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RATE OF APPROACH TO MINIMA AND SINKS

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RATE OF APPROACH TO MINIMA AND SINKS

by

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I Dedicate This Dissertation to My Parents

Katherine Godlewski - Wisniewski

and

Julius George Wisniewski

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Helena S. Wisnieski
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INTRODUCTION

The dynamical systems herein are diffeomorphisms and flows on compact manifolds of finite dimension. Determining the asymptotic rate of approach to sinks amounts to comparing the Riemannian measure of the entire manifold, which we take normalized to be one, to the measure of the set of points whose orbits remain outside a neighborhood of the sinks after N iterations for diffeomorphisms, or time T for flows. For Axiom A systems this rate is bounded by expression of the form: $K \exp(-CN)$.

As an example of the results obtained, consider a gradient system: $dx(t)/dt = \text{grad } F(x)$ defined on a compact manifold M, with $F : M \to R$ a Morse function. Let $\phi_t(x)$ be the flow corresponding to the integral curves of this system. Let $f(x) = \phi_1(x)$ be the time one diffeomorphism.

Then for the gradient system we can state the following:

<u>Corollary:</u> For S = the set of initial points whose orbits under f remain outside a neighborhood of the sinks of f after N iterations, we get:

$\mu(S) \leq K \exp(-DN)$, with K,D constants greater than zero.

For Morse - Smale diffeomorphisms we find that the number D in the preceeding Corollary, may be any number smaller than :

$$C = \min_{P} \{ 1/m \log \operatorname{Jac} D_P f^m \mid W^u (P) \} ,$$

where the minimum is taken over all non-sink P in the in the non-wandering set of f, and m is the least integer such that $f^m P = P$.

More Specifically we get the following Theorem :

Theorem 1.2.1 : Let f be a Morse-Smale diffeomorphism on the C^∞ compact manifold M, with $\Omega(f)$ consisting only of fixed points. Let V be a neighborhood of the sinks of f. Let μ be the measure induced by the Riemannian metric on M. Then given $\delta > 0$, and $P \in \Omega(f)$, not a sink there is a neighborhood U of P such that for :

$$U_N = \{ x \in U : f^k (x) \notin V \text{ for } k \leq N \} ,$$

we get the following:

$$\mu (U_N) \leq K (1 + \delta)^N \exp (- CN), \text{ where } K > 0,$$

$$\text{and } C = \log \min_{P_i} \{ \operatorname{Jac} D_{P_i} (f \mid W^u (P_i))\} , \text{ for}$$

non-sink fixed points of f, P_i.

This theorem is proved in Section 2, of Chapter 1.

In Section 3 we extend this theorem to the entire manifold M, so that there is no restriction on the location of the initial points of trajectories. We accomplish this by using a fine filtration adapted to f. We also prove that the results carry over to the case of Morse-Smale flows.

In Chapter 2 we prove the result for C^2 Axiom A diffeomorphisms and flows with no cycles. We begin Chapter 2 by constructing an example without the transversality condition, that shows the exponential constant can actually be less than the Jacobian of the unstable part of the periodic orbits.

The determination of the exponential constants, in the general case, is more complicated as there is a slower overall rate of decay.

The following is a special case of Theorem 2.2.2 for C^2 Axiom A diffeomorphisms with no cycles.

Theorem: Let M be a compact C^∞ Riemannian manifold. Let μ be the measure induced by the Riemannian metric on M. Let f be a C^2 Axiom A diffeomorphism with no cycles. Let V be an open set in M which:

a) contains all sinks and attractors for f

b) satisfies $f(V) \subset V$

Then defining: $U_N = \{ x \in M : f^k(x) \not\in V \text{ for } k \leq N\}$

we get for all $\delta > 0$ there exists $K = K(\delta)$, and

$C > 0$ such that:

$$\mu(U_N) \leq K \exp(-C(1-\delta)N).$$

The precise form of the constant C will be given in section 2.

It should be noted that these theorems apply as long as we know a volume estimate for the basic sets. For Morse-Smale we know such an estimate when f is C^1; however, for the general Axiom A case we need f to be C^2. These points will be discussed further in Chapter 2 - Section 4. Also Chapter 2 - Section 4 contains a discussion of related cases and some open questions.

CHAPTER 1

Morse - Smale Diffeomorphisms and Flows

§1 Background

Let M be a compact C^{∞} Riemannian manifold. For $r \geq 1$, let Diff(M) be the set of C^r diffeomorphisms of M.

Definition: Let $f \in$ Diff (M) and let $x \in$ M. x is a non-wandering point of f, $x \in \Omega(f)$, if for every open neighborhood U of x there exists an $n \in Z - \{ 0 \}$ such that $f^n(U) \cap U \neq \emptyset$.

Consider $f \in$ Diff(M) which satisfies:

(a) $\Omega(f)$ is finite

(b) $\Omega(f)$ is hyperbolic

(c) Transversality condition

Condition (a) implies that $\Omega = \Omega(f)$ consists of periodic points. Then (b) means that if $x \in \Omega$ and $f^m(x) = x$, for some $m > 0$, then the derivative $D f^m(x) : T_x(M) \to T_x(M)$ has its eigenvalues not equal to one in absolute value. From condition (b) one has defined stable and unstable manifolds for each $x \in \Omega$ denoted by $W^s(x)$, $W^u(x)$ respectively.

Then condition (c) means that for each $x, y \in \Omega$, $W^s(x)$, $W^u(y)$ have transversal intersection in M.

A diffeomorphism which satisfies conditions (a) through (c) above is called Morse - Smale, [6]

As mentioned in the introduction, the problem herein considered is the rate of approach of orbits (trajectories) under f (ϕ_t), to the sinks (attractors) on M. To make this idea of rate precise, we cast the problem in terms of volumes. Let P be a fixed point for f which is either a source or a saddle and let U be a neighborhood of P on M. If $x \in U$, then unless x is on the stable manifold of P, the orbit of P under f will leave U. This is simply the familiar fact from the stable manifold theorem that for U small enough

$\bigcap_{n=0}^{\infty} f^n (U)$ is the local stable manifold of P. The volume lemmas of Bowen-Ruelle [2] and Fried - Shub [5] add to this statement that the volume of $\bigcap_{n=0}^{N} f^n (U_N)$ decays exponentially with the rate, by this we mean the exponential constant being related to the logarithm of the Jacobian determinant of the unstable part of f at P.

Now these results are local in nature; they only concern the rate at which orbits leave a neighborhood of P. The related global question is the following: Let V be a neighborhood of the sinks of f. Let U_n be the subset of U whose orbits remain outside of V through iteration N (by at least N); more precisely we let:

$$U_N = \{ x \in U : f^k(x) \notin V \text{ for } k \leq N \}.$$

The global question is then: Does vol (U_N) decay exponentially with N ?. We answer this question in the affirmative, with the following theorem.

Theorem 1.2.1: Let f be a Morse-Smale Diffeomorphism on the C^∞ compact manifold M, with $\Omega(f)$ consisting only of fixed points. Let V be a neighborhood of the sinks of f. Let μ be the measure induced by the Riemannian metric on M. Given $\delta > 0$, and $P \in \Omega(f)$, not a sink, there is a neighborhood U of P such that, with U_N defined as above, we get the following:

$$\mu (U_N) \leq K (1 + \delta)^N \exp (- C N),$$ where $K > 0$, and $\exp (C) = \min_P \text{Jac } D f^u_P$, the Jacobian determinant of the unstable part of f at P, where P are the non-sink fixed points of f.

This theorem is proved in Section 2, of this chapter.

In section 3 we extend this theorem to the entire manifold M, i.e. there is no restriction on the location of the initial points of trajectories. We accomplish this extension by the use of a fine filtration adapted to f. If we take $U = M - \mathrm{cl}\, V$ and U_N defined as before, the filtration argument shows that there exists a fixed iteration N^* such that if $x \in U$ and $f^N(x) \not\in V$, for $N \geqq N^*$, then $\Theta(x)$ intersects some neighborhood U_P of a fixed point P. By this fact we are able to reduce the volume estimate of U_N to the local estimates of $U_N(P)$.

Specifically we prove Theorem 1.2.1 of the introduction, i.e. $\mu\, (U_N) \leqq K\, (1 + \delta)^N \exp\, (-C\, N)$ with $K > 0$, and $\exp\, (C)$ as before. The theorem states that orbits under f are pushed into the sinks at an exponential rate depending only on the Jacobian determinants of the unstable parts of f at the sources and saddles.

We conclude section 3 by dropping the restriction that $\Omega(f)$ consist only of fixed points and thus prove the theorem for all Morse - Smale diffeomorphisms of M.

In section 4 we consider Morse - Smale flows on M. We now define the analogue of Morse - Smale diffeomorphisms for flows. Let X be a vector field on M, and denote by ϕ_t its induced flow. We say ϕ_t is a Morse - Smale flow if X satisfies the following conditions: [6]

(a) $\Omega(X)$ is the union of a finite number of fixed points $x_1, x_2, \ldots, x_m$ and a finite number of closed orbits $\gamma_1, \gamma_2, \ldots, \gamma_n$ of X.

(b) the x_i, γ_j are all hyperbolic

(c) the stable and unstable manifolds of the x_i, γ_j have transversal intersection.

We conclude this chapter by giving an example of a Morse - Smale diffeomorphism whose rate of volume decay is exactly that given in the theorem, i.e. $K \exp(-CN)$, with $\exp(C) = \min \{\mathrm{Jac}\, D_p f^u\}$ this is done in section 5.

§2 The Fixed Point Case

Throughout this section let f be a Morse-Smale diffeomorphism, and let $\Omega(f)$ consist only of fixed points. We prove the volume theorem for f in two stages. First we consider submanifolds of M in a neighborhood $U(P)$ of $P \in \Omega(f)$ which are of the same dimension as $W^u(P)$ and C^1 close to $W^u(P)$, showing that the volume of these submanifolds in $U_N(P)$ decays exponentially. Secondly we cover $U_N(P)$ by a C^1 system of such manifolds and use Fubini's Theorem to arrive at our result for $U_N(P)$.

Let us begin by precisely defining our neighborhoods $U(P)$. For $P \in \Omega(f)$ we know that $T_P M$ splits into the direct sum decomposition $T_P M = E^u + E^s$ where E^u (resp. E^s) is the subspace of $T_P M$ expanded (contracted) by $D f(P)$. For small ε we can find a C^1 chart $\phi : T_P M \to M$ such that :

1) $\phi(0) = P$, and $D_0 \phi : T_P M \to T_P M$ is the identity

2) $\phi(E^s(\varepsilon)) \subset W^s P$ and

3) $\phi(E^u(\varepsilon)) \subset W^u P$.

The map $F = \phi^{-1} \circ f \circ \phi$ represents f in this coordinate system. [5]

Such a chart can be chosen for each element of of $\Omega(f)$ and we assume that ε suffices for all $p \in \Omega(f)$.

The non-wandering set of f can be partially ordered by the relation: $p \leq q$ if and only if $W^u(q) \cap W^s(p) \neq \emptyset$. We choose a simple ordering consistent with this partial order and index the non-wandering set according to the simple order. That is take: $\Omega(f) = \{ p_1, \ldots, p_k \}$ where $i < j$ implies that $p_i \not> p_j$. We assume that p_1 is the only sink in order to simplify the proofs (modifying the proofs to account for more than one sink is merely a matter of notation).

In the following the slope of a smooth submanifold N of $T_i M(\varepsilon)$ with $\dim(N) \leq \dim(E^u)$ at a point $n \in N$ refers to the supremum of the quotients

$||v^s|| \,/\, ||v^u||$, $||v^u|| \neq 0$, taken over all tangent vectors $v = (v^u, v^s)$ to N at n. To say that N has slope less than or equal to ω relative to E^u_i means that for all $n \in N$, the slope of N at n is less than or equal to ω.

We are now ready to state the following Proposition:

<u>Proposition</u>: 1.2.1: For $i = 2,3, \ldots, r$ there exist constants ε_i, ω_i such that given $\delta > 0$ the following hold:

1) $\varepsilon_i \leq \varepsilon$, $0 < \omega_i \leq \omega < 1$

2) $\omega_i \leq \omega_j / 3$ for $j < i$

3) If L Hom (E^u, E^s), with $\| DL \| \leq \omega_i$, and $v \in T_{p_i} M(\varepsilon_i)$ then :

   (*) Jac D_v $(F|v + \text{graph } L) \geq$ Jac D_o $(F|E^u) / (1 + \delta)$.

4) The graph transform Γ_i defined on C^1 functions g from $E^u_i(\varepsilon_i)$ to $E^s_i(\varepsilon_i)$ by graph $(\Gamma_i(g)) = F_i(\text{graph}(g))$ where F_i is $\phi_i^{-1} \circ f \circ \phi_i$ is such that if the slope of $g \leq \omega_i$ then the slope of $\Gamma_i(g) \leq \omega$

   In this context the slope refers to the slope of the graph.

5) $T_iM(\varepsilon_i)$ is contained in a neighborhood V_i of P_i in T_iM, such that for all $P_k \in \Omega(f)$ either $W^u_k \cap W^s_i = \emptyset$ or $[\phi_i^{-1}(W^u_k) \cap T_iM(\varepsilon_i)]$

   C^1 fibers over the intersection of $\phi_i^{-1}(W^u_k)$ with E^s_i in V_i

with each fiber of dimension no-greater than dim (W_i^u) and slope no greater than $\omega_i / 3$ relative to E_i^u .

6) For $i > j$, if $W_i^u \cap W_j^s \neq \emptyset$, then given a C^1 function of $E_i^u(\varepsilon)$ to $E_i^s(\varepsilon_i)$, with slope $\leq \omega_i$ whenever $f^N \phi_i$ (graph g) $\cap B_j(\varepsilon_j) = \emptyset$,

then:

$\phi_j^{-1} f^N \phi_i$ (graph g), C^1 fibers over $T W_i^u \cap E_j^s$ with the fibers having dimension less than or equal to dim W_j^u and slope less than or equal to ω_j relative to W_j^u.

