76-30,413

JEVICKI, Antal, 1951-
EXTENDED PARTICLES IN QUANTUM FIELD THEORY.

City University of New York, Ph.D., 1976
Physics, elementary particles and high energy

Xerox University Microfilms, Ann Arbor, Michigan 48106

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by

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A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1976

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

5/26/76 date — Bunji Sakita, Chairman of Examining Committee

5/26/76 date — Myriam P. Sarachik, Executive Officer

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Abstract

Extended Particles in Quantum Field Theory

by

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In this thesis a detailed review of the collective coordinate approach to extended particles in quantum field theory is presented. We consider field theories possessing finite energy particlelike classical solutions and give a complete quantum treatment of corresponding extended particles. In order to preserve the symmetries respected by the classical solutions we introduce new dynamical variables (collective coordinates), associated with the symmetry degrees of freedom. This forms the basis of the collective coordinate method which we formulate employing the Feynman path integral quantization procedure. We consider explicitly two spacetime dimensional scalar quantum field theories for simplicity. Applying the method of collective coordinates we develop a systematic weak coupling perturbation expansion about extended particle states. For the one particle sector a set of Feynman rules is derived which can be used to perform perturbative calculations to an arbitrary order in coupling constant. Furthermore, for the sine-Gordon model we present a quantum treatment of soliton scattering and compute the two soliton scattering amplitude both in tree and one loop approximation.

Acknowledgements

I wish to express my sincere and deep gratitude to Professor Bunji Sakita for continuous guidance and encouragement throughout my graduate work. All the results presented in this thesis are contained in papers done in collaboration with Professor J.-L. Gervais and Professor B. Sakita. I would like to thank both of them for their invaluable help without which this work could never have been completed. Thanks are also due to Dr. P. Senjanovic for help and encouragement during the initial stages of this work. I would also like to thank Miss Carol Grossman for her excellent typing of the thesis.

I am indebted to the National Science Foundation for financial support during the period of this research and I also acknowledge the hospitality and financial support extended to me by the Laboratoire de Physique Theorique de l' Ecole Normale Superieure where part of this work was done.

CONTENTS

I. INTRODUCTION AND SUMMARY

In the usual perturbative approach to quantum field theory the particle spectrum is obtained by neglecting first the interaction and then quantizing free fields. It has built into it the assumption that the asymptotic states of a field theory are free fields and so it represents only a partial solution to the full interacting problem. For quantum electrodynamics and weak interactions this method is very successful but it cannot provide a complete computational framework for strong interactions.

It was noticed long ago that nonlinear classical field theories possess particlelike classical solutions[1]. By now a large number of classical solutions are known for a variety of nonlinear field theory models. The simples are the exact classical solutions in two space-time dimensional scalar field theories. In $\lambda\phi^4$ theory with the wrong sign of the mass term we have the "kink" solution and in sine-Gordon theory the soliton and the antisoliton classical solutions. These static solutions have finite energy and the field configuration is topologically different from that of the vacuum classical solution. In the sine-Gordon theory even exact multi-soliton solutions are known as in addition there are periodic time dependent classical solutions. By the application of inverse scattering method to this theory all the classical solutions with finite energy have been derived and it has been shown that this model is classically completely integrable[2].

For classical field theories in four dimensions approximate classical solutions have been found recently. In the Abelian Higgs model a stringlike classical solution was obtained by Nielsen and Olesen[3] which is the analogue of vortex lines in type II superconductors. The string has finite energy per unit length and by introducing Dirac magnetic monopoles one obtains a finite length stringlike classical solution confining magnetic monopoles[4]. In a non-Abelian gauge theory a spherically symmetric finite energy classical solution possessing magnetic charge was found[5] by G. t' Hooft and independently by A. Polyakov. The magnetic charge is of topological origin.

In these theories the classical energy of the static solution is finite and proportional to inverse of the coupling constant. Because of the Lorentz invariance of the theory there exist always time dependent classical solutions which are obtained from the static solution by Lorentz boosts. Since the relativistic energy-momentum relation is satisfied and since the energy density has spatial extent these solutions represent classical extended particles. We adopt the name "soliton" to denote such a particle of finite energy.

The existence of classical extended objects leads to the expectation that the full quantum theory possesses a much richer spectrum than that obtained by ordinary perturbation theory. Over the last few years it has been demonstrated that to these classical objects there correspond particle states in quantized theory. Several different methods dealing with this problem were developed.

First, a semiclassical WKB method was formulated by Dashen, Hasslacher and Neveu[6] to study the particle spectrum in quantum field theory. This method requires as an input the knowledge of classical solutions to the field equations and leads to the existence of the corresponding particle states in the spectrum of the quantized theory. For the case of weak coupling they show that the time independent classical solution is relevant and that there will exist a corresponding quantum particle with large mass. In general, in this method it is necessary to know all the classical periodic solutions to field equations or at least a whole family of periodic classical solutions. Dashen, Hasslacher and Neveu show that the energy spectrum is obtained by imposing the WKB quantization condition on the action of classical motions. But in this method it is not known how in a systematic way to evaluate corrections to the WKB result.

An approach based on coupling constant expansion is formulated by Goldstone and Jackiw[7] using the method of Kerman and Klein. It starts with a set of initial assymptions about the existence of a particle with large mass (soliton) at the quantum level and also about order in coupling constant of the field operator matrix elements in the one soliton sector. Then, saturating the commutation relations and equations of motion it is possible to demonstrate the self-consistency of these assymptions and to calculate the explicit form for the one soliton sector matrix elements. In particular, the static classical solution is recovered in quantum theory as the matrix element of the field operator between localized one soliton states. This gives a very simple understanding of the relation between classical and quantum theories but in this method it becomes complicated to perform higher order calculations. Also, it is not clear how to discuss more difficult questions like, for example, scattering of solitons in this approach.

An entirely different approach to extended particles which is also based on coupling constant expansion is the collective coordinate method[8][9][10][11] It represents the generalization of the collective coordinate method of many-body theory to quantum field theory. It was originally developed to study the static strong coupling model[12] where it leads to a perturbation expansion in inverse of the strong coupling constant.

In this thesis we will present a detailed review of the path integral collective coordinate method. All the results which will be given are obtained in collaboration with J.-L. Gervais and B. Sakita. Throughout this work we consider for simplicity two space-time dimensional scalar field theories. In formulating the collective coordinate approach to extended particles we employ the Feynman path integral quantization procedure. Although collective coordinates can also be introduced using the operator method[13], the relevance of classical particlelike solutions to the quantum theory is best seen in the path integral approach. Furthermore, the path integral method becomes truly advantageous when discussing the quantum scattering of solitons.

To understand the relevance of classical solutions to the quantum theory we start with the observation that in theories with spontaneous symmetry breaking one has to shift the quantum field by the classical vacuum solution in order to develop a perturbation expanison. This means that in the path integral expression for the vacuum generating functional only the paths near the classical vacuum solution contribute. Then in analogy with spontaneous symmetry breaking we are led to the idea that a perturbation expansion about the soliton state can be developed by translating the field by the classical soliton solution and considering small fluctuations around it. But by doing this we break the translational invariance of the theory and consequently, in the spectrum of small oscillations, a zero frequency mode (Goldstone boson) appears leading to an infrared problem. In order to preserve the translational invariance when shifting by the classical solution we introduce a new dynamical variable representing the center of mass coordinate and separate the translational degree of freedom. This means that we translate the field variable by a classical solution in which the constant classical position is promoted to a dynamical variable. The center of mass coordinate is introduced into the path integral expression for the one soliton sector transition amplitude through a δ-function condition. This δ-function condition is arbitrary and we call it a gage condition. It completely eliminates the zero frequency mode from the path integral and consequently, there is no infrared problem.

Now it is straightforward to develop a systematic weak coupling perturbation expansion about the one soliton state and dervie a set of Feynman rules which can be used to perform calculations to arbitrary order in coupling constant. Since we shifted the field by the classical solution physical quantities will have a classical part which is non-analytic in coupling constant and the quantum corrections are given as a power series in coupling constant. So in the leading approximation the soliton mass agrees with the classical mass.

This method can be easily generalized to treat more complicated classical extended particlelike solutions which sespect more than one continuous symmetry(14)(15). Now the arbitrary parameters appearing in the classical solution are to be promoted to dynamical variables and their conjugate variables are the infinitesimal generators of continuous symmetries. We call these new variables collective coordinates and they are introduced in the path integral through a set of δ-function conditions and a canonical transformation generated by the symmetry generators. Half of these δ-function conditions represent constraints and the rest are gauge conditions. The separation of symmetry degrees of freedom assumes that the invariances are present at the quantum level also and furthermore, the zero frequency modes are completely eliminated by the gauge conditions. Now one can develop a consistent perturbation expansion free of the infrared proble. For the presentation of the general method of collective coordinates we essentially follow the discussion given in reference (15).

The main application of the general method we discuss in this thesis is the formulation of soliton-soliton scattering formalism. In sine-Gordon theory exact multi-soliton classical solutions are known and we can generalize the one soliton sector discussion to the multi-soliton sector case. Specifically we consider the soliton-antisoliton scattering and develop a method for computing the scattering amplitude. The collective coordinates introduced in this case are conjugate variables to the total momentum and the energy of the system. The general formalism then leads to a path integral expression for the soliton-antisoliton scattering phase shift. The leading term is given by the classical phase shift which is non-analytic in coupling constant and the quantum corrections are in powers of the coupling constant. We evaluate the first quantum correction.

In chapter II we present a detailed discussion of the Feynman path integral method. Since we use point canonical transformations when we introduce collective coordinates, we give a careful treatment of point canonical transformations in this chapter.

In chapter III we start with the investigation of extended particles in quantum field theory by developing the collective coordinate method for one soliton sector. A systematic weak coupling perturbation expansion is formulated and perturbative calculations of the soliton mass and one soliton sector Green functions are carried out.

The general method of collective coordinates is developed in chapter IV.

Finally, in chapter V we apply this general method to the problem of soliton scattering. A systematic quantum formalism for scattering of solions for the sine-Gordon theory is developed and the scattering phase shifts are calculated both in tree and one loop approximation.

II. FEYNMAN PATH INTEGRAL QUANTIZATION

In this chapter we give a detailed discussion of the Feynman path integral method. Special emphasis is on topics not much discussed in the literature but which appear to be relevant to our work. In section 1 we present a review of the phase space path integral method with special emphasis on the path integral measure. In section 2 we investigate point canonical transformations using the Feynman diagram technique and in section 3 we give a treatment of point transformations in the path integral approach.

II.1 Phase Space Path Integral

In this section we will review the quantization procedure based on phase space path integration. In the literature path integral methods are mainly used in Lagrangian formulation developed originally by R. Feynman, but there are certain advantages to the phase space approach. First of all in the canonical phase space path integral method the integration measure is simple and by integrating over the momentum variables one can deduce the non-trivial measure required in the Lagrangian path integral. Second, there is a direct correspondence between phase space path integrals and ordered Hamiltonian operators. That is the main reason why we choose to use the canonical phase space path integral method throughout this work.

We consider a dynamical system of n-degrees of freedom denoting our canonical variables by $q=(q_1, q_2, \ldots q_n)$ and $p=(p_1, \ldots p_n)$.

The operators $\hat{p}_i$ and $\hat{q}_i$ are represented as follows:

$$\hat{q}_i |q\rangle = q_i |q\rangle$$
$$\hat{p}_i |q\rangle = \frac{\hbar}{i}\frac{\partial}{\partial q_i}|q\rangle \tag{1.1}$$

and the dynamics is described by the Hamiltonian operator $\hat{H}$. In Heisenberg picture the operators evolve in time like

$$\hat{q}_i(t) = e^{\frac{i}{\hbar}\hat{H}t}\,\hat{q}\;e^{-\frac{i}{\hbar}\hat{H}t} \tag{1.2}$$

and the corresponding eigenstates as

$$|q,t\rangle = e^{\frac{i}{\hbar}\hat{H}t}|q\rangle \tag{1.3}$$
$$\hat{q}(t)|q,t\rangle = q\,|q,t\rangle$$

The main object of our discussion is the calculation of the time evolution operator matrix element

$$K(q'',t'';q',t') = \langle q'',t''|q',t'\rangle = \langle q''|\;e^{-\frac{i}{\hbar}\hat{H}(t''-t')}\;|q'\rangle \tag{1.4}$$

usually referred to as the kernel. It can be expressed in the following phase space path integral form:

$$K(q'',t'';q',t') = \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q\,\mathcal{D}p\;\exp\left\{\frac{i}{\hbar}A[p,q]\right\} \tag{1.5}$$

where the functional in the exponent is the action

$$A[p,q] = \int_{t'}^{t''} dt\,\left(\sum_i p_i(t)\,\dot{q}_i(t) - H(p(t),q(t))\right) \tag{1.6}$$

This representation for the evolution operator appears to be very attractive being expressed entirely in terms of classical mechanical ingredients and it looks like it defines a canonicaly invariant quantum theory. But unfortunately at present there is no consistent way of performing general canonical transformations in the path integral formalism. The correct treatment of general point transformations is nevertheless possible and we will present a detailed discussion of this problem in sections 2 and 3 of this chapter.

In the literature one also finds statements that the ordering problem which appears in the operator quantization procedure when the Hamiltonian contains non-comuting factors doesn't appear at all in the path integral method, or, the opposite claims that the path integral formalism appears to be ambiguous and unsafe especially with respect to the ordering problem. Naturally both of these statements are incorrect and in what follows we will present a clear formulation of the phase space path integral method with special emphasis on the ordering problem. We essentially follow the work of Berezin[18].

The above mentioned intricacies are connected with the very definition of the integration over paths. As yet there exists no rigorous mathematical definition of this integral and it is defined by subdividing the time interval $t'' - t'$ into N segments so that

$$\begin{aligned} t'' - t' &= N\epsilon \\ t_k &= t' + k\epsilon \quad k=0,1,\dots,N \\ q(t_k) &= q(k) \\ p(t_k) &= p(k) \end{aligned} \tag{1.7}$$

and replacing the exponent by the integral sum

$$A[p,q] = \sum_{k=0}^{N-1} A\left(p(k), q(k+1), q(k)\right) \tag{1.3}$$

with subsequent finite dimensional integrations

$$\prod_i \prod_{k=1}^{N-1} dq_i(k) \prod_{k=0}^{N-1} \frac{dp_i(k)}{2\pi\hbar}$$

The final answer is then obtained as a limit when $N \to \infty$.

In general different finite dimensional approximations will have different limits and that is precisely the way how the ordering problem appears in the path integral method. So the above definition has meaning only if additional specifications are made concerning the short time action $\mathcal{A}(p(k), q(k+1), q(k))$. These specifications are in direct correspondence with the order of factors in operator approach. Namely the finite dimensional definition of the path integral can be derived by using the completness relation for state vectors $|q(k), t_k\rangle$ so that

$$K(q'',t'';q',t') = \int \prod_i dq_i(1) \int \cdots \int \prod_i dq_i(N-1) \langle q(N), t_N | q(N-1), t_{N-1}\rangle \times$$
$$\times \langle q(N-1), t_{N-1} | q(N-2), t_{N-2}\rangle \cdots \cdots \langle q(1), t_1 | q(0), t_0\rangle \tag{1.9}$$

Here we denoted $q(0) = q'$ and $q(N) = q''$. If the Hamiltonian has a simple form

$$\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{q}) \tag{1.10}$$

the short time matrix element is approximated as follows·

$$\langle q(k+1)| e^{-\frac{i}{\hbar}\epsilon \hat{H}} |q(k)\rangle = \int \frac{dp(k)}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}\left[p(k)\cdot(q(k+1)-q(k)) - \epsilon\left(\frac{p(k)^2}{2} + V(q(k))\right)\right]\right\} \tag{1.11}$$

In this case the final answer appears to be insensitive to how we approximate the short time action. i.e. whether we have $q(k+1)$ or $\frac{q(k+1)+q(k)}{2}$ or some other linear combination in the potential instead of $q(k)$. But for more general Hamiltonians which contain non-commuting factors it is not so. Thus we specify the definition of the path integral in precise correspondence ordering of non-commuting factors in the operator Hamiltonian.

a. $\hat{p}\,\hat{q}$ Ordering Procedure

In this quantization procedure with every classical polynomial function

$$f(p,q) = \sum f_{m_1 m_2 \cdots m_n, m_1' \cdots m_n'} p_1^{m_1'} \cdots p_n^{m_n'} q_1^{m_1} \cdots q_n^{m_n} \tag{1.12}$$

we associate an ordered operator

$$\hat{f} = \sum f_{m_1 m_2 \cdots m_n, m_1' \cdots m_n'} \hat{p}_1^{m_1'} \cdots \hat{p}_n^{m_n'} \hat{q}_1^{m_1} \cdots \hat{q}_n^{m_n} \tag{1.13}$$

such that $\hat{p}$ always stands to the left of $\hat{q}$ This correspondence is one to one. So if our operator Hamiltonian $\hat{H}$ is associated with the classical Hamiltonian function in the above sense we approximate the short time matrix element as follows First

$$\langle q(k+1)|\, e^{-\frac{i}{\hbar}\epsilon\hat{H}}\, |q(k)\rangle \approx \langle q(k+1)|\left(1-\frac{i}{\hbar}\epsilon\hat{H}\right)|q(k)\rangle \tag{1.14}$$

and then inserting a complete set of states to the right-hand side becomes

$$\int\prod_{i=1}^{n}\frac{dp_i(k)}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}p(k)\cdot\left(q(k+1)-q(k)\right)\right\}\times\left\{1-\frac{i}{\hbar}\epsilon H(p(k),q(k))\right\} \tag{1.15}$$

So the path integral for the kernel of this operator theory has the form

$$K(q'',t'';q',t')_{\hat{p}\hat{q}} = \lim_{N\to\infty}\int\prod_i\prod_{k=1}^{N-1}dq_i(k)\prod_{k=0}^{N-1}\frac{dp_i(k)}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\sum_{k=0}^{N-1}\left[p(k)\cdot\left(q(k+1)-q(k)\right)-\epsilon H(p(k),q(k))\right]\right\} \tag{1.16}$$

In the Hamiltonian appearing in exponent of (1.16), one is not allowed to substitute q(k) by, for example q(k+1), since this would give a theory with the $\hat{q}\,\hat{p}$ ordering of factors.

Next we will present a more symmetric specification for the path integral which will be used consistently throughout this work.

b. Symmetric or Weyl Ordering Procedure

The Weyl ordered operator $(\hat{p}^{\ell}\,\hat{q}^{m})_W$ corresponding to the classical polynomial $p^{\ell}q^{m}$ is defined through the relation:

$$\left(\alpha\hat{p}+\beta\hat{q}\right)^{N}=\sum_{\ell,m}\frac{N!}{\ell!\,m!}\,\alpha^{\ell}\beta^{m}\,\left(\hat{p}^{\ell},\hat{q}^{m}\right)_W \tag{1.17}$$

For example we have

$$\left(\hat{p}^{2},\hat{q}^{N}\right)_W=\left(\sum_{\ell,m,n}\hat{q}^{\ell}\hat{p}\hat{q}^{m}\hat{p}\hat{q}^{n}\right)\Big/\left(\sum_{\ell,m,n}1\right) \tag{1.18}$$

In the symmetric or Weyl quantization procedure with every classical polynomial function f(p, q) we associate the following operator:

$$\hat{f}=\sum f_{m_1m_2\cdots m_n,m_1'\cdots m_n'}\left(\hat{p}_1^{m_1'}\hat{q}_1^{m_1}\right)_W\cdots\left(\hat{p}_n^{m_n'}\hat{q}_n^{m_n}\right)_W \tag{1.19}$$

the bracket denoting that this non-commuting factors are Weyl ordered.

This correspondence is also one to one and furthermore there is a simple inverse relation expressing the classical function f(p,q) in terms of the corresponding Weyl ordered operator $\hat{f}$ (19)

$$f(p,q) = \int dv\, e^{\frac{i}{\hbar} p\cdot v} \langle q - \tfrac{v}{2} | \hat{f} | q + \tfrac{v}{2} \rangle \tag{1.20}$$

This transformation is called the Weyl transform and to prove it one first shows by induction that

$$(\alpha p + \beta q)^N = \int dv\, e^{\frac{i}{\hbar} p\cdot v} \langle q - \tfrac{v}{2} | (\alpha \hat{p} + \beta \hat{q})^N | q + \tfrac{v}{2} \rangle \tag{1.21}$$

and so

$$p^{\ell} q^{m} = \int dv\, e^{\frac{i}{\hbar} p\cdot v} \langle q - \tfrac{v}{2} | (\hat{p}^{\ell}, \hat{q}^{m})_w | q + \tfrac{v}{2} \rangle \tag{1.22}$$

Now the relation (1.20) follows

Using relation (1.20) we easily prove the following useful correspondence if

$$\hat{f} \leftrightarrow f(p,q)$$

then

$$\hat{p}_i \hat{f} \leftrightarrow \left(p_i - \frac{i\hbar}{2} \frac{\partial}{\partial q_i}\right) f(p,q)$$

$$\hat{f}\, \hat{p}_i \leftrightarrow \left(p_i + \frac{i\hbar}{2} \frac{\partial}{\partial q_i}\right) f(p,q) \tag{1.23}$$

In order to derive an expression for the short time matrix element we use the following relation

$$\langle q'' | \hat{f} | q' \rangle = \int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar} p (q'' - q')} f\left(p, \frac{q'' + q'}{2}\right) \tag{1.24}$$

which can be proved starting with the identity

$$\langle q'' | \hat{f} | q' \rangle = \int dq_1\, dp_1\, dq_2\, dp_2 \langle q'' | p_1 \rangle \langle p_1 | q_1 \rangle \times$$

$$\times \langle q_1 | \hat{f} | q_2 \rangle \langle q_2 | p_2 \rangle \langle p_2 | q' \rangle \tag{1.25}$$

and making the change of variables

$$\frac{q_1 + q_2}{2} = q \qquad \frac{p_1 + p_2}{2} = p$$

$$q_2 - q_1 = v \qquad p_2 - p_1 = u \tag{1.26}$$

so that the right-hand side becomes

$$\int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar} p(q''-q')} \int dq\, \delta\left(q - \frac{q''+q'}{2}\right) \int dv\, e^{\frac{i}{\hbar} pv} \left\langle q - \frac{v}{2} \right| \hat{f} \left| q + \frac{v}{2} \right\rangle \qquad (1.27)$$

Thus relation (1.24) follows.

Now we are ready to deduce the approximation for the short time matrix element

$$\langle q(k+1), t_{k+1} | q(k), t_k \rangle \approx \langle q(k+1) | \left(1 - \frac{i}{\hbar} \epsilon \hat{H}\right) | q(k) \rangle \qquad (1.28)$$

We consider a theory with the operator Hamiltonian $\hat{H}$ associated with the classical Hamiltonian function $H(p,q)$ in accordence with the Weyl quantization procedure (which means that $H(p,q)$ is the Weyl transform of $\hat{H}$). Then using relation (1.24) we have

$$\langle q(k+1), t_{k+1} | q(k), t_k \rangle \approx \int \frac{dp(k)}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar} \left[p(k)\cdot(q(k+1) - q(k)) - \right.\right.$$
$$\left.\left. - \epsilon H\left(p(k), \frac{q(k+1)+q(k)}{2}\right) \right] \right\} \qquad (1.29)$$

and thus the precise form of the phase space path integral follows:

$$K(q''t''; q't')_W = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} dq(k) \prod_{k=0}^{N-1} \frac{dp(k)}{2\pi\hbar} \times$$
$$\times \exp\left\{ \frac{i}{\hbar} \sum_{k=0}^{N-1} A\left(p(k), q(k+1), q(k)\right) \right\} \qquad (1.30)$$

with the "midpoint" short time approximation for the action in the exponent:

$$A(p(k), q(k+1), q(k)) = p(k)\cdot(q(k+1) - q(k)) - \epsilon H\left(p(k), \frac{q(k+1)+q(k)}{2}\right) \qquad (1.31)$$

Summarizing our result we conclude that the path integral

$$\int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}p\mathcal{D}q\, \exp\left\{ \frac{i}{\hbar} A[p,q] \right\} \qquad (1.32)$$

with additional specification that the short time action is to be evaluated at the midpoint $\bar{q}(k) \equiv \frac{q(k+1)+q(k)}{2}$ describes a theory with operator Hamiltonian $\hat{H}$, obtained by Weyl quantization procedure from the classical Hamiltonian function $H(p,q)$ appearing in the action of (1.32). It is this midpoint specification of the path integral which we accept and use throughout this work.

