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I. THEORY OF COEXISTENCE OF SUPERCONDUCTIVITY AND CHARGE DENSITY WAVES. II. TRANSIENT OPTICAL PROPAGATION THROUGH A THIN SLAB

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I. "THEORY OF COEXISTENCE OF SUPERCONDUCTIVITY AND CHARGE DENSITY WAVES"

II. "TRANSIENT OPTICAL PROPAGATION THROUGH A THIN SLAB"

by

Joseph Malinsky

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

THEORY OF COEXISTENCE OF CHARGE-DENSITY-WAVES AND SUPER-CONDUCTIVITY AND TRANSIENT OPTICAL PROPAGATION THROUGH A DIELECTRIC SLAB.

by

Joseph Malinsky

Adviser: Professor J.L. Birman

Ginzburg-Landau-Gor'kov equations and the expression for the current for coexisting superconducitivity-charge-density-wave (CDW) systems near the superconducting transition temperature have been derived from a microscopic theory. An additional current due to the sliding CDW and the CDW-induced anisotropy lead to interesting physical consequences as far as thermodynamic and electrodynamic properties of this system are concerned. For instance, Meissner effect is sensitive to the contribution from both currents. We propose measurements of CDW-induced anisotropy which can be used to prove the microscopic mechanism of coexistance.

In the second part of this thesis we propose a microscopic theory of electromagnetic transients propagating through a thin dielectric slab. This theory is time

dependent, allowing us to calculate the energy and momentum transfer to the medium which comprises an array of Lorentz harmonic oscillators. In the microscopic level for the first time, we recover the first Sommerfeld precursor.

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I thank at last, but not least, Mrs. E. De Crescenzo, Mr. Y. Budansky and Mr. J. Pajuelo for their encouragement and technical help during the preparation of my thesis.

Dedication

I dedicate this thesis to my colleagues left behind in the U.S.S.R. who, in spite of all difficulties, persevere in continuing the affair with their first love - physics. I dedicate this thesis also to my family - my wife, Julia, my children, and my so distant parents and sister.

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Part I

Theory of the Coexistence of Charge-Density-Waves and Superconductivity

Chapter 1. Historical Background, Experimental Evidence for Coexistance.

A. Introduction.

For a one-dimensional metal R. Peierls showed that the energy of the electrons can always be lowered by opening an energy gap in the vicinity of the Fermi energy. This effect transfers all the occupied states to the lower energy gap. Also a structural phase transition occurs which gives rise to the formation of a charge density wave. See Figure 1.1.

In three dimensional systems the process of gap opening may also take place in the directions where the Fermi surface is "nested". "Nesting" means that we have similar (parallel) areas of Fermi surface which are spanned by a certain wave vector, see Figure 1.2. Under this condition we expect to observe the coexistance of a charge density wave and superconductivity.

B. Experimental Investigations.

Transition metal dichalcogenides (TX_2) are good candidates for observations of charge-density-waves because of the two-dimensional character of their structure [1]. In the 2H polytype of $NbSe_2$, the onset of an incommensurate CDW at $T_p = 33.5K$ has been established by neutron experiments [2]. Experimental physicists established that this compound is of particular interest since it exhibits both the highest superconducting temperature $T_c (= 7\text{-}7.3K)$ and the lowest $T_p = 33^oK$ observed in the TX_2 family. In 1975 C. Berthier, P. Molinie and D. Jerome [3] presented evidence for a connection between the CDW and the pressure enhancement of superconductivity in 2H - $NbSe_2$. They concluded that elimination of the charge density wave under pressure leads to an increase of the superconducting gap in this compound.

Recently some compounds have been found (like $Eu_yMo_6S_8$, $y \sim 1$) which exhibit interplay among superconductivity, charge-density-waves and magnetism [4].

In 1980 [5] R. Sooryakumar and M.V. Klein showed the coexistence of charge-density-wave and superconductivity by a Raman scattering experiment on 2H - $NbSe_2$. The sample was immersed in superfluid helium in the presence of a variable magnetic field. Coexistance or rather competition between charge-density-wave and superconductivity was shown for example at H_{c_2} where the superconductivity peak disappeared and the CDV peak was enhanced in the Raman spectrum. In 2H-$NbSe_2$ there are ordinary phonon Raman lines at 234 and 243 cm^{-1} [5]. Also there are the Raman lines induced by charge-density-waves [1] below T_p at approximately 40 cm^{-1}. Sooryakumar and Klein discovered two additional peaks at approximately 16 cm^{-1} which are present below the superconducting transition temperature T_c. These values agree with the superconducting energy gap, $2\Delta = 17.2 \pm .4$ cm^{-1} as measured in a different sample by infrared transmission [2]. These new lines decrease in intensity, and shift in frequency when large magnetic fields ($\sim$40KG) are applied. This indicates that they are associated with the superconductivity of the material. A typical diagram is shown in Figure 1.3. Also the frequency spectrum showed well-defined sharp lines which clearly indicate that they cannot be directly related to the excitation of superconducting quasiparticles. This sharpness is characteristic of boson-like excitations (phonons, magnons). The pronounced magnetic field dependence clearly indicates that the superconducting properties of the sample must be involved.

As has been mentioned above, coexistence of superconductivity, charge-density-waves and magnetism was also detected in $Eu_yMo_6S_8$. It is of general physical interest to understand the nature of the coexistence of quite different physical phenomena like superconductivity and CDW. This should be understood on a microscopic level. In addition we must investigate any possible interesting macrascopic properties of such materials and their thermodynamic and electromagnetic properties. In this thesis an investigation of these matters will be given.

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Chapter 2. Review of Previous Theories of Superconductivity and Charge-Density-Wave Coexistence.

A. Introduction.

In this chapter a brief review of previous theories will be given. Special emphasis will be given to the ideas of Balseiro and Falicov who proposed an electron-phonon coupling mechanism, and of Littlewood and Varma who criticized the treatment by Balseiro and Falicov, and proposed a gauge-invariant theory. Other theories will also be mentioned.

In contrast to the present work no previous theory derived the macroscopic current or macroscopic Ginzburg-Landau-Gor'kov equations starting from a microscopic model.

B. Balsiero-Falicov Theory.

In the paper "Phonon Raman Scattering in Superconductors" by C.A. Balseiro and L.M. Falicov [1], they suggested that electron-phonon coupling leads to a complex bound excitation which is Raman active, with a discrete frequency lower than twice the BCS gap 2Δ and with an appreciable scattering intensity. They proposed that these excitations are the lines found by Sooryakumar and Klein at approximately the energy-gap frequency in superconducting 2H-$NbSe_2$. Namely they suggested"as a very likely explanation that the observed lines are a direct consequence of the interaction between $\vec{q} = 0$ phonons of low frequency (at 40 cm^{-1}, caused by charge-density-waves) and the superconducting electrons."

In order to investigate this mechanism, they used the Hamiltonian

$$H = H_{BCS} + H_{ph} + H_{int} \tag{2.1}$$

which describes a Bardeen-Cooper-Schrieffer model of interacting electrons, a single $\vec{q} = 0$ phonon of frequency ω_0

$$H_{ph} = \hbar\omega_0 b^+ b \tag{2.2}$$

and an interaction term

$$H_{int} = g \, \Sigma_{k\sigma} \, c^+_{k\sigma} c_{k\sigma} (b+b^+) \tag{2.3}$$

where $b^+(b)$ creates (destroys) a phonon and $c^+_{k\sigma}$ $(c_{k\sigma})$ is the creation (destruction) operator for an electron of wave vector k and spin σ.

They used Bogoluibov canonical transformation methods and a perturbation calculation which hybridizes the single-phonon state with an electron-hole pair. See Figure 2.1. They found the complete one-phonon spectral function and showed that for $\hbar\omega_0 > 2\Delta$ the phonon intensity is highly peaked about $\hbar\omega_0$, but with a continuous distribution which extends over the excited pair energy range, $\hbar\omega > 2\Delta$. Also they showed the existence of a bound state of energy λ which splits off from the quasiparticle continuum: $\lambda < 2\Delta$. Figure 2.2.

We only cite the most interesting conclusions following from the Balsiero-Falicov model:

a) A bound state consisting of a phonon-quasiparticle-pair always exists in their model and its energy λ is smaller than 2Δ;

b) The spectral intensity of the bound state is only appreciable when $\hbar\omega_0$ is of the same order of magnitude as 2Δ. This explains why the effect has been observed in 2H-$NbSe_2$, a CDW superconductor (ordinary Raman-active phonons have frequencies = (10 - 100) $<$ energy-gap frequency)). In particular an incommensurate CDW has some low frequency phonon modes which are Raman active because of "folding" of the Brillouin Zone: this originates because opposite parts of the Fermi Surface are spanned by a reciprocal vector which becomes the new reciprocal lattice vector and defines a new Brillouin Zone. This is the effect of "nesting": Figure 1.2.

C. Littlewood-Varma Theory.

P.B. Littlewood and C.M. Varma [2] attempted to develop a gauge-invariant theory of "dynamic interactions of a charge-density-wave and superconductivity." Using the results of the calculation of energy bands by J.E. Inglesfield [3] and earlier work by L.F. Mattheiss [4], Littlewood and Varma emphasized that in a CDW state, there exist special zone-center optic phonon modes. These CDW amplitude modes (CDW-AM) are accompanied by an oscillation of the CDW gap, which leads to a variation of the average density of states at the Fermi surface, Figure 2.4. Furthermore, Littlewood and Varma argued that if a material in a CDW state undergoes a superconducting transition, the excitation of the CDW-AM also leads to a time dependent perturbation of the superconducting gap, wince the latter depends on the density of states at the Fermi surface, N(0). They studied the effect of this coupling.

In the study of the dynamics the leading order coupling of the amplitude $\vec{u}$ of the CDW-AM to the superconducting gap is given by

$$\Delta = \Delta_0 + \Delta_1 \vec{u} \qquad (2.4)$$

where

$$\Delta_1 = \frac{\Delta_0}{\lambda_0 N(0)} \cdot \frac{dN(0)}{du} \tag{2.5}$$

where λ_0 is the BCS coupling constant.

They used the Hamiltonian

$$H = H_{ph} + H_{el} + H_{int} \; .$$

where in H_{ph} they considered only the $q \simeq 0$ CDW-AM and for the Hamiltonian they took

$$H_{ph} = \hbar\omega_0 b^+ b$$

where H_{el} is the electronic part which gives rise to superconductivity. In the Nambu notation [5]

$$H_{el} = \sum_k \epsilon_k \psi^+_{\vec{k}} \tau_3 \psi_k + \frac{1}{2} \sum_{\vec{k}\vec{k}'\vec{q}} V(\vec{k},\vec{k}',\vec{q}) (\psi^+_{\vec{k}+\vec{q}} \tau_3 \psi_{\vec{k}}) (\psi^+_{\vec{k}-q} \tau_3 \psi_{\vec{k}}) \tag{2.5}$$

where electron creation and annihilation operators are written as two component vectors.

$$\psi_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow} \end{pmatrix}, \qquad \psi^+_{\vec{k}} = (c^+_{\vec{k}\uparrow}, c_{-\vec{k}\downarrow}) \tag{2.6}$$

The τ's are Pauli matrices; $V(k,k',q)$ contains both the Coulomb repulsion and the effective attraction mediated via phonons. Thus

$$H_{el} = H_0 + H_1 \tag{2.7}$$

(2.8) $$H_0 = \sum_k \psi_k^+(\epsilon_k\tau_3 + \Delta_0\tau_1)\psi_k$$

(2.9) $$H_1 = \frac{1}{2}\sum_{kk'q} V(kk'q)(\psi_{k+q}^+\tau_3\psi_k)(\psi_{k'-q}^+\tau_3\psi_{\vec{k}'}) - \sum_k \psi_k^+\Delta_0\tau_1\psi_k.$$

They wrote H_{int} in the form:

(2.10) $$H_{int} = H_{int}^{\Delta} + H_{int}^{\rho}$$

where H_{int}^{Δ} is the part due to the effect arising from $\Delta = \Delta_0 + \Delta_{1u}$ which is [5]

(2.11) $$H_{int}^{\Delta} = g(b+b^+)\sum_k \psi_k^+\tau_1\psi_k = g = \Delta_1\left(\frac{\hbar}{2NM\omega_0}\right)^{1/2}$$

and

(2.12) $$H_{int}^{\rho} = \lim_{\vec{q}\to 0} g(b_q+b_{-q}^+)\sum_k \psi_{k+q}^+\tau_3\psi_k \qquad [5].$$

Littlewood and Varma considered H_{int}^{Δ} as the crucial part for the coexistance problem. (Below we will very briefly compare the Littlewood-Varma and Balseiro-Falicov theories.) The phonon propagator for the q = 0 CDW-AM is modified from its unperturbed form denoted $D_0(\nu)$ in the absence of superconductivity ($\Delta_0 = 0$) to

(2.13) $$D^{-1}(\nu) = D_0^{-1}(\nu) - \Sigma(\nu) \qquad [5]$$

where $\Sigma(\nu)$ is the phonon self energy. In the lowest order $\Sigma = \Sigma_0$ is represented by $g\tau_1$  $g\tau_1$ and within the BCS approximation is equal to

$$(2.14) \qquad \Sigma_0 = \frac{-ig^2}{(2\pi)^4}\int d^3k \int d\omega \; \mathrm{Tr}[\tau_1 G(\vec{k},\omega+\nu)\tau_1 G(\vec{k},\omega)]$$

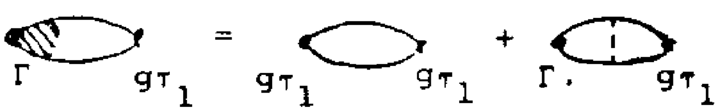
where

$$(2.15) \qquad G^{-1}(\vec{k},\omega) \text{ is the BCS propagator}$$

and

$$(2.16) \qquad G^{-1}(\vec{k},\omega) = \omega I - \epsilon_k \tau_3 - \Delta_0 \tau_1 .$$

Littlewood and Varma went beyond the BCS approximation and calculated the "Ladder Corrections" to the phonon self-energy due to H_1. Diagramatically they are



and give

$$(2.17) \qquad \Sigma(\nu) = -i\int \frac{d^3k d\omega}{(2\pi)^4} \mathrm{Tr}[\Gamma(\vec{k}\omega;\vec{k},\omega+\nu)G(\vec{k},\omega+\nu)g\tau_1 G(\vec{k},\omega)]$$

where Γ calculated in the ladder approximation is

$$(2.18) \qquad \Gamma(\vec{k},\omega+\nu;\vec{k},\omega)$$

$$= g\tau_1 + i\int \frac{d^3k d\omega}{(2\pi)^4} \tau_3 G(\vec{k},\omega'+\nu)\Gamma(\vec{k}',\omega'+\nu;\vec{k}',\omega')G(\vec{k}',\omega')\tau_3 V_k .$$

They found

$$(2.19)\qquad \Sigma(\nu) = \frac{\Sigma_0(\nu)}{1 + \frac{\lambda_0}{2g^2N(0)}\,\Sigma_0(\nu)}$$

and

$$\mathrm{Re}\ \Sigma_0(\nu) = -2N(0)g^2\left[\frac{1}{\lambda_0} - \left(\frac{4\Delta_0^3-\nu^2}{\nu^2}\right)^{1/2}\tan^{-1}x\right] \quad \text{for } \nu < 2\Delta_0$$

$$= -2N(0)g^2\left[\frac{1}{\lambda_0} - \frac{1}{2}\left(\frac{\nu^2-4\Delta_0^2}{\nu^2}\right)^{1/2}\ln\left|\frac{1+x}{1-x}\right|\right],$$

$$\text{for } \nu > 2\Lambda_9$$

where

$$(2.20)\qquad x = \left|\frac{\nu^2}{\nu^2-4\Delta_0^2}\right|^{1/2}\left[1 + \frac{\Delta_0}{\hbar\omega_D}\right]^{-1/2}.$$

One can see that $\Sigma(\nu)$ is divergent for any value of the coupling constant g for $\nu \simeq 2\Delta_0$. This means that a pole necessarily appears in the phonon spectral weight [5]

$$(2.21)\qquad S(\nu) = -\frac{1}{\pi}\,\mathrm{Im}\ D(\nu)$$

at a frequency ν_g below 2Δ. Introduce a dimensionless coupling strength: $\alpha = \frac{4g^2N(0)}{\lambda^2\hbar\omega_0}$. For $\alpha << 1$

$$(2.22)\qquad \hbar\nu_g = 2\Delta\left[1 - \frac{2\alpha^2}{\pi^2}\left(1 - \frac{4\Delta_0^2}{(\hbar\omega_0)^2}\right)^{-1}\right]$$

with spectral weight

$$S(\nu_g) = \frac{8\alpha^2}{\pi^2} \cdot \frac{2\Delta/\hbar\omega_0}{[1-(\frac{2\Delta}{\hbar\omega_0})^2]^3} . \tag{2.23}$$

Littlewood and Varma identified this mode with the new "gap" mode observed by Sooryakumar and Klein.

Littlewood and Varma argued that Falicov and Balsiero used H^{ρ}_{int} which represents the usual coupling of phonons to the long-wavelength components of the electronic charge density. In the BCS approximation, the self-energy of a phonon due to H^{ρ}_{int} is

$$\Sigma_{\rho}(\nu) = -ig'^2 \int \frac{d^3k\,d\omega}{(2\pi)^4} \mathrm{Tr}[\tau_3 G(\vec{k},\omega+\nu)\tau_3 G(\vec{k},\omega)] . \tag{2.24}$$

Littlewood and Varma noticed that this quantity occurs in the calculation of the longitudinal response of the superconductor. It produces a plasmon-like high frequency excitation. Thus they argue this is not relevant to the Sooryakumar and Klein experiment as Falicov and Balsiero assumed.

D. <u>Other theories</u>.

After Balsiero and Falicov and Littlewood and Varma several more articles appeared. For example D. A. Browne and K. Levine [6] investigated the coupling of a coexisting incommensurate charge-density-wave superconductor. They use a gauge-invariant Random Phase Approximation that correctly includes screening. They treat both the amplitude and phase modes of the coupled system. Unlike previous work by Balsiero and Falicov, and Littlewood and Varma, the SC and CDW ordering were treated fully self-consistently, on an equal footing, so they obtained qualitative agreement with Raman scattering data on $NbSe_2$.

X.L. Lei. C.S. Ting and J.L. Birman [7] took anisotropy of the materials into account by using the usual Bogoluibov transformation they investigated SC and CDW coupling self-consistantly,and calculated Raman line location and intensity.K. Machida also presented an investigation of the interplay between superconductivity and charge-density-wave based on a partial gapping model appropriate for anisotropic materials. Various thermodynamic quantities such as the specific heat jump at the superconducting transition and the anisotropic penetration depth were derived [8].

An interesting treatment was proposed by J.L. Birman and A. Solomon [9] who were trying to understand the problem of coexistance of charge-density-wave and superconductivity from a dynamical group theoretical point of view. Using the Cartan algebra [10] they analysed the possibility that a Hamiltonian shall contain symmetries characteristic of both superconductivity (SU(2)) and charge-density-wave (O(3)). This method is sufficient for diagonalization of complicated Hamiltonians and also illuminates the role of symmetry in coexistance problems and can be applied to other systems. Probably, in this way we may have interesting interplay between solid state physics and recent trends in supersymmetries in particle physics. We will not discuss other articles but the reader should be aware that work on coexistance of CDW and SC continues actively [11].

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Chapter 3. Microscopic Theory of Superconducting and Charge Density Wave Currents.

A. Introduction.

The supercurrent in a pure superconducting system causes interesting macroscopic effects. For an idealized pure charge-density-wave system with an incommensurate CDW vector $|\vec{Q}| = 2k_F$, and neglecting pinning, a sliding mode produces a different type of persistent current. This idea was proposed by Frohlich [1] and later discussed by Bardeen [2] and Lee, Rice and Anderson [3].

The complete description of macroscopic electromagnetic properties of a CDW-SC system is only possible when the Ginzburg-Landau-Gor'kov (GLG) equations for the order parameters have been obtained. These equations couple self-consistently to the electromagnetic fields through the current.

Thus the objective of my thesis is the systematic derivation of the macroscopic current as well as the Ginzburg-Landau-Gor'kov equations for the order parameter starting from a microscopic model for the coexisting system. Owing to the coexisting CDW, the current in the system depends on time derivatives of the order parameter in addition to spatial derivatives. Therefore a closed time path Green function [CTPCF] formalism has been used [4]. I want to emphasize a feature of the present work: all the coupling parameters in the macroscopic equations are obtained from the microscopic theory; no parameters are inserted by hand. I shall start from a microscopic model system with local electron-phonon interation, and separated BCS-local interaction present.

In this chapter I will discuss the consequences of this microscopic model which assumes partial gapping of electrons in the vicinity of the Fermi surface and I shall present the derivation of the expression for the current in coexisting systems. In the next chapter Ginzburg-Landau-Gor'kov equations will be derived. In the concluding chapter 5, I will present an investigation of electromagnetic and thermodynamic properties of charge-density-wave and superconducting systems based on the Ginzburg-Landau-Gor'kov equations obtained.

The major steps in the analysis of this chapter are as follows. The Hamiltonian is set up using a microscopic model. Then the generating function is set up. Gauge invariance is shown; permitting simplification of the analysis by omitting the vector potential in the next steps. The effective action is taken at mean-field level and is expanded with respect to the order parameters, keeping only lowest order in inhomogeneity (space and time derivatives). The electromagnetic field is then restored using gauge invariance. By the functional differentiation of the action with respect to the vector potential we obtain the current. Finally, using the assumed band structure we obtain the explicit form of the current in the coexisting system with coefficients depending on microscopic parameters.

B. The Band Structure Model.

As we previously discussed, the coexistance of superconductivity and charge density waves has been found in layer-like compound structures, namely 2H - $NbSe_2$. Therefore we have to use (where needed) the band structure of these materials in our calculations. As we mentioned, D. Matheiss and P. Inglesfeld made detailed calculations of this band structure. Also for simplicity of the calculations we assume that only one charge-density-wave has been created by Peierls instability. This makes our model quasi-one-dimensional. This specific direction will correspond to the vector $\vec{Q}$. I want to emphasize a specific feature of coexistant superconductivity and charge-density-wave: competition between 3-dimensional phenomenon of superconductivity and the quasi-one-dimensional charge-density-wave. Therefore we expect a non-trivial interplay between their order parameters and also a non-trivial gauge effect. This competition should be especially interesting in the case of incommensurability of $\vec{Q}$ with the lattice parameter (we will discuss this here). When doing integrations in energy or momentum space we will subdivide the Fermi surface into two parts: I- where nesting occurs and II-where only Cooper pairing occurs.

C) The Model Hamiltonian and Effective Action.

We take as a general Hamiltonian

$$(3.1)\qquad -H = \frac{1}{2}uD_0^{-1}u + \psi^{+}\left(i\hbar\frac{\partial}{\partial t}-\frac{1}{2m}\left(\left(-i\hbar\vec{\nabla}-\frac{e}{c}\right)^{2}+\mu\right)\psi\right.$$

$$+G\psi^{+}\psi u + g\psi_{\uparrow}^{+}\psi_{\downarrow}^{+}\psi_{\downarrow}\psi_{\uparrow},$$

where

$$D_0^{-1}(x_1,x_2) = \rho\left(\hbar^2\frac{\partial^2}{\partial t_1^2} - \hbar^2\omega^2(q_1)\right)\delta(x_1-x_2)$$

G is the electron-phonon coupling constant with $g = \frac{\lambda}{2}$ as the BCS local interaction. For the ion displacement $u(x)$ we separate into left and right running waves:

$$(3.2)\qquad u(x) = u_{ac}(x) + \tilde{u}(x)e^{\frac{i}{\hbar}\vec{Q}\cdot\vec{x}} + \tilde{\tilde{u}}(x)e^{-\frac{i}{\hbar}\vec{Q}\cdot\vec{x}} .$$

where $u_{ac}(x)$ corresponds to the acoustic part. For the superconducting order parameter take [5]

$$\chi(x) \equiv \langle\psi_{\downarrow}(x)\psi_{\uparrow}(x)\rangle = \chi(x) + \tilde{\chi}_{op}(x)e^{\frac{i}{\hbar}\vec{Q}\cdot\vec{x}} + \tilde{\tilde{\chi}}_{op}(x)d^{-\frac{i}{\hbar}\vec{Q}\cdot\vec{x}} .$$

$$\chi^{+}(x) = \langle\psi_{\uparrow}^{+}(x)\psi_{\downarrow}^{+}(x)\rangle = \chi^{+}(x)+\chi_{op}^{+}(x)e^{-\frac{i}{\hbar}\vec{Q}\cdot\vec{x}} + \tilde{\tilde{\chi}}_{op}e^{\frac{i}{\hbar}\vec{Q}\cdot\vec{x}} .$$

Because of the introduction of order parameters the mean-field approximation is being used. The electronic wave function in the Nambu representation is

$$(3.3)\qquad \psi(x) = \begin{pmatrix} \tilde{\psi}_{\uparrow}(x)e^{\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} \\ \tilde{\psi}_{\downarrow}^{+}(x)e^{-\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} \\ \tilde{\tilde{\psi}}_{\uparrow}(x)e^{-\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} \\ \tilde{\tilde{\psi}}_{\uparrow}(x)e^{-\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} \end{pmatrix} \Rightarrow \bar{\psi}(x) \equiv \begin{pmatrix} \tilde{\psi}_{\uparrow}(x) \\ \tilde{\psi}_{\downarrow}^{+}(x) \\ \tilde{\tilde{\psi}}_{\uparrow}(x) \\ \tilde{\tilde{\psi}}_{\downarrow}^{+}(x) . \end{pmatrix}$$

Here we use Heisenberg representation with $x = (\vec{r},t)$ and

$$(3.4a)\qquad \psi(x) = \begin{pmatrix} e^{\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} & 0 & 0 & 0 \\ 0 & e^{-\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} & 0 \\ 0 & 0 & 0 & e^{\frac{i}{\hbar}\frac{\vec{Q}}{2}\cdot\vec{x}} \end{pmatrix}\begin{pmatrix} \tilde{\psi}_{\uparrow}(x) \\ \tilde{\psi}_{\downarrow}^{+}(x) \\ \tilde{\tilde{\psi}}_{\uparrow}(x) \\ \tilde{\tilde{\psi}}_{\downarrow}^{+}(x) \end{pmatrix}$$

The advantage of using the Nambu representation is that we can use the same perturbation theory as for a normal metal [6].

In equation (3.2) and (3.3)

(3.4b)
$$u(x) = \frac{1}{\sqrt{2}}(u_1(x) - iu_2(x))$$
$$\tilde{\tilde{u}}(x) = \frac{1}{\sqrt{2}}(u_1(x) + iu_2(x))$$
$$\chi(x) = \frac{1}{\sqrt{2}}(\chi_1(x) - i\chi_2(x))$$
$$\chi^+(x) = \frac{1}{\sqrt{2}}(\chi_1(x) + i\chi_2(x))$$

have been used where $\frac{\vec{Q}}{2}$ is the Fermi momentum in the nesting direction. Now we can immediately write down the generating functional [see appendix B]

(3.5)
$$\bar{Z}[J,h] = \exp \frac{i}{\hbar}[-\frac{1}{g}h^+h] \cdot \int [d\psi^+][d\psi][du_i][d\chi_i]$$
$$\exp \frac{i}{\hbar}\{\frac{1}{2}uD_0^{-1}u - \lambda\chi_i\chi_i + \psi^+(i\hbar \frac{\partial}{\partial t} - \frac{1}{2m}(-i\hbar\nabla + \rho_3\frac{\vec{Q}}{2} - \frac{e}{c}\tau_3\vec{A})^2\tau$$
$$+ u\tau_3 + \frac{1}{\sqrt{2}}Gu_i\rho_i\tau_3 + \frac{g}{\sqrt{2}}\chi_i\tau_i)\psi + j_iu_i + h_i\chi_i\}$$

After integration over the electron variables we have:

(3.6)
$$\bar{z} = \exp\frac{i}{\hbar}[-\frac{1}{g}h^+h] \cdot \int [du_i][d\chi_i] \cdot$$
$$\cdot \exp \frac{i}{\hbar}\{\frac{1}{2}uD_0^{-1}((i))u - \frac{1}{2}g\chi_i^+ \chi_i - i\hbar \mathrm{Tr} \ln G^{-1} + J_iu_i + h_i\chi_i\}$$

where c_i correspond to the electron branch subspace and τ_i to the Nambu spin-space, and

$$(3.7) \qquad D_0^{-1}(\omega) = D_0^{-1}(x_1,x_2) = -\rho\hbar^2\left(\frac{\partial^2}{\partial t^2}-\omega_a^2\right)\delta^4(x_1-x_2)$$

$$G^{-1}(x_1,x_2) = [i\hbar\frac{\partial}{\partial t_1}-\frac{1}{2m}(-ih\sigma_1+\rho_3\frac{\vec{Q}}{2}-\frac{e}{c}\tau_3\vec{A}(x_1))^2\tau_3$$

$$+\ \mu\tau_3 + \frac{1}{\sqrt{2}}G(u_i(x_1)\rho_i\tau_3+\frac{1}{\sqrt{2}}g\chi_i(x_1)\tau_i]\delta^4(x_1-x_2).$$

At this point we drop the electromagnetic field. Later in the work we will use the gauge invariance of the theory to restore the field.

The effective action is then

$$(3.8) \qquad Ieff[u,\chi] = \frac{1}{2}uD_0^{-1}(\omega)u - \frac{1}{2}g\chi_i\chi_i - ih\ \mathrm{Tr}\ \ln\ G^{-1}.$$

The Free Energy density in the mean field approximation is (see Appendix B)

$$(3.9) \qquad F[u,\chi] \sim \frac{1}{2}uD_0^{-1}u - \frac{1}{2}g\chi_i\chi_i + \frac{1}{\beta}\ \Sigma_n\ \mathrm{Tr}\ \ln\ G^{-1}(i\omega_n).$$

Here the summation over Matsubara frequences ω_n is used and the formulae det U = exp Tr ln U have been used and $\int C_i^* dc_i e^{c_i^* A_{ij} c_j} \sim (\det A)^{1/2}$ (see Appendix B) were used.

In the mean-field approximation the equations for the order parameter are (see Appendix B)

$$\frac{\delta\, I_{eff}[u,\chi]}{\delta u_i(x)} = 0$$

(3.10)

$$\frac{\delta\, I_{eff}[u,\chi]}{\delta \chi} = 0$$

We should now introduce a microscopic band model.

Following the quasi-one-dimensional approximation of Levin et al [8] and Bilbro-McMillan [9], we divide the Fermi surface into two regions. In region I, the Fermi surface is nested, with wave vector $\vec{Q}$, and the energy bands have approximately particle-hole symmetry near the Fermi surface, so that the band energy satisfies the relation $\varepsilon_s(\vec{p}) = [\varepsilon(\vec{p}+\frac{\vec{Q}}{2})+\varepsilon(\vec{p}-\frac{\vec{Q}}{2})]/2 \approx 0$. We will also neglect $\vec{p}$ compared to $\vec{Q}$ in region I because $|\vec{p}| \ll Q \sim 2p_F$. In region II we neglect the CDW coupling and use inversion symmetry.

Using the model given in Section B

$$\rho_1 G^r(p_0, \varepsilon_a(\vec{p}))\rho_1 = G^r(p_0, -\varepsilon_a(p))$$

in region I and

$$\tau_1(G^r(p_0, \varepsilon(p))\tau_1 = G^r(p_0, -\varepsilon(p)) \tag{3.11}$$

where $\varepsilon_a(p)$ is the energy of electrons where nesting occurs. Result(3.11) is the consequence of assumed model and commutation relations of Pauli matrices. The Green function in the Nambu representation is

$$(3.12)\quad G^{-1}(p) = p_0 - \epsilon_s(\vec{p})\tau_3 - \epsilon_a(\vec{p})\rho_3\tau_3 + \frac{1}{\sqrt{2}}\Lambda_i\tau_i + \frac{1}{\sqrt{2}}w_i\rho_i t_i$$

where

$$\epsilon_s(\vec{p}) = \frac{\frac{(\vec{p}+\frac{\vec{Q}}{2})^2}{2m} - \mu + \frac{(\vec{p}-\frac{\vec{Q}}{2})^2 - \mu}{2m}}{2} = \frac{\epsilon(\vec{p}+\frac{\vec{Q}}{2}) + \epsilon(\vec{p}-\frac{\vec{Q}}{2})}{2}.$$

Now we introduce gap functions

$$\Lambda_i \equiv g x_i; \quad \Delta = \frac{1}{2}(\Delta_1^2+\Delta_2^2)$$

$$W_i = G u_i; \quad W = \frac{1}{2}(W_1^2+W_2^2).$$

The retarded Green function is now:

$$(3.13)\quad G_r(\vec{p}) = \frac{1}{[(p_0+i\eta)^2 - \epsilon_s^2(\vec{p}) - \epsilon_a^2(\vec{p}) - W^2 - \Delta^2]^2 - 4\epsilon_s^2(p)(W^2+\epsilon_a^2(\vec{p}))}$$

$$\times \{p_0(p_0^2 - \epsilon_s^2(p^2) - \epsilon_a^2(p) - W^2 - \Delta^2) + 2\epsilon_s(p)p_0(\rho_3\epsilon_a(\vec{p}) - \frac{1}{\sqrt{2}}\rho_i W_i)$$

$$- 2\epsilon_s(\vec{p})\frac{1}{\sqrt{2}}\Lambda_i\tau_i(\rho_3\epsilon_a(\vec{p}) - \frac{1}{\sqrt{2}}\rho_i W_2)$$

$$+ \tau_3\epsilon_s(\vec{p})(p_0^2 - \epsilon_s^2(\vec{p}) + \epsilon_a^2(\vec{p}) + W^2 - \Delta^2)$$

$$- \frac{1}{\sqrt{2}}\tau_i\Lambda_i(p_0^2 - \epsilon_s^2(p) - \epsilon_a^2(\vec{p}) - W^2 - \Delta^2)$$

$$+ \tau_3(\rho_3\varepsilon_a(\vec{p})-\frac{1}{\sqrt{2}}\rho_i W_i)(p_0^2+\varepsilon_s^2(p)-\varepsilon_a^2(\vec{p})-W^2-\Delta^2)\}.$$

Here we have taken the Fourier series with respect to discrete variables and

$$G(x_1,x_2) = \frac{1}{\beta}\Sigma_n \int \frac{d^3\vec{p}}{(2\pi)} e^{i\vec{p}\,(\vec{r}_1-\vec{r}_2)-i\omega_n(\tau-\tau')} G(\vec{p},\omega_n),$$

with

$$\omega_n = (2n+1)\frac{\pi}{\beta}$$

$$G(\vec{p},\tau_1,\tau_2) = -\langle T\psi_p(\tau_1)\tilde{\psi}_p(\tau_2)\rangle$$

$$\psi_p = \begin{pmatrix} \psi_{p\uparrow} \\ \psi^+_{-p\downarrow} \end{pmatrix}, \qquad \tilde{\psi}_p = (\psi^+_{p\uparrow},\psi_{-p\downarrow})$$

In order to be able to obtain macroscopic equations, in what follows we will be using the technique of Kadanoff and Baym [11] for slow and fast variables [see Appendix B].