<u>Proof</u> : The proof is by induction. Take $\bar{\varepsilon}_2 \leq \varepsilon$ and $0 < \omega_2 \leq \omega$ so that (*) holds in $T_{P_2}M(\bar{\varepsilon}_2)$. Furthermore take $\varepsilon_2' \leq \bar{\varepsilon}_2$ so that if $g : E_2^u(\varepsilon_2') \to E_2^s(\varepsilon_2')$ is a C^1 function with slope $\leq \omega_2$ then the graph transform of g by f, $\Gamma_{P_2} g$, [5] , also has slope $\leq \omega_2$. This satisfies (1) through (4) of the proposition , for i = 2. To do (5) we use a result of Palis (Lemma 1.11 of [6]) :

Lemma 1.11 [6]. Let f be a Morse - Smale diffeomorphism of M and let $P \in \Omega(f)$ have unstable index (= dimension) r. Fix a cell neighborhood B^r of P in LU(P). There exists a neighborhood V of P, $V = \delta I^{m-r} \times B^r$ such that $W^u(P) \cap V = \emptyset$, or $W^u(P_i) \cap V$ is an r-cell fiber bundle over $W^u(P_i) \cap V \cap LS(P)$ with the fibers C^1 close to B^r. $(P_i \in \Omega(f))$.

Note: LU(P) is the local unstable manifold of P.

We remark that although this lemma does not necessarily guarantee that the fiber bundle is a C^1 fiber bundle, the following lemma (proved in Appendix 1) insures that the fibering can be chosen to be C^1.

Lemma 1.2.1 : Let E_1, E_2, E_3 be finite dimensional Euclidian Spaces and $W \subset E_3$ an open set. Let $\dim(E_1) < \dim(E_3) < \dim(E_1 \times E_2)$ and let $G : W \to (E_1 \times E_2)$ be a diffeomorphism of W onto its range G(W). Furthermore let G(W) have transversal intersection with E_2. Then there is an open set $W^1 \subset W$ containing $G^{-1}(E_2)$ such that $G(W')$ is a C^1 fiber bundle over

$G(W') \cap E_2$ with fibers transversal to E_2 and having dimension equal to $\dim E_1$.

We now resume the Proof of the Theorem. By the preceeding Lemmas we can take $\varepsilon_2 \leq \varepsilon_1'$ so that $T_{P_2}M(\varepsilon_2)$ is contained within the neighborhood V of P_2 guaranteed by Palis's Lemma 1.11. Thus if

$\phi(W_k^u) \cap T_{P_2}M(\varepsilon_2)$ is not empty we get that it C^1 fibers over its intersection with E_2^s, with all of its fibers having slope no greater than $\omega_2/3$. (Note by taking $T_{P_2}M(\varepsilon_2)$ properly contained in V we can apply our Lemma 1.2.1 to get the C^1 fiber structure). Thus (5) is satisfied, and this completes the proof for $i = 2$, since P_i is the sink.

Now assume that ε_j, ω_j have been chosen as to satisfy (1) through (6) for $j = 1, \ldots, i-1 < r$. We show how to choose ε_i and ω_i.

First take $\bar{\varepsilon}_i \leq \varepsilon$ and $0 < \bar{\omega}_i \leq \omega < 1$ so that the Jacobian inequality (*) holds in $T_i M(\bar{\varepsilon}_i)$. Then take $\omega_i \leq \bar{\omega}_i$ and $\omega_i \leq \omega_{i-1}/3$ (this implies $\omega_i \leq \omega_j/3$ for $j < i$). Next choose $\varepsilon_i' \leq \varepsilon_i$ so that the graph transform Γ_i

satisfies $||Dg|| \leq \omega_i$ implies
$||D_i(g)|| \leq \omega_i$ for C^1 $g : E^u(\varepsilon'_i) \to E^s(\varepsilon'_i)$.

For the next step observe that if any orbit of f leaves the neighborhood $\phi_i(T_iM(\varepsilon'_i))$ of P_i on M then there is a bounded number of iterations N_i by which:

1) the orbit is in a neighborhood of the sink P_1 or
2) the orbit has entered (and possibly left) one of the neighborhoods $\phi_j(T_jM(\varepsilon_j))$.
Thus we suppose that N is a C^1 submanifold of $T_i M(\varepsilon'_i)$ and that for some $k \leq N_i$
$f^k \phi_i N \cap B_j(\varepsilon_j) \neq \emptyset$.

By taking N in a tubular neighborhood of $E^u(\varepsilon'_i$ we get that $\phi_j^{-1} f^k \phi_i N \cap T_jM(\varepsilon_j)$ inherits a C^1 fiber structure over its intersection with E^s_j in V_j from $\phi_j^{-1} W^u_i$. If we now choose the width of the tubular neighborhood of $E^u(\varepsilon'_i)$ small enough, say ε^*_i, we can insure since $k \leq N_i$, that the slope of the fibers of $\phi_j^{-1} f^k \phi_i N - \phi_j^{-1}(W^u_i)$ relative to E^u_j is no more than ω_i larger than the slope of N relative to E^u_i. Because we are assuming via the induction hypothesis that the slope of ϕ_j^{-1} (W_i^u) relative to E_j^u is not larger than $\omega_j/3$, the slope of the fibers of $\phi_j^{-1} f^k \phi_i N$, relative to E_j^u is bounded by $\omega_j/3 + \omega_i$ + (slope N relative E_i^u). Beginning with an N with slope $\leq \omega_i$, relative to E_i^u, gives that the slope of the fibers of $\phi_j^{-1} f^k \phi_i N$, relative to E_j^u, is less than or equal to: $\omega_j/3 + 2\omega_i \leq \omega_j$. Using ε_i^* for the neighborhood of P_i in T_iM satisfies (6). So we only now need to prove (5). But here we argue as for $i = 2$ to get $\varepsilon_i \leq \varepsilon_i^*$ so that (5) is satisfied.

Thus these ε_i, ω_i have been chosen to as satisfy (1) through (6), and the proof of Proposition 1.2.1 is complete.

We are now ready to prove the main result of this section.

<u>Theorem 1.2.1</u> : Let $\delta > 0$ be given and let V be a neighborhood of the sink P. Then for $i = 2,\dots,r$ there is a compact neighborhood $U(P_i)$ such that with μ a measure derived from the Riemannian metric and : $U_N(P_i) = \{ x \in U(P_i) : f^k(x) \notin V$ for $k = 0,\dots, N \}$. Then :
$\mu(U_N(P_i)) \leqq K(1+\delta)^N \exp(-CN)$, where $K > 0$ is independent of N and
$\exp(C) = \min_{i=2,\dots,r} \{ \operatorname{Jac} D f^u(P_i) \}$.

<u>Proof</u> : Given $\delta > 0$, let $\delta_1 < \delta$. Then we can find for all $i = 2,\dots, r$ a pair $(\varepsilon_i, \omega_i)$ which satisfy (1) through (6) of proposition 1.2.1 with δ'
We take $\varepsilon > 0$ such that $\varepsilon < \min \varepsilon_i$ and consider the compact neighborhood of 0 in $T_i M$, $\mathrm{cl}\, T_i M(\varepsilon) = \mathrm{cl}\,(E_i^u(\varepsilon) + E_i^s(\varepsilon))$. Then take
$U(P_i) = \phi_i(\mathrm{cl}\, T_i M(\varepsilon))$. These are our neighbor - hoods of the fixed points. Now fix i and let

$$S_N = \phi_i^{-1}(U_N(P_i)) \quad ;$$

$S_N \subset \mathrm{cl}\, T_i M(\varepsilon)$. We prove that :

$$\nu(S_N) \leqq K(1+\delta)^N \exp(-CN), \text{ where}$$

ν is the ordinary Lebesque measure in $T_i M$.
Passing back to the measure μ on M involves only mul - tiplication by constants bounded away from 0 and ∞ .

For $v \in \mathrm{cl}\, E_i^s(\varepsilon)$ consider the linear variety :

$H_v = \{ (u,v) \in T_i M : u \in E_i^u \}$. We consider the subgraphs $S_N \cap H_v$ for $v \in \mathrm{cl}\, E_i^s(\varepsilon)$. These are compact sets which cover S_N in the sense that :

$$S_N = S_N \cap \left[\bigcup_{v \in E_i^s(\varepsilon)} H_v \right] .$$

We now appeal to the following Lemma in order to get an area estimate for the subgraphs $(S_N \cap H_v)$.

Lemma 1.2.2 : For $i \geqq 1$ if $g : \mathrm{cl}\, E_i^u(\varepsilon) \to \mathrm{cl} E_i^s(\varepsilon)$ is a C^1 function with slope no greater than ω_i , then : area (graph $g \cap S_N$) $\leqq K (1 + \delta)^N \exp(-C N)$, where $K > 0$, $C = \min_{i \geqq j > 1} \{\mathrm{Jac}\, D f^u (P_j) \}$, and area (.) refers to r dimensional measure along graph g , where $r = \dim (E^u)$.

This is proved in detail in the appendix. However we shall give indication of the steps here. If we consider the set (graph $g \cap S_N$) , then this decomposes into the finite union (not disjoint) of compact subgraphs : $\bigcup_{m=0}^{N} S(g,m)$ whose images are in $U(P_i)$ for exactly m iterations. For those m iterations we may estimate the change in area ($F^m (S (g,m))$) via the Jacobian inequality (*) . This inequality is of the form required to produce the estimate :

area ($S (g,m)$) $\leqq K (1 + \delta')^m \exp (-C_i m)$,

where $C_i = \mathrm{Jac}\, D f^u (P_i)$. Further, for iterates larger than m , iterations of $S(g,m)$ or

$f^k \phi_i F^m S(g,m)$ for $k > 0$, pass within a bounded number of iterations to other $U(P_j)$. Because of Proposition 1.2.1 these iterates fiber into a C^1 system of C^1 submanifolds (graphs) with slopes $\leqq \omega_j$. Thus the same Jacobian estimation technique can be applied so that the area of these iterates of $S(g,m)$ can be estimates. Then translating all these area estimates back to $U(P_i)$ we get :

$$\text{area}\,(S(g,m)) \leqq K(1+\delta')^N \exp(-CN)$$

The number of $S(g,m)$ grows algebraically with N but since $\delta' < \delta$ we can redefine K so that :

$$\text{area}\,(\text{graph } g \cap S_N) \leqq K(1+\delta)^N \exp(-CN).$$

Now applying the Lemma we continue the proof, of the Theorem. Since $S_N \cap H_v$ is contained in a linear variety, area (.), here denoted ν^u is an r dimensional Lesbeque measure. Thus we get :

$\nu(S_N \cap H_v) \leqq K(1+\delta)^N \exp(-CN)$. Now by application of Fubini's Theorem :

$$\nu(S_N) = \int_{v \,\varepsilon\, E_i^s(\varepsilon)} \nu^u (S_N \cap H_v)\, d\nu^s$$

OR : $\nu(S_N) \leqq K(1+\delta)^N \exp(-CN)\, \nu^s(\mathrm{cl}\, E_i^s(\varepsilon))$

$$\leqq K(2\varepsilon)\dim E^s (1+\delta)^N \exp(-CN).$$

Then redefine K as $K(2\varepsilon)\dim E^s$, we then get :

$\nu(S_N) \leqq K(1+\delta)^N \exp(-CN)$. Hence :

$\mu\ (U_N\ (\ P_i)) \leqq\ K\ \ (\ 1 + \delta\ )^N\ \exp\ (\ -\ C\ N\ ),$ where again K is redefined. This completes the proof of Theorem 1.2.1.

§ 3 Extentions of Main Results Via Filtrations

In this section we extend the main result of Section 2 in the following two directions :

(a) First we show that using filtration arguments Theorem 1.2.1 holds for $\mu(U_N)$, where :
$U_N = \{ x \in M - \mathrm{cl}\ V : f^k x \notin V$, for $k = 0,\dots, N$.

(b) Then we remove the restriction that f have only fixed points, thus proving the main result for all Morse- Smale diffeomorphisms of M.

<u>Definition</u> : Let $f \in \mathrm{Diff}(M)$, a filtration for f is a sequence of compact manifolds, with boundary, $M = M_k \supset M_{k-1} \supset \cdots \quad M_1 \supset M_o = \emptyset$, such that $f(M_k) \subset \mathrm{int}\ M_k$. Given a filtration,
$K_i = \bigcap_{n \in Z} f^n (M_i - \mathrm{int}\ M_{i-1})$ is the maximal invariant set contained in $M_i - M_{i-1}$. If $K_i = \Omega \bigcap (M_i - M_{i-1})$, for all i, we say that the filtration is a fine filtration for f . Finally, if we are given closed invariant sets which are disjoint $\Lambda_1 , \dots, \Lambda_k$, we say
$M = M_k \supset M_{k-1} \supset \cdots \quad M_1 \supset M_o = 0$ is a filtration for $\Lambda_1 ,\dots, \Lambda_k$ if $\Lambda_i = K_i$.

Note : We will sometimes denote $M_i - \mathrm{int}\ M_j$ by $M[i,j]$.

We now state the main result of this section .

Theorem 1.3.1 : Let f be a Morse-Smale diffeomorphism on M. Then there is a filtration $\mathcal{M} = \{M_i\}_{i=1}^{r}$ such that :

(a) All the sinks of f are in M_1 .

(b) $S(i,j,N) = f^{-N}(f^{N}(M_i - \text{int } M_j))$

Or $S(i,j,N) = M_i - f^{-N}(\text{int } M_j)$

(the latter one will be the one we shall use)

Then given $\delta > 0$ we have that :

$\mu(S(i,j,N)) \leq K(1+\delta)^{N} \exp(-CN)$, with $K > 0$ independent of N and

$\exp(C) = \min\{\text{Jac } Df^{u}(P_k)\}$, where the minimum is taken over all non-sink fixed points P_k .

Proof : This Theorem is proved from the main result of section 2 using filtrations, and induction.

