'-1}(\ell)$, ... $U_{(A'-1)^m}(\ell)$, ... are all distinct.

Suppose A is not ergodic. Then there is an $f \in L^2(V^n/L)$ such that $U_A f = f$ and $f \neq$ constant. Thus there exists an $\ell_0 \neq 0$ such that $p_{\ell_0} \neq 0$. But Bessel's inequality gives us

$$\sum_{\ell \in L} |p_\ell|^2 \leq \int_{V^n/L} |f(x)|^2 dx < \infty,$$ and we also have

$$\sum_{\ell \in L} |p_\ell|^2 \geqslant p^2_{\ell_0} + p^2_{U_{A'-1}(\ell_0)} + \dots + p^2_{U_{(A'-1)^m}(\ell_0)} + \dots = \infty,$$

which is impossible.

<u>Definitions</u>. Let A be a linear transformation of V^n which is nonsingular and maps L into L. Then A can be represented by a matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad a_{ij} \text{ integers.}$$

For any vector $v = (v_1, v_2, \dots v_n)$, let $\{v\} = (\{v_1\}, \{v_2\}, \dots, \{v_n\})$ where the symbol $\{\ \}$ denotes the fractional part. The elements of V^n/L may be identified with the points $\bar{v}$ in the unit cube of V^n, with $(v_1, \dots, 0, \dots v_n)$ and $(v_1, \dots, 1, \dots v_n)$ identified whenever the zero and one components are in identical positions. Define $T(\bar{v}) = \{A(\bar{v})\} = (\{a_{11}v_1 \dots a_{1n}v_n\}, \dots, \{a_{n1}v_1 \dots a_{nn}v_n\})$. We call T a generalized automorphism of V^n/L. It is shown in [1] that each point $\bar{v}$ of the unit cube has an inverse image under T and that the measure of the complete inverse image of a set equals the measure of the set.

<u>Theorem [1]</u> Let T be a generalized automorphism of the torus V^n/L. Then if T has no eigenvalues which are roots of unity, then T is ergodic.

Proof. For $f \in L^2(V^n/L)$, let $f(x) = \sum_{\ell} p_\ell e^{2\pi i \ell \cdot x}$ and $U_A f(x) = \sum_{\ell} p' e^{2\pi i \ell \cdot x}$. Then $p'_{A'(\ell)} = \int_0^1 \int_0^1 f(A(x)) e^{-2\pi i A'(\ell) \cdot x} = \int_0^1 \int_0^1 f(A(x)) e^{-2\pi i A(x) \cdot \ell} = p_\ell$, since we can replace x with $A^{-1}(x)$

and apply the change in variables formula for integration. The remainder of the proof is as in Theorem 1,p.5.

Definitions. Let V^n/L be a torus, A a generalized automorphism of V^n/L, and $v_0 \in V^n/L$. We call $T: vL \rightarrow v_0A(v)L$, $v \in V^n$, a generalized affinity of V^n/L. For p a positive integer, define $K_p(A) = \text{kernel}(A'^p - I) \cap L$, where A' denotes the adjoint of A, and I denotes the identity. We say x_0 is rationally independent over $K_1(A)$ if whenever $n \in K_1(A)$ and the inner product (x_0,n) is rational, then $n = 0$.

Theorem 3. Let T be a generalized affinity of the torus V^n/L. Then if there exists an integer $p > 1$ such that $K_1(A) \neq K_p(A)$, then T isn't ergodic. If $0 \neq K_1(A) = K_p(A)$ for every $p \geq 1$, then T is ergodic if and only if x_0 is rationally independent over $K_1(A)$.

Proof. Let $U_{T^p}f(v) = f(T^p(v))$, where $f \in L^2(V^n/L)$ and $v \in V^n$. As in Theorem 1, to prove the first statement it suffices to show there is a character of V^n/L which has finite orbit under U_T. Define $B_p = A^{p-1} + A^{p-2} + \dots + A + I$, and assume $n \in L$, $A'^p(n) = n$, $A'(n) \neq n$, $n \neq 0$. For $n' = (A' - I)n$, we have $n' \neq 0$, $B_p' n' = 0$, and $A'^p(n') = n'$. Once again, let ϕ_n denote the character of V^n such that $\phi_n(x) = e^{2\pi i(n,x)}$. Then

$$U_{T^p}\phi_{n'}(x) = \phi_{n'}(B_p x_0)\phi_{n'}(A^p(x)) =$$

$$\phi_{B_p' n'}(x_0)\phi_{A'^p n'}(x) = \phi_{n'}(x),$$ so $\phi_{n'}$ has finite orbit.

