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THEORETICAL EXAMINATION OF HIGH ENERGY PARTICLE CORRELATION FUNCTIONS

by

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Abstract

THEORETICAL EXAMINATION OF
HIGH ENERGY PARTICLE CORRELATION FUNCTIONS

by

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We examine the theoretical predictions made for various observables (and particularly, the average multiplicity and integrated correlation functions) by several models of high energy hadron-hadron interactions in Part I.

In Part II, we concentrate on a specific model for the purpose of fitting the experimental data. Experimental results indicate that the two-body correlation function (f_2^{ch}) is positive at laboratory momenta above about 50 GeV/c. To obtain positive f_2, we examine multiperipheral models modified to include more than one different vertex type, with more than one particle emitted at some vertices. We will show that in such models, $f_2 = c_2 \ln(s)$ + constant at high energy, and c_2 can be positive, s being the square of the center of mass energy of the incident particles. Furthermore, $\langle n \rangle$, the average multiplicity, and all correlation functions f_j go as $c_j \ln(s)$ at high s. As a result, the model features a special form of Koba-Nielsen-Olesen (KNO)

scaling.

We have made a numerical analysis of the model for clusters of up to eight particles emitted at the vertices, and we have calculated the average multiplicity, the Mueller two-body correlation function, and the Mueller three-body correlation function. We obtain a favorable, though crude, fit to the experimental data.

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   Note that several of the models listed (one M=3 model, the M=4,5,6, and 7 models, and the first M=8 model) are special in that only one cluster type is allowed, m=M. None of these "single cluster models" is capable of fitting the experimental data for both $\langle n \rangle$ and f_2. The last three models are reasonable fits, and all of them include clusters of 8 particles plus at least one cluster type of smaller size than 8.

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PART I

INTRODUCTION

Correlation functions are experimentally observable quantities for which we can obtain predictions from many high energy models. These functions can thus be used to distinguish between the validity of such models, and focus our attention on a successful class of models. In the present endeavor, we will restrict ourselves primarilly to the average multiplicity and the Mueller two-body and three-body integrated correlation functions, although some discussion of other observables will be found in Appendix A.

Before moving on to the model of our greatest interest (the experimental and theoretical background for which may be found in Ref. 1-5), we will present a survey of various models for hadron-hadron collisions, discussing the general characteristics of their predictions for various observables. Because of the immense quantity and variety of high energy models currently under scruting by the physics community, any survey must be subject to a sizeable set of constraints. It is our intent in this survey to present a reasonable cross section of models which have been exciting people in recent years, particularly those models which are in some way related to the model to be discussed in detail in Part II, either by contrast or by similarity.

Thus, we now present some of the results pertaining to the average multiplicity and the correlation functions of several models in order to emphasize the areas where there is consistency with the experimental data, and also those areas where there are defects in the models. Many of the calculations leading to these results are discussed in considerable detail in Appendix A.

In the diffractive fragmentation model, an impact parameter representation is used to treat the target and projectile as spatially extended objects which "go through" each other with attenuation.[6-11] To obtain the average multiplicity and the correlation functions, it is necessary to assume a definite form for the excitation spectrum of the target and the projectile, as well as other assumptions. For example, Jacob and Slansky make use of Mueller's analysis of inclusive processes in obtaining $\langle n \rangle \sim \ln(s)$ and also $f_n \sim (\sqrt{s})^{n-1}$, as shown in Table 1. We will see that many models, including our own multiperipheral cluster model, provide that $\langle n \rangle \sim \ln(s)$. The result $f_n \sim (\sqrt{s})^{n-1}$ is more unusual. For more details, see Appendix A, Sections 1 and 5, and Figure 1.

The statistical thermodynamic bootstrap model rests on the assumption that the various multiparticle production possibilities are governed only by their statistical weights (See Ref. 12,13,14). Every particle is treated as a "fireball", and each fireball is a statistical equilibrium of all kinds of fireballs. Each constituent fireball is itself such a statistical equilibrium, thus providing the bootstrap principle of the model. A particle observed to be a proton, for example, is a specific state realized by this statistical equilibrium. A mass spectrum is defined to match the experimentally observed particles and resonances up to the limits of current experiments, and a partition function is calculated. A functional expression for the mass spectrum can be inferred from the bootstrap assumption, and a "temperature" can be defined. The mass spectrum is:

$$\rho(m) \rightarrow am^{\frac{-5}{2}} \exp(\frac{m}{T_o}) \quad (1.1)$$

where "a" and "T_o" are constants. It is observed that there is an upper limit to the temperature (T_o) such that any energy added to the system when it has reached this limit (i.e., when the temperature T is at T_o) is used for particle creation. Near this limit, the transverse momentum distribution is of the Boltzmann type:

$$w(p_\perp) \approx c \cdot p_\perp^{\frac{3}{2}} \cdot \exp(\frac{-p_\perp}{T_o}) , \quad (1.2)$$

where "c" is a constant. Finally, the average multiplicity goes as ln(s), as in the previous model, but the statistical independence of the events leads to a Poisson distribution, where $f_2=0$. This result contradicts experimental data for pp-scattering above laboratory momenta of about 50 GeV/c, where $f_2>0$. For the derivations of many of these results, see Appendix A, Section 2.

The Chew-Pignotti multiperipheral bootstrap model consists of the simple AFS model, with the resonant amplitudes governed by Regge theory.[15,16,17] An assumption is made regarding the momentum transfer dependence of these amplitudes (See (A.39)). If all side particles of the chain are meson trajectories, the results $\langle n\rangle \sim \ln(s)$ and $f_2=0$ are again obtained. More details may be found in Appendix A, Section 3, and Figures 2-5.

Caneschi and Schwimmer show that the addition of absorption to the multiperipheral bootstrap model improves the self-consistency of the model (and satisfies the Froissart bound), and provides a rather interesting two-body correlation function.[17-21] They find $\langle n\rangle \sim \ln(s)$ and $f_2 \sim (\ln s)^2$. Indeed, they show that the leading term of f_n is $(\ln s)^n$. This behavior may be suggested by recent experimental data. (See Ref. 1 and 5). These results are found in Appendix A, Section 4.

Mueller has developed a technique for dealing with inclusive processes by generalizing the optical theorem to three-body reactions.[22,23,24] The one-particle inclusive distribution ρ_{AB}^{C} equals the product of $(1/s)$ and the imaginary part of the forward three-body elastic amplitude for $A\bar{C}B \rightarrow A\bar{C}B$. The Mueller treatment features scaling, limiting fragmentation, and the existence of a central plateau in the single particle distribution function vs. rapidity (See Figure 6). Finally, the result

$$\langle n\rangle \sigma_{AB} \gtrsim \ln(s), \tag{1.3}$$

(where "$\gtrsim$" here means "rises no slower than")
which arises from the Mueller analysis and the assumption $\alpha(t) \cong 1$, indicates that if σ_{AB} is constant at high energies, then $\langle n\rangle \gtrsim \ln(s)$. If σ_{AB} increases no slower than

ln(s), this behavior of $\langle n \rangle$ is still possible, but it is not necessary. Further details are contained in Appendix A, Section 5, and Figures 6-11.

The above results are summarised in Table 1, which, for completeness, also includes results of the simple AFS model, and those of our cluster model of Part II.

Having established a background for our investigation of high energy hadron-hadron interactions in the form of this series of models, we will proceed to examine a model which we believe to be a promising one. The motivation for this choice follows.

Data available at the present time[1] for high energy multi-particle production seems to favor two main characteristics. First, the increase in multiplicity with s is slow, probably

$$\langle n \rangle \sim \ln(s),$$

where $\langle n \rangle$ is the mean number of particles produced at a center of mass energy whose square is s.

Secondly, the distribution of transverse momenta of the produced particle is bounded by about 350 MeV/c and is rather insensitive to s. These characteristics are well described by multiperipheral-type models[2].

More recent analyses of the data along lines suggested by Mueller[3] have shown that the two-body integrated correlation function, $f_2 = \langle n(n-1) \rangle - \langle n \rangle^2$, is positive at 200 GeV/c lab momentum, while the simple multiperipheral model leads to f_2 zero or negative.

We are thus led to choose a multiperipheral model which would retain the first two characteristics regarding multiplicity and transverse momentum, but one which is modified to provide positive f_2 at high energies. To devise such a model, we consider a multiperipheral model in which clusters[4] of $1, 2, \ldots, M$ particles are produced in arbitrary number and arbitrary order along the multiperipheral chain. We will show that we can obtain an integral equation for the forward absorptive amplitude for this model which is identical to the Amati-Fubini-Stanghellini (AFS) equation, where the resonant amplitude is replaced by the sum of the m-particle cluster resonant amplitudes. We derive this equation by following the procedure set up by AFS for the original model, and we perform the first iteration of the equation. From this first iteration, we calculate the eigenvalue equation relating the coupling constants to one another. We will proceed to arrive at expressions for the average multiplicity and the correlation functions.

We will then display the results of numerical calculations of the multiplicity and correlation functions, demonstrating that f_2 can be made positive by including the new vertices. We observe that this is not possible using a ρ-dominance model in which only ρ-type clusters, of any mass, are produced along the multiperipheral chain (See Appendix D).

Since the integral equation which we obtain is essentially that of the original AFS model, the AFS conclusions concerning the asymptotic form of the solution, the multiplicity dependence on energy, leading particle effect, etc., are shown to hold immediately for our model of an arbitrary number, type and order of clusters.

In particular, we note that since

$$f_k \xrightarrow[s\to\infty]{} c_k \ln(s),$$

we have

$$\frac{\langle n^q\rangle}{\langle n\rangle^q} = 1 + \mathcal{O}\left(\frac{1}{\ln(s)}\right) ,$$

which exhibits a special form of Koba-Nielsen-Olesen (KNO) scaling,[5] a point which we will return to at a later time in this study.

PART II

A MULTIPERIPHERAL CLUSTER MODEL

1. THE INTEGRAL EQUATION AND THE FIRST ITERATION

As explained in the Introduction, we will follow the procedure of Amati, Fubini and Stanghellini in their analysis of the simple multiperipheral model[2] and adapt the model to include clusters of different multiplicities. Thus, our model also consists exclusively of identical scalar particles.

Consider the invariant amplitude corresponding to the inelastic diagram of Figure 12a):

$$T(pp', s_o \dots s_{N-1}, m_1 \dots m_N)$$
$$= \frac{T^R_{m_1}(-pq_1, s_o) \ \dots \ T^R_{m_N}(q_{N-1}p', s_{N-1})}{(q_1{}^2 - \mu^2) \ \dots \ (q_{N-1}{}^2 - \mu^2)} \tag{2.1}$$

where $T^R_m(qq', s_i)$ is the invariant amplitude for particles of momenta q, $-q'$, interacting to form a cluster of type m, μ is the particle mass, m_j denotes the type of vertex present in each link j of the chain, $s_j = (q_{j+1} - q_j)^2$ with $q_o = -p$ and $q_N = p'$, and where N is the total number of clusters in the diagram. The cross section corresponding to Figure 12a) is thus:

$$\sigma(s, m_1 \dots m_N) = \frac{1}{\Delta^{\frac{1}{2}}(s, p^2, p'^2)} \int \dots \int \frac{d^4q_1 \dots d^4q_{N-1}}{(8\pi^4)^{N-1}}$$
$$\times \ \frac{\Delta^{\frac{1}{2}}(s_o, q_o{}^2, q_1{}^2)\,\sigma^R_{m_1}(s_o) \dots \Delta_{N-1}, q_{N-1}{}^2, q_N{}^2)\,\sigma^R_{m_N}(s_{N-1})}{(q_1{}^2 - \mu^2)^2 \ \dots \ (q_{N-1}{}^2 - \mu^2)^2}$$

where Δ is the well known triangle function,

$\Delta(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$,

where $\sigma^R_{m_j}(s_{j-1})$ is the cross section for the production of a cluster of type m_j and center of mass energy $\sqrt{s_{j-1}}$, and $s = (p+p')^2$. We now apply the optical theorem to each link of the chain and to the chain as a whole:

$$\Delta^{\frac{1}{2}}(s_{j-1}, q_{j-1}^2, q_j^2)\,\sigma^R_{m_j}(s_{j-1}) = A^R_{m_j}(s_{j-1})$$

$$\Delta^{\frac{1}{2}}(s, p^2, p'^2)\,\sigma(s, m_1 \ldots m_N) = A(pp',\ m_1 \ldots m_N).$$

We thus obtain an expression for the absorptive part of the elastic diagram (Figure 13):

$$A(pp', m_1 \ldots m_N) = \int \ldots \int \frac{d^4q_1 \ldots d^4q_{N-1}}{(8\pi^4)^{N-1}}\ \frac{A^R_{m_1}(s_o) \ldots A^R_{m_N}(s_{N-1})}{(q_1{}^2-\mu^2)^2 \ldots (q_{N-1}{}^2-\mu^2)^2} \tag{2.2}$$

where $A^R_{m_j}(s_{j-1})$ represents the absorptive part of the elastic diagram of Figure 14. Note that $A(pp', m_1 \ldots m_N)$ corresponds to a particular ordering of vertices along the multiperipheral chain.

It is useful to note here two distinctions in notation between our treatment and the original AFS[2] treatment. Firstly, we are using the particle multiplicities rather than the number of clusters to index our amplitudes, since our theory features clusters of differing cluster multiplicities, unlike the original AFS model. Secondly, we have written

$\sigma(s,m_1 \ldots m_N)$ in terms of triangle functions so as to present an invariant form, whereas AFS[2] wrote this cross section in the center of mass system. Since the intermediate links of the chain are off-shell, we believe the more general triangle form to be more appropriate.

We will now introduce that feature of our calculation which permits us to apply the AFS treatment to a model with clusters of different multiplicities. We have a counting problem, since at each level of cluster production, permutations of the positions of clusters of different types make different contributions to the elastic amplitude. We must adopt a notation to handle this problem. Let n_m be the number of m-type clusters along the chain. We seek a recursion relation for the elastic diagrams, and so we shall first sum over permutations of the vertex types $m_1 . m_2, \ldots, m_N$, while holding $n_1, n_2, \ldots,$ constant, forming the subsum of diagrams:

$$A^s(pp', n_1 n_2 \ldots) = \sum_{\substack{\text{permutations} \\ \text{of } m_1 \ldots m_N}} A(pp', m_1 m_2 \ldots m_N) \ .$$

We will write a recursion relation for the A^s. Consider the expression:

$$\int \frac{d^4q}{8\pi^4(q^2-\mu^2)^2} \Big[A^s(pq; n_1-1, n_2, n_3, \ldots) \; A_1^R((q-p')^2) + A^s(pq; n_1, n_2-1, n_3, \ldots) \; A_2^R((q-p')^2) + \ldots \Big] .$$

Note that, as a consequence of (2.2), the first term of this expression is an N-cluster amplitude where the first N-1 vertices have been symmetrized and the last cluster is of type 1, the second term is an N-cluster amplitude where the first N-1 vertices have been symmetrized and the last cluster is of type 2, etc. Hence, the total expression represents an N-cluster amplitude where all N clusters have been symmetrized, and so we have the recursion relation:

$$A^{S}(pp';n_1,n_2,\dots)$$

$$=\int\frac{d^4q}{8\pi^4(q^2-\mu^2)^2}\Big[A^{S}(pq;n_1-1,n_2,\dots)\ A_1^{R}((q-p')^2)$$

$$+\ A^{S}(pq;n_1,n_2-1,\dots)\ A_2^{R}((q-p')^2)+\dots\Big]. \tag{2.3}$$

To obtain the absorptive forward elastic amplitude $A(p,p')$, we need only sum the subsums over all values of the numbers of clusters $n_1,n_2,\dots$:

$$A(p,p') = \sum_{n_1 n_2\dots} A^{S}(pp';n_1,n_2,\dots).$$

If we now sum (2.3) over all values of $n_1,n_2,\dots$, we obtain the AFS integral equation for the total absorptive amplitude:

$$A(p,p') = \int\frac{d^4q}{8\pi^4(q^2-\mu^2)^2}\ A(p,q)\Big[\sum_m A_m^{R}((q-p')^2)\Big]$$

where the kernel has been modified (relative to the AFS form) through the replacement of A^R by the sum $\sum_m A_m^R$, and since we shall be interested in asymptotic results, we have omitted the inhomogeneous term. Our result is clearly valid for an arbitrary number of types of clusters.