We know from [10] that there is a fine filtration $\mathcal{M}^{*} = \{M_k^{*}\}_{k=1}^{r}$ adapted to f . We immediately alter this to ; $\mathcal{M} = \{M_i\}_{i=1}^{r}$, by letting $M_1 = \bigcup M_k^{*}$, for $k = 1,\dots,\ell$, with ℓ the number of sinks of f on M.

Thus assertion (a) is proved. Now in order to prove assertion (b) we first quote a Proposition from [9] , about filtrations .

Proposition [9] : Let $M = M_k \supset M_{k-1} \quad \cdots \supset M_1 \supset M_0 = \emptyset$ be a filtration for $f \in \mathrm{Diff}^r(M)$. Then :

(a) $\Omega \subset \bigcup_{i=1}^{k} \Lambda_i$

(b) M is the disjoint union of the $W^s(\Lambda_i)$ and $W^u(\Lambda_i)$

(c) There are no cycles and $i > j$ implies $\Lambda_j \not> \Lambda_i$

(d) $M_i \subset \bigcup_{j \leqq i} W^s(\Lambda_j)$

(e) given arbitrarily small neighborhoods U_i of Λ_i, there exist m_i and $n_i > 0$ such that :

$$f^{n_i} M_i - f^{-m_i}(\mathrm{int}\, M_{i-1}) \subset U_i \,.$$

<u>Proof</u> : This can be found in [9] , however since part (e) is directly applied I will give its proof here :

As $\bigcap_{\infty < m < \infty} f^m(M[i,i-1]) = \Lambda_i$, there exists $n_i, m_i > 0$ such that :

$$U_i \supset \bigcap_{-m_i \leqq m \leqq n_i} f^m(M[i,i-1]) \supset f^{n_i} M_1 - f^{-m_i}(\mathrm{int} M_{i-1})$$

We will now use this proposition in proving the next two propositions which are needed to prove (2).

Proposition 1.3.1 : If $x \in M$ but $x \in f^{-n}(U(P_i))$. then there exists an iteration L_i depending only on $U(P_i)$ for which $f^{L_i}(x)$ int M_{i-1}.

Proof : From proposition [9], part (e) we have that there exists n_i and m_i such that $f^{n_i}(M_i) - f^{m_i}(\text{int} M_{i-1})$ U(P). This yields $M \subset f^{-n_i}(U(P)) \cup f^{-m_i - n_i}(\text{int } M_{i-1})$.
If $x \in M_i$ but $x \in f^{-n_i}(U(P_i))$. Then :

$x \in f^{-(m_i + n_i)}(\text{ int } M_{i-1})$. This is equivalent to :

$f^{m_i + n_i}(x) \in \text{int } M_{i-1}$. Therefore take $L_i = m_i + n_i$ and the proposition holds.

Next we will relate these two propositions to the set $S(i,j,N)$. We obtain :

(a) $\{P_i\}$ is the largest invariant set in $M[i,i-1]$ and

(b) For any neighborhood U of P_i there are m_i and n_i, fixed integers, so that $U \supset f^{n_i} M_i - f^{-m_i} \text{int } M_{i-1}$.

Next applying proposition 1.3.1 we find that :

$f^{n_i}(S(i,i-1,N + L_i)) \subset U(P_i)$. But also since

$f^{n_i}(S(i,i-1,N + L_i)) \subset S(i,i-1, N + m_i)$ we have :

$f^{n_i}(S(i,i-1,N + L_i)) \subset U_{N + m_i}(P_i)$ and then :

$S(i,i-1, N + L_i) \subset f^{-n_i}(U_{N+m_i}(P_i))$.

We now have the preliminaries to begin the proof of Theorem 1.3.1 . Let $i = 2$. We consider $S(2,1,N)$.
$S(2,1,N) = S(2,1,(N-L_2) + L_2)$, so

$$S(2,1,(N-L_2) + L_2) \subset f^{-n_2} U_{N-n_2}(P_2)$$

Thus : $\mu(S(2,1,N)) \leq K^* \mu(U_{N-n_2}(P_2))$

$$\leq K^* K(1+\delta)^{N-n_2} \exp(-C(N-n_2))$$

for $N \geq L_2$.

Now by redefining K we get :

$$\mu(S(2,1,N)) \leq K(1+\delta)^N \exp(-CN).$$

Now assuming that we have the estimate :

$\mu(S(i-1,1,N)) \leq K(1+\delta)^N \exp(-CN)$ for $i < r$. Then consider $S(i,1,N)$.

$$S(i,1,N) = S(i,i-1,N) \cup S(i-1,1,N) \cup A_N$$

where $A_N = \{ x \in M[i,i-1] : f^N(x) \in \text{int } M_{i-1} \}$.

Now $\mu(S(i-1,1,N))$ is known by hypothesis and $\mu(S(i,i-1,N))$ is gotten by a trivial modification of the $S(2,1,N)$ case. So we consider A_N. Next write $A_N = D \cup E$, where :

$$D = \{ x \in A_N : f^{L_i}(x) \in M[i,i-1] \}$$

$$E = \{ x \in A_N : f^{L_i}(x) \in \text{int } M_{i-1} \}$$

Then : $f^{L_i}(E) \subset S(i-1,1,N-L_i)$ and $E \subset f^{-L_i}(S(i-1,1,N-L_i))$ so that

$\mu(E) \leq K' \mu(S(i-1,1,N-L_i))$ where $K' = \max \{ \text{Jac } f^{-L_i} \}$ over $M[i,i-1]$. Thus $\mu(E) \leq K(1+\delta)^N \exp(-CN)$, by modifying K.

Finally for D we have :

$D \subset f^{-n_i} U(P_i)$ so that : $f^{n_i} D \subset U_{N-n_i}(P_i)$.

Hence :

$$\mu(D) \leqq K^* \mu(U_{N-n_i}(P_i))$$

Or : $\mu(D) \leqq K^* K(1+\delta)^{N-n_i} \exp(-C(N-n_i))$, for $N \geqq L_i$.

Then by redefining our constants K, we get :

$$\mu(S(i,1,N)) \leqq \mu(S(i,i-1,N)) + \mu(S(i-1,j,N)) + \mu(D) + \mu(E)$$

$$\leqq K(1+\delta)^N \exp(-CN)$$

as desired. This completes the proof of the Theorem.

The natural question at this point is does the theorem hold for periodic orbits of f ? The following Corollary answers this in the affirmative.

Corollary 1.3.1 : Theorem 1.3.1 holds for all Morse-Smale diffeormophisms on M.

Proof : We let ℓ be the least common multiple of the periods of γ_j , where γ_j are the periodic orbits of f , for j=1,...,r , (note that here fixed point means period 1).

Next let $g = f^{\ell}$. Then g is Morse -Smale with only fixed points. Then Theorem 1. holds for g . That is there is a filtration $\mathcal{M} = \{M_i\}_{i=1}^{r}$ such that all the sinks of g are in M_1 and $\mu(S(i,1,N) \leqq K(1+\delta)^N \exp(-CN)$ for any $\delta > 0$ and some positive K. Here C is given by :

$$C = \log \min \{ \operatorname{Jac} D_{p_j} g \mid W^u(p_j) \quad \text{for } j = 2,\dots,r \} .$$

Where the p_j are the fixed points for g. Then we get ;

$$\mu(S_f(i,1,N\ell)) \leq K(1+\delta)^N \exp((-C/\ell) N\ell)$$
$$K(1+\delta)^N \exp((-C/\ell) N\ell)$$

Now certainly $S_f(i,1,N\ell+k) \subset S_f(i,1,N\ell)$ for $k=1,\ldots,\ell-1$. Thus by redefining K we get

$$\mu(S_f(i,1,N)) \leq K(1+\delta)^N \exp((-C/\ell) N) \text{ for all } N.$$

The proof is finished by noting that C/ℓ is the minimum of the quantities $1/m \log \mathrm{Jac}\, D_{p_j} f^m\, W^u(p_j)$ where the minimum is taken over all p_j in the non-wandering set for f which are not sinks and $f^m p_j = p_j$.

§4 Morse - Smale Flows

The goal of this section is to prove a corresponding Theorem for Morse-Smale flows.

For x a fixed point of a flow ϕ_t we define :

$\phi^u(x) = \log \text{Jac } D\phi_1 \mid E^u_x \to E^u_x$.

For γ a closed orbit of ϕ_t with period τ we define :

$$\phi^u(\gamma) = 1/\tau \left(\min_{y \in \gamma} \{ \log \text{Jac } D\phi_\tau \mid E^u_y \to E^u_y \} \right) .$$

Theorem 1.4.1 : Let ϕ_t be a Morse-Smale flow on M. There is a filtration $\mathcal{M} = \{M_i\}_{i=1}^{r}$ such that :

(a) All attractors for ϕ_t are in M_1 .

(b) For all $\delta > 0$,

$$\mu(S(i,1,T)) \leq K(1+\delta)^T \exp(-CT)$$

for $i > 2$, $K > 0$ independent of T, and

$$C = \min_{x_j, \gamma_j} \{ \phi^u(x_j), \phi^u(\gamma_j) \} \quad \text{where} \quad x_j, \gamma_j$$

are respectively the fixed points and closed orbits of ϕ_t which are not sinks or attractors.

Proof : We show how to modify our previous arguments to this case.

If x is a fixed point of ϕ_t , then our previous work suffices to describe the behavior of trajectories in the neighborhood of x by considering the time one diffeomorphism $g = \phi_1$.

If γ is a closed orbit of ϕ_t which is not an attractor, then $T_\gamma M$ splits continuously into $E + E^s + E^u$ where E is the one-dimensional bundle tangent to the flow and dimension $E^u \geq 1$. For $g = \phi_\tau$ the time τ diffeomorphism where τ is the period of γ, γ is a closed, infinite, non-hyperbolic set in $\Omega(g)$. However since $D\,g(y)$ expands E^u_y more rapidly than E^u_y for any $y \varepsilon \gamma$, then given $\varepsilon > 0$ we can find δ, $0 > \delta > \varepsilon$, so that if $g^k(x) \; \varepsilon \; B_y(\varepsilon)$ for $0 \leq k \leq n$ but $g^{n+1}(x) \; \varepsilon \; B_y(\varepsilon)$, then $d(g^{n+1}(x), \gamma) > \delta$. By compactness we can cover $B_\gamma(\delta)$ by a finite number of $\{B_y(\varepsilon)\}_{y \varepsilon \gamma}$.

The effect of this construction is to permit use of the previously established volume and area estimates in the neighborhood of γ by considering only a finite number of neighborhoods covering γ. If N is a C^1 submanifold of M in cl $B_\gamma(\delta)$, then N decomposes into a finite number of submanifolds determined by its intersections with $B_y(\varepsilon)$. Let N_y be a typical one. Then $\phi^{-1}_y N_y$ is a C^1 submanifold of $T_y M(\varepsilon)$. If in addition it is the graph of a C^1 function of a closed subset of $E^u_y(\varepsilon)$ into $E^{cs}_y(\varepsilon) = E^s_y(\varepsilon) + E_y(\varepsilon)$ with prescribed slope, then under iteration by $G = \phi^{-1}_y \circ g \circ \phi_y$, it remains the graph of a C^1 function of the unstable space E^u_y into the center stable space E^{cs}_y. That is the graph transform properties from before apply here [2]. Locally γ behaves like a finite number of non-wandering points as far as the area

Lemma (Lemma 1.2.2) is concerned.

The only other issue is that of transversal intersection. The stable and unstable manifold of any two closed orbits, including fixed points, have transversal intersection. In the case of a closed orbit there is a third direction - the flow direction. We actually need each unstable manifold to intersect each center stable manifold transversally. This occurs since the stable and center stable manifolds intersect along the flow. Hence the analogue of Proposition 1.2.1 is true for Morse-Smale flows. Also by our previous remark the area Lemma 1.2.2 has an analogue. We thus prove our volume estimates locally using fixed time diffeomorphisms $g = \phi_\tau$ and finish the proof by normalizing the estimates by dividing the exponential constants by τ when necessary.

§5 An Example With Transversal Intersection

Consider the torus $T^2 \subset R^3$ which is tilted back with respect to the horizontal plane. Let the gradient field on it be of the form :

$$\dot{X} = - \text{grad } (h)$$

where h is the height function of points in relation to the horizontal plane. So the flow is downward. Next consider the Dehn twist. More specifically fix time t to τ, and consider the diffeomorphism x_τ . Now add a diffeomorphism g described as follows : let g = id on the torus except in a small band. In this band g moves points along their level curves (which are circles). g rotates the level curves in the following manner : the uppermost curve rotates 0 radians, the lowermost rotates 2π radians, and the curves in between rotate from 0 to 2π going down the band. The diffeomorphism g is C^1. The effect of h is to push g down causing repetition of the pattern resulting in the transversal intersection, which can be seen in Fig.1.5.1 . The local intersection is illustrated in Fig.1.5.2 .

We note that in the figures P_4 is the source, P_3 and P_2 are the saddles, and P_1 is the sink.