Now suppose there exists an $n \in K_1(A)$, $n \neq 0$, such that $(x_0,n) = q/p$. Then $\phi_n(x) = e^{2\pi i(x,n)}$ is a character with finite orbit, as follows:

$$U_{T^p}(\phi_n)(x) = \phi_n(B_p(x_o))\ \phi_n(A^p(x))$$

$$\phi_{(A'^{p-1}+\dots+A'+I)n}(x_o)\ \phi_{A'^p(n)}(x)$$

$$\phi_{pn}(x_o)\ \phi_n(x) = \phi_q\phi_n(x) = \phi_n(x).$$

Conversely, suppose T is not ergodic. This implies that there is an $n \in L$, $n \neq 0$, and an integer p such that (by [9] p. 431)

$$U_{T^p}\ \phi_n(x) = \phi_n(x_o + A(x_o) + \dots + A^{p-1}(x_o))\ U_{A^p}\ \phi_n(x)$$

$= U_{A^p}(\phi_n(x)) = \phi_n(x)$. This implies

$\phi_n(x) = U_{A^p}(\phi_n(x)) = \phi_{A'^p(n)}(x)$, so that

$n = A'^p(n)$ or $n \in K_p(A) = K_1(A)$. It also implies that

$$1 = \phi_n(x_o)\ (x_o + A(x_o) + \dots + A^{p-1}(x_o)) = \phi_{(A'^{p-1}\ \dots\ A'\ I)n}(x_o)$$

$= \phi_{pn}(x_o)$. Hence, the inner product

$(x_o, pn) = p(x_o, n) \equiv 0 \mod 1$, so that (x_o, n) is rational.

Definitions. Let N be a connected, simply-connected nilpotent Lie group, Γ a discrete subgroup of N such that N/Γ is compact. For $x\Gamma \in N/\Gamma$, $n_o \in N$, and A a generalized automorphism of N/Γ, define $T: x\Gamma \to n_o A(x)\Gamma$ as a generalized affinity of N/Γ. For $f \in L^2(N/\Gamma)$ define $U_T(f)(x) = f(T(x))$. We will call an eigenfunction of the unitary operator U_T an eigenfunction of T acting on N/Γ. Let $N \supset N^1 \supset \dots \supset N^k \supsetneq N^{k+1} = e$, $N^{i+1} = [N, N^i]$, be the lower central series of N.

Theorem ([7]). Let T, N, N^1 be as defined above. Then T is ergodic on N/Γ if and only if the map induced by T on

$N/N^1\Gamma \to N/N^1\Gamma$ is ergodic. If T is ergodic, all its eigenfunctions corresponding to eigenvalue 1 factor through $N/N^1\Gamma$.

Proof. It suffices to show that whenever an eigenfunction of T: $N/N^\ell\Gamma \to N/N^\ell\Gamma$ factors through $N/N^1\Gamma$, the same is true for T: $N/N^{\ell+1}\Gamma \to N/N^{\ell+1}\Gamma$. (By T we mean the map induced by T). Replacing N by $N/N^{\ell+1}$ and Γ by $N^{\ell+1}\Gamma/N^{\ell+1}$ and noting that $(N/N^{\ell+1})^{\ell+1} = e$, we see we may assume $\ell = k$. Therefore if $0 \neq f' \in L^2(N/\Gamma)$ and $U_T(f') = f'$, it suffices to show f' factors through $N/N^k\Gamma$. Define the unitary representation, U_{N^k} of N^k acting on $L^2(N/\Gamma)$ as follows:

for $g \in N^k$ and $x \in N/\Gamma$, $U_g f(x) = f(g^{-1}x)$.

Note that U_{N^k} acts as a compact abelian group $N^k\Gamma/\Gamma$. Hence $L^2(N/\Gamma)$ is a direct sum of 1-dimensional subspaces invariant under U_{N^k}. Thus for ϕ a character of N^k annihilating $N^k \cap L$ f' can be written as an orthogonal sum

$f' = \sum_\phi f_\phi \ni U_g f_\phi = \phi(g) f$. Now

$U_T(f') = \sum_\phi Tf_\phi$ which, by hypothesis, $= f' = \sum_\phi f_\phi$.