The remainder of the calculation leading to the eigenvalue equation parallels the original AFS procedure[2], with the replacement mentioned above. We extend the incident pion out from the mass shell, $p'^2 \equiv -u$:

$A(p,p') \equiv \iint A(s,u)\ \delta(s-(p+p')^2)\ \delta(u+p'^2)\ ds\ du.$

As did AFS[2], we make use of the function Q:

$Q(s,u;s',u';s_o) \equiv \int d^4p''\ \delta[(p'-p'')^2-s_o]\ \delta[(p''+p)^2-s]\ \delta(p''^2+u').$

Our integral equation may now be written:

$$A(s,u) = \int \frac{ds_o}{8\pi^4}\left[\sum_m A_m^R(s_o)\right]\iint ds'du' \frac{Q(s,u;s',u';s_o)\ A(s',u')}{(u'+\mu^2)^2} .$$

At high s, this becomes:

$$A(s,u) = \frac{1}{s}\int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right]\int_{s'_{min}}^{s'_{max}} ds' \int_{u'_{min}}^{u'_{max}} du'\ \frac{A(s',u')}{(u'+\mu^2)^2}$$

where Q has provided the limits,

$$u'_{\substack{max\\min}} = \frac{s-s'}{2} \pm \sqrt{\left(\frac{s-s'}{2}\right)^2 - s's_o - \frac{s'}{s}u(s-s')}$$ and s'_{max} is given

by the vanishing of this radical, so $s'_{max} \approx s-2\sqrt{ss_o}$. We note that except for very high s', $u'_{max} \sim s-s'$ and $u'_{min} \sim x(u + \frac{s_o}{1-x})$. Since $s_o \ll s$, $s'_{max} \approx s-2\sqrt{ss_o} \approx s$. The presence of $(u'+\mu^2)^{-2}$ indicates that only small u' contributes, so $u'_{max} = s-s' \rightarrow \infty$ is allowable, since the error incurred is neglegible. Furthermore, $s'_{min} \approx 0$. Thus:

$$A(s,u) = \frac{1}{s}\int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right]\int_o^s ds' \int_{\frac{s'}{s}(u+\frac{s_o}{1-s'/s})}^{\infty} du' \; \frac{A(s',u')}{(u'+\mu^2)^2} \;. \tag{2.4}$$

Since this is the AFS equation with one function of s_o replaced by a sum of such functions, the kernel is still invariant under the scale transformation $s \rightarrow cs$, $s' \rightarrow cs'$, as was the original equation. The energy dependence of the elastic amplitude must therefore be:

$$A(s,u) = s^{\alpha}\, \varphi_{\alpha}(u) \;. \tag{2.5}$$

This form in (2.4) provides us with an integral equation for $\varphi_{\alpha}(u)$:

$$\varphi_{\alpha}(u) = \int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right]\int_o^1 dx \int_{x(u+\frac{s_o}{1-x})}^{\infty} du' \; \frac{x^{\alpha}\, \varphi_{\alpha}(u')}{(u'+\mu^2)^2}$$

where $x \equiv s'/s$. Consider the integration over x:

$$I \equiv \int_o^1 dx\; x^{\alpha} \int_{x(u+\frac{s_o}{1-x})}^{\infty} du' \; \frac{\varphi_{\alpha}(u')}{(u'+\mu^2)^2} \;.$$

Integrate by parts over x:

$$I = \left[\frac{x^{\alpha+1}}{\alpha+1}\int_{u'_{min}}^{\infty} du' \frac{\varphi_\alpha(u')}{(u'+\mu^2)^2}\right]_{x=0}^{x=1} - \int_0^1 dx\, \frac{x^{\alpha+1}}{\alpha+1}\frac{d}{dx}\int_{u'_{min}}^{\infty} du'\, \frac{\varphi_\alpha(u')}{(u'+\mu^2)^2},$$

where $u'_{min} \equiv x(u+\frac{s_o}{1-x})$. At x=1, the limits of u' in the surface term are equal. At x=0, since the integral

$$\int du'\, \frac{\varphi_\alpha(u')}{(u'+\mu^2)^2}$$

is well-behaved at positive u', there is no surface contribution. We are then left with:

$$\begin{aligned} I &= - \int_0^1 dx\, \frac{x^{\alpha+1}}{\alpha+1}\, \frac{d}{dx} \int_{u'_{min}}^{\infty} du'\, \frac{\varphi_\alpha(u')}{(u'+\mu^2)^2} \\ &= - \int_0^1 dx\, \frac{x^{\alpha+1}}{\alpha+1} \left(\frac{du'}{dx}\,\frac{d}{du'}\int du'\, \frac{\varphi_\alpha(u')}{(u'+\mu^2)^2}\right)_{u'=u'_{min}} \\ &= - \int_0^1 dx\, \frac{x^{\alpha+1}}{\alpha+1} \left(u+\frac{s_o}{1-x} + \frac{xs_o}{(1-x)^2}\right) \frac{\varphi_\alpha\left[x(u+s_o/(1-x))\right]}{\left[x(u+\frac{s_o}{1-x})+\mu^2\right]^2}. \end{aligned}$$

We now transform back to $u' = x(u+\frac{s_o}{1-x})$, with our new limits $u'(x=0) = 0$ and $u'(x=1) = \infty$:

$$I = \int_0^{\infty} du'\, \frac{x^{\alpha+1}}{\alpha+1}\, \frac{\varphi_\alpha(u')}{(u'+\mu^2)^2}.$$

We solve $u'=x(u+\frac{s_o}{1-x})$ for its roots in x and obtain:

$$x = \frac{s_o + u + u' \pm \sqrt{(s_o+u+u')^2 - 4uu'}}{2u}.$$

Thus, our integration is:

$$I = \int_0^{\infty} \frac{du'}{\alpha+1} \left[\frac{s_o+u+u' \pm \sqrt{(s_o+u+u')^2-4uu'}}{2u} \right]^{\alpha+1} \frac{\Psi_\alpha(u')}{(u'+\mu^2)^2} .$$

Since $x = s'/s$ lies between 0 and 1, we choose the minus sign to prevent $x > 1$. Furthermore,

$$x = \frac{s_o+u+u' - \sqrt{(s_o+u+u')^2-4uu'}}{2u} = \frac{2u'}{s_o+u+u' + \sqrt{(s_o+u+u')^2-4uu'}} .$$

The integral equation for $\Psi_\alpha(u)$ is thus:

$$\Psi_\alpha(u) = \frac{1}{16\pi^3(1+\alpha)} \int ds_o \left[\sum_m A_m^R(s_o) \right]$$
$$\times \int_0^{\infty} du' \left[\frac{2u'}{s_o+u+u' + \sqrt{(s_o+u+u')^2-4uu'}} \right]^{\alpha+1} \frac{\Psi_\alpha(u')}{(u'+\mu^2)^2} . \quad (2.6)$$

To solve the equation and obtain the eigenvalue equation, we adopt the spectral representation introduced by Ceolin, et. al.[25]:

$$\frac{\Psi_\alpha(u)}{(u+\mu^2)^2} = \int \frac{\rho(a)}{u+a} \, da . \quad (2.7)$$

The integral equation for Ψ_α becomes an integral equation for $\rho(a)$; we revert back to the form:

$$\Psi_\alpha(u) = \int \frac{ds_o}{16\pi^3} \left[\sum_m A_m^R(s_o) \right] \int_0^1 dx \, x^\alpha \int_{x(u+\frac{s_o}{1-x})}^{\infty} du' \frac{\Psi_\alpha(u')}{(u'+\mu^2)^2} .$$

This with our definition of $\rho(a)$ becomes:

$$\int \frac{\rho(a)}{u+a}\, da = \int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right] \int_o^1 \frac{dx\; x^\alpha}{(u+\mu^2)^2} \int_{x(u+\frac{s_o}{1-x})}^{\infty} du' \int \frac{\rho(a')}{u'+a'}\, da'.$$

Now $\displaystyle\int_{x(u+\frac{s_o}{1-x})}^{\infty} du' \;\frac{1}{u'+a'} = \int_o^{\infty} dy\; \frac{1}{u'+a'+y}\Bigg|_{u'=x(u+\frac{s_o}{1-x})}$,

and so we may write:

$$\int \frac{\rho(a)}{u+a}\, da = \int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right] \int_o^1 \frac{dx\; x^\alpha}{(u+\mu^2)^2} \int_o^{\infty} dy \int \frac{\rho(a')}{x(u+\frac{s_o}{1-x})+a'+y}\, da'$$

or simply:

$$\int \frac{\rho(a)}{u+a}\, da = \int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right] \int_o^1 \frac{dx\; x^{\alpha-1}}{(u+\mu^2)^2} \int_o^{\infty} dy \int \frac{\rho(a')}{u+\frac{a'+y}{x}+\frac{s_o}{1-x}}\, da'.$$

Now we make use of the fact that:

$$\frac{1}{u+\frac{a'+y}{x}+\frac{s_o}{1-x}} = \int \frac{da}{u+a}\quad \delta\left[a - \left(\frac{a'+y}{x}\right) - \frac{s_o}{1-x}\right],\ \text{so that:}$$

$$\int \frac{\rho(a)}{u+a}\, da = \frac{1}{(u+\mu^2)^2} \int \frac{da}{u+a} \int da'\, \rho(a')$$

$$\times \left[\int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right] \int_o^1 dx\; x^{\alpha-1} \int_o^{\infty} dy \quad \delta\left[a-\left(\frac{a'+y}{x}\right)-\frac{s_o}{1-x}\right]\right]$$

which may be written as:

$$\int \frac{\rho(a)}{u+a}\, da = \frac{1}{(u+\mu^2)^2} \int \frac{da}{u+a} \int K(a,a')\; \rho(a')\; da'$$

where:

$$K(a,a') = \int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right]\int_o^1 dx\ x^{\alpha-1}\int_o^\infty dy\ \delta\left[a-\left(\frac{a'+y}{x}\right)-\frac{s_o}{1-x}\right].$$

Performing the integration over y yields:

$$K(a,a') = \int \frac{ds_o}{16\pi^3}\left[\sum_m A_m^R(s_o)\right]\int_o^1 dx\ x^{\alpha}\ \theta\left(a-\frac{a'}{x}-\frac{s_o}{1-x}\right).$$

Return to the integral equation for $\rho(a)$ itself:

$$\int \frac{\rho(a)}{u+a}\, da = \frac{1}{(u+\mu^2)^2}\int \frac{da}{u+a}\int K(a,a')\ \rho(a')\ da'.$$

Note that:

$$\frac{1}{(u+\mu^2)^2(u+a)} = \frac{1}{(u+a)(a-\mu^2)^2} + \frac{1}{(u+\mu^2)^2(a-\mu^2)} - \frac{1}{(u+\mu^2)(a-\mu^2)^2}$$

$$= \frac{1}{(u+a)(a-\mu^2)^2} + \int \frac{d\bar{a}\ \delta'(\bar{a}-\mu^2)}{(u+\bar{a})(a-\mu^2)} - \frac{1}{(u+\mu^2)(a-\mu^2)^2}$$

$$= \frac{1}{(u+a)(a-\mu^2)^2} + \int \frac{d\bar{a}\ \delta'(\bar{a}-\mu^2)}{(u+\bar{a})(a-\mu^2)} - \int \frac{d\bar{a}\ \delta(\bar{a}-\mu^2)}{(u+\bar{a})(a-\mu^2)^2}$$

where the first step results from the fact that

$$\int \delta'(x-x_o)f(x)dx = -f'(x_o),$$

the prime denoting differentiation with respect to the argument. The integral equation thus becomes:

$$\int \frac{\rho(a)}{u+a}da = \int da\left[\frac{1}{(u+a)(a-\mu^2)^2} + \int \frac{d\bar{a}\ \delta'(\bar{a}-\mu^2)}{(u+\bar{a})(a-\mu^2)} - \int \frac{d\bar{a}\ \delta(a-\mu^2)}{(u+\bar{a})(a-\mu^2)^2}\right]$$

$$\times \int K(a,a')\ \rho(a')\ da'.$$

Replacing the dummy variable of integration (a) in the first term on the right by $\bar{a}$, we obtain:

$$\int \frac{\rho(a)}{u+a} da = \int \frac{d\bar{a}}{u+\bar{a}} \Big[\delta'(\bar{a}-\mu^2) \iint \frac{K(a,a')}{a-\mu^2} \rho(a')\, da\, da'$$

$$- \delta(\bar{a}-\mu^2) \iint \frac{K(a,a')}{(a-\mu^2)^2} \rho(a')\, da\, da'$$

$$+ \frac{1}{(\bar{a}-\mu^2)^2} \int K(\bar{a},a')\, \rho(a')\, da' \Big] .$$

The expression in brackets must be $\rho(\bar{a})$:

$$\rho(\bar{a}) = \delta'(\bar{a}-\mu^2) \iint \frac{K(a,a')}{a-\mu^2} \rho(a')\, da\, da'$$

$$- \delta(\bar{a}-\mu^2) \iint \frac{K(a,a')}{(a-\mu^2)^2} \rho(a')\, da\, da'$$

$$+ \frac{1}{(\bar{a}-\mu^2)^2} \int K(\bar{a},a')\, \rho(a')\, da' .$$

Define the constants:

$$N_1 \equiv \iint \frac{K(a,a')}{(a-\mu^2)} \rho(a')\, da\, da' ,$$

$$N_2 \equiv \iint \frac{K(a,a')}{(a-\mu^2)^2} \rho(a')\, da\, da' .$$

Thus, we may write:

$$\rho(a) = N_1 \delta'(a-\mu^2) - N_2 \delta(a-\mu^2) + \frac{1}{(a-\mu^2)^2} \int K(a,a')\, \rho(a')\, da'.$$

We choose to take $N_1=1$, so that:

$$\rho(a) = \delta'(a-\mu^2) - \delta(a-\mu^2)\int\frac{K(\bar{a},a')}{(\bar{a}-\mu^2)^2}\ \rho(a')\ d\bar{a}\ da'$$

$$+ \frac{1}{(a-\mu^2)^2}\int K(a,a')\ \rho(a')\ da' \tag{2.8}$$

As a first approximation, we will take:

$$\rho_0(a) \equiv \delta'(a-\mu^2).$$

Our choice for N_1 then gives the eigenvalue equation:

$$1 = \iint\frac{K(a,a')}{a-\mu^2}\ \rho(a')\ da\ da' \approx \iint\frac{K(a,a')}{a-\mu^2}\ \delta'(a-\mu^2)\ da\ da'.$$

Integrate by parts over a', noting that $\delta(a'-\mu^2) = 0$ at $\pm\infty$:

$$1 = -\iint\frac{K'(a,a')}{a-\mu^2}\ \delta(a'-\mu^2)\ da\ da'$$

where $K'(a,a') \equiv \dfrac{\partial K(a,a')}{\partial a'}$. Therefore:

$$1 = -\int\frac{K'(a,\mu^2)}{a-\mu^2}\ da$$

where $K'(a,\mu^2) \equiv \left(\dfrac{\partial K(a,a')}{\partial a'}\right)_{a'=\mu^2}$.

Consider $K'(a,\mu^2)$, referring back to our equation for K:

$$K(a,a') = \int \frac{ds_o}{16\pi^3} \left[\sum_m A_m^R(s_o) \right] \int_o^1 dx \; x^{\alpha} \, \Theta(a - \frac{a'}{x} - \frac{s_o}{1-x})$$

$$K'(a,a') = -\int \frac{ds_o}{16\pi^3} \left[\sum_m A_m^R(s_o) \right] \int_o^1 dx \; x^{\alpha-1} \, \delta(a - \frac{a'}{x} - \frac{s_o}{1-x})$$

$$K'(a,\mu^2) = -\int \frac{ds_o}{16\pi^3} \left[\sum_m A_m^R(s_o) \right] \int_o^1 dx \; x^{\alpha-1} \, \delta(a - \frac{\mu^2}{x} - \frac{s_o}{1-x}).$$

This in our cigenvalue equation provides:

$$1 = -\int \frac{da}{a-\mu^2} \left[-\int \frac{ds_o}{16\pi^3} \left[\sum_m A_m^R(s_o) \right] \int_o^1 dx \; x^{\alpha-1} \, \delta(a - \frac{\mu^2}{x} - \frac{s_o}{1-x}) \right] .$$

Use of the delta function gives us:

$$1 = \int \frac{ds_o}{16\pi^3} \left[\sum_m A_m^R \; (s_o) \right] \int_o^1 dx \; x^{\alpha-1} \; (\frac{\mu^2}{x} + \frac{s_o}{1-x} - \mu^2)^{-1}$$

or more simply:

$$1 = \int \frac{ds_o}{16\pi^3} \left[\sum_m A_m^R(s_o) \right] \int_o^1 dx \; \frac{x^{\alpha}(1-x)}{s_o x + \mu^2(1-x)^2} \quad . \qquad (2.9)$$

This first iteration of the eigenvalue equation will be used to relate the coupling constants to each other (and to alpha) and to calculate the average multiplicity and the correlation functions.

2. COUPLING CONSTANTS, MULTIPLICITY AND CORRELATION FUNCTIONS

The method for calculating the correlation functions used here is due in large measure to Mueller[3]. It has the advantage of compact notation, and thereby simplifies any numerical analysis. In Appendix **C**, we present an alternate (but equivalent) method, which is an extension of the AFS[2] multiplicity calculation to higher order correlation functions.

Let us denote the normalized cross sections for producing n_m type m clusters of m particles each by:

$$\frac{\sigma^s_{n_1 n_2 \ldots}}{\sigma_{total}} \quad .$$

Let us now replace the coupling constants $g_m{}^2$ by $g_m{}^2(1+h)^m$. Then clearly, every $\mathcal{N} = \sum_m m n_m$ particle cross section is replaced by:

$$\sigma^s_{n_1 n_2 \ldots} \rightarrow (1+h)^{\mathcal{N}} \, \sigma^s_{n_1 n_2 \ldots} \equiv \sigma^s_{n_1 n_2 \ldots}(h) \qquad (2.10)$$

where it is understood that h→0 at the end of the calculation, and the multiplicity is:

$$\langle n \rangle = \frac{1}{\sigma_{tot}} \sum_{\mathcal{N}} \frac{\partial}{\partial h} \, \sigma^s_{n_1 n_2 \ldots}(h) = \frac{\partial \alpha}{\partial h} \ln s + \frac{\partial \ln \varphi_\alpha}{\partial h} . \qquad (2.11)$$

Asymptotically, in accordance with Ref. **3**:

$$\langle n\rangle \rightarrow \frac{\partial\alpha}{\partial h}\ \ln(s) + b\ , \tag{2.12}$$

$$f_k \rightarrow \frac{\partial^k\alpha}{\partial h^k}\ \ln(s) + b_k\ . \tag{2.13}$$

Note that in general, multiperipheral models do not yield Poisson distributions ($f_2=0$).