Fig. 1.5.1

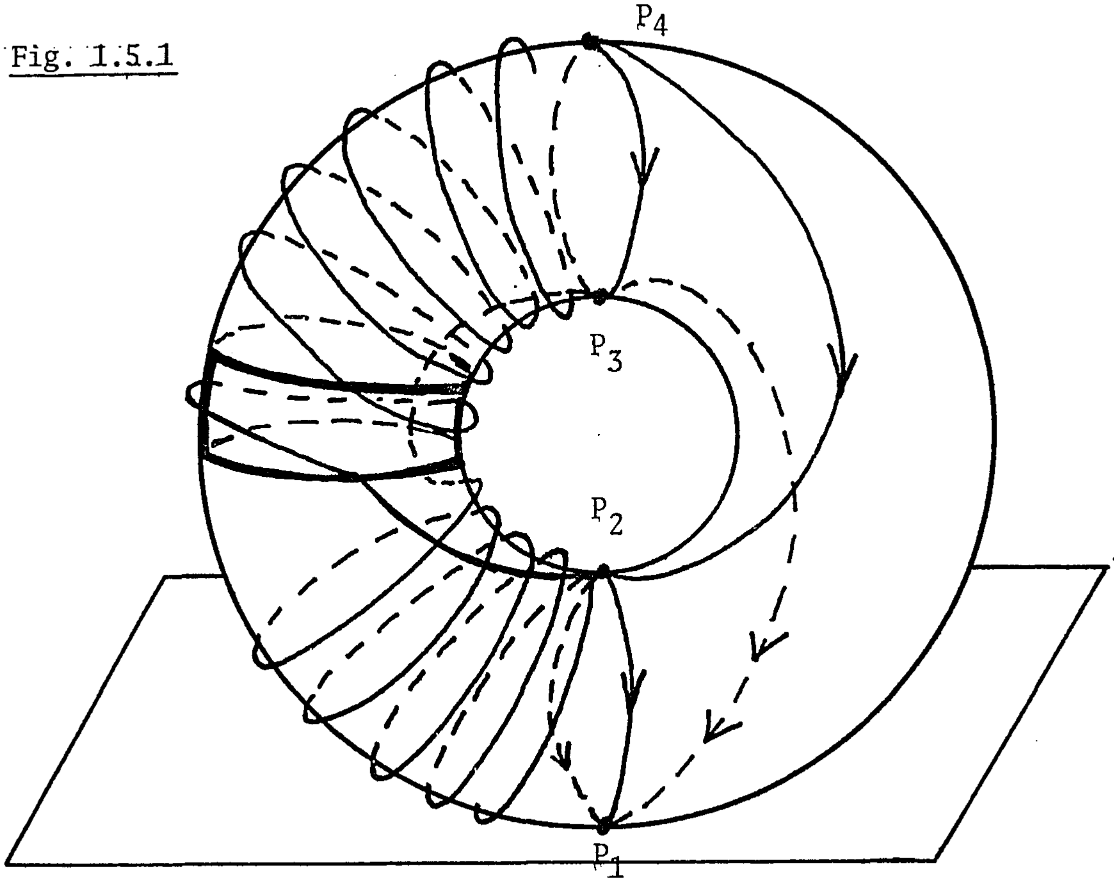
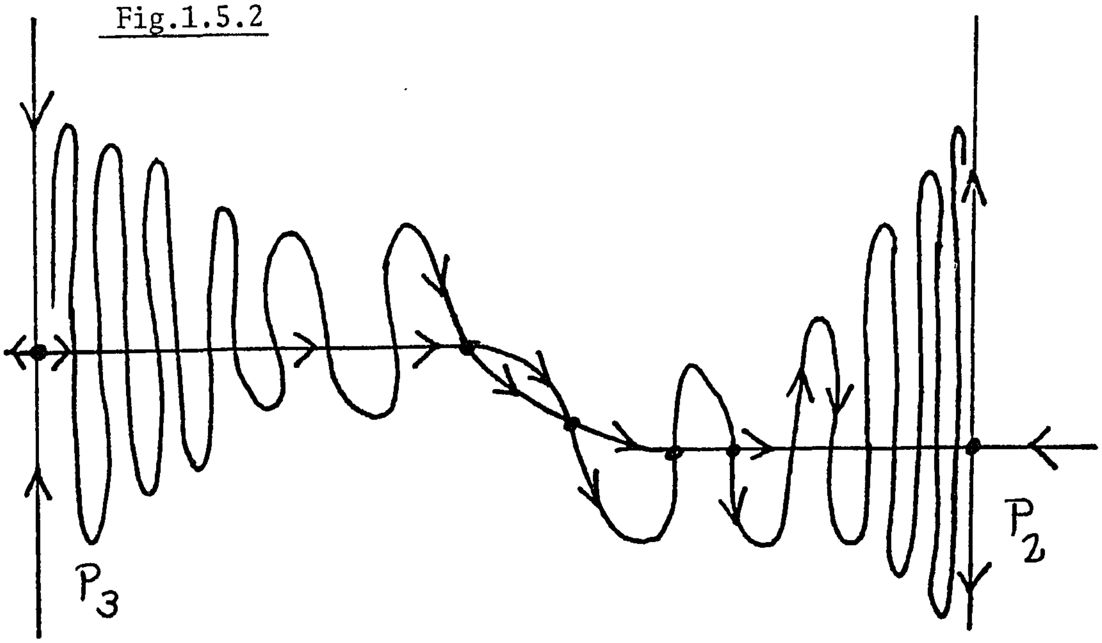


Fig.1.5.2



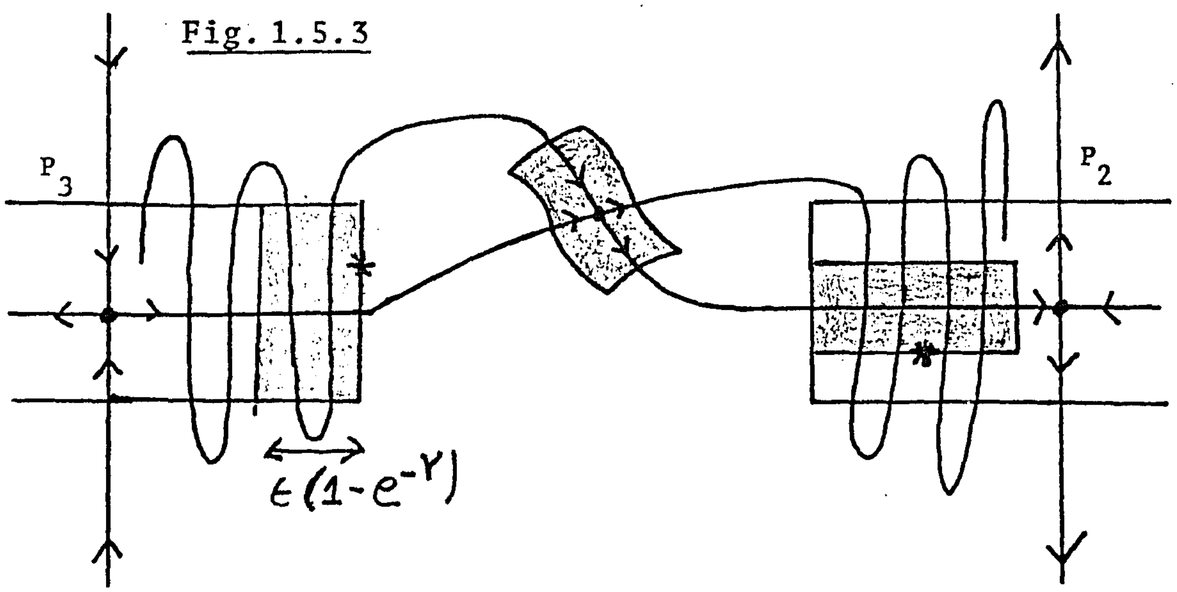
In reference to Fig. 1.5.2 we point out that there are an infinite number of such intersections tending forward to P_2. The λ Lemma of Palis [6] tells us that $W^u(P_3)$ in a neighborhood of P_2 becomes close to $W^u(P_2)$ in both distance and slope. In fact it contains $W^u(P_2)$ in its closure.

Next in linearized neighborhoods of P_3 and P_2 the local diffeomorphisms can be given by :

$$(**) \quad \begin{aligned} X &= X_0 \exp(\gamma) \\ Y &= Y_0 \exp(-\alpha) \end{aligned} \qquad\qquad \begin{aligned} X &= X_0 \exp(-\gamma) \\ Y &= Y_0 \exp(\beta) \end{aligned}$$

respectively. The logarithms of the eigenvalues are $\gamma, -\alpha, -\gamma, \beta$ For sets moving from P_3 to P_2, the transversality property causes the two unstable directions to become alligned. This is shown in Fig. 1.5.3, where we only consider one point of transversal intersection outside the neighborhoods, for clarity.

Fig. 1.5.3



We see by the transversality property the height of the set which comes into the neighborhood of P_2 is independent of the number of iterations for which it stays in a neighborhood of P_3. So only $W^u(P_2)$ influences the set in the neighborhood. In Chapter 2 we will see that without the transversality condition this is not the case.

Using the diffeomorphisms given in (**) we will show that the area of the set whose orbits remain in the neighborhood of P_3 for exactly n iterations and in the neighborhood of P_2 for at least m iterations is :

$$4\,\varepsilon^2 \exp(-\beta m) \exp(-\gamma n)$$
$$= 4\,\varepsilon^2 \exp(C) \exp(-C N) \qquad \text{where } C = \min\{\gamma\ \beta\}$$

and $N = n + 1 + m$.

To see how this estimate was obtained consider the following illustrations in Fig. 1.5.4, and Fig. 1.5.5.

Fig. 1.5.4

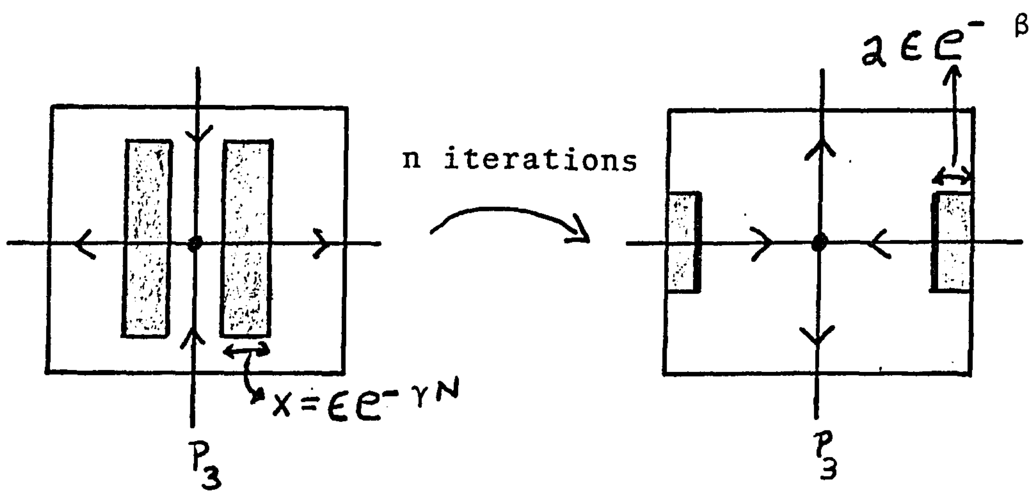
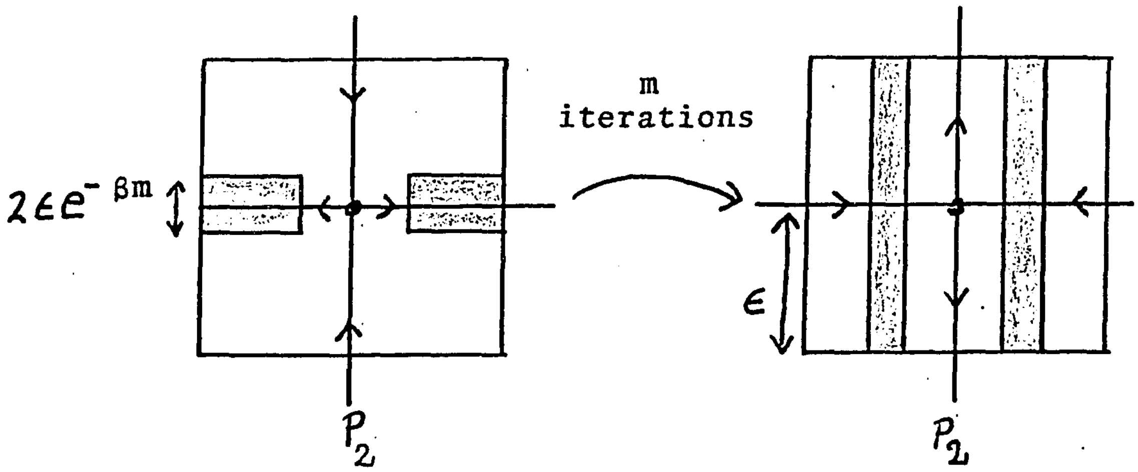


Fig.1.5.5



We note that from Fig.1.5.4 to Fig.1.5.5 is one iteration, which gives a total of N iterations overall.