We first give some useful mathematical results. Let $\Gamma[Q]$ be a functional of a slowly varying order parameter $Q(X)$ such as $u(x)$ or $\chi_i(x)$. For example

$$\Gamma[Q] = \int d\vec{r}dt\ \Gamma(u(\vec{r},t),\chi_i(\vec{r},t)).$$

Then we may write

$$\frac{\delta\Gamma[Q]}{\delta Q(x)} = \frac{\delta\Gamma[Q]}{\delta Q(x)}\Big|_{Q\to Q(x)} + \int dy\, \frac{\delta^2\Gamma[Q]}{\delta Q(x)\,\delta Q(y)}\Big|_{Q\to Q(X)}$$

$$(Q(y)-Q(x)) + \frac{1}{2}\int dy\int dz\, \frac{\delta^3\Gamma[Q]}{\delta Q(x)\,\delta Q(y)\,\delta Q(x)}\Big|_{Q\to Q(x)}$$

$$(Q(y)-Q(x))(Q(z)-Q(x)) + \ldots .$$

Now introduce relative coordinates (x-y), and center-of-mass coordinates: (x+y/2). Then for a function of two variables F(x,y) we have

$$(3.14)\quad F(\vec{x},\vec{y}) = F\left(\frac{\vec{x}+\vec{y}}{2},\vec{x}-\vec{y}\right) = \int \frac{d^3q}{(2\pi\hbar)^3} e^{-\frac{i}{\hbar}\vec{q}\cdot(\vec{x}-\vec{y})} F\left(\frac{\vec{x}+\vec{y}}{2},\vec{q}\right).$$

Then we may write:

$$(3.15)\quad \int dyF(x,y)(Q(y)-Q(x))$$

$$= \int dy\int \frac{d^3q}{(2\pi h)^3} e^{-\frac{i}{\hbar}q(x-y)} F\left(\frac{x+y}{2},q\right)(Q(y)-Q(x))$$

$$= \int dy\int \frac{d^3q}{(2\pi h)^3} e^{-\frac{i}{\hbar}q(x-y)} \left\{F(x,q)+\frac{y-x}{2}\,\frac{\partial F(x,q)}{\partial x}\right\}$$

$$\cdot\left\{(y-x)\frac{\partial Q(x)}{\partial x} + \frac{1}{2}(y-x)^2\,\frac{\partial^2 Q}{\partial x}\right.$$

$$= \int dy\int \frac{d^3q}{(2\pi h)^3} e^{-\frac{i}{\hbar}q(x-y)} \left\{i\hbar\,\frac{\partial F(x,q)}{\partial q}\cdot\frac{\partial Q}{\partial x}+\frac{(i\hbar)^2}{2}\left(\frac{\partial^2 F(x,q)}{\partial q\partial q}\cdot\frac{\partial^2 Q}{\partial x\partial x}\right.\right.$$

$$\left. + \frac{\partial^3 F}{\partial x\partial q\partial q}\cdot\frac{\partial Q}{\partial x}\right)$$

$$= i\hbar\left[\frac{\partial F(x,q)}{\partial q}\right]_{q=0}\cdot\frac{\partial Q}{\partial x} - \frac{\hbar^2}{2}\left[\frac{\partial^2 F[x,q]}{\partial q\partial q}\right]_{q=0}\cdot\frac{\partial^2 Q}{\partial x\partial x}$$

$$- \frac{\hbar^2}{2}\left[\frac{\partial^3 F(x,q)}{\partial x \partial q \partial q}\right]_{q=0} \cdot \frac{\partial Q}{\partial x}$$

Here the expansion is with respect to $\hbar \frac{\partial}{\partial q} \cdot \frac{\partial}{\partial x}$, and x and q are understood as all components of $\vec{x}$ and $\vec{q}$. We will use this treatment later in our derivation of the current and Ginzburg-Landau-Gor'kov effective action. Before going on to a derivation of the macroscopic current and effective action, it is very useful to separate the phase and amplitude part of the order parameter. We define

(3.16)
$$g_X = |\Lambda| \exp(-i\theta)$$
$$Gu = |w| \exp[-i\phi].$$

Let us examine the object:

(3.17)
$$\left(\frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\tau_i, \frac{1}{\Delta}\frac{\Lambda_i}{\sqrt{2}}\epsilon_{ij}\tau_j, \tau_3\right)$$

where by definition

(3.18)
$$\Delta^2 = \frac{\Delta_1^2 + \Delta_2^2}{2} \Rightarrow \begin{array}{l} \Delta_1 = \sqrt{2}\,\Lambda \cos \theta \\ \Lambda_2 = \sqrt{2}\,\Delta \sin \theta \end{array}$$

Clearly:

(3.19)
$$\left(\frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\tau_i\right)\left(\frac{i}{\Delta}\frac{\Lambda_j}{\sqrt{2}}\tau_j\right) = \frac{1}{\Delta^2}\Lambda_i\Delta_j \cdot \frac{1}{2}\tau_i\tau_j = 1$$

δ_{ij}

Or

$$(3.20)\qquad \left(\frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\epsilon_{ij}\tau_j\right)^2 = 1 \rightarrow \frac{1}{\Delta^2}(\Delta_1\tau_2-\Delta_2\tau_1)^2 = \frac{\Delta_1^2+\Delta_2^2}{2\Delta^2} = 1.$$

Using $\tau_3^2 = 1$, where

$$\epsilon_{12} = -\epsilon_{21} = 1$$

$$\epsilon_{11} = 0 = \epsilon_{12} = 0$$

is the antysymmetric unit tensor. Also we notice:

$$(3.21)\qquad \left(\frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\tau_i\right)\left(\frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\epsilon_{ij}\tau_j\right) =$$

$$(3.22)\qquad \left(\frac{1}{\Delta}\frac{\Delta_1\tau_1+\Delta_2\tau_2}{\sqrt{2}}\right)\cdot\left(\frac{1}{\Delta}\frac{\Delta_1\tau_2-\Delta_2\tau_1}{\sqrt{2}}\right) = \tau_3\frac{\Delta_1^2+\Delta_2^2}{2\Delta^2} = \tau_3$$

$$(3.23)\qquad \left(\frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\epsilon_{ij}\tau_j\right)\cdot\tau_3 = \frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\tau_i$$

$$(3.24)\qquad \tau_3\cdot\left(\frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\tau_i\right) = \frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\epsilon_{ij}\tau_j.$$

Thus, we will define an object

$$(\tau_\Delta, \tau_\Theta, \tau_3),$$

where

$$(3.25)\qquad \tau_\Delta = \frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\tau_i$$

$$(3.26)\qquad \tau_\Theta = \frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\epsilon_{ji}x_i.$$

So,

(3.27) $$\frac{\partial}{\partial x}\frac{\Delta_i}{\sqrt{2}} = \frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\frac{\partial\Delta}{\partial x} + \frac{1}{\Delta}\frac{\Delta_j}{\sqrt{2}}\epsilon_{ji}\Delta\frac{\partial\theta}{\partial x}$$

(3.28) $$\frac{\partial}{\partial x}\frac{\Delta_i}{\sqrt{2}}\tau_i = \frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\tau_i\frac{\partial\Delta}{\partial x}+\frac{1}{\Delta}\frac{\Delta_j}{\sqrt{2}}\epsilon_{ji}\tau_i\Delta\frac{\partial\theta}{\partial x} = \tau_\Delta\frac{\partial\Delta}{\partial x} + \tau_\theta\Delta\frac{\partial\theta}{\partial x}$$

(3.29) $$\frac{\partial^2}{\partial x\partial x}\frac{\Delta_i}{\sqrt{2}} = \frac{1}{\Delta}\frac{\Delta_i}{\sqrt{2}}\left(\frac{\partial^2\Delta}{\partial x^2}-\Delta\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial x}\right)+\frac{1}{\Delta}\frac{\Delta_j}{\sqrt{2}}\epsilon_{ji}\left(\Delta\frac{\partial^2\theta}{\partial x^2}+\frac{\partial\Delta}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial x}\frac{\partial\Delta}{\partial x}\right).$$

Then the functional derivatives with respect to the phase and amplitude part of the gap order paratemers become

(3.30) $$\frac{\delta}{\delta\theta(x)} = \frac{\Delta_i}{\sqrt{2}}\epsilon_{ij}\frac{\delta}{\delta\left(\frac{\Delta_j(x)}{\sqrt{2}}\right)} = \Delta_i\epsilon_{ij}\frac{\delta}{\delta\Delta_j}$$

(3.31) $$\Delta(x)\frac{\delta}{\delta\Delta(x)} = \frac{\Delta_i(x)}{\sqrt{2}}\frac{\delta}{\delta\left(\frac{\Delta_i(x)}{\sqrt{2}}\right)} = \Delta_i\frac{\delta}{\delta\Delta_i(x)}.$$

Now we use the mean field approximation and take into account only the lowest nonvanishing derivatives (slow variation approximation). We then have for the functional derivative of the generating functional with respect to Δ:

$$(3.32)\quad \frac{\delta\Gamma}{\delta\frac{\Delta_i(x)}{\sqrt{2}}} = \left.\frac{\delta\Gamma}{\delta\left(\frac{\Lambda_i(x)}{\sqrt{2}}\right)}\right|_{\substack{\Lambda_i\to\Lambda_i(x)\\ W_a\to W_a(x)}}$$

$$+\ i\hbar\left[\frac{\partial\Gamma_{ij}}{\partial q}\right]_{\substack{x\\ q=0}}\cdot\frac{\partial}{\partial x}\left(\frac{\Delta_j}{\sqrt{2}}\right)-\frac{\hbar^2}{2}\left[\frac{\partial^2\Gamma_{ij}}{\partial q\partial q}\right]_{\substack{x\\ q=0}}:\frac{\partial^2}{\partial x\partial x}\left(\frac{\Delta_j}{\sqrt{2}}\right)$$

$$-\ \frac{\hbar^2}{2}\left[\frac{\partial^3\Gamma_{ij}}{\partial x\partial q\partial q}\right]_{\substack{x\\ q=0}}\cdot\frac{\partial}{\partial x}\left(\frac{\Lambda_j}{\sqrt{2}}\right)+i\hbar\left[\frac{\partial\Gamma_{ib}}{\partial q}\right]_{\substack{x\\ q=0}}\cdot\frac{\partial}{\partial x}\left(\frac{W_b}{\sqrt{2}}\right)$$

$$-\ \frac{\hbar^2}{2}\left[\frac{\partial^2\Gamma_{ib}}{\partial q\partial q}\right]_{\substack{x\\ q=0}}:\frac{\partial^2}{\partial x\partial x}\left(\frac{W_b}{\sqrt{2}}\right)-\frac{\hbar^2}{2}\left[\frac{\partial^3\Gamma_{ib}}{\partial x\partial q\partial q}\right]_{\substack{x\\ q=0}}\cdot\frac{\partial}{\partial x}\left(\frac{W_b}{\sqrt{2}}\right)$$

$$-\ \frac{\hbar^2}{2}\left[\frac{\partial^2\Gamma_{ijk}}{\partial q_1\partial q_2}\right]_{\substack{x\\ q_1=q_2=0}}\cdot\frac{\partial}{\partial x}\left(\frac{\Lambda_j}{\sqrt{2}}\right)\frac{\partial}{\partial x}\left(\frac{\Delta_k}{\sqrt{2}}\right)$$

$$-\ \frac{\hbar^2}{2}(\left[\frac{\partial^2\Gamma_{ijc}}{\partial q_1\partial q_2}\right]_{\substack{x\\ q_1=q_2=0}}\cdot\frac{\partial}{\partial x}\left(\frac{\Delta_j}{\sqrt{2}}\right)\frac{\partial}{\partial x}\left(\frac{W_c}{\sqrt{2}}\right)$$

$$+\ \left[\frac{\partial^2\Gamma_{ibk}}{\partial q_1\partial q_2}\right]_{\substack{x\\ q_1=q_2=0}}\cdot\frac{\partial}{\partial x}\left(\frac{W_b}{\sqrt{2}}\right)\frac{\partial}{\partial x}\left(\frac{\Delta_k}{\sqrt{2}}\right)$$

$$-\ \frac{\hbar^2}{2}\left[\frac{\partial^2\Gamma_{ibc}}{\partial q_1\partial q_2}\right]_{q_1=q_2=0}\cdot\frac{\partial}{\partial x}\left(\frac{W_b}{\sqrt{2}}\right)\frac{\partial}{\partial x}\left(\frac{W_b}{\sqrt{2}}\right) = 0.$$

Next, taking the functional derivative with respect to W,

$$(3.33)\quad \frac{\delta\Gamma}{\delta\left(\frac{W_a(x)}{\sqrt{2}}\right)} = \left.\frac{\delta\Gamma}{\delta\left(\frac{W_a(x)}{\sqrt{2}}\right)}\right|_{\substack{\Delta_i\to\Delta_i(x)\\ W_a\to W_a(x)}}$$

$$+ \; i\hbar[\frac{\partial\Gamma_{aj}}{\partial q}]_{\substack{x\\q=0}} \cdot \frac{\partial}{\partial x}(\frac{\Lambda_j}{\sqrt{2}}) \; - \; \frac{\hbar^2}{2}[\frac{\partial^2\Gamma_{aj}}{\partial q\partial q}]_{\substack{x\\q=0}} : \frac{\partial^2}{\partial x\partial x}(\frac{\Lambda_j}{\sqrt{2}})$$

$$- \; \frac{\hbar^2}{2}[\frac{\partial^3\Gamma_{aj}}{\partial x\partial q\partial q}]_{\substack{x\\q=0}} \cdot \frac{\partial}{\partial x}(\frac{\Lambda_j}{\sqrt{2}}) \; + \; i\hbar[\frac{\partial\Gamma_{ab}}{\partial q}]_{\substack{x\\q=0}} \cdot \frac{\partial}{\partial x}(\frac{W_b}{\sqrt{2}})$$

$$- \; \frac{\hbar^2}{2}[\frac{\partial^2\Gamma_{ab}}{\partial q\partial q}] : \frac{\partial^2}{\partial x\partial x}(\frac{W_b}{\sqrt{2}}) \; - \; \frac{h^2}{2}[\frac{\partial^3\Gamma_{ab}}{\partial x\partial q\partial q}]_{\substack{x\\q=0}} \cdot \frac{\partial}{\partial x}(\frac{W_b}{\sqrt{2}})$$

$$- \; \frac{\hbar^2}{2}[\frac{\partial^2\Gamma_{ajk}}{\partial q_1\partial q_2}]_{\substack{x\\q_1=q_2=0}} : \frac{\partial}{\partial x}(\frac{\Lambda_j}{\sqrt{2}})\frac{\partial}{\partial x}(\frac{\Lambda_k}{\sqrt{2}})$$

$$- \; \frac{\hbar^2}{2}(\,[\frac{\partial^2\Gamma_{ajc}}{\partial q_1\partial q_2}]_{\substack{x\\q_1=q_2=0}} : \frac{\partial}{\partial x}(\frac{\Lambda_j}{\sqrt{2}})\frac{\partial}{\partial x}(\frac{W_c}{\sqrt{2}})$$

$$+ \; [\frac{\partial\Gamma_{abk}}{\partial q_1\partial q_2}]_{\substack{x\\q_1=q_2=0}} : \frac{\partial}{\partial x}(\frac{W_b}{\sqrt{2}})\frac{\partial}{\partial x}(\frac{\Lambda_k}{\sqrt{2}})$$

$$- \; \frac{\hbar^2}{2}[\frac{\partial^2\Gamma_{abc}}{\partial q_1\partial q_2}] : \frac{\partial}{\partial x}(\frac{W_b}{\sqrt{2}})\frac{\partial}{\partial x}(\frac{W_c}{\sqrt{2}}) \; = \; 0$$

Direct differentiation of Γ gives

$$\left.\frac{\delta\Gamma}{\delta\left(\frac{\Lambda_i(x)}{\sqrt{2}}\right)}\right|_{\substack{\Lambda\to\Delta(x)\\W\to W(x)}} = -\frac{2}{g}\frac{\Delta_i}{\sqrt{2}}+\frac{1}{\beta}\Sigma_n\mathrm{Tr}\{\tau_i G(i\omega_n;\vec{x},\vec{x})\}$$

where $\beta = (k_B T)^{-1}$

$$(3.34) \qquad = -\frac{2}{g}\frac{\Lambda_i}{\sqrt{2}} - \frac{i\hbar}{2}\,\mathrm{Tr}\{\tau_i G_c(x,x)\}$$

and

$$(3.35) \qquad \left.\frac{\delta\Gamma}{\delta\left(\frac{W_a(x)}{\sqrt{2}}\right)}\right|_{\substack{\Delta\to\Lambda(x)\\ W\to W(x)}} = \frac{2}{G}D_0^{-1}(Q)\frac{W_a}{\sqrt{2}}+\frac{1}{\beta}\Sigma_n \mathrm{Tr}\{\rho_a\tau_3 G(i\omega_n,\vec{x},\vec{x})\}$$

$$(3.35) \qquad (= \frac{2}{G}D_0^{-1}(Q)\frac{W_a}{\sqrt{2}} - \frac{ih}{2}\mathrm{Tr}\{\rho_a\tau_3 G_c(x,x)\})$$

where

$$D_0^{-1}[Q] = -\rho\hbar^2\omega_Q^2$$

$$(3.36) \qquad \Gamma_{ij}[q]\big|_{\substack{\Lambda\to\Lambda(x)\\ W\to W(x)}} = -\frac{2}{g}\delta_{ij}-\frac{1}{\beta}\Sigma_n\int\frac{d^3p}{(2\pi\hbar)^3}\mathrm{Tr}\{\tau_i G(i\omega_n$$

$$+i\nu,\frac{\vec{q}}{2}+\vec{p})\tau_j G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})\} \quad i\nu \to q_0 + i\eta$$

$$(3.37) \qquad \Gamma_{ib}[q]\big|_{\substack{\Lambda\to\Lambda(x)\\ W\to W(x)}} = -\frac{1}{\beta}\Sigma_n\int\frac{d^3p}{(2\pi\hbar)^3}\mathrm{Tr}\{\tau_i G(i\omega_n+i\nu,\frac{\vec{q}}{2}$$

$$+\vec{p})\rho_b\tau_3 G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})\} \quad i\nu \to q_0 + i\eta.$$

$$(3.38) \qquad \Gamma_{aj}[q]\big|_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}} = -\frac{1}{\beta}\Sigma_n\int\frac{d^3p}{(2\pi\hbar)^3}\mathrm{Tr}\{\rho_a\tau_3 G(i\omega_n+i\nu,\frac{\vec{q}}{2}$$

$$+\vec{p})\tau_j G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})\} \quad i\nu \to q_0 + i\eta$$

$$(3.39)\quad \Gamma_{ab}[q]\Big|_{\substack{\Delta\to\Delta(x)\\ W\to W(x)}} = \frac{2}{G}\rho\,(q_0^2-\hbar^2\omega_Q^2)$$

$$-\frac{1}{\beta}\Sigma_n\int\frac{d^3p}{(2\pi\hbar)^3}\mathrm{Tr}[\rho_a\tau_3G(i\omega_n+i\nu,\frac{\vec{q}}{2}+\vec{p})\rho_b\tau_3G(i\omega_n,\frac{\vec{q}}{2}+\vec{p})\quad i\nu\to q_0+i\eta$$

G_a, G_r, G_c will later be shown to be the advanced, retarded and causal Green functions, connected by [10]

$$(3.40)\qquad G_c(p) = \tanh\frac{\beta p_0}{2}(G_r(p)-G_a(p))$$

with Fourier transform

$$(3.41)\qquad G_c(x,y) = \int\frac{d^3\vec{p}}{(2\pi\hbar)^3}e^{-\frac{i}{\hbar}\vec{p}\cdot(\vec{x}-\vec{y})}G_c(p).$$

We'll also use

$$(3.42)\quad \frac{1}{\beta}\Sigma_n\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}[\alpha G(i\omega_n+i\nu,\frac{\vec{q}}{2}+\vec{p})\beta G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})]\Big|_{i\nu\to q_0+i\eta}$$

$$= -\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}[dG_r(\frac{q_0}{2}+p_0,\frac{\vec{q}}{2}+\vec{p})\beta G_c(-\frac{q_0}{2}+p_0,-\frac{\vec{q}}{2}+\vec{p})$$

$$+ dG_c(\frac{q_0}{2}+p_0\frac{\vec{q}}{2}\ \vec{})\beta G_r(-\frac{q_0}{2}+p_0,-\frac{\vec{q}}{2}+\vec{p})].$$

Also by definition

$$(3.43)\quad \Gamma_{ijk}[q_1q_2]\Big|_{\substack{\Delta\to\Delta(x)\\ W\to W(x)}} = \frac{1}{\beta}\Sigma_n\int\frac{d^4p}{(2\pi\hbar)^4}(\mathrm{Tr}[\tau_iG(i\omega_n+i\nu_1+i\nu_2,\vec{p}$$

$$+\vec{q}_1+\vec{q}_2)\tau_jG(i\omega_n+i\nu_2,\vec{p}+\vec{q}_2)\tau_kG(i\omega_n,\vec{p})]_{\substack{i\nu_1\to q_{10}+i\eta\\ i\nu_2\to q_{20}+i\eta}}$$

$$+ \mathrm{Tr}\{\tau_i G(i\omega_n+i\nu_2+i\nu_1,\vec{p}+\vec{q}_2+\vec{q}_1)\tau_k G(i\omega_n+i\nu_2,\vec{p}+\vec{q}_2)\tau_x G(i\omega,\vec{p})\}$$

$$i\nu_1 \to q_{10} + i\eta$$
$$i\nu_2 \to q_{20} + i\eta$$

All other Γ_{ijk} can be obtained just by substitution of τ and ρ matrices as follows. For example: in Γ_{ijk} we have

$$\tau_i \cdots \tau_j \cdots \tau_k$$
$$\tau_i \cdots \tau_k \cdots \tau_j$$

in Γ_{ijc} we have

$$\tau_i \cdots \tau_j \cdots \rho_c\tau_3$$
$$\tau_i \cdots \rho_c\tau_3 \cdots \tau_j$$

in Γ_{ibk} we have

$$\tau_i \cdots \rho_b\tau_3 \cdots \tau_k$$
$$\tau_i \cdots \tau_k \cdots \rho_b\tau_3$$

and also the relation:

(3.44) $$\frac{1}{\beta}\Sigma_n \int \frac{d^4p}{(2\pi\hbar)^4}(\mathrm{Tr}\{\alpha G(i\omega_n+i\nu_1+i\nu_2,\vec{p}+\vec{q}_1+\vec{q}_2)\beta G(i\omega_n+i\nu_2,\vec{p}+\vec{q}_2)\gamma G(i\omega_n,\vec{p})\}_{\substack{i\nu_1\to q_{10}+i\eta\\ i\nu_2\to q_{20}+i\eta}}$$

$$+ \mathrm{Tr}\{\alpha G(i\omega_n+i\nu_1+i\nu_2,\vec{p}+\vec{q}_2+\vec{q}_1)\gamma G(i\omega_n+i\nu_1)\vec{p}\}$$

$$+ \vec{q}_1)\beta G(i\omega_n, \vec{p})\}\Big|_{\substack{i\nu_1 \to q_{10}+i\eta \\ i\nu_2 \to q_{20}+i\eta}}$$

$$= -\frac{i\hbar}{2}\int \frac{d^4p}{(2\pi\hbar)^4} Tr\{\alpha G_r \beta G_r \gamma G_c + dG_r \beta G_c \gamma G_a + dG_c \beta G_a \gamma G_a$$

$$+ \alpha G_r \gamma G_r \beta G_c + dG_r \gamma G_r \beta G_c + dG_c \gamma G_a \beta G_c\}.$$

D. Electrical Current Density.

Now we are ready to find the electrical current. In an appendix to this chapter we show the gauge invariance of this theory. Therefore we can restore the vector potential in the effective action by the substitution $\vec{\pi} \to \vec{\pi} - \frac{e}{c}\vec{A}$ as appropriate. Then the electrical current density of the system $j(x)$ has the following general form:

$$\frac{1}{c} j(x) = [\frac{\delta\Gamma[\chi_1 u]}{\delta A(x)}]_{t_+ = t_-} \tag{3.45}$$

where $t_{\pm}$ means the two time branches in the CTPGF language [4].

One can immediately see that the total current will contains 3 contributions:

$$\frac{1}{c} j(x) = \frac{1}{c} j^{(1)}(x) + \frac{1}{c} j^{(2)}(x) + \frac{1}{c} j^{(3)}(x) \tag{3.46}$$

where

(3.47) $$\frac{1}{c}j^{(1)}(x) = -\frac{i\hbar}{2}\left(\frac{e}{mc}\right)\frac{\hbar}{i}\mathrm{Tr}\{\nabla G_c(x,x)\} \text{ where } \nabla = \frac{1}{2}(\vec{\nabla}-\overleftarrow{\nabla})$$

(3.48) $$\frac{1}{c}j^{(2)}(x) = -\frac{i\hbar}{2}\left(\frac{e}{mc}\right)\frac{\vec{Q}}{2}\mathrm{Tr}\{\rho_3 G_c(x,x)\}$$

(3.49) $$\frac{1}{c}j^{(3)}(x) = \frac{i\hbar}{2}\,\frac{e^2}{mc^2}A(x)\mathrm{Tr}\{\tau_3 G_c(x,x)\}$$

(3.50) $$j(x) = \frac{1}{c}j^{(1)}(x)\Big|_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}} + \int dy\left[\frac{\delta}{\delta\left[\frac{\Lambda_1(x)}{\sqrt{2}}\right]}\frac{1}{c}j^{(1)}(x)\right]_{\substack{\Delta\to\Lambda(x)\\ W\to W(x)}}$$

$$\cdot\frac{\Lambda_j(y)-\Delta_j(x)}{\sqrt{2}} + \int dy\left[\frac{\delta}{\delta\left[\frac{W_b(y)}{\sqrt{2}}\right]}\frac{1}{c}j^{(1)}(x)\right]_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}}$$

$$\cdot\frac{W_b(y)-W_b(x)}{\sqrt{2}} + \frac{1}{c}j^{(2)}(x)\Big|_{\substack{\Delta\to\Lambda(x)\\ W\to W(x)}}$$

$$+ \int dy\left[\frac{\delta}{\delta\left[\frac{\Lambda_j(y)}{\sqrt{2}}\right]}\frac{1}{c}j^{(2)}(x)\right]_{\substack{\Lambda\to\Lambda(x)\\ W\to W(x)}}\cdot\frac{\Lambda_j(y)-\Delta_j(x)}{\sqrt{2}}$$

$$+ \int dy\left[\frac{\delta}{\delta\left[\frac{W_b(y)}{\sqrt{2}}\right]}\frac{1}{c}j^{(2)}(x)\right]_{\substack{\Delta\to\Delta(x)\\ W\to W(x)}}\cdot\frac{W_b(y)-W_b(x)}{\sqrt{2}}$$

$$+ \frac{1}{c}j^{(3)}(x)\Big|_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}}.$$

In addition for the system under discussion, especially considering the order parameters, gauge invariance means that the effective action $I_{eff}[\chi,u]$ is invariant under the transformations

$$\vec{A}(x) \to \vec{A}(x) + \nabla\Lambda(x)$$

(3.51) $$\chi(x) \to e^{i\frac{2e}{c}\Lambda(x)}\,\chi(x)$$

$$u(x) \to u(x).$$

One of the consequences of this gauge invariance is that the order of $\frac{\hbar}{i}\vec{\nabla}$ is the same as $\frac{e}{c}\vec{A}$. i.e. $(\frac{\hbar}{i}\frac{\partial\psi}{\partial x}$ is the same order as $\frac{e}{c}A\psi)$.

At this stage it is useful to define functions which we will call F, such that

(3.52) $$\left[\frac{\delta}{\delta\left[\frac{\Delta_j(y)}{\sqrt{2}}\right]}\frac{1}{c}j^{(1)}(x)\right]_{\substack{\Delta\to\Lambda(x)\\ W\to W(x)}} = F_j^{(1)}(x,y)$$

$$= \int\frac{d^4q}{(2\pi h)^4}e^{-\frac{i}{\hbar}q\cdot(x-y)}F_j^{(1)}\left(\frac{x+y}{2},q\right)$$

(3.53) $$\left[\frac{\delta}{\delta\left[\frac{W_b(y)}{\sqrt{2}}\right]}\frac{1}{c}j^{(1)}(x)\right]_{\substack{\Lambda\to\Lambda(x)\\ W\to W(x)}} = F_b^{(1)}(x,y)$$

$$= \int\frac{d^4q}{(2\pi h)^4}e^{-\frac{i}{\hbar}q\cdot(x,y)}F_b^{(1)}\left(\frac{x+y}{2},q\right)$$

(3.54) $$\left[\frac{\delta}{\delta\left[\frac{\Delta_j(y)}{\sqrt{2}}\right]}\frac{1}{c}j^{(2)}(x)\right]_{\substack{\Delta\to\Lambda(x)\\ W\to W(x)}} = F_j^{(2)}(x,y)$$

$$= \int \frac{d^4q}{(2\pi\hbar)^4} e^{\frac{i}{\hbar}q(x-y)} F_j^{(2)}\left(\frac{x+y}{2},q\right)$$

$$(3.55) \quad \left[\frac{\delta}{\delta\left[\frac{W_b(y)}{\sqrt{2}}\right]} \frac{1}{c} j^{(2)}(x)\right]_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}} = F_b^{(2)}(x,y)$$

$$= \int \frac{d^4q}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}q(x-y)} F_b^{(2)}\left(\frac{x+y}{2},q\right).$$

Then we can write for the current

$$(3.56) \quad \frac{1}{c} j(x) = \frac{1}{c} j^{(1)}(x)\Big|_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}} + i\hbar\left[\frac{\partial F_1^{(1)}(x,q)}{\partial q}\right]_{q=0} \frac{\partial}{\partial x}\left(\frac{W_b(x)}{\sqrt{2}}\right)$$

$$(3.57) \quad + i\hbar\left[\frac{\partial F_b^{(1)}(x,q)}{\partial q}\right]_{q=0} \frac{\partial}{\partial x}\left(\frac{W_b(x)}{\sqrt{2}}\right)$$

$$(3.58) \quad + \frac{1}{c} j^{(2)}(x)\Big|_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}} + i\hbar\left[\frac{\partial F_j^{(2)}(x,q)}{\partial q}\right]_{q=0} \frac{\partial}{\partial x}\left(\frac{j(x)}{\sqrt{2}}\right)$$

$$(3.59) \quad + i\hbar\left[\frac{\partial F_b^{(2)}(x,q)}{\partial q}\right]_{q=0} \frac{\partial}{\partial x}\left(\frac{W_b(x)}{\sqrt{2}}\right) + \frac{1}{c} j^{(3)}(x)\Big|_{\substack{\Lambda\to\Delta(x)\\ W\to W(x)}}$$

where

$$(3.60) \quad \vec{F}_j^{(1)}(q) = -\frac{e}{mc}\cdot\frac{1}{\beta}\Sigma_n \int \frac{d^4p}{(2\pi\hbar)^4}\vec{p}\ \mathrm{Tr}\{G(i\omega_n+i\nu,\frac{\vec{q}}{2}$$

$$+ \vec{p})\tau_j G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})\}_{i\nu\to q_0+i\eta}$$

$$= \frac{e}{mc}\ \frac{i\hbar}{2}\cdot \int \frac{d^4p}{(2\pi\hbar)^4}\vec{p}\{G_r(\frac{q_0}{2}+p_0,\frac{\vec{q}}{2}+\vec{p})\tau_j G_c(-\frac{q_0}{2}+p_0,-\frac{\vec{q}}{2}+\vec{p})$$

$$+ G_c(\frac{q_0}{2}+p_0,\frac{\vec{q}}{2}+\vec{p})\tau_j G_a(-\frac{q_0}{2}+p_0,-\frac{\vec{q}}{2}+\vec{p})\}$$

for $F_b^{(1)}(q)$ we have the same expression only with an interchange of $\tau_j \to \rho_b\tau_3$:

$$(3.61)\quad F_j^{(2)}(q) = -\frac{e}{mc}\cdot\frac{\vec{Q}}{2}\cdot\frac{1}{\beta}\Sigma_n\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\rho_3 G(i\omega_n+i\nu,\frac{\vec{q}}{2}+\vec{p})\tau_j G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})\}_{i\nu\to q_0+i\eta}$$

$$(3.62)\quad = \frac{e}{mc}\,\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\rho_3 G_r(\frac{q_0}{2}+p_0,\frac{\vec{q}}{2}+\vec{p})\tau_j G_c(-\frac{q_0}{2}+p_0,-\frac{\vec{q}}{2}+\vec{p})$$

$$+ \rho_3 G_c(\frac{q_0}{2}+p_0,\frac{\vec{q}}{2}+\vec{p})\tau_j G_a(-\frac{q_0}{2}+p_0,-\frac{\vec{q}}{2}+\vec{p})\}.$$

$F_b^{(2)}$ is obtained from this expression by interchanging

$$\tau_j \to \rho_b\tau_3.$$

Now we use the procedure which has already been outlined in this chapter (see (3.13)-(3.19)) to separate phase and amplitude part of order parameters. Namely we introduce

$$(3.63\quad \frac{\Delta_1(x)}{\sqrt{2}} = \Lambda\cos\varphi(x) \qquad \frac{W_1}{\sqrt{2}} = W\cos\Phi(x)$$

$$(3.63)\quad \frac{\Delta_2(x)}{\sqrt{2}} = \Lambda\sin\varphi(x) \qquad \frac{W_2}{\sqrt{2}} = W\sin\Phi(x).$$

We have

$$\frac{\partial}{\partial x}\frac{\Lambda_i}{\sqrt{2}}\tau_i = \tau_\Lambda \frac{\partial \Delta}{\partial x} + \tau_\Theta \Delta \frac{\partial \Theta}{\partial x}. \tag{3.64}$$

$$\frac{\partial}{\partial x}\frac{W_i}{\sqrt{2}}\rho_i \tau_3 = \tau_3 \rho_W \frac{\partial W}{\partial x} + \tau_3 \rho_\Phi W \frac{\partial \Phi}{\partial x} \tag{3.65}$$

where we have used invariant SU(2) combinations

$$\begin{aligned} \tau_\Delta &= \frac{1}{\Delta}\frac{\Lambda_i}{\sqrt{2}}\tau_i \\ \tau_\Theta &= \frac{1}{\Delta}\frac{\Lambda_j}{\sqrt{2}}\epsilon_{ji}\Lambda_i \end{aligned} \tag{3.66}$$

$$\rho_W = \frac{1}{W}\frac{W_i}{\sqrt{2}}\rho_i \tag{3.67}$$

$$\rho_\Phi = \frac{1}{W}\frac{W_j}{\sqrt{2}}\epsilon_{ji}\rho_i . \tag{3.68}$$

We then have for the current

$$\frac{1}{c}\, j(x) = \frac{1}{c}\, j^{(1)}(x)\Big|_{\substack{\Delta\to\Delta(x)\\ W\to W(x)}} + i\hbar\left(\left[\frac{\partial F_\Theta^{(1)}(q)}{\partial q}\right]_{q=0}\cdot \Delta\frac{\partial \Theta}{\partial x} + \left[\frac{\partial F_\Delta^{(1)}(q)}{\partial q}\right]_{q=0}\cdot\frac{\partial \Delta}{\partial x}\right) \tag{3.69}$$

$$+ \frac{1}{c}\, j^{(2)}(x)\Big|_{\substack{\Delta\to\Delta(x)\\ W\to W(x)}} + i\hbar\left(\left[\frac{\partial F_\Phi[q]}{\partial q}\right]_{q=0}\cdot W\frac{\partial \Phi}{\partial x} + \left[\frac{\partial F_W^{(2)}(q)}{\partial q}\right]\cdot\frac{\partial W}{\partial x}\right). \tag{3.70}$$

The F function now becomes

$$(3.71)\quad F_{\Lambda}^{(1)}(q) = -\frac{e}{mc}\frac{1}{\beta}\Sigma_{n}\int\frac{d^4p}{(2\pi\hbar)^4}\vec{p}\,\mathrm{Tr}\{G(i\omega_n+i\nu,\frac{\vec{q}}{2}+\vec{p})\tau_{\Lambda}G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})\}_{i\nu\to q_0+i\eta}$$

$F_{\theta}^{(1)}(q)$ is the same with the change of τ_{Δ} to τ_{θ}:

$$(3.72)\quad F_{W}^{(2)}(q) = -\frac{e}{mc}\frac{\vec{Q}}{2}\frac{1}{\beta}\Sigma_{n}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\rho_3 G(i\omega_n+i\nu,\frac{\vec{q}}{2}+\vec{p})\tau_3\rho_W G(i\omega_n,-\frac{\vec{q}}{2}+\vec{p})\}_{i\nu\to q_0+i\eta}$$

$F_{\varphi}^{(2)}(q)$ is the same with the change of $\tau_3\rho_W$ to $\tau_3\rho_{\varphi}$.

Now we analyse these expressions, using symmetry arguments. First of all

$$(3.73)\quad F_j^{(1)}(q_0,\vec{q}=0) + F_j^{(2)}q_0,\vec{q}=0)$$

$$= \frac{e}{mc}\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{(\vec{p}+\rho_3\frac{\vec{Q}}{2})G_r(\frac{q_0}{2}+p_0,\vec{p})\tau_j G_c(-\frac{q_0}{2}+p_0,\vec{p})$$

$$+ G_c(\frac{q_0}{2}+p_0,\vec{p})\tau_j G_a(-\frac{q_0}{2}+p_0,\vec{p})\}$$

$$= \frac{e}{mc}\frac{i\hbar}{2}\int\frac{dp_0}{(2\pi\hbar)}\int\frac{d^3p}{(2\pi\hbar)^3}\mathrm{Tr}\{(\vec{p}+\rho_3\frac{\vec{Q}}{2}][G_r(q_0+p_0,\vec{p})\tau_j G_c(p_0,\vec{p})$$

$$- G_c(p_0,\vec{p})G_a(-q_0+p_0,\vec{p})]\}$$

$$= \frac{e}{mc}\frac{i\hbar}{2}\int\frac{dp_0}{(2\pi\hbar)}\left(\int_{I}\frac{d^3p}{(2\pi\hbar)^3}+\int_{II}\frac{d^3p}{(2\pi\hbar)^3}\right)\mathrm{Tr}\{(\vec{p}+\rho_3\frac{\vec{Q}}{2})[G_r(q_0+p_0,\vec{p})\tau_j G_c(p_0,\vec{p})$$

$$+ G_c(p_0,\vec{p})\tau_j G_a(-q_0+p_0,\vec{p})]\}$$

So we obtain the contributions from I and II

where

$$\text{(3.74)} \quad \text{From I} = \frac{e}{mc}\frac{i\hbar}{2}\int_{I}\frac{dp_0}{2\pi\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\mathrm{Tr}\{(\vec{p}+\rho_3\frac{\vec{Q}}{2})[G_r(q_0 + p_0,\vec{p})\tau_j G_c(p_0,\vec{p})+G_c(p_0,\vec{p})\tau_j G_a(-q_0+p_0,\vec{p})]\}$$

$$\text{(3.75)} \quad \text{From II} = \frac{e}{mc}\frac{i\hbar}{2}\int\frac{dp_0}{2\pi\hbar}\int_{II}\frac{d^3p}{(2\pi\hbar)^3}\mathrm{Tr}\{(\vec{p}+\rho_3\frac{\vec{Q}}{2})[G_r(q_0 + p_0,\vec{p})\tau_j G_c(p_0,\vec{p})+G_c(p_0,\vec{p})\tau_j G_a(-q_0+p_0,\vec{p})]\}$$

where region I corresponds to the nesting direction and therefore:

$$\text{(3.76)} \qquad G^{-1}(p_0,\vec{p}) = p_0 - \epsilon_a(p)\rho_3\tau_3 + \Delta\tau_1 + W\rho_1\tau_3.$$

Region II corresponds to pure superconducting electrons, thus here W = 0 and

$$\text{(3.77)} \qquad G^{-1}(p_0,\vec{p}) = p_0 - \epsilon_s(\vec{p})\tau_3 - \epsilon_a\rho_3\tau_3 + \Delta\tau_1.$$

Using equations (3.71)-(3.77), we arrive at the following expressions after lengthy calculation:

$$(3.78)\quad \left[\frac{\partial F_{\ominus}[\chi,q]}{\partial \vec{q}}\right]_{q=0} =$$

$$= \frac{e}{2m^2\beta}\langle\vec{p}\vec{p}\rangle_{II}\, \Sigma_n N_2(0) \int_{II} d\epsilon_{\vec{p}} \left[\frac{2i\Delta(x)}{[(i\omega_n)^2-\epsilon_p^2-|\Delta(x)|^2]^2}\right]$$

$$- \frac{e}{2m^2\beta}\vec{Q}\vec{Q}\Sigma_n \int_I N_1(0)\, d\epsilon^a_{\vec{p}} \left[\frac{2i\Delta(x)}{[(i\omega_n)^2-(\epsilon^a_p)^2-|\Delta(x)|^2-|W(x)|^2]^2}\right]$$

and

$$(3.79)\quad \left[\frac{\partial F_{\phi}[x,q]}{\partial q}\right]_{q=0}$$

$$= -i\frac{e\vec{Q}}{2m}|W(x)|^2 \int_I N_1(0)d\epsilon^a_p \left[\frac{\tanh[\frac{\beta}{2}\sqrt{(\epsilon^a_p)^2+|\Delta(x)|^2+|w(x)|^2}]}{[(\epsilon^a_p)^2+|\Delta(x)|^2+|W(x)|^2]^{3/2}}\right].$$

All other $\frac{\partial F_\lambda}{\partial q^\mu}$ ($\lambda = |\Delta|, |W|, \ominus, \phi$) are exactly equal to zero due to the symmetry properties of the Green function. These symmetry properties follow from the behavior of the Green function under independent rotation of the Nambu and CDW space.

Thus the current in the coexisting SC and CDW system is equal to

$$(3.80)\quad j(x) = \frac{e\hbar}{2i}\left(\Pi^{II}_{\ominus}[T]\left\langle\frac{\vec{p}\vec{p}}{p_F}\right\rangle_{II}+\Pi^{I}_{\ominus}[T]\,\hat{e}\hat{e}\right)\cdot(\Delta^*(x)\nabla\Delta(x)-\nabla\Delta^*\Delta(x))$$

$$- \frac{(2e)^2}{2mc}\left(n^{II}_s[T]\left\langle\frac{\vec{p}\vec{p}}{p_F}\right\rangle_{II}+n^I_s[T]\,\hat{e}\hat{e}\right)\cdot\vec{A}(x)$$

$$+ \hbar \frac{e}{m} \frac{\vec{Q}}{2} |W(x)|^2 \Pi_{\varphi}^{(0)}[T] \frac{\partial \Phi}{\partial t}$$

where $\hat{e} = \frac{Q}{|\vec{Q}|}$ is a unit vector in the nesting direction. $n_s^{I,II}[T] = m|\Lambda|^2 \Pi_\theta^{I,II}[T]$ is a quantity proportional to the density of superconducting electrons.