We now define:

$$F_m^{(k)} \equiv \int\frac{ds_o}{16\pi^3}\ A_m^R(s_o)\int_o^1 dx\ \frac{x^{\alpha(h)}\ (1-x)\ (\ln x)^k}{s_o x + \mu^2(1-x)^2}\ , \tag{2.14}$$

$$\phi_\nu^{(k)} \equiv \sum_m m(m-1)\ldots(m-\nu+1)F_m^{(k)}(1+h)^{m-\nu} \tag{2.15}$$

where $\phi_o^{(o)} \equiv \sum_m F_m^{(o)}$ **, and**

where we shall take h=0 at the end of the calculation. The eigenvalue equation (2.9) now has the form:

$$1 = \phi_o^{(o)}\ . \tag{2.16}$$

It further follows that:

$$\frac{\partial F_m^{(k)}}{\partial h} = F_m^{(k+1)}\left(\frac{\partial\alpha}{\partial h}\right)$$

and so:

$$\frac{\partial\phi_\nu^{(k)}}{\partial h} = \phi_{\nu+1}^{(k)} + \phi_\nu^{(k+1)}\frac{\partial\alpha}{\partial h}\ . \tag{2.17}$$

Differentiating the eigenvalue equation(2.16) implicitly with respect to alpha gives, asymptotically:

$$\frac{\langle n \rangle}{\ln(s)} = \frac{\partial \alpha}{\partial h} = - \frac{\phi_1^{(o)}}{\phi_o^{(1)}} \quad , \tag{2.18}$$

and differentiation of this yields:

$$\frac{f_2}{\ln(s)} = \frac{\partial^2 \alpha}{\partial h^2} = \frac{1}{(\phi_o^{(1)})^2} \left[\left(\phi_1^{(o)} \phi_o^{(2)} - \phi_o^{(1)} \phi_1^{(1)} \right) \frac{\partial \alpha}{\partial h} + \left(\phi_1^{(o)} \phi_1^{(1)} - \phi_o^{(1)} \phi_2^{(o)} \right) \right] . \tag{2.19}$$

Differentiation of (2.16) three times obtains:

$$\frac{f_3}{\ln(s)} = \frac{\partial^3 \alpha}{\partial h^3} = \frac{-1}{\phi_o^{(1)}} \left[\phi_3^{(o)} + 3 \phi_1^{(1)} \frac{\partial^2 \alpha}{\partial h^2} + 3 \phi_o^{(2)} \frac{\partial \alpha}{\partial h} \frac{\partial^2 \alpha}{\partial h^2} + 3 \phi_1^{(2)} \left(\frac{\partial \alpha}{\partial h} \right)^2 + 3 \phi_2^{(1)} \frac{\partial \alpha}{\partial h} + \phi_o^{(3)} \left(\frac{\partial \alpha}{\partial h} \right)^3 \right] . \tag{2.20}$$

We may continue this process, obtaining expressions for $(f_k/\ln(s))$ in terms of lower order f_k and an additional integral.

We must now consider what form of resonant amplitudes $A_m^R(s_o)$ we will use in (2.18), (2.19) and (2.20).

3. RESONANT AMPLITUDES

For the production of an m-particle cluster along the chain, we have chosen a vertex corresponding to an m+2-point interaction with a cutoff. We observe that the coupling constant, multiplicity and the two and three body correlation functions are reasonably independent of the cutoff (See Figure 15), **when $s_{o_{max}}$ is near $200(GeV)^2$.**

For the cases m=1 and m=2, we will find analytic expressions using Feynman Rules. We will also show that these expressions may be obtained by invoking unitarity through the optical theorem. For m=2, we will adopt our cutoff.

For m=3, we will use Feynman Rules to write $A_3^R(s_o)$ in terms of a single integration, and perform a numerical analysis to approximate the function. We will show that the result is really just the asymptotic form of three-body phase space for $s_o \gg 9\mu^2$, whether calculated directly or by the optical theorem.

For $m > 3$, we will take the asymptotic form of m-body phase space:

$$A_m^R(s_o) = g_m{}^2 \left(\frac{s_o}{\mu^2}\right)^{m-2} , \quad s_o \leq s_{o_{max}} \tag{2.21}$$

where a numerical factor has been absorbed into the coupling constant.

(i). Calculation of $A_1^R(s_o)$.

a). Direct Calculation of $A_1^R(s_o)$.

Consider the case of two particles of momenta p,q, interacting to form a virtual particle of momentum p+q and center of mass energy squared $s_o = (p+q)^2$, which decays into two particles of momenta P=p and Q=q. The S-matrix by Feynman rules is:

$$S = \frac{(-ig)^2}{\sqrt{(2V)^4 \omega_p^2 \omega_q^2}} \frac{i}{2\pi V(s_o - \mu^2)} (2\pi V)^2 \delta^4(p+q-P-Q)$$

where the interaction has been taken as $g\varphi^3$, and

$$S = 1+R = 1 + i(2\pi V)\delta^4(p+q-P-Q) \frac{M}{\sqrt{(2V)^4 \omega_p^2 \omega_q^2}}$$

so that $\mathrm{Im}(M) = 2|\vec{p}|\sqrt{s_o}\,\sigma(\succ)$. We obtain:

$$M = \frac{g^2}{\mu^2 - s_o}$$

for the invariant matrix element. The absorptive part is:

$$A_1^R(s_o) = \mathrm{Im}(M) = \mathrm{Im}\left(\frac{g^2}{\mu^2 - s_o}\right) = \mathrm{Im}\left(\frac{g^2 P}{\mu^2 - s_o} + i\pi g^2 \delta(s_o - \mu^2)\right)$$

$$= \pi g^2 \delta(s_o - \mu^2).$$

More explicitly,

$$\mathrm{Im}\left(\frac{1}{\mu^2-s_o-i\epsilon}\right) = \mathrm{Im}\,\frac{\mu^2-s_o+i\epsilon}{(\mu^2-s_o)^2+\epsilon^2} = \frac{\epsilon}{(\mu^2-s_o)^2+\epsilon^2} \to \begin{cases} 0 \text{ if } s_o \neq \mu^2. \\ \infty \text{ if } s_o = \mu^2. \end{cases}$$

Furthermore,

$$\int_{-\infty}^{\infty} ds_o\ A_1^R(s_o)\ f(s_o) = \lim_{\epsilon \to o} g^2 \int_{-\infty}^{\infty} \frac{d(\frac{y}{\epsilon})}{(\frac{y}{\epsilon})^2 + 1}\ f(y+\mu^2)$$

$$= \lim_{\epsilon \to o} g^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\ f(\epsilon \tan\theta + \mu^2) = g^2 \pi f(\mu^2)$$

where we analytically continued s_o to negative values, and we defined $s_o \equiv y+\mu^2$ and $\frac{y}{\epsilon} \equiv \tan\theta$. Thus:

$$A_1^R(s_o) = \pi g^2 \delta(s_o-\mu^2).$$

The coupling constant "g" arises from taking the interaction $g\varphi^3$ and has units of mass (the Lagrangian density $\mathcal{L}$ is $\mathcal{L} \sim \frac{L}{d^3x} \sim Lk^3 \sim k^4$ and $\varphi \sim \frac{d^3k}{\omega} a_k \sim \frac{d^3k}{k}\,\frac{1}{k} \sim k$, so $\mathcal{L} = g\varphi^3 \sim gk^3 \sim k^4 \Rightarrow g \sim k$, where L is the Lagrangian and φ the pion field). We define a dimensionless coupling constant $g_1 \equiv (g\sqrt{\pi}/\mu)$, so that:

$$A_1^R(s_o) = (g_1\mu)^2 \delta(s_o-\mu^2) \tag{2.22}$$

where we have also absorbed the factor $\sqrt{\pi}$ into the new coupling constant. Now we will calculate $\sigma(\succ\!\!-)$ and show that unitarity leads to the same result (2.22).

b) Calculation of $A_1^R(s_o)$ from $\sigma(\succ\!\!-)$.

The optical theorem to order g^2 for boson-boson scattering in terms of the invariant matrix element M and in the center of mass system yields:

$$A_1^R(s_o) = \mathrm{Im}(M) = 2|\vec{p}|\sqrt{s_o}\,\sigma(\succ\!\!-). \tag{2.23}$$

This relates the elastic diagram $\succ\!\!-\!\!\prec$, a part of the elastic amplitude, to the cross section for $\succ\!\!-$, a part of the inelastic (total) cross section. We now calculate $\sigma(\succ\!\!-)$ for two particles of momenta p,q, interacting to form a particle of momentum p'. The S-matrix is:

$$S = 1 + R = 1 + i(2\pi V)\delta^4(p+q-p')\tilde{\mathcal{T}}$$

where $$\tilde{\mathcal{T}} = \frac{-g}{2^{\frac{3}{2}}V^{\frac{3}{2}}\sqrt{\omega_p\omega_q\omega_{p'}}}$$

and $$d\sigma(\succ\!\!-) = \frac{|\mathcal{T}(\succ\!\!-)|^2}{v_{rel}}(2\pi V)\delta^4(p+q-p')\,d^3p'.$$

In the center of mass system, $v_{rel} = \dfrac{|\vec{p}|\sqrt{s_o}}{\omega_p^2}$, so that:

$$d\sigma(\succ\!\!-) = \frac{g^2}{8\omega_{p'}}\,\frac{2\pi}{|\vec{p}|\sqrt{s_o}}\,\delta^4(p+q-p')\,d^3p'.$$

Performing the three-space integration:

$$\sigma(\succ\!\!-) = \frac{\pi g^2}{4\omega_{p'}|\vec{p}|\sqrt{s_0}}\,\delta(\omega_p+\omega_q-\omega_{p'}) = \frac{\pi g^2}{4\omega_{p'}|\vec{p}|\sqrt{s_0}}\,\delta(\sqrt{s_0}-\omega_{p'}).$$

In the center of mass system, $\vec{p}+\vec{q}=0$, so that $\vec{p}'=0$ and $\omega_{p'}=\mu$:

$$\sigma(\succ\!\!-) = \frac{\pi g^2}{4\mu|\vec{p}|\sqrt{s_0}}\,\delta(\sqrt{s_0}-\mu).$$

Note that if taken literally, $\sqrt{s_0}=\mu$ violates energy conservation, since $\sqrt{s_0}=2\omega_p \geq 2\mu$. Recall that the incident particles will actually represent side links of the multiperipheral chain and are therefore off-shell. This gives us the right to analytically continue s_0 to values which are unphysical for the simple diagram $\succ\!\!-$, by failing to write $\delta(\sqrt{s_0}-\mu) = 0$. Rewriting our result in terms of s_0 rather than $\sqrt{s_0}$, we have:

$$\sigma(\succ\!\!-) = \frac{\pi g^2}{4\mu|\vec{p}|\sqrt{s_0}}\;\frac{\delta(s_0-\mu^2)}{\frac{1}{2}s_0^{-\frac{1}{2}}} = \frac{\pi g^2}{2\mu|\vec{p}|}\,\delta(s_0-\mu^2).$$

The optical theorem provides that:

$$A_1^R(s_0) = 2|\vec{p}|\sqrt{s_0}\;\sigma(\succ\!\!-) \;=\; \frac{\pi g^2\sqrt{s_0}}{\mu}\,\delta(s_0-\mu^2) = \pi g^2\delta(s_0-\mu^2)$$

which is the same result as found directly by Feynman rules in part a).

(ii). Calculation of $A_2^R(s_o)$.

a). Calculation of $A_2^R(s_o)$ Directly.

Consider the invariant matrix element M for two particles of momenta p,q, interacting to form two virtual particles of momenta k,k', which finally interact to form particles of momenta P=p, Q=q:

$$R = \frac{i(2\pi V)\delta^4(p+q-P-Q)M}{4V^2\omega_p\omega_q} .$$

Feynman rules yield for R:

$$R = \frac{(-ig')^2}{4V^2\omega_p\omega_q}\iint d^4k\, d^4k' \frac{(2\pi V)^2\delta^4(p+q-k-k')\delta^4(P+Q-k-k')}{(k^2-\mu^2)(k'^2-\mu^2)}\left(\frac{i}{2\pi V}\right)^2 .$$

Thus, the matrix element is just:

$$\begin{aligned} M &= \frac{-ig'^2}{2\pi V}\int\frac{d^4k'}{(k'^2-\mu^2)\left[(p+q-k')^2-\mu^2\right]} \\ &= \frac{-ig'^2}{2\pi V}\int\frac{d^4k'}{(k'^2-\mu^2)\left[k'^2-2(p+q)\cdot k'+s_o-\mu^2\right]} \end{aligned}$$

where $s_o = (p+q)^2$ and we now let V = 1. Note the well-known relation:

$$\frac{1}{ab} = \int_o^1 \frac{dx}{\left[ax+b(1-x)\right]^2} .$$

Allowing $a=k'^2-\mu^2$ and $b = k'^2-2(p+q)\cdot k'+s_o-\mu^2$, we have:

$$\frac{1}{ab} = \int_0^1 dx \, \frac{1}{\left[(k'^2-\mu^2)x+(k'^2-2\{p+q\}\cdot k'+s_o-\mu^2)(1-x)\right]^2} \quad .$$

The matrix element thus becomes:

$$M = \frac{-ig'^2}{2\pi} \int_0^1 dx \int \frac{d^4k'}{\left[k'^2-2(1-x)(p+q)\cdot k'-\mu^2 x+(s_o-\mu^2)(1-x)\right]^2} \quad .$$

Multiplying by $(-1)^2$ in the denominator and introducing an imaginary part:

$$M = \frac{-ig'^2}{2\pi} \int_0^1 dx \int \frac{d^4k'}{\left[-k'^2+2(1-x)(p+q)\cdot k'+s_o x-s_o+\mu^2-i\epsilon\right]^2} \quad .$$

Let $p' = (1-x)(p+q)$ and $s' = s_o x - s_o + \mu^2$ for the purposes of this calculation only, so that:

$$A_2^R(s_o) = \mathrm{Im}(M) = \mathrm{Im}\left[\frac{-ig'^2}{2\pi} \int_0^1 dx \int \frac{d^4k'}{\left[-k'^2+2p'\cdot k'+s'-i\epsilon\right]^2}\right] .$$

Consider the $\mathbf{k}'$-integration. Let $k=k'-p'$ and $s=s'+p'^2$. Note that contour integration, with the contour closed either in the upper half plane or the lower half plane, yields:

$$\int_{-\infty}^{\infty} \frac{dk_o}{(-k^2+s-i\epsilon)} = \frac{i\pi}{\sqrt{|\vec{k}|^2+s}} \quad .$$

Differentiate this with respect to s on both sides:

$$-\int_{-\infty}^{\infty} \frac{dk_o}{(-k^2+s-i\epsilon)^2} = \frac{-i\pi}{2(|\vec{k}|^2+s)^{3/2}} \quad .$$

Now complete the four-space integration by integrating both sides over $\vec{k}$:

$$\int \frac{d^4k}{(-k^2+s-i\epsilon)^2} = \frac{i\pi}{2}\int \frac{d^3k}{(|\vec{k}|^2+s)^{3/2}} = 2\pi^2 i \int_0^\infty \frac{z^2\,dz}{(z^2+s)^{3/2}}$$

where $z \equiv |\vec{k}|$. Note that $s=s'+p'^2=x^2s_o+\mu^2-xs_o$ and so $s=s_o(x-\frac{1}{2})^2-\frac{s_o}{4}+\mu^2$. The roots of the equation $s=0$ are

$$x = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}}\,.$$

In addition, $s>0$ and $\sqrt{s}$ real if

$$x<\tfrac{1}{2}-\tfrac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}} \quad \text{or if } x>\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}}\,,$$

while $s<0$ and $\sqrt{s}$ imaginary if

$$\tfrac{1}{2}-\tfrac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}} < x < \tfrac{1}{2}+\tfrac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}}\,.$$

The functional dependence of s on x is displayed in Figure 16. Now return to our integral. We have:

$$\int \frac{d^4k}{(-k^2+s-i\epsilon)^2} = 2\pi^2 i \int_0^\infty \frac{\left(\frac{z^2}{s}\right)d\left(\frac{z}{\sqrt{s}}\right)}{\left(\frac{z^2}{s}+1\right)^{3/2}}$$

where $s \neq 0$ for the imaginary part of M; in fact, $s<0$.

Thus:

$$\int \frac{d^4k'}{(-k'^2+2p'\cdot k'+s'-i\epsilon)^2} = 2\pi^2 i \int_0^{\infty} \frac{\left(\frac{z^2}{s}\right) d\left(\frac{z}{\sqrt{s}}\right)}{\left(\frac{z^2}{s}+1\right)^{3/2}} \quad .$$

Change variables. Let $\frac{z}{\sqrt{s}} = \tan\theta$, so that $d(\frac{z}{\sqrt{s}}) = \sec^2\theta\, d\theta$ and $\frac{z^2}{s} + 1 = \sec^2\theta$. Therefore:

$$\int \frac{d^4k'}{(-k'^2+2p'\cdot k'+s'-i\epsilon)^2} = 2\pi^2 i \int_{z=0}^{z=\infty} \frac{\sin^2\theta}{\cos\theta}\, d\theta$$

$$= 2\pi^2 i \int_{z=0}^{z=\infty} (\sec\theta - \cos\theta)\, d\theta$$

$$= 2\pi^2 i \Big[\ln(\sec\theta+\tan\theta) - \sin\theta\Big]_{z=0}^{z=\infty} \quad .$$

Adopting a cutoff at $z=\Lambda$, we have:

$$\int \frac{d^4k'}{(-k'^2+2p'\cdot k'+s'-i\epsilon)^2} = 2\pi^2 i \left[\ln\left(\frac{\sqrt{\Lambda^2+s}+\Lambda}{\sqrt{s}}\right) - \frac{\Lambda}{\sqrt{\Lambda^2+s}}\right].$$

Utilizing this in our expression for $A_2^R(s_o)$, we obtain:

$$A_2^R(s_o) = \left\{ \operatorname{Im}\left(\frac{-ig'^2}{2\pi}(2\pi^2 i) \int_0^1 dx \left[\ln\left(\frac{\sqrt{\Lambda^2+s}+\Lambda}{\sqrt{s}}\right) - \frac{\Lambda}{\sqrt{\Lambda^2+s}}\right]\right)\right\}$$

or more simply:

$$A_2^R(s_o) = \operatorname{Im}\left\{ g'^2\pi \int_0^1 dx \left[\ln\left(\frac{\sqrt{\Lambda^2+s}\ \Lambda}{\sqrt{s}}\right) - \frac{\Lambda}{\sqrt{\Lambda^2+s}}\right]\right\}.$$

This is nonzero only where $\sqrt{s}$ is imaginary, when

$$\tfrac{1}{2} - \tfrac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}} < x < \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}} \quad .$$

We can then write:

$$A_2^R(s_o) = \mathrm{Im}\left[\pi g'^2 \int_{x_-}^{x_+} dx(-\ln\sqrt{s}\,)\right]$$

where $x_\pm \equiv \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}}$. Note that $\ln\sqrt{s} = \ln(|\sqrt{s}|e^{i\phi})$ $= \ln|\sqrt{s}| + i\phi$, and $\sqrt{s} = \pm i|\sqrt{s}| \Rightarrow e^{i\phi} = \pm 1 \Rightarrow \phi = \pm\frac{\pi}{2}$, so that $\mathrm{Im}(\ln\sqrt{s}) = \pm\frac{\pi}{2}$. Our expression becomes:

$$A_2^R(s_o) = \frac{\pi^2 g'^2}{2}\int_{x_-}^{x_+} dx = \frac{\pi^2 g'^2}{2}\sqrt{\frac{s_o-4\mu^2}{s_o}}\,.$$

Now $\mathcal{L} \sim k^4$ and $\mathcal{L} = g'\varphi^4 \sim g'k^4$, so that this coupling constant is already dimensionless. To simplify it, define $g_2{}^2 = \frac{\pi^2 g'^2}{2}$, so that our result is:

$$A_2^R(s_o) = g_2{}^2\sqrt{\frac{s_o-4\mu^2}{s_o}}\,. \tag{2.24}$$

A cutoff will be adopted later on.

b). Calculation of $A_2^R(s_o)$ from $\sigma(\times)$.