We now can add these estimates to get the " total estimate " for $n = 0,1,2,\ldots, N$, which implies $m = N ,\ldots, 0$. This gives :

$$\sum_{n=0}^{N} 4\,\varepsilon^2 \exp(C) \exp(-C N)$$

$$= 4\,\varepsilon^2 \exp(C) \left((N+1) \exp(-C N) \right) .$$

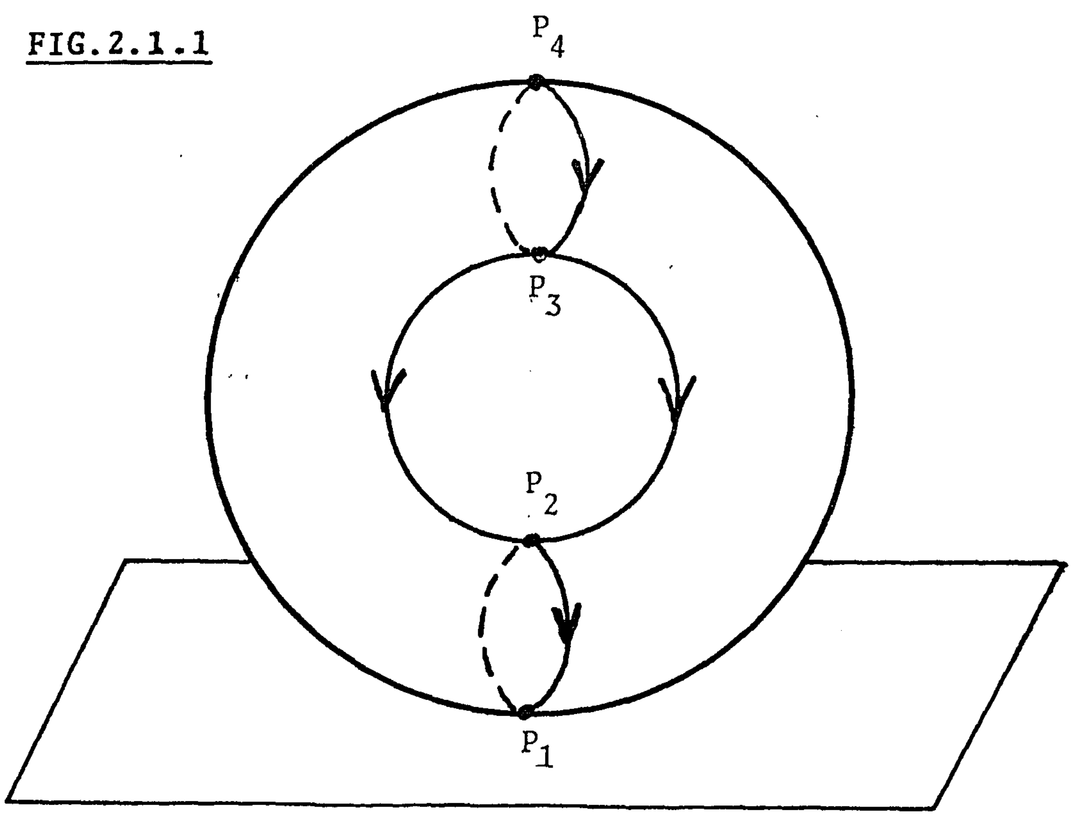
CHAPTER 2

§ 1. An Example Without Transversal Intersection

Let us now suppose that we no longer have the transversality condition of the Morse-Smale diffeomorphisms and flows, what can be said about the rate established in Chapter 1 ? We will begin by considering the gradient system, defined previously, on the torus $T^2 \subset R^3$. However suppose, in contrast to the previous case, that T^2 is not tilted back. Thus we have a non-transversal saddle connection,

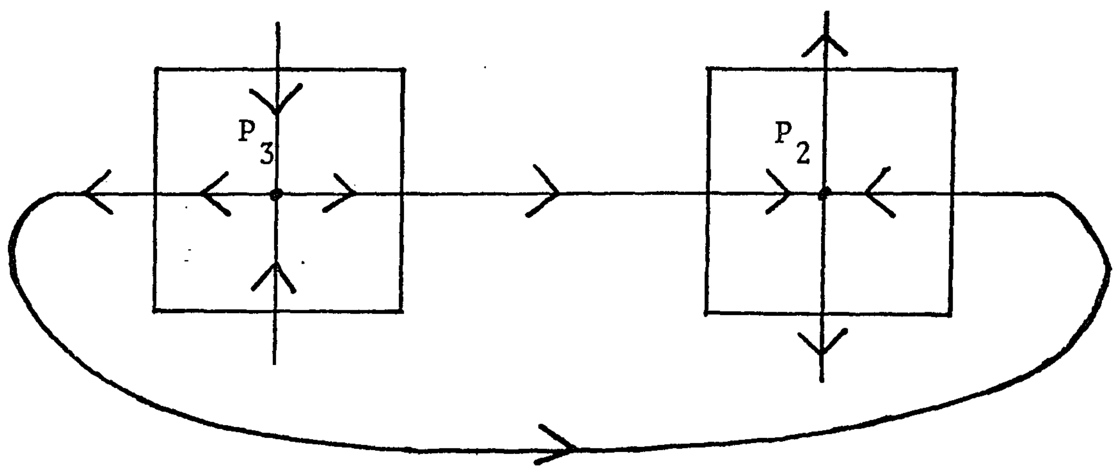
$$(W^u (P_3) = W^s (P_2)), \text{ Fig.2.1.1.}$$

FIG.2.1.1



We perturbe ϕ_t (the gradient system associated with the height function), to allow a flow f^t so that f^t has the same fixed points as ϕ_t, f^t is linear in a neighborhood of P_3 and in a neighborhood of P_2. There is a tubular neighborhood of $W^u(P_3)$, outside of the chosen neighborhoods of P_3 and P_2 in which the flow is parallel to $W^u(P_3)$. Refer to FIG. 2.1.2.

FIG.2.1.2



Next in linearized neighborhoods of P_3 and P_2 consider the following ordinary differential equations:

In the Neighborhood of P_3 In the Neighborhood of P_2

$$\dot{X} = \gamma X \qquad\qquad X = -\gamma X$$

$$\dot{Y} = -\alpha Y \qquad\qquad \dot{Y} = \beta Y$$

where $\gamma, \alpha, \beta > 0$.

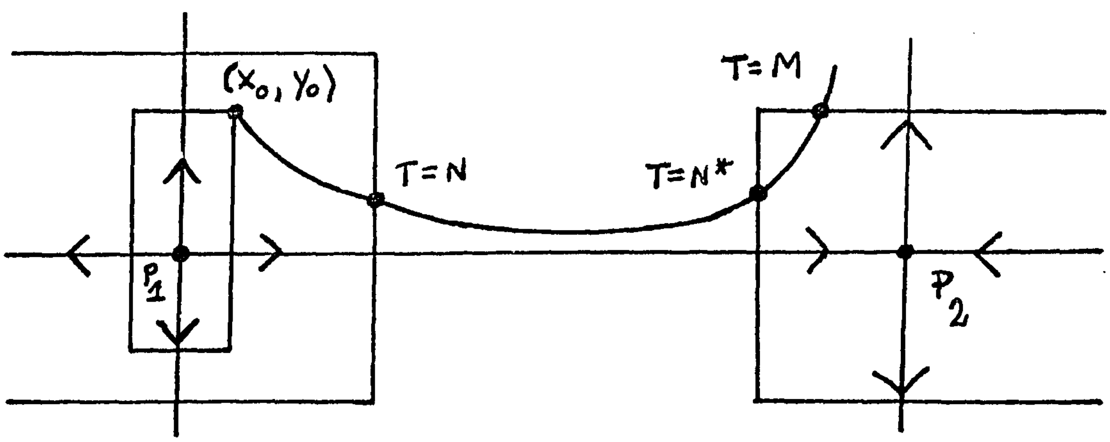
Now consider a point (X_o, Y_o) in the neighborhood of P_3. Its orbit in this neighborhood is given by :

$$X = X_o \exp(\gamma T)$$

$$Y = Y_o \exp(-\alpha T)$$

If this orbit remains in the neighborhood through time N, then since $X = X \exp(\gamma N)$, and $Y = Y_o \exp(-\alpha N)$ we get that $X_o \leqq \varepsilon / \exp(\gamma N)$. Refer to Fig. 2.1.3.

<u>FIG. 2.1.3</u>



We assumed that the flow is parallel to W^u (P_3) between the two neighborhoods, thus the orbit of (x_o, y_o) when it first enters the neighborhood of P_2 has coordinates (x,y) satisfying:

$$|X| = \varepsilon \quad \text{and} \quad |Y| = |Y_o| \exp(-\alpha N).$$

This corresponds to time $N^* = N + S$, where S is a constant giving the time of the flow between the two neighborhoods. Refer to Fig. 2.1.3. Now the orbit while in the neighborhood of P_2 will satisfy:

$$|X| = \varepsilon \exp(-\gamma(T - N^*))$$

$$|Y| = |Y_o| \exp(-\alpha N) \exp(\beta(T - N^*)).$$

Where T is the time measured from the initial point (x_o, y_o).

Then if this orbit remains in the neighborhood of P_2 through time M we get : $|Y| \leqq \varepsilon$

Or $\quad Y_o \exp(-\alpha N) \exp(\beta(M-N^*)) \leqq \varepsilon$

Or $\quad Y_o \leqq \exp(\alpha N) \exp(\beta(N^* - M))\varepsilon.$

Thus the area of the set of points whose orbit stay in the neighborhood of P_3 for time N and do not leave the neighborhood of P_2 is given by :

$|X_o| \cdot |Y_o|$, or is less than or equal to :

$$[\varepsilon \exp(-\gamma N)] \cdot [\varepsilon \exp(\alpha N) \exp(\beta (N^*-M)]$$

$$= \varepsilon^2 \exp(\alpha N) \cdot \exp(-\gamma N) \cdot \exp(\beta (N^*-M))$$

$$= \varepsilon^2 \exp(\beta S) \cdot \exp((\alpha - \gamma) N) \cdot \exp(\beta (N-M))$$

Refer to Fig. 2.1.3

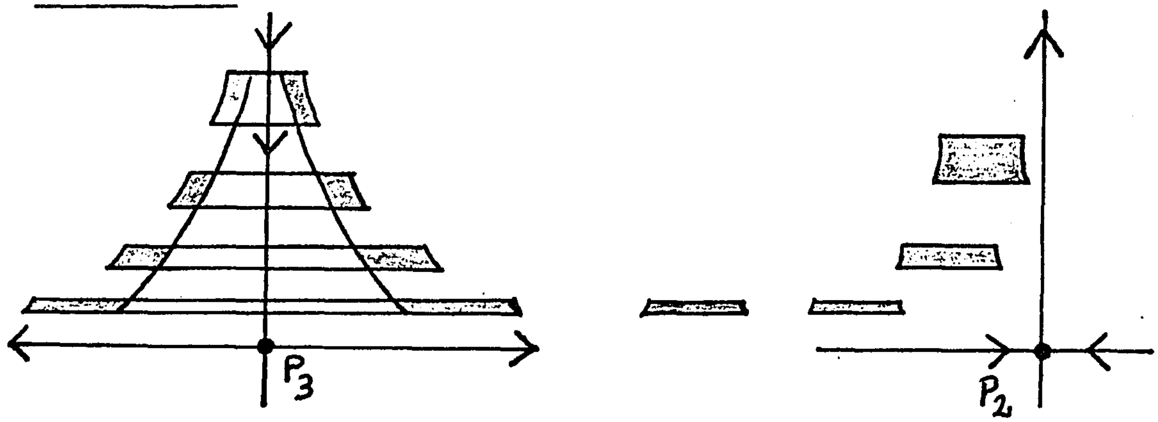
Now $\gamma - \alpha < \gamma$ because $\alpha > 0$. If in addition we take $\beta > \gamma - \alpha$ then we have that the area of our set is larger than that in the Morse-Smale case, i.e.

$\gamma - \alpha < \min\{\gamma, \beta\}$. Thus the rate of decay is slower than in the Morse-Smale case.

The point illustrated here is the unstable direction in the neighborhoods of P_3 transforms to the stable direction in the neighborhood of P_2.

This allows the distortion of volume in the neighborhood of P_3 ,(due to the stable part of f), to affect the rate at which the set leaves P_2 . More specifically the skrinkage of volume in the stable direction at P_3 must be overcome by the unstable part of f at P_2 . Thus the number of iterations required for a set to leave the neighborhood of P_2 depends on the number of iterations during which the set remained in the neighborhood of P_2 , in contrast to the Morse-Smale case. This is illustrated below :

FIG. 2.1.4



Hence by the preceeding asymptotic estimate we see that the overall decay rate is slower than in the Morse-Smale case. That is to say that the exponential constant C is less than the minimum of γ and β , where exp (γ) and exp (β) are, respectively, the jacobians of the unstable parts of Df^1 at the two saddles.

This example shows that the result of Chapter 1 must be modified in order to hold for Axiom A, no cycle diffeomorphisms and flows in general.

2 The Results For C^2 Axiom A Diffeomorphisms With No Cycles

The goal of this section is to establish a result similiar to that obtained for Morse-Smale diffeomorphisms in the more general case of Axiom A, no cycle diffeomorphisms. We will prove a theorem for C^2 diffeomorphisms, and will discuss the conditions under which the theorem holds for C^1 diffeomorphisms in section 4. We begin with a theorem which relates local volume estimates to global estimates.

Definition: Let $\mathcal{M} = \{M_i\}_{i=0}^{r}$ be a filtration adapted to f and let Λ_k be the largest invariant set in M_k - int M_k-1. Then

$B_{\Lambda_k}(\varepsilon,N) = \{x \varepsilon M : d(f^k x, \Lambda_k) \leqq \varepsilon$ for $K = 0,1,\dots,N\}$

Proposition: 2.2.1 for small ε there exist L_i, n_i as in proposition 1.3.1 such that

$$S(i,i-1, N+L_i) \subset f^{-n_i}(B_{\Lambda_k}(\varepsilon,N))$$

where $S(i,i-1, n) = M_i - f^{-n}(\text{int } M_{i-1})$.

Proof: Because $B_{\Lambda_k}(\varepsilon)$ is a neighborhood of Λ_k we know from proposition [9] that there exist L_i, n_i such that if $x \varepsilon M_i$ - int M_{i-1} and $f^{L_i}x \varepsilon$ int M_{i-1}, then $x \varepsilon f^{-n_i}(B_{\Lambda_k}(\varepsilon))$.

If $x \in S(i,i-1, N+L_i)$, then $x \in f^{-(N+L_i)}(\text{int } M_i - 1)$. Hence $x \in f^{-N_i} B_{\Lambda_i}(\varepsilon)$ But also we have $x \in f^{-(k+L_i)}(\text{int } M_{i-1})$ where $0 \leq k \leq N$ because $f(M_{i-1}) \subset \text{int } M_{i-1}$. Thus $f^k x \in f^{-N_i} B_{\Lambda_i}(\varepsilon)$ or $f^{k+N_i} x \in B_{\Lambda_i}(\varepsilon)$ for $0 \leq k \leq N$.

This means that $f^{N_i} x, f^{N_i+1} x, \ldots, f^{N_i+N} x$ are all in $B_{\Lambda_i}(\varepsilon)$. By definition this means $f^{N_i} x \in B_{\Lambda_i}(\varepsilon)$ or $x \in f^{-N_i} B_{\Lambda_i}(\varepsilon)$
Hence $S(i,i-1, N+L_i) \subset f^{-N_i} B_{\Lambda_i}(\varepsilon,N)$

<u>Theorem</u> 2.2.1 Let f be a diffeomorphism of M with $\mathcal{M} = \{M_i\}_{i=0}^{r}$ a filtration adapted to f with Λ_k the largest invariant set in $M_k - \text{int } M_k - 1$.