Now we will show $T(f_\phi) = f_\phi$: Let γ = A restricted to N^k. Then $T(f_\phi) = f_\phi$. We may assume, by considering T^n rather than T, that A has no proper roots of unity as eigenvalues. Hence either $\phi\gamma = \phi$ in which case $T(f_\phi) = f_\phi$ as we want, or else $\phi\gamma^n$ are all distinct. However, using Bessel's inequality as in Theorem 1, this second possibility is ruled out.

Hence we have reduced the problem to showing f_ϕ factors through $N/N^k\Gamma$. Dividing N/Γ by the connected component of the kernel of ϕ , we may assume without loss of generality that N^k is one-dimensional. We may also assume $|f_1| = 1$.

Now let Z be the center of N and consider the unitary representation of $Z \cap N^{k-1}$ given by

$$U_n f(x) = f(n^{-1}x), \quad n \in Z \cap N^{k-1}.$$

Then as in our previous work, for θ a character of $Z \cap N^{k-1}$ annihilating $Z \cap N^{k-1} \cap L$, we have

$$f_\phi = \sum f_\theta. \quad T(f_\theta) = f_\theta. \quad U_n f_\theta = \theta(n) f.$$

$$f_\phi = \sum_\theta f_\theta, \quad T(f_\theta) = f_\theta, \quad U_n f_\theta = \theta(n) f,$$

where we ma[illegible]n reduced to showing θ annihilates N^k. A brief summary of the rest of the proof in [7] follows:

Extend θ to a continuous map from N to the complex numbers by defining $\theta(g) = \int U_g(f) \cdot f \, dm$. Let

$G = \{ g \in N^{k-1} \ni U_g f = \theta(g) f$, and let G^o be the identity component of G. We will show that $N^k = [N, G^o]$. Let

$V_1 = \{ v \in L.A.(N^{k-1}) : A^n(v) \to 0 \text{ mod } L.A.(N^k) \text{ as } n \to \infty$

$V_2 =$ " " " $n \to -\infty$

$V_3 = \{ v \in L.A.(N^{k-1}) : (A-I)^n(v) = 0 \text{ mod } L.A.(N^k)$ for some positive integer n.

It is shown on p.761 [7] that $G^o \supset \exp V_i$, $i = 1,2,3$. Since N^k is assumed to be 1-dimensional, if $[N, G^o] \neq N^k$, then $[N, G^o] = e$. However, this alternative implies

$$L.A.(Z) \cap L.A.(N^{k-1}) = L.A.(G^o) \supset V_1 \oplus V_2 \oplus V_3$$

and A acting on L.A. $(N^{k-1}/Z \cap N^{k-1})$ has eigenvalues of absolute value 1 which are not roots of unity. But this contradicts the fact that A acting on $N^{k-1}/Z \cap N^{k-1}$ preserves a lattice and thus must have a root of unity as an eigenvalue. Hence $[N, G^o] = N^k$, and we will show θ annihilates $[N, G^o]$: Letting h_t be a 1-parameter subgroup of N and $g \in G^o$,

$$\theta(h_t)\,\theta(g) = \theta(h_t g h_t^{-1} g^{-1} g h_t) = \theta(h_t g h_t^{-1} g^{-1})\theta(g)\theta(h_t)$$

and $\theta(h_t g h_t^{-1} g^{-1}) = 1$ for all t. Hence θ annihilates N^k.

Remark. We have thus related the ergodicity of a generalized affinity on a compact nilmanifold to the ergodicity of the induced map on a torus, and we may therefore apply the results of Theorem 3.

We shall now discuss the work of L.Auslander in [1] on relating the ergodicity of an automorphism to the ergodicity of a group-induced flow.

Definitions. Let N, L, and A be as previously defined and let A(t) be a 1-parameter group through A. Let Y denote N/L X $[0,1]$ with (x,0) identified with (A(x),1), $x \in N/L$. Then (u x dx), the product of the measure u induced on N/L by Haar measure on N, with Lebesgue measure dx on $[0,1]$, induces a measure on Y. We now define a group of measure-preserving transformations on Y called the skew product flow: For $x \in N/L$, $s \in [0,1]$, and t a real number $m \leq t < m+1$, m an integer, $t(x,s) = (A^m(x), (s+t) \bmod 1)$ when $s + (t \bmod 1) < 1$

$t(x,s) = (A^{m+1}(x), (s+t) \bmod 1)$ when $s + (t \bmod 1) \geq 1$

We say t(x,s) acts ergodically on Y if the only measurable subsets U of Y such that $(t,s)(U) = U$ all t,s have the property that the measure of U is zero or equals the measure of Y.