Typical integrals for superconducting and charge density wave system are

$$\Pi_\theta^{I}[T] = -\frac{1}{m^2}\left(\frac{Q}{2}\right)^2 \int_I N_1(0)\, d\epsilon_p^a \left[\frac{1}{E_a(\vec{p})} \frac{d}{dE_a(p)}\left(\frac{\tanh \frac{\beta E_a(p)}{2}}{E_a(p)}\right)\right]$$

$$(3.81) \quad \Pi_\theta^{II}[T] = -\frac{2\mu}{3m^2} \int N_2(0)\, d\epsilon_p \left[\frac{1}{E(p)} \frac{d}{dEp}\left(\frac{\tanh \frac{\beta E(p)}{2}}{E(p)}\right)\right]$$

$$\Pi_\varphi^{(0)}[T] = \int_I N_i(0)\, d\epsilon_p^a \left[\frac{\tanh(\beta E_a(\vec{p}))}{E_a^3(p)}\right].$$

$E(p)$ is equal to $E_a(\vec{p})$ with $W = 0$. $N_1(0)$ and $N_2(0)$ are the density of states corresponding to regions I and II. If nesting takes place along one fixed direction, then region I will occupy only a small angular region in the Brillouin Zone. We may then approximate the tensor $\langle \vec{p}\vec{p} \rangle_F / p_F^2$ by $I - \alpha\hat{e}\hat{e}$ to describe the anisotropy of the Fermi surface with $0 < \alpha \leq 1$, is a microscopic parameter.

The final result for the current is:

$$(3.82) \quad \vec{j}(x) = \frac{e\hbar}{2i}[\Pi_{\theta}^{II}[T]I + (\Pi_{\theta}^{I}[T] - \alpha\Pi_{\theta}^{II}[T]\hat{e}\hat{e}$$

$$\cdot(\Delta^*(x)\nabla\Delta(x) - \nabla\Delta^*(x)\Delta x)$$

$$- \frac{(2e)^2}{2mc}[n_s^{II}[T]I + (n_s^{(I)}(T) - \alpha n_s^{(II)}(T))\hat{e}\hat{e}]\cdot\vec{A}(x)$$

$$+ \hbar\frac{e}{m}\frac{\vec{Q}}{2}|W(x)|^2\Pi_{\varphi}^{(0)}[T]\cdot\frac{\partial\Phi}{\partial t}.$$

The first two terms are just the usual Ginzburg-Landau current for the superconductor including anisotropic coefficients which reflect the influence of the coexisting CDW, i.e. a CDW-induced anisotropy of superconductivity. The last term is most interesting and is proportional to the time derivative of the phase of CDW order parameter. It is natural to interpret that term as the contribution from the sliding Goldstone mode due to the existence of an incommensurate CDW.

If the superconducting current is absent the last term corresponds to a pure CDW system. We can rewrite this expression for the current as

$$(3.83) \qquad j(x) = c\vec{v}_F N(0)\hbar\frac{\partial\Phi}{\partial t}\cdot f(\beta)$$

where

$$f(\beta) = \frac{\left(\int_I \frac{d\epsilon \tanh \frac{\beta E(\vec{p})}{2}}{E^3(\vec{p})}\right)}{\left(\int d\epsilon \frac{1}{E_p^3}\right)}$$

E. <u>Relation to Lee-Rice-Anderson Phenomenological Result</u>.

The density of states $N_1(0) \sim \frac{N}{\epsilon_F}$ with N the density of electrons and ϵ_F the Fermi energy in the last expression for the current has just the same form as Lee and Rice's [7] phenomenological result up to a temperature dependent correction factor $f(\beta)$. This result is exact in our theory up to all terms of the first order in inhomogeneity. Worth special mention is the absence of any other gradients and time derivatives besides $\nabla\theta$ and $\frac{\partial\Phi}{\partial t}$, and also the absence of direct $\nabla\theta \cdot \nabla\Phi$ or $\nabla\theta \cdot \frac{\partial\Phi}{\partial t}$ coupling. Higher order terms will be proportional $\hbar$ and are much smaller, compared to those just discussed. This situation is a consequence of the quasi-1-dimensional nature of the CDW and has been incorporated in the microscopic model assumed by Levin et al [8] and Bilbro-McMillan [9] via a partition of the Fermi surface. It results from the independence of the rotation of the SC Nambu spin space and the CDW pseudo-spin space. We want to emphasize that in the theory presented we neither make any phenomenological assumptions nor introduce any phenomenological coefficients.

Appendix to Chapter 3:

Gauge Invariance

Gauge transformations (see [6])

For the superconducting order parameter (3.84)

$$\psi(x) \rightarrow e^{\frac{i}{\hbar}\frac{e}{c}\Lambda(x)\tau_3}\psi(x)$$

For the electromagnetic field $\vec{A}(x) \rightarrow \vec{A}(x) + \vec{\nabla}\Lambda(x)$.

Then:

$$(3.85) \quad \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix} = \begin{bmatrix} \cos\frac{2e}{\hbar c}\Lambda(x) & \sin\frac{2e}{\hbar c}\Lambda(x) \\ -\sin\frac{2e}{\hbar c}\Lambda(x) & \cos\frac{2e}{\hbar c}\Lambda(x) \end{bmatrix} \begin{bmatrix} \chi_1(x) \\ \chi_2(x) \end{bmatrix}$$

$$= \left[e^{i\frac{2e}{\hbar c}\Lambda(x)\tilde{\tau}_2} \right] \begin{bmatrix} \chi_1(x) \\ \chi_2(x) \end{bmatrix}$$

where $\tilde{\tau}_2$ defined in the $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ space. The combination

$$(3.86) \quad \begin{aligned} \frac{\chi_1 - i\chi_2}{\sqrt{2}} &= e^{i\frac{2e}{\hbar c}\Lambda(x)} \frac{\chi_1 - i\chi_2}{\sqrt{2}} \\ \frac{\chi_1 + i\chi_2}{\sqrt{2}} &= e^{-i\frac{2e}{\hbar c}\Lambda(x)} \frac{\chi_1 + i\chi_2}{\sqrt{2}} \end{aligned}$$

which gives us

$$(3.87) \quad \left(\frac{\hbar}{i}\nabla - \frac{e}{c}\tau_3 A(x)\right)\psi(x) \Rightarrow \left(\frac{\hbar}{c}\nabla - \frac{2e}{c}\,\tilde{\tau}_2 A(x)\right)\begin{pmatrix}\chi_1(x)\\ \chi_2(x)\end{pmatrix}$$

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Chapter 4. The Ginzburg-Landau-Gor'kov Effective Action and Equations for the Order Parameters.

A. Introduction.

The Ginzburg-Landau action will be calcuated as follows, using the previously obtained effective action. In outline, we will follow a similar approach as we used in Chapter 3. However in the present case both the order parameters and the electromagnetic field are present. Thus we require, in addition, the evaluation of the functional derivatives of the action with respect to the order parameters. Setting these derivatives equal to zero will give us the Ginzburg-Landau-Gor'kov equations. As in Chapter 3 we make use of gauge invariance. The coefficients in these Ginzburg-Landau-Gor'kov equations are evaluated using the assumed band structure.

In the derivation we assume the temperature is close to T_c, therefore in the final integrands we make an expansion in $(\frac{T-T_c}{T_c})$. Also because typically $T_p > T_c$ we can use a "frozen CDW" Ansatz and take the CDW order parameters at their static values at T_c (amplitude = const). Further we assume that the superconducting order parameter and the electromagnetic field vary slowly over distances of the order equal to a superconducting correlation length, and we will carry out the separation of the amplitude and phase part of the order parameter.

B. Functional Derivatives of the Generating Functional.

We write the functional derivative of the generating functional (see chapter 3) as

$$(4.1)\qquad \frac{\delta\Gamma}{\delta\left(\frac{\Lambda_i(x)}{\sqrt{2}}\right)} = \left.\frac{\delta\Gamma}{\delta\left(\frac{\Lambda_i(x)}{\sqrt{2}}\right)}\right|_x + i\hbar\left[\frac{\partial\Gamma_{ij}(q)}{\partial q}\cdot\frac{\partial}{\partial x}\left(\frac{\Delta_j(x)}{\sqrt{2}}\right)\right)$$

$$-\frac{\hbar^2}{2}\left[\frac{\partial^2\Gamma_{ij}[q]}{\partial q\partial q}\right]_{q=0} : \frac{\partial^2}{\partial x\partial x}\left(\frac{\Delta_j(x)}{\sqrt{2}}\right)$$

$$-\frac{\hbar^2}{2}\ \frac{\partial^3\Gamma_{ij}(q)}{\partial x\partial q\partial q}\cdot\frac{\partial}{\partial x}\ \frac{\Lambda_j(x)}{\sqrt{2}} - \frac{\hbar^2}{2}\left[\frac{\partial^2\Gamma_{ijk}[q]}{\partial q_1\partial q_2}\right]_{q_1=q_2=0} :$$

$$\frac{\partial}{\partial x}\left(\frac{\Delta_j(x)}{\sqrt{2}}\right)\frac{\partial}{\partial x}\left(\frac{\Delta_k(x)}{\sqrt{2}}\right)\ \frac{\delta\Gamma}{\delta\Delta(x)} = \frac{1}{\Lambda(x)}\ \frac{\Delta_i(x)}{\sqrt{2}}\ \frac{\delta\Gamma}{\delta\left(\frac{\Lambda_i(x)}{\sqrt{2}}\right)};$$

$$\frac{1}{\Delta(x)}\ \frac{\delta\Gamma}{\delta\theta(x)} = \frac{1}{\Delta(x)}\ \frac{\Delta_i(x)}{\sqrt{2}}\ \epsilon_{ij}\ \frac{\partial\Gamma}{\partial\left(\frac{\Delta_j(x)}{\sqrt{2}}\right)}$$

The first term is:

$$(4.2)\qquad \left.\frac{\delta\Gamma}{\delta\Delta(x)}\right|_x = \frac{1}{\Delta(x)}\cdot\frac{\Delta_i(x)}{\sqrt{2}}\ \left.\frac{\delta\Gamma}{\delta\frac{\Delta_i(x)}{\sqrt{2}}}\right|_x$$

$$= \frac{1}{\Lambda(x)}\ \frac{\Delta_i(x)}{\sqrt{2}}\left(-\frac{2}{g}\ \frac{\Lambda_i}{\sqrt{2}} - \frac{i\hbar}{2}\mathrm{Tr}\{\tau_i G_c(x,x)\}\right)$$

$$= -\frac{2}{g}\Lambda(x) - \frac{i\hbar}{2}\mathrm{Tr}\{\tau_\Delta G_c(x,x)\}$$

The phase part is

$$(4.3)\quad \frac{1}{\Lambda(x)}\frac{\delta\Gamma}{\delta\theta(x)}\bigg|_x = \frac{1}{\Lambda(x)}\frac{\Delta_j(x)}{\sqrt{2}}\epsilon_{ji}\frac{\delta\Gamma}{\delta\frac{\Delta_i(x)}{\sqrt{2}}}\bigg|_x$$

$$= \frac{1}{\Lambda(x)}\frac{\Delta_j(x)}{\sqrt{2}}\epsilon_{ji}\left(-\frac{2}{g}\frac{\Delta_i(x)}{\sqrt{2}} - \frac{i\hbar}{2}\mathrm{Tr}\{\tau_i G_c(x,x)\}\right)$$

$$= 0 - \frac{i\hbar}{2}\mathrm{Tr}\{\tau_\theta G_c(x,x)\}$$

The next term is

$$(4.4)\quad \frac{1}{\Lambda(x)}\frac{\Delta_i(x)}{\sqrt{2}}i\hbar\left[\frac{\partial\Gamma_{ij}(q)}{\partial q}\right]_{q=0}\cdot\frac{\partial}{\partial x}\left(\frac{\Delta_j(x)}{\sqrt{2}}\right)$$

$$= i\hbar\frac{1}{\Lambda(x)}\frac{\Delta_i(x)}{\sqrt{2}}\left[\frac{\partial}{\partial q}\left(-\frac{2}{g}\delta_{ij}-\Pi_{ij}(q)\right)\right]_{q=0}$$

$$\cdot\left(\frac{1}{\Lambda}\frac{\Delta_j}{\sqrt{2}}\frac{\partial\Lambda}{\partial x} + \frac{1}{\Lambda}\frac{\Delta_\ell}{\sqrt{2}}\epsilon_{\ell j}\Lambda\frac{\partial\theta}{\partial x}\right)$$

$$= -i\hbar\frac{1}{\Lambda(x)}\frac{\Delta_i(x)}{\sqrt{2}}\left[\frac{\partial\Pi_{ij}(q)}{\partial q}\right]_{q=0}\left(\frac{1}{\Lambda}\frac{\Delta_j}{\sqrt{2}}\frac{\partial\Delta}{\partial x}+\frac{1}{\Lambda}\frac{\Delta_\ell}{\sqrt{2}}\epsilon_{\ell j}\Lambda\frac{\partial\theta}{\partial x}\right)$$

$$= -i\hbar\left(\left[\frac{\partial\Pi_{\Lambda\Lambda}(q)}{\partial q}\right]_{q=0}\frac{\partial\Delta}{\partial x} + \left[\frac{\partial\Pi_{\Lambda\theta}}{\partial q}\right]_{q=0}\Lambda\frac{\partial\theta}{\partial x}\right).$$

The phase part is

$$(4.5)\quad \frac{1}{\Lambda(x)}\frac{\Delta_s(x)}{\sqrt{2}}\epsilon_{s_i}\; i\hbar\left[\frac{\partial\Gamma_{ij}[q]}{\partial q}\right]_{q=0}\cdot\frac{\partial}{\partial x}\left(\frac{\Delta_j(x)}{\sqrt{2}}\right)$$

$$= i\hbar \frac{1}{\Delta} \frac{\Delta_s(x)}{\sqrt{2}} \epsilon_{s_i} [\frac{\partial}{\partial q}(-\frac{2}{g} \delta_{ij} - \Pi_{ij}(q))]_{q=0}$$

$$\cdot (\frac{1}{\Delta} \frac{\Delta_j}{\sqrt{2}} \frac{\partial \Delta}{\partial x} + \frac{1}{\Delta} \frac{\Delta_\ell}{\sqrt{2}} \epsilon_{\ell_j} \Delta \frac{\partial \theta}{\partial x}$$

$$= -i\hbar \frac{1}{\Delta(x)} \frac{\Delta_s}{\sqrt{2}} \epsilon_{s_i} [\frac{\partial \Pi_{ij}(q)}{\partial q}]_{q=0} (\frac{1}{\Delta} \frac{\Delta_j}{\sqrt{2}} \cdot \frac{\partial \Delta}{\partial x} + \frac{1}{\Delta} \frac{\Delta_\ell}{\sqrt{2}} \epsilon_{\ell_j} \frac{\partial \theta}{\partial x})$$

$$= -i\hbar ([\frac{\partial \Pi_{\theta\Delta}}{\partial q}]_{q=0} \frac{\partial \Delta}{\partial x} + [\frac{\partial \Pi_{\theta\theta}(q)}{\partial q}]_{q=0} \cdot \Delta \frac{\partial \theta}{\partial x}).$$

All other terms can be transformed analogously. We shall write only the final expressions:

$$(4.6) \quad \frac{1}{\Delta} \frac{\Delta_i(x)}{\sqrt{2}} (-\frac{\hbar^2}{2} [\frac{\partial^2 \Gamma}{\partial q \partial q}]_{q=0} : \frac{\partial^2 \Delta_j(x)}{\partial x^2 \sqrt{2}}$$

$$= \frac{\hbar^2}{2} [\frac{\partial^2 \Pi_{\Delta\Delta}(q)}{\partial q \partial q}]_{q=0} \cdot (\frac{\partial^2 \Delta}{\partial x^2} - \Delta \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x})$$

$$+ \frac{\hbar^2}{2} [\frac{\partial^2 \Pi_{\Delta\theta}(q)}{\partial q \partial q}]_{q=0} \cdot (\Delta \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \Delta}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial x} \frac{\partial \Delta}{\partial x})$$

where q and x mean the components of vectors $\vec{q}$ and $\vec{x}$

$$(4.7) \quad \frac{1}{\Delta(x)} \frac{\Delta_s(x)}{\sqrt{2}} \epsilon_{s_i} (-\frac{\hbar^2}{2} [\frac{\partial^2 \Gamma}{\partial q \partial q}]_{q=0} : \frac{\partial^2}{\partial x^2} \frac{\Delta_j(x)}{\sqrt{2}})$$

$$= \frac{\hbar^2}{2} [\frac{\partial^2 \Pi_{\theta\Delta}(q)}{\partial q \partial q}]_{q=0} \cdot (\frac{\partial^2 \Delta}{\partial x^2} - \Delta \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x})$$

$$+ \frac{\hbar^2}{2} [\frac{\partial^2 \Pi_{\theta\theta}(q)}{\partial q \partial q}]_{q=0} \cdot (\Delta \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \Delta}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial x} \frac{\partial \Delta}{\partial x})$$

$$\frac{1}{\Delta(x)}\ \frac{\Delta_i(x)}{\sqrt{2}}\left(-\frac{\hbar^2}{2}\left[\frac{\partial^2\Gamma_{ijk}(q_1q_2)}{\partial q_1\partial \epsilon_2}\right]_{q_1=q_2=0}:\frac{\partial}{\partial x}\ \frac{\Delta_j(x)}{\sqrt{2}}\ \frac{\partial}{\partial x}\ \frac{\Delta_k(x)}{\sqrt{2}}\right.$$

$$= -\frac{\hbar^2}{2}\{[\frac{\partial^2\Gamma_{\Delta\Delta\Delta}}{\partial q_1\partial q_2}]_{q_1=q_2=0}:\frac{\partial\Delta}{\partial x}\ \frac{\partial\Delta}{\partial x}+[\frac{\partial^2\Gamma_{\Delta\Delta\theta}}{\partial q_1\partial q_2}]_{q_1=q_2=0}\ \frac{\partial\Delta}{\partial x}\Delta\frac{\partial\theta}{\partial x}$$

$$+ [\frac{\partial^2\Gamma_{\Delta\theta\Delta}}{\partial q_1\partial q_2}]_{q_1=q_2=0}:\Delta\frac{\partial\theta}{\partial x}\ \frac{\partial\Delta}{\partial x} + [\frac{\partial^2\Gamma_{\Delta\theta\theta}}{\partial q_1\partial q_2}]_{p_1=p_2=0}:\Delta\frac{\partial\theta}{\partial x}\Delta\frac{\partial\theta}{\partial x}$$

$$(4.8)\quad \frac{1}{\Delta(x)}\ \frac{\Delta_s(x)}{\sqrt{2}}\epsilon_{s_i}\left(-\frac{\hbar^2}{2}\ \frac{\partial^2\Gamma_{ijk}[q_1q_2]}{\partial q_1\partial q_2}:\frac{\partial}{\partial x}\ \frac{\Delta_j(x)}{\sqrt{2}}\ \frac{\partial}{\partial x}\ \frac{\Delta_x(x)}{\sqrt{2}}\right.$$

$$= -\frac{\hbar^2}{2}\{[\frac{\partial^2\Gamma_{\theta\Delta\Delta}}{\partial q_1\partial q_2}]_{q_1=q_2=0}:\frac{\partial\Delta}{\partial x}\ \frac{\partial\Delta}{\partial x} + [\frac{\partial^2\Gamma_{\theta\Delta\theta}}{\partial q_1\partial q_2}]_{q_1=q_2=0}:\frac{\partial\Delta}{\partial x}\Delta\frac{\partial\theta}{\partial x}$$

$$+ [\frac{\partial^2\Gamma_{\theta\theta\Delta}}{\partial q_1\partial q_2}]_{q_1=q_2=0}:\Delta\frac{\partial\theta}{\partial x}\ \frac{\partial\Delta}{\partial x} + [\frac{\partial^2\Gamma_{\theta\theta\theta}}{\partial q_1\partial q_2}]_{q_1=q_2=0}:\Delta\frac{\partial\theta}{\partial x}\Delta\frac{\partial\theta}{\partial x}\}$$

Next we assume that T_c is much smaller that T_p which is true in 2H-NbSe$_2$ as previously noted. We are also interested in temperatures close to T_c so we can freeze the amplitude of the charge-density-wave order parameter.

Keeping terms of the lowest order in derivatives and order parameters we have to the first order $\frac{\hbar}{i}\ \frac{\partial}{\partial x}\ \frac{\partial}{\partial q}$ in derivatives and $\Delta\varphi$ in order parameters

$$(4.9)\quad -\frac{2}{g}\Delta(x)-\frac{i\hbar}{2}\cdot \mathrm{Tr}\{\tau_\Delta G_c(x,x)\}-i\hbar[\frac{\partial\Pi_{\Delta\theta}[q]}{\partial q}]_{q=0}\cdot\Delta\frac{\partial\theta}{\partial x}$$

$$+ \frac{\hbar^2}{2}[\frac{\partial^2\Pi[q]}{\partial q\partial q}]\Big|_{q=0}:\Delta\frac{\partial^2\theta}{\partial x\partial x}$$

$$- i\hbar\left[\frac{\partial\Pi_{\Delta\varpi}}{\partial q}\right]_{q=0}\cdot W\frac{\partial\varpi}{\partial x}+\frac{\hbar^2}{2}\left[\frac{\partial^2\Pi_{\Delta\varpi}[q]}{\partial q\partial q}\right]_{q=0}:W\frac{\partial^2\varpi}{\partial x\partial x}+\ldots = 0$$

$$- i\hbar\left[\frac{\partial\Pi_{\theta\theta}}{\partial q}\right]_{q=0}\cdot\Lambda\frac{\partial\theta}{\partial x}+\frac{\hbar^2}{2}\left[\frac{\partial^2\Pi_{\theta\theta}}{\partial q\partial q}\right]_{q=0}:\Lambda\frac{\partial^2\theta}{\partial x\partial x}$$

$$- i\hbar\left[\frac{\partial\Pi_{\varpi\varpi}}{\partial q}\right]_{q=0}\cdot W\frac{\partial\varpi}{\partial x}+\frac{\hbar^2}{2}\left[\frac{\partial^2\Pi_{\varpi\varpi}}{\partial q\partial q}\right]_{q=0}:W\frac{\partial^2\varpi}{\partial x\partial x}+\ldots = 0$$

and also

$$(4.10)\quad \frac{2}{G}D_0^{-1}[Q]-\frac{i\hbar}{2}\mathrm{Tr}\{\tau_3\sigma_W G_c(x,x)\}-i\hbar\left[\frac{\partial\Pi_{W\theta}}{\partial q}\right]_{q=0}\cdot\Lambda\frac{\partial\theta}{\partial x}$$

$$+\frac{\hbar^2}{2}\left[\frac{\partial^2\Pi_{W\theta}}{\partial q\partial q}\right]_{q=0}:\Lambda\frac{\partial^2\theta}{\partial x\partial x}-i\hbar\left[\frac{\partial\Pi_{W\varpi}[q]}{\partial q}\right]_{q=0}\cdot W\frac{\partial\varpi}{\partial x}$$

$$+\frac{\hbar^2}{2}\left[\frac{\partial^2\Pi_{W\varpi}[q]}{\partial q\partial q}\right]_{q=0}:W\frac{\partial^2\varpi}{\partial x\partial x}+\ldots = 0$$

$$- i\hbar\left[\frac{\partial\Pi_{\varpi\theta}(q)}{\partial q}\right]\cdot\Lambda\frac{\partial\theta}{\partial x}+\frac{\hbar^2}{2}\left[\frac{\partial^2\Pi_{\varpi\theta}(q)}{\partial q\partial q}\right]_{q=0}:\frac{\partial^2\theta}{\partial x\partial x}$$

$$- i\hbar\left[\frac{\partial\Pi_{\varpi\varpi}}{\partial q}\right]_{q=0}\cdot W\frac{\partial\varpi}{\partial x}+\frac{\hbar^2}{2}\left(-\frac{2}{G}\frac{\partial^2 D_0^{-1}}{\partial q\partial q}+\frac{\partial^2\Pi_{\varpi\varpi}[q]}{\partial q\partial q}\right]_{q=0}W\frac{\partial^2\varpi}{\partial x\partial x} = 0.$$

C. <u>The Functions</u> Π: <u>Definition and Derivations</u>.

The Π functions and various derivatives of them will be defined in equations (4.11)-(4.35) below.

$$(4.11)\quad \Pi_{\Delta\varpi}[q_0,\vec{q}] = \frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_1 G_r(q_0+p_0,\vec{q}+\vec{p})\tau_2 G_c(p_0,\vec{p})$$

$$+ \tau_1 G_c(p_0,\vec{p})\tau_2 G_a(-q_0+p_0,-\vec{q}+\vec{p})\}$$

where

(4.12) In region I: $G_r^{-1}(p_0,\vec{p}) = p_0-\epsilon_a\tau_3+\Delta\tau_1+W\rho_1\tau_3$.

(4.13) In region II: $G_r^{-1}(p_0,\vec{p}) = \frac{1+\rho_3}{2}(p_0-\epsilon(\vec{p}+\frac{\vec{Q}}{2})\tau_3$

$$+ \Delta\tau_1+\frac{1-\rho_3}{2}(p_0-\epsilon(\vec{p}-\frac{\vec{Q}}{2})\tau_3+\Delta\tau_1).$$

Then we have

(4.14) $\Pi_{\Delta\rho}[q_0,\vec{q}=0] = -\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_1 G_r(q_0+p_0,\vec{p})\tau_2 G_c(p_0,\vec{p})$

$$+ \tau_1 G_c(p_0,\vec{p})\tau_2 G_a(-q_0+p_0,\vec{p}) = I \cdot \tau_1\rho_3$$

$$G_r^{-1}(p_0,\vec{p}) \to G_r^{-1}(p_0,-\vec{p})$$

$$\tau_2 \to -\tau_2$$

(4.15) $II \to \tau_1 \quad G_r^{-1}(p_0,\epsilon) \to G_r^{-1}(p_0,-\epsilon)$

$$\tau_2 \to -\tau_2.$$

Using the properties of nesting permits us to change from a (new) extended zone to a (new) reduced Brillouin zone by shifting the values of $\vec{k}$ by a (new) reciprocal lattice vector. Then $\epsilon(\vec{p})$ becomes spherical. Taking into account the properties of Pauli matrices we have

$$\text{in I}: \tau_2 \quad G^{-1}(i\omega_n,\vec{p}) = -G^{-1}(i\omega_n,\vec{p})$$

$$\tau_1 \to -\tau_1$$

$$\text{in II}: \tau_2 \quad G^{-1}(i\omega_n,\vec{p}) = -G^{-1}(-i\omega_n,\vec{p})$$

$$\tau_1 \to -\tau_1$$

Hence:

$$(4.16) \quad \Pi_{\Delta\theta}[q_0=0,\vec{q}] = \frac{1}{\beta}\Sigma_n \int \frac{d^4p}{(2\pi\hbar)^4}\{\tau_1 G(i\omega_n,\vec{q}+\vec{p})\tau_2 G(i\omega_n,\vec{p})\} = 0$$

Then:

$$(4.17) \quad \frac{\partial \Pi_{\Delta\theta}[q_0,\vec{q}]}{\partial q}\Big|_{\vec{q}=0} = -\frac{i\hbar}{2}\int_I \frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_1 G_r(p_0$$

$$+ p_0,\vec{p})\frac{\partial \varepsilon_a(p)}{\partial \vec{p}}\rho_3\tau_3 G_r(q_0+p_0,\vec{p})\cdot\tau_2 G_c(p_0,\vec{p})$$

$$+ \tau_1 G_c(p_0,\vec{p})\tau_2 G_a(-q_0+p_0,\vec{p})\frac{\partial \varepsilon_a(\vec{p})}{\partial \vec{p}}\rho_3\tau_3 G_a(-q_0+p_0,\vec{p})\}.$$

Again shifting $\vec{k}$ as before and using:

$$(4.18) \quad \text{In region I} \quad \rho_1: G_r^{-1}(p_0,\vec{p}) \to G_r^{-1}(p_0,-\vec{p})$$

$$\rho_3\tau_3 \to -\rho_3\tau_3, \quad \frac{\partial \varepsilon_a(\vec{p})}{\partial \vec{p}} = \frac{\vec{Q}}{2m} \text{ independent of } \vec{p}$$

(4.19) In region II τ_1: $G_r^{-1}(-p_0,\epsilon) \to G_r^{-1}(p_0,-\epsilon)$

$$\tau_2 \to -\tau_2$$

$$\tau_3 \to -\tau_3$$

$$\frac{\partial\epsilon}{\partial p} = \frac{\vec{p}}{m} \to -\frac{\vec{p}}{m}.$$

Then

$$(4.20)\quad -\frac{i\hbar}{2}\int_{II}\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_1 G_r(q_0+p_0,\vec{p})\frac{\partial\epsilon(\vec{p})}{\partial\vec{p}}\tau_3 G_r(q_0 + p_0,\vec{p})\tau_2 G_c(p_0,\vec{p}) + \tau_1 G_c(p_0,\vec{p})\tau_2 G_a(-q_0+p_0,\vec{p})\frac{\partial\epsilon(\vec{p})}{\partial\vec{p}}\tau_3 G_a(-q_0+p_0,\vec{p})\} = 0$$

Next:

$$(4.21)\quad \Pi_{\theta\varphi}[q_0,\vec{q}] = -\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_2 G_r(q_0+p_0,\vec{q}+\vec{p})\rho_3\tau_3 G_c(p_0,\vec{p}) + \tau_2 G_c(p_0,\vec{p})\rho_2\tau_3 G_a(-q_0+p_0,-\vec{q}+\vec{p})\}$$

In region I $G_r^{-1}(p_0,\vec{p}) = p_0 - \epsilon_a(\vec{p})\rho_3\tau_3 + \Delta\tau_1 + W\rho_1\tau_3$.

In region II $$G_r^{-1}(p_0,\vec{p}) = \frac{1+\rho_3}{2}(p_0-\tau_3\epsilon(\vec{p}+\frac{\vec{Q}}{2})+\tau_1\Delta) + \frac{1-\rho_3}{2}(p_0-\tau_3\epsilon(p-\frac{\vec{Q}}{2}) + \tau_1\Delta).$$

Further

Region I $\quad \rho_1 \quad G_r^{-1}(p_0,\vec{p}) = G_r^{-1}(p_0,-\vec{p})$

$$\rho_2\tau_3 \to -\rho\tau_3$$

Region II $\quad \rho_3 \quad G_r^{-1}(p_0,\vec{p})$ invariant

$$\rho_2\tau_3 \to -\rho_2\tau_3$$

$$(4.22)\quad \Pi_{?\varpi}[q_0,\vec{q}=0] = -\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_2G_r(q_0+p_0,\vec{p})\rho_2\tau_3G_c(p_0,\vec{p})$$

$$+\tau_2G_c(p_0,\vec{p})\rho_2\tau_3G_a(-q_0+p_0,\vec{p})\} = 0$$

Also using

In Region I $\quad \tau_2 \quad G^{-1}(i\omega_n,\vec{p}) \cdot -G^{-1}(-i\omega_n,\vec{p})$

$$\rho_2\tau_3$$

In Region II $\quad \rho_3 \quad G^{-1}(i\omega_n,\vec{p})$ invariant

$$\rho_2\tau_3 \to -\rho_2\tau_3$$

it follows that

$$(4.23)\quad \Pi_{?\varpi}[q_0=0,\vec{q}]$$

$$= \frac{1}{\beta}\Sigma_n\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_2G(i\omega_n,\vec{q}+\vec{p})\rho_2\tau_3G(i\omega_n,\vec{p})\} = 0$$

$$(4.24)\quad \frac{\partial\Pi_{?\varpi}[q_0,\vec{q}]}{\partial\vec{q}}\Big|_{\vec{q}=0} + \frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\tau_2G_r(q_0+p_0,\vec{p})\rho_3\tau_3\cdot$$

$$\cdot \frac{\partial \epsilon_a(\vec{p})}{\partial p} G_r(q_0+p_0,\vec{p})\rho_2\tau_3 G_c(\vec{p},p)$$

$$+ \tau_2 G_c(p_0,\vec{p})\rho_2\tau_3 G_a(-q_0+p_0,\vec{p})\rho_3\tau_3 \frac{\partial \epsilon_a(\vec{p})}{\partial \vec{p}} Gr(-q_0+p_0,\vec{p})]$$

Region I $\qquad \rho_1$

$\tau_1\rho_3$

Region II $\qquad \rho_3 \qquad G^{-1}(i\omega_n,\vec{p}) \sim$ invariant

$\rho_2\tau_3 \to -\rho_2\tau_3$

(4.25) $\Pi_{\varphi\varphi}[q_0,\vec{q}]$

$$= -\frac{i\hbar}{2}\int \frac{d^4p}{(2\pi\hbar)^4}\{Tr(\frac{\tau_2 p_0+q_0+\epsilon_a(\vec{p}+\vec{q})\rho_3\tau_3-\Delta\tau_1-W\rho_1\tau_3}{(-_0+q_0+i\eta)^2 - \epsilon_a^2(\vec{p}+\vec{q})-\Delta^2-W^\Delta}\ \rho_2\tau_3 \cdot$$

$$\cdot(p_0+\epsilon_a(\vec{p})\rho_3\tau_3-\Delta\tau_1-W\rho_1\tau_3)$$

$$+ \rho_2\tau_3\frac{p_0-q_0+\epsilon_a(\vec{p}-\vec{q})\rho_2\tau_3-\Delta\tau_1-W\rho_2\tau_3}{(p_0-q_0-i\eta)^2-\epsilon a^2(\vec{p}-\vec{q})-\Delta^2-W^2}\ \tau_2$$

$$\cdot(p_0+\epsilon_a(\vec{p})\rho_3\tau_1-\Delta\tau_1-W\rho_1\tau_3)\}\}(-2\pi i)\,\epsilon(p_0)\tanh\frac{\beta p_0}{2}\delta(p_0^2-\epsilon_a^2(p)$$

$$-W^2-\Delta^2) = 0.$$

All the Pauli matrices (2×2) have $Tr_\tau = 0$, $Tr_\rho = 0$.

(4.26) $$\Pi_{\varphi\varphi}[q_0,\vec{q}] = -\frac{i\hbar}{2}\frac{d^4p}{(2\pi\hbar)^4}Tr\{\rho_2\tau_3 G_r(q_0+p_0,\vec{q}+\vec{p})\rho_2\tau_3 G_c(p_0,\vec{p})$$

$$+ \rho_2\tau_3 G_c(p_0,\vec{p})\rho_2\tau_3 G_c(-q_0+p_0,-\vec{q}+\vec{p})\,]$$

$$= -\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{(\rho_2\tau_3 G_r(q_0+p_0,\vec{q}+\vec{p})\rho_2\tau_3$$

$$+ \rho_2\tau_3 G_a(-q_0+p_0-\vec{q}+\vec{p})\rho_2\tau_3)G_c(p_0,\vec{p})\}$$

$$= -\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\{\rho_2\tau_3\frac{p_0+q_0+\epsilon_a(\vec{p}+\vec{q})\rho_3\tau_3-\Delta\tau_1-W\rho_1\tau_3}{(p_0+q_0+i\eta)^2\,\epsilon_a^2(\vec{p}+\vec{q}) - \Delta^2 - W^2}\rho_2\tau_3\cdot$$

$$\cdot(p_0+\epsilon_a(p)\rho_3\tau_3-\Delta\tau_1-W\rho_1\tau_3)+\rho_2\tau_3\frac{p_0-q_0+\epsilon_a(\vec{p}-\vec{q})\rho_3\tau_2-\Delta\tau_1-W\rho_1\tau_3}{(p_0-q_0-i\eta)^2-\epsilon_a^2(\vec{p}-\vec{q})-\Delta^2-W^2}\rho_2\tau_3$$

$$\cdot p_0+\epsilon_a(\vec{p})\rho_3\tau_3-\Delta\tau_1-W\rho_1\tau_3)\}\cdot(-2\pi i)\,\epsilon(p_0)\tanh\frac{\beta p_0}{2}\delta(p_0^2-\epsilon_a^2(\vec{p})-W^2-\Delta^2)$$

$$= -\frac{i\hbar}{2}\int\frac{d^4p}{(2\pi\hbar)^4}\mathrm{Tr}\frac{p_0+q_0-\epsilon_a(\vec{p}+\vec{q})\rho_3\tau_3+\Delta\tau_1+W\rho_1\tau_3}{(p_0+q_0+i\eta)^2-\epsilon_a^2(\vec{p}+\vec{q}) - \Delta^2 W^2}$$

$$(p_0+\epsilon_a(\vec{p})\rho_3\tau_3-\Delta\tau_1-W\rho_1\tau_3)$$

$$+\frac{p_0-q_0-\epsilon_a(\vec{p}-\vec{q})\rho_3\tau_3\Delta\tau_1-W\rho_1\tau_3}{(p_0-q_0-i\eta)^2-\epsilon_a^2(\vec{p}-\vec{q}) - \Delta^2-W^2}(p_0+\epsilon_a(\vec{p})\rho_3\tau_3-\Delta\tau_1-W\rho_1\tau_3)\,] \tag{4.27}$$

$$\cdot(-2\pi i)\,\epsilon(p_0)\tanh\frac{\beta p_0}{2}\delta(p_0^2-\epsilon_a^2(\vec{p})-W^2-\Delta^2).$$

After some additional calculation, introducing $E_p^2 \equiv \epsilon_a^2 + \Delta^2 + W^2$ we finally obtain an expression for $\Pi_{\varphi\varphi}$ as:

$$(4.28)\quad \Pi_{\varphi\varphi}[q_0,\vec{q}] = -\int \frac{d^4p}{(2\pi\hbar)^4}\,\frac{\tanh\frac{\beta E_p}{2}}{E_p}$$

$$\cdot\Big[\frac{q_0E_p+\epsilon_a^2(\vec{p})-\epsilon_a(\vec{p}+\vec{q})\,\epsilon_a(\vec{p})}{(q_0+i\eta)^2+2(q_0+i\eta)E_p+\epsilon_a^2(\vec{p})-\epsilon_a^2(\vec{p}+\vec{q})}$$

$$+\frac{q_0E_p+\epsilon_a^2(\vec{p})-\epsilon_a(\vec{p}-\vec{q})\,\epsilon_a(\vec{p})}{(q_0+i\eta)^2+2(q_0+i\eta)E_p+\epsilon_a^2(\vec{p})-\epsilon_a^2(\vec{p}-\vec{q})}$$

$$+\frac{-q_0E_p+\epsilon_a^2(\vec{p})-\epsilon_a(\vec{p}+\vec{q})\,\epsilon_a(\vec{p})}{(p_0+i\eta)^2-2(q_0+i\eta)E_p+\epsilon_a^2(\vec{p})-\epsilon_a^2(\vec{p}+\vec{q})}$$

$$+\frac{-q_0E_p+\epsilon_a^2(\vec{p})-\epsilon_a(\vec{p}-\vec{q})\,\epsilon_a(\vec{p})}{(q_0+i\eta)^2-2(q_0+i\eta)E_p+\epsilon_a^2(\vec{p})-\epsilon_a^2(p-q)}\Big].$$

Using this expression we can obtain:

$$(4.29)\quad \Pi_{\varphi\varphi}[q_0,\vec{q}=0] = \int \frac{d^4p}{(2\pi\hbar)^4}\cdot\frac{8E_p\,\tanh\frac{\beta E_p}{2}}{(q_0+i\eta)^2-4E_p^2}$$

and also

$$(4.30)\quad \Big[\frac{\partial^2\Pi_{\varphi\varphi}(\vec{q})}{\partial q_0^2}\Big]_{\vec{q}=0} = -\int \frac{d^4p}{(2\pi\hbar)^4}\,\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}$$

and also

$$(4.31)\quad \Pi_{\varphi\varphi}[q_0=0,\vec{q}] = -\int_I \frac{d^2p}{(2\pi\hbar)^4}\,\frac{\tanh\frac{\beta E_p}{2}}{E_p}$$

$$\cdot 2\,\epsilon_a(\vec{p})\Big[\frac{1}{\epsilon_a(\vec{p})+\epsilon_a(\vec{p}+\vec{q})}+\frac{1}{\epsilon_a(\vec{p})+\epsilon_a(\vec{p}-\vec{q})}\Big].$$

In the nested direction $\epsilon_a(\vec{p}) \simeq \frac{\vec{p}}{m}.\frac{\vec{Q}}{2}$. Therefore $\frac{\partial \epsilon_a(\vec{p}+\vec{q})}{\partial \vec{q}} = \frac{\vec{Q}}{2m}$ is independent of $\vec{q}$. So the second derivative becomes

$$(4.32) \quad \frac{\partial^2 \Pi_{\varphi\varphi}[q]}{\partial \vec{q} \partial \vec{q}}\Big|_{q=0} = -\frac{1}{m^2} \frac{\vec{Q}}{2} \frac{\vec{Q}}{2} \int_I \frac{d^4 p}{(2\pi\hbar)^4} \frac{\tanh \frac{\beta E_p}{2}}{E_p} . \frac{1}{\epsilon_a^2(\vec{p})}$$

$$= -\frac{1}{m^2} \frac{\vec{Q}}{2} \frac{\vec{Q}}{2} N(0) \int_I d\epsilon_a(\vec{p}) \frac{\tanh \frac{\beta E_p}{2}}{E_p} . \frac{1}{\epsilon_a^2(\vec{p})}$$

$$= -\frac{1}{m^2} \frac{\vec{Q}}{2} \frac{\vec{Q}}{2} N(0) \int d\epsilon_a(\vec{p}) \frac{1}{E_p} \frac{d}{dE_p} \left(\frac{\tanh \frac{\beta E_p}{2}}{E_p} \right)$$

Here we have used $\int_I N(0) d\epsilon_a(\vec{p}) = \int \frac{d^4 p}{(2\pi\hbar)^4}$.