Assume that $\mu(B_{\Lambda_k}(\varepsilon,N)) \leq D_k \exp(-C_k N)$,

with D_k and C_k positive constants for $k=2,3,\ldots,r$.

Then for $i > j$, $i \leq r$, $j \geq 1$, there exist positive constants $w_{i,j}$, $D_{i,j}$ and $C_{i,j}$ such that $N \geq w_{i,j}$ implies $\mu(S(i,j,N)) \leq D_{i,j} \exp(-C_{i,j}N)$.

<u>Proof</u>: Given $i > 1$ we will first prove the result for $j = i-1$ and then extend the result by induction.

By Proposition 2.2.1 we know that $S(i,i-1,\ n + L_i)$ $\subset f^{-n_i}\ (B_{\Lambda_i}(\varepsilon,n))$. But

$$\mu(f^{-n_i}(B_{\Lambda_i}(\varepsilon,n))) = \int_{B_{\Lambda_i}(\varepsilon,n)} \mathrm{Jac}\, f^{-n_i}\, d\mu$$

$$\leq J^*_i \ \mu(B_{\Lambda_i}(\varepsilon,n)$$

where $J^*_i = \max\limits_{M[i,i-1]} (\mathrm{Jac}\ |f^{-n_i}|)$.

Thus we have : $\mu(S(i,i-1,n + L_i) \leq J^*_i\ (B_{\Lambda_i}(\varepsilon,n))$.

$$\leq J^*_i\, D_i \exp(-C_i n),$$

where the last step follows from the hypothesis. Now take $N = n + L_i$ and require $N \geq L_i$, in which case we have:

$$\mu(S(i,i-1,N) \leq J^*_i\ D_i \exp(C_i L_i) \exp(-C_i N).$$

Let $D_{i,i-1} = J^*_i\ D_i \exp(C_i L_i)$

and $C_{1,1-1} = C_1$, and $w_{i,i-1} = L_i$; we have the desired result.

Next assume, by way of induction, that the statement holds for

$\mu(S(i-1,j,N))$, for $i > j$, $j \geq 1$, and $N \geqq w_{i-1,j}$.

Now consider: $S(i,j,N)$. Recall from Chapter 1 that:

$S(i,j,N) = S(i-1,j,N) \cup S(i,i-1,N) \cup A_N$, where

$A_N = \{x \in M[i,i-1] : f^N(x) \in M[i-1,j]\}$.

Furthermore decompose $A_N = D \cup E$, where :

$D = \{x \in A_N : f^{L_i}(x) \in M[i,i-1]\}$ and

$E = \{x \in A_N : f^{L_i}(x) \in M[i-1,j]\}$. Thus for $N \geqq L_i$

we have:

1) $S(i,j,N) = S(i-1,j,N) \cup S(i,i-1,N) \cup D \cup E$.

We have volume estimates for the first two directly from the induction hypothesis.

2) $\mu S(i-1,j,N)) \leqq D_{i-1,j} \exp(-C_{i-1,j} N)$ for $N \geqq w_{i-1,j}$

A similar result holds for $\mu(S(i,i-1,N))$.

For E, observe that $f^{L_i}(E) \subset S(i-1,j, N-L)$,

so that: $\mu(f^{L_i}(E))$ can be estimated.

$\mu(f^{L_i}(E)) \leqq J_i^{L_i}(E)$, where $J_i = \min\limits_{M[i,i-1]} \{|\text{Jac } f|\}$.

Thus we have :

3) $\mu(E) \leqq J_i^{-L_i} \mu(S(i-1,j,N-L_i)) \leqq J_i^{-L_i} D_{i-1,j} \exp(-C_{i-1,j} N)$,

for $N \geqq L_i + w_{i-1,j}$.

Next I claim the following :

Lemma 2.2.1 :

For $\delta > 0$ small, and for all $N \geqq 2(L_i + w_{i-1,j})$

we find that :

$$\mu(D) \leqq K(\exp(-C_i^{\#}(1-\delta)N)$$

where : $C_i^{\#} = \min\{C_i/q \;, C_{i-1,j}(1-\delta)\}$. and $q \in Z^+$ is chosen so that q satisfies :

$$q \geq (1 + (C_i^* / C_{i-1,j}] / \delta \qquad \text{, where}$$

$$C_i^* = \log(\min_{P_i \in M[i,i-1]} |\mathrm{Jac}\; D_{P_i} f|) \quad .$$

The proof of this lemma will be given in Appendix II.

We are now ready to state the " total " estimate which is given by the following form :

$$\mu(S(i,j,N)) \leqq \mu(S(i-1,j,N) + \mu(S(i,i-1,N)) + \mu(D) + \mu(E)$$

The preceeding estimate is obtained by using Lemma 2.2.1 and the estimates (2), and (3) . Thus we find :

$$\mu(S(i,j,N)) \leqq K_{i,j} \exp(-C_{i,j}(1-\delta)N) \;,$$

for $N \geq \max\{w_{i,i-1} \;, 2(L_i + w_{i-1,j})\}$, where the $C_{i,j}$ are defined recursively by $C_{i,j} = C_i^{\#}$, where $C_i^{\#}$ has been previously defined.

This completes the proof of the Theorem.

We see from Theorem 2.2.1 that whenever the diffeomorphism f has a filtration $\mathcal{M}$ such that the sets Λ_k have local volume estimates, Λ_k not an attractor, then a global estimate holds. We now show that this is the case for C^2 Axiom A diffeomorphisms with no cycles.

<u>Definition</u> : Let $f \in \mathrm{Diff}(M)$. Then f satisfies Axiom A if and only if :

(a) $\Omega(f)$ is hyperbolic

(b) $\Omega(f) = \mathrm{cl}\,\mathrm{Per}\, f$

<u>Definition</u> : We say that a diffeomorphism f has an Ω decomposition if $\Omega(f)$ may be written as the finite disjoint union of closed invariant sets for f, $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_k$. Moreover, if $f|\Omega_i$ is topologically transitive for all i, i.e. $f|\Omega_i$ has a dense orbit for all i, then we say that f has a spectral decomposition.

Now given $f \in \mathrm{Diff}^r(M)$ with an Ω - decomposition, we may define a relation on the Ω_i as follows : $\Omega_i > \Omega_j$ if $(W^u(\Omega_i) - \Omega_i) \cap (W^s(\Omega_j) - \Omega_j) \neq \emptyset$, i.e., there is an x which comes from Ω_i and goes to Ω_j.

<u>Definition</u> : Let $f \in \mathrm{Diff}^r(M)$ have an Ω - decomposition $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_k$. We say that f has no cycles if

$\Omega_{i_0} > \Omega_{i_1} > \ldots > \Omega_{i_j} = \Omega_{i_0}$ is impossible for any $j \geq 1$.

If f has an Ω - decomposition and no cycles we may reorder the Ω_i by defining $\Omega_i > \Omega_j$ iff there exists a sequence $\Omega_i > \Omega_{k_1} > \dots > \Omega_{k_i} > \Omega_j$ and finally we may reindex the Ω_i such that $\Omega = \Omega_1 \cup \dots \cup \Omega_k$ and $i < j$ implies $\Omega_i \not> \Omega_j$. Henceforth, we will assume that the Ω_i are indexed as above for any f with an Ω - decomposition and the no cycle property.

Spectral Decomposition Theorem (Smale) : If f satisfies Axiom A, f has a spectral decomposition.

Definition : We say that f is an Axiom A diffeomorphism with no cycles if :

(a) f satisfies Axiom A and

(b) the spectral decomposition of $\Omega(f)$ has no cycles.

Now in [9] we find the following :

Theorem : Let $f \in \mathrm{Diff}^r(M)$. Then f has a fine filtration if and only if f has an Ω decomposition with no cycles.

Therefore, we know that if f is an Axiom A, no cycle diffeomorphism, then f has a fine filtration $\mathcal{M} = \{M_i\}_{i=0}^{r}$ such that the largest invariant set Λ_k in M_k - int M_{k-1} is the corresponding set in the spectral decomposition of $\Omega(f)$. Λ_k is called a basic set for f [1] .

The following is a simple modification of a Theorem found in

Bowen and Ruelle [2]. The difference is that they prove an estimate for $\bigcup_{x \varepsilon \Lambda_j} B_x(\varepsilon, N)$ which is not the same as $B_{\Lambda_j}(\varepsilon, N)$.

Theorem 2.2.2 : Let f be a C^2 Axiom A diffeomorphism, with no cycles. Let Λ_j be a C^2 basic hyperbolic set for f. Then for ε small and K_j, C_j both positive constants we find :

$$\mu (B_{\Lambda_j} (\varepsilon , N)) \leqq K_j \exp (- C_j N) .$$

Before proving this Theorem, we will prove the following proposition.

Proposition 2.2.2 : For all $\gamma > 0$, there exists $\varepsilon > 0$ such that $B_{\Lambda_j} (\varepsilon , N) \subset \bigcup_{x \varepsilon \Lambda_j} (B_x (\varepsilon + \gamma , N))$.

Proof : Let $\gamma > 0$ be given. Take $\beta = \gamma/2$. By Bowen's Shadow Lemma [1], the local product structure of Λ_j yields α such that every α pseudo orbit in Λ_j is β shadowed by some y^* in Λ_j. Furthermore take $\varepsilon = \alpha / (K_f + 1)$,

where : $K_f = \max\limits_{\substack{x,y \,\varepsilon\, M \\ x \neq y}} \{ d(f(x)), f(y)) / d(x,y) \}$.

If $x \,\varepsilon\, B_{\Lambda_j} (\varepsilon , N)$ there exists $y_0 , y_1 , \ldots , y_N \,\varepsilon\, \Lambda j$, such that : $d (f^k (x) , y_k) \leqq \varepsilon$, for $0 \leqq k \leqq N$.

<u>Claim</u> : $y_o, y_1, \ldots, y_k$, is an α pseudo orbit.

<u>Proof</u> :

$$d(y_{k+1}, f(y_k)) \leq d(y_{k+1}, f^{k+1}(x)) + d(f^{k+1}(x), f(y_k))$$
$$\leq d(y_{k+1}, f^{k+1}(x)) + d(f(f^k(x), f(y_k)))$$
$$\leq \varepsilon + K_f \, (d(f^k(x), y_k))$$
$$\leq \varepsilon + K_f \varepsilon = (K_f + 1)\varepsilon$$

Therefore : $d(y_{k+1}, f(y_k)) \leq \alpha$, and hence $\{y_k\}_{k=0}^{N}$ is an α pseudo orbit.

So there exists $y^* \varepsilon \Lambda_j$ which shadows $y_o, \ldots, y_k$ i.e. $d(f^k(y^*), y_k) \leq \gamma$ for $0 \leq k \leq N$.

But then :

$$d(f^k(y^*), f^k(x)) \leq d(f^k(y^*), y_k) + d(y_k, f^k(x))$$
$$\leq (\gamma + \varepsilon) \qquad \text{for } 0 \leq k \leq N .$$

Therefore : $x \varepsilon B_{y^*}(\gamma + \varepsilon, N)$. Hence :

$$B_{\Lambda_j}(\varepsilon, N) \subset \bigcup_{x \varepsilon \Lambda_j} B_x(\gamma + \varepsilon, N) \qquad \text{as desired.}$$

We now return to Theorem 2.2.2 and begin its proof .

<u>Proof</u> : Let γ be small enough so that we can apply a local estimate $B_x(2\gamma, N)$, i.e., the one obtained from the Local Volume Lemma of [1] , mentioned in Chapter 1. Now let $\varepsilon \leq \gamma$

be small enough so that by Proposition 2.2.2 we have :

$$B_{\Lambda_j}(\varepsilon, N) \subset \bigcup_{x\,\varepsilon\Lambda_j} B_x(\gamma + \varepsilon, N).$$

Since $\gamma + \varepsilon \leqq 2\gamma$, and Λ_j is not an attractor, and is a C^2 basic hyperbolic set, then for $(\varepsilon + \gamma)$ sufficiently small, we get by application of Proposition 4.8 and Theorem 4.11 of [1], that the following is true :

$$\limsup \left(1/N \log \mu \left(\bigcup_{x\,\varepsilon\Lambda_j} B_x(\gamma + \varepsilon, N) \right) \right) = P(f|\Lambda_j, \phi^{(u)}) < 0 .$$

$P(f|\Lambda_j, \phi^{(u)})$ here is the topological pressure of f restricted to the basic set for the continuous function :

$$\phi^{(u)}(x) = -\log \lambda(x) \quad \text{where} \quad \lambda(x) \text{ is the}$$

Jacobian of the linear map $Df : E^u_x \to E^u_{f(x)}$ using inner products derived from the Riemannian metric [1] . Again by Theorem 4.11 of [1] we know that $P(f|\Lambda_j, \phi^{(u)}) = 0$ is equivalent to Λ_j being an attractor ($f(U) \subset U$ for some U a neighborhood of Λ_j) .

Now let the above lim sup $= \delta$, then for all $\alpha > 0$ we get : $1/N \log \mu\left(\bigcup_{x\,\varepsilon\,\Lambda_j} B_x(\gamma + \varepsilon, N) \right) \leqq (\delta + \alpha)$ for all but finitely many N. Now if one chooses α such that $(\delta + \alpha) < 0$, we get :

$$\log \mu \left(\bigcup_{x\,\varepsilon\,\Lambda_j} B_x(\gamma + \varepsilon, N) \right) \leqq N(\delta + \alpha) < 0, \quad \text{or}$$

$$\mu \bigcup_{x \in \Lambda_j} B_x (\varepsilon + \gamma , N) \leqq \exp (N(\delta + \alpha)).$$ Hence

$$\mu(B_{\Lambda_j} (\varepsilon, N)) \leqq \mu(\bigcup_{x \in \Lambda_j} B_x (\varepsilon + \gamma , N))$$

$$\leqq \exp (N (\delta + \alpha))$$

for all but a finitely many N.

In this case we have that $\delta = P(f|_{\Lambda_j}, \phi^{(u)})$ and $(\delta + \alpha) < 0$.