As an example we consider a theory given by the following ordered operator Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{i,j}' g^{-1/4}(\hat{q})\,\hat{p}_i\, g^{ij}(\hat{q})\, g^{1/2}(\hat{q})\,\hat{p}_j\, g(\hat{q})^{-\frac{1}{4}} \tag{1 33}$$

where g^{ij} denotes the inverse of g_{ij} and $g=\det g_{ij}$. This operator is not Weyl ordered and consequently $H(p\ q)$ appearring in the phase space path integral expression of the kernel is not

$$\frac{1}{2}\sum_{i,j}' p_i p_j\, g^{ij}(q) \tag{1 34}$$

To find the classical Hamiltonian function we first reorder $\hat{H}$ so that

$$\hat{H} = \frac{1}{2}\sum' \hat{p}_i\, g^{ij}(\hat{q})\,\hat{p}_j - \frac{\hbar^2}{2}\, g(\hat{q})^{-\frac{1}{4}}\left[g^{ij}(\hat{q})\; g(\hat{q})^{\frac{1}{2}}\left(g^{-\frac{1}{4}}(\hat{q})\right)_{,i}\right]_{,j} \tag{1 35}$$

and then use the correspondence relation (1 23) to find

$$\frac{1}{2}\hat{p}_i\, g^{ij}(\hat{q})\,\hat{p}_j \leftrightarrow \frac{1}{2}\left(p_i - \frac{i\hbar}{2}\frac{\partial}{\partial q_i}\right)\left(p_j + \frac{i\hbar}{2}\frac{\partial}{\partial q_j}\right) g^{ij}(q) \tag{1 36}$$

so for the classical Hamiltonain function we obtain

$$H(p,q) = \frac{1}{2}\sum_{i,j}' p_i p_j\, g^{ij}(q) + \Delta V(q) \tag{1.37}$$

where

$$\Delta V(q) = \frac{\hbar^2}{8}\, g^{ij}(q)_{,ij} - \frac{\hbar^2}{2}\, g(q)^{-\frac{1}{4}}\left[g^{ij}(q)\, g(q)^{\frac{1}{2}}\left(g^{-\frac{1}{4}}(q)\right)_{,i}\right]_{,j}$$

II.2 Canonical Transformations and Feynman Diagrams

In this section we will make some observations concerning the problem of performing canonical transformations in functional method. We restrict ourselves to point canonical transformations only.

In the literature on functional methods one often finds the statement that point transformations can be performed by simply changing integration variables in functional integrals.[20][21] Since there exists a direct correspondence between functional integrals and Feynman diagrams, all manipulations carried out with functional integrals can be explicitly checked using the diagram technique. Thus, in this section we will use the Feynman diagram technique to question the statement about point transformations made above. Namely, we will demonstrate by explicit calculations on a simple example that in general it is not correct to perform point canonical transformations by just making naive changes of variables in the functional integral. The fact is that after this naive change of variables we end up with a quantum theory entirely dirrerent from the original one. These two theories are only identical at the tree and one loop level. The difference between them appears already at the two loop level. That is what we are going to demonstrate in what follows.

Although point canonical transformations can be discussed in Lagrangian functional formalism we prefer to use the phase space path integral method throughout this section. Let us consider a simple example of N free harmonic oscilators coupled to external sources:

$$H_J(p,q)=\sum_{a=1}^{n}\left(\frac{1}{2}p_a^2+\frac{1}{2}\omega^2 q_a^2-J_a q_a\right) \tag{2.1}$$

The generating functional is defined as usual and is represented in path integral form by:

$$Z(J)=\int\prod_{a=1}^{n}\mathcal{D}p_a\mathcal{D}q_a\exp\left\{\frac{i}{\hbar}\int dt\left[p\cdot\dot{q}-H_J(p,q)\right]\right\} \tag{2.2}$$

With the small negative imaginary part added to ω^2 we obtain the answer for this path integral as:

$$Z(J)=\exp\left\{-\frac{1}{2}\sum_{a,b}\int dt\,dt'\,J_a(t)\Delta_F^{ab}(t-t')J_b(t')\right\} \tag{2.3}$$

where we have the Feynman propagators

$$\Delta_F^{ab}(t-t')=\delta_{ab}\,\Delta_F(t-t')$$
$$\Delta_F(t-t')=\int\frac{d\nu}{2\pi}\,e^{i\nu(t-t')}\frac{i}{\nu^2-\omega^2+i\epsilon} \tag{2.4}$$

Now, we perform a general point transformation from the old variables p_a, q_a to new canonical variables P_i, Q_i as follows:

$$q_a=F^a(Q)$$
$$p_a=F^a_{,i}(Q)\,g^{ij}(Q)\,P_j \tag{2.5}$$

We denote

$$F^a_{,i}(Q)=\frac{\partial F^a(Q)}{\partial Q_i}$$
$$g_{ij}(Q)=\sum_{a=1}^{n}F^a_{,i}(Q)F^a_{,j}(Q) \tag{2.6}$$

and $g^{ij}(Q)$ represents the inverse matrix of $g_{ij}(Q)$. The Jacobian of this transformation is one and thus by simply changing integration variables in the path integral expression for the generating functional we arrive at the new path integral:

$$\bar{Z}(J)=\int\prod_{i=1}^{n}\mathcal{D}P_i\mathcal{D}Q_i\exp\left\{\frac{i}{\hbar}\int dt\left[P\cdot\dot{Q}-\bar{H}_J(P,Q)\right]\right\} \tag{2.7}$$

with the new Hamiltonian $\bar{H}_J$ given by

$$\bar{H}_J(P,Q)=H_J\big(p(P,Q),q(Q)\big) \tag{2.8}$$

We denoted this path integral expression for the generating functional by a differnt symbol $\bar{Z}(J)$ since it is not clear whether the naive change of variables is a correct step or not. Indeed in what follows we are going to demonstrate that contrary to naive expectations $\overline{Z}(J)$ is not equal to $Z(J)$.

Since our Hamiltonian expressed in terms of new canonical variables P_i and Q_i contains in general complicated interaction terms the only way we can calculate the new generating functional is by perturbation theory. To find the Feynman rules of this perturbation expansion we write as usual:

$$\overline{Z}(J) = \exp\left\{-\frac{i}{\hbar}\int dt\, \overline{H}_{int}\left(\frac{1}{i}\frac{\delta}{\delta K}, \frac{1}{i}\frac{\delta}{\delta J}\right)\right\} Z_0(J,K) \Big|_{K=0} \tag{2.9}$$

where the free generating functional is given by

$$Z_0(J,K) = \int \prod_{i=1}^{n} \mathcal{D}P_i \mathcal{D}Q_i \exp\left\{\frac{i}{\hbar}\int dt \left[P\cdot\dot{Q} - \frac{1}{2}\left(P\cdot P + \omega^2 Q\cdot Q\right) + J\cdot Q + K\cdot P \right]\right\} \tag{2.10}$$

We introduced additional sources K_i coupled to momentum variables P_i since the interaction Hamiltonian contains derivative interactions. The free generating functional $Z_0(J,K)$ is easily found to be

$$Z_0(J,K) = \exp\left\{-\frac{1}{2}\int dt\,dt' \left(J_i(t) - \dot{K}_i(t)\right)\Delta_F^{ij}(t-t')\left(J_j(t') - \dot{K}_j(t')\right) + \frac{i}{2}\int dt\, K_i(t) K_i(t) \right\} \tag{2.11}$$

giving the propagators:

$$\begin{aligned} \langle 0| T\{\hat{Q}_i(t), \hat{Q}_j(t')\}|0\rangle &= \delta_{ij}\,\Delta_F(t-t') \\ \langle 0| T\{\hat{P}_i(t), \hat{Q}_j(t')\}|0\rangle &= \delta_{ij}\,\frac{d}{dt}\Delta_F(t-t') \\ \langle 0| T\{\hat{P}_i(t), \hat{P}_j(t')\}|0\rangle &= \delta_{ij}\left[\frac{d}{dt}\frac{d}{dt'}\Delta_F(t-t') - i\,\delta(t-t')\right] \end{aligned} \tag{2.12}$$

represented graphically by Fig. 1. The Feynman rules are completed by an infinite set of vertices obtained by expanding $F^a(Q)$ and $g^{ij}(Q)$ in powers of Q_i:

$$\begin{aligned} F^a(Q) &= F^a + F^a_{,i} Q_i + \frac{1}{2!} F^a_{,ij} Q_i Q_j + \frac{1}{3!} F^a_{,ijl} Q_i Q_j Q_l + \cdots \\ g^{ij}(Q) &= g^{ij} + g^{ij}_{,l} Q_l + \frac{1}{2!} g^{ij}_{,lm} Q_l Q_m + \cdots \end{aligned} \tag{2.13}$$

For simplicity we choose $F^a = 0$ and $F^a_{,i} = \delta_{ai}$. The Hamiltonian from which we read of the vertices is now:

$$\bar{H}_J(P,Q) = \frac{1}{2}\sum_{i,j} P_i P_j \left(\delta_{ij} + g^{ij}_{,\ell} Q_\ell + \frac{1}{2!} g^{ij}_{,\ell m} Q_\ell Q_m + \cdots \right)$$
$$+ \frac{1}{2}\omega^2 \left[Q \cdot Q + F^\ell_{,ij} Q_i Q_j Q_\ell + \left(\frac{1}{3} F^\ell_{,ijm} + \frac{1}{4} F^a_{,ij} F^a_{,\ell m} \right) Q_i Q_j Q_\ell Q_m + \cdots \right]$$
$$- \sum_{a=1}^{n} J_a \left(Q_a + \frac{1}{2!} F^a_{,ij} Q_i Q_j + \cdots \right) \qquad (2.14)$$

The first few in this series of vertices are represented graphically at Fig. 2.

Now we have a loop expansion for $\bar{Z}(J)$. One can easily see that due to cancellations of graphs when calculating $\bar{Z}(J)$ at tree and one loop level the result is equal to $Z(J)$. As an example of such cancellations consider the one loop tadpole diagrams shown at Fig. 3.a, 3.b, and 3.c. Their respective contributions are:

$$(a) = -\frac{i}{4\omega}\left(F^\ell_{,ii} + 2 F^i_{,\ell i} \right) \int dt\, J_\ell(t)$$
$$(b) = \frac{i}{2\omega} F^i_{,i\ell} \int dt\, J_\ell(t) \qquad (2.15)$$
$$(c) = \frac{i}{4\omega} F^\ell_{,ii} \int dt\, J_\ell(t)$$

where we used the identity

$$g^{ij}_{,\ell}(Q) = -g^{ii'}(Q)\, g^{jj'}(Q)\, g_{i'j',\ell}(Q) \qquad (2.16)$$

Obviously the sum of these three terms is zero. For a general discussion the reader is referred to reference (21).

But, unfortunately these cancellations do not persist already at the next two loop level. Consider for example all the two loop bubble diagrams whown at Fig. 4.a, 4.b, and 4.c. After some calculation using the integrals

$$\int dt\, \Delta_F^3(t) = -\frac{1}{12}\frac{i}{\omega^4}$$
$$\int dt\, \dot{\Delta}_F^2(t)\,\Delta_F(t) = \frac{1}{12}\frac{-i}{\omega^2} \qquad (2.17)$$

we find the contributions of these graphs:

$$(a) = \frac{i\hbar}{8}\left(-g_{ii,\ell}\, g_{jj,\ell} - \frac{1}{4} F^a_{,ii} F^a_{,jj} + \frac{1}{2} F^a_{,ij} F^a_{,ij} \right)$$

$$(b) = \frac{i\hbar}{8}\,\frac{1}{4}\, F^a_{,ii} F^a_{,jj}$$

$$(c) = \frac{3i\hbar}{8}\left(\frac{1}{2}F^{a}_{,ij}F^{a}_{,ij} + F^{e}_{,ij}\,F^{i}_{,je}\right) \tag{2.18}$$

The sum of these three terms does not give zero but rather:

$$(a)+(b)+(c) = \frac{i\hbar}{8}\;F^{e}_{,ij}\,F^{i}_{,je} \neq 0 \tag{2.19}$$

This explicit calculation shows that $\bar{Z}(J)$ is indeed different from $Z(J)$. So the new path integral (2.7) with the Hamiltonian $\bar{H}_J(P,Q)$ dies not define the same theory as the original one. This result means that it is incorrect to formally perform point transformations in path integral and that more care is needed when doing so

In the following section we will present a careful treatment of this problem and it will be shown that it leads to additional potential terms in the action of the new path integral (2.7). They are of the form:

$$\Delta V(Q) = \frac{\hbar^2}{8}\left\{ F^{a}_{,i}(Q)\,F^{a}_{,jem}(Q)\,g^{ij}(Q)\,g^{em}(Q) - \frac{1}{2}\left(\frac{1}{g(Q)}\,g_{,e}(Q)\right)_{,m} g^{em}(Q) \right\} \tag{2.20}$$

This means that the modified Hamiltonian

$$\bar{H}'_J(P,Q) = \bar{H}_J(P,Q) + \Delta V(Q) \tag{2.21}$$

represents the same theory as the original one $H_J(p\ q)$. Indeed we see that this additional term gives at the two loop level the following contribution

$$-\frac{i\hbar}{8}\,F^{e}_{,ij}\,F^{i}_{,ej} \tag{2.22}$$

which precisely cancels out the sum of two loop bubble diagrams (2.19)

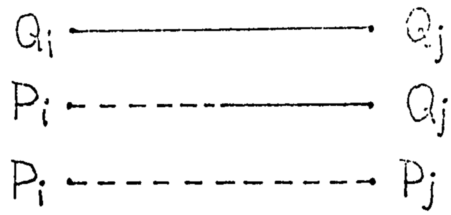

Fig. 1

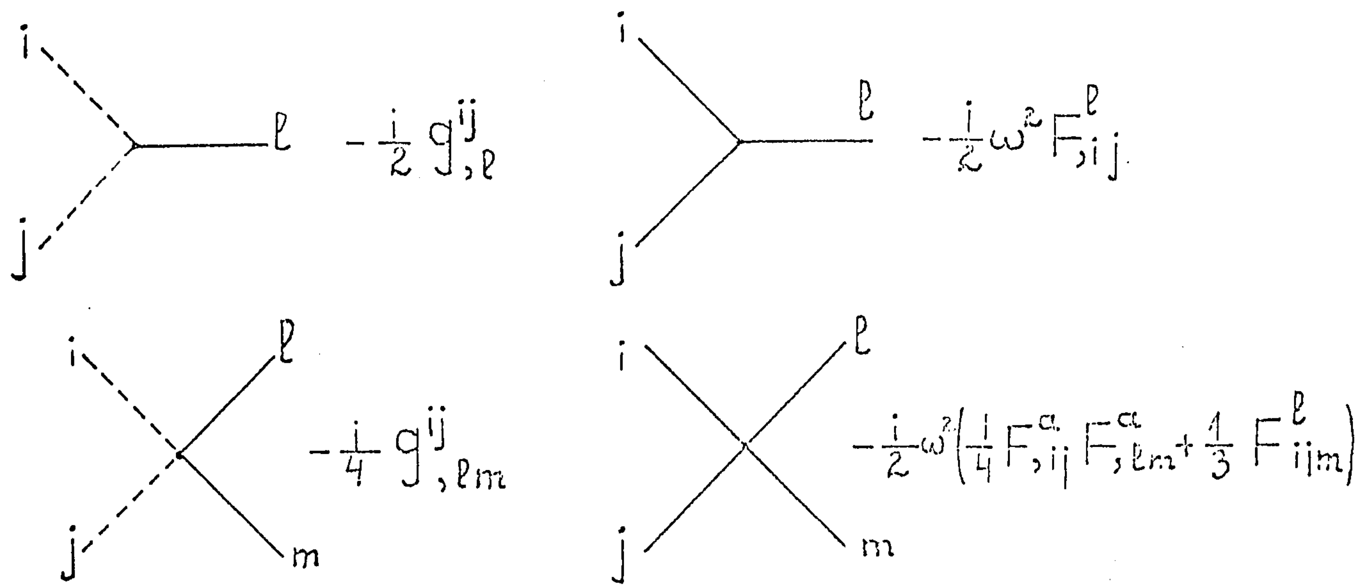

Fig. 2

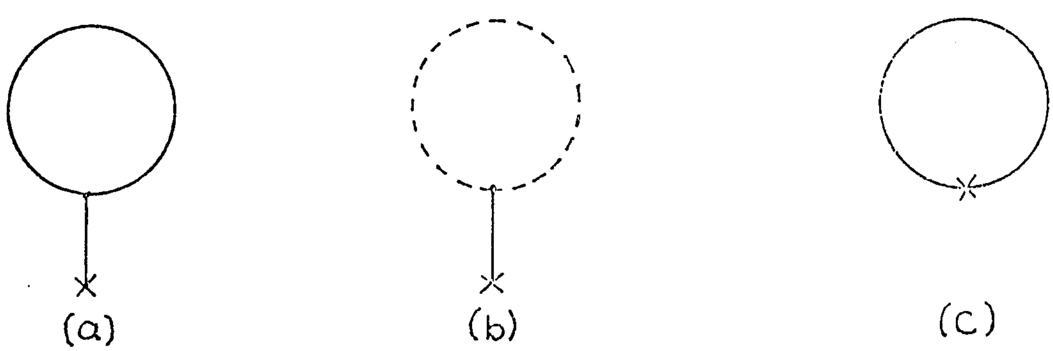

Fig. 3

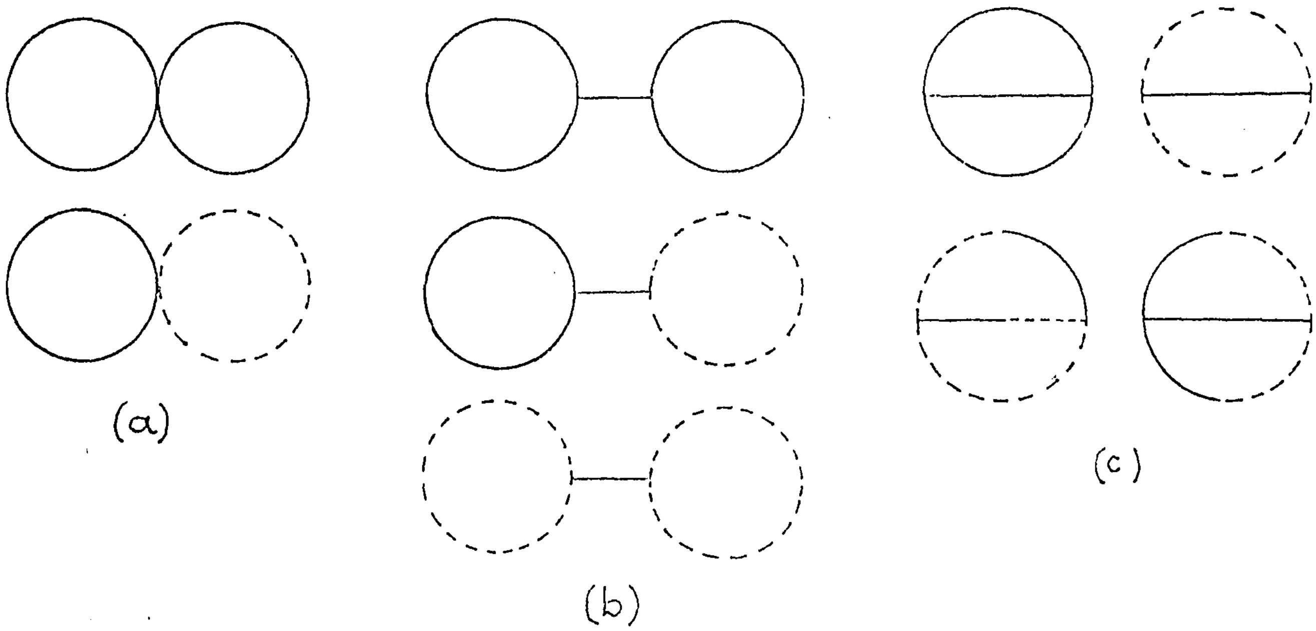

Fig.4

II.3 Point Canonical Transformations In Path Integral Method

The path integral method is often considered as of heuristic value only, with the understanding that all results derived in this approach are to be checked by parallel calculations in the operator formalism. There is also a belief that path integrals can safely be used only through their correspondence with the diagram technique, so that manipulations with path integrals just represent manipulations with Feynman diagrams.

We don't accept such limited definitions of the path integral method, especially in view of the fact that this method proves to be extremely useful in developing non-perturbative techniques. Although as of yet there exists no rigorous mathematical definition of integration over paths, there is still a precise enough formulation of the method which allows us to derive conclusions as rigorously as with the operator formalism. This is the original definition introduced by Feynman who defined the path integral as a limit of finite dimensional integrations. It was discussed in detail in section 1.

In this section we make use of this precise definition in order to demonstrate how it allows for a rigorous treatment of point canonical transformations in the path integral method [17].

We consider a general theory described by the Lagrangian:

$$L(q,\dot{q}) = \frac{1}{2}\sum_a \mathring{q}_a(t) - V(q(t)) \tag{3.1}$$

The kernel to go from some initial point $q' = (q'_1, q'_2, \ldots)$ at time t' to some final point $q'' = (q''_1, q''_2, \ldots)$ at time t'' is given by the following path integral:

$$\tag{3.2}$$

the right-hand side being defined as the limit of finite dimensional integrations. Namely, we subdivide the time t"-t' into N equal intervals denoting $t_k = t' + \epsilon k$ and $q_a(k) \equiv q_a(t_k)$. Then the precise expression for this path integral reads:

$$\lim_{N\to\infty} \int \prod_a \prod_{k=1}^{N-1} \frac{d q_a(k)}{(2\pi i \hbar \epsilon)^{\frac{1}{2}}} \prod_{k=0}^{N-1} \exp\left\{ \frac{i}{\hbar} A\left(q(k+1), q(k) \right) \right\} \tag{3.3}$$

with $q_a(0) = q'_a$ and $q_a(N) = q''_a$. In the exponent we have the short time action:

$$A\left(q(k+1), q(k)\right) = \frac{1}{2\epsilon} \sum_a \left(q_a(k+1) - q_a(k) \right)^2 - \epsilon V(q(k)) \tag{3.4}$$

We now perform a general point canonical transformation $q_a(t) = F^a(Q(t))$ to new variables $Q = (Q_1, Q_2, \ldots)$. This means that in the finite dimensional integral we have to change variables substituting $q_a(k) = F^a(Q(k))$. Then the new integration measure is given by:

$$\prod_a d q_a(k) = \left[\det g_{ij}(Q(k)) \right]^{1/2} \prod_i d Q_i(k) \tag{3.5}$$

and the new short time action is simply:

$$\tilde{A}\left(Q(k+1), Q(k) \right) = \frac{1}{2\epsilon} \sum_a \left(F^a(Q(k+1)) - F^a(Q(k)) \right)^2 - \epsilon V\left(F(Q(k)) \right) \tag{3.6}$$

At this point it is tempting to use the expansion

$$F^a(Q(k+1)) - F^a(Q(k)) = F^a_{,i}(Q(k))\, \Delta Q_i(k) + \\ + \frac{1}{2!} F^a_{,ij}(Q(k))\, \Delta Q_i(k)\, \Delta Q_j(k) + \cdots \tag{3.7}$$

and approximate the short time action (3.6) as

$$\tilde{A}\left(Q(k+1), Q(k) \right) \approx \frac{1}{2\epsilon} \sum_{ij} g_{ij}(Q(k))\, \Delta Q_i(k)\, \Delta Q_j(k) - \\ - \epsilon V(F(Q(k))) \tag{3.8}$$

thus dropping the higher order terms in $\Delta Q_i(k) = Q_i(k+1) - Q_i(k)$.

But the above approximation appears to be incorrect and would lead us to erroneous results. In fact, it is equivalent to the naive change of variables discussed in section 2. The mistake which would be made by making this approximation has its origin in the fact that higher order terms in the short time action of the form $\frac{\Delta Q^3}{\epsilon}$ and $\frac{\Delta Q^4}{\epsilon}$ still contribute to the path integral and thus cannot be dropped. Namely, due to the stohastic nature of path integrals we have that $\Delta Q \neq O(\epsilon)$ but rather $\Delta Q^2 = O(\epsilon)$. This observation is due to Edwards and Gulyaev who investigated the change from Cartesian to polar co-ordinates in the path integral formalism[22].

So in order to correctly approximate the short time action we have to keep all terms effectively of $O(\epsilon)$. We choose to expand the functions $F^a(Q(k+1))$ and $F^a(Q(k))$ about the midpoint $\overline{Q}_i(k) = \frac{Q_i(k+1)+Q_i(k)}{2}$ and thus it is the symmetric ("midpoint") definition of the path integral which we adopt. Then keeping terms of up to the fourth order in ΔQ we obtain the following approximation for the short time action:

$$\begin{aligned}\overline{A}\left(Q(k+1), Q(k)\right) = \frac{1}{2\epsilon}\Big[& g_{ij}(\overline{Q}(k))\,\Delta Q_i(k)\,\Delta Q_j(k) + \\ & + \frac{1}{12} F^a_{,i}(\overline{Q}(k)) F^a_{,jkl}(\overline{Q}(k))\,\Delta Q_i(k)\,\Delta Q_j(k)\,\Delta Q_k(k)\,\Delta Q_l(k)\Big] \\ & - \epsilon\, V(F(\overline{Q}(k)))\end{aligned} \tag{3.9}$$

We now have the correct path integral expression for the kernel in terms of new variables Q_i:

$$\begin{aligned}\left[\det g_{ij}(Q'')\,\det g_{ij}(Q')\right]^{-\frac{1}{4}} \lim_{N\to\infty} \int \prod_i \prod_{k=1}^{N-1} \frac{dQ_i(k)}{(2\pi i \epsilon\hbar)^{1/2}} \times \\ \times \prod_{k=0}^{N-1} J(Q(k+1), Q(k)) \exp\left\{\frac{i}{\hbar}\overline{A}(Q(k+1), Q(k))\right\}\end{aligned} \tag{3.10}$$

with the Jacobian given by:

$$J(Q(k+1), Q(k)) = \left[\det g_{ij}(Q(k+1))\,\det g_{ij}(Q(k))\right]^{\frac{1}{4}} \tag{3.11}$$

In order to have the symmetric path integral we still need to expand the Jacobian about $\bar{Q}_i(k)$ keeping quadratic terms in $\Delta Q_i(k)$ since they still contribute being of $O(\epsilon)$. Using the identity

$$\det(A+B) = \det A \, \det(1 + A^{-1}B) = \det A \left\{ 1 + \mathrm{Tr}(A^{-1}B) + \frac{1}{2}(\mathrm{Tr}\,A^{-1}B)^2 - \frac{1}{2}\mathrm{Tr}(A^{-1}B)^2 + \cdots \right\} \tag{3.12}$$

we obtain after some calculation

$$J(Q(k+1),Q(k)) \approx [\det g_{ij}(\bar{Q}(k))]^{\frac{1}{2}} \left[1 + \frac{1}{16}\left(g^{ij}(\bar{Q})\, g_{ij,\ell m}(\bar{Q}) + g^{ij}_{\ ,\ell}(\bar{Q})\, g_{ij,m}(\bar{Q}) \right) \Delta Q_\ell \, \Delta Q_m \right] \tag{3.13}$$

Although now we have a correct path integral expression for the kernel in terms of the new variables Q_i it is not very useful since in the action of (3. 9) there are additional terms of the form $\frac{\Delta Q^4}{\epsilon}$ and also in the Jacobian terms of the form $(\Delta Q)^2$. One would prefer to eliminate these terms in favor of a potential like term of the form $\Delta V(Q)\epsilon$. That can be done and a detailed discussion of this problem was already given by McLaughlin and Schulman in connection with the path integral quantization in curved spaces where the same problem appears.[23] We will not repeat their arguments here but only give a short derivation of our results.