Theorem. Let A(t) act by translation on $A(t) \cdot N/A(m) \cdot L$. Then A is ergodic on N/L if and only if A(t) acts ergodically on $A(t) \cdot N/A(m) \cdot L$.

Proof. We first note that A is ergodic on N/L if and only if the reals act ergodically as the skew product flow, since U is an invariant set under A if and only if $U \times [0,1]$ is invariant under the skew product flow. Now, as in Lemmas 9&10 [1], $A(t)\cdot N/A(m)\cdot L$ is homeomorphic to Y defined previously, and A(t) acts by translation on $A(t)\cdot N/A(m)\cdot L$ as the skew product flow. q.e.d.

<u>Remark</u>. We do a similar construction when analyzing ergodic properties of affinities. In that case, we form the semidirect product $N\cdot A(t)/\Gamma\cdot A(m)$, which is homeomorphic to $N/\Gamma \times [0,1]$ with (n,1) identified with (n,0), $n \in N/\Gamma$. For $n_0 \in N$, the affinity $T\colon n\Gamma \rightarrow n_0 A(n)\Gamma$ is given by the component in N/Γ of translation by (n_0, A) in $N\cdot A(t)/\Gamma\cdot A(m)$. If we can put a 1-parameter group $(n_0(t), A(t))$ through (n_0, A) then translation by $(n_0, A(t))$ in $N\cdot A(t)/\Gamma\cdot A(m)$ will act as a "slanted" skew product flow. This flow will be ergodic on $N\cdot A(t)/\Gamma\cdot A(m)$ if and only if T is ergodic on N/Γ.

<u>Definitions</u>. Let N_3 be the set of matrices

$$N_3 = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad x_i \text{ real } i = 1,2,3$$

which we may denote by the set of triples $\{(x_1, x_2, x_3)\}$.

Let $L = \{(m_1, m_2, 1/2\, m_3)\}$, m_i integers, $i = 1,2,3$.

Then N_3/L is a compact nilmanifold.

<u>Theorem ([1])</u>. Let A be an automorphism of N_3 which maps L onto itself such that A on L.A.(N_3) is given by

$$\begin{pmatrix} e^{\lambda} & 0 & 0 \\ 0 & e^{-\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \lambda \neq 0.$$

Then A induces an ergodic map on N/Γ .

Proof. Using the skew product flow, it suffices to show $A(t)$ acts ergodically on $A(t)\cdot N/A(m)\cdot L$. Now translation by $A(t)$ induces a unitary representation of $A(t)$ on $L^2(A(t)\cdot N/A(m)\cdot L)$ given by

$$U_{A(t)}f(x) = f(A(t)(x)) \text{ where } f \in L^2(A(t)\cdot N/A(m)\cdot L).$$

This may be extended to a unitary representation, $U_{A(t)\cdot N_3}$. Now $A(t)$ is ergodic if and only if $U_{A(t)}f = f$ implies f is constant. But it can be proved that the following Mautner property holds:

$$U_{A(t)}f = f \text{ implies } U_{A(t)\cdot N_3}f = f.$$

But since $A(t)\cdot N_3$ acts transitively on $A(t)\cdot N_3/A(m)\cdot L$, we have that A is ergodic on N/Γ .

L.Auslander has also used these ideas of skew product flow and Mautner property to analyze the ergodic properties of affinities of the torus. We will, in Part II and Part III, generalize these methods to obtain ergodic properties of generalized automorphisms and of generalized affinities of any compact nilmanifold.

PART I

1. SOME PRELIMINARIES, continued.

Theorem 1. Let N be a connected, simply-connected nilpotent Lie group, Γ a discrete subgroup of N such that N/Γ is compact. Let A' be an automorphism of N sending Γ into Γ. Then $\Gamma/A'(\Gamma)$ is finite.

Proof. Let N_Q be the rational nilpotent subgroup of N containing Γ. Let $\Gamma' = A'(\Gamma)$. Since $\Gamma' \subset \Gamma$, the rational algebraic hull of Γ', a.h.$(\Gamma') \subseteq N_Q$. But by the fundamental extension property of automorphisms of nilpotent groups $A' : \Gamma \rightarrow \Gamma'$, which is an isomorphism, is uniquely extendable to an automorphism $(A')^* : N_Q \rightarrow$ a.h.(Γ'). It then follows that $(A')^* = A'$ and a.h.$(\Gamma') = N_Q$. Thus we see that Γ and Γ' are discrete cocompact subgroups of N_Q and so are commensurable.