Remark : Thus the Theorem for f a C^2 Axiom A , diffeomorphism with no cycles is proven. This can be stated as :

Theorem 2.2.3 : Let M be a compact Riemannian manifold . Let μ be the measure induced by the Riemannian metric on M. Let f be a C^2 Axiom A diffeomorphism with no cycles, defined on M. Then f has a filtration $\mathcal{M} = \{M_i\}_{i=1}^{r}$ for which every sink of f is in M_1 . Let $S(i,j,N)$ be as defined before. Then for all $\delta > 0$ there exists $K = K (\delta)$ such that :

$$\mu (S(i,j, N)) \leqq K \exp (-C_{i,j}(1 - \delta) N) ,$$

for large N , where $C_{i,j}$ are defined recursively by :

$$C_{i,j} = \min \{ C_i / q , C_{i-1,j}(1 - \delta) \} ,$$

where C_i can be taken to be any number smaller than

$- P (f|_{\Lambda_i} , \phi^{(u)})$, where $P(f|_{\Lambda_i}, \phi^{(u)})$ was defined earlier, and $q \in Z^+$ is chosen so that q satisfies :

$$q \geqq (1 + C_i^* / C_{i-1,j}) / \delta , \text{ where}$$

$$C_i^* = \log (\min_{M[i,i-1]} \{ |\text{Jac } D_{p_i} f| \}).$$

We note that by taking i to r, j to 1 in Theorem 2.2.3 we obtain the Theorem stated in the introduction.

To obtain our constants $C_{i,j}$ we begin with $C_{1,1} = C_1$ and proceed to calculate $C_{2,1}$, $C_{3,1}$ etc. , up to $C_{r,1}$ using $C_i < -P (f , \phi^u)$.

§3 The Results For C^2 Axiom A Flows With No Cycles

An analogous result for Axiom A flows , to Theorem 2.2.3 is as follows :

Theorem 2.3.1 : Let ϕ_t be a C^2 Axiom A flow on M with no cycles. Then :
$\mu(S\ (i,j,T\ )\ ) \leqq D_{i,j} \exp\ (-\ C_{i,j}\ T\ )$ for all $i > j$, $i \leqq r$, $j \geqq 1$, $D_{i,j}$ and $C_{i,j}$ positive constants, and $T \geqq w_{i,j}$, with the $C_{i,j}$ defined recursively as in Theorem 2.2.3.

Proof : We know from [9] that ϕ_t has a filtration which we again assume to be $\mathcal{M} = \{M_i\}_{i=1}^{r}$ with all the attractors in M_1 . Here $S(i,j,T\ )$ is defined analogously to $S(i,j,N\ )$.

Fix $t = 1$ and consider the diffeomorphism ϕ_1 . This is strictly speaking not an Axiom A Diffeomorphism since ϕ_1 may not be hyperbolic on all of $\Omega\ (\phi_1) = \ (\phi_t)$. However it does have a spectral decomposition into basic sets Λ_k . Furthermore the basic sets Λ_k have local product structure for flows [8] , and have a Shadowing Lemma for ϕ_1 [3] .

By Proposition 4.4 and Theorem 5.6 of [2] the basic sets Λ_k for ϕ_1 have a volume estimate similarly related to topological pressure , $P\ (\ F\ |_{\Lambda i}\ ,\ \phi^{(u)}\ )$, of of the flow $F = (\phi_t : M \to M)$ and $\phi^{(u)}$, with $\phi^{(u)}(x)$ defined as $-d\,\lambda_t(x)/dt$, where $\lambda_t(x)$ is the Jacobian of $Df^t : E^u_x \to E^u_{f^t_x}$. We refer the reader to [2] for details.

Thus by Theorem 2.2.1 we get :

$\mu(S(i,j,N)) \leq D_{i,j} \exp(-C_{i,j}N)$.

But for T between N and $N+1$ we get :

$S(i,j,T) \subset S(i,j,N)$ whence :

$\mu(S(i,j,T)) \leq D_{i,j} \exp(-C_{i,j}N)$.

But then :

$\mu(S(i,j,T)) \leq D_{i,j} \exp(C_{i,j}(T-N)) \exp(-C_{i,j}T)$

We thus define $D^*_{i,j}$ to be : $D_{i,j} \exp(C_{i,j})$ and have :

$\mu(S(i,j,T)) \leq D^* \exp(-C_{i,j}T)$, where the $C_{i,j}$ are defined recursively as in Theorem 2.2.3 and C_i can be taken to be any positive number smaller than $-P(F, \phi^{(u)})$.

§4 A Discussion Concerning C^1 vs C^2, and Some Open Questions

Since we have proved a Theorem in Chapter 1 for a class of C^1 diffeomorphisms, the question arises as to under what conditions Theorem 2.2.2 holds for C^1 diffeomorphisms of M ? We know from Fried and Shub [5], that for each $x \in \Lambda$ a basic set for a diffeomorphism f we have a local volume lemma. However [5] does not provide a local estimate for $B_\Lambda(\varepsilon, N)$ only for $B_x(\varepsilon, N)$. While this suffices to yield their results on entropy, it does not yield the uniform $B_\Lambda(\varepsilon, N)$ estimate required herein. In fact Bowen [4] has given an example of a C^1 horseshoe with $\mu(W^s_\Lambda) > 0$ for Λ a basic set which is not an attractor.

For a C^1 diffeomorphism, not Morse-Smale, which has finite Theorem 2.2.2 holds. In general, however what additional hypothesis beyond Axiom A and no- cycles is needed to insure that Theorem 2.2.2 holds for a diffeomorphism of M is an open question.

A related open question concerns extending Theorem 1.3.1 to a larger class then Morse-Smale diffeomorphisms and flows, and still getting the exponential constant to be as in this Theorem. It may be that Axiom A, and strong transversality suffices.

APPENDIX I

Lemma 1.2.2 : Let E_1 , E_2 and E_3 be finite dimensional Euclidian Spaces and $W \subset E_3$ an open set. Let $\dim (E_1) < \dim (E_3) < \dim (E_1 \times E_2)$. Let $G : W \to E_1 \times E_2$ be a C^1 diffeomorphism of W onto its range $G(W)$. Further let $G(W)$ have transversal intersection with E_2 . Then $G(W)$ is a C^1 manifold in $E_1 \quad E_2$ of dimension equal to $\dim (E_1)$ and there is an open set $W^1 \subset W$ containing $G^{-1} (E_2)$ such that $G(W^1)$ is a C^1 Fiber bundle over $G(W^1) \subset E_2$ with fibers transversal to E_2 and having dimension equal to $\dim E_1$.

Proof : We write $G(W) = (g (w) , f(w))^T$ for $w \in W$ where $g : W \to E_1$ and $f: W \to E_2$ are the coordinate functions relative to the cross product $E_1 \times E_2$.

Now let $w_o \in W$ be such that $g (w_o) = 0$. Since $G (W)$ intersects E_2 transversally we know that $D G|_{w_o} (E_3) \supset E_1$; that is $D g|_{w_o}$ maps E_3

onto E_1. Thus we can find subspaces E_4 and E_5 of E_3 such that $E_3 = E_4 \times E_5$, $\dim (E_5) = \dim (E_1)$ and $(\partial g / \partial E_5 |_{w_o}) : E_5 \to E_1$ is an invertible linear operator. Hence by the implicit function theorem, there are open sets $U \subset E_4$ and $V \subset E_5$ with $w_o \in U \times V \subset W$ and a C^1 function $h : U \to V$ such that $g^{-1} (0) \cap (U \times V) = (x_4 , h (x_4))$. Moreover if $w \in U \times V$ is such that $g(w) = 0$, then w is a point on graph h.

This means that $g^{-1}(0) \cap (U \times V)$ is a C^1 manifold of $E_3 = E_4 \times E_5$ of dimension equal to $\dim(E_4)$. Now fix $u_o \in U$ and consider the vertical line in $U \times V$ given by $u = u_o$ for $w = (u,v) \in U \times V$. This set has a unique point of intersection with the graph of h and clearly is a C^1 manifold of E_3 with dimension equal to $\dim (E_5)$.

Hence $U \times V$ has been C^1 fibered over graph h. The same argument can be repeated for any $w \in W$ with $g(w) = 0$; the second conclusion of the implicit function theorem forces the two C^1 fiber spaces to agree on their intersections. That is let $w_1 \in W$ be such that $g (w_1) = 0$; we get $U_1 \subset E_4$ and $V_1 \subset E_5$ such that $g^{-1}(0) \cap (U_1 \times V_1)$ is the graph of a C^1 function $h_1 : U_1 \to V_1$ and if $w \in U_1 \times V_1$ is such that $g(w) = 0$, then w appears on graph h_1.

Now let $w \in (U_1 \times V_1) \cap (U \times V)$ and suppose $g(w) = 0$. Then $w = (u, h(u)) = (u, h_1(u))$ which forces $h(u) = h_1(u)$. Thus h and h_1 agree on their common domain. Therefore we can extend the C^1 fiber structure to $U^1 \subset E_4$ where U^1 is the projection onto E_4. Then we define $W^1 \subset W$ by $W^1 = U^1 \times V^1$ where V^1 is the projection of W onto E_5.

Now we consider $G(W^1)$. Since we are assuming that G is a diffeomorphism onto its range, we get that $G(g^{-1}(0))$ is a C^1 manifold in $G(W^1)$ and also that $G(\{(u,v) \mid u = u_0 \in U^1\})$ is a C^1 manifold of $G(W^1)$. By assumption the intersection of the two is transversal. Now since $G(W^1)$ is a C^1 manifold in $E_1 \times E_2$ we have that each of $G(\{(u,v) \mid u = u_0 \in U^1\})$ is a C^1 manifold of $E_1 \times E_2$ and also that $G(g^{-1}(0))$ is a C^1 manifold of $E_1 \times E_2$. Thus $G(W^1)$ has a C^1 fiber structure over the intersection of E_2 and $G(W^1)$ with the fibers having dimension equal to the dimension of E_1. Note also that $G(W) \cap E_2$ is characterized by $g(x) = 0$. Thus W^1 contains $G^{-1}(E_2)$.

Lemma 1.2.3 : For $i > 2$, if $g : cl\, E^u_{\cdot}(\varepsilon_i) \rightarrow E^s(\varepsilon_i)$ is a C^1 function with $\|D g\| \leq \omega_i$, then area (graph $g \cap S_N$) $\leq K (1 + \delta)^N \exp (-C N)$, where $K > 0$, $C = \min_{i \leq j \leq 2} \{\text{Jac } D f^u (P_j)\}$, and area refers to the induced r dimensional measure along graph g, where r = dimension (E^u).

Proof : Before we begin the proof of this Lemma we need the following definitions from [6] .

Definition : Let $f \in$ Diff(M) and P be a hyperbolic fixed point of f. We denote by LS(P) and LU(P) the local stable and unstable manifold of P. Let B^s be a cell neighborhood of P in LS (P), such that $f(\partial B^s) \subset \text{int } B^s$. The existance of such a cell B^s follows from the fact that f/L (P) is a contraction. The embedded annulus in LS(P) whose boundaries are B^s , $f(\partial B^s)$ is called a fundamental domain $G^s(P)$ of $W^s(P)$. We have $W^s(P) = \bigcup_{n \in Z} f^n (G^s(P)) \bigcup P$. Any neighborhood $N^s(P)$ of $G^s(P)$ in M, disjoint from LU(P) , is called a fundamental neighborhood associated with $W^s(P)$.

Dually we can define $G^u(P)$ and $N^u(P)$.

We now begin the proof of Lemma 1.2.3.

The area measures are induced by the Riemannian metric induced by the inner product $\langle v,w\rangle = (v^u,w^u) + (v^s,w^s)$. The proof is by induction on i. If i=2, the statement is essentially proved in [5]. For completeness and for use in the induction step we reprove it here. If $x \in S_N$, then $F_x^k \in \mathrm{cl}\ T_2M(\varepsilon)$ for $k=0,\ldots, N-\ell$, where ℓ is a fixed integer representing the maximum number of iterations for an orbit leaving $U(P_2)$ to enter V the neighborhood of the sink(s) on M, (recall $F = \phi_2^{-1} \circ f \circ \phi_2$). The same statement thus applies to $F^k(\text{graph } g \cap S_N)$. Moreover we know that successive iterates of (graph $g \cap S_N$) are themselves graphs of C^1 functions of cl $E_2{}^u(\varepsilon)$ to cl $E_2{}^s(\varepsilon)$ with slope $\leq \omega_2$. This is because ε was chosen to insure that the graph transform of g, $\Gamma_2 g$ is a C^1 function with slope $\leq \omega_2$.

Thus $F^{N-\ell}$(graph $g \cap S_N$) is indeed the graph of a C^1 function of cl $E^u(\frac{\varepsilon}{2})$ into cl $E^s(\varepsilon_2)$ with slope $\leq \omega_i$. Thus we have :

(I) area ($F^{N-\ell}$(graph $g \cap S_N$)) $\leq$ K , where K is a constant depending only on ε and ω_2. So :

$$K = \int_{\text{graph } g \cap S_N} \text{Jac } D_v \left(F^{N-\ell} \,|\, v + E^u_{P_2} \right) d\mu_r \quad .$$

But the Jacobian inequality (*) holds in $\text{cl } T_{P_2} M(\varepsilon)$. That is if $L \in \text{Hom}(E^u_{P_2}, E^s_{P_2})$ with $\|DL\| \leqq \omega_2$ and $v \in \text{cl } T_{P_2} M(\varepsilon)$ we have :

$\text{Jac } D_v \left(F \,|\, v + \text{graph } L \right) \geqq (1+\delta')^{-1} \text{Jac } D_o(F \,|\, E^u_{P_2})$.