Unitarity to order g'^2 in φ^4 interactions demands that:

$$A_2^R(s_o) = \operatorname{Im} M(\rangle\!\!\langle) = 2|\vec{p}|\sqrt{s_o}\,\sigma(\times). \qquad (2.25)$$

Consider $\sigma(\times)$, the cross section for two particles of momenta p,q, to interact, with the final state featuring two particles of momenta p' and q'. The term in the S-matrix is given by:

$$R = \frac{-ig'(2\pi V)\delta^4(p+q-p'-q')}{4V^2\sqrt{\omega_p\omega_q\omega_{p'}\omega_{q'}}} \quad .$$

The differential cross section is:

$$d\sigma(\times) = \frac{|\mathcal{T}(\times)|^2}{v_{rel}}\,(2\pi)\,\delta^4(p+q-p'-q')\,d^3q'\,d^3p'$$

where $\mathcal{T}(\times) = -g'/4\sqrt{\omega_p\omega_q\omega_{p'}\omega_{q'}}$ and we have taken V = 1. In the center of mass system of the incident particles, the cross section becomes:

$$d\sigma(\times) = \frac{g'^2}{16\omega_p^2\omega_{p'}^2}\,\frac{\omega_p^2}{|\vec{p}|\sqrt{s_o}}\,(2\pi)\,\delta^4(p+q-p'-q')\,d^3q'\,d^3p'$$

or more simply:

$$d\sigma(\times) = \frac{\pi g'^2}{8}\,\frac{\delta^4(p+q-p'-q')}{|\vec{p}|\sqrt{s_o}\,\omega_{p'}^2}\,d^3q'\,d^3p'.$$

Performing the integration over q' leaves:

$$d\sigma(\times) = \pi g'^2 \frac{\delta(2\omega_p - 2\omega_{p'})}{8|\vec{p}|\sqrt{s_o}\,\omega_{p'}^2} d^3p'$$

$$= \pi g'^2 \frac{\delta(\omega_p - \omega_{p'})}{16|\vec{p}|\sqrt{s_o}\,\omega_{p'}^2} d^3p'.$$

The angular integration is straight forward:

$$d\sigma(\times) = \frac{\pi^2 g'^2 \delta(\omega_p - \omega_{p'}) |\vec{p}'|^2 d|\vec{p}'|}{4|\vec{p}|\sqrt{s_o}\,\omega_{p'}^2} .$$

Converting to an integral over $\omega_{p'}$, where $\omega_{p'}^2 = |\vec{p}'|^2 + \mu^2$ and $\omega_{p'} d\omega_{p'} = |\vec{p}'|\, d|\vec{p}'|$, yields:

$$d\sigma(\times) = \frac{\pi^2 g'^2 \delta(\omega_p - \omega_{p'}) |\vec{p}'|\, \omega_{p'} d\omega_{p'}}{4|\vec{p}|\sqrt{s_o}\,\omega_{p'}^2}$$

or more simply:

$$d\sigma(\times) = \frac{\pi^2 g'^2 \delta(\omega_p - \omega_{p'}) \sqrt{\omega_{p'}^2 - \mu^2}\, d\omega_{p'}}{4|\vec{p}|\sqrt{s_o}\,\omega_{p'}} .$$

The integral can now be completed:

$$\sigma(\times) = \frac{\pi^2 g'^2 \sqrt{\omega_p^2 - \mu^2}}{4|\vec{p}|\sqrt{s_o}\,\omega_p} = \frac{\pi^2 g'^2}{4\sqrt{s_o}\,\omega_p} = \frac{\pi^2 g'^2}{2s_o} .$$

This result is now ripe for placement in our optical theorem:

$$A_2^R(s_o) = 2|\vec{p}|\sqrt{s_o}\left(\frac{\pi^2 g'^2}{2s_o}\right) = \frac{|\vec{p}|}{\sqrt{s_o}}(\pi^2 g'^2) = \pi^2 g'^2 \sqrt{\frac{\omega_p^2 - \mu^2}{s_o}} .$$

Since $\omega_p = \frac{\sqrt{s_o}}{2}$, we again find the result:

$$A_2^R(s_o) = \frac{\pi^2 g'^2}{2} \sqrt{\frac{s_o - 4\mu^2}{s_o}} = g_2^2 \sqrt{\frac{s_o - 4\mu^2}{s_o}},$$

as we had in (2.24).

(iii). Calculation of $A_3^R(s_o)$ by Approximation Techniques.

a). Direct Calculation of $A_3^R(s_o)$.

Consider the amplitude for two particles of momenta p,q, to interact forming three virtual particles of momenta k_1, k_2, and k_3, which interact to form two particles of momenta P=p and Q=q. The S-matrix is 1+R, where R is given by:

$$R = (-ig'')^2 \iiint \frac{d^4k_1 d^4k_2 d^4k_3}{4\omega_p\omega_q \; V^2} \frac{1}{(k_1{}^2-\mu^2)(k_2{}^2-\mu^2)(k_3{}^2-\mu^2)} \left(\frac{i}{2\pi V}\right)^3$$

$$\times \; (2\pi V)^2 \delta^4(p+q-k_1-k_2-k_3)\delta^4(P+Q-k_1-k_2-k_3).$$

The integration over k_3 is trivial, yielding:

$$R = \frac{ig''^2\delta^4(p+q-P-Q)}{8\pi\omega_p\omega_q} \iint \frac{d^4k_1 d^4k_2}{(k_1{}^2-\mu^2)(k_2{}^2-\mu^2)(k_3{}^2-\mu^2)}$$

where $k_3 = p+q-k_1-k_2$. The invariant matrix element is thus:

$$M = \frac{g''^2}{16\pi^2 V^2} \iint \frac{d^4k_1 d^4k_2}{(k_1{}^2-\mu^2)(k_2{}^2-\mu^2)(k_3{}^2-\mu^2)} \; .$$

We will make use of the well-known identity:

$$\frac{1}{abc} = 2\int_o^1 dx \int_o^{1-x} dy \; \frac{1}{\left[a+(b-a)x+(c-a)y\right]^3}$$

with $a=k_1{}^2-\mu^2$, $b=k_2{}^2-\mu^2$, $c=k_3{}^2-\mu^2$, where $k_3=p+q-k_1-k_2$.

We have:

$$M = \frac{g''^2}{8\pi^2 V^2}\int_0^1 dx \int_0^{1-x} dy \iint \frac{d^4k_1 d^4k_2}{\left[k_1{}^2-\mu^2+(k_2{}^2-k_1{}^2)x+(k_3{}^2-k_1{}^2)y\right]^3}$$

where $k_3{}^2=(p+q)^2+(k_1+k_2)^2-2(p+q)\cdot(k_1+k_2)$,

and so $k_3{}^2-k_1{}^2=s_o+k_2{}^2+2k_1\cdot k_2-2(p+q)\cdot(k_1+k_2)$. Consider the integration over k_2:

$$I \equiv \int \frac{d^4k_2}{\left[k_1{}^2-\mu^2+(k_2{}^2-k_1{}^2)x+(s_o+k_2{}^2+2k_1\cdot k_2-2\{p+q\}\cdot\{k_1+k_2\})y\right]^3} .$$

Rearranging the denominator results in:

$$I = \int \frac{d^4k_2}{\left[(x+y)k_2{}^2+(2k_1-2\{p+q\})\cdot k_2 y+k_1{}^2-\mu^2-k_1{}^2x+s_o'y\right]^3} ,$$

where $s_o' \equiv s_o-2\{p+q\}\cdot k_1$. Extracting $\left[-(x+y)^3\right]$ from the denominator, we obtain:

$$I=\frac{-1}{(x+y)^3}\int d^4k_2\left[-k_2{}^2-\frac{2y}{x+y}(k_1-p-q)\cdot k_2-\left(\frac{k_1{}^2(1-x)-\mu^2+s_o'y}{x+y}\right)\right]^{-3} .$$

Define the following:

$$P \equiv \frac{-y}{x+y}(k_1-p-q)$$

$$s \equiv -\left(\frac{k_1{}^2(1-x)-\mu^2+s_o{}'y}{x+y}\right)$$

so that our integral is of the form:

$$\int \frac{d^4k_2}{(-k_2{}^2+2k_2\cdot P+s-i\epsilon)^3} = \frac{\pi^2 i}{2(s+P^2)} \quad ,$$

thereby yielding the result:

$$I=\frac{-\pi^2 i}{2(x+y)^3}\left[-\left(\frac{k_1{}^2(1-x)-\mu^2+s_o{}'y}{x+y}\right)+\frac{y^2}{(x+y)^2}(k_1-p-q)^2\right]^{-1}$$

which simplifies to:

$$I = \frac{\pi^2 i}{2(x+y)}\left[[(1-x)(x+y)-y^2]k_1{}^2-2xy(p+q)\cdot k_1-\mu^2(x+y)+s_o xy\right]^{-1},$$

where we have used our definition of $s_o{}'$. Return to our matrix element with V=1:

$$M=\frac{-ig''^2}{16}\int_0^1 dx\int_0^{1-x}\frac{dy}{x+y}\int d^4k_1\left[-[(1-x)(x+y)-y^2]k_1{}^2+2xy(p+q)\cdot k_1 +\mu^2(x+y)-s_o xy\right]^{-1} .$$

To perform the integration over k_1, we first note again the result from contour integration, that:

$$\int_{-\infty}^{\infty} \frac{dk_o}{-k^2+s-i\epsilon} = \frac{i\pi}{\sqrt{|\vec{k}|^2+s}} \quad .$$

This time, we do not differentiate as we did in studying $A_2^R(s_o)$. Completing the integral over d^4k leads to:

$$\int \frac{d^4k}{-k^2+s-i\epsilon} = i\pi \int \frac{d^3k}{\sqrt{|\vec{k}|^2+s}} = \frac{4\pi^2 i}{\sqrt{s}} \int_o^{\infty} \frac{z^2 dz}{\sqrt{\frac{z^2}{s}+1}}$$

$$= 4\pi^2 is \int_o^{\infty} \frac{\frac{z^2}{s}\, d\left(\frac{z}{\sqrt{s}}\right)}{\sqrt{\frac{z^2}{s}+1}} \quad .$$

Define $\frac{z}{\sqrt{s}} \equiv \tan\theta$ to obtain:

$$\int \frac{d^4k}{-k^2+s-i\epsilon} = 4\pi^2 is \int_o^{\frac{\pi}{2}} \frac{\sin^2\theta}{\cos^3\theta} d\theta = 4\pi^2 is \int_o^{\frac{\pi}{2}} (\sec^3\theta - \sec\theta)\, d\theta$$

$$= 4\pi^2 is \left[\int_o^{\frac{\pi}{2}} \sec^3\theta\, d\theta - \ln(\sec\theta + \tan\theta) \Big|_o^{\frac{\pi}{2}} \right] .$$

For the integral of $\sec^3\theta$, redefine $\eta \equiv \tan\theta$ to get:

$$\int \frac{d^4k}{-k^2+s-i\epsilon} = 4\pi^2 is \left[\frac{\eta}{2}\sqrt{\eta^2+1} + \tfrac{1}{2}\ln(\eta + \sqrt{\eta^2+1}) \Big|_o^{\infty} - \ln(\sec\theta + \tan\theta) \Big|_o^{\frac{\pi}{2}} \right] .$$

Using $\eta = \tan\theta$ in the last term results in:

$$\int \frac{d^4k}{-k^2+s-i\epsilon} = 4\pi^2 is \left[\frac{\eta}{2}\sqrt{\eta^2+1} - \tfrac{1}{2}\ln(\eta + \sqrt{\eta^2+1}) \right]_o^{\infty}$$

$$= 4\pi^2 is \left[\frac{\Lambda'}{2}\sqrt{\Lambda'^2+1} - \tfrac{1}{2}\ln(\Lambda' + \sqrt{\Lambda'^2+1}) \right]$$

after adopting a cutoff of Λ' for the divergent integral. To facilitate application of this result to our integration over k_1, we take $k \rightarrow k_1 - P'$ and $s \rightarrow s' + P'^2$:

$$\int \frac{d^4k'}{-k'^2 + k'\cdot P' + s' - i\epsilon} = 4\pi^2 i(s'+P'^2)\left[\frac{\Lambda'}{2}\sqrt{\Lambda'^2+1} - \tfrac{1}{2}\ln(\Lambda' + \sqrt{\Lambda'^2+1})\right]$$

where $P' \equiv \dfrac{xy(p+q)}{(1-x)(x+y)-y^2}$ and $s' \equiv \dfrac{\mu^2(x+y)-s_o xy}{(1-x)(x+y)-y^2}$ so that

$$s'+P'^2 = \frac{\mu^2(x+y)-s_o xy}{(1-x)(x+y)-y^2} + \frac{x^2y^2s_o}{[(1-x)(x+y)-y^2]^2}$$

$$= \frac{[\mu^2(x+y)-s_o xy][(1-x)(x+y)-y^2] + x^2y^2s_o}{[(1-x)(x+y)-y^2]^2} .$$

The integral is thus:

$$\frac{1}{[(1-x)(x+y)-y^2]}\int \frac{d^4k'}{-k'^2+2k'\cdot P'+s'-i\epsilon} = \frac{4\pi^2 i(s'+P'^2)}{[(1-x)(x+y)-y^2]}$$

$$\times \left[\frac{\Lambda'}{2}\sqrt{\Lambda'^2+1} - \tfrac{1}{2}\ln(\Lambda' + \sqrt{\Lambda'^2+1})\right].$$

Note that Λ' is the cutoff for $\eta = z/\sqrt{s} = |\vec{k}|/\sqrt{s}$. The cut-off for $|\vec{k}|$ is $\Lambda \equiv |\vec{k}_{max}| = \Lambda'\sqrt{s}$, where $|\vec{k}_{max}| = |\vec{k}_{1max} - \vec{P}'|$ and $s = s'+P'^2$. Our matrix element is now:

$$M = \frac{-ig''^2}{16}\int_0^1 dx \int_0^{1-x} \frac{dy}{(x+y)} \frac{4\pi^2 is}{[(1-x)(x+y)-y^2]}$$

$$\times \left[\frac{\Lambda}{2\sqrt{s}}\sqrt{\frac{\Lambda^2}{s}+1} - \tfrac{1}{2}\ln\left(\frac{\Lambda}{\sqrt{s}} + \sqrt{\frac{\Lambda^2}{s}+1}\right)\right]$$

or simply:

$$M = \frac{\pi^2 g''^2}{16} \int_0^1 dx \int_0^{1-x} \frac{dy}{(x+y)\left[(1-x)(x+y)-y^2\right]}$$
$$\times \left[\frac{\Lambda}{2}\sqrt{\Lambda^2 + s} - \frac{s}{2}\ln\left(\frac{\Lambda + \sqrt{\Lambda^2 + s}}{\sqrt{s}}\right)\right].$$

Since $A_3^R(s_o) = \mathrm{Im}(M)$, we get a contribution only where $s < 0$. We seek the roots of $s=0$, which are the roots of the equation:

$$\left[\mu^2(x+y)-s_o xy\right]\left[(1-x)(x+y)-y^2\right] + x^2y^2s_o = 0.$$

If $s_o \gg \mu^2$, this becomes:

$$-xy\left[(1-x)(x+y)-y^2\right] + x^2y^2 = 0.$$

Consider this as a third order equation in y. One root is $y=0$. If $y \neq 0$, we have a second order equation:

$$\begin{aligned} 0 &= -x-y+x^2+2xy+y^2 \\ &= -(x+y) + (x+y)^2 \\ &= (x+y)(x+y-1) \end{aligned}$$

whose roots are $y=-x$ or $y=1-x$. The three roots of $s=0$ are thus $y=-x, 0, 1-x$. Since s is continuous, it cannot change sign between these roots. Consider the sign of s between roots. Since the denominator of $s = s'+P'^2$ is squared, the sign of s is the same as the sign of its numerator, which is equal to $[\mu^2(x+y)-s_o xy][(1-x)(x+y)-y^2] + x^2y^2s_o$, or in the high s_o limit, $s_o xy(x+y)(x+y-1)$. The sign of s is thus determined by the sign of the function $\phi \equiv xy(x+y)(x+y-1)$. Let ε be a small positive number, and consider the following regions of y:

$$y=-x-\varepsilon \Rightarrow \phi = -x(x+\varepsilon)\varepsilon(\varepsilon+1) < 0$$
$$y=-x+\varepsilon \Rightarrow \phi = x(x-\varepsilon)\varepsilon(1-\varepsilon) > 0$$
$$y=-\varepsilon \Rightarrow \phi = \varepsilon x(x-\varepsilon)(1-x+\varepsilon) > 0$$
$$y=\varepsilon \Rightarrow \phi = -\varepsilon x(x+\varepsilon)(1-x-\varepsilon) < 0$$
$$y=1-x-\varepsilon \Rightarrow \phi = x(1-x-\varepsilon)(1-\varepsilon)(-\varepsilon) < 0$$
$$y=1-x+\varepsilon \Rightarrow \phi = x(1-x+\varepsilon)(1+\varepsilon)\varepsilon > 0.$$

Thus, $s<0$ for $y<-\varepsilon$ and $0<y<1-x$, while $s>0$ for $-x<y<0$ and $y>1-x$. However, our region of integration is from 0 to 1-x, and s is negative for this entire region. Thus, we need only write $s = -|s|$ and integrate from 0 to 1-x to obtain $A_3^R(s_o) = \mathrm{Im}(M)$;

$$A_3^R(s_o) = \frac{\pi^2 g''^2}{16}\,\mathrm{Im}\int_o^1 dx\int_o^{1-x}\frac{dy}{(x+y)[(1-x)(x+y)-y^2]}\left(\frac{s}{2}\ln\sqrt{s}\right)$$
$$= \frac{\pi^2 g''^2}{64}\,\mathrm{Im}\int_o^1 dx\int_o^{1-x}\frac{dy}{(x+y)[(1-x)(x+y)-y^2]}(s\ln s).$$

Now $\mathrm{Im}(\ln s) = \mathrm{Im}(\ln[-|s|]) = \mathrm{Im}(\ln[-1]) = \pm\pi$ (or just note that $s \equiv |s|e^{i\theta} = -|s| \Rightarrow \theta = \pm\pi$ and $\ln(s) = \ln|s| + i\theta$ so that $\mathrm{Im}(\ln s) = \theta = \pm\pi$). We are left with:

$$A_3^R(s_o) = \frac{\pi^3 g''^2}{64}\int_o^1 dx \int_o^{1-x} \frac{dy}{(x+y)\left[(1-x)(x+y)-y^2\right]}(s).$$

Replacing s with its dependence on the integration variables, we have, with $s_o \gg \mu^2$:

$$A_3^R(s_o) = \frac{\pi^3 g''^2}{64}(s_o)\int_o^1 dx \int_o^{1-x} dy \frac{(-xy)\left[(1-x)(x+y)-y^2\right]+x^2y^2}{(x+y)\left[(1-x)(x+y)-y^2\right]^3} .$$

The integrand no longer depends on s_o. It is apparent that for $s_o \gg \mu^2$, $A_3^R(s_o)$ is linear in s_o. It is dominated by the phase space:

$$A_3^R \sim s_o\sigma(\times) \sim \frac{s_o\delta^4(p+q-k_1-k_2-k_3)}{\omega_p\omega_q\omega_1\omega_2\omega_3}\, d^3k_1 d^3k_2 d^3k_3$$

$$A_3^R \sim \frac{s_o(\omega^{-4})}{\omega^5}(\omega^9) \sim s_o . \qquad (2.26)$$

When we make use of $A_3^R(s_o)$ to calculate the multiplicity and correlation functions, our region of integration will be from $(3\mu)^2$ to s_{omax}, where $(3\mu)^2 = 9\mu^2 \approx 0.176\text{GeV}^2$ and $s_{omax} = 200\ \text{GeV}^2$. Thus, the major portion of the integral obeys the condition $s_o \gg \mu^2$. We will also find that deviation from the phase space approximation even at the lower end of the region of integration is very slight.