2. GENERALIZED MAUTNER LEMMA

We will begin by introducing some groups and semigroups and proving a preliminary result before stating our main assertion.

Let $G_1 = Z \ltimes R$ be given by the matrix representation

$$G_1 = \begin{pmatrix} e^{n\lambda} & y \\ 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \lambda \neq 0 \text{ a fixed real number} \\ n \in Z,\ y \in R. \end{array}$$

By G_1^+ we will mean the subsemi-group $Z^+ \ltimes R$ where Z^+ denotes the positive integers. Let (X, μ) be a finite measure space. Assume further that the semi-group G_1^+ acts as a semi-group of measure-preserving transformations of (X,μ). Let $f \in L^2(X,u)$. Then clearly if

$$U_g(f) = f \circ g \qquad g \in G_1$$

then U_g is an isometry of $L^2(X,\mu)$ and $U_{g_2}U_{g_1} = U_{g_1}U_{g_2}$.

<u>Lemma</u>. $\quad U_nU_t = U_{n^{-1}tn}U_n$.

Proof. $\quad U_nU_tf = f \circ t \circ n = f \circ n \circ (n^{-1}tn) = U_{n^{-1}tn}U_nf$

because $t \circ n = n(n^{-1}tn)$ in G_1.

<u>Generalized Mautner Lemma</u>: Let G_1 be as above and assume that G_1^+ acts as a semi-group of measure-preserving transformations on (X,μ). Let $\Psi \in L^2(X,\mu)$. If

$$U_n\Psi = \Psi \qquad \text{for all } n \in Z^+,$$

then

$$U_t\Psi = \Psi \qquad \text{for all } t \in R.$$

Proof. Let
$$\phi_n\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{-n} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{n} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & e^{-n\lambda}t\\ 0 & 1\end{pmatrix}.$$

Then $\lim_{n\to\infty}\phi_n(t) = e$, the identity of G_1. Now for $t \in R$ and $\Psi \in L^2(X,u)$,

$$\langle U_t\Psi, \Psi\rangle = \langle U_nU_t\Psi,\ U_n\Psi\rangle$$

since U_n is a Hilbert space isometry. Hence by the lemma,

$$\langle U_t\Psi,\Psi\rangle = \langle U_{n^{-1}tn}U_n\Psi, U_n\Psi\rangle = \langle U_{n^{-1}tn}\Psi, \Psi\rangle$$

since $U_n\Psi = \Psi$ by hypothesis. Since $\lim_{n\to\infty} U_{n^{-1}tn} = e$, we have

$$\langle U_t \Psi, \Psi \rangle = \langle \Psi, \Psi \rangle .$$

We may proceed as in [3] to complete the argument.

Consider the semi-group G_2^+ consisting of all pairs (m,z), $m \in Z^+$ and z in the set C of complex numbers, and multiplication is defined by

$$(m_1, z_1)(m_2, z_2) = (m_1 + m_2, (e^{(1-i\sigma)m_2})z_1 + z_2) \text{ where}$$

$\sigma > 0$ and real and $i = \sqrt{-1}$. Let U be a representation of the semi-group G_2^+ analogous to the one we discussed for G_1^+. Then if Ψ is such that

$$U_m \Psi = \Psi \qquad \text{all } m \in Z^+$$

then

$$U_z \Psi = \Psi \qquad \text{all } z \in C.$$

This assertion is proven exactly as above.

PART II

PROOF OF THEOREM A

Theorem A. Let A be a generalized automorphism of the nilmanifold N/Γ. Then A acts ergodically on N/Γ if and only if $N = (U\Gamma)^-$, where U is the unstable subgroup of N relative to A.