We start by approximating the exponential in (3.10) by:

$$\exp\left\{ \frac{i}{2\hbar\epsilon} g_{ij}(\bar{Q}(k))\, \Delta Q_i \Delta Q_j - \epsilon V(F(\bar{Q}(k))) \right\} \times \left\{ 1 + \frac{i}{24\hbar\epsilon} F^a_{,i}(\bar{Q})\, F^a_{,j\ell m}(\bar{Q})\, \Delta Q_i\, \Delta Q_j\, \Delta Q_\ell\, \Delta Q_m \right\} \tag{3.14}$$

so that the effective Jacobian has the following form:

$$g(\bar{Q})^{1/2} \left\{ 1 + \frac{1}{16}\left[g^{ij}(\bar{Q})\, g_{ij,\ell m}(\bar{Q}) + g^{ij}_{\ ,\ell}(\bar{Q})\, g_{ij,m}(\bar{Q}) \right] \Delta Q_\ell \Delta Q_m + \frac{i}{24\hbar\epsilon} F^a_{,i}(\bar{Q})\, F^a_{,j\ell m}(\bar{Q})\, \Delta Q_i\, \Delta Q_j\, \Delta Q_\ell\, \Delta Q_m \right\} \tag{3.15}$$

Then making use of the following integrals

$$\int \prod_{i=1}^{n} dx_i \exp\left[\frac{i}{2\hbar\epsilon} g_{ij} x_i x_j \right] x_\alpha x_\beta = (2\pi i \epsilon\hbar)^{\frac{n}{2}} g^{-\frac{1}{2}} (i\hbar\epsilon)\, g^{\alpha\beta}$$

$$\int \prod_i dx_i \exp\left[\frac{i}{2\hbar\epsilon} g_{ij} x_i x_j \right] x_\alpha x_\beta x_\gamma x_\delta = (2\pi i\hbar\epsilon)^{\frac{n}{2}} g^{-\frac{1}{2}} (i\hbar\epsilon)^2 \cdot (g^{\alpha\beta} g^{\gamma\delta} + g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\gamma\beta}) \tag{3.16}$$

we substitute the additional terms in the Jacobian (3.15) by potential like terms of the form

$$\epsilon \Delta V(\bar{Q}) = \epsilon \hbar^2 \Big\{ -\frac{1}{16}\left(g^{ij}(\bar{Q})\, g_{ij,\ell m}(\bar{Q}) + g^{ij}{}_{,\ell}(\bar{Q})\, g_{ij,m}(\bar{Q})\right) g^{\ell m}(\bar{Q}) + \frac{1}{24} F^{a}_{,i}(\bar{Q})\, F^{a}_{,j\ell m}(\bar{Q}) \big(g^{ij}(\bar{Q})\, g^{\ell m}(\bar{Q}) + g^{i\ell}(\bar{Q})\, g^{jm}(\bar{Q}) + g^{im}(\bar{Q})\, g^{j\ell}(\bar{Q}) \big) \Big\} \tag{3.17}$$

Now using the identity

$$\frac{1}{g(Q)}\, g_{,\ell}(Q) = g^{ij}(Q)\, g_{ij,\ell}(Q) \tag{3.18}$$

we simplify this expression to

$$\Delta V(\bar{Q}) = \frac{\hbar^2}{8} \Big\{ F^{a}_{,i}(\bar{Q})\, F^{a}_{,j\ell m}(\bar{Q})\, g^{ij}(\bar{Q})\, g^{\ell m}(\bar{Q}) - \frac{1}{2}\Big(\frac{1}{g(\bar{Q})}\, g_{,\ell}(\bar{Q}) \Big)_{,m} g^{\ell m}(\bar{Q}) \Big\} \tag{3.19}$$

Next introducing the Christoffel symbols defined by

$$F^{a}_{,ij}(Q) = \Gamma^{\ell}_{ij}(Q)\, F^{a}_{,\ell}(Q) \tag{3.20}$$

we find that

$$F^{a}_{,i}(Q)\, F^{a}_{,j\ell m}(Q)\, g^{ij}(Q)\, g^{\ell m}(Q) = \Gamma^{i}_{i\ell,m}(Q)\, g^{\ell m}(Q) + g^{\ell m}(Q)\, \Gamma^{j}_{i\ell}(Q)\, \Gamma^{i}_{jm}(Q) \tag{3.21}$$

which together with the identity

$$\Gamma^{i}_{i\ell,m}(Q) = \frac{1}{2}\Big(\frac{1}{g(Q)}\, g_{,\ell}(Q) \Big)_{,m} \tag{3.22}$$

leads us to the very compact form for the additional potential term:

$$\Delta V(\bar{Q}) = \frac{\hbar^2}{8}\, \Gamma^{i}_{j\ell}(\bar{Q})\, \Gamma^{j}_{im}(\bar{Q})\, g^{\ell m}(\bar{Q}) \tag{3.23}$$

So our final path integral expression for the kernel in terms of the new variables Q_i reads:

$$K(q''t''; q't') = [g(Q'')\, g(Q')]^{-1/4} \lim_{N\to\infty} \int \prod_i \prod_{k=1}^{N-1} \frac{dQ_i(k)}{(2\pi i \hbar \epsilon)^{1/2}} \cdot \prod_{k=0}^{N-1} g(\bar{Q}(k))^{1/2} \exp\Big\{ \frac{i}{\hbar} A_{eff}(\bar{Q}(k), \Delta Q(k)) \Big\} \tag{3.24}$$

with the following effective short time action:

$$A_{eff}(\bar{Q}(k), \Delta Q(k)) = \frac{1}{2\epsilon} \sum_{i,j} g_{ij}(\bar{Q}(k))\, \Delta Q_i(k)\, \Delta Q_j(k) - \epsilon\big[V(F(\bar{Q}(k))) + \Delta V(\bar{Q}(k)) \big] \tag{3.25}$$

This is the main result of this section and the following observations are in order. First, the effective short time action (3.25) differs from the "naive" one by just the additional potential term $\Delta V(Q)$ proportional to $\hbar^2$. This then explains why the formal change of variables breaks down starting at the two loop level. Second, everywhere in the above path integral we have the midpoint coordinate $\bar{Q}_i(k)$ and not $Q_i(k)$ or $Q_i(k+1)$ due to the fact that we adopted the symmetric definition for the path integral. Had we choosen to use a different definition the form of the additional potential term would naturally be different.

Next it is instructive to write down the equivalent phase space path integral which has the form:

$$[g(Q'')g(Q')]^{\frac{1}{4}} \lim_{N\to\infty} \int \prod_i \prod_{k=1}^{N-1} dQ_i(k) \prod_{k=0}^{N-1} \frac{dP_i(k)}{2\pi\hbar} \times$$

$$\times \exp\left\{ \frac{i}{\hbar} \sum_{k=0}^{N-1} \left[P(k)\cdot\Delta Q(k) - H(P(k), \bar{Q}(k)) \right] \right\} \tag{3.26}$$

with the following classical Hamiltonian function

$$H(P,Q) = \frac{1}{2} \sum_{i,j} g^{ij}(Q) P_i P_j + V(F(Q)) + \Delta V(Q) \tag{3.27}$$

We now observe that the operator Hamiltonian corresponding to the above phase space path integral contains non-commuting factors. That is precisely the reason why the careful treatment of this section was necessary. Namely, there is a unique ordering of non-commuting factors so that the final operator Hamiltonian describes the same quantum theory as the original one.

We can easily demonstrate that the operator Hamiltonian corresponding to the above midpoint phase space path integral is indeed the correct one. Using the results of section 1 we conclude that this operator Hamiltonian has the following form

$$\hat{H} = \frac{1}{2} \left(\hat{P}_i \hat{P}_j g^{ij}(\hat{Q}) \right)_w + V(F(\hat{Q})) + \Delta V(\hat{Q}) \tag{3.28}$$

The non-commuting factors in (3.28) are Weyl ordered in view of the fact that in (3.26) we have the symmetric path integral.

Next we perform a point transformation in the operator formalism. Starting with the original Hamiltonian in coordinate representation

$$-\frac{\hbar^2}{2}\sum_a \frac{\partial^2}{\partial q_a^2} + V(q) \tag{3.29}$$

and making the change of variables $q_a = F^a(Q)$ we get for this differential operator

$$-\frac{\hbar^2}{2}\sum_{i,j} \frac{1}{g(Q)^{1/2}}\frac{\partial}{\partial Q_i} g^{ij}(Q)\, g(Q)^{1/2}\frac{\partial}{\partial Q_j} + V(F(Q)) \tag{3.30}$$

with the following scalar product

$$(\psi_1,\psi_2) = \int \prod_i dQ_i\, g(Q)^{1/2}\, \psi_1^*(F(Q))\, \psi_2(F(Q)) \tag{3.31}$$

Redefining the Hilbert space so to eliminate the measure from this scalar product we write the operator Hamiltonian (3.30) in the form:

$$\hat{H} = \frac{1}{2}\, g(\hat{Q})^{-\frac{1}{4}}\, \hat{P}_i\, g^{ij}(\hat{Q})\, g(\hat{Q})^{\frac{1}{2}}\, \hat{P}_j\, g(\hat{Q})^{-\frac{1}{4}} + V(F(\hat{Q})) \tag{3.32}$$

Finally we reorder the non-commuting factors in (3.32) and using the relation

$$\hat{P}_i\, g^{ij}(\hat{Q})\, \hat{P}_j = \{\hat{P}_i \hat{P}_j\, g^{ij}(\hat{Q})\}_W + \frac{\hbar^2}{4}\, g^{ij}_{\ ,ij}(\hat{Q}) \tag{3.33}$$

rewrite the operator Hamiltonian (3.32) as

$$\hat{H} = \frac{1}{2}\{\hat{P}_i \hat{P}_j\, g^{ij}(\hat{Q})\}_W + V(F(\hat{Q})) + \Delta V'(\hat{Q}) \tag{3.34}$$

with

$$\Delta V'(\hat{Q}) = \frac{\hbar^2}{4}\left[\frac{1}{2} g^{ij}_{\ ,ij}(\hat{Q}) - 2 g(\hat{Q})^{-\frac{1}{4}}\left(g^{ij}(\hat{Q})\, g^{1/2}\left(g(\hat{Q})^{-\frac{1}{4}}\right)_{,i}\right)_{,j}\right. \tag{3.35}$$

Now it is straightforward to show that the additional potential terms $\Delta V'(Q)$ and $\Delta V(Q)$ are indeed equal which finishes the proof that our final path integral corresponds to the correctly ordered operator theory.

III. EXTENDED PARTICLES IN TWO-DIMENSIONAL FIELD THEORIES

In this chapter we start with the discussion of extended particles in field theory. For simplicity we consider two space-time dimensional scalar field theories. First in section III.1 we study classical particle like solutions and point out several difficulties in conection with their quantum interpretation. Then in section III.2 we present a method which leads to a complete quantum treatment of extended particles associated with these classical particle like solutions. In section III.3 we develop a systematic perturbation expansion about extended particle states. Finally in section III.4 we show how one is to compute Green's functions in the extended particle sector.

III.1 Particle-like Classical Solutions

Let us consider as a specific example the two-dimensional scalar $\frac{\lambda}{4}\phi^4$ theory described by the Lagrangian

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{2}\partial_\mu\phi(x)^2 - V(\phi) \\ V(\phi) &= -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \end{aligned} \tag{1.1}$$

where the mass term appears with opposite sign. This model is often used to illustrate the phenomenon of spontaneous symmetry breaking. Namely, the potential functional

$$\overline{V}[\phi] = \int dx \left(\frac{1}{2}\phi'(x) - \frac{1}{2}m^2\phi^2(x) + \frac{\lambda}{4}\phi^4(x) \right) \tag{1.2}$$

has a local maximum at $\phi = 0$ and absolute minima at $\phi = \pm\frac{m}{\sqrt{\lambda}}$. The perturbation expansion is then developed by expanding about either of this absolute minima and this is achieved by first translating the field by the classical value $\pm \frac{m}{\sqrt{\lambda}}$.

We now observe that the equations of motion possess other classical solutions besides $0, \pm \frac{m}{\sqrt{\lambda}}$. There are two static solutions $\pm\phi_0(x)$ where

$$\phi_0(x) = \frac{m}{\sqrt{\lambda}} \, \mathrm{th} \, \frac{mx}{\sqrt{2}} \tag{1.3}$$

This is the so called "kihk" solution and obviously for $x \to \pm\infty$ it approaches the two degenerate vacuua $\pm \frac{m}{\sqrt{\lambda}}$. Then the energy of this classical solution differs by a finite amount from the vacuum energy:

$$\begin{aligned} M_0 &= \bar{V}[\phi_0] - \bar{V}\left[\frac{m}{\sqrt{\lambda}}\right] = \\ &= \int dx \left[\tfrac{1}{2}\phi_0'^2(x) - \tfrac{1}{2}m^2\left(\phi_0^2(x) - \frac{m^2}{\lambda}\right) + \frac{\lambda}{4}\left(\phi_0^4(x) - \frac{m^4}{\lambda^2}\right)\right] = \\ &= \frac{2\sqrt{2}}{3}\,\frac{m^3}{\lambda} \end{aligned} \tag{1.4}$$

Next we demonstrate that the kink is a classically stable solution, namely, that it is a local minimum of the potential functional $\bar{V}[\phi]$. First we note that due to the translational invariance of our theory a general static solution has the form $\phi_0(x+x_0)$ where x_0 is an arbitrary constant. Translating the field by this static kink solution

$$\phi(x) = \phi_0(x+x_0) + \eta(x) \tag{1.5}$$

we obtain

$$\begin{aligned} \bar{V}[\phi] &= \bar{V}[\phi_0] + \frac{1}{2}\int dx\, \eta(x)\left[-\frac{d^2}{dx^2} - m^2 + 3m^2\,\mathrm{th}^2\frac{m(x+x_0)}{\sqrt{2}}\right]\eta(x) \\ &+ \lambda\frac{m}{\sqrt{\lambda}}\int dx\, \mathrm{th}\,\frac{m(x+x_0)}{\sqrt{2}}\,\eta^3(x) + \frac{\lambda}{4}\int dx\, \eta^4(x) \end{aligned} \tag{1.6}$$

So in order to study small perturbations about the kink solution we need to solve the following eigenequations:

$$\left[-\frac{d^2}{dx^2} + V''(\phi_0(x+x_0))\right]\psi_n(x) = \omega_n^2\,\psi_n(x) \tag{1.7}$$

Since $\phi_0(x+x_0)$ satisfies the classical equations of motion

$$-\frac{d^2}{dx}\phi_0(x+x_0)+V'(\phi_0(x+x_0))=0 \tag{1.8}$$

we immediately conclude that

$$\psi_0(x)=\frac{1}{\sqrt{M_0}}\frac{d}{dx_0}\phi_0(x+x_0) \tag{1.9}$$

is a zero frequency eigenfunction of (1.7) . Obviously, $\psi_0(x)$ does not have nodes and it is the lowest energy eigenfunction of this Schrodinger equation. Then all the other eigenfrequencies are necessarily positive which implies the stability of the kink classical solution. The zero frequency mode associated with translation symmetry expresses the fact that there is an infinite family of solutions obtained from $\phi_0(x+x_0)$ by translation.

Next we observe that the static solution can be boosted by velocity u to obtain a time dependent solution

$$\phi_u(x,t)=\phi_0\left(\frac{x+ut+x_0}{\sqrt{1-u^2}}\right) \tag{1.10}$$

The energy and momenta of this solution are

$$\begin{aligned} E_u &= \int dx\left(\tfrac{1}{2}\dot{\phi}_u(x,t)+\tfrac{1}{2}\phi_u'(x,t)+V(\phi_u(x,t))-V\left(\tfrac{m}{\sqrt{\lambda}}\right)\right) \\ &= \frac{M_0}{\sqrt{1-u^2}} \\ P_u &= \int dx\ \dot{\phi}_u(x,t)\,\phi_u'(x,t) = \frac{M_0 u}{\sqrt{1-u^2}} \end{aligned} \tag{1.11}$$

and so the energy-momentum relation

$$E_u^2-P_u^2=M_0^2 \tag{1.12}$$

follows expressing the particle like property of this time dependent classical solutions.

In general, if the potential has the form

$$V(\phi(x,t)) = \frac{1}{\lambda}\, U(\sqrt{\lambda}\,\phi(x,t)) \tag{1.13}$$

the classical solution $\phi_n(x,t)$ will be proportional to $1/\sqrt{\lambda}$ and the mass $M_0 \sim \frac{1}{\lambda}$. So for weak coupling we have a heavy particle at the classical level and the question is whether there is a related particle in the full quantum theory also.

In order to study the quantum theory one is immediately led to the ideas of treating the static solution $\phi_0(x+X_0)$ in complete analogy with spontaneous symmetry breaking. That is to translate the field by the static solution and develop a perturbation expansion in a straightforward way. So we consider a one extended particle sector generating functional

$$Z(J;\text{E.P.}) = \int \mathcal{D}\Pi\,\mathcal{D}\phi\; \Psi^*[\phi(\cdot,+\infty)]\,\Psi[\phi(\cdot,-\infty)]\,\exp\left\{\frac{i}{\hbar}\left[A[\Pi,\phi] + \int dx\, J\phi\right]\right\} \tag{1.14}$$

Translating the field $\phi(x,t) = \phi_0(x+X_0) + \eta(x,t)$ we get for the action:

$$A[\Pi,\phi] = A[0,\phi_0] + \frac{1}{2}\int dx\,dt\left[\Pi^2 + \eta\left(-\frac{d^2}{dx^2} + V''(\phi_0(x+X_0))\right)\eta\right] + \sum_{\ell=3} \frac{1}{\ell!}\int dx\,dt\; V^{(\ell)}(\phi_0(x+X_0))\,\eta^{\ell}(x,t) \tag{1.15}$$

the propagator of this perturbation expansion is now

$$\Delta(t-t';x,x') = \sum_n \psi_n(x)\int\frac{d\nu}{2\pi}\, e^{i\nu(t-t')}\,\frac{i}{\nu^2-\omega_n^2+i\epsilon}\,\psi_n^*(x') \tag{1.16}$$

But there are two serious problems with this straightforward approach. First, since we translated the field by a space dependent solution the translational invariance is not preserved. Second, the progator (1.16) contains a zero frequency mode $\psi_0(x)$ leading to an infrared problem. This is actually the Goldstone boson associated with the spontaneous breaking of translational symmetry. Naturally we cannot accept such an interpretation and in the next section we develop a consistent treatment free of infrared problem.

III.2 Translational Invariance and Collective Coordinates

In general we will use the name soliton for the extended particle associated with a classical particle like solution. To give a consistent quantum treatment of solitons we must pay special attention to translational invariance of our theory. As the first step it is important to specify more precisely the initial and final soliton states when defining the one soliton sector generating functional. So we write

$$Z(J; p_f, p_i) = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \int \mathcal{D}\pi \mathcal{D}\phi\, \Psi^*_{p_f}[\phi(\cdot, t_f)]\, \Psi_{p_i}[\phi(\cdot, t_i)] \times \exp\left\{ \frac{i}{\hbar} \int dt\, dx \left[\pi(x,t)\dot{\phi}(x,t) - \mathcal{H}(\pi, \phi) + J(x,t)\phi(x,t) \right] \right\} \tag{2.1}$$

where p_i and p_f denote the initial and final soliton momenta. Obviously, we treat the soliton states in analogy with the vacuum state, namely, using wave functionals defined as

$$\begin{aligned} \Psi_p[\phi(\cdot)] &= \langle \phi(\cdot) | p \rangle \\ \hat{\phi}(x) | \phi(\cdot) \rangle &= \phi(x) | \phi(\cdot) \rangle \end{aligned} \tag{2.2}$$

The external source $J(x,t)$ appearing in this generating functional serves to generate Green's functions of the fundamental scalar field $\phi(x,t)$ in the one soliton sector. Then using reduction formula we can create an arbitrary number of fundamental quanta in addition to the initial and final solitons.

Translational invariance imposes strong conditions on the form of this generating functional. For example if the external source is zero, we have the one soliton transition amplitude

$$\langle p_f | e^{-\frac{i}{\hbar}\hat{H}T} | p_i \rangle = \int \mathcal{D}\pi \mathcal{D}\phi \, \Psi^*_{p_f}[\phi(\cdot, t_f)] \Psi_{p_i}[\phi(\cdot, t_i)] \exp\left\{\frac{i}{\hbar} A[\pi, \phi]\right\} \tag{2.3}$$

which can be written in the form

$$\langle p_f | e^{-\frac{i}{\hbar}\hat{H}T} | p_i \rangle = (2\pi)\,\delta(p_f - p_i)\,F(p_i) \tag{2.4}$$

the momentum conservation condition appearing as a consequence of translational invariance. Before we develop a method for calculating the one soliton sector generating functional it is important to assure the translational invariance of our theory. To achieve this we first separate our the center of mass motion. This is done by introducing new dynamical variables $X(t)$ and $p(t)$ corresponding to the center of mass coordinate and the total momentum:

$$P[\pi, \phi] = \int dx \, \pi(x,t)\, \phi'(x,t) \tag{2.5}$$

These variables we call collective coordinates and they are introduced into the one soliton matrix element (2.3) (or the generating functional (2.1)) using the following identities:

$$\begin{aligned} &\int \mathcal{D}X(t) \prod_t \delta\left(Q[\pi(x-X(t)), \phi(x-X(t),t)]\right) \frac{\partial Q}{\partial X(t)} = 1 \\ &\int \mathcal{D}p(t) \prod_t \delta\left(p(t) - P[\pi, \phi] \right) = 1 \end{aligned} \tag{2.6}$$

The last identity which we call the constraint, serves to identify the variable $p(t)$ with the total momentum of the system while the first represents the gauge condition associated with the constraint. Q can be arbitrary. We notice that $\frac{\partial Q}{\partial X}$ is given by the Poisson bracket:

$$\frac{\partial Q}{\partial X} = \left\{ Q[\pi, \phi], P[\pi, \phi] \right\}_{P.B.} \tag{2.7}$$

In order to transfer the collective coordinates from the δ -conditions into the action we perform the following change of variables:

$$\begin{aligned} \phi(x,t) &= \tilde{\Phi}(x+X(t),t) \\ \pi(x,t) &= \tilde{\pi}(x+X(t),t) \\ \rho &= x+X(t) \end{aligned} \tag{2.8}$$

introducing new field variables $\tilde{\phi}$ and $\tilde{\pi}$. Then using the constraint we get

$$\int dt\, dx \left(\pi\dot{\phi} - \mathcal{H}(\pi,\phi)\right) = \int dt \left[p(t)\dot{X}(t) + \int d\rho \left(\tilde{\pi}\dot{\tilde{\phi}} - \mathcal{H}(\tilde{\pi},\tilde{\phi})\right)\right] \tag{2.9}$$

and since the initial and final soliton states are momentum eigenstates

$$\psi_{p_i,p_f}[\phi(\cdot,t_{i,f})] = e^{i p_{i,f} X(t_{i,f})}\, \psi_{p_i,p_f}[\tilde{\phi}(\cdot,t_{i,f})] \tag{2.10}$$

From the first expression we see that $X(t)$ is the conjugate variable to $p(t)$. Our new expression for the one soliton transition amplitude then reads:

$$\begin{aligned} &\int \mathcal{D}p\,\mathcal{D}X\, e^{-ip_f X(t_f)} e^{ip_i X(t_i)} \int \mathcal{D}\tilde{\pi}\,\mathcal{D}\tilde{\phi}\; \psi^*_{p_f}[\tilde{\phi}(\cdot,t_f)]\,\psi_{p_i}[\phi(\cdot,t_i)] \times \\ &\times \prod_t \delta(p(t) - P[\tilde{\pi},\tilde{\phi}])\,\delta(Q[\tilde{\pi},\tilde{\phi}])\,\{Q[\pi,\phi],P[\pi,\phi]\}_{P.B.} \times \\ &\times \exp\left\{\int_{t_i}^{t_f} dt \left[p(t)\dot{X}(t) + \int d\rho \left(\tilde{\pi}\dot{\tilde{\phi}} - \mathcal{H}(\tilde{\pi},\tilde{\phi})\right)\right]\right\} \end{aligned} \tag{2.11}$$

As $X(t)$ appears only in the term $p\dot{X}$, we can immediately integrate over $X(t)$ and $p(t)$ which leads to

$$\begin{aligned} \langle p_f,t_f | p_i,t_i\rangle &= (2\pi)\,\delta(p_f - p_i) \int \mathcal{D}\tilde{\pi}\,\mathcal{D}\tilde{\phi}\; \psi^*_p[\tilde{\phi}(\cdot,t_f)]\,\psi_p[\tilde{\phi}(\cdot,t_i)] \times \\ &\times \delta(p - P)\,\delta(Q)\,\{Q,P\}_{P.B.}\, \exp\left\{i\int\left[\tilde{\pi}\dot{\tilde{\phi}} - \mathcal{H}(\tilde{\pi},\tilde{\phi})\right]\right\} \end{aligned} \tag{2.12}$$

where $p = p_i = p_f$. So we succeeded in extracting the momentum conservation condition in front of our path integral. Now one can develop a perturbation expansion about the one soliton state withour the danger of violating the translational invariance of the theory. We note that for the more complicated case of the one soliton generating functional (2.1) it is not possible to integrate out the collective coordinates due to the source term in the action which depends on $X(t)$:

$$\int dx\, J(x,t)\,\phi(x,t) = \int d\rho\, J(\rho - X(t),t)\,\tilde{\Phi}(\rho,t) \tag{2.13}$$

In this case the collective coordinates will appear explicitly in our calculations and their main function is to ensure the translational invariance. We continue by observing that the path integral (2.12) has its main contribution from the stationary point of the action with constraints. So we consider the following variational equation

$$\delta\left\{\int dt\left[\int d\rho\left(\tilde{\Pi}\dot{\tilde{\Phi}} - \mathcal{H}(\tilde{\Pi},\tilde{\Phi})\right) - \alpha(t)\left(p - P[\tilde{\Pi},\tilde{\Phi}]\right)\right]\right\} = 0 \tag{2.14}$$

where $\alpha(t)$ is a Lagrange multiplier. One obtains, for the lowest energy stationary point $(\dot{\phi}_c = 0)$, exactly the soliton solution

$$\phi_p(\rho) = \phi_0\left(\sqrt{1+p^2/M_0^2}\,(\rho + X_0)\right)$$

$$\Pi_p(\rho) = \frac{p}{\sqrt{1+p^2/M_0^2}}\,\phi_p'(\rho) \tag{2.15}$$

where ϕ_0 is the solution of the static soliton solution

$$-\phi_0'' + \left.\frac{\partial V(\phi)}{\partial \phi}\right|_{\phi_0} = 0 \tag{2.16}$$

The corresponding classical energy is found to be $\sqrt{M_0^2 + p^2}$ and this is how the classical particle like solution enters the quantum theory. Naturally (2.15) is a solution of the variational problem if the gauge condition $Q[\tilde{\Pi},\tilde{\Phi}]$ is chosen in such a way as to allow this classical solution. This is a time independent solution and also there exists a time dependent solution

$$\phi_{cl}(x,t) = \phi_0\left(\sqrt{1+p^2/M_0^2}\left(x + pt/\sqrt{1+p^2/M_0^2} + x_0\right)\right)$$
$$\Pi_{cl}(x,t) = \dot{\phi}_{cl}(x,t)$$
$$\alpha(t) = 0 \qquad (2.17)$$

which corresponds to a time dependent gauge condition.