By direct differentiation one can show that

$$(4.33) \quad \frac{\partial \Pi_{\varphi\varphi}[q_0 \vec{q}]}{\partial q}\Big|_{\vec{q}=0} = 0 \quad \text{and} \quad \frac{\partial^2 \Pi_{\varphi\varphi}}{\partial q_0 \partial \vec{q}}\Big|_{q=0} = 0$$

The calculation of $\Pi_{\theta\theta}[q_0,\vec{q}]$ is similar and we only cite the result. In the nesting direction:

$$\Pi^I_{\theta\theta}(q_0,\vec{q}] = \Pi_{\varphi\varphi}[q_0,q]$$

$$(4.34) \quad \left[\frac{\partial^2 \Pi^I_{\theta\theta}}{\partial q_0^2}\right]_{q=0} = -\int \frac{d^4 p}{(2\pi\hbar)^4} \frac{\tanh \frac{\beta E_p}{2}}{E_p^3}$$

$$(4.35) \quad \left[\frac{\partial^2 \Pi^I_{\theta\theta}}{\partial \vec{q} \partial \vec{q}}\right]_{q=0} = -\frac{1}{m^2} \frac{\vec{Q}}{2} \frac{\vec{Q}}{2} \int_I N(0) d\epsilon_a(\vec{p}) \frac{1}{E_p} \frac{d}{dE_p} \left(\frac{\tanh \beta E_p}{E_p} \right)$$

$$= -\frac{1}{m^2}\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}\int_I N(0)\,dE_{\vec{p}}\frac{1}{\sqrt{E_p^2-\Delta^2-W^2}}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right)$$

(4.36)
$$\left[\frac{\partial^2\Pi^I_{\theta\theta}}{\partial q_0\partial\vec{q}}\right]_{q=0} = 0$$

Here

(4.37)
$$E_p^2 = \epsilon_a^2(\vec{p})+\Delta^2+W^2 \quad \text{and} \quad \int N(0)\,d\epsilon_a(p) = \int\frac{d^4p}{(2\pi\hbar)^4}.$$

In the unnested direction:

(4.38)
$$\Pi^{II}_{\theta\theta}[q_0,\vec{q}=0] = \int_{II}\frac{d^4p}{(2\pi\hbar)^4}\tanh\frac{\beta E_p}{2}\frac{8E_p}{(q_0+i\eta)^2-4E_p^2}$$

and

(4.39)
$$\left[\frac{\partial^2\Pi^{II}_{\theta\theta}[q_0,\vec{q}=0]}{\partial q_0^2}\right]_{q_0=0} = -\int\frac{d^4\vec{p}}{(2\pi h)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}$$

Here $E_p^2 = \epsilon^2(p) + \Delta^2$,

(4.40)
$$\frac{\partial^2\Pi^{II}_{\theta\theta}[q_0=0,\vec{q}]}{\partial\vec{q}\partial\vec{q}}\Big|_{q=0} = -\int\frac{d^4p}{(2\pi\hbar)^4}\tanh\frac{\beta E_p}{2}.$$

$$\left[-\frac{1}{\epsilon(p)}\frac{\overleftrightarrow{I}}{m}+\frac{\vec{p}\vec{p}}{mm}\frac{1}{\epsilon^2(p)}\right].$$

In Equation (4.40) the first term gives 0 because the integrand is an odd function of $\epsilon(p)$. In the second term in (4.40) we can calculate

(4.41)
$$-\int_{II}\frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p}\left[\frac{1}{\epsilon^2(p)}\frac{1}{m}\frac{\langle\vec{p}\vec{p}\rangle}{m}\right.$$

$$(4.42) \quad = \int \frac{d^4p}{(2\pi\hbar)^4} \frac{\tanh\frac{\beta E_p}{2}}{E_p} \left(\frac{1}{\epsilon^2(\vec{p})} \cdot \frac{1}{m}\left\langle\frac{\vec{p}\vec{p}}{m}\right\rangle\right)$$

$$= -\frac{1}{m^2}\langle\vec{p}\vec{p}\rangle\int_{II} \frac{d^4p}{(2\pi\hbar)^4} \frac{\tanh(\frac{\beta E_p}{2})}{E_p} \cdot \frac{1}{\epsilon^2(\vec{p})}$$

$$= -\frac{1}{m^2}\langle\vec{p}\vec{p}\rangle N(0)\int_{II} d\epsilon(p)\frac{1}{\epsilon^2(p)} \frac{\tanh\frac{\beta E_p}{2}}{E_p} .$$

Then after some calculation this is reduced to

$$(4.43) \quad = -\frac{1}{m^2}\langle\vec{p}\vec{p}\rangle N(0)\int_{II} dE_p \frac{1}{\sqrt{E_p^2-\Delta^2}} \frac{1}{dE_p} \tanh\left(\frac{\frac{\beta E_p}{2}}{E_p}\right)$$

D. <u>Summary of the π functions</u>.

Let us now summarize the result of the calculation of all second derivatives of π-functions

$$\frac{\partial^2\pi_{coco}[q]}{\partial q_0^2}\bigg|_{q=0} = -\int \frac{d^4p}{(2\pi\hbar)^4} \frac{\tanh\frac{\beta E_p}{2}}{E_p^3}$$

$$(4.44) \quad \frac{\partial^2\Pi_{coco}[q]}{\partial\vec{q}\partial\vec{q}}\bigg|_{q=0} = -\frac{N(0)}{m^2}\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}\int d\epsilon_a(\vec{p})\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right)$$

$$= -\frac{1}{m^2}\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}N(0)\int_E dE_p \frac{1}{\sqrt{E_p^2-\Delta^2-W^2}}\frac{d}{dE_p}\left(\frac{\tanh(\frac{PE_p}{2})}{E_p}\right)$$

$$\left[\frac{\partial^2\Pi_{\theta\theta}[q]}{\partial q_0^2}\right]_{q=0} = -\int_I \frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3} - \int_{II}\frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}$$

(4.45) $$\left[\frac{\partial^2 \Pi_{\theta\theta}[q]}{\partial\vec{q}\partial\vec{q}}\right]_{q=0} = -\frac{1}{m^2}\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}N(0)\int dE_p \frac{1}{\sqrt{E_p^2-W^2-\Delta^2}}\frac{d}{dE_p}\cdot\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right) - \frac{1}{m^2}\langle\vec{p}\vec{p}\rangle N(0)\int_{II} dE_p \frac{1}{\sqrt{E_p^2-\Delta^2}}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right)$$

where in area I (nesting direction)

(4.46) $$E_p^2 = \epsilon_a^2(\vec{p}) + W^2 + \Delta^2$$

where in area II (unnesting direction) $E_p^2 = \epsilon^2(\dot{p}) + \Delta^2$

(4.47) $$\langle\vec{p}\vec{p}\rangle = \frac{2}{3}\mu m\overleftrightarrow{I}.$$

Here, μ-Fermi-energy $\overleftrightarrow{I}$-unit tensor.

We obtain equations governing the phase of the charge-density-wave:

(4.48) $$\left[\frac{\partial^2 \Pi_{\varphi\varphi}[q]}{\partial q_0^2}\right]_{q=0}\frac{\partial^2\phi}{\partial t^2} - \left[\frac{\partial^2 \Pi_{\varphi\varphi}(q)}{\partial\vec{q}\partial\vec{q}}\right]_{q=0}:\frac{\partial^2\varphi}{\partial\vec{x}\partial\vec{x}} = 0.$$

and the phase of the superconducting order parameter

(4.49) $$\left[\frac{\partial^2 \Pi_{\theta\theta}[q]}{\partial q_0^2}\right]_{q=0}\cdot\frac{\partial^2\theta}{\partial t^2} - \left[\frac{\partial^2 \Pi_{\theta\theta}[q]}{\partial\vec{q}\partial\vec{q}}\right]_{q=0}:\frac{\partial^2\theta}{\partial\vec{x}\partial\vec{x}} = 0$$

Let us change notation in order to simplify:

$$\Pi_{\varphi}^{(0)} = \int_I \frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}$$

$$(4.50) \qquad \Pi_{\varphi}^{(s)} = -\frac{1}{m}\left|\frac{\vec{Q}}{2}\right|^2 N(0)\int_{I} d\epsilon_a(\vec{p})\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right).$$

Then we have

$$(4.51) \qquad \left[\frac{\partial^2 \Pi_{\varphi\varphi}[q]}{\partial q_0^2}\right]_{q=0} = -\int_{I}\frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3} = \Pi_{\varphi}^{(0)}$$

and

$$(4.52) \qquad \left[\frac{\partial^2 \Pi_{\varphi\varphi}[q]}{\partial\vec{q}\partial\vec{q}}\right]_{q=0} = \Pi_{\varphi}^{(s)}\frac{\vec{Q}\ \vec{Q}}{|\vec{Q}|\,|\vec{Q}|}.$$

Also let us call

$$\Pi_{\theta}^{(0)} = -\int_{I}\frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3} - \int_{II}\frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}$$

$$(4.53) \qquad \Pi_{\theta}^{(Is)} = -\frac{1}{m^2}\left|\frac{\vec{Q}}{2}\right|^2\int_{I} N(0)\,d\epsilon_a(\vec{p})\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right)$$

$$I\Pi_{\theta}^{(IIs)} = -\frac{1}{m^2}\langle\vec{p}\vec{p}\rangle\int_{II} N(0)\,d\epsilon(\vec{p})\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right).$$

Therefore

$$\left[\frac{\partial^2 \Pi_{\theta\theta}[q]}{\partial q_0^2}\right]_{q=0} = \Pi_{\theta}^{(0)}$$

(4.54)

$$\left[\frac{\partial^2 \Pi_{\theta\theta}[q]}{\partial\vec{q}\partial\vec{q}}\right] = \overleftrightarrow{I}\,\Pi_{\theta}^{(IIs)} + \frac{\vec{Q}\ \vec{Q}}{|Q|\,|Q|}\Pi_{\theta}^{(Is)}$$

If we choose the $\hat{z}$ direction for the $\vec{Q}$ then we have the equations

$$(4.55)\qquad \Pi_{\varphi}^{(0)} \frac{\partial^2 \phi}{\partial t^2} - \Pi_{\varphi}^{(s)} \frac{\partial^2 \phi}{\partial z^2} = 0$$

and

$$(4.56)\qquad \Pi_{\theta}^{(0)} \frac{\partial^2 \theta}{\partial t^2} - \Pi_{\theta}^{(Is)} \frac{\partial^2 \theta}{\partial z^2} - \Pi_{\theta}^{(IIs)} \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}\right) = 0.$$

In the presence of the electromagnetic field, using gauge invariance, we have

$$(4.57)\qquad \frac{h}{i} \nabla \theta(x) \to \frac{h}{i} \nabla \theta(x) - i\frac{2e}{c}\vec{A}(x)$$

Then the current is

$$(4.58)\qquad \frac{1}{c}\vec{J}(x) = \frac{e}{mc}(-i)8\frac{\langle \vec{p}\vec{p}\rangle}{2m}\Delta^2\left(-i\hbar\frac{\partial \theta}{\partial \vec{x}} - i\frac{2e}{c}\vec{A}(x)\right)\frac{1}{\beta}\Sigma_n \int \frac{d^4p}{(2\pi\hbar)^4}$$

$$\cdot \frac{1}{[\omega_n^2 + \epsilon^2 + \Delta^2 + W^2]} + \frac{e}{mc}\,\frac{\vec{Q}}{2}W^2\hbar\frac{\partial \phi}{\partial t}\int \frac{d^4p}{(2\pi\hbar)^4}\,\frac{\tanh \frac{\beta}{2}\sqrt{\epsilon_a^2 + W^2 + \Delta^2}}{(\epsilon_a^2 + W^2 + \Delta^2)^{3/2}}.$$

Using known results with Fermi frequences we have

$$(4.59)\qquad \frac{1}{\beta}\Sigma_n \frac{1}{\omega_n^2 + \epsilon_a^2 + \Delta^2 + W^2} = \frac{1}{\beta}\Sigma_n \frac{1}{\omega_n^2 + E_p^2} = \frac{1}{2E_p}\tanh\frac{\beta E_p}{2}$$

and

$$(4.60)\qquad \frac{1}{\beta}\Sigma_n \frac{1}{(\omega_b^2 + \epsilon^2 + \Delta^2 + W^2)} = \frac{1}{\beta}\Sigma_n \frac{1}{(\omega_n^2 + E_p^2)^2} = \frac{1}{2E_p}\,\frac{d}{dE_p}\,\frac{1}{\beta}\Sigma \frac{1}{\omega_n^2 + E_p^2}$$

$$= -\frac{1}{2E_p}\frac{d}{dE_p}\frac{1}{\beta}\Sigma\frac{1}{\omega_n^2+E_p^2} = -\frac{1}{2E_p}\frac{d}{dE_p}\frac{\tanh\frac{\beta E_p}{2}}{2E_p} = -\frac{1}{4E_p}\frac{d}{dE_p}\frac{\tanh\frac{\beta E_p}{2}}{E_p}.$$

So

$$(4.61)\quad \frac{1}{c}\vec{J} = \frac{e}{mc}4i\frac{\langle\vec{p}\vec{p}\rangle}{2m}\Delta^2\frac{\hbar}{i}\frac{\partial\Theta}{\partial\vec{x}}\int\frac{d^4p}{(2\pi\hbar)^4}\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}\right)$$

$$+\frac{e}{mc}\frac{\vec{Q}}{2}w^2\hbar\frac{\partial\vartheta}{\partial t}\int_I\frac{d^4p}{(2\pi\hbar)^4}\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}$$

$$(4.62)\quad +\frac{2e^2}{mc^2}\langle\frac{\vec{p}\vec{p}}{m}\rangle\Delta^2\ \vec{A}(\vec{x})\int\frac{d^4p}{(2\pi\hbar)^4}\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right).$$

Now we define the density of superconducting electrons in the unnested direction (region II)

$$(4.63)\quad n_s^{(II)}[T] = -\frac{2}{3}u\Delta^2\int_{II}\frac{d^4p}{(2\pi\hbar)^4}\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\right).$$

After some calculations for the density of states at the Fermi-level we have

$$(4.64)\quad N_{II}(0) = \frac{\Omega_{II}p_F^2}{(2\pi\hbar)^3}\cdot\frac{1}{\frac{d\epsilon(p_F)}{dp_F}} = \frac{\Omega_{II}p_F m}{(2\pi h)^3}$$

we get

$$(4.65)\quad n_s^{II}[T] = n_s^{II}[T=0]+\frac{2u}{3}\Delta^2\int_{II}\frac{d^4p}{(2\pi\hbar)^4}\frac{1}{E_p}\frac{d}{dE_p}\left(\frac{1-\tanh\frac{\beta E_p}{2}}{E_p}\right).$$

So we have

(4.66)
$$\frac{1}{c}\vec{J}(x) = \frac{(-2e)^2}{(2m)c^2}\Big\{n_s^{II}(T)\overleftrightarrow{I} - \Big(\frac{\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}}{m}\Big)\Delta^2\int\frac{d^4p}{(2\pi\hbar)^4}\,\frac{1}{E_p}\,\frac{d}{dE_p}\,\cdot$$
$$\cdot\Big(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\Big)\Big\}\cdot\vec{A}(\vec{x}) - \frac{e}{mc}2i\Big\{n_s^{II}(T)\overleftrightarrow{I} - \Big(\frac{\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}}{m}\Big)\Delta^2\int\frac{d^4p}{i(2\pi\hbar)^4}\,\frac{1}{E_p}\cdot$$
$$\cdot\frac{d}{dE_p}\Big(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\Big)\Big\}\frac{\hbar}{i}\,\frac{\partial\Theta}{\partial\vec{x}} + \frac{e}{mc}\,\frac{\vec{Q}}{2}W^2\int\frac{d^4p}{i(2\pi\hbar)^4}\,\frac{\tanh\frac{\beta E_p}{2}}{E_p^3}\cdot\hbar\frac{\partial\Theta}{\partial t}$$

where

(4.67)
$$n_s^{II}(T) = -\frac{2\mu}{3}\Delta^2\int\frac{d^4p}{(2\pi\hbar)^4}\,\frac{1}{E_p}\,\frac{d}{dE_p}\Big(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\Big)$$

$$\Big\langle\frac{\vec{p}\vec{p}}{m}\Big\rangle_{\text{Fermi}} = \frac{2}{3}\mu\overleftrightarrow{I}.$$

In the Coulomb gauge

(4.68)
$$\Big(\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \nabla^2\Big)\vec{A}(x) = \frac{4\pi}{c}\,\vec{j}(x).$$

We then arrive at the equation for the electromagnetic field and the current:

(4.69)
$$\Big(\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}-\nabla^2\Big)\vec{A}(\vec{x}) + \frac{4\pi(2e)^2}{2mc^2}\Big\{n_s^{II}(T)\overleftrightarrow{I} - \Big(\frac{\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}}{m}\Big)\Delta^2\int\frac{d^4p}{(2\pi\hbar)^4}$$
$$\cdot\,\frac{1}{E_p}\,\frac{d}{dE_p}\Big(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\Big)\Big\}$$
$$= -\frac{e}{mc}2i\Big\{n_s^{II}(T)\overleftrightarrow{I} - \Big(\frac{\frac{\vec{Q}}{2}\frac{\vec{Q}}{2}}{m}\Big)\Delta^2\int\frac{d^4p}{i(2\pi\hbar)^4}\,\frac{1}{E_p}\,\frac{d}{dE_p}\Big(\frac{\tanh\frac{\beta E_p}{2}}{E_p}\Big)\frac{\hbar}{i}\,\frac{\partial\Theta}{\partial\vec{x}}$$

$$+ \frac{e}{mc} \frac{\vec{Q}}{2} W^2 \int \frac{d^4p}{(2\pi\hbar)^4} \frac{\tanh \frac{\beta E_p}{2}}{E_p^3} \hbar \frac{\partial \Phi}{\partial t} \}$$

E. Final Derivation of Ginzburg-Landau Equations.

Now we proceed to obtain Ginzburg-Landau equations for the superconducting charge-density-wave order parameter.

Taking derivatives of Γ (see Chapter 3 equation (8), in the vicinity of T_c where $|\Delta| = 0$ we obtain

$$(4.70) \quad -\frac{1}{g} + \int_I \frac{d^4p}{(2\pi\hbar)^4} \frac{\tanh \frac{\beta_c\sqrt{\epsilon_a^2+|W|^2}}{2}}{\sqrt{\epsilon_a^2+|W|^2}} + \int_{II} \frac{d^4p}{(2\pi\hbar)^4} \frac{\tanh \frac{\beta_c \epsilon(p)}{2}}{\epsilon(p)} = 0$$

Also one can show

$$(4.71) \quad \frac{\delta\Gamma}{\delta\Delta^*(x)} = \frac{1}{2} \frac{1}{|\Delta(x)|} \Delta(x) \frac{\Delta_i(x)}{\sqrt{2}} \frac{\delta\Gamma}{\delta\frac{\Delta_i(x)}{\sqrt{2}}} + \frac{\hbar^2}{4} [\frac{\partial^2 \Pi_{\alpha\beta}[q]}{\partial q \partial q}]_{q=0}$$

$$= \frac{\partial^2 \Delta}{\partial x \partial x}$$

The first term here can be transformed into

$$(4.72) \quad \Delta(x) \{ \frac{T_c - T}{T} [N_2(0) + \frac{\beta_c}{2} \int_I N_1(0) d\epsilon_a(p) \operatorname{sech} \frac{\beta_0\sqrt{\epsilon_a^2(p) + W^2}}{2}] \}$$

$$- \Delta(x) |\Delta(x)|^2 \{ \int_I \frac{d^4p}{(2\pi\hbar)^4} (- \frac{1}{2\epsilon_a(\vec{p})} \frac{d}{d\epsilon_a(p)} (\frac{\tanh \frac{\beta\sqrt{\epsilon_a^2(p) + |W|^2}}{2}}{\sqrt{\epsilon_a^2 + |W|^2}})$$

$$+ \int_{II} \frac{d^4p}{(2\pi\hbar)^4} \left(-\frac{1}{2q(\vec{p})}\right) \frac{d}{dq(p)} \tanh\left(\frac{\beta\epsilon(p)}{}\right) \}$$

The second term we will write

$$\frac{\hbar^2}{4}\left[\frac{\partial^2 \Pi_{\theta\theta}[q]}{\partial q \partial q}\right]_{q=0} : \frac{\partial^2 \Delta}{\partial x \partial x}$$

$$= \frac{\hbar^2}{4}\left\{-\Pi_{\theta}^{(0)}[T]\frac{1}{c^2}\frac{\partial^2 \Delta(x)}{\partial t^2} + (\overleftrightarrow{I}\Pi_{\theta}^{II}[T] + \hat{e}\hat{e}\Pi_{\theta}^{I}[T]) : \frac{\partial^2 \Delta(x)}{\partial \vec{x} \partial \vec{x}}\right.$$

where $\hat{e} = \frac{\vec{Q}}{|\vec{Q}|}$ is a unit vector in the nesting direction.

So we have the Ginzburg-Landau equations for SC and CDW systems:

$$(4.73) \quad \left\{-\frac{\hbar^2}{4}\Pi_{\theta}^{(0)}[T]\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{2ie}{\hbar c}\phi(x)\right)^2\right.$$

$$+ \frac{\hbar^2}{4}(\overleftrightarrow{I}\Pi_{\theta}^{II}[T] + \hat{e}\hat{e}\Pi_{\theta}^{(E)}[T] : (\nabla - \frac{2ie}{\hbar c}A(x))^2 \} \Delta(x)$$

$$+ \left\{\frac{T_c - T}{T_c}\left[N_2(0) + \frac{\beta_c}{2}\int_{I} N_1(0)\, d\epsilon_a(\vec{p})\, \mathrm{sech}^2 \frac{\beta_c\sqrt{\epsilon_a^2(\vec{p}) + |W|^2}}{2}\right]\right.$$

$$- \left[\int_{I} N_1(0)\, d\epsilon_a(p)\left(-\frac{1}{2\epsilon_a(p)}\right)\frac{d}{d\epsilon_a(p)}\left(\frac{\tanh\frac{\beta\sqrt{\epsilon_a^2 + |W|^2}}{2}}{\sqrt{\epsilon_a^2 + |W|^2}}\right)\right.$$

$$+ \int_{II} N_2(0)\, d\epsilon(p)\left(-\frac{1}{2\epsilon(p)}\right)\frac{1}{d\epsilon(p)}\left(\frac{\tanh\frac{\beta\epsilon(p)}{2}}{\epsilon(p)}\right)|\Delta(x)|^2\}\cdot$$

$$\cdot\Delta(x) = 0$$

For the current we had

$$(4.74)\quad \frac{1}{c}J(x) = \frac{e\hbar}{2ic}(\Pi_\theta^{II}[T]\overleftrightarrow{I}+\Pi_\theta^{I}[T]\hat{e}\hat{e})\cdot$$

$$\cdot(\Delta^*(x)\nabla\Delta(x)-\nabla\Delta^*(x)\cdot\Delta(x))$$

$$-\frac{(2e)^2}{(2m)c^2}(n_s^{II}[T]I + n_s^{(I)}(T)\hat{e}\hat{e})A(x)$$

$$+\frac{e}{mc}\frac{\vec{Q}}{2}|W(x)|^2\Pi_\omega^{(0)}[T]\cdot\hbar\frac{\partial\vartheta}{\partial t},$$

where

$$(4.75)\quad n_s^{II}[T] = m|\Delta(x)|^2\Pi_\theta^{II}[T] = m|\Delta|^2\Pi_\theta^{II}[T]$$

$$(4.76)\quad n_s^{I}[T] = m|\Delta(x)|^2\Pi_\theta^{I}[T] = m|\Delta(x)|^2\Pi_\theta^{I}[T]$$

$$(4.77)\quad \Pi_\omega^{(0)}[T] = \int\frac{d^4p}{(2\pi\hbar)^4}\,\frac{\tanh\beta E(p)}{E(p)}$$

$$(4.78)\quad \Pi_\theta^{(0)}[T] = \int_I\frac{d^4p}{(2\pi\hbar)^4}\,\frac{\tanh\beta E(p)}{E(p)}+\int_{II}\frac{d^4p}{(2\pi\hbar)^4}\,\frac{\tanh\frac{\beta E(p)}{2}}{E(p)}$$

In region I $E_a(p) = \sqrt{\epsilon^2(p)+|W|^2}$

In region II $E(p) = \epsilon(p)$

$$(4.79)\quad \Pi_\theta^{II}[T]I = -\frac{1}{m^2}\langle\vec{p}\vec{p}\rangle\int_{II}N_2(0)d\epsilon(\vec{p})\frac{1}{E(p)}\,\frac{d}{dE(p)}\left(\frac{\tanh\frac{\beta E(p)}{2}}{E(p)}\right)$$

$$(4.80)\quad \Pi_{\theta}^{I}[T] = -\frac{1}{m^2}\frac{|Q|^2}{2}\int_I N_1(0)\,d\varepsilon_a(p)\frac{1}{E_a(p)}\frac{d}{dE_a(p)}\cdot$$

$$\cdot\ \left(\frac{\tanh\frac{\beta E_a(p)}{2}}{E_a(p)}\right.$$

$$(4.81)\qquad \frac{\langle\vec{p}\vec{p}\rangle}{m} \simeq \frac{2}{3}\mu\overleftrightarrow{I}.$$

The solutions of the Ginsburg-Landau-Gor'kov equations will be given in the next chapter (5).

Chapter 5: Analysis of Ginzburg-Landau-Gorkov Equations and Electromagnetic Properties of Coexisting Charge-Density-Wave and Superconductivity Systems.

A. Introduction.

In this chapter we will discuss the electromagnetic and thermodynamic properties of the coexisting systems based on the Ginzburg-Landau-Gor'kov equations previously obtained. First we study the one dimensional geometry, and we obtain spatial dependence of the superconducting order parameter and magnetic field. We also find the penetration depth for both of them. Next we investigate the stability of the system described by the one dimensional free energy. This will allow us to find the critical current and we estimate the magnitude of the sliding charge-density-wave current at the critical value. The expression for the order parameter will then exhibit the coexistence-competition of the two effects. A proposed experiment involving reversal of the magnetic field will show a change in the superconducting order parameter which is linear in the applied magnetic field and sliding current. This new effect is a consequence of the theory. We also demonstrate quantitative changes in the Meissner effect: it becomes anisotropic. From a measurement of the temperature dependent anisotropic penetration depth, the electrom density in the nesting and non-nesting directions can be determined. Analysis of the 3 dimensional case gives the correlation length in all directions; the anisotropic upper critical field as function of the angles between direction of applied magnetic field and principal axes of the effective mass tensor. We calculate effects due to fluctuations just above T_c namely: temperature-dependent magnetic susceptibility and the magnetic moment of a thin film. Also the helicoidal solution for superconducting order parameter is obtained in thin films.

In an Appendix to this chapter we will discuss the threshold energies for polarized infrared absorbtion in a coexisting system.

B. General Discussion of the one Dimensional Case.

Suppose we have a semiinfinite sample occupying the positive x-axis ($x \geq 0$) with the magnetic field parallel to the boundary. Then we can choose the vector potential in the form $\vec{A} = (0, A(x), 0)$ so that

$$(5.1)\qquad \vec{H} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & A(x) & 0 \end{pmatrix} = (0,0,\frac{\partial A(x)}{\partial x}).$$

Now assume that the current is also parallel to the boundary, but parallel to the y-direction. Maxwell's (Ampere) equation is

$$(5.2)\qquad [\frac{4\pi}{c}\vec{J} = \nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2\vec{A} = -\nabla^2\vec{A}]$$

so $\vec{J}\|\vec{A}\|\hat{j}$. Both $\vec{J}$ and $\vec{A}$ may both be parallel. So $\Delta(x)$ may be assumed also to be a function of x only.

The Ginzburg-Landau equations for this case are

$$\frac{\hbar^2}{4}(\overleftrightarrow{I}\Pi_\theta^{II}[T]+\hat{e}\hat{e}\Pi_\theta^{I}[T]):(\nabla-\frac{2ie}{\hbar c}\vec{A}(x))^2\Delta(x)$$

$$= \frac{T_c-T}{T_c}[N_2(0)+\frac{\beta_c}{2}\int_I N_1(0)d\varepsilon_a(\vec{p})\,\mathrm{sech}^2\,\frac{\beta_c\sqrt{\varepsilon_a^2(p)+|W|^2}}{2}]\Delta(x)$$

$$+ [\int_I N_1(0)d\varepsilon_a(\vec{p})(-\frac{1}{2\varepsilon_a(\vec{p})})\frac{d}{d\varepsilon_a}(\frac{\tanh\frac{\beta\sqrt{\varepsilon_a^2(\vec{p})+|W|^2}}{2}}{\sqrt{\varepsilon_a^2(\vec{p})+|W|^2}})$$

$$+ \int_{II} N_2(0)d\varepsilon(p)(-\frac{1}{2\varepsilon(\vec{p})})\frac{d}{d\varepsilon(p)}\frac{\tanh\frac{\beta\varepsilon(\vec{p})}{2}}{\varepsilon(\vec{p})}]|\Delta(x)|^2\Delta(x)$$

$$\nabla^2\vec{A}(x) - 4\pi\frac{(2e)^2}{2mc^2}(n_s^{II}[T]\overleftrightarrow{I}+n_s^{I}[T]\hat{e}\hat{e})\cdot\vec{A}(\vec{x})$$

$$+ \frac{4\pi e}{mc}\frac{\vec{Q}}{2}|W(x)|^2\Pi_\varphi^{(0)}[T]\hbar\frac{\partial\phi}{\partial t} = 0.$$

For the chosen geometry:

$$(\nabla - \frac{2ie}{\hbar c}A(x))\Delta(x) = \hat{i}\frac{\partial\Delta}{\partial x} + \hat{j}(-\frac{2ie}{\hbar c}A(x))\Delta(x)$$

and

$$(5.3)\qquad \overleftrightarrow{I}:(\nabla-\frac{2ie}{\hbar c})^2\Delta(x) = (\frac{\partial^2}{\partial x^2} + (\frac{2ie}{\hbar c})^2A^2(x))\Delta(x)$$

$$(5.4)\qquad \hat{e}\hat{e}:(\nabla - \frac{2ie}{\hbar c})^2\Delta(x) = (\frac{2ie}{\hbar c})^2A^2(x)\Delta(x)$$

and we also denoted:

$$(5.5)\qquad n_s^{II}[T] = m|\Delta(x)|^2\Pi_{\theta}^{II}[T]$$

$$(5.6)\qquad n_s^{I}[T] = m|\Delta(x)|^2\Pi_{\theta}^{I}[T].$$

Ginzburg-Landau-Gor'kov equations for the lD-situation are:

$$(5.7)\qquad \frac{\hbar^2}{4}\{\Pi_{\theta}^{II}[T]\frac{\partial^2}{\partial x^2} + (\Pi_{\theta}^{II}+\Pi_{\theta}^{I})(\frac{2ie}{\hbar c})^2A^2(x)\}\Delta(x)$$

$$= -\frac{T_c-T}{T_c}[N_2(0)+\frac{\beta_c}{2}\int_I N_1(0)d\epsilon_a(\vec{p})\,\mathrm{sech}^2\,\frac{\beta_c}{2}\sqrt{\epsilon_a^2(\vec{p})+|W|^2}\,]\Delta(x)$$

$$+ [\int_I N_1(0)d\epsilon_a(\vec{p})(-\frac{1}{2\epsilon_a(\vec{p})})\frac{d}{d\epsilon_a(\vec{p})}(\frac{\tanh\frac{\beta\sqrt{\epsilon_a^2(\vec{p})+|W|^2}}{2}}{\sqrt{\epsilon_a^2(p)+|W|^2}})$$

$$+ \int_{II} N_2(0)d\epsilon(\vec{p})(-\frac{1}{2\epsilon(\vec{p})})\frac{d}{d\epsilon(\vec{p})}(\frac{\tanh\frac{\beta\epsilon(\vec{p})}{2}}{\epsilon(\vec{p})})]\,|\Delta(x)|^2\Delta(x)$$

$$(5.8)\quad \frac{\partial^2 A(x)}{\partial x^2} - 4\pi\frac{(2e)^2}{(2m)c}m|A(x)|^2(\Pi_\theta^{II}[T]\overleftrightarrow{I}+\Pi_\theta^{I}[T]\hat{e}\hat{e})\cdot A(x)$$

$$+\frac{4\pi e}{me}\frac{\vec{Q}}{2}|W(x)|^2\Pi_\varphi^{(0)}[T]\hbar\frac{\partial\Phi}{\partial t} = 0.$$

This can be rewritten as:

$$(5.9)\quad \left[\frac{\partial^2}{\partial x^2} - \left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)\frac{(2e)^2}{\hbar^2c^2}A^2(x)\right]\psi(x)$$

$$= -\frac{T_c-T}{T}\cdot\frac{N_2(0)+\frac{\beta_c}{2}\int N_1(0)d\epsilon_a(\vec{p})\operatorname{sech}\frac{\beta c}{2}\sqrt{\epsilon_a^2(\vec{p})+|W|^2}}{\frac{\hbar^2}{4}\Pi_\theta^{II}[T]}\psi(x)$$

$$+\int_I N_1(0)d\epsilon_a(\vec{p})\left(-\frac{1}{2\epsilon_a(\vec{p})}\right)\frac{d}{d\epsilon_a(\vec{p})}\left(\frac{\tanh\beta\sqrt{\epsilon_a^2(\vec{p})+|W|^2}}{\sqrt{\epsilon_a^2(\vec{p})+|W|^2}}\right)$$

$$\frac{+\int N_a(0)d\epsilon(p)\frac{1}{2\epsilon(\vec{p})}\frac{d}{d\epsilon(p)}\left(\frac{\tanh\frac{\beta\epsilon(p)}{2}}{\epsilon(\vec{p})}\right)|\psi(x)|}{\frac{\hbar^2}{4}\Pi_\theta^{II}[T]\cdot m(\Pi_\theta^{I}[T]+\Pi_\theta^{II}[T])}$$

and for the vector potential the equation is:

$$(5.10)\quad \frac{\partial^2 A(x)}{\partial x^2} - \frac{4\pi(2e)^2}{(2m)c^2}|\psi(x)|^2A(x) + \frac{4\pi e}{mc}\frac{Q}{2}|W|^2\Pi_\varphi^{(0)}[T]\hbar\frac{\partial\Phi}{\partial t} = 0$$

where we have introduced

$$(5.11)\qquad |\psi(x)|^2 = m(\Pi_\theta^{I}[T] + \Pi_\theta^{II}[T])|\Delta(x)|^2.$$

Now to put the Ginzburg-Landau-Gor'kov equation in convenient form we introduce

$$(5.12)\qquad \alpha = 2\frac{T_c - T}{T}\cdot\frac{N_2(0) + \frac{\beta_c}{2}\int_I N_1(0)\,d\epsilon_a(\vec{p})\,\mathrm{sech}\frac{2\beta_c\sqrt{\epsilon_a^2(\vec{p})+|W|^2}}{2}}{2m\Pi_\theta^{II}[T]}$$

$$(5.13)\qquad \beta = 4\frac{\int_I N_1(0)\,d\epsilon_a(\vec{p})\left(-\frac{1}{2\epsilon_a(\vec{p})}\right)\frac{d}{d\epsilon_a(\vec{p})}\left(\frac{\tanh\beta\sqrt{\epsilon_a^2(\vec{p})+|W|^2}}{\sqrt{\epsilon_a^2(p)+|W|^2}}\right)}{m^*\Pi_\theta^{II}[T]\,(\Pi_\theta^{I}[T]}$$

$$\frac{+\int_{II} N_2(0)\,d\epsilon(p)\left(-\frac{1}{2\,\epsilon(p)}\right)\frac{d}{d\epsilon(p)}\frac{\tanh\frac{\beta\epsilon p}{2}}{\epsilon(p)}}{+\Pi_\theta^{II}[T])}.$$

Thus

$$(5.14)\qquad \left[\frac{\partial^2}{\partial x^2} - \left(1+\frac{\Pi^{I}[T]}{\Pi^{II}[T]}\right)\frac{(2e)^2}{\hbar^2 c^2}A^2(x)\right]\psi(x)$$

$$= \frac{2(2m)}{\hbar^2}\alpha\psi(x) + \frac{2(2m)}{\hbar^2}\beta|\psi|^2\psi(x)$$

$$(5.15)\qquad \frac{\partial^2 A(x)}{\partial x^2} - \frac{4\pi(2e)^2}{(2m)c^2}|\psi(x)|^2 A(x) + \frac{4\pi e}{mc}\frac{Q}{2}|W|^2\Pi_\varphi^{(0)}[T]\frac{\partial\phi}{\partial t} = 0.$$