Since the coefficient of s_o in the present form of $A_3^R(s_o)$ is a rather awful integral, we will await numerical analysis for the unitarity calculation of $A_3^R(s_o)$ which follows.

b). Calculation of $A_3^R(s_o)$ from $\sigma(\succ\!\!\!\in)$.

We have already seen that the phase space of $\sigma(\succ\!\!\!\in)$, when entered into the optical theorem, provides the result just calculated directly, that $A_3^R(s_o) \sim s_o$ for $s_o \gg \mu^2$. Now we will perform a more exact calculation to verify the validity of this approximation.

Consider the cross section for two particles of momenta p,q, interacting to form three particles of momenta k_1, k_2, k_3. The term in the S-matrix $S = 1 + R$ is:

$$R(\succ\!\!\!\in) = \frac{(-ig'')(2\pi V)\delta^4(p+q-k_1-k_2-k_3)}{\sqrt{32V^5\omega_p\omega_q\omega_1\omega_2\omega_3}} \quad .$$

The differential cross section is thus:

$$d\sigma(\succ\!\!\!\in) = \frac{|\mathcal{T}(\succ\!\!\!\in)|^2}{v_{rel}}(2\pi V)\delta^4(p+q-k_1-k_2-k_3)\; d^3k_1 d^3k_2 d^3k_3$$

where $$\mathcal{T}(\succ\!\!\!\in) = \frac{-g''}{\sqrt{32V^5\omega_p\omega_q\omega_1\omega_2\omega_3}}$$

and in the center of mass system of the incident particles,

$$v_{rel} = \frac{|\vec{p}|\sqrt{s_o}}{\omega_p^2} \quad .$$

We can now write the differential cross section as:

$$d\sigma(\succ\!\!\!\in) = \frac{g''^2}{32\omega_p^2\omega_1\omega_2\omega_3}\frac{\omega_p^2}{|\vec{p}|\sqrt{s_o}}(2\pi)\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\,\delta(\omega_1+\omega_2+\omega_3-\sqrt{s_o})$$
$$\times d^3k_1 d^3k_2 d^3k_3$$

where we have let V=1. Performing the integration over $\vec{k}_3$ brings us to the result:

$$d\sigma(\succ\!\prec) = \frac{\pi g''^2}{16|\vec{p}|\sqrt{s_o}\,\omega_1\omega_2}\frac{\delta\left(\omega_1+\omega_2+\sqrt{(\vec{k}_1+\vec{k}_2)^2+\mu^2}-\sqrt{s_o}\right)}{\sqrt{(\vec{k}_1+\vec{k}_2)^2+\mu^2}}d^3k_1d^3k_2 \;.$$

We can convert the integrations over $|\vec{k}_1|$ and $|\vec{k}_2|$ into integrals over ω_1,ω_2, by noting $\omega_1^{\,2}=|\vec{k}_1|^{\,2}+\mu^2$ and $\omega_2^{\,2}=|\vec{k}_2|^2+\mu^2$:

$$d\sigma(\succ\!\prec) = \frac{\pi g''^2|\vec{k}_1||\vec{k}_2|}{16|\vec{p}|\sqrt{s_o}}\;\frac{\delta\left(\omega_1+\omega_2+\sqrt{(\vec{k}_1+\vec{k}_2)^2+\mu^2}-\sqrt{s_o}\right)}{\sqrt{(\vec{k}_1+\vec{k}_2)^2+\mu^2}}$$

$$\times\; d\omega_1 d\omega_2 d\Omega_1 d\Omega_2.$$

If we select as our polar axis the direction of $\vec{k}_1$, the integration over Ω_1 is trivial, and we have:

$$d\sigma(\succ\!\prec) = \frac{\pi g''^2|\vec{k}_1||\vec{k}_2|}{16|\vec{p}|\sqrt{s_o}}(8\pi^2)\int_{-1}^{+1}du\frac{\delta\left(\omega_1+\omega_2+\sqrt{\omega_1^{\,2}+\omega_2^{\,2}-\mu^2+2|\vec{k}_1||\vec{k}_2|u}-\sqrt{s_o}\right)}{\sqrt{\omega_1^{\,2}+\omega_2^{\,2}-\mu^2+2|\vec{k}_1||\vec{k}_2|u}}$$

$$\times\; d\omega_1 d\omega_2$$

where $u \equiv \cos\theta_2 \equiv \hat{k}_1\cdot\hat{k}_2$. Now change variables,

$$x \equiv \omega_3 = \sqrt{\omega_1^{\,2}+\omega_2^{\,2}-\mu^2+2|\vec{k}_1||\vec{k}_2|u}\;,\quad \text{so that } du = \frac{x\,dx}{|\vec{k}_1||\vec{k}_2|}\;.$$

The result is:

$$d\sigma(\succ\!\prec) = \frac{\pi^3 g''^2}{2|\vec{p}|\sqrt{s_o}}\int_{x_-}^{x_+}dx\;\delta(x+\omega_1+\omega_2-\sqrt{s_o})\;d\omega_1 d\omega_2$$

where $x_{\pm} = \sqrt{\omega_1^{\,2}+\omega_2^{\,2}-\mu^2\pm 2|\vec{k}_1||\vec{k}_2|} = \sqrt{(|\vec{k}_1|\pm|\vec{k}_2|)^2+\mu^2}$. We must insure that u has physical values, so we shoud actually write:

$$d\sigma(\gtrless) = \frac{\pi^3 g''^2}{2|\vec{p}|\sqrt{s_0}}\int_{x}^{x_+} dx\; \theta(1-u^2)\,\delta(x+\omega_1+\omega_2-\sqrt{s_0})\; d\omega_1 d\omega_2$$

where $u = \dfrac{x-\omega_1^{\,2}-\omega_2^{\,2}+\mu^2}{2\sqrt{(\omega_1^{\,2}-\mu^2)(\omega_2^{\,2}-\mu^2)}}$.

Now we can integrate over x:

$$d\sigma(\gtrless) = \frac{\pi^3 g''^2}{2|\vec{p}|\sqrt{s_0}}\;\theta(1-u^2)\; d\omega_1\; d\omega_2$$

where $u = \dfrac{(\sqrt{s_0}-\omega_1-\omega_2)^2-\omega_1^{\,2}-\omega_2^{\,1}+\mu^2}{2\sqrt{(\omega_1^{\,2}-\mu^2)(\omega_2^{\,2}-\mu^2)}} = \dfrac{s_0-2\sqrt{s_0}(\omega_1+\omega_2)+2\omega_1\omega_2+\mu^2}{2\sqrt{(\omega_1^{\,2}-\mu^2)(\omega_2^{\,2}-\mu^2)}}$.

To apply an approximation techique to the remaining integrations, we seek finite limits for ω_1 and ω_2. In order that the technique provide a good approximation, we want limits which do not contain large "blank" areas where the integrand is zero. We have $\omega_1+\omega_2+\omega_3=\sqrt{s_0}$ and $\vec{k}_1+\vec{k}_2+\vec{k}_3=0$, where $\omega_j^{\,2}=\vec{k}_j^{\,2}+\mu^2$. Thus:

$$\omega_1 = \sqrt{\vec{k}_1^{\,2}+\mu^2} = \sqrt{\vec{k}_2^{\,2}+\vec{k}_3^{\,3}+2\vec{k}_2\cdot\vec{k}_3+\mu^2}.$$

ω_1 is greatest for $\vec{k}_2 \parallel \vec{k}_3$, when $\vec{k}_2\cdot\vec{k}_3 = |\vec{k}_2||\vec{k}_3|$. Then:

$$\omega_1 = \sqrt{\omega_2^{\,2}+\omega_3^{\,2}-\mu^2+2\sqrt{(\omega_2^{\,2}-\mu^2)(\omega_3^{\,2}-\mu^2)}}.$$

To maximize ω_1 with respect to ω_2 and ω_3, we must keep in mind that $\omega_1+\omega_2+\omega_3 = \sqrt{s_o}$, so that:

$$\frac{d\omega_1}{d\omega_3} + \frac{\partial\omega_2}{\partial\omega_3} + 1 = 0.$$

Differentiating ω_1 with respect to ω_3 yields:

$$\begin{aligned}
\frac{d\omega_1}{d\omega_3} &= \frac{d}{d\omega_3}\sqrt{\omega_2{}^2+\omega_3{}^2-\mu^2+2\sqrt{(\omega_2{}^2-\mu^2)(\omega_3{}^2-\mu^2)}} \\
&= \frac{\partial}{\partial\omega_3}\sqrt{\omega_2{}^2+\omega_3{}^2-\mu^2+2\sqrt{(\omega_2{}^2-\mu^2)(\omega_3{}^3-\mu^2)}} \\
&\quad+\left[\frac{\partial}{\partial\omega_2}\sqrt{\omega_2{}^2+\omega_3{}^2-\mu^2+2\sqrt{(\omega_2{}^2-\mu^2)(\omega_3{}^2-\mu^2)}}\right]\frac{\partial\omega_2}{\partial\omega_3} \\
&= \frac{\omega_3}{\omega_1}\left[1+\sqrt{\frac{\omega_2{}^2-\mu^2}{\omega_3{}^2-\mu^2}}\right]+\frac{\omega_2}{\omega_1}\left[1+\sqrt{\frac{\omega_3{}^2-\mu^2}{\omega_2{}^2-\mu^2}}\right]\frac{\partial\omega_2}{\partial\omega_3} \quad .
\end{aligned}$$

Using our equation for $(d\omega_1/d\omega_3)$ in terms of $(\partial\omega_2/\partial\omega_3)$ in this result brings us to:

$$\frac{d\omega_1}{d\omega_3}\left[1+\frac{\omega_2}{\omega_1}\left(1+\sqrt{\frac{\omega_3{}^2-\mu^2}{\omega_2{}^2-\mu^2}}\right)\right]=\frac{\omega_3}{\omega_1}\left(1+\sqrt{\frac{\omega_2{}^2-\mu^2}{\omega_3{}^2-\mu^2}}\right)-\frac{\omega_2}{\omega_1}\left(1+\sqrt{\frac{\omega_3{}^2-\mu^2}{\omega_2{}^2-\mu^2}}\right).$$

From this, we observe that

$$\frac{d\omega_1}{d\omega_3} = 0 \Longrightarrow \omega_2=\omega_3.$$

Thus, ω_1 is maximized by $|\vec{k}_2| = |\vec{k}_3|$ and $\vec{k}_2 \parallel \vec{k}_3$, and so we have $\vec{k}_2 = \vec{k}_3$. Returning to our condition,

$$\sqrt{s_o} = \omega_1+\omega_2+\omega_3 = \omega_1+2\omega_2$$

and $\sqrt{s_o}-2\omega_2 = \omega_1 = \sqrt{|\vec{k}_1|^2+\mu^2} = \sqrt{|2\vec{k}_2|^2+\mu^2} = \sqrt{4(\omega_2^2-\mu^2)+\mu^2}$.
Squaring, this becomes:

$$s_o+4\omega_2^2-4\sqrt{s_o}\,\omega_2 = 4(\omega_2^2-\mu^2)+\mu^2$$
$$s_o-4\sqrt{s_o}\,\omega_2 = -3\mu^2$$
$$4\sqrt{s_o}\,\omega_2 = s_o+3\mu^2$$

$$\omega_2 = \frac{\sqrt{s_o}}{4} + \frac{3\mu^2}{4\sqrt{s_o}} \quad \text{and} \quad \omega_1 = \sqrt{s_o}-2\omega_2 = \frac{\sqrt{s_o}}{2} - \frac{3\mu^2}{2\sqrt{s_o}} .$$

Our lower limit will simply be μ. Since the conditions $\sqrt{s_o} = \omega_1+\omega_2+\omega_3$ and $\vec{k}_1+\vec{k}_2+\vec{k}_3 = 0$ are symmetrized in our two integration variables ω_1 and ω_2, we may use these limits for ω_2 also:

$$\sigma(\times) = \frac{\pi^3 g''^2}{2|\vec{p}|\sqrt{s_o}} \int_\mu^{\omega_{max}} d\omega_1 \int_\mu^{\omega_{max}} d\omega_2\ \Theta(1-u^2)$$

where

$$\omega_{max} \equiv \frac{\sqrt{s_o}}{2} - \frac{3\mu^2}{2\sqrt{s_o}} .$$

The optical theorem now yields:

$$A_3^R(s_o) = \pi^3 g''^2 \int_\mu^{\omega_{max}} d\omega_1 \int_\mu^{\omega_{max}} d\omega_2\ \Theta(1-u^2).$$

Two methods were used to approximate $A_3^R(s_o)$. The first method involved performing the double integration by applying a quadrature technique twice, once for each integral, using the IBM 360 computer at the City College. The result is seen in Figure 17a. The second method requires only one integral approximation (of the same type, using the IBM 360), since we can perform the integral over ω_2 as follows.

We seek to calculate ω_2 as a function of ω_1 at the limits of the ω_2 integration. Energy and momentum conservation demand that:

$$\omega_3 = \sqrt{s_o} - \omega_1 - \omega_2, \text{ and}$$

$$\omega_3 = \sqrt{\vec{k}_1^{\,2} + \vec{k}_2^{\,2} + 2|\vec{k}_1||\vec{k}_2|u + \mu^2}$$

where $u = \pm 1$ at the limits of the ω_2 integration, since it is the $\Theta(1-u^2)$ which restricts that integration by requiring that $u^2 \leq 1$. Combining the two conservation equations to eliminate ω_3, we have:

$$\sqrt{\vec{k}_1^{\,2} + \vec{k}_2^{\,2} \pm 2|\vec{k}_1||\vec{k}_2| + \mu^2} = \sqrt{s_o} - \omega_1 - \omega_2$$

at the limits of the ω_2 integration. Squaring both sides, we find:

$$\vec{k}_1^{\,2}+\vec{k}_2^{\,2}\pm 2|\vec{k}_1||\vec{k}_2|+\mu^2 = s_o+\omega_1^2+\omega_2^2+2\omega_1\omega_2-2\sqrt{s_o}(\omega_1+\omega_2).$$

Writing $\vec{k}_j^{\,2} = \omega_j^2 - \mu^2$ yields:

$$\pm 2\sqrt{\omega_1^2-\mu^2}\sqrt{\omega_2^2-\mu^2} = s_o+\mu^2+2\omega_1\omega_2-2\sqrt{s_o}(\omega_1+\omega_2) \equiv \phi \ .$$

Squaring the left side, we have $\phi^2 = 4(\omega_1^2-\mu^2)(\omega_2^2-\mu^2)$, or:

$$\omega_2^2 = \frac{\phi^2}{4(\omega_1^2-\mu^2)} + \mu^2 \ .$$

But ϕ is also equal to $s_o+\mu^2+2\omega_1\omega_2-2\sqrt{s_o}(\omega_1+\omega_2)$, so that:

$$\omega_2 = \frac{\phi-(s_o+\mu^2-2\sqrt{s_o}\,\omega_1)}{2(\omega_1-\sqrt{s_o})} \ .$$

Define $B = s_o+\mu^2-2\sqrt{s_o}\,\omega_1$, which depends explicitly on ω_1 and not on ω_2. Thus, the last equation becomes:

$$\omega_2 = \frac{\phi-B}{2(\omega_1-\sqrt{s_o})} \ .$$

We need $\phi=\phi(\omega_1)$ to find $\omega_2=\omega_2(\omega_1)$. But we can find $\phi=\phi(\omega_1)$ by eliminating ω_2 in our two equations:

$$\frac{\phi^2}{4(\omega_1^2-\mu^2)} + \mu^2 = \omega_2^2 = \left(\frac{\phi-B}{2(\omega_1-\sqrt{s_o})}\right)^2 \ .$$

Defining $\alpha = \left[4(\omega_1-\sqrt{s_o})^2\right]^{-1}$ and $\beta = \left[4(\omega_1^2-\mu^2)\right]^{-1}$, we have:

$$\beta\phi^2 + \mu^2 = \alpha(\phi-B)^2$$

which can be written as:

$$\phi^2\left(\frac{\alpha-\beta}{\alpha}\right) - 2B\phi - \frac{\mu^2}{\alpha} + B^2 = 0.$$