At this point, we observe that, due to the property of our potential, $\phi_p(x)$ is of the order of $\lambda^{-\frac{1}{2}}$; accordingly, M_0 is of the order of λ^{-1} . We can develop the perturbation expansion in λ around the classical solution

$$\tilde{\Phi}(\rho,t) = \phi_p(x) + \chi(x,t)$$
$$\tilde{\Pi}(\rho,t) = \Pi_p(x) + \varpi(x,t) \qquad (2.18)$$

Here χ and ϖ are considered order zero and represent small quantum fluctuations around the classical solution. In the case when the initial and final states contain only one soliton, the shift (2.15) (2.17) gives, in the relativistic form for the soliton energy and we may develop a relativistic perturbation theory for the one soliton transition amplitude.

On the other hand, if the momentum is considered $O(1)$, one can alternatively shift by the zero momentum classical solution $\phi_0(\rho)$, because our method only requires that the function used in the shift be a classical solution to leading order, so that, in the final expression, one does not get zeroth order terms which are linear in $\chi(\rho,t)$. Since this last case appears to be more appropriate when calculating Green's functions in the one soliton sector we will discuss it in detail.

But before translating the field by these static solutions we comment on the choice of gauge condition $Q[\pi,\phi]$. Although arbitrary choice would lead us to a consistent perturbation expansion free of infrared divergences, we prefer to use a linear gauge condition:

$$Q[\phi(x-X(t),t)] = \int dx\, f(x)\, \phi(x-X(t),t)$$
$$\frac{\partial Q}{\partial X(t)} = -\int dx\, f(x)\, \phi'(x-X(t),t) \tag{2.19}$$

in order to eliminate the zero energy mode in a simplest possible way. Here $f(x)$ is still an arbitrary function and, identifying it later with the zero frequency eigenfunction, we will completely eliminate the zero energy mode from our functional integral. Now we linearize the constraint which is quadratic in fields, by making the following change of variables

$$\tilde{\pi}(\rho,t) = \underset{\sim}{\pi} + f(\rho)\, \frac{p - \int \underset{\sim}{\pi}(\rho,t)\left(\tilde{\phi}'(\rho,t) - f(\rho)\right)d\rho}{\int f(\rho)\, \tilde{\phi}'(\rho,t)\, d\rho} \tag{2.20}$$

Then the constraint becomes:

$$\delta\left(p - \int \tilde{\pi}\,\tilde{\phi}'\,d\rho\right) = \delta\left(\int d\rho\, f(\rho)\, \underset{\sim}{\pi}(\rho,t)\right)$$

Computing the Jacobian of this transformation, we get

$$\det\left(\frac{\delta\tilde{\pi}}{\delta\underset{\sim}{\pi}}\right) = \prod_t \left(\int d\rho\, f(\rho)\, \tilde{\phi}'(\rho,t)\right)^{-1} \tag{2.21}$$

so, it exactly cancels out the $\{Q,\mathcal{P}\}$ term which is given by (2.19). Now the Hamiltonian becomes more complicated:

$$H = \int \rho\, \mathcal{H} = \frac{\left(p - \int \underset{\sim}{\pi}(\tilde{\phi}' - f)\right)^2}{2\left(\int f\tilde{\phi}' d\rho\right)^2} + \int d\rho\left[\frac{1}{2}\underset{\sim}{\pi}^2 + \frac{1}{2}\tilde{\phi}'^2 + V(\tilde{\phi})\right] \tag{2.22}$$

We used the normalization condition $\int d\rho\, f(\rho)^2 = 1$. The transition amplitude is now of the form

$$\langle p_f, t_f \,|\, p_i, t_i \rangle = (2\pi)\,\delta(p_f - p_i) \int \mathcal{D}\underset{\sim}{\Pi}\,\mathcal{D}\tilde{\Phi}\;\; \Psi_p^{*}[\tilde{\Phi}(\cdot,t_f)]\,\times$$
$$\times\, \Psi_p[\tilde{\Phi}(\cdot,t_i)]\, \prod_t \;\delta\Big(\int d\rho\, f(\rho)\, \tilde{\Phi}(\rho,t)\Big)\; \delta\Big(\int d\rho\, f(\rho)\, \underset{\sim}{\Pi}(\rho,t)\Big)\,\times$$
$$\times\, \exp\Big\{ i \int dt \Big[\int d\rho\, \underset{\sim}{\Pi}(\rho,t)\,\dot{\tilde{\Phi}}(\rho,t) \;-\; H(\underset{\sim}{\Pi},\tilde{\Phi},p)\Big]\Big\} \tag{2.23}$$

and since both the gauge condition and the constraint are linear in fields, one can easily develop a perturbation expansion.

So we continue our discussion by making the shift

$$\tilde{\Phi}(\rho,t) = \phi_0(\rho) + \chi(\rho,t)$$
$$\underset{\sim}{\Pi}(\rho,t) = \varpi(\rho,t) \tag{2.24}$$

with the corresponding choice of f :

$$f(\rho) = \Psi_0(\rho) = \frac{1}{\sqrt{M_0}}\, \phi_0'(\rho) \tag{2.25}$$

The Hamiltonian becomes

$$H = \frac{\left(p - \int \varpi\chi'\, d\rho\right)^2}{2M_0\left(1 + \frac{\xi}{M_0}\right)^2} + \int d\rho \Big[\tfrac{1}{2}\varpi^2 + \tfrac{1}{2}\chi'^2 + \sum_{\ell=2} \tfrac{1}{\ell!} V^{(\ell)}(\phi_0)\,\chi^\ell \Big] + M_0$$
$$\xi = \int d\rho\, \phi_0'(\rho)\, \chi'(\rho,t) \tag{2.26}$$

At this point we must face the fact that the above effective Hamiltonian is derived by making formal changes of variables in path integral. In the previous chapter we learned how a formal change of variables leads to incorrect results and also that the more careful treatment gives additional potential terms in the action of the path integral. Now it is a straightforward task to compute the explicit form of this additional term for this specific case. We introduced the collective coordinate $X(t)$ through the following point canonical transformation:

$$\phi(x,t) = \phi_0(x + X(t)) + \chi(x + X(t), t) \tag{2.27}$$

where $\phi_0(x+X)$ is the classical soliton solution. The χ-field describing the small oscillations satisfies the gauge condition

$$\int dx\ \psi_0(x+X(t))\,\chi\,(x+X(t)\,,t)=0 \tag{2.28}$$

and ψ_0 is the lowest energy wave function in a complete set $\{\psi_i(x)\,;\,i=0,1,2\,\cdots\}$. We write the following expression for the χ-field:

$$\chi(x+X(t),t)=\sum_{n=1}^{\infty}\psi_n(x+X(t))\,Q_n(t) \tag{2.29}$$

leaving out the zero frequency mode due to the gauge condition (2.28).

We denote $X(t)=Q_0(t)$ and then using the notation of section II.2 we have

$$F^x(Q)=\phi_0\,(x+Q_0(t))+\sum_{n=1}^{\infty}\psi_n(x+Q_0(t))\,Q_n(t) \tag{2.30}$$

and so

$$g_{ij}(Q)=\int dx\ F^x_{,i}(Q)\,F^x_{,j}(Q)\qquad\qquad i,j=0,1,2\,\cdots$$

$$\sum_a F^a_{,i}(Q)F^a_{,jkm}(Q)=\int dx\ F^x_{,i}(Q)\,F^x_{,jkm}(Q) \tag{2.31}$$

The matrix $g_{ij}(Q)$ is explicitly given by

$$g_{00}(Q)=\int dx\,\left(\phi_0'(x)+\chi'(x,t)\right)^2\equiv(\phi',\phi')$$

$$g_{0n}(Q)=g_{n0}(Q)=\int dx\ \psi_n(x)\left(\phi_0'(x)+\chi'(x,t)\right)\equiv(\psi_n,\phi')$$

$$g_{nn'}(Q)=\delta_{nn'}\qquad\qquad n,n'=1,2,3\,\cdots \tag{2.32}$$

and its determinant is easily calculated to be

$$g(Q)=\det g_{ij}(Q)=(\psi_0,\phi')^2 \tag{2.33}$$

We also need the inverse matrix $g^{ij}(Q)$ which can be read off directly from the phase space path integral derived earlier

$$g^{oo}(Q) = \frac{1}{(\psi_o, \phi')^2}$$

$$g^{on}(Q) = g^{no}(Q) = -\frac{(\psi_n, \phi')}{(\psi_o, \phi')^2}$$

$$g^{nn'}(Q) = \delta_{nn'} + \frac{(\psi_n \phi')(\phi' \psi_{n'})}{(\psi_o, \phi')^2} \tag{2.34}$$

Next we simply use the general formula (3. 19) obtained in section II.3 to calculate the additional potential term for this specific case. After some calculation we find:

$$\Delta V = \frac{\hbar^2}{8}\left[-\frac{\sum_1' |(\psi_n, \psi_m')|^2}{(\psi_o, \phi')^2} - 3\frac{(\psi_o' \psi_o')^2}{(\psi_o, \phi')^2} + 2\frac{(\psi_o' \phi'')}{(\psi_o, \phi')^2} + \frac{(\psi_o' \phi')^2}{(\psi_o, \phi')^4} \right] \tag{2.35}$$

Now, the one soliton sector generating functional is expressed in the following phase space path integral form

$$Z(J; p_f, p_i) = \int \mathcal{D}p \mathcal{D}X \, e^{-ip_f X(t_f)} e^{ip_i X(t_i)} \int \mathcal{D}\varpi \mathcal{D}\chi$$
$$\delta(\int \psi_o \chi) \, \delta(\int \psi_o \varpi) \psi^*_{p_f} \psi_{p_i} \exp\left\{ \frac{i}{\hbar} \int dt \left[p(t) \dot{X}(t) - \right.\right.$$
$$\left.\left. - H_{eff}(p, \varpi, \chi) + \int d\rho \, J(\rho - X(t), t)(\phi_o(\rho) + \chi(\rho, t)) \right] \right\} \tag{2.36}$$

with the understanding that this is a symmetric path integral, namely the one with the additional specification that the short time action is to be evaluated at the midpoint. The total effective Hamiltonian reads:

$$H_{eff}(p, \varpi, \chi) = M_o + \frac{(p(t) - \int \varpi \chi')^2}{2(\psi_o, \phi')^2} + \sum_{\ell=2}' \frac{1}{\ell!} V^{(\ell)}_{(\phi_o)} \chi^\ell$$
$$+ \Delta V(\chi) \tag{2.37}$$

In comparison with the effective Hamiltonian derived before by making formal changes of variables in path integral it contains the additional potential term $\Delta V(\chi)$. Since it is proportional to $\hbar^2$ it starts contributing at the two loop level.

III,3 Perturbation Theory and Feynman Rules

Based on the formalism developed in the previous section we can now formulate a systematic perturbation expansion in powers of the coupling constant. For definiteness we consider the $\frac{\lambda}{4}\phi^4$ theory and the expansion is a weak coupling perturbation expansion. We will present in this section a detailed derivation of Feynman rules. Using these rules, one can then make perturbative computations of energy, matrix elements and Green's functions to arbitrary orders in coupling constant.

Since we are translating the field by the static solution in this perturbation expansion Lorentz invariance is not manifest, but one can show that higher order corrections in coupling constant, sum up to restore the Lorentz invariance.[9] So we consider the following generating functional

$$Z(J,K) = \int \mathcal{D}\chi \mathcal{D}\varpi \, \delta\left(\int \psi_0 \chi\right) \delta\left(\int \psi_0 \varpi\right) e^{i\int dt \left[\int d\rho \left(\varpi\dot{\chi} + J\chi + K\varpi\right) - H\right]} \tag{3.1}$$

where H is the total Hamiltonian given by Eq. (2.37) and J and K are external sources. We separate the Hamiltonian into a quadratic χ -field part and an interaction part given by:

$$H_0 = M_0 + \int d\rho \left[\frac{1}{2}\varpi^2 + \frac{1}{2}\chi'^2 - \frac{1}{2}\left(m^2 - 3\lambda\phi_0^2(\rho)\right)\chi^2\right] \tag{3.2}$$

$$H' = \frac{\left(p - \int d\rho \, \varpi \chi'\right)^2}{2M_0\left(1 + \xi/M_0\right)^2} + \int d\rho \left(\lambda\phi_0\chi^3 + \frac{\lambda}{4}\chi^4\right) + \Delta V(\chi) \tag{3.3}$$

where $\xi = (\phi_0', \chi')$. Then the generating functional can be written in the form:

$$Z(J,K) = \exp\left\{-i\int dt\, H'\left(\frac{1}{i}\frac{\delta}{\delta J}, \frac{1}{i}\frac{\delta}{\delta K}\right)\right\} Z_0(J,K) \tag{3.4}$$

where $Z_0(J,K)$ is the free generating functional

$$Z_0(J,K) = \int \mathcal{D}\chi\, \mathcal{D}\varpi\, \delta\left(\int \psi_0 \chi\right) \delta\left(\int \psi_0 \varpi\right) \times$$
$$\times \exp\left\{i\int dt\, d\rho \left[\varpi\dot{\chi} - \frac{1}{2}\varpi^2 - \frac{1}{2}\chi'^2 + \frac{1}{2}(m^2 - 3\lambda\phi_0^2)\chi^2 + J\chi + K\varpi\right]\right\} \tag{3.5}$$

This quadratic functional integral can easily be evaluated by expanding the fields χ and ϖ in terms of eigenfuctions $\psi_n(\rho)$ which are solutions of the following eigenequation:

$$\Omega^2 \psi_n(\rho) = \omega_n^2 \psi_n(\rho)$$
$$\Omega^2 = -\frac{d^2}{d\rho^2} - m^2 + 3\lambda\phi_0^2(\rho) \tag{3.6}$$

There are two discreet eigenvalues for n=0 and n=1, and the $\omega_0 = 0$ eigenfunction is just $\psi_0 = \frac{1}{\sqrt{M_0}}\phi_0'$. There is also a continuous spectrum for $\omega(q)^2 = q^2 + 2m^2$. The normalized eigenfunctions are

$$\psi_1(\rho) = \frac{1}{N_1} \frac{sh\frac{m\rho}{\sqrt{2}}}{ch^2\frac{m\rho}{\sqrt{2}}}, \quad \omega_1 = m\sqrt{\frac{3}{2}}$$
$$\psi_q(\rho) = \frac{1}{N_q} e^{iq\rho}\left[3th^2\frac{m\rho}{\sqrt{2}} - 3i\frac{q\sqrt{2}}{m} th\frac{m\rho}{\sqrt{2}} - 1 - 2\frac{q^2}{m^2}\right]$$
$$N_q^2 = \frac{2L}{m^4}(q^2+2m^2)(2q^2+m^2) - \frac{12\sqrt{2}}{m^3}(q^2+m^2) \tag{3.7}$$

Here L is the length of the box. We use the box normalization and periodic boundary conditions. Introducing the notation

$$\chi_n(t) = \int d\rho\, \chi(\rho,t)\psi_n(\rho) \equiv (\chi, \psi_n)$$
$$\varpi_n(t) = \int d\rho\, \varpi(\rho,t)\psi_n(\rho) \equiv (\varpi, \psi_n) \tag{3.8}$$

we have

$$\chi(\rho,t) = \sum_n{}' \chi_n(t)\, \psi_n^*(\rho)$$
$$\varpi(\rho,t) = \sum_n{}' \varpi_n(t)\, \psi_n^*(\rho) \tag{3.9}$$

and the ϖ integral is now

$$\int \prod_n \mathcal{D}\varpi_n \, \delta(\varpi_0) \exp\left\{ i \int dt \sum_n{}' \left[\varpi_n^* (\dot{\chi}_n + K_n) - \frac{1}{2} \varpi_n^* \varpi_n \right] \right\}$$
$$= \exp\left\{ i \int dt \, \frac{1}{2} \sum_n{}' (\dot{\chi}_n^* + K_n^*)(\chi_n + K^n) \right\} \tag{3.10}$$

Here the sum $\sum'$ is such that the zero frequency mode (n=0) is omitted.

Next one has the following χ-integral

$$\int \prod_n \mathcal{D}\chi_n \, \delta(\chi_0) \exp\left\{ i \int dt \sum_n{}' \left[\frac{1}{2} \chi_n^* i G_n^{-1} \chi_n + (J_n - \dot{K}_n) \chi_n^* \right] \right\} \tag{3.11}$$

where

$$i G_n^{-1} = \left(-\partial_t^2 - \omega_n^2 + i\epsilon \right) \delta(t-t') \tag{3.12}$$

and the answer is

$$\exp\left\{ i \int dt \, \frac{1}{2} \sum_n{}' (J_n^* - \dot{K}_n^*) \, i G_n \, (J_n - \dot{K}_n) \right\} \tag{3.13}$$

Now in the (ρ, t) representation the Green's function is given by

$$G(t-t'; \rho\rho') = \sum_n{}' \psi_n(\rho) \int \frac{d\nu}{2\pi} e^{i\nu(t-t')} \frac{i}{\nu^2 - \omega_n^2 + i\epsilon} \psi_n^*(\rho') \tag{3.14}$$

defining

$$\Delta(t-t'; \rho\rho') = -i\, \delta(t-t') \sum_n{}' \psi_n(\rho) \psi_n^*(\rho') \tag{3.15}$$

We can write the final form for the generating functional

$$Z_0(J,K) = \exp\left\{ -\int dt \, d\rho \int dt' \, d\rho' \left[\frac{1}{2} \left(J(\rho,t) - \dot{K}(\rho,t) \right) G(t-t'; \rho\rho') \cdot \right.\right.$$
$$\left.\left. \times \left(J(t',\rho') - \dot{K}(\rho',t') \right) + \frac{1}{2} K(\rho,t) \, \Delta(t-t'; \rho\rho') \, K(\rho',t') \right] \right\} \tag{3.16}$$

From this free generating functional one can deduce the

Feynman propagators. We see that there are three types of propagators, i.e., by differentiating with respect to sources J and K we get the $\chi - \chi$, $\chi - \varpi$ and the $\varpi - \varpi$ propagators respectively:

$$\frac{1}{i}\frac{\delta}{\delta J(\rho,t)}\frac{1}{i}\frac{\delta}{\delta J(\rho',t')}Z_0(J,K)\Big|_{J=K=0} = G(t-t';\rho\rho') \tag{3.17}$$

$$\frac{1}{i}\frac{\delta}{\delta J(\rho,t)}\frac{1}{i}\frac{\delta}{\delta K(\rho',t')}Z_0(J,K)\Big|_{J=K=0} = \partial_{t'}G(t-t';\rho\rho') \tag{3.18}$$

$$\frac{1}{i}\frac{\delta}{\delta K(\rho,t)}\frac{1}{i}\frac{\delta}{\delta K(\rho',t')}Z_0(J,K)\Big|_{J=K=0} = \partial_t\partial_{t'}G(t-t';\rho\rho') + \Delta(t-t';\rho\rho') \ . \tag{3.19}$$

It is important to note that since in (3.14) and (3.15) the zero frequency mode is excluded, these propagators avoid the infrared divergences associated with it. This is the consequence of the subsidiary conditions, i.e. δ-function conditions in (3.1).

The vertices of our perturbation theory are determined by the interaction part H'. Besides the ordinary vertices $\lambda\phi_0\chi^3$ and $\frac{\lambda}{4}\chi^4$, we have an infinite series of vertices coming from the non-local terms in H'

$$\frac{(p-\xi\varpi\chi')^2}{2\Pi_0(1+\xi/\Pi_0)^2}+\frac{\hbar^2}{8}\left[-\frac{\Sigma|(\psi_n,\psi_m')|^2}{(\psi_0,\phi')^2}-3\frac{(\psi_0'\psi_0')}{(\psi_0,\phi')^2}+2\frac{(\psi_0',\phi'')}{(\psi_0,\phi')^3}+\frac{(\psi_0'\,\phi')^2}{(\psi_0,\phi')^4}\right] \tag{3.20}$$

Since ξ/Π_0 is of the order $\lambda^{1/2}$ our perturbation expansion will be in the powers of $\lambda^{1/2}$. Expanding $1/(1+\xi/\Pi_0)$ one gets the first set of vertices proportional to P^2:

$$-\frac{p^2}{2\Pi_0}\left(-2\frac{\xi}{\Pi_0}+3\frac{\xi^2}{\Pi_0^2}-4\frac{\xi^3}{\Pi_0^3}+\cdots\right) \tag{3.21}$$

which are of the order of $\lambda^{3/2}$, λ^2, ... successively.

The second set of vertices is proportional to P and are given by

$$- \frac{P}{M_0} \int \mathcal{D}\chi' \, d\rho \left(1 - 2\frac{\xi}{M_0} + 3\frac{\xi^2}{M_0^2} - 4\frac{\xi^4}{M_0^2} + \cdots \right) \tag{3.22}$$

It is important to observe here that these vertices are local in time but non-local in the space variable.

It is trivial to generalize these Feynman rules to arbitrary two-dimensional field theory described by the $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$, which has a classical solitary wave solution $\phi_0(x)$. Then the propagators have the same forms as those given by Eq. (3.14) and Eq. (3.15), but now with ψ_n and ω_n^2 obtained from the following eigenequation:

$$\left[-\frac{d^2}{d\rho^2} + V''(\phi_0(\rho)) \right] \psi_n(\rho) = \omega_n^2 \, \psi_n(\rho) \tag{3.23}$$

The χ — field vertices are given by the cubic and higher terms in the expansion of the potential

$$U(\phi_0 + \chi) = \sum_{\ell=0} \frac{1}{\ell!} U^{(\ell)}(\phi_0) \, \chi^\ell \tag{3.24}$$

and depend on the specific form of the potential. Finally, we observe that the meson-soliton non-local vertices remain the same as that given by Eq. (3.20).