This can be rewritten as:

$$(5.16)\qquad \left[\frac{\partial^2}{\partial x^2} + \frac{2(2m)\alpha}{\hbar^2}\left[1-\frac{(2e)^2}{2(2m)c^2\alpha}\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)A^2(x)\right]\psi(x)$$

$$- \frac{2(2m)}{\hbar^2}\beta|\psi(x)|^2\psi(x) = 0$$

$$(5.17) \quad \frac{\partial^2 A(x)}{\partial x^2} - \frac{4\Pi(2e)^2}{(2m)c^2}|\psi(x)|^2 A(x) + \frac{4\pi e}{mc}\frac{Q}{2}|W|^2 \Pi_{\varphi}^{(0)}[T]\frac{\partial \phi}{\partial t} = 0.$$

Now we introduce the definitions:

$$\psi_0 = \psi|_{A=0} = \sqrt{\frac{\bar{\alpha}}{\beta}} \quad \text{and} \quad \psi = \frac{\psi}{\psi_0} = \psi\sqrt{\frac{\beta}{\alpha}}.$$

We have

$$(5.18) \quad \{\frac{\partial^2}{\partial x^2} + \frac{2(2m)}{\hbar^2}\alpha[1 - \frac{(2e)^2}{2(2m)c^2\alpha}(1+\frac{\Pi_{\theta}^{T}[T]}{\Pi_{\theta}^{TI}[T]})A^2(x)]\}\psi(x)$$

$$- \frac{2(2m)}{\hbar^2}\alpha|\psi(x)|^2\psi(x) = 0$$

$$(5.19) \quad \frac{\partial^2 A}{\partial x^2} - \frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}|\psi(x)|^2 A(x) + \frac{4\pi e\hbar}{mc}\frac{Q}{2}|W|^2 \Pi_{\varphi}^{(0)}[T]\frac{\partial \phi}{\partial t} = 0.$$

In order to rewrite these equations in dimensionless form we make a change of variables to:

$$\tilde{x} = \frac{x}{\delta_0} \text{ (dimensionless length) where } \frac{1}{\delta_0^2} = \frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}$$

$$(5.20) \quad \frac{\partial}{\partial x} = \frac{1}{\delta_0}\frac{\partial}{\partial \tilde{x}}; \quad \frac{\partial^2}{\partial x^2} = \frac{1}{\delta_0^2}\frac{\partial^2}{\partial \tilde{x}^2} = \frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}\frac{\partial^2}{\partial \tilde{x}^2}.$$

Thus

$$(5.21)\quad \frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}\frac{\partial^2}{\partial\tilde{x}^2}$$

$$+\frac{2(2m)}{\hbar^2}\alpha[1-\frac{(2e)^2}{2(2m)c^2\alpha}(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]})A^2(\tilde{x})]\psi(x) - \frac{2(2m)}{\hbar^2}\alpha|\psi|^2$$

$$(5.22)\quad \frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}\frac{\partial^2 A(x)}{\partial\tilde{x}^2} - \frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}\cdot|\psi(\tilde{x})|^2 A(x)$$

$$+\frac{4\pi e\hbar}{mc}\frac{\Omega}{2}|W|^2 \Pi_\varphi^{(0)}[T]\frac{\partial\Phi}{\partial t} = 0.$$

The dimensionless vector potential is given via

$$(5.23)\qquad a^2(x) = \frac{(2e)^2}{2(2m)c^2\alpha}(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]})A^2(x)$$

Now we can rewrite our equations in the following form.

$$(5.24)\quad [\frac{\partial^2}{\partial\tilde{x}^2}+\frac{2(2m)\alpha}{\hbar^2}\frac{1}{\frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}}[1-a^2(\tilde{x})]]\psi(\tilde{x})$$

$$-\frac{2(2m)\alpha}{\hbar^2}\cdot\frac{1}{\frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}}|\psi(\tilde{x})|^2\psi(\tilde{x})$$

$$(5.25)\quad \frac{\partial^2 a(\tilde{x})}{\partial \tilde{x}^2} - |\psi|^2 a(\tilde{x}) + \frac{\frac{4\pi e\hbar}{mc}\,\frac{Q}{2}|W|^2 \Pi_{\omega}^{(0)}[T]\frac{\partial \xi}{\partial t}}{\frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta}\,\frac{1}{\left[\frac{(2e)^2}{2(2m)c^2\alpha}\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)\right]^{1/2}}} = 0.$$

Introducing the usual notation $K^2 = \frac{(2m)^2 c^2 \beta}{2\pi(2e)^2\hbar^2}$ for a dimensionless constant

$$(5.26)\quad \mathfrak{J} = \frac{\frac{4\pi e\hbar}{mc}\,\frac{Q}{2}|W|^2 \Pi_{\omega}^{(0)}[T]\frac{\partial \xi}{\partial t}}{\frac{4\sqrt{2}\,\pi(2e)^*\alpha^{3/2}}{(2m)^{1/2}c\beta}}\left[\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)\right]^{1/2}$$ for a

dimensionless current,

we arrive at coupled dimensionless equations:

$$(5.27)\quad -\frac{1}{K^2}\frac{\partial^2\psi}{\partial\tilde{x}^2} + a^2(\tilde{x})\psi(\tilde{x}) = \psi(\tilde{x}) - |\psi(\tilde{x})|^2\psi(\tilde{x})$$

$$(5.28)\quad \frac{\partial^2 a(\tilde{x})}{\partial\tilde{x}^2} - |\psi(\tilde{x})|^2 a(\tilde{x}) + \mathfrak{J} = 0$$

where

$$(5.29)\quad \tilde{x} = \frac{x}{\delta_0};\ \text{and}\ \frac{1}{\delta_0^2} = \frac{4\pi(2e)^2}{(2m)c^2}\cdot\frac{\alpha}{\beta};\ \text{and}\ \psi(\tilde{x}) = \frac{\psi(\tilde{x})}{\psi_0}\Big|_{A=0} = \sqrt{\frac{\alpha}{\beta}}$$

$$(5.30)\quad K^2 = \frac{\beta}{2\pi}\left(\frac{2mc}{2e\hbar}\right) = \frac{2(2e)^2}{\hbar c}H_{cr}^2\delta_0^4$$

$$(5.31)\quad a(x) = \left[\frac{(2e)^2}{2(2m)c^2\alpha}\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)A(x)\right]^{1/2}$$

$$= \frac{1}{\sqrt{2}\,H_c\delta_0}\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi^{II}[T]}\right)^{\frac{1}{2}} A(x).$$

Finally the dimensionless magnetic field is:

$$(5.32)\qquad h(x) = \frac{1}{\sqrt{2}H_c}\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)^{\frac{1}{2}} H(x)$$

$$(5.33)\quad \mathfrak{J} = \frac{\frac{e}{m}\frac{Q}{2}|W|^2\Pi_m^{0}\frac{\partial\phi}{\partial t}}{\left[\frac{2}{(2m)}\right]^{\frac{1}{2}}(2e)\frac{\alpha^{3/2}}{\beta}}\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)^{\frac{1}{2}}$$

$$= \frac{e}{m}\frac{Q}{2}\,\frac{|W|^2\Pi_m^{(0)}[T]\frac{\partial\phi}{\partial t}}{\frac{cH_{c_r}}{2\sqrt{2}\pi\delta_0}}\left(1+\frac{\Pi_\theta^{I}[T]}{\Pi_\theta^{II}[T]}\right)$$

where $H_{c_r}^2 = \frac{4\pi\alpha^2}{\beta}$.

C. <u>The Free Energy of the Coexisting System</u>.

Later we shall discuss solutions of these equations and various physical consequences. Now we turn to the free energy corresponding to the 1-D case. The form of the Ginzburg-Landau-Gorkov equations immediately suggests the following free energy density:

$$(5.34)\quad F = F_0 - |\psi|^2 + \frac{1}{2}|\psi|^4 + |(\frac{1}{iK}\vec{\nabla}-\vec{a})\psi|^2 + (\nabla\times\vec{a})^2 - 2\tilde{J}\ a$$

$$= F_0 - \psi^*\psi + \frac{1}{2}(\psi^*\psi)^2 + \frac{1}{K^2}\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x} + a^2\psi^*\psi + (\frac{\partial a}{\partial x})^2 - 2\tilde{J}\cdot a.$$

All quantities here are expressed in dimensionless form as given in equations (5.29)-(5.33) above. In order to investigate the stability of the physical system, we have to analyse the behavior of the free energy as a function of the order parameter.

We start with the case of a homogeneous superconductor. Then for the free energy (all gradient terms vanish)

$$(5.35)\quad F = F_0 - \psi^*\psi + \frac{1}{2}(\psi^*\psi)^2 + a^2\psi^*\psi - 2\tilde{J}a.$$

ψ is complex, so we can write:

$$(5.36)\qquad \psi = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) = |\psi|e^{-i\theta}$$

$$(5.37)\qquad \psi^* = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) = |\psi|e^{i\theta}.$$

If we take $|\psi| \equiv \psi \equiv$ real ψ:

$$(5.38)\qquad F = F_0 - |\psi|^2 + \frac{1}{2}|\psi|^4 + a^2|\psi|^2 - 2\tilde{J}\ a.$$

If we take $\psi = \frac{1}{\sqrt{2}}\psi_1$ and $\psi_2 = 0$ then:

$$(5.39) \qquad F = F_0 - \frac{1}{2}\psi_1^2 + \frac{1}{4}\psi_1^4 + \frac{1}{2}a^2\psi_1^2 - 2\tilde{J}\ a.$$

The free energy here is a function of two variables: the superconducting order parameter ψ_1 and the electromagnetic field a. The extremum conditions are

$$(5.40) \qquad \begin{aligned} \frac{\partial F}{\partial \psi_1} &= \psi_1(-1 + a^2 + \psi_1^2) = 0 \\ \frac{\partial F}{\partial a} &= a\psi_1^2 - 2\tilde{J} = 0. \end{aligned}$$

$\psi_1 = 0$ is a trivial solution so we have

$$(5.41) \qquad \begin{aligned} a^2 + \psi_1^2 &= 1 \\ a &= \frac{2\tilde{J}}{\psi_1^2}. \end{aligned}$$

Depending on the magnitude of $\tilde{J}$ we have three different situations:

1) no roots
2) one root
3) 2 roots.

Introduce variables

$$(5.42) \qquad \begin{aligned} y &= a^2 \\ x &= \psi_1^2 \end{aligned}$$

We then solve the simultaneous equations

$$x + y = 1 \tag{5.43}$$
$$y = \frac{4\tilde{J}^2}{x^2}$$

Let us call the situation when we have only one root "critical". The equation of the tangent to $y = \frac{4\tilde{J}^2}{x^2}$ is

$$y - y_0 = \frac{d}{dx_0}\left(\frac{4\tilde{J}^2}{x_0^2}\right)(x-x_0) = -2\frac{4\tilde{J}^2}{x_0^3}(x-x_0) = -\frac{8\tilde{J}^2}{x_0^3}(x-x_0)$$

$$y = -\frac{8\tilde{J}^2}{x_0^3}x + \frac{8\tilde{J}^2}{x_0^2} + y_0 = -\frac{8\tilde{J}^2}{x_0^3}x + \frac{8\tilde{J}^2}{x_0^2} + \frac{4\tilde{J}^2}{x_0^2} \tag{5.44}$$

so

$$y = -\frac{8\tilde{J}^2}{x_0^3}x + \frac{12\tilde{J}^2}{x_0^2} \tag{5.45}$$

so for the tangent passing through (0,1) and (1,0) we have

$$\frac{8\tilde{J}^2}{x_0^3} = 1 \text{ and } \frac{12\tilde{J}^2}{x_0^2} = 1. \tag{5.46}$$

Dividing these equations we have $x_0 = \frac{2}{3}$; $\tilde{J}^2_{cr} = \frac{1}{27}$; $y_0 = \frac{1}{3}$.

Thus we obtain finally

$$\psi^2_{1,cr} = \frac{2}{3}$$
$$\tilde{J}^2_{cr} = \frac{1}{27} \tag{5.47}$$
$$a^2_{cr} = \frac{1}{3}$$

At this point we can estimate the magnitude of the CDW-sliding current when it reaches the critical value. We take the $H_{cr} = 10^3$ Gauss, and $\delta_0 = 10^{-5}$cm. This gives

$$(5.48) \quad \tilde{J}_{CDW_{cr}} \sim \frac{1}{\sqrt{27}} \cdot \frac{H_{cr}}{2\sqrt{2\pi}} \cdot \frac{1}{\delta_0} \sim \frac{1}{65.1} \cdot H_{cr}$$

$$= \frac{1}{65.1} \times \frac{10^3}{4\pi}\left(\frac{a}{m}\right) \cdot \frac{10^3}{10^{-5}\,\mathrm{cm}} = \frac{1}{65.1} \times \left(\frac{10}{4\pi}\right) \cdot \frac{10^3}{10^{-5}} \frac{a}{\mathrm{cm}^2}$$

$$\simeq 10^7 \frac{\mathrm{amp}}{\mathrm{cm}^2}.$$

Note this is of the same order of magnitude as the critical current measured first in thin films [1].

In order to determine the stability of the solution we found for the free energy we examine the Hessian

$$(5.49) \quad \begin{bmatrix} \frac{\partial^2 F}{\partial \psi_1^2} - \lambda & \frac{\partial^2 F}{\partial \psi_1 \partial a} \\ \frac{\partial^2 F}{\partial a \partial \psi_1} & \frac{\partial^2 F}{\partial a^2} - \lambda \end{bmatrix} = \begin{bmatrix} a^2 - 1 + 3\psi_1^2 - \lambda & 2a\psi_1 \\ 2a\psi_1 & \psi_1^2 - \lambda \end{bmatrix}$$

The characteristic equation, corresponding to this Hessian is

$$\lambda^2 - \lambda(a^2 - 1 + 4\psi_1^2) + \psi_1^2(a^2 - 1 + 3\psi_1^2) - 4a^2\psi_1^2 = 0.$$

On the curve $a^2 + \psi_1^2 = 1$ it becomes:

$$\lambda^2 - 3\psi_1^2\lambda + \psi_1^2(2\psi_1^2) - 4(1-\psi_1^2)\psi_1^2 = 0$$

or

$$\lambda^2 - 3\psi_1^2\lambda + (6\psi_1^2-4)\psi_1^2 = 0$$

(5.50) $$\lambda = \frac{1}{2}[3\psi_1^2 \pm \sqrt{16\psi_1^2 - 15\psi_1^4}].$$

We can draw conclusions: 1) because $\psi_1 < 1$, the roots λ are real; and 2) because $\lambda_1\lambda_2 = \psi_1^2(6\psi_1^2 - 4)$. If $6\psi_1^2 > 4$ or $\psi_1^2 > \frac{2}{3}$ (supercritical) then

(5.51) $$\lambda_1 > 0 \qquad \lambda_2 > 0$$

which corresponds to the stable solution.

It is of general physical interest to check experimentally whether or not it is possible to observe this solution which presumably corresponds to a metastable state.

D. <u>One Dimensional Case. Perturbation Theory Solution</u>.

Now we turn to the one-dimensional case which will be solved by perturbation theory assuming a slow spatial variation of the superconducting order parameter in a weak magnetic field. We measure smallness via the dimensionless magnetic field times K^2:

$a_0 = 0$ (no magnetic field

$a = a_1 + a_2$

$\psi = \alpha + \varphi$

(α- superconducting order parameter in th; absence of $\vec{H}$)

then in equations (5.27)-(5.28) we get:

$$(5.52) \quad -\frac{1}{K^2}\varphi'' + (a_1^2+a_2^2+2a_1a_2)(\alpha + \varphi) = (\alpha + \varphi) + (\alpha + \varphi)^3$$

$$(5.53) \quad a_1'' + a_2'' = (\alpha^2+2\alpha\varphi+\varphi^2)(a_1+a_2) + \mathfrak{J}.$$

$\mathfrak{J}_{cr}$ is equal to $\frac{1}{27}$ and we assume therefore that $\tilde{J}$ is of first order. Then for the order of perturbation theory

0 - order gives $\alpha - \alpha^3 = 0$ so $\alpha = 1$

first order $a_1'' = \alpha^2 a_1 + \mathfrak{J}.$

These have an immediate solution $a_1 = -\frac{h_0}{\alpha} e^{-\alpha x} - \frac{\mathfrak{J}}{\alpha^2}$

$$\alpha = 1.$$

Thus $a_1 = -h_0 e^{-x} - \tilde{J}.$

$$(5.54) \quad \text{In first order: } -\frac{1}{K^2}\varphi'' + \alpha a_1^2 = (1 - 3\alpha^2)\varphi$$

$$a_2'' = \alpha^2 a_2.$$

This gives

$$\omega'' - 2K^2\omega = (h_0 e^{-x} + \tilde{J})^2. \tag{5.55}$$

The solution of the homogeneous equation is:

$$y_1 = e^{\sqrt{2}\,Kx}$$

$$y_2 = e^{-\sqrt{2}\,Kx}.$$

Using a usual Lagrange method one finds

$$\psi = 1 + \frac{K}{\sqrt{2}} e^{K\sqrt{2}\,x}\{\int^x e^{-K\sqrt{2}\,x}(h_0 e^{-x} + \tilde{J})^2 dx + C_1\}$$
$$- \frac{k}{\sqrt{2}} e^{-K\sqrt{2}\,x}\{\int^x e^{K\sqrt{2}\,x}(h_0 e^{-x} + \tilde{J})^2 dx + C_2\} \tag{5.56}$$

where C_1 and C_2 are constants of integration to be determined from the boundary conditions. In what follows we will use the Ginzburg-Landau boundary conditions namely $(\nabla\psi)_{\text{boundary}} = 0$. In our case this corresponds to $\frac{\partial\psi}{\partial x}\big|_{x=0} = 0$. Omitting simple intermediate calculations we write the final result:

$$\psi = 1 + \frac{K h_0^2}{\sqrt{2}(2-K^2)}\left(\frac{k}{\sqrt{2}} e^{-2x} - e^{-\sqrt{2}\,Kx}\right)$$
$$+ \frac{2\tilde{J}h_0}{\sqrt{2}(2K^2-1)}\left(Ke^{-x} - \frac{1}{\sqrt{2}}e^{-K\sqrt{2}\,x}\right) - \tilde{J}^2. \tag{5.57}$$

The first two terms are exactly the same as in the original article by Ginzburg and Landau [2],[3]. The remaining terms are new. In the absence of a magnetic field $\psi = 1 - \tilde{J}^2$. This corresponds to competition between CDW and SC. In a purely superconducting system obviously $\psi = 1$. Also, quite interesting, is the term proportional to $\tilde{J}h_0$, which changes its sign with change of the direction of the $\tilde{J}_{CDW}$. So

$$(5.58) \qquad \Delta\psi = \psi_{\rightarrow} - \psi_{\leftarrow} = \frac{4\tilde{J}\ h_0}{\sqrt{2}\,(2x^2-1)}(Ke^{-x} - \frac{1}{\sqrt{2}}\ e^{-K\sqrt{2}\ x}).$$

We think that this result may be used as a check of the validity of the theory presented here. From (58) we see that in a weak magnetic field $\Delta\psi$ is a linear function of h_0 as well as of $\tilde{J}$. One has to measure $\Delta\psi$ as a function of h_0.

E. The Meissner Effect.

For the magnetic field we have $h = h_0 e^{-x}$ which corresponds to a Meissner effect with screening length δ_0 (see (5.29)). We can also look at the Meissner effect from a slightly different side. Consider an idealized experiment with the semi-infinite sample in $z < 0$ and assume the resting direction is along the x-axis and the magnetic field $\vec{B}$ is parallel to the x-y plane. Then

$$(5.59) \qquad (B_x, B_y, B_z) = \left(-\frac{\partial A_y(z)}{\partial z}, \frac{\partial A_x(z)}{\partial z}, 0\right),$$

and Maxwell's equations will have the form

$$(5.60) \qquad \frac{\partial^2 A_x(z)}{\partial z^2} - \frac{4\pi(2e)^2}{2mc^2}(n_s^{(I)}[T] + (1-\alpha)n_s^{(II)}(T))A_x(z)$$

$$= -\frac{2\pi e\hbar}{mc} Q|W|^2 \Pi_\phi^{(0)}[T]\frac{\partial\phi}{\partial t}$$

and

$$(5.61) \qquad \frac{\partial^2 A_y(z)}{\partial z^2} - \frac{4\pi(2e)^2}{2mc^2} n_s^{II}(T) A_y(z) = 0$$

with $A_z = 0$.

If B is parallel to the x-axis, i.e. the nesting direction, the penetration depth is determined by $n_s^{II}[T]$, which is the density of pairing electrons contributed by region II. This is essentially the ordinary Meissner effect. If the magnetic field B is parallel to the y-axis, i.e. perpendicular to the nesting direction, there will still be a Meissner effect--but with a different penetration depth which is determined by $n_s^{(I)}[T] + (1-a)n_s^{(II)}[T]$, where $n_s^{(I)}[T]$ has the physical meaning of the density of Cooper pairs in the nesting direction. The Levin-Bilbro-McMillan model counts the density of states in the two different regions in an average sense.

Along the nesting direction, due to the Peierls mechanism, it is difficult to pair electrons. Therefore, according to equations (5.60)-(5.61) $n_s^I[T]$ is proportional to the density of states. So the "experimental" value of $n_s^{(I)}[T]$ may be thus much smaller than the value estimated from the coexistance transition temperature T_c. Several experiments can now be proposed in order to test the proposed microscopic theory given here.

In the perpendicular geometry if perfect diamagnetism is found, it would indicate the existence of a bulk pairing supercurrent which cancels the sliding CDW current automatically. Therefore, observation of the Meissner effect in this case will give us interesting experimental information about whether there is direct coupling between the phase of the superconducting order parameter and the phase of the charge-density-wave order parameter.

A temperature-dependent measurement of the two penetration depths (parallel and perpendicular geometry) can give valuable information about $n_s^{(I)}[T]$ and $n_s^{(II)}[T]$, since it is presumably insensitive to temperature.

F. General 3 Dimensional Case.

Now we turn to the general 3-D case. Let us rewrite the Ginzburg-Landau-Gov'kov equations by placing them into conventional form:

$$\frac{\hbar^2}{4m} : \left(\nabla - \frac{2ie}{\hbar c} A\right)^2 \psi + a\psi(x) + b|\psi|^2\psi = 0 \qquad (5.62)$$

where $\frac{\overleftrightarrow{1}}{4m} \simeq I_\theta^{II}[T] + \hat{e}\hat{e}\, I_\theta^{I}[T]$ is the effective mass tensor. In the absence of an external electromagnetic field, and after diagonalization of the mass tensor, we have in the principal axes

$$-\frac{\hbar^2}{4m_i}\frac{d^2\psi}{dx_i^2} + a\psi + b|\psi|^2\psi = 0. \qquad (5.63)$$

We can define three principal temperature dependent coherence lengths.

$$\zeta_i = \frac{\hbar^2}{m_i a} \qquad i = 1,2,3. \qquad (5.64)$$

We now calculate the upper critical magnetic field M_{c2} in the CDW-SC system. Choosing $\vec{Q}$ along the x-axis for the tensor of effective masses we have for example

$$\frac{\overleftrightarrow{1}}{4m} = (\Pi_\theta^{II} + \Pi_\theta^{I})\hat{i}\hat{i} + \Pi_\theta^{II}(\hat{j}\hat{j} + \hat{k}\hat{k}). \qquad (5.65)$$

Suppose now that a magnetic field is along the z direction. We can choose the gauge

(5.66) $$\vec{A} = (-yB, 0, 0) \quad \text{so} \quad \vec{B} = (0, 0, \vec{B})$$

The Ginzburg-Landau-Gor'kov equation becomes:

(5.67) $$\frac{1}{4m_{xx}}\{-\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{2i\hbar e}{c}(-yB)\frac{\partial}{\partial x} + \frac{e^2 y^2 B^2}{c^2}\}\psi(x,y,z)$$

$$+ \frac{1}{4m_{yy}}(-\hbar^2 \frac{\partial^2 \psi(x,y,z)}{\partial y^2}) + \frac{1}{4m_{zz}}(-\hbar^2 \frac{\partial^2 \psi(x,y,z)}{\partial z^2})$$

$$= a\psi .$$

Now taking $\psi(x,y,z) = e^{i(k_x x + k_z z)} f(y)$, we obtain

(5.68) $$-\hbar^2\{-\frac{k_x^2}{4m_{xx}} - \frac{k_z^2}{4m_{zz}} + \frac{1}{4m_{yy}}\frac{d^2}{dy^2}\}f(y) + \frac{\hbar k_x eyB}{2m_{xx}c}f(y)$$

$$+ \frac{e^2 B^2}{4m_{xx}C^2}f(y) = af(y).$$

Now shift variables

(5.69) $$y = y' - \frac{c\hbar kx}{eB}.$$

We then get an equation for f(y')

(5.70) $$-\frac{\hbar^2}{4m_{yy}}\frac{d^2}{dy'^2}f(y') + \frac{e^2 B^2 y'^2}{4m_{xx}C^2}f(y') = af(y').$$

The straightforward solution of this equation is:

$$(5.71) \qquad B = \frac{ca(m_{yy}m_{xx})^{\frac{1}{2}}}{2(n+\frac{1}{2})\hbar e} - \frac{\hbar^2 k_z^2 (m_{xx}m_{yy})^{\frac{1}{2}}}{2m_{zz}}$$

so the upper critical magnetic field is

$$B_{c2_{\perp}} = \frac{2ac}{e\hbar}(m_{yy}m_{xx})^{\frac{1}{2}}.$$

A similar calculation for a magnetic field in the x-y plane (for example $\vec{B}$ in the y direction) leads to the result

$$(5.72) \qquad B_{c2_{\parallel}} = \frac{2ac}{eh}\,(m_{xx}m_{zz})^{\frac{1}{2}}.$$

Comparing the two results we obtain

$$(5.73) \qquad \frac{B_{c2_{\parallel}}}{B_{c2_{\perp}}} = \left(\frac{m_{\perp}}{m_{\parallel}}\right)^{\frac{1}{2}}$$

The possibility of estimating the effective mass anisotropy in the CDW-SC system by measurements of the ratio of B_{c2} provide an important test of this theory.

One also can discuss the general case when the magnetic field is in an arbitrary direction. In this case the calculation of the upper critical magnetic field by solving equation (5.62) is also straightforward. We only give a final expression for H_{c2}:

$$(5.74)\quad H_{c2}\ \alpha\left(\frac{\sin^2\phi}{m_\perp}+\frac{\cos^2\phi}{m_{11}}\right)^{\frac{1}{2}}\left(\frac{\cos^2\theta\ \sin^2\phi}{m_{11}}+\frac{\sin^2\theta(1+\cos^2\phi)}{m_\perp}\right)^{\frac{1}{2}}$$

where θ is the angle made by $\vec{B}$ with the normal to the flat surface and angle φ is the angle between $\vec{Q}$ and the projection of $\vec{B}$ onto the surface ($\vec{Q}$ is assumed to lie in the surface). The above result goes over to that of Kats [4] obtained for layered superconductors in the limit $\varphi \to 0$.

$$(5.75)\qquad B_{c2} \sim \frac{1}{\cos^2\theta + \frac{m_\perp}{m_{11}}\sin^2\theta}$$

We now discuss some magnetic properties just above T_c. Here we have the "frozen" CDW and simultaneously superconducting fluctuations. We will calculate the diamagnetic susceptibility. First calculate the change of the free energy due to fluctuations, following a procedure suggested by Lifshitz and Pitaevsky [5]. Assuming $\vec{B}$ in the z direction we have

$$(5.76)\qquad \Delta F = -V\frac{2ek_BTB}{(2\pi\hbar)^2c}\Sigma_n\int_{-\infty}^{+\infty}\frac{\pi k_BT}{\hbar\omega_{0x}(n+\frac{1}{2})\left(\frac{m_{xx}}{m_{yy}}\right)^{1/2}+\frac{p_z^2}{2m_{zz}}+a}dp_z$$

where $\omega_{0x} = \frac{eB}{m_{xx}c}$, k_B-Boltzmann constant, V is the volume. Now using the Poisson summation formula and neglecting an unimportant constant term we obtain:

$$(5.77)\quad \Delta F = \frac{1}{48}\cdot\frac{\pi e^2 T_c^2 B^2 V}{hc^2}\cdot\frac{1}{(m_{xx}m_{yy})^{1/2}} \int_{-\infty}^{\infty} \frac{dp_z}{a+\frac{p_z^2}{2m_{zz}}}$$

$$= \frac{B^2 V}{48}\,\frac{e^2 T_c \Pi^2 \sqrt{2}}{\hbar c^2}\left(\frac{m_{zz}}{m_{xx}m_{yy}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{a}}\ .$$

For the diamagnetic susceptibility we have

$$(5.78)\qquad \chi = -\frac{1}{V}\,\frac{\partial^2 \Delta F}{\partial B^2} = \frac{e^2 T_c}{24\pi\hbar c^2 (\tilde{\alpha} m_{\perp})^{1/2} (T-T_c)^{1/2}}$$

where we substituted $a = \tilde{\alpha}(T-T_c)$. We next evaluate the magnetic moment of a small drop of radius less than ξ_z. Then in the previous sum only the term with eigenvalue $n = 0$ is of importance, corresponding to the solution ψ = const. and is equal to $E_0' = \frac{e\hbar B}{m_{xx}c}\left(\frac{m_{xx}}{m_{yy}}\right)$. So

$$(5.79)\qquad M = -\frac{T_c}{a}\,\frac{\partial E_0'}{\partial B} = \frac{T_c \hbar e}{ac}\cdot\frac{1}{(m_{xx}m_{yy})^{1/2}}\ .$$

Similarly the magnetic moment of a thin film thickness $d \ll \xi_z$ in a magnetic field perpendicular to its plane can be evaluated for $T > T_c$ and $T - T_c \ll T_c$.

In this case

$$\Delta F = \frac{AB}{\pi\hbar c}\Sigma_n \frac{\pi k_B T}{\hbar\omega_{0x}(n+\frac{1}{2})(\frac{m_{xx}}{m_{yy}})^{1/2}+a} \tag{5.80}$$

$$= \frac{AB^2 k_B T_c}{24\hbar c}\frac{\hbar e}{ac}(m_{xx}m_{yy})^{-1/2} .$$

Hence the magnetic moment of this foil is

$$M = -\frac{\partial\Delta F}{\partial B} = -\frac{A\ k_B T e^2 B}{12c^2 a(m_{xx}m_{yy})^{1/2}} . \tag{5.81}$$

G. Helicoidal Solution of Ginzburg-Landau Equations in Thin Films.

We will now examine solutions of the Ginzburg-Landau equations of the Ambegaokar-Langer type [6] relating to a film of thickness $L \ll \delta$ and $L \ll \xi$ and picking the vector potential $\vec{A}$, with $A^2 \ll \delta^2 H_{cr}^2$.

For the case of such a thin film let us assume for the superconducting order parameter: $\psi_x = f\ e^{ikx}$. The superconductor occupies a thin slab of the thickness L. If $a^2 \ll 1$ then the solution of the first equation (5.27) is just

$$f = 1 - \frac{k^2}{K^2} . \tag{5.82}$$

The solution of the second equation (5.28) for the vector potential was found using the boundary condition that the magnetic field is continuous. It is:

$$(5.83)\quad A = \frac{h_0 \sinh\sqrt{1-\frac{k^2}{K^2}}\,x}{\sqrt{1-\frac{k^2}{K^2}}\cosh\sqrt{1-\frac{k^2}{K^2}}\cdot\frac{L}{2}} - \frac{L}{2}\,\frac{[\mathfrak{J}+\frac{k}{K}(1-\frac{k^2}{K^2})]}{2\sqrt{1-\frac{k^2}{K^2}}\sinh\sqrt{1-\frac{k^2}{K^2}}}\cosh\sqrt{1-\frac{k^2}{K^2}}\,x$$

$$+ \frac{x^2}{2}[\tilde{\mathfrak{J}} + \frac{k}{K}(1 - \frac{k^2}{K^2})].$$

Now we can find the expression for the total current

$$(5.84)\quad J_{total} = h_0 \frac{\sinh\sqrt{1-\frac{k^2}{K^2}}\,x}{\cosh\sqrt{1-\frac{k^2}{K^2}}\;\frac{L}{2}}\sqrt{1-\frac{k^2}{K^2}}$$

$$- \frac{L}{2}\,\frac{[\mathfrak{J}+\frac{k}{K}(1-\frac{k^2}{K^2})]}{\sinh\sqrt{1-\frac{k^2}{K^2}}\cdot\frac{1}{2}}\sqrt{1-\frac{k^2}{K^2}} + \tilde{\mathfrak{J}} + \frac{k}{K}(1-\frac{k^2}{K^2}).$$

We can see a few interesting terms in (5.84) brought about by the coexisting system. In the long wave limit ($k \to 0$) we obtain

$$(5.85)\qquad \text{Total } \mathfrak{J} \sim h_0\,\frac{\sinh x}{\cosh\frac{L}{2}} - \frac{L\mathfrak{J}}{2}\,\frac{\cosh x}{\sinh\frac{L}{2}} + \mathfrak{J}.$$

The first term is the usual result. The two additional ones are new and may be used to compare experimental results in this superconducting films.

In conclusion we want to emphasize the large variety of interesting experimental manifestations of coexistence of charge-density-wave and superconductivity at the macroscopic level. We hope to stimulate experimental work in this area in order to "illuminate" the validity of the theory of "partial gapping".

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<u>Supplement to Chapter 4</u>: <u>Threshold for (Polarized) Infrared Absorbtion</u>.

Now we will discuss the problem of the absorbtion of an electromagnetic field by a coexistant SC and CDW system. Following Pokrovsky and Rivkin [1] we will calculate the threshold for this absorbtion.

From energy conservation we may write

$$\text{(S.1)} \qquad \hbar\omega = E_{\vec{p}} + E_{\vec{p}-\vec{q}}$$

where

$$\text{(S.2)} \qquad E_p = \begin{cases} \sqrt{\epsilon_p^2 + |\Delta_p|^2} & \text{unnested direction} \\ \text{or } \sqrt{\epsilon_p^2 + |\Delta_p|^2 + W^2} & \text{nested direction} \end{cases}$$

Consequently

$$\text{(S.3)} \qquad E_{p-q} = \begin{cases} \sqrt{(\epsilon_{p-q} - \vec{v}\cdot\vec{q})^2 + |\Delta_{p-q}|^2} \\ \sqrt{(\epsilon_{p-q} - \vec{v}\cdot\vec{q})^2 + |\Delta_{p-q}|^2 + |W_{p-q}|^2} \end{cases}$$

Here $\omega, \vec{q}$ are the photon energy and momentum and $\vec{v}$ is the electron velocity. Using $p >> q$ we can solve equation (1) which yields

$$\text{(S.4)} \qquad \epsilon_{p_{1,2}} = \frac{1}{2}[\vec{v}\cdot\vec{q} \pm \hbar\omega\left(\frac{\hbar^2\omega^2 - (\vec{v}\cdot\vec{q})^2 - 4\Delta^2}{\hbar^2\omega^2 - (\vec{v}\cdot\vec{q})^2}\right)^{\frac{1}{2}}],$$

and for the nested direction:

$$\varepsilon_{p_{1,2}} = \frac{1}{2}[\vec{v}\cdot\vec{q} \pm \hbar\omega(\frac{\hbar^2\omega^2 - (\vec{v}\cdot\vec{q})^2 - 4|\Delta|^2 - 4|W|^2}{\hbar^2\omega^2 - (\vec{v}\cdot\vec{q})^2})^{\frac{1}{2}}] \tag{S.5}$$

Since ε_p is real, the threshold frequency ω_T is equal to

$$(\hbar\omega_T)^2 = \min[(\vec{v}\cdot\vec{q})^2 + 4|\Delta|^2] \qquad \text{nonnested direction} \tag{S.6}$$

$$(\hbar\omega_T)^2 = \min[(\vec{v}\cdot\vec{q})^2 + 4|\Delta|^2 + 4W^2] \qquad \text{for nested direction} \tag{S.7}$$

Clearly measurements of ω_T may be used for estimation of charge-density-wave gap in the coexisting system. This can be compared to measurements of the CDW gap W for $T_p > T > T_c$ and to measurement of the BCS gap Δ by other means: for instance , infrared absorption, when the CDW gap has been suppresssd. This measurement will also permit estimation of the strength of the competition between superconductivity and charge-density wave gaps.

We must notice here that light is supposed to be polarized along the nested direction in order to be able to excite electrons involved in the competition.

Changing the orientation of a sample or rather the direction of polarization of light an experimentalist would see different thresholds.

Reference

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Appendix A. A "Naive" Macroscopic Theory of Coexistence of a Superconducting-like System with a Scalar Boson Field.

The aim of this Appendix is to demonstrate, using a "simplest" model the coexistance of a superconducting-like system (described by a Ginzburg-Landau effective action), and a scalar boson field. The latter may correspond to the deviation of an unperturbed lattice from commensurate to incommensurate case.

Assuming there is a slowly varying order parameter $\psi(r)$ characteristic of superconductivity, and an order parameter φ of the scalar field we write a phenomenological free energy as

$$(A.1) \quad F = F_0 + \int \left[\frac{\hbar^2}{4m}|\nabla\psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right] d^3\vec{r}$$
$$+ \int \frac{|\nabla\varphi|^2}{2m^*} d^3\vec{r} + \int \beta\varphi^2(\vec{r})\, d^3\vec{\gamma} + \int \gamma\varphi|\psi(\vec{r})|^2 d^3\vec{r}$$

where the first three terms correspond to a conventional Ginzburg-Landau free energy. The next two terms correspond to the kinetic and potential energy associated with a bosonic field. In the last term note the linear coupling of the two order parameters.

We can also discuss a more sophisticated model of coexistence where for small ion deviations $u(\vec{r})$, the elastic energy H_e has the well-known form [1]

$$(A.2) \qquad H_e = \int [\frac{Ku_{ii}^2}{2} + \mu(u_{ij} - \frac{1}{3}\delta_{ij}u_{\alpha\alpha})^2 - \sigma_{ij}u_{ij}] d^3\vec{r}$$

where the strain tensor is:

$$u_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

For a purely superconducting system

$$H[\psi] = \int [\frac{|\nabla\psi|^2}{4m} + a|\psi|^2 + \frac{b}{2}|\psi|^4] d^3\vec{r}.$$

In the case of quadratic striction [2]

$$H_{int} = -g\int |\psi|^2 u_{ii}(\vec{x}) d^3\vec{r}.$$

So the Gibbs thermodynamic potential $\Phi(\sigma_{\alpha,\beta}T)$ is

$$(A.3) \qquad \Phi = \Phi_0 - T \ln \sum_{\vec{u},\psi} \exp[-(H[\psi]+H_e+H_{int})/T].$$

Taking Fourier components of $u_{\alpha\beta}$ [2]

$$\frac{\partial u_\alpha}{\partial x_\beta} = u_{\alpha\beta}^{(0)} + \frac{1}{\sqrt{V}} \Sigma\, e^{i\vec{k}\cdot\vec{x}} ik_\beta u_\alpha(\vec{k}).$$

Here $u_{\alpha\beta}^0$ is the tensor of uniform deformation, and V is the volume. After some calculation we obtain:

$$(A.4) \quad (\Phi-\Phi_0) = -\frac{g^2}{2(\lambda+2\mu)}\int |\psi|^4(\vec{x})\, d^3\vec{r} + \frac{g^2}{2V(\lambda+2\mu)}[\int \psi^2(\vec{x}) d^3\vec{r}]^2$$

we introduced here the Lamé coefficients λ [1]

$\lambda = K - \frac{2\mu}{3}$.