Proof. We may apply the reasoning in Theorem 3.5 [3], substituting our stronger Mautner Lemma to conclude that A is ergodic on N/Γ if and only if A induces an ergodic transformation on $N/(\Gamma U)^-$. Now, again as in [3], we have that $(\Gamma U)^-_0$, the identity component of $(\Gamma U)^-$, is normal in N and so we may form the groups

$$M = N/(\Gamma U)^-_0 \quad \text{and} \quad \Delta = \Gamma/(\Gamma U)^-_0 \cap \Gamma$$

and conclude that M/Δ is a compact nilmanifold. Further, A induces a generalized automorphism B of M/Δ. Since U is the unstable subgroup of A, we have easily that B has determinant ± 1 and so B is an automorphism of M/Δ and all the eigenvalues of B have absolute value one. Thus A is ergodic on N/Γ if and only if B is ergodic on M/Δ. But it is a classical result that the eigenvalues of B must be roots of unity and no non-trivial automorphism of a nilmanifold can be ergodic if it has eigenvalues that are roots of unity. Hence B is ergodic if and only if M/Δ is a point on $N = (\Gamma U)^-$.

PART III

PROOF OF THEOREM B

It will be convenient to introduce the group $G = N \rtimes A^m$ for $m \in Z$. Then the affine motion of $g \in G$ on N is given by the component N of gn for $n \in N$.

Lemma 1. Let A_s denote the semi-simple part of A acting on N and let $g = n_0 A$, $n_0 \in N$. Then we may choose an inner automorphism of G by an element of N such that if $n^{-1} g n = g' = n_0' A$, $A_s(n_0') = n_0'$.

(This is essentially Theorem 2.2 of [4], p.68.)

We will henceforth always assume that this has been done.

Lemma 2. Let $g = n_0 A$ and let M be a normal subgroup of N invariant under A. Then the image of g in G/M defines an affine action on N/M.

(The proof is trivial).

Theorem. Let $g \in G$. Then the action of g on N/Γ is ergodic if and only if the action of g induced on $N/(\Gamma U)^-$ is ergodic.

Proof. Consider $gNg^{-1} = B_u B_s$ where B_u is unipotent and B_s is semi-simple. Then B_u lies on a one parameter group of unipotent automorphisms $B_u(t)$ and we may form

$$M = B_u(t) \rtimes N.$$

Note that $B_s = A_s$. Choose $X \in L(M)$ such that

1. $A_s(X) = \lambda X$ λ real.

2. X is in the lowest possible term of the

lower central series with property 1.

Since B_u and A_s commute, $B_u(X) = X'$ also has the property 1. But then $B_u(X) = X + X''$ where X'' is in a lower term of the lower central series than X and obviously

$$A_s(X + X'') = A_s(X) + A_s(X'') = \lambda X + A_s(X'')$$

and $A_s(X + X'') = \lambda(X + X'') = \lambda X + \lambda X''$.

Hence $A_s(X'') = \lambda X''$, which contradicts the definition of X unless $X'' = 0$. This proves that g and exp(X) generate a group isomorphic to G_1". We may repeat the argument in Theorem 3.2 pg.76 of [4] to prove our theorem. (Of course the case where λ is imaginary with positive real part goes the same way.)

Lemma. In order for $g \in G$ to act ergodically on N/Γ it is necessary that A be unipotent when acting on $N/(\Gamma U)^-$.

Proof. This may be seen by observing that if g acts ergodically, its action on $N/(\Gamma U)^- = M/\Delta$ must be ergodic, which implies its action on $M/\Delta[M,M]$ is ergodic, as in Theorem 4.2, pg 65 [4]. Since $M/[M,M]$ is abelian, it is well-known that A must have no roots of unity when acting on $M/[M,M]$, if A is to be ergodic. But then A is unipotent since we have divided by U, the unstable subgroup of N relative to A.

We are now in a position to prove Theorem B of the Introduction. But this amounts now to nothing more than verifying- using the skew product flow as in [1] condition 2

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AUTOBIOGRAPHICAL STATEMENT

Evelyn G. Mayer was born in New York City in 1946. She graduated from George Washington High School in 1963 and entered the City College of New York, where she received the B.S. degree in 1966. In 1966 she was accepted in the Doctoral program in Mathematics at the University of Pennsylvania, where she was awarded a N.A.S.A. Doctoral Traineeship. Following a death in the family in 1967, she transferred to the City University of New York Graduate Center, at the kind suggestion of Prof. Abraham Schwartz of City College. She obtained a part-time teaching position at City College, and passed her written Doctoral examinations in 1968. At first she received financial aid from the N.S.F. and from the C.U.N.Y. Graduate Center. From 1970 to 1973, wanting to broaden her experience, she taught Mathematics at Queens College, Hunter College, and John Jay College of Criminal Justice of the City University of New York.