As an explicit calculation we will next evaluate the first quantum correction to the soliton mass M_0. Because by the ultraviolet divergences appearing in this calculation we now have to face the problem of renormalization in the one soliton sector. This two-dimensional field theories are made finite by normal ordering which gives infinite counterterms in the Hamiltonian appearing in our path integral. The fact is that these counterterms which make the non-soliton sector of our field theory finite renormalize the one soliton sector. Thus, one needs no new counterterms to cancel all the divergences which appear. This is similar to the situation with spontaneously broken field theories, where the divergence structure of a renormalizable theory is not affected by the occurance of spontaneous symmetry breakdown, so that the same counterterms which renormalize the theory with the unbroken vacuum are enough to renormalize the corresponding theory with the spontaneously broken vacuum.

First subtracting the classical vacuum energy we write our Hamiltonian density as

$$\mathcal{H}(\pi,\phi) = \frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 \tag{3.25}$$

To cancel the divergent self-energy diagram we next normal order this Hamiltonian. To normal order an interaction Hamiltonian we must specify the mass of the free Hamiltonian through which the creation and annihilation operators are defined. Choosing a different normal ordering mass has the same effect as choosing a different renormalization point. We choose this mass to be $\sqrt{2}\,m$ and then the Hamiltonian becomes

$$\hat{\mathcal{H}} = N_{\sqrt{2}m}\left\{\frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}\hat{\Phi}^2 + \frac{\lambda}{4}\left(\hat{\Phi}^2 - \frac{m^2}{\lambda}\right)^2\right\} \tag{3.26}$$

Next we use the identity

$$U(\hat{\Phi}) = N_\mu\left\{\exp\left(\frac{1}{2}\Delta\frac{d^2}{d\Phi^2}\right)U(\hat{\Phi})\right\} \tag{3.27}$$

where Δ is found from

$$\hat{\phi}^2(x) = N_\mu \hat{\phi}^2(x) + \Delta \tag{3.28}$$

to read

$$\Delta = \frac{1}{L}\sum_n{}' \frac{1}{2\omega(k_n)} = \frac{1}{2\pi}\int dk \frac{1}{2\omega(k)}$$

$$k_n = \frac{2\pi n}{L} \quad , \; n = 0, \pm 1, \pm 2, \cdots \tag{3.29}$$

and write the Hamiltonian density in the form

$$\mathcal{H} = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\hat{\phi}'^2 + \frac{\lambda}{4}\left(\hat{\phi}^2 - \frac{m^2}{\lambda}\right)^2 - \frac{1}{2}\delta m^2\left(\hat{\phi}^2 - \frac{m^2}{\lambda}\right) - \sum_n{}' \frac{1}{2}\omega(k_n)$$

$$\delta m^2 = 3\lambda \sum_n{}' \frac{1}{2L\omega(k_n)} = \frac{3\lambda}{4\pi}\int_{-\Lambda}^{\Lambda} \frac{dk}{\omega(k)} \tag{3.30}$$

So in deriving the effective one soliton Hamiltonian we have to start with this Hamiltonian density containing infinite renormalization counterterms. For the one loop calculations we shift the field as before with $\phi_0(\rho)$ still being the solution of the equation of motion without the mass counterterm. Then we end up with an additional term in the final effective Hamiltonian of the form

$$\delta H = -\frac{1}{2}\delta m^2 \int\left[\left(\phi_0^2 - \frac{m^2}{\lambda}\right) + 2\phi_0\chi + \chi^2\right]d\rho - \frac{1}{2}\sum_n{}' \omega(k_n) \tag{3.31}$$

which will be sufficient to cancel out all the divergences which appear in the one soliton sector calculations.

To find the one loop quantum correction to the soliton mass we have to keep only the quadratic terms in ϖ, χ in the action and also the constant renormalization counterterm. Thus we write:

$$\exp\left[-\frac{i}{\hbar}M_1 T\right] = \int \mathcal{D}\varpi \mathcal{D}\chi\, \delta\left(\int \psi_0 \chi\right)\delta\left(\int \psi_0 \varpi\right) \exp\left\{\frac{i}{\hbar} A^{(2)}[\varpi, \chi]\right\}$$

$$A^{(2)}[\varpi,\chi] = \int\left[\varpi\dot{\chi} - \frac{1}{2}\left(\varpi^2 + \chi'^2 - m^2\chi^2 + 3\lambda\phi_0^2\chi^2\right) + \frac{1}{2}\delta m^2\left(\phi_0^2 - \frac{m^2}{\lambda}\right) + \frac{1}{2}\sum_n{}' \omega(k_n)\right] \tag{3.32}$$

so that the explicit form of the first quantum correction is given by the expression[6]

$$M_1 = \sum{}' \frac{1}{2}\omega_n - \frac{1}{2}\delta m^2 \int d\rho\left(\phi_0^2 - \frac{m^2}{\lambda}\right) - \sum_n{}' \frac{1}{2}\omega(k_n) \tag{3.33}$$

Due to the periodic boundary conditions we have for the scattering states

$$q_n L + \delta(q_n) = 2\pi n \tag{3.34}$$

where the phase shift $\delta(q)$ is defined by

$$\lim_{L\to\infty} \psi_q\left(\tfrac{L}{2}\right) = \exp\left(i\left[q\tfrac{L}{2} + \tfrac{1}{2}\delta(q)\right]\right)$$

$$\lim_{L\to\infty} \psi_q\left(-\tfrac{L}{2}\right) = \exp\left(-i\left[q\tfrac{L}{2} + \tfrac{1}{2}\delta(q)\right]\right) \tag{3.35}$$

and found to be

$$\delta(q) = \begin{cases} 2\pi - 2\tan^{-1}\left(\frac{q\sqrt{2}}{m}\right) - 2\tan^{-1}\left(\frac{q}{m\sqrt{2}}\right) & q>0 \\ -2\pi - 2\tan^{-1}\left(\frac{q\sqrt{2}}{m}\right) - 2\tan^{-1}\left(\frac{q}{m\sqrt{2}}\right) & q<0 \end{cases} \tag{3.36}$$

Now the sum

$$\sum_n{}' \tfrac{1}{2}\omega_n - \sum_n{}' \tfrac{1}{2}\omega(k_n) = \tfrac{1}{2}\omega_1 + \tfrac{1}{2}\sum_n{}'\left(\omega(q_n) - \omega(k_n)\right) \tag{3.37}$$

is approximated by

$$\tfrac{1}{2}\omega_1 + \tfrac{1}{2}\sum_n{}'\left[\omega\left(k_n - \tfrac{1}{L}\delta(k_n)\right) - \omega(k_n)\right] \approx \tfrac{1}{2}\omega_1 +$$

$$+ \tfrac{1}{2}\left(-\tfrac{1}{L}\right)\sum_n{}' \frac{d\omega(k_n)}{dk_n}\delta(k_n) \tag{3.38}$$

The second term is in integral form written as

$$-\frac{1}{4\pi}\int dk\, \delta(k)\frac{d\omega(k)}{dk} = -\frac{1}{4\pi}\left[\omega(k)\,\delta(k)\right]\Big|_{-\infty}^{\infty} + \frac{1}{4\pi}\int dk\,\frac{d\delta(k)}{dk}\,\omega(k) =$$

$$= -\frac{3m}{\pi\sqrt{2}} - \frac{3m}{\pi\sqrt{2}}\int_0^{\Lambda}\frac{dk}{\sqrt{k^2+2m^2}} - \frac{3m}{\pi\sqrt{2}}\int_0^{\infty}\frac{m^2\,dk}{(m^2+2k^2)(k^2+2m^2)^{1/2}} \tag{3.39}$$

Obviously, the logarithmically divergent part is precisely cancelled by the mass counterterm in (3.32) and we end up with the final answer for the first quantum correction

$$M_1 = -\frac{3m}{\pi\sqrt{2}} + \frac{m}{2\sqrt{6}} \tag{3.40}$$

III.4 Green's Functions in the One Soliton Sector

With the systematic perturbation theory developed in Section III.3 we can make perturbative calculations of other quantities besides the soliton energy. Of special interest are the Green's functions in the one soliton sector. The $\hat{\Phi}$ field matrix elements between the one soliton and many meson states $\langle p', \{k_i\} | \hat{\Phi} | p, \{k_i\} \rangle$ were first considered by Goldstone and Jackiw(7) in their treatment of these two-dimensional extended particle theories. The name meson stands for the fundamental quanta of this scalar field theories. Goldstone and Jackiw formulated a self consistent method, assuming that the connected matrix elements between m and n mesons have an expansion in powers of $\lambda^{\frac{1}{2}}$ and that the leading order is $\lambda^{\frac{m+n-1}{2}}$. These assumptions can now easily be justified.

Let us start with the matrix element $\langle p' | \hat{\Phi}(x,0) | p \rangle$. Performing the canonical transformation described in the introduction, we get

$$\langle p' | \hat{\Phi}(x,0) | p \rangle = \langle p' | \Phi_0(x + \hat{X}(0)) | p \rangle + \langle p' | \hat{\chi}(x + \hat{X}(0), 0) | p \rangle \tag{4.1}$$

where $\hat{X}$ is the coordinate operator. The first term is of the order $\lambda^{-\frac{1}{2}}$, while the second term gives, at most, λ^{0} contributions. Inserting the identity, we can easily evaluate the leading term:

$$\langle p' | \Phi_0(x + \hat{X}) | p \rangle = \int \frac{dy}{2\pi} e^{i(p-p')y} \Phi_0(x+y) \tag{4.2}$$

This essentially classical part of the matrix element was the initial Ansatz of Goldstone and Jackiw. Next, one can compute the first quantum correction coming from the second term in (4.1). From the path integral representation of $\langle p' | \hat{\chi}(x + \hat{X}, 0) | p \rangle$ we see that one first has to find the "vacuum" expectation value of $\chi(\rho,0)$ and then making the substitution $\rho = x + \hat{X}$ to evaluate this operator between the soliton states. Thus the first quantum correction is given by the tadpole graph of order $\lambda^{1/2}$ and we have:

$$\langle p'|\chi(x+\hat{X},0)|p\rangle \approx \langle p'|\int d\rho' \tilde{G}(0;x+\hat{X},\rho')(3\lambda\phi_0(\rho'))G(0;\rho'\rho')|p\rangle \tag{4.3}$$

This contribution is again of the form:

$$\int \frac{dy}{2\pi} e^{i(p-p')y} f(x+y) \tag{4.4}$$

with
$$f(x+y) = \int d\rho\, \tilde{G}(0;x+y,\rho)\,(3\lambda\phi_0(\rho))\,G(0;\rho\rho)$$

$$\tilde{G}(0;\rho\rho') = \int dt\, G(t;\rho\rho') \tag{4.5}$$

It is logarithmically divergent and the divergence is cancelled out by the contribution coming from the mass counterterm $\frac{1}{2}\delta m^2 \phi_0 \chi$ which also gives a tadpole graph of the order $\lambda^{+\frac{1}{2}}$.

Next, we will compute the leading term of the one meson matrix element $\langle p'|\hat{\Phi}|p,\omega_n\rangle$, where p is the total momentum of the soliton and meson and $\omega_n = \sqrt{k^2+2m^2}$ is the meson energy. This matrix element is equal to

$$\langle p'|\phi_0(x+\hat{X}(0))|p,\omega_n\rangle + \langle p'|\chi(x+\hat{X}(0),0)|p,\omega_n\rangle \tag{4.6}$$

and here the first classical term has no contribution to the leading order. To evaluate the second term we expand the field $\chi(\rho,t)$ in terms of the eigenfunctions ψ_n:

$$\chi(\rho,t) = \hat{Q}(t)\psi_1(\rho) + \sum_n{}'' \frac{1}{\sqrt{2\omega_n}}\{\hat{a}_n(t)\psi_n(\rho) + \hat{a}_n^+(t)\psi_n^*(\rho)\} \tag{4.7}$$

where the n=0 mode is omitted because of the δ-function constraints. There the n=1 mode describes the internal soliton degree of freedom and not a quantum particle. Therefore, the soliton can have energetically excited states. The continuum modes correspond to the meson degrees of freedom since, neglecting the interaction $\hat{a}_n$ satisfies the equation $\ddot{\hat{a}}_n(t) = -\omega_n^2 \hat{a}_n(t)$ with $\omega_n^2 = k_n^2 + 2m^2$. Thus, in the first approximation the one meson matrix element is

$$\langle p' | \sum_{n'}'' \frac{1}{\sqrt{2\omega_{n'}}} \left\{ \psi_{n'}(x+\hat{X}(0))\, \hat{a}_{n'}(0) + \psi^*_{n'}(x+\hat{X}(0))\, \hat{a}^+_{n'}(0) \right\} | p, \omega_n \rangle =$$
$$= \frac{1}{\sqrt{2\omega_n}} \langle p' | \psi_n(x+\hat{X})\, \hat{a}_n | p, \omega_n \rangle = \frac{1}{\sqrt{2\omega_n}} \langle p' | \psi_n(x+\hat{X}) | p \rangle \tag{4.8}$$

so that

$$\langle p' | \hat{\Phi}(x,0) | p, \omega_n \rangle = \int \frac{dy}{2\pi}\, e^{i(p-p')y}\, \psi_n(x+y) \tag{4.9}$$

The n-point Green's functions in the one soliton sector can also be computed perturbatively. For simplicity we will calculate the two-point function and it is then trivial to generalize the result to the arbitrary n-point function. Let us consider the Fourier transformed form:

$$G(p', k'; p, k) = \int dx'dt'\, e^{i(\omega(k')t' - k'x')} \int dt dx\, e^{-i(\omega(k)t - kx)}$$
$$\cdot \langle p' | T\{ \hat{\Phi}(x',t'), \hat{\Phi}(x,t) \} | p \rangle \tag{4.10}$$

Making the canonical transformation in the operator form, the time-ordered product is equal to

$$T\{ \phi_0(x'+\hat{X}(t'))\, \phi_0(x+\hat{X}(t)) \} + T\{ \phi_0(x'+\hat{X}(t')), \chi(x+\hat{X}(t), t) \}$$
$$+ T\{ \chi(x'+\hat{X}(t'), t'), \phi_0(x+\hat{X}(t)) \} + T\{ \chi(x'+\hat{X}(t'), t'), \chi(x+\hat{X}(t), t) \} \tag{4.11}$$

where $\hat{X}$ is now the coordinate operator. We will first evaluate the contribution from the first term which is of the order $O(\lambda^{-1})$ while the others give smaller order contributions. The time ordered product can be split into two parts

$$\Theta(t'-t)\,\phi_0(x'+\hat{X}(t'))\,\phi_0(x+\hat{X}(t)) + \Theta(t-t')\,\phi_0(x+\hat{X}(t))\,\phi_0(x'+\hat{X}(t')) \tag{4.12}$$

Inserting the identity $\int dy\,|y,t\rangle\langle y,t| = 1$ into the matrix element of the first part

$$\Theta(t'-t)\ \langle p'|\ \phi_0(x'+\hat{X}(t'))\,\phi_0(x+\hat{X}(t))\ |p\rangle \tag{4.13}$$

for both t' and t we see that it is equal to

$$\int dy'dy\, e^{-i(p'y'-E(p')t')}\phi_0(x+y')\langle y',t'|y,t\rangle\,\phi_0(x+y)\,e^{i(py-E(p)t)} \tag{4.14}$$

and after translation of the time and space variables, we get the following contribution to the Green's function:

$$\begin{aligned}&\delta(p'+k'-p-k)\,\delta(E(p')+\omega(k')-E(p)-\omega(k))\int dx_1\, e^{-ik'x_1}\phi_0(x_1)\cdot\\ &\cdot\int dx\, e^{-i(p'+k')x}\int dt\, e^{i(E(p')+\omega(k'))t}\langle x,t|0,0\rangle\cdot\\ &\qquad\cdot\int dx_2\, e^{-ikx_2}\phi_0(-x_2)\end{aligned} \tag{4.15}$$

The presence of the δ-functions shows that transitional invariance is indeed respected by the formalism. Recognizing the non-relativistic propagator we get for the factor multiplying the δ-functions:

$$\tilde{\phi}_0(k')\ \frac{i}{E(p')+\omega(k')-M_0-\frac{(p'+k')^2}{2M_0}}\ \tilde{\phi}_0(k) \tag{4.16}$$

There is also a similar term coming from the second part of the ordered product (4.12):

$$\tilde{\phi}_0(k)\ \frac{i}{E(p')-\omega(k')-M_0-\frac{(p'-k')^2}{2M_0}}\ \tilde{\phi}_0(k') \tag{4.17}$$

This is the classical part of Green's function and the first quantum correction can also be computed in a similar way. It is of the order $0(\lambda^{\circ})$ and comes from the last three terms in Eq. (4.11). We will just demonstrate for example how one computes the corrections coming from the last term of Eq. (4.11) which is the Fourier transform of:

$$\langle p' | T \{ \hat{\chi}(x' + \hat{X}(t'), t'), \hat{\chi}(x + \hat{X}(t), t) \} | p \rangle \tag{4.18}$$

This rather unconventional form can be understood if we write down the corresponding path integral expression, since the canonical transformation was originally carried out in the path integral formalism. Then we see that one has first to find the "vacuum" expectation value of $T\{\chi(\rho', t'), \chi(\rho, t)\}$ and next after substitution of $\rho = x + \hat{X}(t)$ and $\rho' = x + \hat{X}(t')$ to evaluate this operator between the one soliton states. Since the operators $\hat{X}(t)$ and $\hat{X}(t')$ do not commute, this operator between the soliton states has to be time ordered. So, in the first approximation, we have that the $0(\lambda^{\circ})$ contribution to the matrix element (4.18) is given by:

$$\langle p' | T\, G(t'-t; x' + \hat{X}(t'), x + \hat{X}(t)) | p \rangle \tag{4.20}$$

where $G(t'-t; \rho', \rho)$ is the propagator. This gives the following contribution to the Green's function:

$$\delta(p'+k'-p-k)\, \delta(E(p') + \omega(k') - E(p) - \omega(k)) \cdot$$
$$\cdot \int dx\,dt\; e^{i(\omega(k')t - k'x)} \langle p' | T G(t; x + \hat{X}(t), \hat{X}(0)) | p \rangle \tag{4.21}$$

Now we can continue in the same way as in the preceeding calculation, obtaining for the $\theta(t)$ part of the time ordered product:

$$\sum_n{}' \tilde{\psi}_n(-k') \int \frac{d\omega}{2\pi} \frac{i}{\omega^2 - \omega_n^2 + i\epsilon} \cdot \frac{i}{\omega + E(p') + \omega(k') - M_0 - \frac{(p'+k')^2}{2M_0}} \tilde{\psi}_n(-k) \tag{4.22}$$

and similarly for the $\Theta(-t)$ part of the time ordered product.

IV GENERAL METHOD OF COLLECTIVE COORDINATES

The method we have developed for the one soliton sector can easily be generalized. In the previous chapter we showed that it is necessary to introduce a new dynamical variable (center-of-mass coordinate) in order to assure the translational invariance of the quantum theory when perturbing about the classical one soliton solution. So if in general we like to perturb about a classical solution respecting more independent conservation laws it will be necessary to introduce more collective coordinates. For definiteness we consider the following general Lagrangian:

$$\mathcal{L}(\phi) = \frac{1}{2}\sum_{i=1}^{n}\left(\partial_\mu \phi_i(x,t)\right)^2 - \frac{1}{\lambda} V\left(\sqrt{\lambda}\,\phi_i(x,t)\right)$$

Here λ plays the role of a coupling constant and rescaling the field

$$\underset{\sim}{\phi}_i = \sqrt{\lambda}\,\phi_i$$

we can write this Lagrangian in the form

$$\mathcal{L}(\phi) = \frac{1}{\lambda}\left\{\frac{1}{2}\sum_{i=1}^{n}\left(\partial_\mu \underset{\sim}{\phi}_i(x,t)\right)^2 - V\left(\underset{\sim}{\phi}_i(x,t)\right)\right\}$$

so that a general classical solution to the Euler-Lagrange equations reads

$$\phi^i_{cl}(x,t) = \lambda^{-\frac{1}{2}}\,\varphi_i(x,t\,;\,c_1 c_2 \cdots c_n)$$

Here $\{c_\alpha\}$ are arbitrary constants and we have a family of classical solutions which all have the same energy. These constants signalize the existence of a continuous symmetry group G with n generators. The infinitesimal generators of G which we denote by P_α are constants of motion and therefore, take arbitrary values P_α for the classical solution. Then, starting from a fixed classical solution with given P_α we can generate an infinite set of classical solutions with the same P_α by applying an arbitrary transformation of G with constant parameters.

Now if we attempt to develop a perturbation expansion by shifting the fields by the classical solution

$$\phi_i(x,t) = \phi^i_{cl}(x,t) + \eta_i(x,t) \qquad i = 1, \cdots, N$$

some difficulties will appear as in the one soliton case. Namely, at the quantum level this shift will break the invariance under the continuous symmetry group G. Consequently, as always is the case with spontaneous symmetry breaking, zero frequency modes (Goldstone bosons) will appear leading to an infrared problem. It is then obvious that a consistent quantum treatment can be developed only if we preserve the continuous symmetries at the quantum level. This is achieved by introducing additional dynamical variables (collective coordinates) analogous to the center-of-mass variable in the one soliton example.

First we discuss how one is to introduce collective coordinates at the classical level. Let $\phi(x)$ and $\pi(x)$ be a canonical field and its conjugate momenta respectively. Let $H[\pi,\phi]$ be the Hamiltonian of a system under consideration. We consider a group G of transformations generated by a set of n generators $P_\alpha[\pi,\phi]$ through Poisson brackets. We assume that the Lie algebra closes, namely

$$\{ P_a[\pi,\phi], P_b[\pi,\phi] \}_{P.B.} = C^d_{ab}\, P_d[\pi,\phi] \tag{1}$$

For arbitrary group element specified by parameters $\{X_\alpha\}$, the transform of $A[\pi,\phi]$ is given by

$$\tau_{[X]}(A) \equiv A_{[X]} = \sum_{n=0}^{\infty} \frac{1}{n!} \{G_{[X]}, \{G_{[X]}, \{ \cdots \cdots \{G_{[X]}, A\} \cdots \} \tag{2}$$

where

$$G_{[X]} = \sum_{\alpha} X_\alpha P_\alpha[\pi, \phi] \tag{3}$$

Since G is a Lie group, we have

$$\tau_{[Z]} = \tau_{[Y]} \tau_{[X]} \tag{4}$$

where

$$Z_a = f_a(Y_1, \cdots Y_n; X_1, \cdots, X_n) \equiv f_a(Y, X) \tag{5}$$

Let us define V^a_b and its inverse U^a_b by

$$V^a_b(X) = \left.\frac{\partial f_a(Y, X)}{\partial Y_b}\right|_{Y=0}$$

$$V^a_b U^b_c = \delta_{ac} \tag{6}$$

The structure constant C^d_{ab} is related to V and U by

$$C^d_{ab} = - V^e_a V^f_b \left(\frac{\partial U^d_f}{\partial X_e} - \frac{\partial U^d_e}{\partial X_f} \right) \tag{7}$$

Considering an infinitesimal transformation, one obtains

$$\{P_a, \phi_{[X]}\} = V^b_a(X) \frac{\partial \phi[X]}{\partial X_b} \tag{8}$$

Now that we have enough machinery, we introduce $X^{(t)}_\alpha$ as a dynamical variable through making the change of variable

$$\phi(x,t) = \tilde{\phi}_{[X(t)]}(x,t)$$

$$\pi(x,t) = \tilde{\pi}_{[X(t)]}(x,t) \tag{9}$$

As we shall see below, classically, we will be able, in certain cases, to determine $X_\alpha(t)$ such that $\tilde{\phi}(x,t)$ and $\tilde{\pi}(x,t)$ are time independent; then the motion of ϕ and π is governed by the time development of X's through (9), i.e., ϕ and π move collectively. Thus, we call $X_\alpha(t)$ collective coordinates, while one may say that and $\tilde{\phi}, \tilde{\pi}$ are fields in the body fixed coordinate system(or moving co-cordinate system).

Next, we insert (9) into the Hamiltonian to obtain a new Hamiltonian as a function of $\tilde{\phi}$, $\tilde{\pi}$ and X's ;

$$\tilde{H}[\tilde{\pi}, \tilde{\phi}, X] = H[\tilde{\pi}_{[X]}, \tilde{\phi}_{[X]}] \tag{10}$$

Let us now consider a new system with Hamiltonian $\tilde{H}$, canonical variables X_α, $\tilde{\phi}$, and canonical momenta p_α and $\tilde{\pi}$.

We now show that this new system is equivalent to the old one if we impose the constraints

$$\Psi_\alpha[\tilde{\pi}, \tilde{\phi}, p, X] = p_\alpha - P_\beta[\tilde{\pi}, \tilde{\phi}]\, U_\alpha^\beta(X) = 0 \tag{11}$$

It is obvious that we have to impose n constraints since the new system has n more dynamical variables than the old one.