Now we return to(A.1)and try to understand what are the consequences suggested by the model (A.1). We notice that an alternative form of interaction like

$$\int \gamma\varphi^2 |\psi|^2 d^3\vec{r}$$

has to be ruled out because it is inconsistent with coexistence of CDW and SC. Now we show that free energy (A.1) leas to interesting physical consequences.

Minimization of (A.1) with respect to φ gives the relation

$$\text{(A.5)} \qquad \frac{\delta F}{\delta\varphi} = \gamma|\psi|^2 + 2\beta\varphi = 0$$

with the solution

$$\text{(A.6)} \qquad \varphi = -\frac{\gamma}{2\beta}|\psi|^2.$$

For a spatially homogeneous medium we have

$$\text{(A.7)} \qquad F = F_0 + \int [a|\psi|^2 + (\frac{b}{2} - \frac{\gamma^2}{4\beta})|\psi|^4] d^3\vec{r} \ .$$

As derived by Gor'kov $a = \alpha(T-T_V)$ (see Appendix B)

$$\text{(A.8)} \qquad d = \frac{6\Pi^2 T_c}{7\zeta(3)\mu} = \frac{7.04T_c}{\mu} \qquad b = \frac{\alpha T_c}{n} \quad |\psi|^2 = \frac{\rho_s}{m}$$

$$n = \frac{\rho}{m} \qquad n = \frac{p_F^3}{3\pi^2\hbar^3} \qquad \mu = \frac{p_F^2}{2m}.$$

At equilibrium $|\psi| = -\frac{a}{b-\frac{\gamma^2}{2\beta}} = \frac{\alpha(T_c-T)}{b-\frac{\gamma^2}{2\beta}}$. Thus the free energy becomes

$$\delta F = F - F_0 = -\frac{V\alpha^2}{2b-\frac{\gamma^2}{\beta}}(T_c-T)^2 \tag{A.9}$$

and the specific heat

$$c = c_0 + \frac{V^2\alpha T_c}{b-\frac{\gamma^2}{2\beta}}\,. \tag{A.10}$$

In the vicinity of T_c, $F - F_0$ is small. Due to the theorem of small increments [3] the same quantity gives the difference in Gibbs' potential $\Phi - \Phi_0$. On the other hand this quantity should be equal to $-VH_c^2/8\pi$, where H_c is the critical magnetic field, which destroys superconductivity.

Thus near the transition point we find

$$H_c = \left(\frac{4\pi a^2}{b-\frac{\gamma^2}{2\beta}}\right)^{1/2} = \left(\frac{4\pi\alpha^2}{b-\frac{\gamma^2}{2\beta}}\right)^{1/2}(T_c-T).$$

The measurement of this field will give an estimate of $\frac{\gamma^2}{2\beta}$.

In the presence of the magnetic field we can immediately generalize equation (A.1) to

$$\text{(A.11)} \qquad F = F_0 + \int\left\{\frac{B^2}{8\pi}+\frac{\hbar^2}{4m}\left|\left(\nabla-\frac{2ie}{\hbar c}\right)^2\psi\right| + a|\psi|^2 + \left(\frac{b}{2}-\frac{\gamma^2}{2\beta}\right)|\psi|^2\psi\right\}d^3\vec{r} = 0.$$

We use Maxwell's equation $\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J}$ where the current density is equal to

$$\text{(A.12)} \qquad \vec{J} = \frac{e\hbar}{i2m}(\psi^*\nabla\psi-\psi\nabla\psi^*) - \frac{2e^2}{mc}|\psi|^2\vec{A}$$

and boundary conditions

$$\text{(A.13)} \qquad \hat{n}\cdot(-i\hbar\vec{\nabla}\psi - \frac{2e}{c}\vec{A}\psi) = 0.$$

In the presence of a weak external magnetic field we then obtain a London-like equation. For the penetration depth we find:

$$\text{(A.14)} \qquad \delta = \left[\frac{mc^2(b-\frac{\gamma^2}{2\beta})}{8\pi e^2|a|}\right]^{\frac{1}{2}} = \left[\frac{mc^2(b-\frac{\gamma^2}{2\beta})}{8\pi e^2\alpha(T_c-T)}\right]^{\frac{1}{2}}.$$

The correlation radius of fluctuations of the order parameter is, as in Ginzburg-Landau theory

$$\text{(A.15)} \qquad \xi(T) = \frac{\hbar}{2(m|a|)^{1/2}} = \frac{\hbar}{2(m\alpha)^{1/2}(T_c-T)^{1/2}}$$

The Ginzburg-Landau parameter is

$$\text{(A.16)} \qquad K = \frac{\delta(T)}{\xi(T)} = \frac{mc\left(b-\frac{\gamma^2}{2\beta}\right)^{1/2}}{(2\pi)^{1/2}|e|\hbar} .$$

As is well known [2] k = 1 corresponds to the boundary of two kinds of superconductivity: type I and type II. So we see that the bosonic field can shift the boundary between those 2 types of superconductivity.

Let us also note an interesting fact if we assume that the Hamiltonian (by analogy with the free energy) has the form:

$$\text{(A.17)} \qquad \int [-\psi^+ \frac{\nabla^2}{2m}\psi + \frac{\lambda}{2}(\psi^+(\psi^+\psi)\psi)] d^3\vec{r} + \int \gamma\varphi(\psi^+\psi) d^3\vec{r}$$

where $\psi^+\psi = \psi^+_a\psi_a$. Here and $\psi(r)$ and $\psi^+(r)$ are in the Schroedinger representation and satisfy the usual commutation relations. Then, we can find the energy of excitation as

$$\text{(A.18)} \qquad E = \sqrt{(\epsilon-\epsilon_F)^2 + \lambda(\psi\psi) + \gamma(\varphi^*\varphi)}$$

which clearly corresponds to our result in the microscopic theory (see Equation (3.78)):

$$E = \sqrt{(\epsilon-\epsilon_F)^2 + \Delta^2 + W^2} .$$

Also one can analyse the shape of the vortices in these type II superconductor. We get a system of coupled equations, describing the nonhomogeneous systems, namely

$$-\frac{\hbar^2}{4m}\nabla^2\psi + a|\psi| + \frac{b}{2}|\psi|^3 + 2\gamma\varphi|\psi(r)| = 0$$

(A.19) and

$$-\frac{\hbar^2}{2m}\nabla^2\varphi + 2\beta\varphi + \gamma|\psi|^2 = 0.$$

The second equation is acually a Helmholtz type equation and we can immediately write [4]

(A.20) $$\varphi = \gamma\int G_H(r,r')|\psi(r')|^2 d^3\vec{r}'$$

where $G_H(\vec{r},\vec{r}')$ is an appropriate Green function for the Helmholtz equation. We then obtain a nonlinear integro-differential equation to be solved for $|\psi|$:

(A.21) $$-\frac{\hbar^2}{4m}\nabla^2\psi + a|\psi| + \frac{b}{2}|\psi|^3 - 2\gamma^2|\psi|\int G(\vec{r},\vec{r}')|\psi(\vec{r}')|d^3r' = 0.$$

For the case of an infinite 3-D sample in the vicinity of the transition point neglecting the cubic term and using the explicit form of G(r,r'):

(A.22) $$-\frac{h^2}{4m}\nabla^2\psi + a|\psi| - 2\gamma^2|\psi(r)|\int \frac{e^{-\frac{\sqrt{4\beta m^*}}{\hbar}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} |\psi(\vec{r}')|^2 d^3\vec{r}' = 0.$$

Here it is useful to make a substitution assuming that the superconducting order parameter $|\psi|$ is a slowly varying function of coordinates compared with Green function G. We can bring $\psi(r)$ outside the integration and we obtain

$$-\frac{\hbar^2}{4m}\nabla^2\psi + a|\psi| + \frac{b}{2}|\psi|^3 - 2\gamma^2|\psi|^3\langle G\rangle = 0, \tag{A.23}$$

where $\langle G\rangle$ is the average of the Green function over the range of integration.

$$\langle G\rangle \equiv \int \frac{e^{-\frac{\sqrt{4\beta m^*}|\vec{r}-\vec{r}\,'|}{\hbar}}}{|\vec{r}-\vec{r}\,'|}dr = 4\pi\int_0^\infty \frac{e^{-\Omega r}}{r}r^2 dr \tag{A.24}$$

where $\Omega = \frac{\sqrt{4\beta m^*}}{\hbar}$. This is then

$$= 4\pi\int_0^\infty e^{-\Omega r}\, r dr = 4\pi\int_0^\infty r\, e^{-\Omega r}dr \tag{A.25}$$

$$= 4\pi r\frac{e^{-\Omega r}}{-\Omega}\Big|_0^\infty - 4\pi\int_0^\infty \frac{e^{-\Omega r}}{-\Omega}dr = \frac{4\pi}{\Omega}\,\frac{e^{-\Omega r}}{-\Omega}\Big|_0^\infty \tag{A.26}$$

$$= \frac{4\pi}{\Omega^2} = \frac{\pi h^2}{\beta m^*}$$

Finally, we have

$$-\frac{\hbar^2}{4m}\nabla^2\psi + a|\psi| + \frac{b}{2}|\psi|^3 - \frac{2\gamma^2\pi\hbar^2}{\beta m^*}|\psi|^3 = 0. \tag{A.27}$$

Equations similar to (A.27) have been investigated by Ginzburg and Pitaevsky [5], Pitaevsky [6] and Gross [7] for the case of an axially symmetrical vortex line. It can be shown that in this case the solution is of the form

$$\psi = \left(\frac{a}{\frac{b}{2} - \frac{2\gamma^2 \hbar^2}{\beta m^*}}\right)^{1/2} f\left(\frac{r}{r_0}\right) e^{i\phi} \tag{A.28}$$

where $r_0 = \frac{\hbar}{\sqrt{4ma}}$. Introducing the variable $\xi = \frac{r}{r_0}$ we have the equation for f:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) - \frac{f}{\xi^2} + f - f^3 = 0. \tag{A.29}$$

The graph of this function is given in [5]

Our conclusion: a bosonic field does not effect the effective radius of a vortex line, but modifies the shape of the effective function f. The case when $\frac{b}{2} = \frac{2\gamma^2 \hbar^2}{\beta m^*}$ is critical and can be used for estimations.

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Appendix B. The Path Integral Method in the Theory of Superconductivity.

The path integral technique is now widely used in statistical mechanical problems. In this Appendix (following Sakita) we will outline its application to pure supercoductivity. The objective is to derive the current and GLG equations [1].

The partition function for a system with Hamiltonian H is

$$Z = \mathrm{Tr}e^{-\beta H} = \int dq_0 < q_0|e^{-\beta H(\hat{p},\hat{q})}|q_0>$$

$$= \int dq_0 < q_0|\lim_{N\to\infty}(1 - \epsilon H(p.q))^N|q_0>$$

$$= \lim_{N\to\infty}\int dq_0\int dq_1 \ldots \int q_{N-1}<q_0|1-\epsilon H|q_{N-1}>$$

$$<q_{N-1}|1-\epsilon H|q_{N-2}>|q_{N-(N-1)}><q_1|1-\epsilon H|q_0>$$

but

$$<q_{i+1}|1-\epsilon H(p,q)|q_i> = \int\frac{dp_i}{2\pi}\, e^{ip_i(q_{i+1}-q_i)}(1-\epsilon H).$$

So

$$Z = \int\ldots\int \Pi_{i=0}^{N-1}\, dq_1\, \Pi^{N-1}\frac{dp_i}{2\pi}e^{i\Sigma p_i(q_{i+1}-q_i)-\epsilon\Sigma_{i=0}^{N-1}\mu}.$$

Now rename the variables $q_0 = q_N$; $\epsilon = d\tau$; $q_{i+1}-q_i = \epsilon\frac{dq}{d\tau} = \dot{q}d\tau$. Finally, for the partition function:

$$Z = \int\int DqDp\, e^{-i\int_0^\beta d\tau p(\tau)\dot{q}(\tau) - \int_0^\beta d\tau H(p(\tau),q(\tau))}$$

$$= \int\int DqDpe^{S}$$

where

$$S = -\int_0^\beta \mathcal{L} d\tau \quad \text{and} \quad \mathcal{L} = -ip(\tau)\dot{q}(\tau) + H(p(\tau),q(\tau)).$$

Now we use the Lagransian density in a continuous (field-theoretical) form.

$$q \to \phi(x,\tau)$$

$$p \to \Pi(x,\tau)$$

$$H = \int dx \mathcal{H}(x)$$

$$Z = \int\int D\Pi D\phi \exp[\int_0^\beta d\tau \int_V dx[i\Pi(x,\tau)\dot{\phi}(x,\tau) - \mathcal{H}[\Pi(x,\tau),\phi(x,\tau)]]$$

The boundaty condition is: $\phi(x,0) = \phi(x,\beta)$.

A. Perturbation Expansion of the Partition Function.

Writ $Z = \mathrm{Tr}e^{-\beta K}$, where K is the grand-canonical Hamiltonian ($K = H-\mu N$). In the important case of a two body interaction Hamiltonian K becomes

$$\hat{K} = \hat{H}-\mu\hat{N} = \int d^3x \hat{\psi}^+(x)\left(-\frac{\hbar^2\nabla^2}{2m}-\mu\right)\psi(x)$$

$$+ \frac{1}{2}\int d^3x_1 d^3x_2 \psi^+(x_1)\psi^+(x_2)V(x_1-x_2)\psi(x_2)\psi(x_1).$$

Introducing the Fourier components of the fields and potential

$$\psi(x) = \frac{1}{\sqrt{V}}\, \Sigma\, \varphi(\vec{k})e^{i\vec{k}\cdot\vec{x}}$$

$$V(x) = \Sigma_k\, e^{i\vec{k}\vec{x}}\; v(\vec{k})$$

we get

$$K = \Sigma_k \omega_k \varphi^+(k)\varphi(k) + \frac{1}{2V}\Sigma_{q,p,k}\; (\vec{q})\varphi^+(\vec{q}+\vec{k})\varphi^+(\vec{p}-\vec{k})\varphi(\vec{p})\varphi(\vec{k})$$

$$= K_0 + H_{int}(\varphi^+,\varphi)$$

with

$$H_0 = \Sigma_k(\frac{\hbar^2k^2}{2m} - \mu)\varphi^+(k)\varphi(k)\,;\;\; \omega_k = \frac{\hbar^2k^2}{2m} - \mu.$$

Now the corresponding path integral is

$$Z = \int \prod_{k,\tau} d\varphi^*(\vec{k},\tau)d\varphi(k,\tau)\exp\{-\int_0^\beta d\tau[\sum_k \varphi^*(k,\tau)(\frac{\partial}{\partial\tau}+\omega_k)\varphi(k,\tau)$$

$$+ H_{int}(\varphi^*,\tau),\varphi(\;.\tau)]\}.$$

Now denoting all the arguments in the exponential as the function (-S) we define

$$Z[\eta,\eta^*] \equiv \int\prod_{\vec{k}\tau} d\varphi^*(k,\tau)dp(k,\tau)\exp\{-S-\int_0^\beta d\tau\sum_k [\eta^*(\vec{k},\tau)\varphi(\vec{k},\tau)$$

$$+ \varphi^*(\vec{k},\tau)\eta(\vec{k},\tau)]\}$$

and

$$Z_0[\eta,\eta^*] = \int_{\vec{k},\tau} d\varphi^*(k,\tau)d\varphi(k,\tau)\exp\{-S_0-\int_0^\beta d\tau\Sigma_k[\eta^*(k,\tau)\varphi(k,\tau)$$

$$+ \varphi^*(k,\tau)\eta(\vec{k},\tau)]\}$$

where

$$S_0 + \int_0^\beta H_{int}(\varphi^*(\ ,\tau),\varphi(\ ,\tau)) = S.$$

This is Schwinger's trick.

Now we have the following relation between the two last formulae:

$$Z[\eta,\eta^*] = \exp[-\int_0^\beta H_{int}(\frac{\delta}{\delta\eta},\frac{\delta}{\delta\eta^*})]Z_0[\eta,\eta^*].$$

The proof is almost trivial using the relation:

$$\frac{\delta\eta(k,\tau)}{\delta\eta(k,\tau')} = \delta_{kk'}\delta(\tau-\tau')$$

and the obvious formula:

$$\varphi(x_1)\ldots\varphi(x_n) = \frac{\delta}{\delta\eta(x_1)}\ldots\frac{\delta}{\delta\eta(x_n)}\exp[\int\varphi(x)\eta(x)dx]\Big|_{\eta=0}.$$

For example an arbitrary functional $\Phi(\varphi)$ of $\varphi(x)$ can be written as

$$\Phi(\varphi) = \Phi(\frac{\delta}{\delta\eta(x)})\exp[\int \varphi(x)\eta(x)dx]\Big|_{\eta=0}$$

also

$$\exp[-\int V(x)dx] = \exp[-\int V(\frac{\delta}{\delta\eta})dx]\exp[\int\varphi\eta dx]\Big|_{\eta=0}.$$

In order to calculate $Z_0[\eta,\eta^*]$ we note that:

$$\varphi(k,0) = \varphi(k,\beta) \quad \text{for bosons}$$

and

$$\varphi(k,0) = -\varphi(k,\beta) \quad \text{for fermions}$$

(the same is true for η). Therefore we introduce the following two series

$$\varphi(k,\tau) = \Sigma_n \varphi_n(\vec{k})\, e^{-i\xi_n\tau} \frac{1}{\sqrt{\beta}}$$

$$\eta(k,\tau) = \Sigma_n \eta_n(k)\, e^{-i\xi_n\tau} \frac{1}{\sqrt{\beta}}$$

$$\xi_n = \begin{cases} 2n\frac{\pi}{\beta} & \text{for bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{for fermions} \end{cases}$$

After introducing these two series we find

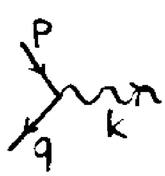
$$Z_0[\eta,\eta^*] = Z_0[0,0]\exp[\sum_{k,n} \eta_n^*(k)\frac{-1}{i\xi_n-\omega_k}\eta_n(k)]$$

and note

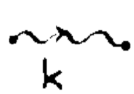
$$Z_0[0,0] = \mathrm{Tr}\; e^{-\beta k_0}.$$

Using the last expression we can develop the perturbation series.

We will not be discussing here details of the calculation of Z. We only exhibit the Feynman diagrams using

vertex: 

$\frac{1}{\sqrt{\beta V}}\,\delta^{(4)}(q+k-p)$

Propagator 

$-\mathcal{V}(\vec{k})$

Propagator: $\xrightarrow[p]{}$ $\qquad G(p) = \frac{-1}{i\xi_n - \omega_p}.$

2. The Effective Action.

Consider $\eta = \eta^* = 0$. Define $W[J]$ by

$$Z[j] = e^{W[J]}.$$

$W[J]$ is the generating functional for connected graphs.

Define $\phi(x)$ by

$$\phi(x) = \frac{\delta W[j]}{\delta j(x)}.$$

The Legendre transform of $W[j]$ is given by

$$\Gamma[\phi] = \int dx j(x)\phi(x) - W[j]$$

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = j(x).$$

The physical relevance of $\Gamma[\phi]$ is clear from the relation

$$Z = Z[j]\big|_{j=0} = e^{\int dx j(x)\phi(x) - \Gamma[\phi]}\big|_{j=0} = e^{-\Gamma[\phi]\big|_{j=0}}.$$

Thus $\Gamma[\phi]$ is related to the free energy of the system and the condition

$$\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\bigg|_{j=0} = 0$$

gives the equalibrium configuration for the system. The key theorem states

$$\Gamma[\Phi] = \frac{1}{2}\int dxdy\,\Phi(x)\Delta^{-1}(x-y)\Phi(y) - \kappa[\Phi]$$

where

$$\Delta^{-1}(x-y) = \int d^3k\, e^{ik(x-y)}[-v(k)]^{-1}$$

and $\kappa[\Phi]$ is the generating functional of one-particle irreducible graphs . For proof see the proof in: (E. Abers, B. Lee, Gauge Theories, page 98 (1973)).

3. <u>Calculation of Path Integrals</u> [2].

Suppose $Q(x)$ is a quadratic form in one variable

$$Q(x) = \frac{1}{2}ax^2 - bx = -\frac{b^2}{2a} + \frac{1}{2}a(x-x_0)^2$$

where

$$x_0 = \frac{b}{a}.$$

Then

$$\int_{-\infty}^{\infty} dx e^{-Q(x)} = \left(\frac{2\pi}{a}\right)^{1/2} e^{b^2/2a}.$$

If $Q(u)$ is a quadratic form in n variables $\{u_1, u_2, \ldots, u_n\}$

$$Q(u) = \frac{1}{2}(u, Au) - (b, u)$$

where b is a constant n-vector. A is a symmetric nonsingular $n \times n$ matrix, and $(b,u) = \Sigma_{i=1}^{n} b_i u_i$. We can also write $Q(u) = -\frac{1}{2}(b, A^{-1}b) + \frac{1}{2}((u-u_0), A(u-u_0))$

$$u_0 = A^{-1}b.$$

Then

$$\int [du]\, e^{-Q[u]} = \frac{e^{1/2(b,A^{-1}b)}}{\sqrt{\det A}}$$

where

$$[du] = \frac{1}{(2\pi)^{n/2}}\, du_1 \ldots du_n.$$

For the real field $\phi(x)$, define a quadratic form $Q[\phi]$

$$Q[\phi] = \frac{1}{2}(\phi, A\phi) - (b,\phi).$$

Then

$$\int [d\phi]\, e^{-Q[\phi]} = N\frac{e^{1/2(b,A^{-1}b)}}{(\det A)^{1/2}}$$

where N is a normalization constant.

4. Application to Superconductivity.

The Hamiltonian for the superconducting system is

$$H = H_0 + H_{int} = \Sigma_{s=\uparrow,\downarrow} \int d\vec{r}\, \psi_s^+(\vec{r}) \left(-\frac{\nabla^2}{2m} - \mu\right) \psi_s(\vec{r})$$

$$- g\int d\vec{r}\, \psi_\uparrow^+(\vec{r})\psi_\downarrow^+(\vec{r})\psi_\downarrow(\vec{r})\psi_\uparrow(\vec{r}).$$

We can introduce the electromagnetic interaction in a gauge-invariant way using the minimum substitution $\vec{\nabla} \to \vec{\nabla} - ie\vec{A}$. Hence for the Hamiltonian H:

$$H = \Sigma_{s=\uparrow,\downarrow}\int d\vec{r}\psi_s^+(\vec{r})[-\frac{1}{2m}(\nabla - ieA)^2 - \mu]\psi_s(\vec{r})$$

$$- g\int d\vec{r}\psi_\uparrow^+(\vec{r})\psi_\downarrow^+(\vec{r})\psi_\downarrow(\vec{r})\psi_\uparrow(\vec{r}).$$

H is invariant under a gauge transformation:

$$\psi_s(\vec{r}) \to e^{ie\Lambda(\vec{r})}\psi_s(r)$$

$$\vec{A}(r) \to \vec{A}(\vec{r}) + \vec{\nabla}\Lambda(\vec{r}).$$

The partition function is given by

$$Z = \int\Pi_{\vec{x}}[d\psi(\vec{x})d\psi^*(\vec{x})]\exp[-\int_0^\beta d\tau\int_V d\vec{x}\psi_s^+(\frac{\partial}{\partial\tau} - \frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2)\psi_s]$$

$$\cdot\exp[g\int_0^\beta d\tau\int_V d\vec{x}\psi_\uparrow^+(\vec{x})\psi_\downarrow^+(x)\psi_\downarrow(\vec{x})\psi_\uparrow(\vec{x})].$$

Now we introduce an auxiliary field $\Phi(\tau,\vec{x})$

$$N = \int\int[d\Phi^*][d\Phi]e^{-x^2\Phi^*\Phi}$$

and examine

$$\frac{1}{N}\int\int d\Phi^* d\Phi e^{(-x^2\Phi^*\Phi + \sqrt{g}\ x(\Phi\psi_\uparrow^+\psi_\downarrow^+ + \Phi^*\psi_\downarrow\psi_\uparrow))}.$$

Making the shift

$$\Phi \to \Phi + \frac{\sqrt{g}}{x}\psi_\downarrow\psi_\uparrow$$

$$\Phi^* \to \Phi^* + \frac{\sqrt{g}}{x}\psi_\uparrow^+\psi_\downarrow^*$$

we get back $e^{g\psi_\uparrow^+\psi_\downarrow^+\psi_\downarrow\psi_\uparrow}$ after Gaussian integration. So

$$Z = \int [d\psi^*][d\psi]\exp(\int d^4x\mathcal{L})$$

$$= \frac{1}{N}\int [d\psi^*][d\psi]\ [d\phi^*][d\phi]\exp(d^4x\mathcal{L})$$

with

$$\mathcal{L} = -\psi_s^+(\frac{\partial}{\partial\tau} - \frac{(\vec{\nabla}-ie\vec{A})}{2m} - \mu)\psi_s$$

$$+ \sqrt{g}\ x(\phi\psi_\uparrow^+\psi_\downarrow^+ + \phi^*\psi_\downarrow\psi_\uparrow) - \kappa^2\phi^*\phi.$$

We use the boundary condition $\phi(\beta,\vec{x}) = \phi(0,\vec{x})$.

One can immediately draw Feynmann diagrams for this interaction. We will not dwell on this here.

Introducing a source for the ϕ-field as

$$Z[j,j^*] = \frac{1}{N}\int [d\psi][d\psi^*][d\phi][d\phi^*]\exp[-\int d^4x\psi_s^+(\frac{\partial}{\partial\tau} - \frac{(\vec{\nabla}-ieA)^2}{2m})\psi_s]$$

$$\cdot\exp[-x^2\int \phi^*\phi d^4x + \sqrt{g}\ x\ \int d^4x(\psi_\uparrow^+\psi_\downarrow^+\phi + \psi_\downarrow\psi_\uparrow\phi^* + j^*\phi+\phi^*j)].$$

The partition function is then

$$Z = Z[j,j^*]_{j=j^*=0}.$$

The generating functional for the connected Green's function $W[j,j^*] = \ln Z[j,j^*]$. The Lengendre transform of $W[j,j^*]$ is

$$\Gamma[\varphi,\varphi^{*}{}'\vec{A}] = \int d4x(j\phi^{*}+\phi j^{*}) - W[j,j^{*}].$$

The partition function is

$$Z = Z[j.j^{*}]_{j=j^{*}=0} = \exp[-\Gamma(\varphi,\varphi^{*}{}'\vec{A})]\,|_{j=j^{*}=0}$$

which implies that Γ has to satisfy

$$\frac{\delta\Gamma}{\delta\varphi(x)}\Big|_{j=j^{*}=0} = 0 \qquad \frac{\delta\Gamma}{\delta\varphi^{*}}\Big|_{j=j^{*}=0} = 0.$$

These are equations for the order parameter.

5. Alternate Derivation of GLG from BCS Hamiltonian.

Again taking

(B.1) $$Z = e^{-\beta\Omega} = \mathrm{Tr}\, e^{-\beta k}$$

with Ω-free energy:

(B.2) $$\Omega = \frac{1}{|g|}\int |\Delta(r)|^{2} d\vec{r}$$

$$-\frac{1}{\beta}\int d^{4}x\int d^{4}x' \langle T_{\tau}\psi_{\uparrow}^{+}(x)\psi_{\downarrow}^{+}(x)\psi_{\downarrow}(x')\psi_{\uparrow}(x')\rangle \Delta(\vec{r}_1)\Delta(\vec{r}_2)$$

$$= \frac{1}{|g|}\int|\Delta(\vec{r})|^{2}d\vec{r} - \frac{1}{\beta}\int d^{4}x\int d^{4}x' G^{2}(x-x')\Delta(r_1)\Delta(r_2)$$

where

(B.3) $$G(\vec{r},\tau) = T\sum_{\omega} e^{-i\omega\tau}\int\frac{e^{i\vec{p}\cdot\vec{r}}}{i\omega-\xi_p}\frac{d^{3}\vec{p}}{(2\pi)^{3}} = \frac{1}{\beta}\sum_{\omega} G\omega(\vec{r})e^{-i\omega\tau}$$

is the Green function of a normal metal. Integrating over τ and going over to the variables $\vec{R} = \frac{1}{2}(\vec{r}_1+\vec{r}_2)$, $\vec{r} = \vec{r}_1 - \vec{r}_2$ we get

$$\text{(B.5)} \quad \Omega = \frac{1}{|g|}\int|\Delta(\vec{R})|^2 d\vec{R} - \frac{1}{\beta}\Sigma_{\omega}\int d\vec{R}\int d\vec{r}\, G_{\omega}(\vec{r})G_{-\omega}(\vec{r}) \times \Delta(\vec{R}+\frac{\vec{r}}{2})\Delta^*(\vec{R}-\frac{\vec{r}}{2}).$$

Now let us make an expansion in gradients

$$\text{(B.6)} \quad \Delta(\vec{R}\pm\frac{\vec{r}}{2}) = \Delta(\vec{R}) \pm \frac{r_i}{2}\frac{\partial\Delta}{\partial R_i} + \frac{1}{8}r_i r_j \frac{\partial^2\Delta}{\partial R_i \partial R_j}$$

$$\text{(B.7)} \quad \Omega = \frac{1}{|g|}\int|\Delta(\vec{R})|^2 d\vec{R} - \frac{1}{\beta}\Sigma_{\omega}\int d\vec{R}\int d\vec{r}\, G_{\omega}(\vec{r})G_{-\omega}(\vec{r}) \cdot (|\Delta(\vec{R})|^2 - \frac{1}{2}r_i r_j \frac{\partial\Delta}{\partial R_i}\frac{\partial\Delta^*}{\partial R_j})$$

Now using (B.4) and $\frac{1}{\beta}\Sigma\frac{2\varepsilon}{\omega^2+\varepsilon^2} = \tanh\frac{\varepsilon}{2}\beta$ we may write

$$\text{(B.8)} \quad \frac{1}{\beta}\Sigma_{\omega}\int d\vec{r}\, G_{\omega}(\vec{r})G_{-\omega}(\vec{r}) = \int \frac{d\vec{p}}{(2\pi)^3}\frac{1}{2\xi_p}\tanh\frac{\xi}{2}\beta \approx \frac{1}{|g|} - N(0)(1 - \frac{T}{T_c}).$$

Here we used $\left(1 - \frac{T}{T_c}\right) \ll 1$, and expanded the integrand with respect to $\left(\frac{T_c-T}{T_c}\right)$ and kept only the first order term. We also used the equation which defines critical temperature, namely [4]:

$$\int_0^{\omega_D/2T_c} \frac{\tanh x}{x} dx = \frac{1}{|g|N(0)}.$$

Calling

$$A_{ij} = \Sigma_\omega \int d\vec{r} G_\omega(\vec{r}) G_{-\omega}(\vec{r}) r_i r_j$$

$$= \Sigma_\omega \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3} G_\omega(p) G_{-\omega}(p') \int r_i r_j e^{i(\vec{p}+\vec{p}')\cdot\vec{r}} d\vec{r}$$

$$= -\Sigma \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3} G_\omega(p) G_{-\omega}(p') \frac{\partial^2}{\partial p_i' \partial p_j'} \delta(\vec{p}+\vec{p}')$$

$$= -\Sigma_\omega \int \frac{d^3\vec{p}}{(2\pi)^3} G_\omega(\vec{p}) \frac{\partial^2 G_{-\omega}(-\vec{p})}{\partial p_i \partial p_j}$$

$$= \Sigma_\omega \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\partial G_\omega(\vec{p})}{\partial p_i} \cdot \frac{\partial G_{-\omega}(-\vec{p})}{\partial p_j} .$$

Now because $G_\omega(\vec{p}) = G_\omega(-\vec{p}) = \frac{1}{i\omega-\xi_p}$,

$$A_{ij} = \frac{1}{m^2} \Sigma_\omega \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{p_i p_j}{(i\omega-\xi_p)^2 (i\omega+\xi_p)^2}$$

$$= \frac{1}{m^2} \Sigma_\omega \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{p_i p_j}{(\omega^2+\xi_p^2)^2}$$

an important contribution (same order ξ). Thus we can bring slowly varying terms at $\xi = 0$ outside the integral i.e. on the Fermi level (in momentum), and extend the limits of integration over ξ to infinity. Then we obtain

$$\text{(B.9)} \quad A_{ij} = \frac{2}{3}\delta_{ij}\,{}^{2}_{0}N(0)\int_0^\infty \frac{d\xi}{(\omega^2+\xi^2)^2} = \frac{\pi}{6}\delta_{ij}N(0)v_0^2\Sigma\frac{1}{|\omega|^3}$$

$$= \frac{7\zeta(3)v_0^2N(0)}{24\pi^2T_c}\delta_{ij}.$$

So

$$\frac{1}{\beta_c}\Sigma_\omega\int d\vec{r}\,r_i r_j G_\omega(r)G_{-\omega}(r) = \frac{7\zeta(3)N(0)v_0^2}{24\pi^2T_c^2}\quad .$$

where v_0-velocity at Fermi surface.

Eventually we get the Ginzburg-Landau effective functional

$$\text{(B.10)} \qquad \Omega = N(0)\int\{\frac{T-T_c}{T_c}|\Delta(R)|^2 + \frac{7\zeta(3)v_0^2}{48\pi^2T_c^2}|\nabla\Delta(R)|d\vec{R}$$

Equation (B.10) contains terms of order higher than quadratic. Therefore we have to proceed in an expansion of Ω to the fourth order in Δ. We have to calculate $\frac{\partial^4\Omega}{\partial^2\Delta_p\partial\Delta_{p^*}}$. After simple calculation one obtain $\frac{7\zeta(3)N(0)}{16\pi^2T_c^2}$.

Thus we obtain the Ginzburg-Landau functional

$$\Omega = N(0)\int\frac{T-T_c}{T_c}|\Delta(R)|^2 + \alpha|\nabla\Delta(R)|^2 + \frac{\beta}{2}|\Delta(R)|^4 d\vec{R}$$

where

$$\alpha = \frac{7\zeta(3)}{48\pi^2}\left(\frac{v_0}{T_c}\right)^2$$

and

$$\beta = \frac{7\zeta(3)}{8\pi^2\ T^2}\ .$$

Substituting into (B.1), and taking the logarithm then gives the GLG for energy as a functional of the order parameter. Minimizing this free energy gives the familiar GLG results.

References to Appendix B

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Part 2. Transient Optical Propagation through a Dielectric Slab

Chapter 6: Introduction

i) A typical experiment in a modern picosecond laboratory may be represented as in Figure 6.1. Usually in such an experiment an electromagnetic pulse which is finite in extent is incident on a slab of a certain substance. After the pulse passes through matter, it is usually analysed by sensitive devices to measure how the matter effects the electromagnetic radiation. In the visible region the intensity of this incident pulse $\sim |E_{inc}(\omega)|^2$ peaks at about 680nm and has a spread in frequences: $\Delta\omega = 1 - 10$nm (width at half maximum). We will employ a microscopic, linear theory in the model developed here. Linearity is understood in the sense that in what follows we will keep only linear terms in a dielectric slab s thickness d. In other words we assume that the medium is thin so that, to first order in d the strength of the incident wave is not diminished as it traverses the medium. Therefore each charged particle in the medium experiences the same incident wave. The approximation we use is equivalent to neglecting local field corrections and taking the local exciting field at each particle to be equal to the incident electric field. If the medium is thicker, each charged particle will experience a different incident wave as well as significant radiation from other charges in the medium [1]. As far as we know nobody has yet attempted to calculate the time-dependent electromagnetic field at the output in this situation.

ii) We will formulate the theoretical problems arising here as:

a) to calculate time-dependent electromagnetic field transients which occur after a finite pulse is incident upon a thin slab of dielectric;

b) to analyse the energy and momentum transfer of the electromagnetic field in this time dependent picture.

iii) In contrast to the usual, albeit steady state calculations [4], we will discuss this problem microscopically: specifically no index of diffraction will be incorporated in our calculation [1], [2], [3].

Chapter 7. Historical background. Precursors.

A. Sommerfeld [5] and L. Brillouin [6] discussed the propagation of electromagnetic signals through a dispersive medium with an assumed frequency dependent index of refraction $n(\omega)$. They considered an electromagnetic wave with a well-defined front edge incident normally upon a semi-infinite medium. They investigated the scenario of the arrival of the electromagnetic field at some distance inside the medium. Recent reviews of their results can be seen in many modern texts [2], [4], [7]. The usual way of carrying out this analysis is by considering a Fourier integral representation of the pulse in terms of plane waves, where each wave is characterized by a frequency ω, traveling with a phase velocity $\frac{c}{n(\omega)}$. The field inside the medium may be represented by:

$$E(x,t) = \int_{-\infty}^{\infty} \frac{2A(\omega)}{n(\omega)+1} e^{ik(\omega)x-i\omega t} \tag{7.1}$$

where

$$A(\omega) = \frac{1}{2\pi} \int E_i(0^-,t)e^{i\omega t}\, dt \tag{7.2}$$

Is the Fourier transform of the real incident electric field $E_i(x,t)$ evaluated just outside the medium at $x = 0$.

The quantity $k(\omega) = \frac{\omega}{c}n(\omega)$ is the frequency dependent wave number in the medium. Since the frequency-dependent index of refraction $n(\omega)$ can have values less than one (even complex), the phase velocity of some of the waves can be greater than c. Using the method of stationary phase one can immediately show, using (7.1) that no energy can propagate to a point L inside the medium faster than vacuum speed C. Sommerfeld and Brillouin also showed that some energy, however small, propagates at the vacuum speed of light inside the medium. This part of the wave is called the first or Sommerfeld precursor.

In 1969 P. Pleshko and I. Palocz [8] experimented and observed precursors for microwave frequences. We will examine the effect of a thin slab of medium on the propagation of an electromagnetic pulse. A microscopic approach will now be introduced without employing a frequency dependent index of refraction. We will show that the results of our calculations are in agreement with those by Sommerfeld. As far as we know this is the first microscopic theory of the Sommerfeld precursor.

The propagation of an electromagnetic wave of finite extent through a thin slab of medium consisting of the oscillating dipoles is analysed in this part of my thesis. The time-independent case has been discussed previously in the literature by many authors [1], [2], [3], [9], [10].

We will give results for the electric field at a certain point L outside of the dielectric medium, as the superposition of the incident electric field plus an electric field produced by the oscillating dipoles. Energy and momentum propagation will be discussed in detail. We will confirm Sommerfeld's result that the front edge of the resulting electromagnetic field propagates with vacuum velocity of light even though it passed through the medium.

Chapter 8. Theory: Propagation of an Electromagnetic Pulse Through a Medium Composed of an Array of Dipole Oscillators.