The definitions of α, β and B give us

$$\frac{\alpha-\beta}{\alpha} = \frac{-B}{\omega_1^{\,2}-\mu^2} ,$$

so that:

$$\phi^2\left(\frac{-B}{\omega_1^{\,2}-\mu^2}\right) - 2B\phi + B^2 - \frac{\mu^2}{\alpha} = 0$$

where $B = s_o+\mu^2-2\sqrt{s_o}\,\omega_1$ and $\alpha = \left[4(\omega_1-\sqrt{s_o})^2\right]^{-1}$. We can now solve this quadratic equation for $\phi=\phi\,(\omega_1)$:

$a\phi^2 + b\phi + c = 0$

where $a \equiv -B(\omega_1^{\,2}-\mu^2)^{-1}$, $b \equiv -2B$ and $c \equiv B^2-4\mu^2(\omega_1-\sqrt{s_o})^2$. The solution is:

$$\phi_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \left(\frac{B \pm \sqrt{B^2-ac}}{-B}\right)(\omega_1^{\,2}-\mu^2)$$

where $\omega_{2\pm} \equiv (\phi_\pm - B)\left[2(\omega_1-\sqrt{s_o})\right]^{-1}$. We can now write:

$$A_3^R(s_o)=\pi^3 g''^2\int_\mu^{\omega_{max}} d\omega_1\ (\omega_{2+} - \omega_{2-})$$

where

$$\omega_{2+} - \omega_{2-} = \frac{\phi_+ - \phi_-}{2(\omega_1-\sqrt{s_o})} = \frac{2\sqrt{B^2-ac}}{-B}(\omega_1^{\,2}-\mu^2)\frac{1}{2(\omega_1-\sqrt{s_o})}$$

Thus, the integrand becomes:

$$\omega_{2+}-\omega_{2-} = -\left(\frac{\omega_1^{\,2}-\mu^2}{\omega_1-\sqrt{s_o}}\right)\left(\frac{\sqrt{(s_o+\mu^2-2\sqrt{s_o}\,\omega_1)^2+\frac{B}{\omega_1^{\,2}-\mu^2}(B^2-4\mu^2[\omega_1-\sqrt{s_o}]^2)}}{s_o+\mu^2-2\sqrt{s_o}\,\omega_1}\right).$$

The computer approximation technique can now be applied to the integration over ω_1, and the result is found in Figure 17b, and it is consistent with our previous calculation of $A_3^R(s_o)$. We can write, to an excellent approximation, for $s_o \gg \mu^2$:

$$A_3^R(s_o) = 3.724\ g''^2\mu^2\left(\frac{s_o}{\mu^2}\right)$$

where again, a cutoff will be adopted later on in the calculation. We define a convenient dimensionless coupling constant as $g_3 = (g''\mu)\sqrt{3.724}$ (remember that the Lagrangian density goes as k^4 and $\chi \sim g''\varphi^5 \sim g''k^5$, so that $g'' \sim k^{-1}$). Our third resonant amplitude is then:

$$A_3^R(s_o) = g_3^{\,2}\left(\frac{s_o}{\mu^2}\right). \tag{2.27}$$

(iv). Higher Order Resonant Amplitudes.

We assume that for $m>3$ and $s_o \gg (m\mu)^2$, the m^{th} resonant amplitude. corresponding to two particles of center of mass energy s_o interacting to form m particles which subsequently return to the two particle state, is given by the phase space approximation:

$$A_m^R(s_o) = g_m^{\,2}\left(\frac{s_o}{\mu^2}\right)^{m-2}, \quad s_o \leq s_{omax} \tag{2.28}$$

and $A_m^R(s_o) = 0$ for $s_o > s_{omax}$. We have chosen to define $A_m^R(s_o)$ in terms of the dimensionless coupling constant g_m, which differs from the actual Feynman coupling constant by a constant factor(Note that $\chi \sim k^4$ and $\chi \sim g\varphi^{m+2} \sim gk^{m+2}$, so that $g \sim k^{2-m}$ and the dimensionless coupling constants are given by $g_m \sim g/\mu^{2-m}$).

Before beginning the series of integrations needed to find the average multiplicity and the correlation functions, we will remark on the reasonableness of our choice of resonant amplitudes. Asymptotically, for $m \geq 2$, we have chosen:

$$A_m^R(s_o) = g_m^{\,2}\left(\frac{s_o}{\mu^2}\right)^{m-2} \Theta(s_{omax}-s_o).$$

This function rises with s_o from $s_o=0$ to $s_o=s_{omax}$, and then drops to zero for $s_o > s_{omax}$. It is thus sharply peaked at $s_o=s_{omax}$, much like a delta function, and in this respect it resembles the original AFS model, where $A^R(s_o) \sim \delta(s_o-\mu^2)$

was selected.[2]

In the next section, we will begin our integrations.

(v). Integrations Over s_o.

Recall that the integrals involved in calculating the average multiplicity and correlation functions are:

$$F_m^{(k)} = \int \frac{ds_o}{16\pi^3} A_m^R(s_o) \int_o^1 dx \frac{x^\alpha (1-x)(\ln x)^k}{s_o x+\mu^2(1-x)^2} \quad .$$

The s_o integrations are all of the form:

$$I_m \equiv \int ds_o \frac{A_m^R(s_o)}{s_o x+\mu^2(1-x)^2} \equiv \int ds_o \frac{A_m^R(s_o)}{\alpha s_o+\beta}$$

which may be performed exactly for all eight of the resonant amplitudes which we will use.

a). m=1.

Consider the case m=1:

$$I_1 = \int ds_o \frac{(g_1\mu)^2 \delta(s_o-\mu^2)}{\alpha s_o+\beta} = \frac{(g_1\mu)^2}{\alpha\mu^2+\beta} = \frac{g_1^{\ 2}}{x+(1-x)^2} \quad .$$

The other cases (m$>$1) are not as simple, but all are straight forward.

b). m=2.

The more difficult case of m=2 is:

$$I_2 = g_2{}^2 \int_{4\mu^2}^{s_{omax}} \frac{ds_o}{\alpha s_o+\beta} \sqrt{\frac{s_o-4\mu^2}{s_o}} .$$

Change variables, defining $y = \alpha s_o+\beta$, yielding:

$$I_2 = g_2{}^2 \int_{4\mu^2\alpha+\beta}^{\alpha s_{omax}+\beta} \frac{dy}{\alpha y} \sqrt{\frac{y-\beta-4\mu^2\alpha}{y-\beta}} .$$

Drop the limits for convenience of notation. The integral is of the form:

$$I_2 = g_2{}^2 \int \frac{dy}{\alpha y} \sqrt{\frac{ay+b}{my+n}} = g_2{}^2 \frac{a}{\alpha} \int \frac{dy}{\sqrt{(my+n)(ay+b)}}$$

$$+ g_2{}^2 \frac{b}{\alpha} \int \frac{dy}{y\sqrt{(my+n)(ay+b)}}$$

where $a=m=1$, $b=-\beta-4\mu^2\alpha$, $n=-\beta$, and we have used the fact that $a+\frac{b}{y} = \frac{ay+b}{y} = \frac{(\sqrt{ay+b})^2}{y}$. The first term of I_2 is simply:

$$g_2{}^2 \frac{a}{\alpha} \frac{2}{\sqrt{am}} \ln\left[\sqrt{m(ay+b)} + \sqrt{a(my+n)}\right] .$$

The second term of I_2 is:

$$\int \frac{dy}{y\sqrt{(my+n)(ay+b)}} = -2\int \frac{dz}{n-bz^2} = -\frac{2}{n}\int \frac{dz}{1-\frac{b}{n}z^2}$$

where $z^2 \equiv (my+n)/(ay+b)$. Now $(b/n) = [(\beta+4\mu^2\alpha)/\beta] \geq 1$, so try the substitution $\cosh\theta = \sqrt{\frac{b}{n}}\, z$. We thus have:

$$\int \frac{dy}{y\sqrt{(my+n)(ay+b)}} = \frac{2}{\sqrt{bn}} \int \operatorname{csch}\theta \, d\theta = \frac{-2}{\sqrt{bn}} \ln(\operatorname{csch}\theta + \coth\theta).$$

Returning to the variable z, this becomes:

$$\int \frac{dy}{y\sqrt{(my+n)(ay+b)}} = \frac{-2}{\sqrt{bn}} \ln\left(\frac{1}{\sqrt{\frac{b}{n}z^2-1}} + \frac{\sqrt{\frac{b}{n}}\, z}{\sqrt{\frac{b}{n}z^2-1}}\right)$$
$$= \frac{-2}{\sqrt{bn}} \ln\sqrt{\frac{1+\sqrt{\frac{b}{n}}\, z}{\sqrt{\frac{b}{n}}\, z - 1}}\,.$$

Our full integral is now:

$$I_2 = g_2^{\,2}\, \frac{a}{\alpha}\, \frac{2}{\sqrt{am}} \ln\left[\sqrt{m(ay+b)} + \sqrt{a(my+n)}\right]$$
$$+\, g_2^{\,2}\, \frac{b}{\alpha}\left(\frac{-2}{\sqrt{bn}}\right) \ln\sqrt{\frac{\sqrt{\frac{b}{n}}\, z + 1}{\sqrt{\frac{b}{n}}\, z - 1}}$$

where $z = \sqrt{(y-\beta)/(y-\beta-4\mu^2\alpha)}$ and we must evaluate the expression on the right between $y=4\mu^2\alpha+\beta$ and $y=\alpha s_{omax}+\beta$. First we write:

$$I_2 = \left[g_2^{\,2}\, \frac{2}{\alpha} \ln\left(\sqrt{y-\beta-4\mu^2\alpha} + \sqrt{y-\beta}\right) - g_2^{\,2}\, \frac{2}{\alpha}\sqrt{\frac{\beta+4\mu^2\alpha}{\beta}} \ln\sqrt{\frac{\sqrt{\frac{\beta+4\mu^2\alpha}{\beta}}\, z + 1}{\sqrt{\frac{\beta+4\mu^2\alpha}{\beta}}\, z - 1}}\right]_{4\mu^2\alpha+\beta}^{\alpha s_{omax}+\beta}.$$

If we define $\gamma = \sqrt{(\beta+4\mu^2\alpha)/\beta} = (1+x)/(1-x)$ and

$D = s_{omax}/(s_{omax}-4\mu^2)$, we have:

$$I_2 = g_2^{\,2}\left[\frac{2}{x} \ln\left(\sqrt{\frac{s_{omax}-4\mu^2}{4\mu^2}} + \sqrt{\frac{s_{omax}}{4\mu^2}}\right) - \frac{2\gamma}{x} \ln\sqrt{\frac{\gamma\sqrt{D}+1}{\gamma\sqrt{D}-1}}\right].$$

Furthermore, if $S \equiv (s_{omax}/4\mu^2)$, this is:

$$I_2 = g_2^{\,2}\left[\frac{2}{x}\ln\left[(S-1)^{\frac{1}{2}} + S^{\frac{1}{2}}\right]\right.$$

$$\left.- \frac{1}{x}\left(\frac{1+x}{1-x}\right)\ln\left(\frac{(1+x)D^{\frac{1}{2}} + 1 - x}{(1+x)D^{\frac{1}{2}} - 1 + x}\right)\right].$$

c). m=3.

The third integral to be considered is:

$$I_3 = \frac{g_3^{\,2}}{\mu^2}\int_{9\mu^2}^{s_{omax}} \frac{s_o ds_o}{\alpha s_o + \beta}$$

which is almost trivial to integrate:

$$I_3 = g_3^{\,2}\left[\left(\frac{s_{omax}}{\mu^2} - 9\right)\frac{1}{x} - \frac{(1-x)^2}{x}\ln\left(\frac{\frac{s_{omax}}{\mu^2}x + (1-x)^2}{9x + (1-x)^2}\right)\right].$$

d). m=4,5,6,7 and 8.

The fourth resonant amplitude requires the integration:

$$I_4 = \frac{g_4^{\,2}}{\mu^4}\int_{16\mu^2}^{s_{omax}} ds_o \frac{s_o^{\,2}}{s_o x + \mu^2(1-x)^2}$$

$$= \frac{g_4^{\,2}}{\mu^4}\left[\frac{s_o^{\,2}}{2x} - \frac{\mu^2(1-x)^2 s_o}{x^2} + \frac{\mu^4(1-x)^4}{x^3}\ln\left[s_o x + \mu^2(1-x)^2\right]\right]_{16\mu^2}^{s_{omax}}$$

$$= g_4^{\,2}\left[\frac{(s_{omax}/\mu^2)^2 - 256}{2x} - \frac{(1-x)^2}{x^2}\left(\frac{s_{omax}}{\mu^2} - 16\right)\right.$$

$$\left.+ \frac{(1-x)^4}{x^3}\ln\left(\frac{(s_{omax}/\mu^2)x + (1-x)^2}{16x + (1-x)^2}\right)\right].$$

Similarly, with the definition $S' = (s_{omax}/\mu^2)$, the other integrals of interest to us are:

$$I_5 = g_5^2\left[\frac{S'^3-15{,}625}{2x} - \frac{(1-x)^2}{2x^2}(S'^2-625) + \frac{(1-x)^4}{x^3}(S'-25) - \frac{(1-x)^6}{x^4}\ln\left(\frac{S'x+(1-x)^2}{25x+(1-x)^2}\right)\right],$$

$$I_6 = g_6^2\left[\frac{S'^4-(36)^4}{4x} - \frac{(1-x)^2}{3x^2}[S'^3-(36)^3] + \frac{(1-x)^4}{2x^3}[S'^2-(36)^2] - \frac{(1-x)^6}{x^4}(S'-36) + \frac{(1-x)^8}{x^5}\ln\left(\frac{S'x+(1-x)^2}{36x+(1-x)^2}\right)\right],$$

$$I_7 = \frac{g_7^2}{\mu^{10}x^6}\left[\frac{y^5}{5}-\frac{5}{4}\mu^2(1-x)^2y^4+\frac{10}{3}\mu^4(1-x)^4y^3-5\mu^6(1-x)^6y^2 +5\mu^8(1-x)^8y-\mu^{10}(1-x)^{10}\ln(y)\right]_{y=49\mu^2x+\mu^2(1-x)^2}^{y=s_{omax}x+\mu^2(1-x)^2},$$

$$I_8 = \frac{g_8^2}{\mu^{12}x^7}\left[\frac{y^6}{6}-\frac{6}{5}\mu^2(1-x)^2y^5+\frac{15}{4}\mu^4(1-x)^4y^4-\frac{20}{3}\mu^6(1-x)^6y^3 +\frac{15}{2}\mu^8(1-x)^8y^2-6\mu^{10}(1-x)^{10}y+\mu^{12}(1-x)^{12}\ln(y)\right]_{y_-}^{y_+}$$

where $y_- \equiv 64\mu^2x+\mu^2(1-x)^2$ and $y_+ = s_{omax}x+\mu^2(1-x)^2$.

Now that the integrations involved in calculating the $A_m^R(s_o)$ have been taken care of, and the integrals over s_o have all been performed, only the integrations over **x** remain:

$$F_m^{(k)} = \frac{1}{16\pi^3}\int_0^1 dx\ I_m(x)\ x^\alpha(1-x)(\ln x)^k\ . \tag{2.29}$$

These integrals were accomplished on the IBM 360 computer using a library subroutine for gaussian quadrature in 32 intervals. To obtain sufficient accuracy, the integration interval (zero to one) was split into five pieces, with the subroutine applied to each piece, for a total mesh of 5X32=160. The machine was also used to calculate the average multiplicity and the two and three body correlation functions from these integrals. In the search for a best fit, the PDP 10 computer was also used (the results of the IBM 360 for the integrals were used in the program for the PDP10). The results are discussed in the following chapter.

4. NUMERICAL RESULTS

Least square fits to the experimental data at and above 50 GeV/c in Table 4 of Ref. 1 (Slattery) yield:

$$\langle n^{ch}\rangle = -3.65 + (1.95 \pm 0.035)\ln(s),$$
$$f_2^{ch} = -20.9 + (4.80 \pm 0.19)\ln(s),$$
$$f_3^{ch} = -10.8 + (2.87 \pm 1.18)\ln(s). \tag{2.30}$$

where s is in $(GeV)^2$.

To include the neutral pions, we can use recent π^0 data from Ref. 26 or from Ref. 27. The experimental data is rather ambiguous here. If we select Ref. 26 and apply the 205 GeV/c result to the model generally, we have:

$$\langle n_{\pi^0}\rangle = (0.6 \pm 0.2)\langle n_{\pi^-}\rangle + 1 \pm 0.5$$

and we assume that $\langle n_{\pi^+}\rangle = \langle n_{\pi^-}\rangle$.