In the sense of Dirac [24], those constraints are first class since one can check that

$$\{\Psi_\alpha, \Psi_\beta\} = 0 \qquad , \{\tilde{H}, \Psi_\alpha\} = 0 \tag{12}$$

The existence of first class constraints, precisely, reflects the gauge invariance of the new system under the canonical transformations generated by Ψ_α . Its effective Hamiltonian has the form

$$H_{eff} = \tilde{H}[\tilde{\pi}, \tilde{\phi}, X] + \sum_\alpha \lambda_\alpha \Psi_\alpha[\tilde{\pi}, \tilde{\phi}, p, X] \tag{13}$$

where λ_α are Lagrange multipliers. λ_α is determined from the equation $\dot{X}_\alpha = \frac{\delta H_{eff}}{\delta p_\alpha}$ which gives

$$\lambda_\alpha(t) = \frac{d}{dt} X_\alpha(t) \tag{14}$$

Choosing X_α determines λ_α through this equation and thus fixes the gauge.

For $X_\alpha(t) = 0$, we find back the old system since then $\tilde{\phi} = \phi$, $\tilde{\pi} = \pi$, $\lambda_\alpha = 0$. Therefore, since the physical contents of the theory is gauge independent, the new description is equivalent to the old one.

The quantization of the new system can be performed using Faddeev's path integral quantization method for constrained theories [20]. Namely, one adds n additional gauge fixing conditions

$$Q_\alpha[\tilde{\pi}, \tilde{\phi}, p, X] = 0 \qquad \alpha = 1, 2, \cdots n \tag{15}$$

such that

$$\det \{\Psi_\alpha , Q_\beta\}_{P.B.} \neq 0 \tag{16}$$

and the transition matrix element is expressed according to Faddeev in the form

$$\int \mathcal{D}\tilde{\phi}\, \mathcal{D}\tilde{\pi}\, \mathcal{D}X\, \mathcal{D}p \prod_\alpha \delta(\Psi_\alpha) \prod_\beta \delta(Q_\beta) \det\{\Psi_\alpha, Q_\beta\} \cdot \exp\left\{ i \int dt \left[p_\alpha \dot{X}_\alpha + \int dx\, \tilde{\pi} \dot{\tilde{\phi}} - \tilde{H}[\tilde{\pi}, \tilde{\phi}, p, X] \right] \right\} \tag{17}$$

However, in order to be self-contained and to show the generalization of the one soliton method, we rederive the quantization procedure starting from the original path integral form of the transition matrix element

$$\langle \Psi_f, t_f | \Psi_i, t_i \rangle = \int \mathcal{D}\phi \mathcal{D}\Pi \; \Psi_f[\phi] \, \Psi_i[\phi] \exp\left\{ i \int dt \left[\int dx \, \Pi \dot{\phi} - H \right] \right\} \tag{18}$$

We introduce the collective coordinates into this functional integral through the following δ-function identities:

$$\int \prod_{\alpha,\beta} \mathcal{D}p_\alpha \mathcal{D}X_\beta \prod_t \delta\left(p_\alpha(t) - P_\alpha[\Pi_{[-X(t)]}, \phi_{[-X(t)]}]\right) \cdot \\ \cdot \, \delta\left(Q_\beta[\Pi_{[-X(t)]}, \phi_{[-X(t)]}]\right) J = 1 \tag{19}$$

Here the Jacobian is given as:

$$J = \prod_t \det\left(\frac{\delta Q_\alpha}{\delta X_\beta}\right) = \prod_t \det\left\{Q_\alpha, P_\beta\right\} \tag{20}$$

Then the transition matrix elements read

$$\int \mathcal{D}\phi \mathcal{D}\Pi \prod_{\alpha,\beta} \mathcal{D}X_\alpha \mathcal{D}p_\beta \prod_t \delta\left(p_\alpha(t) - P_\alpha[\phi_{[-X]}, \phi_{[-X]}]\right) \cdot \\ \delta\left(Q_\beta[\Pi_{[-X]}, \phi_{[-X]}]\right) \det\left\{Q_\alpha, P_\beta\right\} \Psi_f^*[\phi(\cdot, t_f)] \Psi_i[\phi(\cdot, t_i)] \\ \exp\left\{ \frac{i}{\hbar} \int dt \left[\int dx \, \Pi \dot{\phi} - H[\Pi, \phi] \right] \right\} \tag{21}$$

In this derivation we consider for simplicity only the case of an abelian canonical group.

In order to transfer the collective coordinates from the δ-funcion conditions into the action of our path integral we perform the following change of variables:

$$\phi(x,t) = \tilde{\phi}^{(x,t)}_{[X(t)]} \\ \Pi(x,t) = \tilde{\Pi}^{(x,t)}_{[X(t)]} \tag{22}$$

Obviously, the Jacobian is one since (22) is a canonical transformation.

Next, we consider the term

$$\int_{t_i}^{t_f} dt\, dx\, \Pi \dot{\phi} = \int_{t_i}^{t_f} dt\, dx\, \tilde{\Pi}_{[X(t)]}(x,t) \frac{d}{dt} \tilde{\phi}_{[X(t)]}(x,t) \equiv F_{[X]} \tag{23}$$

For an infinitesimal change δX , one can verify that

$$F_{[\delta X]} = \int_{t_i}^{t_f} dt\, p_\alpha(t) \frac{d}{dt} \delta X_\alpha(t) + \int_{t_i}^{t_f} dt\, dx\, \tilde{\Pi}(x,t) \frac{d}{dt} \tilde{\phi}(x,t)$$
$$+ \delta X_\alpha(t) \left[P_\alpha[\tilde{\pi}, \tilde{\phi}] - \int dx \tilde{\Pi} \frac{\delta P_\alpha}{\delta \tilde{\Pi}} \right] \Big|_{t_i}^{t_f} \tag{24}$$

and then knowing

$$\frac{\delta F_{[X]}}{\delta X_\alpha(t)} \tag{25}$$

and

$$F_{[0]} = \int dt\, dx\, \tilde{\Pi}(x,t) \frac{d}{dt} \tilde{\phi}(x,t) \tag{26}$$

we find

$$\int_{t_i}^{t_f} dt\, dx\, \Pi(x,t)\, \dot{\phi}(x,t) = \int dt\, p_\alpha(t)\, \dot{X}_\alpha(t) + \int dt\, dx\, \tilde{\Pi} \frac{d}{dt} \tilde{\phi}$$
$$+ X_\alpha(t) \left(P_\alpha[\tilde{\Pi}, \tilde{\phi}] - \int dx\, \tilde{\Pi} \frac{\delta P_\alpha}{\delta \tilde{\Pi}} \right) \Big|_{t_i}^{t_f} \tag{2.7}$$

From this expression we see that $p_\alpha(t)$ and $X_\alpha(t)$ represent conjugate canonical variables. The new Hamiltonian is now given by

$$\tilde{H}[\tilde{\pi}, \tilde{\phi}, X] = H[\tilde{\pi}_{[X]}, \tilde{\phi}_{[X]}] \tag{28}$$

and the final path integral expression for the transition matrix element reads:

$$\int D\tilde{\phi} D\tilde{\pi} \prod_{\alpha,\beta} Dp_\alpha DX_\beta \prod_t \delta(p_\alpha - P_\alpha)\, \delta(Q_\beta) \det\{Q, P\}$$
$$\tilde{\Psi}_f^*[\tilde{\phi}]\, \tilde{\Psi}_i^*[\tilde{\phi}] \exp\left\{ i \int dt \left[p_\alpha \dot{X}_\alpha + \int dx\, \tilde{\Pi} \dot{\tilde{\phi}} - \tilde{H}[\tilde{\pi}, \tilde{\phi}, X] \right] \right\} \tag{29}$$

which agrees with (17).

In this expression we have replaced Ψ_i , Ψ_f by $\tilde{\Psi}_i$, $\tilde{\Psi}_f$, in order to take into account the surface terms appearing in (27) together with the change of argument in $\Psi_{i,f}$. It is likely, though no general proof exists, that $\tilde{\Psi}_{i,f}$ is simply transformed from $\Psi_{i,f}$ by

$$\tilde{\Psi}_{i,f}[\tilde{\phi}] = \left(e^{i\sum_\alpha X_\alpha(t_{i,f})\hat{P}_\alpha}\, \Psi_{i,f}\right)(\tilde{\phi}) \tag{30}$$

namely, it is, as one expects, obtained from $\Psi_{i,f}$, by the unitary transformation associated with the canonical transformation introduced by (22). If P_α are constants of the motion, one finds that the new Hamiltonian

$$\tilde{H} = H[\tilde{\pi}[x], \tilde{\phi}[x]] = H[\tilde{\pi}, \tilde{\phi}] \tag{31}$$

does not depend on the collective coordinates X_α . Then choosing $\Psi_{i,f}$ to be eigenstates of the conserved quantities $\hat{P}_\alpha$ we can as in the one soliton sector case trivially integrate out the collective coordinates obtaining:

$$\prod_\alpha (2\pi)\delta(p'_\alpha - p_\alpha) \int \mathcal{D}\tilde{\phi}\,\mathcal{D}\tilde{\pi}\ \det\{Q, P\} \prod_t \prod_{\alpha,\beta} \delta(Q_\alpha)$$
$$\delta(p_\beta - P_\beta)\,\Psi_f^*[\tilde{\phi}]\,\Psi_i[\tilde{\phi}]\,\exp\left\{\frac{i}{\hbar}\int dt\,dx\left[\tilde{\pi}\dot{\tilde{\phi}} - \mathcal{H}(\tilde{\pi}, \tilde{\phi})\right]\right\} \tag{32}$$

Naturally, this is only valid if there is no external source in the action.

We now want to consider quantum fluctuarions around a stationary point of the functional integral (32). It corresponds to a classical solution associated with the original Hamiltonian. The gauge conditions Q_α which appear in (29) can be chosen arbitrarily, since changing Q_α is equivalent to a gauge transformation. Together with the constraints $p_\beta - P_\beta = 0$, they fix the classical solution. One can always choose Q_α to be such that one is led to a time independent classical solution. Alternatately, the possibility always exists of choosing a time dependent gauge condition which then leads to a perturbation expansion about the time dependent classical solution.

Finally, a comment is in order concerning canonical transformations in path integral. Namely, we derived our final result by performing canonical transformations in path integral and it is this step which raises several questions. The fact is that at present there exists no consistent way of discussing general canonical transformations in quantum theory and this limitation is valid for the path integral method also. The obstacles for performing a general canonical transformation are connected with the very definition of integration over paths which is defined as a limit of finite dimensional integrals so that the action appearing in the path integral is not quite like the classical action. Consequently, performing the change of variables at the classical level may lead to erroneous results. In chapter II we showed that a correct treatment of point canonical transformations is still possible in path integral method. Compared with the formal approach we found additional potential terms which are of the order $\hbar^2$. So for the case of point canonical transformations the results of these sections are correct if we add to the action the additional $\hbar^2$ terms. For the case of more general canonical transformations the formal approach presented in this chapter is probably correct at tree and one loop levels but more care is needed at two loop level.

V. SCATTERING OF SOLITONS

In this chapter we employ the general collective coordinate method formulated in the previous chapter in order to discuss scattering of solitons in the sine-Gordon theory. First in Section V.1 we develop a relativistic perturbation expansion for the one soliton sector using the time dependent one soliton classical solution. In Section V.2 the scattering theory for solitons is formulated and the classical soliton - antisoliton and soliton - soliton phase shifts are computed. Finally, in Section V.3 we evaluate the first quantum corrections to this scattering phase shifts.

V.1 One Soliton Sector

The one soliton sector perturbation expansion was discussed in detail in chapter III but there are several reasons why we elaborate more on this problem in this section. First of all in our previous discussions we established the perturbation expansion by choosing a time independent gauge condition which then led to an expansion about the static classical field. For the discussion of soliton scattering it will be appropriate to perturb about the time dependent two soliton classical solution and so it is instructive to study first the one soliton case by choosing a time dependent gauge condition. Furthermore, for the soliton scattering we like to establish a manifestly relativistic formalism and again it is important to give first a relativistic treatment for the one soliton case. The choice of a time dependent gauge condition leading to a perturbation expression about the moving soliton classical solution appears to be very much appropriate for obtaining in a simple way manifestly relativistic answers. We will demonstrate this in this section by computing the soliton energy at tree and one loop level[16].

Finally, the discussion of the one soliton sector is instructive for learning abount the wave functionals appearing in the path integral expression for the transition amplitude. For the scattering case it is important to use the right initial and final wave functionals. Wrong wave functionals would lead to incorrect result for the scattering amplitude. Though the present discussion is general, we will always refer to the explicit example of the sine-Gordon model where

$$\begin{aligned} \mathcal{L}(\phi) &= \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \\ V(\phi) &= \frac{1}{\gamma}\left[1 - \cos\left(\sqrt{\gamma}\,\phi(x,t)\right)\right] \\ \gamma &= \frac{m^2}{\lambda} \end{aligned} \tag{1.1}$$

since it is only in this case that exact multi-soliton solutions are known. The one soliton and one antisoliton static classical solutions read respectively:

$$\begin{aligned} \phi_s(x) &= \frac{4}{\sqrt{\gamma}} \tan^{-1}\left(e^{x}\right) \\ \phi_{\bar{s}}(x) &= \frac{4}{\sqrt{\gamma}} \tan^{-1}\left(e^{-x}\right) \end{aligned} \tag{1.2}$$

and the moving solutions are simply obtained by boosting them with velocity u. The action is invariant by the change

$$\phi(x,t) \rightarrow \phi(x,t) + \frac{2\pi}{\sqrt{\gamma}} \tag{1.3}$$

and there is an infinite set of equivalent descriptions of any state of the system related by (1.3).

We consider the matrix element of the evolution operator:

$$\langle \Psi_f | \exp\{-i(t_f - t_i)\hat{H}\} | \Psi_i \rangle = \int \mathcal{D}\Pi \mathcal{D}\phi \; \Psi_f^*[\phi(\cdot, t_f)]$$

$$\Psi_i[\phi(\cdot, t_i)] \exp\left\{ i \int_{t_i}^{t_f} dt \int dx \left[\Pi \dot{\phi} - \mathcal{H}(\Pi, \phi) \right] \right\} \tag{1.4}$$

We redefine the field by making a constant translation so that $\phi(\pm\infty, t) = \pm \frac{\pi}{\sqrt{\gamma}}$. After this change ϕ has the same symmetric boundary condition as in $\lambda\phi^4$ theory so that we can directly apply our discussion following the method developed before.

We want to specify the initial and final soliton momenta to be p and p'. Our computation will then show that it is appropriate to have the following form for the initial and final wave functionals:

$$\Psi_p[\phi(\cdot)] = \int dy \; e^{-ipy} \delta\left(Q_p[\phi(x+y)] \right) \frac{\partial Q_p}{\partial y} \times$$

$$\times \tilde{\Psi}_p[\phi(x+y)] \tag{1.5}$$

where $\tilde{\Psi}_p$ and Q_p will be determined below. It is easy to see that for arbitrary $\tilde{\Psi}_p$ and Q_p this is indeed an eigenstate of the momentum operator.

After introducing the collective coordinates $X(t)$ and $p(t)$ and changing variables in order to transfer the collective coordinates from the δ-function conditions into the action we integrate out $X(t)$ and $p(t)$ obtaining the following expression for the transition amplitude:

$$(2\pi)\delta(p'-p) \int \mathcal{D}\tilde{\pi} \mathcal{D}\tilde{\phi} \, \delta(p - P[\tilde{\pi}, \tilde{\phi}]) \, \delta(Q[\tilde{\phi}]) \, \{Q, P\} \times$$

$$\times \Psi_{p'}^*[\phi] \Psi_p[\phi] \exp\left\{ i \int_{t_i}^{t_f} dt \, dx \left[\tilde{\pi}\dot{\tilde{\phi}} - \mathcal{H}(\tilde{\pi}, \tilde{\phi}) \right] \right\} \tag{1.6}$$

The δ-function condition in front of this path integral inforces the momentum conservation condition $p' = p$.

At this point we have to choose a gauge condition $Q[\tilde{\Phi}]$. In our earlier discussion we used a time independent gauge which then led to a perturbation expansion about the time independent classical solution. Now we present a treatment choosing a time dependent gauge condition given by

$$Q[\tilde{\Phi}] = \int dx\, \phi_s'(x,t;u)\,\tilde{\Phi}(x,t)$$
$$\phi_s(x,t;u) = \phi_s\left(\frac{x+ut}{\sqrt{1-u^2}}\right) - \frac{\pi}{\sqrt{\gamma}} \tag{1.7}$$

The minimum of the action with the constraints is then given by the following time dependent classical solution:

$$\tilde{\Phi}_{cl}(x,t) = \phi_s(x,t;u)$$
$$\tilde{\Pi}_{cl}(x,t) = \frac{d}{dt}\phi_s(x,t;u) \tag{1.8}$$

and the momenta p is related to the velocity u through the relation

$$p = \frac{M_0 u}{\sqrt{1-u^2}} \quad , \quad M_0 = \int dx\, \mathcal{H}(\phi_s(x)) = \frac{8}{\gamma} \tag{1.9}$$

Then we should keep in mind that u is not the exact velocity of the soliton since it is given by (1.9) which involves the classical mass and not the true mass of the soliton. We will come back to this point when it is important.

Next, we determine the functionals Q_p which were introduced in (1.5) in such a way that the condition $Q=0$ at the initial or final times matches the condition $Q_p=0$ appearing in the wave functions Ψ_p.

We then have

$$Q_p[\phi(\cdot)] = \int dx\, \phi_s'\left(x\sqrt{1+\frac{p^2}{M_0^2}}\right)\phi(x) \tag{1.10}$$

As a result, in the wave functions Ψ_p , $\Psi_{p'}$ one can integrate immediately over the y, y' variables. The result reads:

$$\langle p', t_f | p, t_i \rangle = (2\pi)\,\delta(p'-p)\,\exp\{-ipuT\}\int \mathcal{D}\tilde{\pi}\,\mathcal{D}\tilde{\phi}\,\delta(Q)$$
$$\delta(p-\mathcal{P}[\tilde{\pi},\tilde{\phi}])\,\tilde{\Psi}_{p'}^{*}[\tilde{\phi}]\,\tilde{\Psi}_p[\tilde{\phi}]\,\exp\{i\int[\tilde{\pi}\dot{\tilde{\phi}} - \mathcal{H}(\tilde{\pi},\tilde{\phi})]\} \tag{1.11}$$

We now establish the perturbation expansion by writing

$$\tilde{\phi}(x,t) = \phi_s(x,t;u) + \chi(x,t)$$
$$\tilde{\pi}(x,t) = \dot{\phi}_s(x,t;u) + \varpi(x,t) \tag{1.12}$$

Here we shall only consider the first quantum correction and we thus disregard in the action all terms which involve powers higher than quadratic in χ and ϖ . In the same spirit we only keep lowest order terms in the constraint condition $p - \mathcal{P}[\tilde{\pi},\tilde{\phi}]$. The result takes the form

$$\exp\left\{-i\frac{M_0u^2}{\sqrt{1-u^2}}T + iA[\pi_{cl},\phi_{cl}]\right\}\cdot\int\mathcal{D}\varpi\,\mathcal{D}\chi\;\delta\left(\int\phi_s'\chi\right)$$
$$\delta\left(\int(\dot{\phi}_s'\chi - \phi_s'\varpi)\right)\varphi_{p'}^{*}\varphi_p\,\exp\{i\,A^{(2)}[\varpi,\chi]\} \tag{1.13}$$

where

$$A^{(2)}[\varpi,\chi] = \frac{1}{2}\int_{t_i}^{t_f}dt\int dx\,[\varpi(\dot{\chi}-\varpi) - \chi(\dot{\varpi} - \chi'' + V_s''\chi)]$$
$$V_s''(x,t) = \left.\frac{\partial^2 V(\phi)}{\partial\phi^2}\right|_{\phi=\phi_s(x,t;u)} \tag{1.14}$$

and

$$\varphi_p[\varpi,\chi] = \exp\{i\int dx\,\varpi\chi\}\,\tilde{\Psi}_p[\phi_{cl}+\chi] \tag{1.15}$$

Observe that we have in front of the path integral the correct phase factor giving the relativistic answer for the classical energy.

In order to compute the first quantum correction to this classical soliton energy one has to evaluate the quadratic path integral in (1.13). This is achieved by first diagonalizing the quadratic action (1. 14). Since $V_s''(x,t)$ is now time dependent, this problem is nontrivial. We solve it in a way which can be generalized to any quadratic action with time dependent potential. The general idea is to consider the complete set of solutions of the equation associated with the action (1. 14) with periodic conditions in a box of length $L = u(t_f - t_i)$. From this we deduce for large L a set of orthonormalized periodic functions $f_n^{\epsilon}(x,t)$, $g_n^{\epsilon}(x,t)$ such that for each time t we can expand χ and ϖ by letting

$$\chi(x,t) = \sum_{n,\epsilon=\pm}' b_n^{\epsilon}(t)\, f_n^{\epsilon}(x,t)$$

$$\varpi(x,t) = \sum_{n,\epsilon=\pm}' b_n^{\epsilon}(t)\, g_n^{\epsilon}(x,t) \qquad (1.16)$$

where

$$b_n^{\epsilon}(t) = \frac{i}{\epsilon}\int dx \left[f_n^{\epsilon *}(x,t)\varpi(x,t) - g_n^{\epsilon}(x,t)\chi(x,t) \right] \qquad (1.17)$$

The equations of motion associated with the quadratic action (1.14) read

$$\mathring{\chi}(x,t) - \varpi(x,t) = 0$$

$$\mathring{\varpi}(x,t) - \chi''(x,t) + V_s''(x,t)\chi(x,t) = 0 \qquad (1.18)$$

We denote the solutions by $\chi_q^{\epsilon}(x,t)$ with $\epsilon = \pm 1$ and they are found to be:

$$\chi_q^{+}(x,t) = e^{-i\left(t\sqrt{q^2+1} - xq\right)}\left[\frac{1 + e^{i\delta(q)}\, e^{2\left(\frac{x+ut}{\sqrt{1-u^2}}\right)}}{1 + e^{2\left(\frac{x+ut}{\sqrt{1-u^2}}\right)}}\right]$$

$$\varpi_q^{+}(x,t) = \frac{d}{dt}\chi_q^{+}(x,t)$$

(1.19)

and

$$\chi_q^-(x,t) = \left[\chi_q^+(x,t) \right]^*$$
$$\varpi_q^-(x,t) = \left[\varpi_q^+(x,t) \right]^* \tag{1.20}$$

Here we denoted:

$$e^{i\delta(q)} = \frac{-1+iq_-}{1+iq_-} \quad , \quad q_- = \frac{q-u\sqrt{1+q^2}}{\sqrt{1-u^2}} \tag{1.21}$$

Next, we put the system in a box by imposing the periodicity conditions

$$\delta(q_n) + q_n L = 2\pi n \tag{1.22}$$

Since the system is periodic in space it is also periodic in time with period $\frac{L}{u}$. We choose $T = \frac{L}{u}$ and introduce the stability angles $\nu(q)$ defined by[25]:

$$\chi_q^\epsilon\left(x, \frac{1}{2}\frac{L}{u}\right) = e^{-i\epsilon\nu(q)} \chi_q^\epsilon\left(x, -\frac{1}{2}\frac{L}{u}\right) \tag{1.23}$$

For large L one finds easily

$$\nu_n \equiv \nu(q_n) = \frac{L}{u}\sqrt{1+q_n^2} + \delta(q_n) \tag{1.24}$$

We now show that the action (1.14) can be decomposed into a set of decoupled harmonic oscillators of energy $\frac{1}{T}\nu_n$. To achieve this we will derive, using the solutions χ_q^ϵ, ϖ_q^ϵ, a complete set of orthogonal functions which for fixed t will diagonalize the action. This is done by introducing a two component vector notation $\begin{pmatrix}\chi \\ \varpi\end{pmatrix}$ and remarking that according to (1.23) the vector

$$V_{nm}^\epsilon(x,t) = \begin{pmatrix} e^{i(\epsilon\nu_n + 2\pi m)\frac{t}{T}} \chi_n^\epsilon(x,t) \\ e^{i(\epsilon\nu_n+2\pi m)\frac{t}{T}} \varpi_n^\epsilon(x,t) \end{pmatrix} \tag{1.25}$$

is periodic in T and is eigenstate of the operator

$$\mathcal{B} = \begin{pmatrix} \frac{\partial}{\partial t} & -1 \\ -\frac{\partial^2}{\partial x^2} + V_S''(x,t) & \frac{\partial}{\partial t} \end{pmatrix} \tag{1.26}$$

with eigenvalue $-\frac{i}{T}(2\pi m + \epsilon \nu_n)$. In the space of functions periodic of period T, $\mathcal{B}$ is hermitian with the inner product

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \, (\chi_1^*, \varpi_1^*) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \chi_2 \\ \varpi_2 \end{pmatrix} \tag{1.27}$$

therefore,

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \; V^{\epsilon_1 *}_{n_1 m_1}(x,t) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} V^{\epsilon_2}_{n_2 m_2} \propto \delta_{\epsilon_1 \epsilon_2} \delta_{n_1 n_2} \delta_{m_1 m_2} \tag{1.28}$$

The integral (1.28) can be written as

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} dt \; e^{\frac{2\pi i t}{T}(m_2 - m_1)} I_{n_1 n_2}(t) \propto \delta_{\epsilon_1 \epsilon_2} \delta_{n_1 n_2} \delta_{m_1 m_2} \tag{1.29}$$

which shows that $I_{n_1 n_2}(t)$ has all the Fourier components equal to zero except for the component of zero frequency. Thus, $I_{n_1 n_2}$ does not depend on t and we have

$$I_{n_1 n_2} = \int dx \left(\chi^{\epsilon_1 *}_{2 n_1}, \varpi^{\epsilon_1 *}_{2 n_1} \right) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \chi^{\epsilon_2}_{2 n_2} \\ \varpi^{\epsilon_2}_{2 n_2} \end{pmatrix} \propto \delta_{n_1 n_2} \delta_{\epsilon_1 \epsilon_2} \tag{1.30}$$

So far we have not yet considered the bound state solution $V_0 = \begin{pmatrix} \phi_S' \\ \dot{\phi}_S \end{pmatrix}$ which as already discussed in many places appears because the system is invariant by space translations. Since V_0 vanishes for large x, its stability angle vanishes and it is an eigenstate of with eigenvalue zero. Our idea is to expand $\begin{pmatrix} \chi(x,t) \\ \varpi(x,t) \end{pmatrix}$ over the eigen-states of $\mathcal{B}$. In this connection, V_0 causes a problem since it is easy to check that

$$\int dx\, V_0^* \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} V_0 = 0 \tag{1.31}$$

that is, V_0 has zero norm. In order to take the subsidiary conditions into account, we add Lagrange multipliers and consider the action

$$\int dt\,dx \left\{ \tfrac{1}{2}\varpi(\dot{\chi} - \varpi) - \tfrac{1}{2}\chi(\dot{\varpi} - \chi'' + V_S''\chi) + \right. \\ \left. + \lambda(t)\,\phi_S'\chi + \mu(t)(\dot{\phi}_S\chi' + \phi_S\varpi) \right\} \tag{1.32}$$

The corresponding wave equations are immediately solved. One finds the solutions

$$\begin{aligned} \chi(x,t) &= \chi_q^{\epsilon}(x,t) - \alpha(t)\,\phi_S'(x,t;u) \\ \varpi(x,t) &= \varpi_q^{\epsilon}(x,t) - \alpha(t)\,\dot{\phi}_S'(x,t;u) \\ \lambda(t) &= 0\,, \quad \mu(t) = \dot{\alpha}(t) \end{aligned} \tag{1.33}$$

and $\alpha(t)$ is determined from the subsidiary condition $\int dx\, \phi_S'(x,t)\chi = 0$ which gives

$$\alpha(t) = \frac{1}{M_0}\int dx \left(\chi_q^{\epsilon}(x,t)\,\phi_S'(x,t;u) \right) \tag{1.34}$$

The second subidiary condition takes the form

$$\int dx \left[\varpi\phi_S' - \dot{\phi}_S'\chi \right] = \int dx \left[\varpi_q^{\epsilon}(x,t)\,\phi_S'(x,t) - \dot{\phi}_S'(x,t)\,\chi_q^{\epsilon}(x,t) \right] \tag{1.35}$$

and this integral indeed vanishes as $V_0(x,t)$ and $\begin{pmatrix} \chi_q^{\epsilon}(x,t) \\ \varpi_q^{\epsilon}(x,t) \end{pmatrix}$ are eigenstates of $\mathcal{B}$ with different eigenvalues.