A. Electric field calculation.

The propagation of an electromagnetic wave through a medium composed of an array of oscillators is determined by the interference between the incident wave and the waves radiated by an array of forced dipoles within the medium. To calculate this let us consider a plane electromagnetic wave, polarized in the $\hat{z}$ direction which at time $t = 0$ is located as shown in Figure 8.1.

The incident electromagnetic pulse traveling in the $\hat{x}$ direction is taken in the theory to be (see Figure 8.1)

$$\vec{E}_{inc}(x,t) = \hat{z}\, E_0 \sin \frac{N\pi}{a}(x - ct) \tag{8.1}$$

where

(8.2) $$-a \_ x - c\dot{t} \leq 0$$

and N is equal to the total number of half cycles in the incident wave. The parameter $\frac{N\pi}{a} = \frac{\omega_N}{c} = \frac{2\pi\nu_N}{c}$ of the theory corresponds to the number of half cycles per unit length of the pulse. For visible light, for example,

(8.3) $$\frac{N}{a} = \frac{\nu}{c} = \frac{10^{16}}{3 \times 10^{10}} = \frac{1}{3} \cdot 10^{6} \text{cm}^{-1}.$$

The Fourier transform of E_{inc} is:

(8.4) $$E_{inc}(x.t) = \int_{-\infty}^{\infty} E_{inc}(\omega) e^{i\frac{\omega}{c}(x-ct)} d\omega$$

where $E_{inc}(\omega)$ can be determined from:

(8.5) $$E_{inc}(x,0) = \int_{-\infty}^{\infty} E_{inc}(\omega) e^{i\frac{\omega}{c}x} d\omega.$$

Substituting $E_{inc}(x,0)$ from (8.3) into (8.7) we get

(8.6) $$E_{inc}(\omega) = \int_{-\infty}^{\infty} \frac{1}{2\pi c} E_{inc}(x,0) e^{-i\frac{\omega}{c}x}$$

$$= \hat{z} \cdot \frac{E_0}{2\pi} \cdot \frac{N\pi c}{a} \cdot \frac{1+e^{i\frac{a\omega}{c}}}{\omega^2 - \frac{N^2 c^2 \pi^2}{a^2}}$$

where N = 1,3,5... is any odd number. The intensity $\sim |E_{inc}(\omega)|^2$ peaks at $\omega = \frac{N\pi c}{a}$ (which can be taken to be equal to λ = 680nm, $\lambda = \frac{2\pi c}{\omega}$). When N becomes large the incident pulse approaches a monochromatic wave of frequency

$$\omega = \frac{N\pi c}{a}.$$

We take the medium to be a "Lorentz" dielectric. Then the response of the medium to the electric field is governed by the equation describing the motion of harmonic oscillators. Let us call $\vec{u}(x,t)$ the displacement of the charge e from its equilibrium position at a point x. Clearly from the symmetry of the problem, $\vec{u}$ is a function of x only. Then the equation of motion for $\dot{u}$ is:

$$\ddot{u}(x,t) + \omega_0^2 \dot{u}(x,t) = \frac{e}{M} \dot{E}_{inc}(x,t). \tag{8.7}$$

In the Lorentz model, the dielectric is composed of an array of bound dipoles of charge e and effective mass M which may not be the mass of a bare electron but M can depend on the material. So e and M are subsequently taken as the charge and mass of the oscillators comprising the medium. For simplicity damping and magnetic effects are neglected assuming $\dot{u}/c << 1$. In this chapter oscillators are considered to be uncoupled and each one only feels the primary (incident) electric field. As we remarked already we take $E_{loc} = E_{inc}$.

Letting

$$\vec{u}(x,t) = \int_{-\infty}^{\infty} \vec{u}(x,\omega)e^{-i\omega t}d\omega \tag{8.8}$$

the expression for displacement has the familiar form [2]:

$$\vec{u}(x,\omega) = \frac{e}{M}\frac{E_{inc}(\omega)}{\omega_0^2-\omega^2}e^{i\frac{\omega}{c}x}. \tag{8.9}$$

If damping is taken into account we can just add $-i\Gamma\omega$ into the denominator [2]. The induced dipole moment due to a wave component of frequency ω is:

$$\vec{p}(x,\omega) = e\vec{u}(x,\omega). \tag{8.10}$$

In what follows we may use the dipole approximation for visible light $\lambda \cong 680$nm. The distance between dipoles in the dielectric is of the order of a few Angstroms. The radiation arriving at a distance L from the front edge of the medium (due to the dipoles) is:

$$E_{Rad}(L,\omega) = \int E_{Rad}(L,\vec{r},\omega)nd^3\vec{r} \tag{8.11}$$

where

$$\vec{E}_{Rad} = k_0^2\frac{e^{ik_0R}}{R}(\frac{\vec{R}}{R}\times\vec{p}(\vec{r},\omega)) \times \frac{\vec{R}}{R} \tag{8.12}$$

is taken in the dipole approximation in the radiation zone [2]. The distance L is assumed to be sufficiently large $L > d$, where d is the thickness of the dielectric slab.

Here $k_0 = \frac{\omega}{c}$. The density of the oscillators n is assumed to be constant and can be taken out of the integrand. The time-dependent electric field arriving at L from the dipoles will be determined by:

$$\vec{E}_{Rad}(L,t) = \int_{-\infty}^{\infty} \vec{E}_{Rad}(L,\omega) e^{-i\omega t} d\omega. \tag{8.13}$$

To calculate the integral in (8.11) we introduce, as usual, cylindrical coordinates ρ, φ, x as in Figure 8.2.

Then

$$\vec{r} = \rho\hat{\epsilon} + x\hat{x}, \qquad \vec{\epsilon} = \cos\varphi\,\hat{z} + \sin\varphi\,\hat{y} \qquad \vec{L} = L\hat{x} \tag{8.14}$$

$$\vec{R} = \vec{L} - \vec{r} = L\hat{x} - \vec{r}$$

and

$$R = \sqrt{\rho^2 + (L-x)^2}$$

The identity

$$\left(\frac{\vec{R}}{R} \times \vec{p}\right) \times \frac{\vec{R}}{R} = p - \frac{\vec{R}}{R}\left(\vec{p}\cdot\frac{\vec{R}}{R}\right) \tag{8.15}$$

with unit vector

$$\frac{\vec{R}}{R} = \frac{\hat{x}(L-x) - \rho\cos\varphi\,\hat{z} - \rho\sin\varphi\,\hat{y}}{\sqrt{\rho^2 + (L-x)^2}} \tag{8.16}$$

is used in the evaluation of (8.11). Integrating over angle φ we get [3]

$$(8.17)\quad \vec{E}_{Rad} = \hat{z}nk_0^2\pi p(\omega)\int_0^d e^{ik_0x}\int_0^\infty \frac{e^{ik_0R}}{R}\left(2 - \frac{\rho^2}{R^2}\right)\rho\, d\rho\, dx.$$

Since

$$R^2 = \rho^2 + (L-x)^2$$

for z = const we get

$$\rho\, d\rho = R dR.$$

Then

$$(8.18)\quad \int_0^\infty \frac{e^{ik_0R}}{R}\left(2 - \frac{\rho^2}{R^2}\right)\rho\, d\rho = \int_{L-x} e^{ik_0R} dR + (L-x)^2\int_{L-x}^\infty \frac{e^{ik_0R}}{R^2} dR.$$

The first integral in equation (8.18) can be evaluated by introducing a convergence factor (which is not significant because the limits of integration are infinite). Then

$$(8.19)\quad \int_{L-x}^\infty e^{ik_0R-dR}\, dR \cong \frac{i}{k_0} e^{ik_0L} e^{-ik_0x}.$$

The second integral in (8.18) can be reduced by integrating by parts. It is

$$(8.20)\quad (L-x)^2\int_{L-x}^\infty \frac{e^{ik_0R}}{R^2} dR = \frac{i}{k_0} e^{ik_0L} e^{-ik_0x}$$

$$\left[1 + \frac{2!}{ik_0(L-x)} + \frac{3!}{ik_0(L-x)^2} + \ldots\right].$$

The series in (8.20) is an asymptotic expansion of the integral [11]. If $L \gg d$, and $k_0 L \gg 1$, only the first term here is important. Finally we obtain

$$\vec{E}_{Rad}(L,\omega) = \frac{2\pi \hat{z} n p(\omega, z=0) d i \omega}{c} e^{i k_0 L} \tag{8.21}$$

Substituting (8.21) into (8.13) we obtain

$$\vec{E}_{Rad}(L,t) = \frac{2\pi e^2 n d i}{Mc} \hat{z} \int_{-\infty}^{\infty} \frac{\omega E_{inc}(\omega)}{\omega_0^2 - \omega^2} e^{-i\frac{\omega}{c}(t-\frac{L}{c})} \, . \tag{8.22}$$

The resultant electric field is given by

$$\vec{E}_{Res}(L,t) = \hat{z}[E_0 \sin \frac{N\pi}{a}(L-ct) + \frac{2\pi e^2 d n E_0}{cM} F(t-\frac{L}{c})] \tag{8.23}$$

where F is the function of $t - \frac{L}{c}$:

$$F(t-\frac{L}{c}) = \frac{i}{E_0} \int_{-\infty}^{\infty} \frac{\omega E_{inc}(\omega)}{\omega_0^2 - \omega^2} e^{-i\omega(t-\frac{L}{c})} \, . \tag{8.24}$$

(8.23) can also be rewritten in a form which will be used in the next chapter

$$\vec{E}_{Res}(L,t) = \hat{z}E_0[-\sin \frac{N\pi c}{a}(t-\frac{L}{c}) + \frac{2\pi e^2 d n}{M} F(t-\frac{L}{c})] \, . \tag{8.25}$$

We can easily obtain an expression for the reflected field by just changing L to $-L$ in the second term of (8.25)

$$\vec{E}_{Refl.}(L,t) = \hat{z}E_0 \frac{2\pi e^2 d n}{cM} F(t+\frac{L}{c}) \, . \tag{8.26}$$

The evaluation of the function F is given in the Appendix to this part II.

One can immediately see even without calculation that F is equal 0 when $t < \frac{L}{c}$, and starts to differ from 0 exactly at $t = \frac{L}{c}$, which is precisely the way the first (Sommerfeld) precursor manifests itself. As far as we know this is shown for the first time using only a microscopic treatment.

As we will show in the Appendix B, for $L \leq tc \leq L + a$ we have

$$(8.27) \qquad F(t-\frac{L}{c}) = \frac{N\pi c}{a} \frac{\{\cos \omega_0(t-\frac{L}{c}) - \cos\frac{N\pi c}{a}(t-\frac{L}{c})\}}{(\omega_0-\frac{N\pi c}{a})^2(\omega_0+\frac{N\pi c}{a})^2} .$$

Somerfeld [1] showed that the front edge of an electromagnetic pulse travels at speed c within a medium. We obtain the same result here for a thin slab: the front edge of the incident pulse travelling at c arrives at point L first. This can be seen by combining (8.25) and (8.27) assuming $t - \frac{L}{c} = T$ is small. Then

$$(8.28) \qquad E_{T\ \mathrm{small}} = \hat{z}E_0\{\frac{N\pi c}{a}T + \frac{\frac{N^3\pi^3c^3}{a^3}\cdot\frac{2\pi e^2 dn}{cM}(\frac{\omega_0^2}{2} - \frac{N^2\pi^2c^2}{a^2})}{(\omega_0-\frac{N\pi c}{a})^2(\omega_0+\frac{N\pi c}{a})^2}T^2\}.$$

Clearly for small T (compared with ω_0^{-1} and $(\frac{N\pi c}{a})^{-1}$)

the first term in (8.28) dominates. This proves that only the incident wave comprises the very front edge of the electromagnetic pulse. The usual view is: the Sommerfeld precursor propagates at a speed c (as in vacuum) because the oscillators which comprise the medium can not respond to high frequences. Therefore high frequency modes propagate as if in vacuum (they do not see the medium) [1].

Chapter 9. Energy and Momentum Transfer Through a Thin Dielectric Slab.

A. Introduction.

In this chapter we will investigate energy and momentum propagation in the model discussed in the previous chapter. Within the approximation employed here we will keep only linear terms in d. We will analyse the physics of this approximation comparing terms which we keep to those which we neglect. We will show how to calculate the time-dependent pressure exerted on the dielectric slab. We will propose some experiments to check the validity of the proposed theory.

B. Energy transport.

Using the expression (8.25) for the electric field obtained in the previous chapter we can turn now to a discussion of energy transfer. For the Poynting vector at point L we obtain from $S = \frac{c}{4\pi}E^2$:

$$(8.29) \qquad S_{Res}(t-\frac{L}{c}) = S_{inc}(t-\frac{L}{c}) - E_0^2\,\frac{e^2 dn}{M}\sin\frac{N\pi c}{a}(t-\frac{L}{c})F(t-\frac{L}{c}).$$

Here once again we assumed that the thickness of the medium d is sufficiently small and therefore we keep only linear term in d.

This approximation in (8.31) is quite safe as now will be shown. Comparing terms we obtain an estimation for d

$$(8.30) \qquad d \ll \frac{a}{N}\cdot\frac{(\omega_0)^2 M}{4\pi e^2 n} = \frac{a}{N}\,\frac{(\omega_0)^2}{(\omega_p)^2}\,.$$

Taking the order of visible light $\frac{N}{a} \sim 10^5\,cm^{-1}$, and $\omega_0 = 10^{16}\,s^{-1}$, with $n = 10^{21}\,cm^{-3}$, and e and M just the electron charge and mass we obtain

$$(8.31) \qquad d \ll 3\;10^{-5}\,cm.$$

Therefore the approximation employed here is satisfactory. We see in (8.29) that energy arrives at point L at a speed c.

The second term in (8.29), which corresponds to interference between incident and radiated electromagnetic waves has a simple and transparent interpretation as the net work done per unit area and unit time by the incident wave on the bound charges of the medium. This term is exactly equal to

$$e\, E_{inc}(z = 0,t)\dot{u}(z=0,t)nd.$$

Therefore we can write:

$$(8.32) \qquad S_{res}(t - \frac{L}{c}) = S_{inc}(t - \frac{L}{c}) - \frac{\text{work}}{\text{AREA}\cdot\text{TIME}}.$$

This proves that the energy-work balance is reasonably taken into account in this model.

C. Momentum Calculation.

In a fashion similar to the energy calculation we can calculate the momentum arriving at time t at point L

$$(8.23) \qquad \dot{g}_{res}(t-\frac{L}{c}) = \dot{g}_{inc}(t-\frac{L}{c})-\hat{x}\,\frac{E_0^2dne^2}{Mc^2}F(t-\frac{L}{c})\sin\frac{N\pi c}{a}(t-\frac{L}{c}).$$

The loss in electromagnetic momentum $\vec{g}_{inc}-\vec{g}_{res}$ should correspond to the radiation pressure exerted on the dielectric slab by light. The time-dependent pressure is

$$(8.34) \qquad P(t-\frac{L}{c}) = c\Delta g = \frac{E_0^2dne^2}{Mc}F(t-\frac{L}{c})\sin[\frac{cN\pi}{a}(t-\frac{L}{c})].$$

As one can anticipate, this pressure is proportional to the square of the amplitude of the incident electric field [2]. One can also estimate the total impulse transferred to the dielectric slab from the entire pulse as

$$(8.35)\qquad \int_{-\infty}^{\infty} Pdt = \int_{0}^{\infty} \frac{E_0^2 dne}{Mc} F(t-\frac{L}{c}) \sin \frac{cN\pi}{a}(t-\frac{L}{c}) d(t-\frac{L}{c})$$

$$= \frac{E_0^2 dne^2}{Mc} \int_0^{\infty} F(T) \sin \frac{\pi Nc}{a} T dT$$

$$= \frac{E_0^2 dne^2}{Mc} \tilde{F}(\frac{N}{a})$$

where

$$\tilde{F}(\frac{N}{a}) = \int_0^{\infty} F(T) \sin \frac{cN\pi}{a} T dT$$

is the sine-transform of function F.

Measurement of the pressure may give some important information about function $F(T)$. If the slab has finite mass m and is able to move, then the mechanical momentum acquired by slab is equal to:

$$(8.36)\qquad Vm = \frac{E_0^2 dne^2}{M} \tilde{F}(\frac{N\pi}{a}).$$

In our calculation of energy and momentum we did not include the reflected light because as one can see immediately from (8.26), the Poynting vector of the reflected light is proportional to d^2 and we neglected all terms of the second degree.

Chapter 9. The Electric Field in the Case when the Interaction Between Dipoles is taken into Account.

In the calculations presented in previous chapters we neglected interactions among bounded charges. The simplest way to improve this and take this interaction into account is to assume that we have only nearest neighbors interactions and that it has an elastic character [13]. Therefore the equation of motion for such an oscillator is (compare (B7))

$$(9.1) \qquad \ddot{\vec{u}}(x,t) + \omega_0^2 \vec{u}(x,t) - \beta \frac{\partial^2 \vec{u}(x,t)}{\partial x^2} = \frac{e}{M} \vec{E}_{inc}(x,t).$$

Here we use the same assumptions as above.

Also introducing

$$\vec{u}(x,t) = \int_{-\infty}^{\infty} \vec{u}(x,\omega) e^{-i\omega t} dt$$

we obtain an equation for $\vec{u}(z,\omega)$

$$(9.2) \qquad (-\omega^2+\omega_0^2)\vec{u}(x,\omega) - \beta \frac{\partial^2 \vec{u}(x,\omega)}{\partial z^2} = \frac{e}{M} \vec{E}_{inc}(\omega) e^{i\frac{\omega}{c}x} .$$

The forced part of its solution is [14]

$$(9.3) \qquad \vec{u}(z,\omega) = \int_0^x \frac{1}{\omega'} \sin \omega'(x-x') E_{inc}(\omega) \frac{e}{M} e^{i\frac{\omega}{c}x'} dx'$$

where

$$(9.4) \qquad (\omega')^2 = \frac{\omega_0^2 - \omega^2}{y}.$$

Finally after integration we get for $\vec{u}(x,\omega)$

$$(9.5) \qquad \vec{u}(x,\omega) = \frac{e}{M}\vec{E}_{inc}(\omega)\frac{e^{-i\omega' x} - e^{i\frac{\omega}{c}x}}{(\omega')^2 - \frac{\omega^2}{c^2}}.$$

For the electric field produced by these interacting oscillators we have

$$(9.6) \qquad E_{Rad}(L,\omega) = -\frac{e^2 in\pi k_0 \hat{z}}{M}\int_0^d dx \frac{e^{i\frac{\omega}{c}x} - e^{-i\omega' x}}{(\omega')^2 - \frac{\omega^2}{c^2}}$$

$$\cdot e^{ik_0 L} e^{-ik_0 x}\left(2 + \frac{2i}{ik_0(L-x)}\right).$$

As in our previous calculations we will keep only linear terms in d (here $d/L \ll 1$). The result for E_{Rad} is,

$$(9.7) \qquad E_R(L,\omega) = \frac{e^2 n\pi k_0}{M\left((\omega')^2 - \frac{\omega_0^2}{c^2}\right)\left(\omega' + \frac{\omega}{c}\right)} \cdot \frac{2d}{L^2}$$

and for time-dependent electric field at point L we have

$$(9.8) \qquad E_R(L,t) = \frac{e^2 \hat{z} n\pi}{m} \cdot \frac{d}{L^2}\int_{-\infty}^{\infty} \frac{\omega E_{inc}(\omega) e^{-i\omega(t-\frac{L}{c})}}{\left((\omega')^2 - \frac{\omega_0^2}{c^2}\right)\left(\omega' + \frac{\omega}{c}\right)} dw.$$

We can see now that the electric field depends on mechanical parameter of the medium. Causality is satisfied: the signal does not arrive at L earlier than the speed of light in vacuum.

In conclusion we note that we presented for the first time microscopical (classical) theory of the propagation of the front of electromagnetic pulse through thin dielectric slab. In this microscopic picture we recovered all the properties of the first (Sommerfeld) precursor. In this model we have the correct energy-work balance. The model allows one to calculate energy and momentum transfer. We believe that this model may serve for future investigations of transients in dielectric medium.

Appendix A. General Integral Equation for electric field.

In this appendix we will derive the integral equation which describes the electric field under quite general conditions: part of the space is occupied by the bound charges taken as Lorentz oscillators. We will show that application of this integral equation justifies the method employed in the main text. Also we will get an estimation for thickness d when our treatment is applicable.

In what follows we will use the dipole approximation, but will not assume that we are only interested in the radiation (far) zone. Therefore the electric field created by a dipole is [2]

$$\text{(A.1)} \quad \vec{E} = k_0^2(\vec{n}\cdot\vec{p})\times\vec{n}\frac{e^{ik_0R}}{R} + [3\vec{n}(\vec{n}\cdot\vec{p})-\vec{p}]\left(\frac{1}{R^3} - \frac{ik_0}{R^2}\right)e^{ik_0R}.$$

Hence instead of equation (8.17) in the text we get

$$\text{(A.2)} \quad E(L,\omega) = E_{inc}(L,\omega) + n\int k_0^2 \frac{e^{ik_0R}}{R}\left(\frac{\vec{R}}{R}\cdot\vec{p}\right)\times\frac{\vec{R}}{R}d^3\vec{r}$$

$$+ n\int\left[3\frac{\vec{R}}{R}\left(\frac{\vec{R}}{R}\cdot\vec{p}\right)-\vec{p}\right]\left(\frac{1}{R^3} - \frac{ik_0}{R^2}\right)e^{ik_0R}d^3\vec{r}$$

where once again we took the density of the dipoles assumed to be constant outside the integrand. We use the same notation as in the text.

Introducing cylindrical coordinates and integrating over the angle φ one gets after some calculations

$$\text{(A.3)} \quad E(L,\omega) = E_{inc}(L,\omega) + n\pi\int_{(x)}\int_0^\infty k_0^2\frac{e^{ik_0R}}{R}\left(2-\frac{\rho^2}{R^2}\right)p(z,\omega)\rho\, d\rho\, dx$$

$$+ n\pi\int_{(x)}\int_0^\infty\left(2-3\frac{\rho^2}{R^2}\right)\left(\frac{1}{R^3}-\frac{ik_0}{R^2}\right)e^{ik_0R}p(z,\omega)\rho\, d\rho\, dx.$$

If the observation point L is inside dielectric we have to exclude this point with its infinitesimally small vicinity. In doing this we have to use the contribution in the integral from this element. Following the idea of Ewald and Oseen [15] we just have to introduce a term $-\frac{4\pi}{3}\,np(L,\omega)$. So we can rewrite equation (A.3) in the form:

$$\text{(A.4)}\quad \left[1+\frac{4\pi}{3}\cdot\frac{e^2 n}{M(\omega_0^2-\omega^2)}\right]\vec{E}(L,\omega)$$

$$= \vec{E}_{inc}+\frac{\pi n e^2\omega^2}{M(\omega_0^2-\omega^2)}\int_{(x)}\int_0^\infty \frac{e^{ik_0R}}{R}\left(2-\frac{\rho^2}{R^2}\right)\vec{E}(x,\omega)\,\rho\, d\rho\, dx$$

$$+\frac{\pi n e^2}{M(\omega_0^2-\omega^2)}\int_{(x)}\int_0^\infty \left(2-3\frac{\rho^2}{R^2}\right)\left(\frac{1}{R^3}-\frac{ik_0}{R^2}\right)e^{ik_0R}\,\vec{E}(x,\omega)\,\rho\, d\rho\, dx.$$

Here we have used: $p(x,\omega) = e\dot{u}(x,\omega) = \frac{e^2}{M}\cdot\frac{\vec{E}(x,\omega)}{\omega_0^2-\omega^2}$. Equation (A.4) is the integral equation which we wanted to obtain. Using equation (A.4) we have to keep in mind that an extra term $\frac{4\pi}{3}\,\frac{e^2 n}{M(\omega_0^2-\omega^2)}$ in the left-hand-side appears only if the observation point L is inside the region occupied by the dielectric. Equation (A.4) just obtained is a quite general integral equation for the electric field $E(L,\omega)$ when the assumptions about medium as an array of Lorentz oscillators have been made. Equation (A.4) is exact within this model and the assumption that the incident electric field is homogeneous in the (y,z) plane.

Now it is a simple matter to show that the result obtained in the text is the result of solution of equation (A.4). To do this we have to drop the term $-\frac{4\pi}{3}\,pn$ and to specify the limit of integration in (A.4) in z from 0 to d. Also we have to assume $L > d$. Because of the latter assumption we may neglect the second integral in (A.4). This can be shown explicitly if it is assumed $kL \gg 1$ as in the text and

$$\text{(A.5)} \qquad \int_{L-z}^{\infty} \frac{e^{ik_0R}}{R^n}\,dR \approx \frac{e^{ik_0R}}{ik_0R^n}\Bigg|_{L-z}^{\infty} + n\int_{L-z}^{\infty} \frac{e^{ik_0R}}{ik_0R^{n+1}}.$$

So our integral equation is reduced now to

$$\text{(A.6)} \qquad E(L,\omega) = E_{inc}(L,\omega) + \frac{-\pi n e^2 \omega^2}{M(\omega_0^2-\omega^2)} \int_0^d \int_0^\infty \frac{e^{ik_0R}}{R}\left(2-\frac{\rho^2}{R^2}\right) E(x,\omega)\rho\, d\rho\, dx.$$

Now solving this equation by iteration we see that the first iteration immediately gives the result cited in the text

$$\text{(A.7)} \quad E(L,\omega) \;R\; E_{inc}(L,\omega) + \frac{\pi n e^2 \omega^2}{M(\omega_0^2-\omega^2)} \int_0^d \int_0^\infty \frac{e^{ik_0R}}{R}\left(2-\frac{\rho^2}{R^2}\right) E(x,\omega)\rho\, d\rho\, dx.$$

and after integration

$$E(L,\omega) = E_{inc}(\omega,L) + \frac{\pi n^2 e^2 \omega d i}{cM(\omega_0^2-\omega^2)} e^{i\frac{\omega}{c}L} E_{inc}(\omega).$$

The next iteration gives

$$\text{(A.8)} \quad E(l,\omega) = E_{inc}(L,\omega) + \frac{\pi n e^2 \omega^2}{M(\omega_0^2-\omega^2)c^2} \int_0^d \int_0^\infty \frac{e^{ik_0R}}{R}\left(2-\frac{\rho^2}{R^2}\right) \cdot$$

$$\cdot \left[E_{inc}(L,\omega) + \frac{\pi n e^2 \omega d i}{cM(\omega_0^2-\omega^2)} E_{inc}\, e^{i\frac{\omega}{c}L}\right] \rho\, d\rho\, dx$$

$$= E_{inc}(L,\omega) + \frac{\pi n e^2 \omega^2 i d}{M(\omega_0^2-\omega^2)c} e^{i\frac{\omega}{c}L} E_{inc}(L,\omega)$$

$$+ \frac{\pi^2 n^2 (e^2)\omega^3 d^2 i}{c^2 M^2 (\omega_0^2-\omega^2)}\, e^{\frac{i\omega L}{c}} \,.$$

Then our result (.27) is valid if

$$\frac{\pi n e^2 d}{cM\omega_0} \ll 1 \quad \text{or} \quad \frac{\omega_p^2 d}{c\omega_0} \ll 1.$$

Taking here $n = 10^{21} \frac{1}{cm^3}$, $e = 4 \cdot 10^{-10}$ c.g.s., $M = 10^{-27} g$, $\omega_0 = 10^{14}$ Hz we obtain as an estimation for d

$$d < 10^5 \text{ A}.$$

So in an experiment people may use $d \sim 10\mu m$. If one to investigate the semiinfinite case not a slab (i.e. the Sommerfeld problem), then the integral equation to be solved is:

$$(A.9)\quad E(L,\omega)\left[1 + \frac{4\pi}{3}\frac{e^2}{M(\omega_0^2-\omega^2)}\right] = E_{inc}(L,\omega)$$

$$+ \frac{\pi n e^2 \omega^2}{M(\omega_0^2-\omega^2)c^2}\int_0^\infty\int_0^\infty \frac{e^{ik_0R}}{R}\left(2-\frac{\rho^2}{R^2}\right)E(x,\omega)\rho\, d\rho\, dx$$

$$+ \frac{\pi n e^2}{M(\omega_0^2-\omega^2)}\int_0^\infty\int_0^\infty \left(2-3\frac{\rho^2}{R^2}\right)\left(\frac{1}{R^3}-\frac{ik_0}{R^2}\right)e^{ik_0R}E(x,\omega)\rho\, d\rho\, dx.$$

We will not discuss the solution of this equation, because it is irrelevant to the problem in our main text.

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Appendix B.

Here we discuss the evaluation of an integral which appears in (.26). Consider:

$$\text{(B.1)} \qquad F(t-\frac{L}{c}) = \frac{i}{E_0}\int_{-\infty}^{\infty} \frac{\omega E_{inc}(\omega)e^{-i\omega(t-\frac{L}{c})}}{\omega_0^2 - \quad} .$$

Here we will discuss the general case, including damping.

We will write (B.1) in the following form:

$$\text{(B.2)} \qquad F(t-\frac{L}{c}) = \frac{i}{E_0}\int_{-\infty}^{\infty} \frac{\omega E_{inc}(\omega)e^{-i\omega(t-\frac{L}{c})}}{(\omega-\omega+)(\omega-\omega-)} .$$

Substituting $E_{inc}(\omega)$ from equation (.8) we obtain:

$$F(t-\frac{L}{c}) = F_1(t-\frac{L}{c}) + F_2(t-\frac{L}{c})$$

where

$$\text{(B.3)} \qquad F_1(t-\frac{L}{c}) = \frac{\omega_N}{2\pi}\int_{-\infty}^{\infty} \frac{i\omega e^{-i\omega(t-\frac{L}{c})}}{(\omega^2-\omega_N^2)(\omega-\omega_+)(\omega-\omega-)}$$

where $\omega_N = \frac{N\pi c}{a}$, $\omega_{\pm} = -i\frac{\Gamma}{2} \pm (\omega_0^2-(\frac{\Gamma}{2})^2)^{1/2}$ F_2 is just like F_1 only with $(t-\frac{L}{c})$ replaced by $(t-\frac{L}{c}-\frac{a}{c})$. The integrand in (B.3) has poles along the real axis when $\omega = \pm\frac{N\pi c}{a}$ and poles in the lower half-plane of the complex ω. The contour of Figure B.1 in the upper half-plane should be used when both $t - \frac{L}{c}$ and $t - \frac{L+a}{c}$ are negative. Carrying out the integration, we get

$$(B.4) \qquad F_1 = \frac{N\pi a}{2c}\mathrm{Re}\,\frac{e^{-i\frac{N\pi a}{c}(t-\frac{L}{c})}}{[(\frac{N\pi a}{c}-\omega_+]\cdot[(\frac{N\pi a}{c})-\omega_-]}$$

$$(B.5) \qquad F_2 = -\frac{N\pi a}{c}\mathrm{Re}\frac{[(\omega_N^2-\omega_0^2)\cos\,\omega_N(t-\frac{L}{c})-\Gamma\omega_N\,\sin\,\omega_N(t-\frac{L}{c})]}{\{[\omega_N-(\omega_0^1-(\frac{\Gamma}{2})^2)^{\frac{1}{2}}]+(\frac{\Gamma}{2})^2\}\{[\omega_N+(\omega_0^2-(\frac{\Gamma}{2})^2)^{\frac{1}{2}}+(\frac{\Gamma}{2})^2]}.$$

If both $t - \frac{L}{c}$ and $t - \frac{L+a}{c}$ are positive then the contour in the lower half-plane should be used and the result is

$$(B.6) \qquad F_1(t-\frac{L}{c}) = -\frac{N\pi a}{c}\left[\frac{2\,\mathrm{Re}\,\omega_+ e^{i\omega_+(t-\frac{L}{c})}}{(\omega_0^2-(\frac{\Gamma}{2})^2)^{\frac{1}{2}}} + \mathrm{Re}\,\frac{e^{-i\frac{N\pi a}{c}(t-\frac{L}{c})}}{(\frac{N\pi a}{c}-\omega_+)(\frac{N\pi a}{c}-\omega_-)}\right].$$

The same expression with $t - \frac{L}{c}$ replaced by $t - \frac{L+a}{c}$ gives $F_2(t-\frac{L}{c})$.

When $t \leq \frac{L}{c}$ F is the sum of: $\{F_1(t-\frac{L}{c} < 0) + F_2(t-\frac{L+a}{c}<0)\}$ which is just the sum (B.4) and (b.5). Therefore F = 0 for $t \leq \frac{L}{c}$ and no radiation wave reaches L before $t = \frac{L}{c}$. For the time interval $L \leq ct \leq L+a$ we have:

$$(B.7) \qquad F = F_1(t-\frac{L}{c}>0) + F_2(t-\frac{L+a}{c} < 0).$$

Combining we obtain equation (.29) in the text (we also pur $\Gamma = 0$ there).

Figure captions

Fig. 1.1 Peierls distortion in a one-dimensional electron gas. (a) Undistorted lattice, (b) distorted lattice with a gap in the single particle excitation spectrum.

Fig. 1.2. Schematic explanation of nesting of the Fermi surface. Two Fermi surfaces are drawn staggered by a vector $\vec{Q}$ in momentum space forming parallel partrons.

Fig. 1.3. Raman spectrum with a sample immersed in superfluid helium and in the presence of a magnetic field. The resolution was 3 cm^{-1}. The curves represent a five-point smoothed computer plot through the original data points. From Sooryakumar , Klein, Phys. Rev. Lett. 45, 660 (1980).

Fig. 2.1. The two processes leading to the formation of a phonon-quasiparticle-pair bound state.

Fig. 2.2. The phonon spectral density for the coupled modes. In these examples the coupling constant is $g^2\omega_0 = .12\ \hbar\omega_0$ and the value of the superconducting energy gap is indicated. The spectral weights of the bound state and the continuum are shown. From C. Balseiro, L.M. Falicov. Phys. Rev. Lett. 45, δ, 662 (1980).

Fig. 2.3. A schematic drawing of the Fermi surface of 2H-$NbSe_2$ and IT-TaS_2 in the $k_z = 0$ plane including the Bragg plane AB induced by a lattice distortion in the CDW state. The inset shows how the Fermi surface changes from the normal state (full line) into the CDW state (dashed line) and with further perturbation of the CDW-AM (dotted line).

Fig. 3.1. Calculated Fermi surface of (a) IT-TaS_2 and (b) 2H-$NbSe_2$ (Matheiss 1973). Figures taken from wilson et al (1975).

Fig. 6.1. Schematic of a typical experiment in a Picosecond laboratory. The incident finite pulse is analysed after it passed through a dielectric slab of thickness d.

Fig. 8.1. Figure shows the situation when finite pulse with N = 5 (defined in equation 8.1) at time t = 0 is incident on a dielectric slab of thickness d.

Fig. 8.2. Standard cylindrical system of co-ordinates used in the evaluation of integrals.

Fig. B.1. Contours in the complex ω-plane used in evaluation of integrals.