Here $0 < \delta' < \delta$ is the constant in the proof of Theorem 1.2.1 We apply this inequality repeatedly to get from (I) the following :

$$K \geqq (1+\delta')^{-N+\ell} \text{Jac } D_o \left(F^{N-\ell} \,\middle|\, E^u_{P_2} \right) \text{ area } (\text{graph } g \cap S_N)$$

so that :

$$\text{area } (\text{graph } g \cap S_N) \leqq K (1+\delta')^{N-} \left(\text{Jac } D_o (F^{N-\ell} \,|\, E^u_{P_2}) \right)^{-1}.$$

Now since f is assumed to have only fixed points,

$$\text{Jac } D_o (F^{N-\ell} \,|\, E^u_{P_2}) = \text{Jac } D_o (F \,|\, E^u_{P_2})^{N-\ell} \quad .$$

Then by redefining K we get :

$$\text{area } (\text{graph } g \cap S_N) \leqq K (1+\delta')^{N} \left(\text{Jac } D_o (F \,|\, E^u_{P_2}) \right)^{-N+\ell}.$$

By observing that the properties of ϕ_2 allow us to show that : $\text{Jac } D f^u(P_2) = \text{Jac } D_o (F \,|\, E^u_{P_2})$ we are done with the $i = 2$ case.

We now assume that our result has been proven for all $i < k$, where $k > 2$ and prove it holds for graphs in $\text{cl } T_k M(\varepsilon)$. So now $S_N = \phi_k^{-1}(U_N(P_k))$. Here the situation for graph $g \cap S_N$ is different because $x \in S_N$ does not imply that $F^{N-\ell}(x) \in cl\ T_k M(\varepsilon)$, where ℓ again is a fixed integer. In fact on M an orbit beginning in $U(P_k)$ may leave $U(P_k)$ and enter some $U(P_j)$ before reaching V. Therefore we must consider $N+1$ seperate subgraphs of (graph $g \cap S_N$). We let B_m be that subgraph of (graph $g \cap S_N$) for which $F^{m+1} B_m \cap cl\ T_k M(\varepsilon) = \emptyset$. But for which $F^m B_m \subset cl\ T_k M(\varepsilon)$

for $0 \leq k \leq m$, (i.e. B_m leaves $T_{P_2}(\varepsilon)$ at exactly the $(m+1)$ iteration.

Thus (graph $g \cap S_N$) $- E_k^s = \bigcup_{m=0}^{\infty} B_m$ where the union is disjoint. We shall modify B_m slightly, replacing B_m by $cl\ B_m$. Then (graph $g \cap S_N$) $- E_k^s = \bigcup_{m=0}^{\infty} B_m$ with the union no longer disjoint but consisting of compact sets. Furthermore we only need to consider a finite union :

$$(\text{graph } g \cap S_N) = A \cup B_0 \cup B_1 \cup \ldots \cup B_{N-1}$$

where A is the closure of that portion of graph ($g \cap S_N$) which remains in $cl\ T_{P_2} M(\varepsilon)$ for at lease N iterations. Since all these sets are disjoint except on their boundaries we have :

$$\text{area} (\text{graph } g \cap S_N) = \text{area} (A) + \sum_{m=0}^{N-1} \text{area} (B_m) .$$

Furthermore we know by the arguments used in the $i=2$ case that :

$$\text{area} (A) \leq K (1 + \delta')^N \exp (- C_k N) \text{, and also that :}$$

$$\text{area} (B_m) \leq (1 + \delta')^m \text{area}(F^m B_m) \exp (-C_k m) \text{, where}$$

$C_k = \text{Jac } D f^u (P_k)$. We now consider :

area ($F^m B_m$) more carefully. The compact set $\phi_k F^m B_m \subset U(P_k)$ passes out of $U(P_k)$ on the next iteration, that is : $f(\phi_k F^m B_m) \cap \text{int } U(P_k) = \emptyset$. Subgraphs of $\phi_k F^m B_m$ may pass under further iterations of f to any $U(P_j)$ for $j < k$ or to V . However since $\phi_k B_m \subset U_N(P_k)$ any subgraph passing to V must be in a B_m with m $N - \ell$ for a fixed integer ℓ . To define the subgraph of ' $\phi_k F^m B_m$ we perform the following procedure. We note that $\phi_k F^m B_m$ is contained in the closed fundamental neighborhood (associate to W^u_k) given by : $D_k = U(P_k) - \text{int } f^{-1} U(P_k)$. We consider :

$$D_k \cap \bigcup_{n=1}^{\infty} f^{-n} (U(P_{k-1}))$$; this is in fact equal to

$$D_k \cap \bigcup_{n=1}^{\ell} f^{-n} (U(P_{k-1}))$$ for some fixed ℓ representing the maximum number of iterations for an orbit $U(P_k)$ to enter $U(P_{k-1})$ for the first time. Hence the intersection of D_k with inverse images of $U(P_{k-1})$ is compact . We also know that the subgraph of $\phi k F^m B_m$ in this intersection is compact.

Now the iterates of $\phi_k F^m B_m \cap \bigcup_{n=1}^{\ell} f^{-N}(U(P_{k-1}))$ under f may not enter $U(P_k)$ at one iteration precisely, but for simplicity of notation we will assume this is the case. In any event we are faced with at most another finite decomposition of $\phi_k F^m B_m$ into subgraphs identified as to by which iteration they enter $U(P_{k-1})$, for the first time.

Hence we assume that $f^{\ell}\phi_k F^m B_m \subset U(P_{k-1})$. Since $F^m B_m$ is the graph of a C^1 function with slope $\leqq \omega_k$, in $\mathrm{cl}\, T_{P_k} M(\varepsilon)$ we know from Proposition 1.2.1 that :

$\phi_k^{-1} f^{\ell}\phi_k F^m B_m \cap \mathrm{cl}\, T_{P_{k-1}} M(\varepsilon)$ is a C^1 fiber bundle over : $\phi_k^{-1} f^{\ell}\phi_k F^m B_m$ intersected with E^s_{k-1} in the fundamental neighborhood associated with W^s given by $V_{k-1} - f(V_{k-1})$. Furthermore the fibers are the graphs of C^1 functions from a closed subset of $\mathrm{cl}\, E^u_{k-1}(\varepsilon)$ into $\mathrm{cl}\, E^s_{k-1}(\varepsilon)$ and all of the slopes $\leqq \omega_{k-1}$. By the induction hypothesis the area ($\dim E^u_{k-1}$) of any fiber is bounded by : $K(1+\delta'')^{N-\ell-m} \exp(-C(N-\ell-m))$ where δ'' is between δ' and δ and C is the appropriate minimum Jacobian. From this we may deduce the area ($\dim E^u_k$) of $\phi_k^{-1} f^{\ell}\phi_k F^m B_m$ by using Fubini's Theorem which applies due to the C^1 fiber structure. We get :

area $(\phi_{k-1}^{-1} f^{\ell}\phi_k F^m B_m) \leqq K(1+\delta'')^{N'} \exp(-CN')$ for $N' = (N-\ell-m)$, and where we have redefined K.

We now translate this back to $\mathrm{cl}\, T_{P_k} M({}_{\varepsilon}\varepsilon)$ and thus find that for a subgraph of $F^m B_m$ its length is bounded by :

$K(1+\delta'')^{N'} \leqq \exp(-CN')$ where again K is redefined to account for bounded factors in the translation, and $N' = (N-\ell-m)$.

We repeat this procedure to determine the areas of the other subgraphs of $F^m B_m$. For P_{k-2} we consider the intersection of :

D_k - int $(D_k \cap \bigcup_{n=0}^{\ell} f^{-n} (U(P_{k-1})))$ and $\bigcup_{n=0}^{\infty} f^{-n}(U(P_{k-2}))$ to determine the portion going directly to $U(P_{k-2})$.

Note that ℓ may change for P_{k-2}. The identical estimates result. That is the area of a subgraph of $F^m B_m$ is bounded by an expression of the form :

$K (1 + \delta'')^{(N-\ell- m)} \exp (C (N - \ell - m)$ (with a possible different K. Since these are finitely many fixed points then the entire area of $F^m B_m$ is of the same form with K modified to account for the number of points and the different ℓ. Then translating the above expression back to the initial iteration $n = 0$ we immediately get :

$$\text{area } (B_m) \leq (1 + \delta')^m K (1 + \delta'') \exp(-C_k m - C N')$$

for $N' = (N - \ell - m)$, where C now is the min of Jac $D f^u (P_j)$ for $1 \leq j \leq k$. We finally redefine K to get :

$$\text{area } (B_m) \leq K (1 + \delta'')^N \exp (-C N)$$

Finally we arrive at :

$$\text{area } (\text{graph } g \cap S_N) \leq \text{area } A + \text{area } B_o + \ldots + \text{area } B_{N-1}$$

$$\leq N K (1 + \delta'')^N \exp (- C N)$$

and since exponential decay dominates algebraic growth we may redefine K to get :

$$\text{area}\ (\text{graph}\ g \cap S_N) \leqq K(1+\delta)^N \exp(-CN)$$

as desired. This completes the proof .

APPENDIX II

<u>Lemma 2.2.1</u> : For $\delta > 0$ small, and for all $N \geq 2(L_i + w_{i-1,j})$ we find that :

$$\mu(D) \leq K(\exp -C_i^{\#}(1-\delta)N), \text{ where}$$

$$C_i^{\#} = \min\{C_i/q, C_{i-1,j}(1-\delta)\}, \text{ and}$$

$q \in Z^+$ is chosen so that it satisfies :

$$q \geq 1/\delta(1+(C_i^{*}/C_{i-1,j}), \text{ with}$$

$$C_i^{*} = \log(\min_{P_i} {}_{M[i,i-1]} |\mathrm{Jac}\, D_{P_i} f|) .$$

<u>Proof :</u> Recall that $D = \{x \in A_N : f^{L_i}(x) \in M[i,i-1]\}$. We now decompose D into $D_1 \cup D_2$ where :

$$D_1 = \{x \in D : f^{N/q+L_i}(x) \notin M[i,i-1]\} \text{ and}$$

$$D_2 = \{x \in D : f^{N/q+L_i}(x) \in M[i,i-1]\} .$$

We have : $D_2 \subset f^{-n_i}(B_{\Lambda_i}(\varepsilon, N/q))$, which yields an estimate for $\mu(D_2)$.

$$\mu(D_2) \leq J_i(B_{\Lambda_i}(\varepsilon, N/q))$$

$$\leq J_i D_i \exp(-C_i/q)N,$$

for $N \leq q\, w_{i,i-1}$. Next let n be such that: $N/q - n \geq 0$, but $N/q - n - 1 \leq 0$, and write :

$D_1 = \bigcup D_1(k)$, for $k = 0,\dots,n$. Where $D_1(k)$ has the property that :

$$f^{N/q + L_i - k}(D_1(k)) \subset M[i-1,j] .$$

It is also a subset of $S(i-1,j,N-n/q-L_i+k)$.

In order to extract the best possible estimate for $D_1(k)$ we consider : $f^{N/q+L_i-k}(D_1(k))$.

Then : $$(f^{N/q+L_i-k}(D_1(k)) = \int_{f^{N/q+L_i-k}(D_1(k))} d\mu$$

So : $$(f^{N/q+L_i-k}(D_1(k)) = \int_{D_1(k)} Jac\, f^{N/q+L_i-k}\, d\mu$$

$$\leq J_i^{*L_i} J_i^{*N/q-k} \int_{D_1(k)} d\mu$$

$$\leq J_i^{*L_i} J_i^{*N/q-k} \mu(D_1(k)) .$$

Thus : $$\mu(D_1(k)) \leq J_i^{*L_i} J_i^{*N/q-k} (S(i-1,j,N-N/q-L_i+k))$$

And : $$\mu(D_1) \leq J_i^{*L_i} \sum_{k=0}^{n} J_i^{*N/q-k} (S(i-1,j,N-N/q-L_i+k)$$

Now using :

$$\mu(S(i-1,j,N-N/q-L_i+k)) \leq D_{i-1,j}\left(\exp(C_{i-1,j}(N-N/q+k)) \cdot \exp(-C_{i-1,j} L_i)\right)$$

We get :

$$\mu(D_1) \leq J_i^{*L_i} \sum_{k=0}^{n} J_i^{*N/q-k}\left(D_{i-1,j} \exp(C_{i-1,j}(N-N/q-k)) \exp(-C_{i-1,j} L_i)\right).$$

So :

$$\mu(D_1) \leqq J_i^{*L_i} \sum_{k=0}^{n} \Big(D_{i-1,j} \exp(C_{i-1,j} L_i) \, J_i^{*N/q-k} \cdot$$

$$\exp(-C_{i-1,j}(N+k-N/q)) \Big) .$$

$$\mu(D_1) \leqq J_i^{*L_i} \sum_{k=0}^{n} D_{i-j,j} \exp(C_{i-1,j} L_i) \exp(C_i^*(n/q-k) \Big)$$

Let $D_k = D_{i-1,j} \exp(C_{i-1,j} L_i)$ and rewriting yields ;

$$\mu(D_1) \leqq J_i^{*L_i} \sum_{k=0}^{n} D_k \exp((-C_{i-1,j})(1-\delta) N) \text{ , for}$$

$N \geqslant (q/(q-1)) \, (L_i + w_{i-1,j})$ which holds if

$N \geq 2 \, (L_i + w_{i-1,j})$, since $q/(q-1) \geqq 2$ for all $q \geqq 1$.

Now majorizing the above sum we get:

$$\mu(D_1) \leqq J_i^{*L_i} D^*(N+1) \exp ((-C_{i-1,j})(1 - \delta) N).$$

But : $\mu(D) = \mu(D_1) + \mu(D_2)$.

Thus we get :

$$\mu(D) \leqq J_i^{*L_i} D_k^*(N+1) \exp((-C_{i-1,j})(1 - \delta) N)$$

$$+ \; J_i \, D_i \exp((-C_i/q)N) \quad .$$

Next let $C_i^{\#} = \min\{ C_i/q \, , \, C_{i-1,j} (1 - \delta) \}$ and using the fact that algebraic growth is dominated by exponential decay, then there exists a constant K^* such that : $\mu(D) \leqq K^*(\exp(-C_i^{\#}(1 - \delta) N)$ for δ small and $N \geqq 2 \, (L_i + w_{i-1,j})$. This completes the proof.

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