To convert the experimental data for charged particles and neutrals into multiplicity and correlation functions for all pions, we define:

$$\langle n_\pi\rangle = \alpha\langle n_\pi^{ch}\rangle + \beta$$

so that

$$\langle n\rangle = \langle n_\pi\rangle + 2 = \alpha\langle n_\pi^{ch}\rangle + \beta + 2 = \alpha\left[\langle n^{ch}\rangle - 2\right] + \beta + 2$$

where $\langle n\rangle$ and $\langle n^{ch}\rangle$ include the two protons. Thus:

$$\langle n\rangle = \alpha\langle n^{ch}\rangle + (\beta - 2\alpha + 2).$$

Now $f_2 = f_2^{\pi} - 2$, and f_2^{π} may be written:

$$\begin{aligned}
f_2^{\pi} &= \langle n_{\pi}(n_{\pi} - 1)\rangle - \langle n_{\pi}\rangle^2 \\
&= \langle(\alpha n_{\pi}^{ch} + \beta)(\alpha n_{\pi}^{ch} + \beta - 1)\rangle - \langle\alpha n_{\pi}^{ch} + \beta\rangle^2 \\
&= \alpha^2\langle n_{\pi}^{ch}(n_{\pi}^{ch} - 1)\rangle + \alpha(\alpha - 1)\langle n_{\pi}^{ch}\rangle - \beta - \alpha^2\langle n_{\pi}^{ch}\rangle^2 \\
&= \alpha^2 f_2^{\pi ch} + \alpha(\alpha - 1)\langle n_{\pi}^{ch}\rangle - \beta.
\end{aligned}$$

In terms of f_2 for all particles, this is:

$$\begin{aligned}
f_2 &= \alpha^2 f_2^{\pi ch} + \alpha(\alpha - 1)\langle n_{\pi}^{ch}\rangle - \beta - 2 \\
&= \alpha^2(f_2^{ch} + 2) + \alpha(\alpha - 1)(\langle n^{ch}\rangle - 2) - \beta - 2 \\
&= \alpha^2 f_2^{ch} + \alpha(\alpha - 1)\langle n^{ch}\rangle - (\beta - 2\alpha + 2).
\end{aligned}$$

We now apply the same analysis to the three body correlation function, $f_3 = f_3^{\pi} - 2$, where:

$$\begin{aligned}
f_3^{\pi} &= \langle n_{\pi}(n_{\pi} - 1)(n_{\pi} - 2)\rangle - 3\langle n_{\pi}\rangle f_2^{\pi} - \langle n_{\pi}\rangle^3 \\
&= \langle(\alpha n_{\pi}^{ch} + \beta)(\alpha n_{\pi}^{ch} + \beta - 1)(\alpha n_{\pi}^{ch} + \beta - 2) \\
&\qquad - 3\langle\alpha n_{\pi}^{ch} + \beta\rangle\left[\alpha^2 f_2^{\pi ch} + \alpha(\alpha - 1)\langle n_{\pi}^{ch}\rangle - \beta\right] - \langle\alpha n_{\pi}^{ch} + \beta\rangle^3
\end{aligned}$$

We seek to manipulate this until we obtain f_3^{π} as a function of $f_3^{\pi ch}$, $f_2^{\pi ch}$ and $\langle n_{\pi}^{ch}\rangle$, to aid us in finding f_3 as a function of f_3^{ch}, f_2^{ch} and $\langle n^{ch}\rangle$. Continuing then:

$$f_3^{\pi} = \langle (\alpha n_\pi^{ch})(\alpha n_\pi^{ch}-1)(\alpha n_\pi^{ch}-2)\rangle$$

$$+ \beta \langle \alpha^2 n_\pi^{ch^2} + \alpha n_\pi^{ch}(\alpha n_\pi^{ch}+\beta-2) - \alpha n_\pi^{ch}\rangle$$

$$+ \beta \langle (\alpha n_\pi^{ch}+\beta-1)(\alpha n_\pi^{ch}+\beta-2)\rangle \quad -3\langle \alpha n_\pi^{ch}\rangle \langle \alpha^2 f_2^{\pi ch}\rangle$$

$$- 3\beta\left[\alpha^2 f_2^{\pi ch} + \alpha(\alpha-1)\langle n_\pi^{ch}\rangle - \beta\right]$$

$$- 3\alpha\langle n_\pi^{ch}\rangle\left[\alpha(\alpha-1)\langle n_\pi^{ch}\rangle - \beta\right]$$

$$- \alpha^3\langle n_\pi^{ch}\rangle^3 - 3\alpha^2\beta\langle n_\pi^{ch}\rangle^2 - 3\alpha\beta^2\langle n_\pi^{ch}\rangle - \beta^3$$

which may be rewritten as:

$$f_3^{\pi} = \alpha^3\langle n_\pi^{ch}(n_\pi^{ch}-1)(n_\pi^{ch}-2)\rangle + 3(\alpha^3-\alpha^2)\langle n_\pi^{ch^2}\rangle$$

$$+ 2(\alpha-\alpha^3)\langle n_\pi^{ch}\rangle + \beta\langle 2\alpha^2 n_\pi^{ch^2} + \alpha(\beta-2)n_\pi^{ch} - \alpha n_\pi^{ch}$$

$$+ (\alpha n_\pi^{ch}+\beta-1)(\alpha n_\pi^{ch}+\beta-2)\rangle$$

$$-3\alpha^3\langle n_\pi^{ch}\rangle f_2^{\pi ch} - 3\beta\left[\alpha^2 f_2^{\pi ch} + \alpha(\alpha-1)\langle n_\pi^{ch}\rangle - \beta\right]$$

$$- 3\alpha\langle n_\pi^{ch}\rangle\left[\alpha(\alpha-1)\langle n_\pi^{ch}\rangle - \beta\right] - \alpha^3\langle n_\pi^{ch}\rangle^3$$

$$- 3\alpha^2\beta\langle n_\pi^{ch}\rangle^2 - 3\alpha\beta^2\langle n_\pi^{ch}\rangle - \beta^3 .$$

Continuing our substitutions:

$$f_3^{\pi} = \alpha^3 f_3^{\pi ch} + 3\alpha^2(\alpha-1)\langle n_\pi^{ch^2}\rangle + 2\alpha(1-\alpha^2)\langle n_\pi^{ch}\rangle$$

$$+\beta\langle 3\alpha^2 n_\pi^{ch^2} + \alpha(\beta-3)n_\pi^{ch} + (\alpha n_\pi^{ch})(2\beta-3) + (\beta-1)(\beta-2)\rangle$$

$$-3\beta\left[\alpha^2 f_2^{\pi ch} + \alpha(\alpha-1)\langle n_\pi^{ch}\rangle - \beta\right] - 3\alpha\langle n_\pi^{ch}\rangle\left[\alpha(\alpha-1)\langle n_\pi^{ch}\rangle - \beta\right]$$

$$-3\alpha^2\beta\langle n_\pi^{ch}\rangle^2 - 3\alpha\beta^2\langle n_\pi^{ch}\rangle - \beta^3$$

$$= \alpha^3 f_3^{\pi ch} + 3\alpha^2(\alpha-1) f_2^{\pi ch} + 3\alpha^2(\alpha-1)\langle n_\pi^{ch}\rangle$$

$$+2\alpha(1-\alpha^2)\langle n_\pi^{ch}\rangle + \beta(\beta-1)(\beta-2) + 3\beta^2 - \beta^3$$

$$= \alpha^3 f_3^{\pi ch} + 3\alpha^2(\alpha-1) f_2^{\pi ch} + \alpha(\alpha-1)(\alpha-2)\langle n_\pi^{ch}\rangle + 2\beta.$$

Finally, we may write f_3 itself:

$$f_3 = \alpha^3 f_3^{\pi ch} + 3\alpha^2(\alpha-1) f_2^{\pi ch} + \alpha(\alpha-1)(\alpha-2)\langle n_\pi^{ch}\rangle + 2(\beta-1)$$

$$= \alpha^3(f_3^{ch}+2) + 3\alpha^2(\alpha-1)(f_2^{ch}+2) + \alpha(\alpha-1)(\alpha-2)(\langle n^{ch}\rangle-2) + 2(\beta-1)$$

$$= \alpha^3 f_3^{ch} + 3\alpha^2(\alpha-1) f_2^{ch} + \alpha(\alpha-1)(\alpha-2)\langle n^{ch}\rangle + 6\alpha^3 - 4\alpha + 2\beta - 2.$$

Now consider the experimental data under our assumptions:

$$\langle n_\pi\rangle = \langle n_{\pi^0}\rangle + 2\langle n_{\pi^-}\rangle = (0.6\pm 0.2)\langle n_{\pi^-}\rangle + 1\pm.5 + 2\langle n_{\pi^-}\rangle$$

or more simply:

$$\langle n_\pi \rangle = (2.6 \pm 0.2)\langle n_{\pi^-} \rangle + 1 \pm 0.5$$

$$= (1.3 \pm 0.1)\langle n_\pi^{ch} \rangle + 1 \pm 0.5$$

so that $\alpha = 1.3 \pm 0.1$ and $\beta = 1 \pm .5$. These values of α and β together with the aforementioned experimental data for charged particles in Ref. 1 yield:

$$\begin{aligned} \langle n \rangle &= -4.35 + (2.54 \pm 0.06)\ln(s), \\ f_2 &= -37.1 + (8.87 \pm 0.4)\ln(s), \\ f_3 &= -46.5 + (13.1 \pm 3.6)\ln(s). \end{aligned} \tag{2.31}$$

If we would have instead chosen to adopt the evidence of Ref. 27, we then would have had $\langle n_{\pi^0} \rangle = \langle n_{\pi^-} \rangle$, so that $\alpha=3/2$ and $\beta=0$, yielding with our charged particle experimental data:

$$\begin{aligned} \langle n \rangle &= -6.48 + 2.93\ln(s), \\ f_2 &= -48.8 + 12.3\ln(s), \\ f_3 &= -93.4 + 25.2\ln(s). \end{aligned} \tag{2.32}$$

In an effort to match the experimental data, we investigated the cluster model for M=1,2,3,4,5,6,7 and 8, where M is the maximum value of m_j allowed in the chain. It was necessary to include clusters of up to eight particles to obtain a reasonable fit to the data.

In Figure 18, we observe that the ln(s) coefficient of f_2 is negative for the M=1 model, which corresponds to the simple φ^3 theory. Figure 19 reveals that we can get closer to the experimental data by increasing the cluster sizes which may be found along the chain. By allowing cluster sizes of up to eight particles, while retaining the smaller clusters as well, we can obtain crude fits for both $\langle n \rangle$ and f_2 simultaneously (See Figure 20). Note that a model in which only one cluster type (m_j=M for all j) is allowed would permit a fit for f_2 at $M \approx 6$, but would not fit $\langle n \rangle$ at the same time (See Table 2). To fit both $\langle n \rangle$ and f_2, we must have more than one vertex type present in the chain, so that $m_j \leq M$. We observe from Figure 20 the theoretical results at $\alpha = 1.00$:

$$
\begin{aligned}
&g_8^{\,2} = 1.38\times10^{-21}, \\
&\langle n \rangle \sim 2.98\ln(s), \\
&f_2 \sim 7.81\ln(s), \\
&f_3 \sim 24.4\ln(s). \qquad\qquad (2.33)
\end{aligned}
$$

By allowing even larger clusters to appear, it may be possible to better fit both $\langle n \rangle$ and f_2, but it would probably push f_3 further from its experimental value.

The need for several different cluster types to fit the data brings up the question of the relative influence of the various cluster sizes. The recursion relation (2.3) may be used to test the strength of the various diagrams. Recall that the partial cross-section for production of $\mathcal{N}$ particles is related to the elastic amplitude with $\mathcal{N}$ intermediate particles:

$$\Delta^{\frac{1}{2}}(s,p^2,p'^2)\,\sigma^{s}(s;n_1,n_2,\ldots) = A^{s}(pp';n_1,n_2,\ldots)$$

where the superscript on σ indicates that we have summed over permutations of the cluster types $m_1,m_2,\ldots\}$ while keeping the numbers of each type $n_1,n_2,\ldots$, fixed. To find a given partial cross-section, we iterate (2.3) by starting with the following expressions on the right side:

$$A^{s}(pq;1,0,0,\ldots) \equiv A_1^R(s_o),$$
$$A^{s}(pq;0,1,0,\ldots) \equiv A_2^R(s_o),$$
$$\text{etc.},$$

where here, $s_o = (p+q)^2$. For example, the cross section for production of $\mathcal{N} = n_m$ particles exclusively by clusters of type m is proportional to:

$$A(0,0,0,\ldots,n_m,0,0,\ldots) \rightarrow \frac{16\pi^3\mu^2\lambda_m^2}{s}\,\frac{(\lambda_m \ln(s))^{n_m-2}}{(n_m-2)!}$$

where $\lambda_m \equiv \frac{1}{16\pi^3\mu^2}\int ds_o\, A_m^R(s_o)$, and where this result was

obtained by rewriting (2.3) in terms of s and u, and the appropriate limits were used (as in (2.4) for the total elastic amplitude).[2] An example of an amplitude where the clusters are of mixed types is:

$$A(2,1,0,0,\ldots) \rightarrow \frac{16\pi^3\mu^2(3\lambda_1{}^2\lambda_2{}^2)(\ln s)}{s}\ .$$

The results of a dozen such calculations are summarized in Table 3. The relative strengths depend on s. Nevertheless, the table reveals that the large clusters do not necessarilly dominate. Indeed, double clusters may be very influential, so that, for example, sixteen-particle production is likely to have many clusters contributing, rather than two clusters of eight particles each being dominant.

Before concluding this discussion of our numerical analysis, we wish to consider another aspect of these results. A fit nearly identical to (2.33) may be obtained while retaining only clusters of types 2 and 8 ($g_2{}^2=100$, $g_8{}^2=1.41\times10^{-21}$, all other $g_m{}^2=0$). This is consistent with our result demanding cluster sizes of at least eight particles, plus at least one other cluster size. The result of this calculation is:

$$\langle n\rangle = 2.98\ln(s) + \text{constant},$$
$$f_2 = 7.92\ln(s) + \text{constant},$$
$$f_3 = 25.2\ln(s) + \text{constant}.$$

Comparison of this with (2.33) reveals that the latter is, in reality, a two-cluster fit also(compare the g_m^2's for the various models in Table 2). Indeed, only one parameter, g_2^2, is needed to obtain our fit for the multiplicity and two-body correlation function (g_8^2 is implied by the eigenvalue equation once alpha is set at 1.00 and g_2^2 is chosen).

Finally, we note that numerical calculations of $\langle n \rangle /\ln(s)$ and $f_2/\ln(s)$ at high s in a model where every cluster is replaced by a single ρ invariably yield negative values of $f_2/\ln(s)$, and in fact, a simple ρ-type cluster is incompatible with the experimental data, independently of the assumed mass of the ρ (See Appendix D).

5. CONCLUSIONS

In both the simple multiperipheral model (M=1) and the present multiple cluster model, we find a special form of KNO scaling due to the fact that $f_k \sim c_k \ln(s)$ for all k, so that

$$\frac{\langle n^k \rangle}{\langle n \rangle^k} = 1. \qquad (2.34)$$

However, the data of Ref. 1 suggests that this ratio may increase with k.

Nevertheless, by fitting the data to linear equations in ln(s), $f_k \sim c_k \ln(s)$, not only can we drive the two body correlation function positive at high energy, but we can obtain theoretical results for $\langle n \rangle$, f_2 and f_3 which crudely match these fits.

Note that our conclusion that very large multiplicity clusters are necessary to fit f_2 and $\langle n \rangle$ is consistent with the large cluster mass found by Hamer and Peierls[4] to be necessary to obtain agreement between the slope of the diffraction peak and the increase of multiplicity of produced pions. It is also consistent with the results of the cluster model of Berger and Fox, who also found that M=8 provides a satisfactory fit for f_2.[4] It would appear that a multiperipheral cluster model can possibly agree with the finer details of the experimental results only if the size of the clusters is very large and if more than one cluster size is allowed to be present. **This implies that any attempt to fit** a cluster model to the pion resonances observed to date must fail, as it would not include clusters of more than 5π. We have found that we need clusters of 8π to fit data for multiplicity and two-body correlation function. The existence of such higher resonances is thus predicted by our cluster model.

Finally, we may consider the implications of the model in other terms. The existence of clusters of size m implies the presence of Lagrangian density terms which go as φ^{m+2}. It may be that clusters larger than m=8 would provide a better fit. Indeed, perhaps the Lagrangian is a sum of infinite terms, and it would be nice if this sum had some closed form. So far as the simplicity of calculations are concerned, it would also be nice if the $A_m^R(s_o)$ were summable to a closed form. Since the present model contains only two significant cluster types (m=2 and m=8) in its fit to the experimental data, it is not yet possible to investigate these questions beyond speculation. Nevertheless, the inclusion of higher m clusters suggests the possibility of a better fit to the experimental data, and the chance that other massive resonances will be predicted by the theory.

APPENDICES

APPENDIX A. DETAILS OF THE SURVEY OF MODELS.

1. DIFFRACTIVE FRAGMENTATION MODEL.

In the multiperipheral model to be discussed in Part II, the virtual state of the elastic two-body high energy interaction is characterized by any discrete number of virtual particles. However, in the diffractive or optical picture of the interaction, the incident particles appear to consist of an infinite number of infinitesimal particles in the virtual state; hence, the name "droplet model".[6] The incident particles thus have finite spatial extension and "go through" each other with attenuation. Such a viewpoint yields an excellent means of studying the elastic interaction by way of an impact parameter representation. We will discuss this aspect of the model before examining its application to inelastic processes.

While the optical model of the nucleus is roughly a quarter of a century old[7], it was first applied to hadron-hadron scattering in the late sixties by Chou and Yang.[6] Since both projectile and target are spatially extended, we are led to apply our impact representation to both particles, rather than just to the target particle.

The elastic differential cross section is

$$\frac{d\sigma}{dt} = \pi\,|a|^2 \quad , \qquad (A.1)$$

where $\sqrt{-t}$ is the three-momentum transfer in the center of mass system, and where the amplitude "a" is expanded in partial waves:

$$a = \lambda^2 \sum_l (2l + 1) P_l(\cos\theta) \frac{(1-S)}{2} \quad . \tag{A.2}$$

At large energies and small angles,

$$\sum_l \longrightarrow \int dl \quad \text{and } P_l(\cos\theta) \longrightarrow J_o(b\sqrt{-t}),$$

where $b \equiv \lambda\,(l + \frac{1}{2})$. Experimentally, the differential cross section approaches a limit f(t) as the incoming energy grows large. This implies that the transmission coefficient S is a function only of the impact parameter b. The amplitude may now be written:

$$a = \int_0^\infty (1 - S) J_o(b\sqrt{-t})\, b\, db. \tag{A.3}$$

Since $J_o(b\sqrt{-t}) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left[ib(\sqrt{-t})\cos\phi\right] \, d\phi$,

we have:

$$a = \frac{1}{2\pi}\int_0^\infty (1-S) \int_0^{2\pi} \exp\left[ib(\sqrt{-t})\cos\phi\right] \, d\phi \;\; b\, db \; . \tag{A.4}$$

Following Chou and Yang further, we introduce a two-dimensional momentum transfer vector $\vec{K}$ and a two-dimensional impact parameter $\vec{b}$ in the plane perpendicular to the incident projectile, with $\vec{K}^2 = -t$ and $\vec{b}^2 = b^2$. Letting ϕ be the angle between $\vec{K}$, $\vec{b}$, we have:

$$a = \frac{1}{2\pi}\iint \left[1 - S(\vec{b})\right] \exp(i\vec{K}\cdot\vec{b})\, d^2b \;=\; \langle 1 - S \rangle \tag{A.5}$$

where $\langle X \rangle$ is used to denote the fourier transform of $X(\vec{b})$. We also use this notation for the inverse fourier transform of functions of $\vec{K}$.