These relations allow us to deduce a solution satisfying the constraint from any solution of the unconstrainedproblem. On the other hand, the zero stability angle mode completely disappears from the expansion if one takes the subsidiary condition into account. We thus introduce a set of orthonormalized eignefunctions:

$$f_n^{\epsilon}(x,t) = \frac{1}{\sqrt{N_n}} e^{i\epsilon\nu_n \frac{t}{T}} \left(\chi_{q_n}^{\epsilon}(x,t) - \frac{1}{M_0} \phi_s'(x,t;u) \int dx\, \chi_{q_n}^{\epsilon} \phi_s' \right)$$

$$g_n^{\epsilon}(x,t) = \frac{1}{\sqrt{N_n}} e^{i\epsilon\nu_n \frac{t}{T}} \left(\varpi_{q_n}^{\epsilon}(x,t) - \frac{1}{M_0} \dot{\phi}_s'(x,t;u) \int dx\, \chi_{q_n}^{\epsilon} \phi_s' \right)$$

$$N_n = 2 \left\{ L\sqrt{1+q_n^2} - \frac{2\sqrt{1-u^2}}{\left(\sqrt{1+q_n^2} - u q_n\right)} \right\} \tag{1.36}$$

One can verify that they satisfy

$$i \int \left[f_{n_1}^{\epsilon_1 *}(x,t)\, g_{n_2}^{\epsilon_2}(x,t) - g_{n_1}^{\epsilon_1 *}(x,t)\, f_{n_2}^{\epsilon_2}(x,t) \right] = \epsilon_1 \delta_{\epsilon_1 \epsilon_2} \delta_{n_1 n_2} \tag{1.37}$$

They are therefore linearly independent. By counting the number of $\begin{pmatrix} f_n^{\epsilon} \\ g_n^{\epsilon} \end{pmatrix}$ and comparing with the vacuum case it is easy to see that they form a complete set in the subspace of functions satisfying the constraint. We can therefore expand $\begin{pmatrix} \chi \\ \varpi \end{pmatrix}$ in terms of $\begin{pmatrix} f_n^{\epsilon} \\ g_n^{\epsilon} \end{pmatrix}$, that is

$$\begin{pmatrix} \chi(x,t) \\ \varpi(x,t) \end{pmatrix} = \sum_{n,\epsilon=\pm}{}' b_n^{\epsilon}(t) \begin{pmatrix} f_n^{\epsilon}(x,t) \\ g_n^{\epsilon}(x,t) \end{pmatrix} \tag{1.38}$$

Since χ and ϖ are real functions and since

$$\left[f_n^{\epsilon}(x,t) \right]^* = f_n^{-\epsilon}(x,t) \quad , \left[g_n^{\epsilon}(x,t) \right]^* = g_n^{-\epsilon}(x,t) \tag{1.39}$$

we have $$\left[b_n^{\epsilon}(t) \right]^* = b_n^{-\epsilon}(t) \tag{1.40}$$

Finally, the action (1.14) can be written as

$$A^{(2)}[\varpi,\chi]=\frac{1}{2}\int_{-\frac{T}{2}}^{T/2}dt\int_{-\frac{L}{2}}^{\frac{L}{2}}dx\,(\chi,\varpi)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\mathcal{B}\begin{pmatrix}\chi\\\varpi\end{pmatrix} \tag{1.41}$$

and one immediately obtains using the expansion (1.41)

$$A^{(2)}[\varpi,\chi]=\int_{-\frac{T}{2}}^{T/2}dt\sum_n\Big\{\frac{1}{2i}\Big[\big(\dot{b}_n^+(t)\big)^*b_n^+(t)-$$
$$-\big(b_n^+(t)\big)^*\dot{b}_n^+(t)\Big]-\frac{\nu_n}{T}b_n^+(t)\,b_n^+(t)\Big\} \tag{1.42}$$

Now, we can determine the wave functionals which are associated with the eigenstates of the Hamiltonian. In particular, the lowest energy eigenstate which is the quantum soliton state is immediately found to be the vacuum of all oscillators, that is

$$\varphi_p[\varpi,\chi]=\exp\Big\{-\frac{1}{2}\sum_n\big(b_n^+(t_{i,f})\big)^*b_n^+(t_{i,f})\Big\} \tag{1.43}$$

This formula determines the one soliton wave functional to lowest order. Since the wave functional (1.43) is the vacuum state of all oscillators, one immediately derives the first quantum correction as the sum of zero point energies $\sum_n'\frac{1}{2}\frac{\nu_n}{T}$. We thus obtain

$$\langle p',\tfrac{T}{2}|p,-\tfrac{T}{2}\rangle=(2\pi)\delta(p'-p)\exp\Big\{-i\Big[T\frac{\Pi_0}{\sqrt{1-u^2}}+\sum_n\frac{1}{2}\nu_n\Big]\Big\} \tag{1.44}$$

This sum of stability angle is to be computed by subtracting the vacuum self-energy and taking into account the renormalization counterterm. One obtains after some calculation

$$\frac{1}{2}\sum_n'\nu_n=\frac{1}{2}\sum_n'T\sqrt{1+q_n^2}\;\Rightarrow\;\Big(-\frac{1}{\pi}\Big)\sqrt{1-u^2}\,T \tag{1.45}$$

and thus the total expression for the soliton energy reads:

$$E_u=\frac{\Pi_0}{\sqrt{1-u^2}}-\frac{1}{\pi}\sqrt{1-u^2} \tag{1.46}$$

Using formula (1.12) we obtain

$$E_u = \sqrt{p^2 + M_0^2} + \frac{M_0 M_1}{\sqrt{p^2 + M_0^2}} \approx \sqrt{p^2 + M^2}$$

$$M = M_0 + M_1 = \frac{8}{\gamma} - \frac{1}{\pi} = \frac{8}{\gamma}\left(1 - \frac{\gamma}{8\pi}\right) \equiv \frac{8}{\gamma'} \tag{1.47}$$

This completes our demonstration that the formalism we presented is indeed relativistic both at tree and one loop level. Finally, it is easy to see that the antisoliton is treated exactly in the same way as the soliton.

V.2 Perturbation Expansion for Soliton Scattering

For the sine-Gordon theory exact multi-soliton classical solutions are known and this fact will enable us to give a quantum treatment of soliton scattering using the general collective coordinate method developed in chapter IV. We will illustrate the formalism in this section on the two soliton scattering example. We explicitly consider the soliton - antisoliton case and the discussion of soliton - soliton scattering is almost identical(16).

The general one soliton and one antisoliton classical solutions are given respectively by

$$\phi_s(x,t\,;u X^\circ) = \frac{4}{\sqrt{\gamma}} \tan^{-1}\left\{ \exp\left[\frac{x + ut + X^\circ}{\sqrt{1-u^2}} \right] \right\}$$

$$\phi_{\bar{s}}(x,t\,;u,X^\circ) = \frac{4}{\sqrt{\gamma}} \tan^{-1}\left\{ \exp\left[\frac{x + ut + X^\circ}{\sqrt{1-u^2}} \right] \right\} \tag{2.1}$$

and the soliton - antisoliton classical solution reads

$$\phi_{S\bar{S}}(x,t\,;u_1u_2X_1^\circ X_2^\circ) = \frac{4}{\sqrt{\gamma}}\tan^{-1}\left\{ ch\,\frac{\theta_1-\theta_2}{2}\,\frac{sh\left[\frac{\omega_1(x,t)-\omega_2(x,t)}{2}\right]}{ch\left[\frac{\omega_1(x,t)+\omega_2(x,t)}{2} - \ln th\,\frac{\theta_1-\theta_2}{2}\right]}\right\} \tag{2.2}$$

where
$$sh\,\theta_e = \frac{u_e}{\sqrt{1-u_e^2}}\ ,\quad ch\,\theta_e = \frac{1}{\sqrt{1-u_e^2}}$$
$$\omega_\ell(x,t) = (x+X_e^\circ)\,ch\,\theta_e + t\,sh\,\theta_e$$
$$\ell = 1,2 \tag{2.3}$$

For large initial and final times this classical solution has the following asymptotics:

$$\phi_{S\bar{S}}(x t\,;u_1u_2X_1^\circ X_2^\circ) \xrightarrow[t\to-\infty]{} \phi_S(x t\,;u_1^- X_1^-) + \phi_{\bar{S}}(x,t\,;u_2^- X_2^-)$$
$$\phi_{S\bar{S}}(x,t;u_1u_2X_1^\circ X_2^\circ) \xrightarrow[t\to+\infty]{} \phi_S(x,t\,;u_1^+X_1^+) + \phi_{\bar{S}}(x,t;u_2^+X_2^+) \tag{2.4}$$

The initial and final velocities and coordinates are connected by the relations

$$u_1^+ = u_1^- = u_1$$
$$u_2^+ = u_2^- = u_2$$
$$\Delta X_1 = X_1^+ - X_1^- = \sqrt{1-u_1^2}\,\ln\left(1 - \frac{4M_0^2}{s}\right)$$
$$\Delta X_2 = X_2^+ - X_2^- = -\sqrt{1-u_2^2}\,\ln\left(1 - \frac{4M_0^2}{s}\right) \tag{2.5}$$

where
$$s = \left(\frac{M_0}{\sqrt{1-u_1^2}} + \frac{M_0}{\sqrt{1-u_2^2}}\right)^2 - \left(\frac{M_0u_1}{\sqrt{1-u_1^2}} + \frac{M_0u_2}{\sqrt{1-u_2^2}}\right)$$
$$M_0 = \frac{8}{\gamma} \tag{2.6}$$

So after the collison the soliton and antisoliton velocities remain the same, only their respective coordinates change. For the two soliton case this is natural due to the total momentum and energy conservation laws but the remarkable fact is that even for N-soliton case the soliton momenta are separately conserved. This happens because there exists an infinite set of conservation laws in the sine-Gordon theory.

We consider the soliton - antisoliton S-matrix element defined as follows:

$$\langle p_1', p_2' | \hat{S} | p_1, p_2 \rangle = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle p_1', p_2' | e^{i\hat{H}_0 t_f} e^{-i\hat{H}(t_f - t_i)} e^{-i\hat{H}_0 t_i} | p_1 p_2 \rangle \tag{2.7}$$

where p_1, p_2 and p_1', p_2' denote the initial and final soliton and anti-soliton momenta, respectively. The free soliton Hamiltonian $\hat{H}_0$ which enters the definition of the S-matrix element is of course not known in this case. We now write this scattering amplitude in the path integral form as:

$$S_{s\bar{s}}(p_1', p_2'; p_1, p_2) = \int \mathcal{D}\Pi \mathcal{D}\phi \; \Psi^*_{p_1' p_2'}[\phi(\cdot, t_f), t_f] \cdot \cdot \Psi_{p_1 p_2}[\phi(\cdot, t_i) t_i] \; \exp\{ i A[\Pi, \phi] \} \tag{2.8}$$

where the time dependent wave functional is given by

$$\Psi_{p_1 p_2}[\phi(\cdot, t), t] \equiv \langle \phi(\cdot) | e^{-i\hat{H}_0 t} | p_1, p_2 \rangle = = e^{-i\left(\sqrt{M^2 + p_1^2} + \sqrt{M^2 + p_2^2}\right)} \Psi_{p_1 p_2}[\phi(\cdot, t)] \tag{2.9}$$

and knowing the form of the one soliton wave functionals we write

$$\Psi_{p_1 p_2}[\phi(\cdot)] = \int dX_1 dX_2 \, e^{i(p_1 X_1 + p_2 X_2)} \delta\left(\int dx \, \phi_s'\left((x+X)\sqrt{1+\frac{p_1^2}{m_0^2}}\right)\phi(x)\right)$$

$$\cdot \, \delta\left(\int dx \, \phi_{\bar{s}}'\left((x+X_2)\sqrt{1+p_2^2/m_0^2}\right)\phi(x)\right) \cdot J_{p_1 p_2} \Psi[\phi(x) - \phi_s - \phi_{\bar{s}}]$$

$$J_{p_1 p_2} = \int dx \, \phi_s'\left((x+X_1)\sqrt{1+\frac{p_1^2}{m_0^2}}\right)\phi(x) \cdot \int dx \, \phi_{\bar{s}}'\left((x+X_2)\sqrt{1+\frac{p_2^2}{m_0^2}}\right)\phi(x) \tag{2.10}$$

The soliton - antisoliton general classical solution involves four constants which are related to the positions and momenta of the soliton and of the antisoliton. According to our general method we introduce two collective coordinates $X(t)$, $\bar{X}(t)$ and their conjugate momenta by using two conserved quantities. The most convenient choice is to introduce the total momentum P as in the one soliton case together with the Hamiltonian itself which will lead to total energy conservation. $X(t)$ is thus, the center-of-mass position while $\bar{X}(t)$ can be considered as related to the time at which the inter-action is taking place which classically fixes the relative distance of the soliton and antisoliton for large time.

These collective coordinates are introduced into the functional integral by inserting the identities:

$$\int \mathcal{D}X \mathcal{D}\bar{X} \mathcal{D}p \mathcal{D}\varepsilon \prod_t \delta\left(Q[\phi^{(x,t)}_{[-X(t),\bar{X}(t)]}]\right)\delta\left(\bar{Q}[\phi^{(x,t)}_{[-X(t),-\bar{X}(t)]}]\right) \cdot$$

$$\cdot \, J \cdot \delta\left(p(t) - P[\pi,\phi]\right) \delta\left(\varepsilon(t) - H[\pi,\phi]\right) = 1 \tag{2.11}$$

where J is the appropriate Jacobian. Following our general scheme we perform the change of variable

$$\phi(x,t) = \tilde{\phi}^{(x,t)}_{[X(t),\bar{X}(t)]} \quad , \quad \pi(x,t) = \tilde{\pi}^{(x,t)}_{[X(t),\bar{X}(t)]} \tag{2.12}$$

and integrate out the variables $X(t)$, $\bar{X}(t)$, $p(t)$, $\mathcal{E}(t)$. In our general discussion given in chapter IV we assumed that the initial and final states are eigenstates of the quantum mechanical operators associated to the conserved quantities. There, it is easily seen that the initial and final wave functionals are indeed eigenstates of $\hat{\mathcal{P}}$. We assume that for large T they also become eigenstates of $\hat{H}$ to leading order and we can therefore apply the general method, to get:

$$
\begin{aligned}
S_{s\bar{s}} = &(2\pi)^2 \delta(p_f - p_i)\, \delta(E_f - E_i) \exp\left\{ iT \sum_{\ell=1}^{2} \sqrt{M^2 + p_\ell^2} \right\} \times \\
&\times \int \mathcal{D}\tilde{\pi} \mathcal{D}\tilde{\phi}\; \delta(Q[\tilde{\phi}])\, \delta(\bar{Q}[\tilde{\phi}])\, \mathcal{J}\, \delta(p_i - \mathcal{P}[\tilde{\pi}, \tilde{\phi}]) \times \\
&\times \delta(E_i - H[\tilde{\pi}, \tilde{\phi}])\, \Psi_f^* \Psi_i \exp\{ i A[\tilde{\pi}, \tilde{\phi}] \}
\end{aligned}
\tag{2.13}
$$

The δ-functions in front of this path integral inforce the total momentum and energy conservation laws. This generalized to the N-soliton case where there will be N independent conserved quantities implying the conservation of individual soliton momenta in complete parallel with the classical behaviour of the N-soliton solutions. This conclusions are based on the assumption that the infinite set of conservation laws of the sine-Gordon equation which was established in the classical theory remains valid in the quantum theory also.

We intend to develop the perturbation expansion about the soliton - antisoliton classical solution

$$
\phi_{s\bar{s}}(xt; u_1 u_2) \equiv \phi_{s\bar{s}}(xt; u_1, u_2, X_1^0 = 0, X_2^0 = 0)
\tag{2.14}
$$

and for this purpose the following gauge conditions are appropriate:

$$
\begin{aligned}
Q[\tilde{\phi}] &= \int dx\, \phi'_{s\bar{s}}(x,t; u_1 u_2)\, \tilde{\phi}(x,t) \\
\bar{Q}[\tilde{\phi}] &= \int dx\, \dot{\phi}_{s\bar{s}}(x,t; u_1 u_2)\, \tilde{\phi}(x,t)
\end{aligned}
\tag{2.15}
$$

At initial and final times the soliton and antisoliton are widely separated. Therefore, Q and $\bar{Q}$ break up into two integrals and we get:

$$\lim_{t\to-\infty} \delta(Q[\tilde{\Phi}])\,\delta(\bar{Q}[\tilde{\Phi}]) \sim \delta\left(\int dx\, \phi_s'(x,t;u_1^-,X_1^-)\,\tilde{\Phi}(x,t)\right) \times$$
$$\times\ \delta\left(\int dx\, \phi_{\bar{s}}'(x,t;u_2^-,X_2^-)\,\tilde{\Phi}(x,t)\right)$$
$$\lim_{t\to+\infty} \delta(Q[\tilde{\Phi}])\,\delta(\bar{Q}[\tilde{\Phi}]) \sim \delta\left(\int dx\, \phi_s'(x,t;u_1^+,X_1^+)\,\tilde{\Phi}(x,t)\right) \times \tag{2.16}$$
$$\times\ \delta\left(\int dx\, \phi_{\bar{s}}'(x,t;u_2^+,X_2^+)\,\tilde{\Phi}(x,t)\right)$$

$$u_1^+ = u_1^- = u_1$$
$$u_2^+ = u_2^- = u_2 \tag{2.17}$$

Now we notice that it is possible to integrate out the variables X_1, X_2 and X_1', X_2' in the expressions for the initial and final wave functionals $\Psi_{p_1p_2}[\phi(\cdot,t_i)]$ and $\Psi^*_{p_1'p_2'}[\phi(\cdot,t_f)]$. Namely, the δ-function conditions appearing in this wave functionals imply that in the integral over, for example, X_1 and X_2 the only contribution comes from the point determined by the equations

$$\int dx\, \phi_s'\left((x+X_1)\sqrt{1+\frac{p_1^2}{M_0^2}}\right)\tilde{\Phi}(x,t_i) = 0$$
$$\int dx\, \phi_{\bar{s}}'\left((x+X_2)\sqrt{1+\frac{p_2^2}{M_0^2}}\right)\tilde{\Phi}(x,t_i) = 0 \tag{2.18}$$

On the other hand, due to the relation (2.16) we see that the gauge conditions at initial time imply the following equations:

$$\int dx\, \phi_s'\left(\frac{x+u_1t_i+X_1^-}{\sqrt{1-u_1^2}}\right)\tilde{\Phi}(x,t) = 0$$
$$\int dx\, \phi_{\bar{s}}'\left(\frac{x+u_2t_i+X_2^-}{\sqrt{1-u_2^2}}\right)\tilde{\Phi}(x,t) = 0 \tag{2.19}$$

Since the velocities and momenta are related by

$$p_1 = \frac{m_0 u_1}{\sqrt{1-u_1^2}}$$

$$p_2 = \frac{m_0 u_2}{\sqrt{1-u_2^2}} \tag{2.20}$$

we conclude comparing (2.18) and (2.19) that

$$X_1 = X_1^- + u_1 t_i$$
$$X_2 = X_2^- + u_2 t_i \tag{2.21}$$

and in the same way for the final time t_f :

$$X_1' = X_1^+ + u_1 t_f$$
$$X_2' = X_2^+ + u_2 t_f \tag{2.22}$$

So afer integrating out this variables the initial and final wave functionals take the simple forms:

$$e^{i(p_1 X_1^- + p_2 X_2^-)} e^{i(p_1 u_1 + p_2 u_2)t_i} \Psi[\tilde{\Phi}(x,t_i) - \phi_s - \phi_{\bar{s}}] \tag{2.23}$$

and

$$e^{-i(p_1 X_1^+ + p_2 X_2^+)} e^{-i(p_1 u_1 + p_2 u_2)t_f} \Psi[\tilde{\Phi}(x,t_f) - \phi_s - \phi_{\bar{s}}] \tag{2.24}$$

Now translating the fields by the soliton - antisoliton classical solution

$$\tilde{\Phi}(x,t) = \phi_{s\bar{s}}(x,t;u_1 u_2) + \chi(x,t)$$
$$\tilde{\Pi}(x,t) = \dot{\phi}_{s\bar{s}}(x,t;u_1 u_2) + \varpi(x,t) \tag{2.25}$$

we obtain the final expression for the scattering amplitude

$$S_{s\bar{s}} = (2\pi)^2 \delta(p_f - p_i)\delta(E_f - E_i) \exp\Big\{ i \Big[\sum_{\ell=1}^{2} \sqrt{M^2 + p_\ell^2}\, T -$$
$$- (p_1 u_1 + p_2 u_2) T - p_1 (X_1^+ - X_1^-) - p_2 (X_2^+ - X_2^-) + A[\dot{\phi}_{s\bar{s}}, \phi_{s\bar{s}}] \Big] \Big\} \times$$

$$\times \int \mathcal{D}\varpi \mathcal{D}\chi\, \Psi^*[\chi(\cdot,t_f)]\, \Psi[\chi(\cdot,t_i)] \prod_t \delta(Q[\chi])\, \delta(\bar{Q}[\chi]) \cdot$$
$$\cdot\ \delta(p_1+p_2-P)\, \delta(E_1+E_2-H)\, J \exp\left\{ iA[\dot{\phi}_{s\bar{s}}+\varpi, \phi_{s\bar{s}}+\chi] - \right.$$
$$\left. - iA[\dot{\phi}_{s\bar{s}}, \phi_{s\bar{s}}] \right\} \tag{2.26}$$

The soliton - antisoliton scattering phase shift is defined by

$$S_{s\bar{s}} = (2\pi)^2 \delta(p_f - p_i)\, \delta(E_f - E_i)\, e^{2i\, \delta_{s\bar{s}}} \tag{2.27}$$

and at the one loop level we intend to work the soliton and the antisoliton energy reads

$$\sqrt{M^2+p_\ell^2} \approx \sqrt{p_\ell^2+M_0^2} + \frac{M_0 M_1}{\sqrt{p_\ell^2+M_0^2}} = \frac{M_0}{\sqrt{1-u_\ell^2}} + M_1\sqrt{1-u_\ell^2}$$
$$\ell = 1,2 \tag{2.28}$$

For simplicity, we use the center-of-mass frame so that

$$p_1 = -p_2 = \frac{M_0 u}{\sqrt{1-u^2}} \tag{2.29}$$

Then the phase factor in front of the path integral gives the classical phase shift[26]

$$2\,\delta_{ce}(u) = \lim_{T\to\infty} \left\{ 2T \frac{M_0(1-u^2)}{\sqrt{1-u^2}} - 2\frac{u^2}{\sqrt{1-u^2}} \Delta(u) + A[\dot{\phi}_{s\bar{s}}, \phi_{s\bar{s}}] \right\} \tag{2.30}$$

where we used the fact that

$$\Delta X_1 = u\Delta(u) = -\Delta X_2$$
$$\Delta(u) = \frac{2}{u}\frac{1}{\sqrt{1-u^2}} \ln(u) \tag{2.31}$$

This classical phase shift is of the order γ^{-1} and by a straightforward calculation one obtains[27]

$$\delta^{ce}_{s\bar{s}}(u) = \frac{4\pi^2}{\gamma} + \frac{16}{\gamma}\int_0^u dx\, \frac{\ln x}{1-x^2} \tag{2.32}$$

In a similar way one calculates the soliton - soliton scattering phase shift[27]

$$\delta^{cl}_{S\bar{S}}(u) = \frac{16}{\gamma}\int_0^u dz \,\frac{1}{1-z^2}\ln z \tag{2.33}$$

The first quantum corrections to this classical phase shifts will be calculated in the next section.