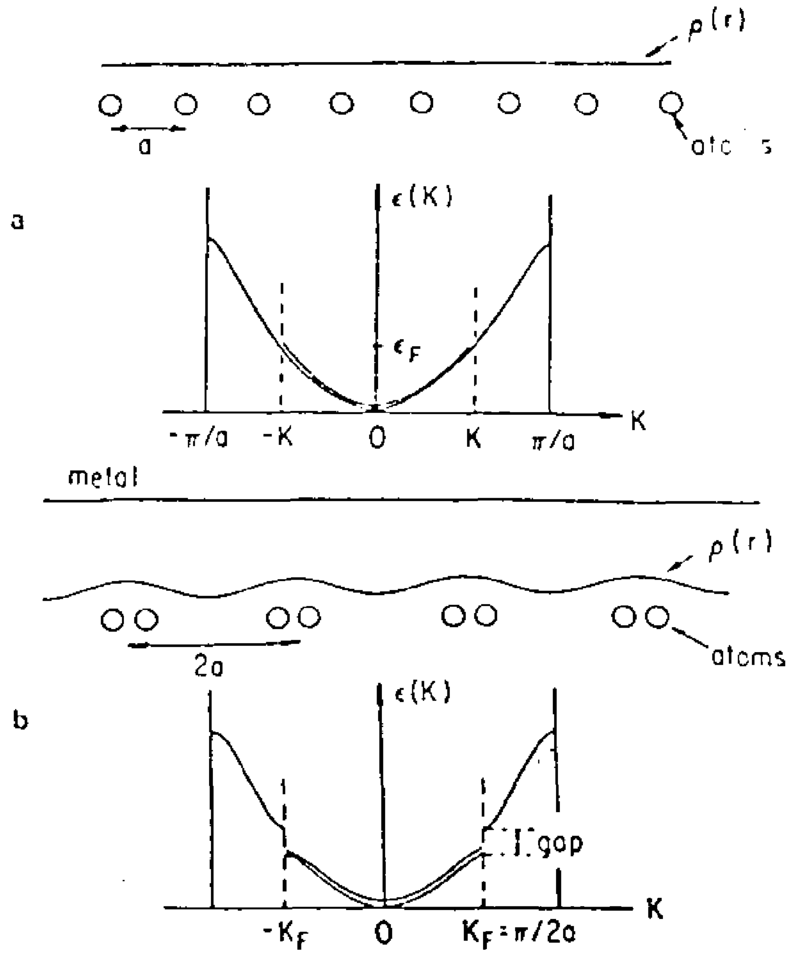
ρ(r)
a
atoms
a
ε(K)
ε_F
-π/a
-K
0
K
π/a
K
metal
ρ(r)
atoms
2a
b
ε(K)
gap
-K_F
0
K_F = π/2a
K

Fig. 1.1

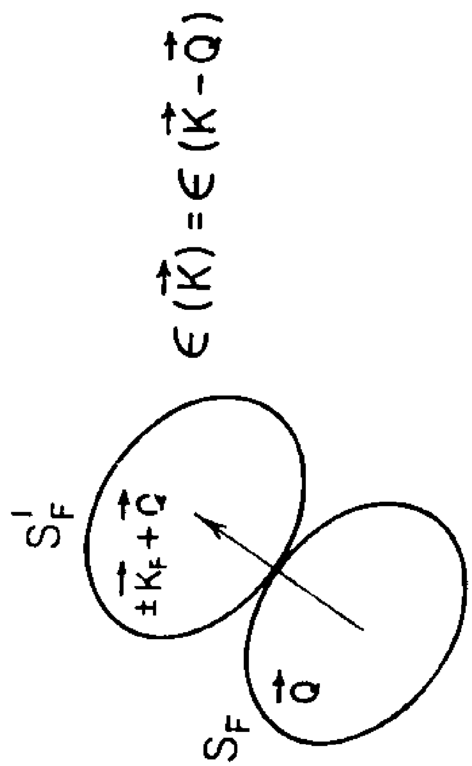

Fig. 1.2

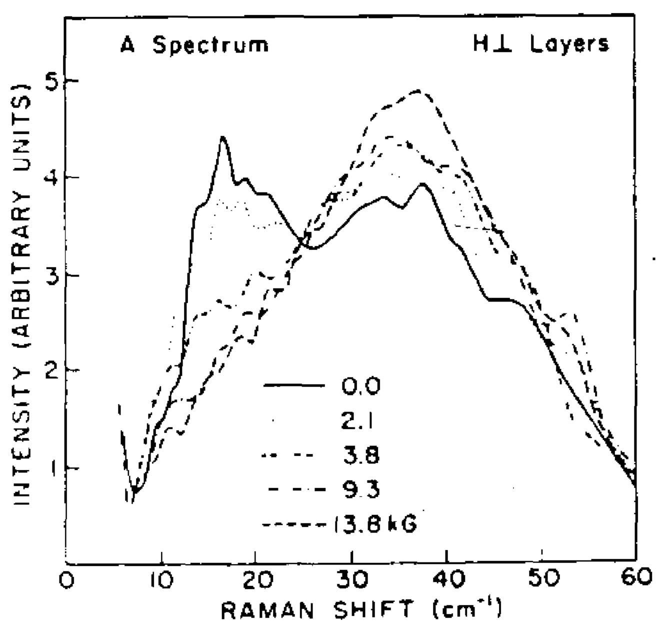
A Spectrum
H⊥ Layers
INTENSITY (ARBITRARY UNITS)
RAMAN SHIFT (cm⁻¹)
0.0
2.1
3.8
9.3
13.8 kG
0
10
20
30
40
50
60
1
2
3
4
5

Fig. 1.3

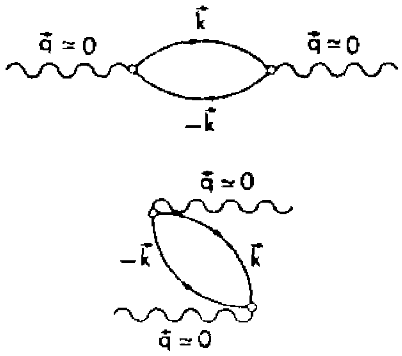
$\vec{k}$
$\vec{q} \simeq 0$
$\vec{q} \simeq 0$
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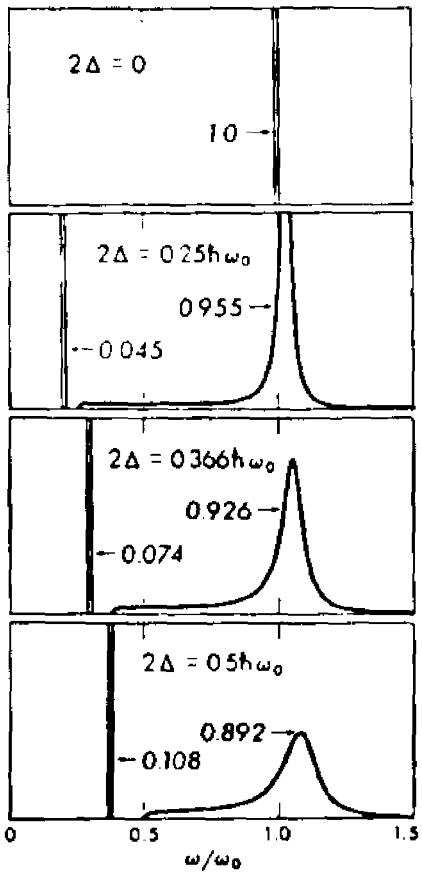
Fig. 2.1


2Δ = 0
1.0
2Δ = 0.25ħω₀
0.955
0.045
2Δ = 0.366ħω₀
0.926
0.074
2Δ = 0.5ħω₀
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0.108
0
0.5
1.0
1.5
ω/ω₀

Fig. 2.2

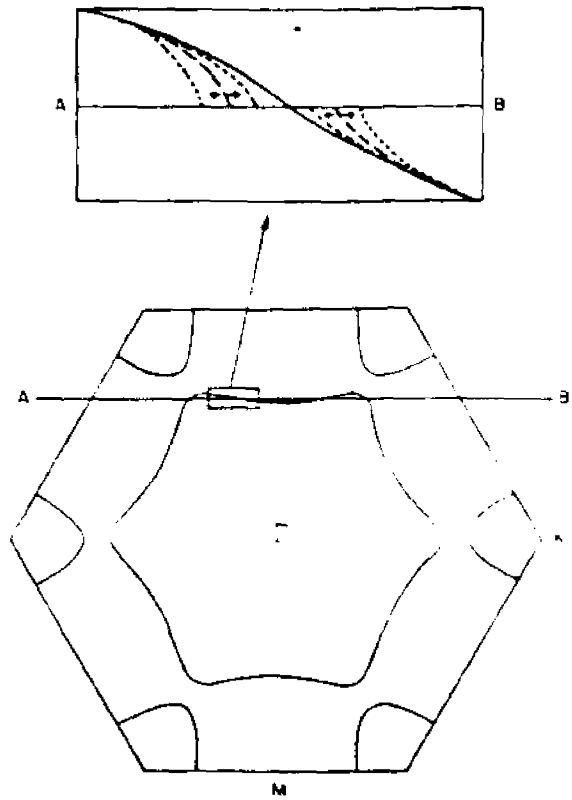

Fig. 2.3

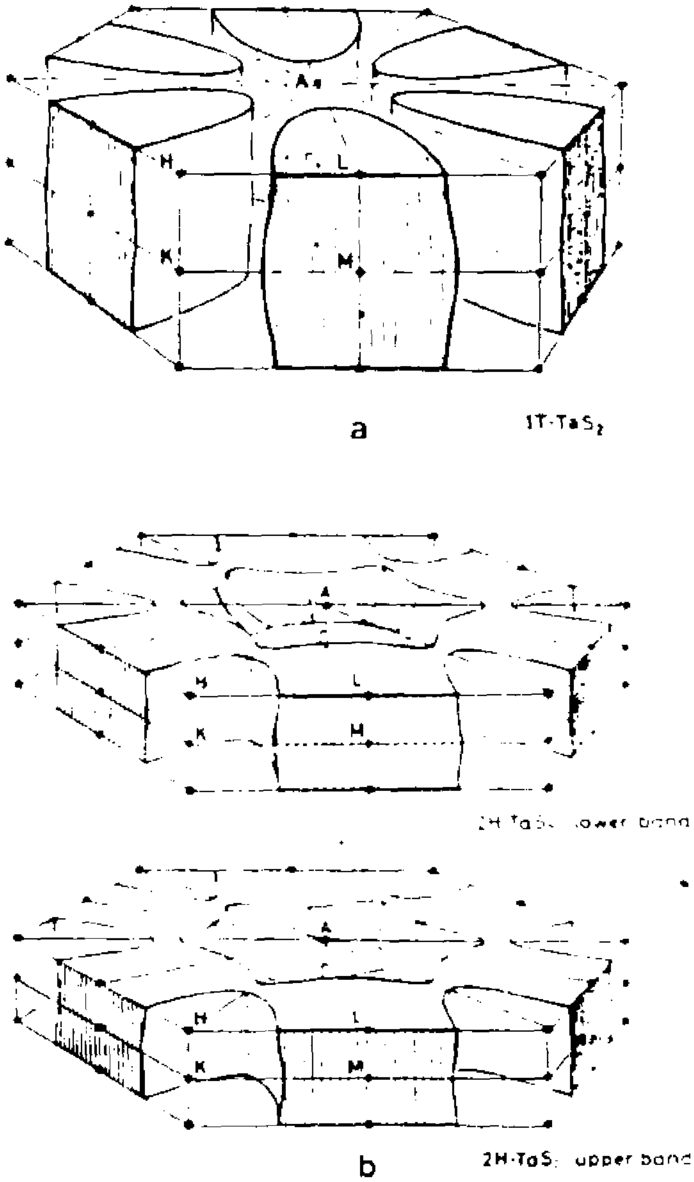

Fig. 3.1

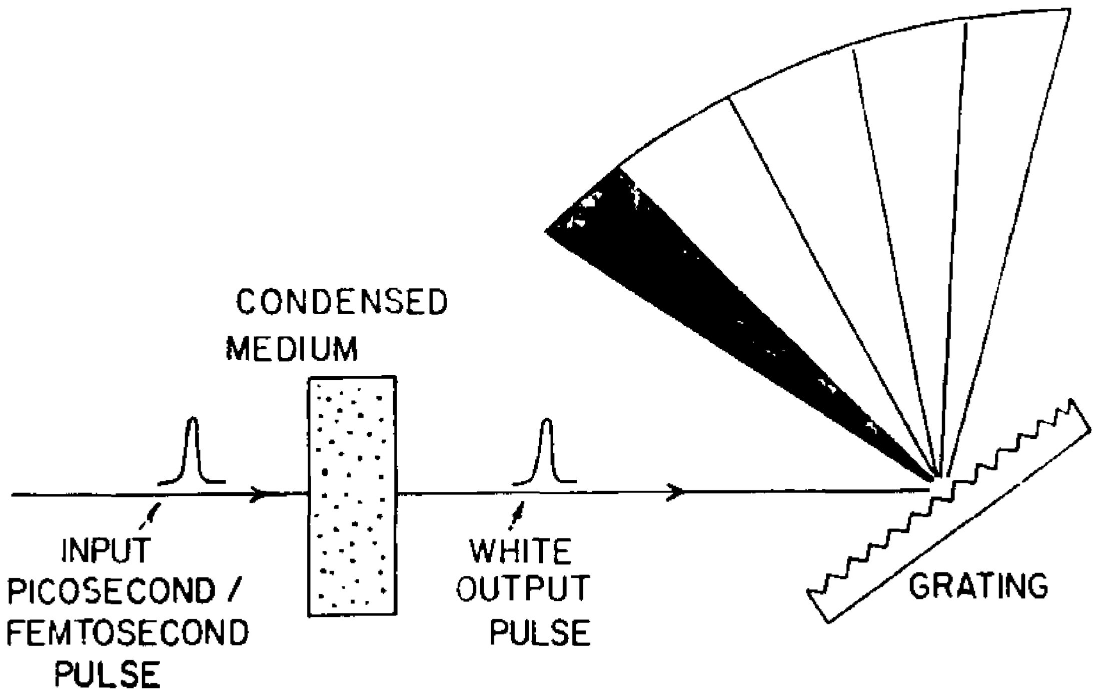
CONDENSED
MEDIUM
INPUT
PICOSECOND /
FEMTOSECOND
PULSE
WHITE
OUTPUT
PULSE
GRATING

Fig. 6.1

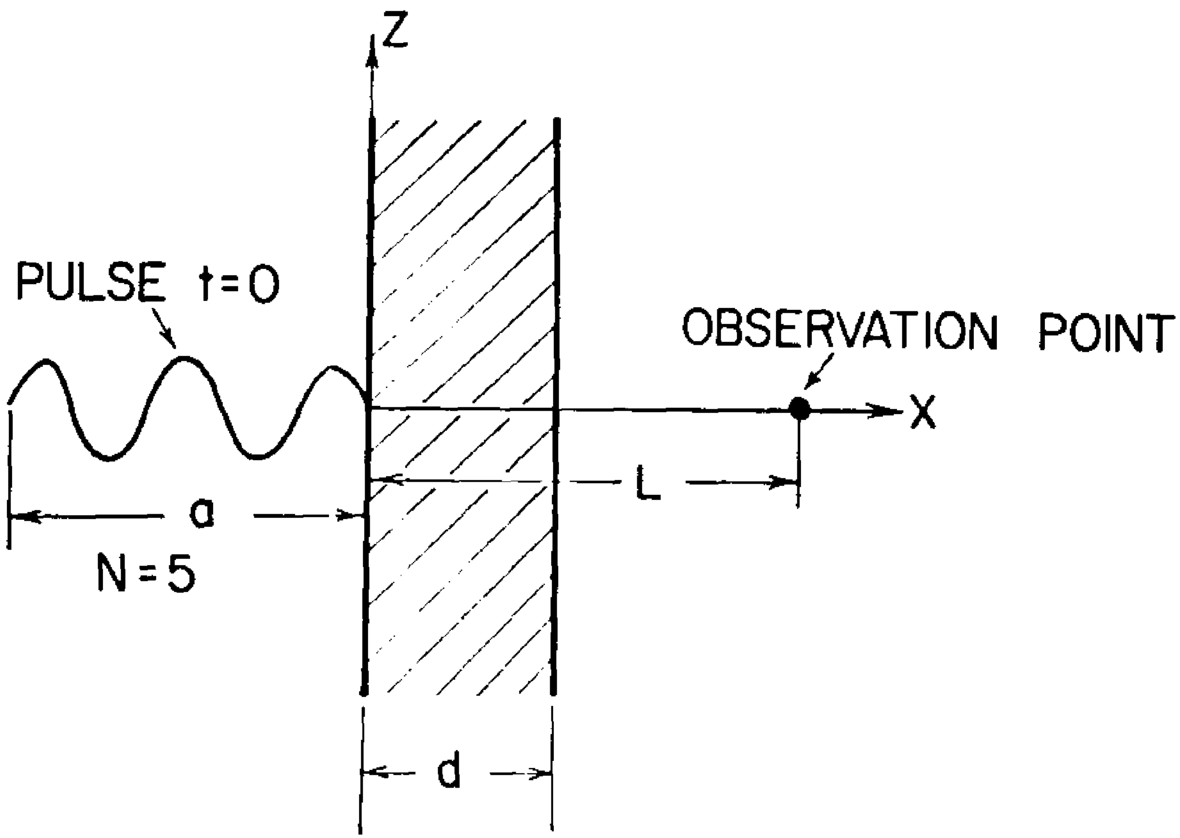

Fig. 8.1

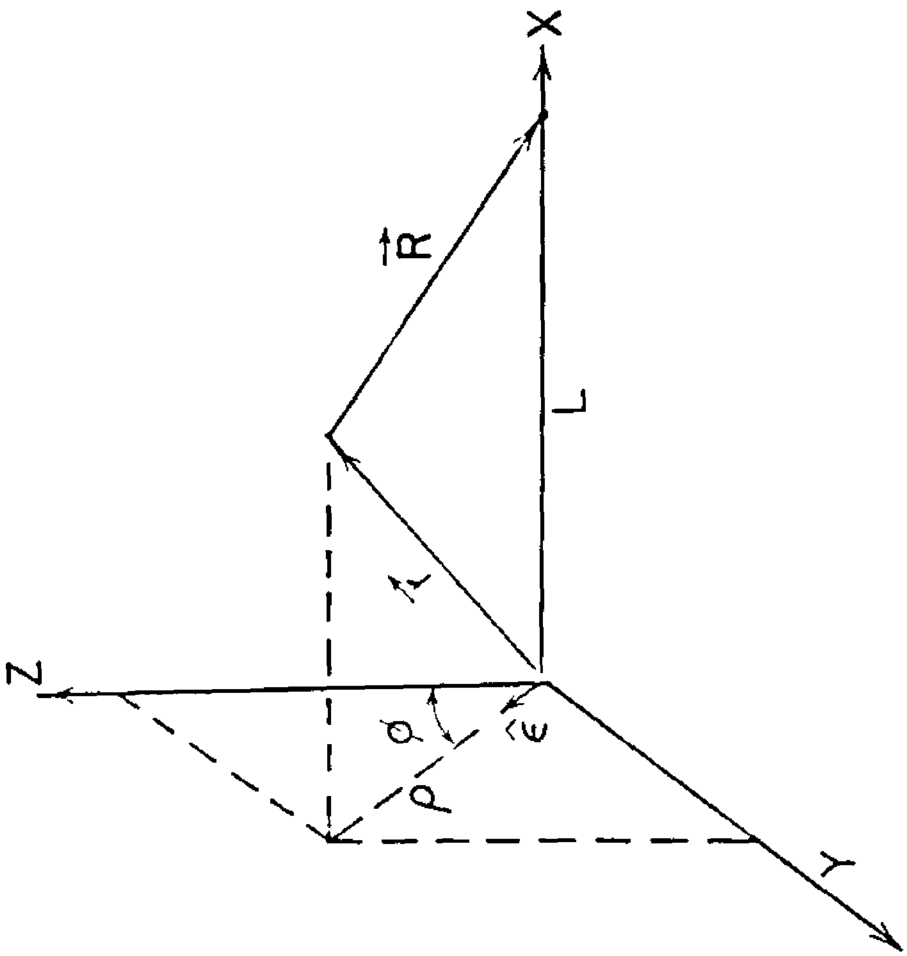

Fig. 8.2

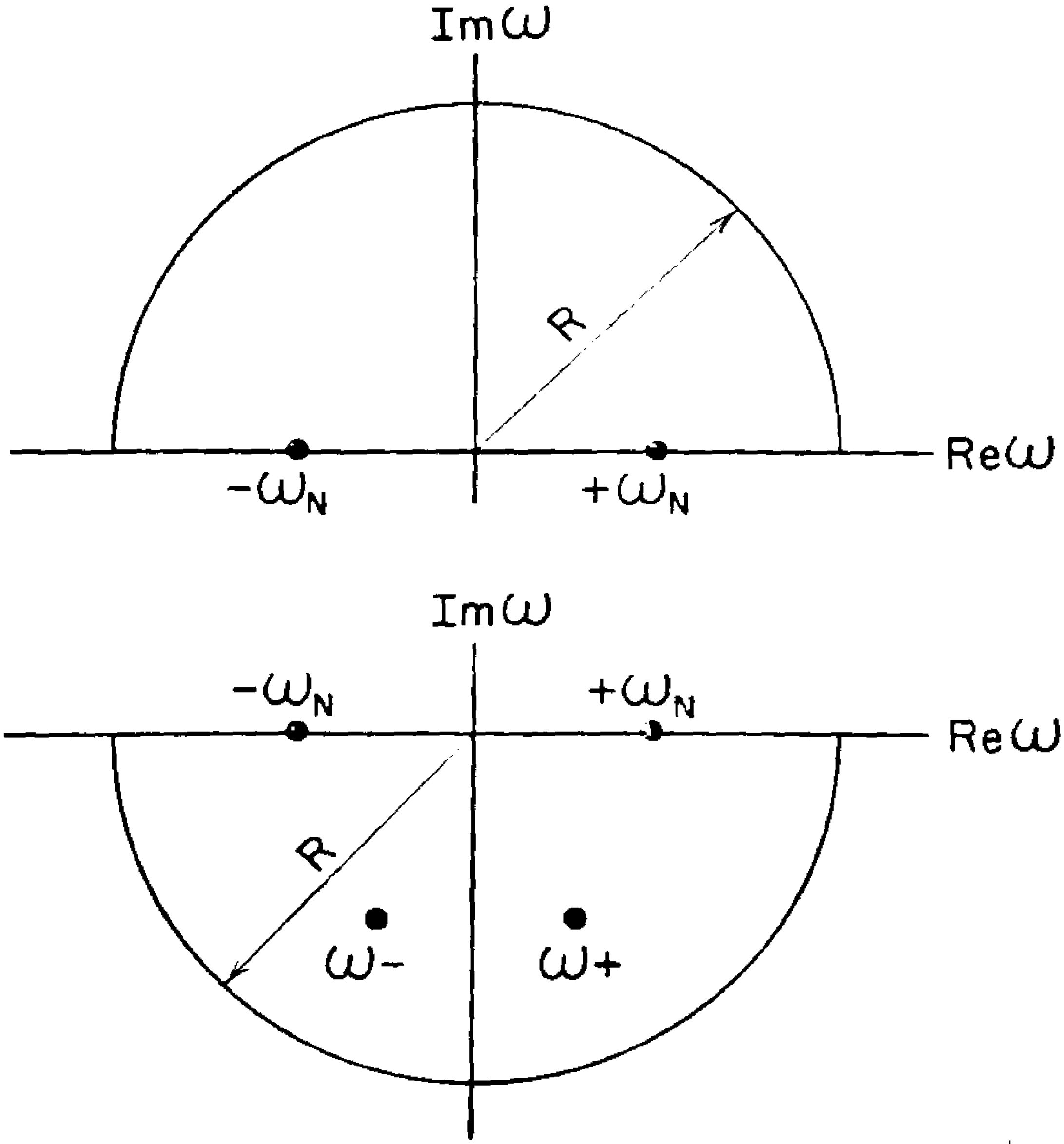

Fig. B.1

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Ginzburg-Landau-Gor'kov Equations, Currents, and Electromagnetic Properties of Coexisting Charge-Density-Wave Superconductors

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Ginzburg-Landau-Gor'kov equations and the expression for the current for coexisting superconductivity–charge-density-wave (CDW) systems near the superconducting transition temperature have been derived from microscopic theory. Two particular features are discussed: an additional current due to the sliding CDW, and the CDW-induced anisotropy, which lead to interesting physical consequences including the Meissner effect with a contribution from both currents. Measurements of CDW-induced anisotropy can be used to probe the microscopic mechanism of coexistence.

PACS numbers: 74.20.Fg, 71.45.Gm, 72.15.Nj

Coexistence of superconductivity (SC) and charge-density waves (CDW) in $2H$-$NbSe_2$ was established in Raman scattering experiments by Sooryakumar and Klein.[1] At present few properties of the coexisting SC-CDW system are known. This Letter reports the results of a microscopically derived calculation of the macroscopic current in the SC-CDW system, and of the Ginzburg-Landau-Gor'kov (GLG) equation for the order parameter, and some novel experimental predictions of the electromagnetic properties of the coexising system. We emphasize one important feature of our work: The coupling parameters in the macroscopic equations are obtained from the microscopic theory; no parameters are inserted by hand. The predicted experiments will also enable a test to be made of the commonly used microscopic model that we used in this work.

We start from a microscopic system with electron-phonon, and separated BCS, local interactions. Because the current now depends on time derivatives of the order parameter in addition to spatial derivatives, we use a closed-time-path Green's-function formalism.[2] The generating functional of the coexistent system is

$$Z[J,h]=\exp(-ih_ih_i/2g)\exp(i\overline{W}[J,h]),$$

$$\exp(i\overline{W}[J,h])=\int[d\Psi][d\Psi^\dagger][du][d\chi]\exp\{i[\tfrac{1}{2}uD_0^{-1}(Q)u-\tfrac{1}{2}g\chi_i\chi_i+\Psi^\dagger G_A^{-1}\Psi+J_iu_i+h_i\chi_i]\},\tag{1}$$

$$G_A^{-1}(x_1,x_2)=\{i\,\partial/\partial t_1-(1/2m)[-i\nabla_1-\rho_3\mathbf{Q}/2-\tau_3e\mathbf{A}(x_1)/c]^2\tau_3+\mu\tau_3$$
$$+Gu_i(x_1)\rho_i\tau_3/\sqrt{2}+g\chi_i(x_1)\tau_i/\sqrt{2}\}\delta^{d+1}(x_1-x_2).\tag{2}$$

It generates the SC order parameter $\chi(x)=(\chi_1-i\chi_2)/\sqrt{2}=\langle\Psi_\downarrow(x)\Psi_\uparrow(x)\rangle$ as well as the optical part of the phonon field $u(x)=\sqrt{2}[u_1\cos(\mathbf{Q}\cdot\mathbf{x})-u_2\sin(\mathbf{Q}\cdot\mathbf{x})]$ when $Z[J,h]$ is varied with respect to $h_i(x)$ and $J_i(x)$, respectively. In Eqs. (1) and (2), $\Psi(x)=(\bar{\Psi}_\uparrow(x),\bar{\Psi}_\downarrow^\dagger(x),\dot{\Psi}_\uparrow(x),\dot{\Psi}_\downarrow^\dagger(x))$ are defined from the electron wave function as

$$\Psi_\sigma(x)=\bar{\psi}_\sigma(x)\exp(i\sigma\mathbf{Q}\cdot\mathbf{x}/2)+\dot{\psi}_\sigma(x)\exp(-i\sigma\mathbf{Q}\cdot\mathbf{x}/2),$$

where the spin index $\sigma=\pm1$ denotes spin up and down, respectively. ρ_1, ρ_2, and ρ_3 are Pauli matrices on the CDW electron branch subspace, and τ_1, τ_2, and τ_3 are the Pauli matrices on the Nambu spin subspace; i,j run over 1,2; dummy indices are summed over. Also $\mathbf{Q}/2=\mathbf{k}_F$ is parallel to the nesting direction of the quasi-one-dimensional CDW, and is not commensurate with the lattice spacing. Here we assume that the anomalous SC pairing Δ_Q is zero.[3,4]

After integration of the fermion variable in (1), the only effective dynamical variables are the SC order parameter $\chi(x)$ and the CDW order parameter $u(x)$. Then gauge invariance takes the form that $\overline{W}$ or $Z[J,h]$ is invariant under $\mathbf{A}(x)\to\mathbf{A}(x)+\nabla\Lambda(x)$, $\chi(x)\to\exp[i2e\Lambda(x)/c]\chi(x)$ and $u(x)$ kept invariant. Making a standard Legendre transformation, we introduce

$$\Gamma[\chi,u]=\overline{W}[J,u]-J_iu_i-h_i\chi_i.\tag{3}$$

Then, the general equations for the order parameters of the coexisting system are

$$[\delta\Gamma[\chi,u]/\delta\chi_i(x)]|_{t_+=t_-}=0,\tag{4}$$

$$[\delta\Gamma[\chi,u]/\delta u_i(x)]|_{t_+=t_-}=0.\tag{5}$$

The electric current density $j(x)$ has the following general form:

$$\mathbf{j}(x) = c[\delta\Gamma[\chi,u]/\delta\mathbf{A}(x)]_{t_+ = t_-}, \tag{6}$$

where $t_\pm$ means the two time branches in the closed-time-path Green's-function language.[2] Equations (4)–(6) are the starting points of the theory. The first two equations will give rise to a GLG type theory; the third equation leads to the expression for currents in the SC-CDW system.

We now assume that the physical order parameter varies slowly in space and time, and also that $\mathbf{A}(x)$ has only a small variation over a SC coherence length. Because of gauge invariance we proceed as follows: At first neglect the vector potential $\mathbf{A}(x)$ and expand Eqs. (4)–(6) with respect to the functional arguments referring to the spatial and temporal inhomogeneity $O(\hbar\,\partial^2/\partial x\,\partial p)$; then finally use gauge invariance to recover its contribution. All of the coefficients of the derivatives can be expressed through the electronic Green's function for the homogeneous system which is easy to calculate in the mean-field approximation. All of the information from the microscopic model is involved in these coefficients.

Here, following the quasi-one-dimensional approximation,[3] we divide the Fermi surface into two regions. In region I, the Fermi surface is nested with wave vector $\mathbf{Q}$, and the energy bands have approximately particle-hole symmetry near the Fermi surface, so that the band energies satisfy the well-known relation $\epsilon_s(\mathbf{p}) = [\epsilon(\mathbf{p}+\mathbf{Q}/2)+\epsilon(\mathbf{p}-\mathbf{Q}/2)]/2 \approx 0$. In region II we neglect the CDW coupling and use inversion symmetry.

In the vicinity of the SC transition temperature T_c, which is assumed to be much lower than the Peierls transition temperature T_P, we may freeze the modulus of the CDW order parameter as a homogeneous (constant) quantity. The equation for the CDW modulus is

$$-\frac{\rho\omega_Q^2}{2} + \int_{\mathrm{I}} N_1(0)\,d\epsilon(p)\,\frac{\tanh[E_a(p)/2T]}{E_a(p)} = 0, \tag{7}$$

where $E_a(p) = [\epsilon^2(p) + |W|^2 + |\Delta|^2]^{1/2}$. We keep the SC order parameter as a small quantity and now make a further expansion with respect to $\Delta(x) = g\chi(x)$. Using Eq. (7) and the approximation that the phase deviation is also a small quantity, after lengthy calculation (details will be given elsewhere), we derive the expression for the current in the CDW-SC system, and GLG equations for the SC order parameter. These are

$$\mathbf{j}(x) = \frac{e\hbar N_1^{(0)}}{2iT_c}\,\tfrac{1}{4}\mathscr{M}\left[\Delta^*(x)\nabla\Delta(x) - \nabla\Delta^*(x)\Delta(x) - \frac{4ei}{\hbar c}\mathbf{A}(x)|\Delta(x)|^2\right] + \frac{e\mathbf{Q}}{2m}|W(x)|^2\Pi_\phi^{(0)}(T)\,\frac{\partial\phi}{\partial t}, \tag{8}$$

$$\frac{\hbar^2}{4}\,\tfrac{1}{4}\mathscr{M}:\left[\nabla - \frac{2ie}{\hbar c}\mathbf{A}(x)\right]^2\Delta(x) = -(T_c - T)\zeta(W)\Delta(x) + \frac{T_c}{2N_2(0)}[\Pi_\phi^{(\mathrm{I})}(T) + \Pi_\phi^{(\mathrm{II})}(T)]|\Delta(x)|^2\Delta(x), \tag{9}$$

where we have defined the inverse-effective-mass tensor $\mathscr{M}$ by

$$\tfrac{1}{4}\mathscr{M} = \frac{T_c Q^2}{4m^2 N_2(0)}\{\mathbf{I}\,\Pi_\phi^{(\mathrm{II})}(T) + \mathbf{ee}[\Pi_\phi^{(\mathrm{I})}(T) - f\,\Pi_\phi^{(\mathrm{II})}(T)]\},$$

and

$$\Pi_\phi^{(0)}(T) = \int_{\mathrm{I}} N_1(0)\,d\epsilon(p)\,\frac{\tanh[E_a(p)/2T]}{E_a^3(p)}, \tag{10}$$

$$\zeta(W) = 1 + \frac{1}{2T_c N_2(0)}\int_{\mathrm{I}} N_1(0)\,d\epsilon(p)\,\mathrm{sech}^2\{[\epsilon^2(p) + |W|^2]^{1/2}/2T_c\}, \tag{11}$$

$$\Pi_\phi^{(\mathrm{I,II})}(T) = -\int_{\mathrm{I,II}} N_{1,2}(0)\,d\epsilon(p)\,\frac{1}{E(p)}\,\frac{d}{dE(p)}\left\{\frac{\tanh[E(p)/2T]}{E(p)}\right\}, \tag{12}$$

in which $E(p)$ is equal to $E_a(p)$ for $\Pi_\phi^{(\mathrm{I})}(T)$ but with $W=0$ for $\Pi_\phi^{(\mathrm{II})}(T)$. $N_1(0)$ and $N_2(0)$ are the density of states corresponding to regions I and II. If nesting takes place along one fixed direction, then region I will occupy a small angular region in the Brillouin zone. We may then approximate the tensor averaged over the region-II Fermi surface: $\langle pp\rangle_{\mathrm{II}}/p_F^2 \approx (\mathbf{I} - f\mathbf{ee})$. Here we have introduced a microscopic parameter f to describe the anisotropy of the Fermi surface, with $0 < f < 1$.

The first term in Eq. (8) is just the Ginzburg-Landau current for the superconductor including an anisotropic coefficient due to the coexisting CDW. The second term is proportional to the time derivative of the phase of the CDW order parameter. We interpret that term as the contribution from the sliding Goldstone mode.[6-8] If the SC current is absent, the second term corresponds to a pure CDW system. To our knowledge this is the first derivation from microscopic theory of the expression for the current due to the sliding CDW, expressed in terms of microscopic quantities. We can rewrite it as

$$\mathbf{j}(x) = e v_F N_1(0)\eta(T)\hbar\, \partial\phi/\partial t, \tag{13}$$

where $\eta(T)$ is a temperature-dependent coefficient with $\eta(T=0)=1$ and could be identified easily from Eqs. (8) and (10). If we interpret the density of states as $N_1(0) = N/\epsilon_F$ with N being the density of the electrons and ϵ_F the Fermi energy, Eq. (13) has just the same form as Lee and Rice's phenomenological result[7] up to a temperature-dependent correction factor $\eta(T)$.

In our approximation, the GLG equation for the SC order parameter decouples from the phase part of the CDW order parameter; the latter will propagate independently. This decoupling appears to be due to a particular feature of the microscopic model, i.e., the division of the Fermi surface into two regions. Comparing our Eqs. (8) and (9) and those for an ordinary superconductor one sees two important differences: an additional persistent current due to the sliding CDW, and a CDW-induced anisotropy.

Now consider a simple geometry. Take the nesting direction Q in the y direction and the magnetic field in the y-z plane and assume that $\Delta(x)$ and $A(x)$ are functions of x only; then

$$(B_x, B_y, B_z) = [0, \; -\partial A_z(x)/\partial x, \; \partial A_y(x)/\partial x].$$

At the same time, assume that $\partial\phi/\partial t = \text{const}$, which is certainly consistent with the equation for the phase part of the CDW order parameter. Then we have a static magnetic situation. At first we consider the simplest case when the temperature is sufficiently low so that the modulus of the SC order parameter is also frozen. Then only Maxwell's equations need to be taken into consideration. If $\mathbf{B}$ is parallel to the nesting direction, the penetration depth is determined by $n_s^{\mathrm{II}}[T]$ $[n_s^{(\mathrm{I,II})}[T] - Q^2|\Delta|^2\Pi_\phi^{(\mathrm{I,II})}(T)/2m]$ which is the density of pairing electrons contributed by region II. This is essentially the ordinary Meissner effect. If the magnetic field $\mathbf{B}$ is perpendicular to the nesting direction, there will still be the Meissner effect but with different penetration depth which is determined by $n_s^{(\mathrm{I})}[T] + (1-f)n_s^{(\mathrm{II})}[T]$, where $n_s^{(\mathrm{I})}[T]$ has the physical meaning of the density of Cooper pairs in the nesting direction. As a result of the Peierls mechanism it is difficult to pair electrons along the nesting direction.

Furthermore, a test of whether there is direct coupling between phases of SC and CDW order parameters is the existence of a Meissner effect in perpendicular geometry. This would indicate the existence of a bulk pairing supercurrent which cancels the sliding CDW current. Additionally, a temperature-dependent measurement of the two penetration depths (parallel and perpendicular geometry) can give information on $n_s^{(\mathrm{I})}[T]$ and $n_s^{(\mathrm{II})}[T]$, since f is presumably insensitive to temperature.

Now fix $\mathbf{A}(x)$ parallel to the y axis and treat the magnetic field and the SC order parameter consistently. Assuming that $f \approx 0$ the coupled GLG equation and static Maxwell equation can be written as follows:

$$-\frac{1}{k^2}\frac{d^2\psi}{dx^2} + a^2(x)\psi(x) = \psi(x) - |\psi(x)|^2\psi(x), \tag{14}$$

$$\frac{d^2a(x)}{dx^2} - |\psi(x)|^2 a(x) + \tilde{j} = 0, \tag{15}$$

where x is a dimensionless coordinate scaled by the penetration depth λ_L with $\lambda_L^2 = mc^2\beta/4\pi e^2\alpha$, while the dimensionless SC order parameter is

$$\psi(x) = [\beta(\Pi_\phi^{(\mathrm{I})} + \Pi_\phi^{(\mathrm{II})})Q^2/2m\alpha]^{1/2}\Delta(x),$$

and

$$a(x) = \left[\frac{e^2}{2mc^2\alpha}\left(1 + \frac{\Pi_\phi^{(\mathrm{I})}(T)}{\Pi_\phi^{(\mathrm{II})}(T)}\right)\right]^{1/2} A(x),$$

$$\tilde{j} = \frac{\hbar Q|W|^2\beta}{4m^{1/2}\alpha^{3/2}}\Pi_\phi^{(0)}(T)\left(1 + \frac{\Pi_\phi^{(\mathrm{I})}(T)}{\Pi_\phi^{(\mathrm{II})}(T)}\right)^{1/2}\frac{\partial\phi}{\partial t},$$

with $\alpha = [4m\zeta(W)N_1(0)/\Pi_\phi^{(\mathrm{I})}(T)](T - T_c)/T_c$, $\beta = 4m^2/Q^4\Pi_\phi^{(\mathrm{II})}(T)$, and $k^2 = \beta^2m^2c^2/2\pi e^2\hbar^2$.

In this geometry the general tensorial GLG equation due to CDW-induced anisotropy simplifies to a scalar equation in dimensionless variables, with the sliding CDW as a distinguishing feature. A dimensionless free energy,

$$F = F_0 - |\psi|^2 + \tfrac{1}{2}|\psi|^4 + \frac{1}{k^2}\frac{\partial\psi^\dagger}{\partial x}\frac{\partial\psi}{\partial x} + a^2|\psi|^2 + \left(\frac{\partial a}{\partial x}\right)^2 - \tilde{j}a, \tag{16}$$

will give Eqs. (14) and (15).

For the homogeneous case, taking ψ to be real, there exists a critical CDW current $\tilde{j}_{cr} = \frac{4}{27}$, while the corresponding critical value of the SC order parameter is $\psi_{cr}^2 = \frac{2}{3}$. Because the superconducting state remains one of perfect diamagnetism, an induced Cooper-pair current must cancel the extra CDW sliding current. Hence the maximum CDW current corresponds to the minimum value of the order parameter. If we take the critical magnetic field as 10^3 G and the penetration depth as 10^{-5} cm, we estimate the critical CDW current to be 10^5 A/cm^2. For $\tilde{J} < \tilde{J}_{cr}$, two solutions are possible. A stability analysis of the solutions based on the free energy forbids the solution with the smaller value of ψ and only a single solution with the larger ψ is stable. However, the free energy at the critical point [for $\psi_{cr} = (\frac{2}{3})^{1/2}$] is smaller than the free energy for $\psi = 0$ i.e., $F_{cr} = F_{\psi=0} - \frac{1}{9}$. It would be interesting if the unstable solution could be observed as a metastable state.

Now take a semi-infinite sample at $x \geq 0$, in the presence of a dimensionless external magnetic field $h_0 = H_0[1 + \Pi_\psi^{(1)}(T)/\Pi_\psi^{(0)}(T)]^{1/2}/\sqrt{2}H_{cr}$, where $H_{cr} = \alpha(4\pi/\beta)^{1/2}$. We shall solve both Eqs. (14) and (15) consistently. Because of the smallness of $\tilde{J}_{cr}$, we solve them using a perturbation expansion. Assuming that $\tilde{J}$ is of the same order as h_0 we find a new interesting term proportional to $\tilde{J}h_0$.[10] If we change the magnetic field from positive to negative, then the corresponding change in ψ will be

$$\Delta\psi = \frac{2\sqrt{2}\tilde{J}h_0}{(2k^2-1)}\left\{k e^{-x} - \frac{1}{\sqrt{2}} e^{-\sqrt{2}kx}\right\}. \qquad (17)$$

Because of the specific nesting direction of the coexisting CDW, the SC-CDW system shows all the properties of a CDW-induced anisotropic superconductor. In terms of H some of the important consequences of the anisotropy are as follows: (i) The ratio of the upper critical fields is $(H_{c2})_\parallel/(H_{c2})_\perp = (m_\perp/m_\parallel)^{1/2}$, where parallel and perpendicular are with respect to the nesting direction. (ii) The angular dependence of the upper critical field is

$$H_{c2} \propto \left[\frac{\sin^2\phi}{m_\perp} + \frac{\cos^2\phi}{m_\parallel}\right]^{1/2}\left[\frac{\cos^2\theta\sin^2\phi}{m_\parallel} + \frac{\sin^2\theta(\cos^2\phi + 1)}{m_\perp}\right]^{-1/2}.$$

(iii) The diamagnetic susceptibility just above T_c is

$$\chi = \frac{e^2 T_c}{24\pi\hbar c^2(\alpha m_\perp)^{1/2}(T - T_c)^{1/2}}.$$

(iv) The magnetic moment of a thin foil of area A is

$$M = -\frac{A K_B T_c e^2 H}{12\pi m_\perp c^2\sqrt{\alpha}(T - T_c)},$$

where H is the applied magnetic field.

Measurement of these quantities can enable the CDW-induced anisotropic coupling to be determined.

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