V.3 Quantum Corrections to Soliton - Antisoliton and Soliton - Soliton Scattering

In order to compute the first quantum corrections to the classical scattering phase shift we use the result of the previous section For simplicity, we choose to work in the center-of-mass frame. Keeping the quadratic terms in $\chi(x,t)$ and $\varpi(x,t)$ in the action of the path integral expression for the soliton - antisoliton scattering amplitude (2.26) we obtain the following expression for the first quantum correction

$$e^{2i\delta^{(1)}_{S\bar{S}}(u)} = e^{i2T\sqrt{1-u^2}M_1}\int \mathcal{D}\varpi\mathcal{D}\chi \;\delta\left(\int \phi'_{S\bar{S}}\chi\right)\delta\left(\int\dot{\phi}_{S\bar{S}}\chi\right)\cdot$$
$$\cdot\,\delta\left(\int\left(\dot{\phi}'_{S\bar{S}}\chi-\phi'_{S\bar{S}}\varpi\right)\right)\delta\left(\int\left(\ddot{\phi}_{S\bar{S}}\chi-\dot{\phi}_{S\bar{S}}\varpi\right)\right)\exp\left\{iA^{(2)}[\varpi,\chi]\right\} \tag{3.1}$$

where

$$A^{(2)}[\varpi,\chi] = \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}}dt\int_{-L}^{L}dx\left[\varpi(\dot{\chi}-\varpi)-\chi\left(\dot{\varpi}-\chi''+V''_{S\bar{S}}\chi\right)\right]$$

$$V''_{S\bar{S}}(x,t) = \left.\frac{\partial^2 V(\phi)}{\partial\phi^2}\right|_{\phi(x,t)=\phi_{S\bar{S}}(x,t)}$$

$$\phi_{S\bar{S}}(x,t) \equiv \phi_{S\bar{S}}(x,t\,;\,u,-u) \tag{3.2}$$

There is still an additional contribution from the renormalization counterterm. The sine-Gordon model can be renormalized by normal ordering[28]. We use the general relation

$$U(\hat{\phi}(x)) = N_m\left\{\exp\left(\Delta \frac{\partial^2}{\partial \phi^2}\right) U(\hat{\phi}(x))\right\} \tag{3.3}$$

where $\hat{\phi}(x)$ is the field operator. Taking $U(\hat{\phi}) = \hat{\phi}^2$ one finds that

$$\Delta = \frac{1}{2} \sum_{n=-N}^{N}{}' \frac{1}{2L} \frac{1}{2\sqrt{K_n^2+m^2}} \qquad K_n = \frac{2\pi n}{2L} \tag{3.4}$$

From (3.3) it follows that

$$\cos(\sqrt{\gamma}\,\hat{\phi}(x)) = e^{-\gamma\Delta} N_m\left\{\cos(\sqrt{\gamma}\,\hat{\phi}(x))\right\} \tag{3.5}$$

and therefore the Lagrangian density which will give finite results in the standard perturbation sector reads

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi(x,t)\right)^2 + \frac{1}{\gamma} e^{\gamma\Delta}\left[\cos(\sqrt{\gamma}\,\phi(x,t)) - 1\right] \tag{3.6}$$

Due to the invariance under the renormalization group the mass of the field $\hat{\phi}(x)$ which serves to define Δ is arbitrary. It is convenient to take it to be equal to the free mass $m^2=1$. Then

$$\Delta = \frac{1}{2}\frac{1}{2L}\sum_{n=-N}^{N}{}' \frac{1}{2\sqrt{K_n^2+1}} = \frac{1}{8\pi}\int_{-\Lambda}^{\Lambda} dk \frac{1}{\sqrt{k^2+1}} \quad , \quad \Lambda = \frac{2\pi N}{2L} \tag{3.7}$$

After the shift of field variables we now obtain from the renormalization counterterm an additional zeroth order term $\Delta\left[\cos(\sqrt{\gamma}\,\phi_{S\bar{S}}) - 1\right]$ By a straightforward computation we find:

$$\int_{-T/2}^{T/2} dt \int dx \left[\cos(\sqrt{\gamma}\,\phi_{S\bar{S}}(x,t)) - 1\right] \underset{\substack{T\to\infty \\ L\to\infty}}{\simeq} -8\left[T\sqrt{1-u^2} + 2u\ln u\right] \tag{3.8}$$

Now we put the system in the "box" $[-L\ ,\ L]$ imposing periodic boundary conditions. Then as discussed by Dashen Hasslacher and Neveu, for large L, $\phi_{s\bar{s}}$ is to be considered as periodic in time also since its evolution will obviously reproduce itself after the time it takes for each particle to go from one end of the box to the other.[25] The corresponding period $\tilde{T}$ must take into account the time delay due to the interaction which is $\Delta(u)$. It is easily seen that there are two collisions per period $\tilde{T}$ as the two particles get two times to be very close. We thus conclude that $\tilde{T} = \frac{2L}{u} + 2\Delta(u)$. On the other hand, in our scattering theory we want to include only one interaction so we let $T = \frac{\tilde{T}}{2}$ and L and T will be related by

$$L = u\,(T - \Delta(u)) \tag{3.9}$$

We next diagonalize the quadratic action (3.2) using exactly the same method as in the one soliton case. We introduce the complete set of solutions of the equations of motion

$$\begin{aligned} &\dot{\chi}(x,t) - \varpi(x,t) = 0 \\ &\dot{\varpi}(x,t) - \chi''(x,t) + V''_{s\bar{s}}(x,t)\,\chi(x,t) = 0 \end{aligned} \tag{3.10}$$

One can show that the general solution which we denote by $\bar{\chi}^{\epsilon}_{p}$, $\epsilon = \pm 1$ is given by

$$\begin{aligned} \bar{\chi}^{+}_{q}(x,t) = {}& \tfrac{1}{2}\, e^{-i(t\omega(q) - xq)} \Big\{ e^{-\frac{x}{\sqrt{1-u^2}}}\, ch\Big(\tfrac{x}{\sqrt{1-u^2}}\Big)\Big[1 \\ &+ e^{i(\delta_+(q)+\delta_-(q))}\, e^{2\frac{x}{\sqrt{1-u^2}}}\Big] + \tfrac{1}{u^2}\, sh\Big(\tfrac{ut}{\sqrt{1-u^2}}\Big)\Big[e^{i\delta_+(q)}\, e^{\frac{ut}{\sqrt{1-u^2}}} \\ &- e^{i\delta_-(q)}\, e^{-\frac{ut}{\sqrt{1-u^2}}}\Big]\Big\} \cdot \Big\{ ch^2\Big(\tfrac{x}{\sqrt{1-u^2}}\Big) + \tfrac{1}{u^2}\, sh^2\Big(\tfrac{ut}{\sqrt{1-u^2}}\Big)\Big\}^{-1} \\ \bar{\varpi}^{+}_{q}(x,t) = {}& \dot{\bar{\chi}}_{q}(x,t) \end{aligned} \tag{3.11}$$

and

$$\bar{\chi}^{-}_{q}(x,t) = \left[\bar{\chi}^{+}_{q}(x,t)\right]^{*}$$

$$\bar{\varpi}^{-}_{q}(x,t) = \left[\bar{\varpi}^{+}_{q}(x,t)\right]^{*} \tag{3.12}$$

where:

$$e^{i\delta_{\pm}(q)} = \frac{-1 + i q_{\pm}}{1 + i q_{\pm}} \quad , \quad q_{\pm} = \frac{q \pm u\,\omega(q)}{\sqrt{1-u^2}}$$

$$\omega(q) = \sqrt{1+q^2} \tag{3.13}$$

The phase shift is easily found to react·

$$\bar{\delta}(q) = \delta_{+}(q) + \delta_{-}(q) \tag{3.14}$$

so that the density of states is determined from the condition

$$\bar{\delta}(q_n) + 2L q_n = 2\pi n \tag{3.15}$$

The stability angles are then easily found to be. for large T.

$$\bar{\nu}_n = \bar{\nu}(q_n) = 2T\sqrt{1+q_n^2} - \left[\delta_{+}(q_n) - \delta_{-}(q_n)\right] \tag{3.16}$$

The problem of finding a complete set of tunction which diagonalizes the action is exactly the same as in the one soliton case except that we have two constraint conditions instead of one By the same method one obtains the set of functions

$$F^{\epsilon}_{n}(x,t) = \frac{1}{\sqrt{N}}\, e^{i\epsilon\bar{\nu}_n \frac{t}{2T}}\left[\bar{\chi}^{\epsilon}_{q_n}(x,t) - \alpha^{\epsilon}_{n}(t)\,\phi'_{s\bar{s}}(x,t) - \beta^{\epsilon}_{n}(t)\,\dot{\phi}_{s\bar{s}}(x,t)\right]$$

$$G^{\epsilon}_{n}(x,t) = \frac{1}{\sqrt{N}}\, e^{i\epsilon\bar{\nu}_n \frac{t}{2T}}\left[\bar{\varpi}^{\epsilon}_{q_n}(x,t) - \alpha^{\epsilon}_{n}(t)\,\dot{\phi}'_{s\bar{s}}(x,t) - \beta^{\epsilon}_{n}(t)\,\ddot{\phi}_{s\bar{s}}(x,t)\right] \tag{3.17}$$

where α^{ϵ}_{n} and β^{ϵ}_{n} are solutions of the system of equations

$$\int_{-L}^{L} dx\, \phi'_{s\bar{s}} \bar{\chi}^{\epsilon}_{q_n} = \alpha^{\epsilon}_n \int_{-L}^{L} dx\, \phi'^{2}_{s\bar{s}} + \beta^{\epsilon}_n \int_{-L}^{L} dx\, \phi'_{s\bar{s}} \dot{\phi}_{s\bar{s}}$$

$$\int_{-L}^{L} dx\, \dot{\phi}_{s\bar{s}} \bar{\chi}^{\epsilon}_{q_n} = \alpha^{\epsilon}_n \int_{-L}^{L} dx\, \phi'_{s\bar{s}} \dot{\phi}_{s\bar{s}} + \beta^{\epsilon}_n \int_{-L}^{L} dx\, \dot{\phi}^{2}_{s\bar{s}} \tag{3.13}$$

The functions are orthonormal: that is, they satisfy, for large L

$$\int_{-L}^{L} dx \left[(F^{\epsilon_1}_{n_1})^* G^{\epsilon_2}_{n_2} - (G^{\epsilon_1}_{n_1})^* F^{\epsilon_2}_{n_2} \right] = \epsilon_1 \delta_{\epsilon_1 \epsilon_2} \delta_{n_1 n_2} \tag{3.19}$$

if we choose

$$\bar{N} = 4 \left(L\sqrt{1+q_n^2} - \frac{1}{\sqrt{1+q_+^2}} - \frac{1}{\sqrt{1+q_-^2}} \right) \tag{3.20}$$

We now expand χ and ϖ as before in terms of these eigenfunctions

$$\chi(x,t) = \sum_{n,\epsilon}' a^{\epsilon}_n(t)\, F^{\epsilon}_n(x,t)$$

$$\varpi(x,t) = \sum_{n,\epsilon}' a^{\epsilon}_n(t)\, G^{\epsilon}_n(x,t) \tag{3.21}$$

The quadratic part of the action will again correspond to decoupled harmonic oscillators. Indeed, one finds

$$A^{(2)}[\varpi,\chi] = \frac{1}{2} \int dx \int dt \left[\varpi(\dot{\chi} - \varpi) - \chi(\dot{\varpi} - \chi'' + V''_{s\bar{s}}\chi) \right] =$$

$$= \int dt \sum_n' \left\{ \frac{1}{2i} \left[(\dot{a}^+_n(t))^* a^+_n(t) - (a^+_n)^* \dot{a}^+_n(t) \right] - \frac{\nu_n}{2T} (a^+_n)^* a^+_n \right\} \tag{3.22}$$

The first quantum correction to the phase shift is now given by

$$2\, \delta^{(1)}_{s\bar{s}}(u) = \lim_{\substack{T\to\infty \\ L\to\infty}} \left\{ -\frac{1}{2} \left(\sum_n' \nu(q_n) - T \sum_n' \omega(k_n) \right) + \right.$$

$$\left. + \Delta \int dt\, dx \left[\cos(\sqrt{\gamma}\, \phi_{s\bar{s}}(x,t)) - 1 \right] + 2TM_1 \sqrt{1-u^2} \right\} \tag{3.23}$$

In order to calculate the sum of angles we define the phase shift $\bar{\delta}(q)$ in such a way that it tends to zero for $q \to \pm\infty$. An easy computation then shows that

$$\bar{\delta}(q) = \begin{cases} 2(\pi - 2\tan^{-1} q\sqrt{1-u^2} & q > 0 \\ 2(-\pi - 2\tan^{-1} q\sqrt{1-u^2} & q < 0 \end{cases} \tag{3.24}$$

and then, $\bar{\delta}(0^{\pm}) = \pm 2\pi$ which is in agreement with Levinson theorem as there are two bound states $\grave{\phi}_{s\bar{s}}$ and $\dot{\phi}_{s\bar{s}}$. The equation (3.15) thus leads to $q_1 = q_{-1} = 0$. Therefore, for the scattering states which are the only ones to be kept in the sum over stability angles, $n = 0$ does not appear and only one of the $n = \pm 1$ contributions should be counted as they both correspond to the same state at threshold. We then consider:

$$\frac{T}{2}\sum_n{}' \frac{\bar{\nu}(q_n)}{2T} - \frac{T}{2}\sum_n{}' \omega(k_n) = -\frac{1}{4}\left[\delta_+(0) - \delta_-(0)\right] - T\,\omega(k_1)$$
$$+\left(\sum_{n=2}^{N}{}' + \sum_{n=-N}^{-2}{}'\right)\left[\frac{T}{2}\left(\omega(q_n) - \omega(k_n)\right) - \frac{1}{4}\left(\delta_+(q_n) - \delta_-(q_n)\right)\right] \tag{3.25}$$

where:

$$\delta_+(q) - \delta_-(q) = -4\tan^{-1}\left(\frac{u}{\sqrt{1-u^2}}\frac{1}{\sqrt{1+q^2}}\right) \tag{3.26}$$

and we have put a cut-off in momenta. Since $\bar{\delta}(q)$ is finite, $q_n - \frac{2\pi n}{2L}$ is very small for large L, we can next expand $\omega(q_n) - \omega(k_n)$ making use of relation (3.15) and write:

$$\omega(q_n) - \omega(k_n) \approx -\frac{1}{2L}\frac{d\omega(k_n)}{dk_n}\bar{\delta}(k_n) + \frac{1}{8L^2}\frac{d}{dk_n}\left[\bar{\delta}^2(k_n)\frac{d\omega(k_n)}{dk_n}\right] \tag{3.27}$$

Note that we have to keep terms of order L^{-2} since we want to determine the contribution to (3.25) which tends to a constant for large T. The discrete sum is approximated by an integral through the general formula:

$$\sum_{n_1}^{n_2} f(n) \simeq \int_{n_1-1}^{n_2} dn\, f(n) + \frac{1}{2}\left[f(n_2) - f(n_1) \right]$$
(3 28)

After integrating by parts we obtain

$$\frac{T}{2}\left(\sum_{n=2}^{N}{}' + \sum_{n=-N}^{-2}{}' \right)\left[\omega(q_n) - \omega(k_n) \right] \simeq T\left(1 + 2\cdot\frac{(-1/\pi)}{\sqrt{1-u^2}} + \right.$$
$$\left. + \frac{1}{4\pi}\int_{-\Lambda}^{\Lambda} dk\, \omega(k)\, \frac{d\bar{\delta}(k)}{dk} \right)$$
(3.29)

In the same way, the sum $\delta_+ - \delta_-$ is approximated by an integral in which we make a partial integration obtaining

$$\left(\sum_{n=2}^{N}{}' + \sum_{n=-N}^{-2}{}' \right)\left[\delta_+(q_n) - \delta_-(q_n) \right] \simeq 2\tan^{-1}\left(\frac{u}{\sqrt{1-u^2}} \right) - \frac{2L}{8\pi}\,\frac{8u}{\sqrt{1-u^2}} -$$
$$- \frac{2L}{2\pi}\int_{-\Lambda}^{\Lambda} dk\, k\left[\frac{d\delta_+(k)}{dk} - \frac{d\delta_-(k)}{dk} \right] + \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} dk\, \frac{d\bar{\delta}(k)}{dk}\left[\delta_+(k) - \delta_-(k) \right]$$
(3 30)

Formulae (3.29) and (3.30) both contain logarithmically divergent parts which will precisely be cancelled by the contribution coming from the renormalization counterterm (3.8).

The total expression for the quantum correction now reads:

$$2\,\delta^{(1)}_{SS}(u) = \lim_{\substack{T\to\infty \\ L\to\infty}} \left\{ T\left(\frac{2}{\pi}\frac{1}{\sqrt{1-u^2}} - \frac{1}{4\pi}\int_{-\Lambda}^{\Lambda} dk\, \omega(k)\, \frac{d\bar{\delta}(k)}{dk} \right) - \right.$$
$$- L\left[\frac{2}{\pi}\frac{u}{\sqrt{1-u^2}} + \frac{1}{4\pi}\int_{-\Lambda}^{\Lambda} dk\, k\left(\frac{d\delta_+}{dk} - \frac{d\delta_-}{dk} \right) \right] + \frac{1}{8\pi}\int_{-\Lambda}^{\Lambda} dk\, \frac{d\bar{\delta}(k)}{dk}\left(\delta_+(k) - \delta_-(k) \right)$$
$$\left. + 2TM_1\sqrt{1-u^2} - \frac{1}{\pi}\left(T\sqrt{1-u^2} + 2u\ln u \right)\int_{-\Lambda}^{\Lambda} dk\, \frac{1}{\omega(k)} \right\}$$
(3 31)

Letting $L = uT - \Delta(u)$ we get the potentially divergent terms as

$$-\frac{T}{4\pi}\int_{-\Lambda}^{\Lambda} dk \left\{ \omega(k) \frac{d\bar{\delta}(k)}{dk} + k\left(\frac{d\delta_+(k)}{dk} - \frac{d\delta_-(k)}{dk} \right) + \frac{4\sqrt{1-u^2}}{\omega(k)} \right\} = 0 \qquad (3.32)$$

and

$$\frac{\Delta(u)}{4\pi}\int_{-\Lambda}^{\Lambda} dk\, k \left(\frac{d\delta_+}{dk} - \frac{d\delta_-(k)}{dk} \right) - \frac{2u}{\pi}\ln u \int_{-\Lambda}^{\Lambda} \frac{dk}{\omega(k)} = -\frac{2}{\pi} \ln u \ln\left(\frac{1+u}{1-u} \right) \qquad (3.33)$$

The finite terms linear in T cancel and we finally obtain the following expression free of divergences:

$$2\,\delta^{(1)}_{s\bar{s}}(u) = \frac{1}{8\pi}\int_{-\Lambda}^{\Lambda} dk\, \frac{d\bar{\delta}(k)}{dk}\left(\delta_+(k) - \delta_-(k) \right) - \frac{2}{\pi}\ln u \ln\left(\frac{1+u}{1-u} \right) + \frac{4u}{\pi}\ln u \qquad (3.34)$$

As the final step we evaluate the integral

$$I(u) = \int_{-\infty}^{\infty} dk\, \frac{d\bar{\delta}(k)}{dk} \left[\delta_+(k) - \delta_-(k) \right] \qquad (3.35)$$

which appears in this formula. We use

$$\frac{d}{dk}\bar{\delta}(k) = -4 \frac{d}{dk} \tan^{-1}\left(k\sqrt{1-u^2} \right) \qquad (3.36)$$

and

$$\delta_+(k) - \delta_-(k) = -4 \tan^{-1}\left(\frac{u}{\sqrt{1-u^2}} \frac{1}{\sqrt{1+k^2}} \right) \qquad (3.37)$$

Then, after partial integration and change of variables

$$z = \tan^{-1}\left(k\sqrt{1-u^2} \right) \qquad (3.33)$$

we get

$$I(u) = 32u \int_0^{\frac{\pi}{2}} dz\, \frac{z \sin z}{\left(1 - u^2\cos^2 z\right)^{1/2}} \qquad (3.39)$$

Next, differentiating with respect to u we obtain

$$\frac{\partial I(u)}{\partial u} = 32 \int_0^{\frac{\pi}{2}} dz \frac{z \sin z}{(1 - u^2 \cos^2 z)^{3/2}} = 16 \frac{1}{u} \ln \frac{1+u}{1-u} \tag{3.40}$$

and then using the fact that $I(0)=0$ the result reads

$$I(u) = 16 \ln u \ln \left(\frac{1+u}{1-u} \right) - 32 \int_0^u dz \frac{\ln z}{1 - z^2} \tag{3.41}$$

We then obtain the following result for the first quantum correction to the soliton - antisoliton scattering phase shift:

$$2 \delta^{(1)}_{s\bar{s}}(u) = - \frac{4}{\pi} \int_0^u \frac{\ln z \, dz}{1 - z^2} + \frac{4u}{\pi} \ln u \tag{3.42}$$

One can show that in the case of soliton - soliton scattering, exactly the same sum over stability angle appears so we have

$$\delta^{(1)}_{ss}(u) = \delta^{(1)}_{s\bar{s}}(u) \tag{3.43}$$

We next compare our results with a very interesting conjecture due to Faddeev, Kulish and Korepin concerning the exact soliton - antisoliton scattering amplitude[11]. They wrote down the "exact" scattering amplitude in such a way that it has the correct bound state masses as poles. We first modify their expression substituting in their formula γ by $\gamma' = \frac{1}{1 + 8\pi/\gamma}$ so that the bound state masses agree with the result found by Dashen, Hasslacher and Neveu using the semi-classical WKB method[25]. Furthermore, multiplying their expression by a constant phase factor the conjectured exact soliton - antisoliton scattering amplitude reads:

$$S_{s\bar{s}} = (2\pi)^2 \delta(E_f - E_i)\delta(p_f - p_i) \prod_{n=1}^{N-1} \left\{ - \frac{\bar{\xi} + e^{i\theta_n}}{1 + \bar{\xi} e^{i\theta_n}} \right\}$$

$$\theta_n = \frac{\pi n}{N}$$

$$\bar{\xi} = \frac{s - 2M^2 + \sqrt{s(s - 4M^2)}}{2M^2} \tag{3.44}$$

where

$$s = (E_1 + E_2)^2 - (p_1 + p_2)^2$$

$$M = M_0 + M_1 = \frac{8}{\gamma'}$$

$$N = \frac{8\pi}{\gamma'} \tag{3.45}$$

Here it is assumed that the coupling constant γ' is such that N is an integer.

We will now show that our result agrees with this formula. Namely, for small γ we expand the right-hand side of (3.44) using the relation (3.28). We find that

$$2\delta_{s\bar{s}}(u) = \frac{8}{\gamma}\left(1 - \frac{\gamma}{8\pi}\right)\int_0^\pi d\theta \ln\left(\frac{\bar{\xi} + e^{i\theta}}{1 + \bar{\xi} e^{i\theta}}\right) + \frac{8\pi^2}{\gamma} + \frac{\pi}{2} \tag{3.46}$$

It is easy to show that

$$\int_0^\pi d\theta \ln\left(\frac{\bar{\xi} + e^{i\theta}}{1 + \bar{\xi} e^{i\theta}}\right) = \frac{2}{i}\int_1^{\bar{\xi}} \frac{dx}{x} \ln\left(\frac{x+1}{x-1}\right) \tag{3.47}$$

Recalling that u was defined by $p = \frac{M_0 u}{\sqrt{1-u^2}}$ we expand

$$\bar{\xi} \simeq \frac{1+u}{1-u} + \frac{\gamma}{4\pi} u \cdot \frac{1+u}{1-u} \tag{3.43}$$

Inserting (3.47) into (3.46) and making the change of variable we get the expression

$$2\,\delta_{S\bar{S}}(u) \simeq \frac{32}{\gamma}\left(1-\frac{\gamma}{8\pi}\right)\int_0^u dz\,\frac{\ln z}{1-z^2} + \frac{4u}{\pi}\ln u + \frac{8\pi^2}{\gamma} \qquad (3.49)$$

It is exactly the sum of the classical soliton - antisoliton phase shift (2.32) and the first quantum correction given by (3.42).

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