We can relate the amplitude "a" to the function $s \equiv -\langle \ln(S) \rangle$ by defining folding integrals:

$$X \otimes Y\Big|_{\vec{b}} = \frac{1}{2\pi}\iint X(\vec{b}-\vec{b}')Y(\vec{b}')\ d^2b' \tag{A.6}$$

$$a \otimes c\Big|_{\vec{K}} = \frac{1}{2\pi}\iint a(\vec{K}-\vec{K}')c(\vec{K}')\ d^2K' \tag{A.7}$$

and noting that

$$\langle X\rangle \langle Y\rangle = \langle X \otimes Y\rangle \text{ and } \langle a\rangle\langle c\rangle = \langle a \otimes c\rangle . \tag{A.8}$$

By (1.5), we have:

$$\langle a\rangle = 1-S = 1-\exp(\ln S) = -\ln S-\frac{1}{2!}(\ln S)^2-\frac{1}{3!}(\ln S)^3-\ldots$$

and

$$-\ln S = -\ln(1-\langle a\rangle) = \langle a\rangle + \frac{1}{2}(\langle a\rangle)^2+\frac{1}{3}(\langle a\rangle)^3+\ldots$$

These two relations, with the identities (1.8), yield:

$$a = s - \frac{1}{2!}s \otimes s + \frac{1}{3!}s \otimes s \otimes s - \ldots \tag{A.9}$$

$$s = a + \frac{1}{2}a \otimes a + \frac{1}{3}a \otimes a \otimes a + \ldots \tag{A.10}$$

We must still relate s to observables. Classically, for the transmission of a wave through a slab of thickness g, the transmission coefficient S is $\exp(-\alpha g)$, where α is a constant. Thus, $\langle s\rangle = -\ln S(\vec{b})$ is proportional to the opaqueness of the hadron at $\vec{b}$. If we define a two-dimensional opaqueness function $D(x,y) = \int_{-\infty}^{+\infty} \rho(x,y,z)\, dz$, where ρ is the density of opaqueness, we can write:

$$\langle s\rangle = -\ln S(\vec{b}) = K_{ij}\iint D_i(\vec{b}-\vec{b}')D_j(\vec{b}')\ d\vec{b}'$$

$$= 2\ K_{ij} D_i \otimes D_j \tag{A.11}$$

where the index on D_i tells us what type of hadron is involved. For pp-scattering, (A.11) becomes:

$$s_{pp} = 2\pi K_{pp} \langle D_p \otimes D_p \rangle = 2\pi K_{pp} \langle D_p \rangle^2.$$

From our definition of D, we have:

$$\langle D \rangle \Big|_{K_x, K_y} = \int_{-\infty}^{+\infty} \langle \rho(x,y,z) \rangle \Big|_{K_x, K_y} dz$$
$$= \int_{-\infty}^{+\infty} \langle \rho(x,y,z) \rangle \Big|_{K_x, K_y} \exp(iK_z z) \Big|_{K_z=0} dz$$
$$= \sqrt{2\pi} \langle \rho \rangle \Big|_{K_x, K_y, 0}.$$

If we identify ρ with the hadronic charge distribution, $\langle D \rangle$ is proportional to the charge form factor of the hadron, $F_1(K^2)$.

To a first approximation, (1.9) or (1.10) indicates

$$a_{pp} = s_{pp} \propto \langle D_{pp} \rangle^2 \propto \left[F_1(K^2)\right]^2,$$

while (1.1) states that at high energies

$$a_{pp} \propto \sqrt{f(t)},$$

so that:

$$f(t) \propto \left[F_1(K^2)\right]^4$$

as suggested by Wu and Yang.[8] Chou and Yang demonstrate remarkably good agreement with experiment for the model.[6]

Before we leave the diffractive model, we must examine its implications for inelastic processes. For such interactions, the model must be expanded with additional assumptions. This is necessitated by the fact that the virtual state of the elastic process does not contain a finite number of virtual particles, as mentioned earlier. Naturally, the final state of the inelastic reaction must contain a finite number of particles, and these two statements must be reconciled in the expanded model. Benecke, et.al. accomplished this by treating the outgoing particles as fragments of the target and the projectile.[9] The projectile and target are still spatially extended objects which pass through each other, but now they may break up into pieces. Mathematically, this statement takes the form of an assumption regarding the high energy partial cross sections of the outgoing particles. It is reasonable to believe that projectile fragments have finite momentum in the projectile rest frame, and that target fragments have finite momentum in the laboratory frame. Benecke, et. al. hypothesize that in the limit of high incoming energy, the partial cross section that a particle of mass m is emitted with momentum $\vec{p}$ is finite, and defined as $\rho_1(\vec{p})\ d^3p$, where $\vec{p}$ is defined in the projectile rest frame for some particles, and in the lab frame for others, and $\rho_1 > 0$. Similarly, partial cross sections $\rho_n \prod_i^n d^3p_i$ for n particles are also assumed to be finite at high energies in the lab or projectile rest frame. The presence of other final particles, whose momentum in the center of mass frame is finite (pionization) is not totally inconsistent with the mathematical statement of this model. Benecke, et. al. reject pionization, as it implies that the colliding particles arrest each other and then evaporate slow pions. This "arrest" appears to be at variance with our motivation for hypothesizing limiting fragmentation, that motivation being the collision of spatially extended objects going through each other, rather than combining with each other. The momentum distribution of secondaries in the diffractive fragmentation model is thus different from that which would be expected from multiperipheral models.

The integrals of the distributions ρ_n are related to average multiplicities in the following manner:

$\int \rho_1 \, d^3p$ = total cross section X average multiplicity per collision of particles of mass m emitted from **target** (projectile),

$\int \rho_2 \, d^3p = \rho_1(p_1)$ X average multiplicity per collision, where a particle of mass m_1 and momentum p_1 is emitted from the target (projectile), of particles of mass m_2 emitted from the target (projectile).

The appropriateness of the words "target" or "projectile" in these equations depends on whether we are examining target fragments or projectile fragments (i.e., slow particles in the lab frame or slow particles in the projectile rest frame). Benecke, et. al. suggest that these integrals (and hence, if there is no pionization, the multiplicities) are divergent. The dependence of multiplicity on the energy is explained by noting that more of these divergent integrals are made accessible at higher energies.

Partial cross sections for particular numbers of emitted particles may be examined by considering the kinematically allowed region for which ρ_1 is defined. Conservation of momentum requires that $p_{inc} = \sum_i p_{\parallel i}$ and energy conservation demands that $\sum_i^n e_i = M_t + E$, where $p_{inc} \equiv$ projectile momentum in the lab system, $p_{\parallel i} \equiv$ i^{th} longitudinal fragment momentum in the lab system, $e_i \equiv$ i^{th} fragment energy in the lab system, $M_t \equiv$ target mass, and $E \equiv$ projectile energy in the lab system. Combining both equations:

$$\sum_{i}^{n} (e - p_{\parallel})_i = M_t + E - p_{inc} ,$$

or at high E:

$$\sum_{i}^{n} (e - p_{\parallel})_i = M_t \qquad (A.12)$$

where only target fragments contribute to the left side at high E, since projectile fragments have infinite energy and momentum in the lab system. Let σ_n be that contribution to ρ_n for which the target is unchanged (elastic scattering and diffractive excitation) and let τ_n be that contribution to ρ_n for which the target breaks into two or more fragments. Then $\rho_n = \sigma_n + \tau_n$, σ_n is the cross section for processes in which there are n target fragments, and τ_n is the cross section for processes in which there are more than n target fragments. σ_n is a delta function defined on the surface in momentum space specified by (A.12), and τ_n is a function defined in the region given by:

$$\sum_{i}^{n} (e - p_{\parallel})_i < M_t . \qquad (A.13)$$

Thus, σ_n is the limiting partial cross section for an n-fold target fragmentation at infinite projectile energy in the lab frame. Experimental fits by Benecke, et. al., are good for single K^- and single π^- distributions in pp collisions, but poor for single p distributions in pp collisions. The model is consistent with observations of low momentum transfer of outgoing particles, since it predicts that large t implies large multiplicity, so that the many outgoing particles can share the transverse momentum, each particle having small transverse momentum.

More detailed study of multiplicity and correlation functions may be undertaken if the model is further extended by additional assumptions. Jacob and Slansky treat the virtual state of the inelastic proton-proton collision as composed of two resonances called "novae" which subsequently decay into our fragments.[10] The exchanged particle in their model is a pomeron, with triple Regge behavior (See Figure 1 and Article 5 of this Part). They suggest an ad hoc formula

$$\rho_\alpha(M) = c_\alpha \frac{\exp[-\beta_\alpha/(M - M_\alpha)]}{(M - M_\alpha)^2} \tag{A.14}$$

for the excitation spectrum $\rho_\alpha(M)$, where

$\sigma_\alpha = \int \rho_\alpha (M)\, dM$,

α labels beam or target, c_α and β_α are characteristic of the nova, and M is the nova mass. If we make the reasonable assumption that the nova mass M is proportional to the number of fragments n, then we have:

$$\sigma_n \propto \frac{1}{n^2} \tag{A.15}$$

where $M_{max} = \frac{1}{2}\sqrt{s}$, so that:

$$\langle n \rangle \propto \ln(s) \quad \text{and} \quad f_n \propto (\sqrt{s})^{n-1} \tag{A.16}$$

where $\langle n \rangle$ is the average multiplicity and f_n is the Mueller n-body correlation function[3,11] (See Appendix **B**). Thus, $f_2 \propto \sqrt{s}$ for this "nova model", a result very different from those of the absorptive multiperipheral model (See **Appendix A**, Article 4) or the multiperipheral cluster model **which we have** discussed in Part II. Furthermore, it **should** be emphasized that we can obtain predictions for $\langle n \rangle$ and f_n in the diffractive fragmentation model only under the framework of additional assumptions, such as those of Jacob and Slansky (e.g., (**A**.14))[10].

In summary, the diffractive fragmentation model satisfies the elastic scattering result

$$\lim_{E\to\infty} \frac{d\sigma}{dt} = f(t)$$

and it is consistent with the behavior of the average multiplicity and with the transverse momentum distributions of the outgoing particles in inclusive experiments. It is nevertheless not fully consistent with all of the experimental data (e.g., the single proton distribution in pp-scattering; see Figure 1 of Ref. 9), and there remain gaps in the model. For example, the model contains no prescription for calculating the average multiplicity and the correlation functions unless additions are made to the theory, such as the triple Regge formalism and (**A**.14).

2. STATISTICAL THERMODYNAMIC MODEL

It was Enrico Fermi who first attempted to describe high energy collisions by treating the various multiparticle production possibilities as though they were governed only by their statistical weights.[12] He even went so far as to apply thermodynamics and introduce the notion of a temperature via Stefan's Law (energy density $\propto T^4$). Nevertheless, it wasn't until nearly two decades later that the ramifications of the thermodynamic picture for such functions as the hadronic mass spectrum and the momentum distribution were formulated by Hagedorn and Ranft.[13,14] The contributions of Hagedorn and Ranft of interest to us here may be summarized as follows:

1) A statement of the basic principle of the model expressed as an asymptotic bootstrap condition.

2) Calculation of the hadronic mass spectrum, and from it, the partition function, allowing the calculation of observables.

3) The establishment of an upper limit on the "temperature" of hadronic matter, leading to an interpretation of high energy hadrons as in a state of "boiling".

4) Calculation of the transverse momentum spectra of products of hadron-hadron collisions.

5) Remark on the average multiplicity of final states.

We will now consider each point in detail.

1) The Principle of the Model.

Let's address ourselves to the first statement. The fundamental assumption of the model has two parts. First, highly excited matter is assumed to be in a statistical equilibrium (actually, it is a special type of pseudo-equilibrium which we will discuss later). Second, each particle or resonance in a highly excited state is treated as a fireball, which consists of fireballs of the same type; i.e., the constituent fireballs also consist of matter in statistical equilibrium. Hagedorn and Ranft[14] state both assumption parts most elegantly in one circuitous sentence:

> "A fireball is
> a statistical equilibrium of an undetermined number of all kinds of fireballs, each of which, in turn, is considered to be "

where, the authors emphasize, "the feedback arrow is most important!" These "fireballs" are specified by a mass spectrum $\rho(m)$, which includes all hadronic matter. At low masses, the hadronic mass spectrum is a discrete set of lines corresponding to common particles; at somewhat higher masses, we have the "resonances"; at very high masses, the resonances are so broad that they form a continuum of fireballs. An alternative, more mathematical, statement of the bootstrap condition will be presented when we discuss the mass spectrum in greater detail, along with the partition function.

We digress here to discuss the nature of our statistical equilibrium. A true equilibrium would imply an isotropic momentum distribution for the products of the reaction. But in high energy collisions, the beam direction is singled out due to its high initial momentum in the lab system. We are naturally led to believe that there is a higher probability that a product will have high longitudinal momentum than that it will have high transverse momentum. This is, indeed, the case experimentally. How, then, can we describe the decaying fireball with statistical thermodynamics? Hagedorn and Ranft resolve this dilemma by superimposing the anisotropy caused by collective motions of the component fireballs in the longitudinal direction on the isotropic velocity distribution conjured up by the statistical treatment.[14] They assume that there is no turbulence (transverse collective motions), so that the transverse momentum distribution need not be weighted in this way. The fireballs are allowed to move collectively only in the longitudinal direction.

There is a second aspect to the application of statistical equilibrium to hadronic matter which will disturb many readers. Where is the "reservoir" which maintains the fireballs at a "temperature" T? We will indeed define a temperature function for hadrons later. Obviously, the heat bath is imaginary, a mere mathematical construct. We use it only to relate the total energy in a fireball to our "T". Inverting this function gives T for any observed energy E in a collision. The temperature is then the most likely temperature of an ensemble of collisions, all of whose energies equal **E**.

A third, more subtle, problem appears in the theory. The first decays of the incident fireballs yield other fireballs, which decay again, etc. Many stages are assumed to pass before the final state of observed particles results. Yet, the calculations of the model are all based on only one decay stage. It is assumed that this is a reasonable approximation, and the rationale is the success of the model in verifying experiment. The nature of the approximation is to treat the products of the first decay as the final state.

Finally, proponents of models based on "fundamental" particles such as quarks may shudder at the bootstrap statement of this model. Nevertheless, the thermodynamic model is not inconsistent with such models. Particular particles, whether they be protons or quarks, are simply specific states of the statistical ensemble, states of mass m_p or m_q, for example. All particles are still fireballs, and there is no "fundamental" fireball. The notion that some particles are fundamental then means that some of the states of the ensemble are constructed from the fundamental states.

Now that we have introduced the physical statementof the model, and discussed its flaws, we turn to some of its virtues.

2) The Mass Spectrum and the Partition Function.

The prime motivation for introducing the hadronic mass spectrum $\rho(m)$ is the necessity to sum over all possible kinds of hadrons and anti-hadrons in calculating the partition function Z. We review Hagedorn's original derivation.[13] The partition function, without the ground state of no particles present, is

$$Z(V_o,T) = \sum_{(\nu)} \exp\left[-\frac{1}{T}\sum_{\alpha\beta} \varepsilon_{\alpha\beta}\nu_{\alpha\beta}\right] - 1 \qquad (A.17)$$

where V_o is the volume of our imaginary black box, T is the parameter which we call the temperature (to be determined by experiment), $\nu_{\alpha\beta}$ is the number of particles present of kind β and momentum $\vec{p}_\alpha$, and $\varepsilon_{\alpha\beta} \equiv \sqrt{\vec{p}_\alpha^{\,2} + m^2}$. The sum is over all matrices $(\nu_{\alpha\beta})$ with elements $\nu_{\alpha\beta} = 0,1,\ldots,$ (but for fermions, $\nu_{\alpha\beta} = 0$ or 1 only). Defining $x_{\alpha\beta} \equiv \exp\left[-\varepsilon_{\alpha\beta}/T\right]$, and distinguishing bosons (label β) from fermions (label φ), we have:

$$1 + Z(V_o,T) = \sum_{(\nu)} \prod_{\alpha\beta} x_{\alpha\beta}^{\nu_{\alpha\beta}} = \left(\sum_{\nu_{11}} x_{11}^{\nu_{11}}\right)\left(\sum_{\nu_{12}} x_{12}^{\nu_{12}}\right)\cdots,$$

$$Z(V_o,T) = \prod_{\alpha\beta} \frac{1}{1-x_{\alpha\beta}} \prod_{\alpha\varphi} (1+x_{\alpha\varphi}) - 1,$$

$$\ln\left[1+Z(V_o,T)\right] = -\sum_{\alpha\beta} \ln(1-x_{\alpha\beta}) + \sum_{\alpha\varphi} \ln(1+x_{\alpha\varphi}).$$

By introducing the mass spectrum $\rho(m)$, the sums over β, γ, may be replaced by an integral:

$$\sum_{\beta,\gamma} \rightarrow \int_0^\infty \rho_{B,F}(m)\,[\ldots]\;dm.$$

Since α labels the momenta $\vec{p}_\alpha$, the sum over α may also be replaced by an integral:

$$\sum_\alpha \rightarrow \int_0^\infty \frac{V_o}{h^3}\,4\pi p^2[\ldots]\;dp = \frac{V_o}{2\pi^2}\int_0^\infty p^2[\ldots]\;dp,$$

where $\hbar \equiv 1$. Our expression for Z is now:

$$\ln\left[1 + Z(V_o,T)\right]$$
$$= \frac{V_o}{2\pi^2}\int_0^\infty p^2\;dp\left[\left(\int_0^\infty dm\left[\rho_F(m)\ln(1+x_{pm})-\rho_B(m)\ln(1-x_{pm})\right]\right)\right].$$

Expanding the logarithm yields:

$$\ln\left[1+Z(V_o,T)\right] = \frac{V_o}{2\pi^2}\sum_1^\infty \frac{1}{n}\int_0^\infty dp\,dm\left[\rho(m;n)\,p^2\,x_{pm}{}^n\right]$$

where $\rho(m;n) \equiv \rho_B(m)-(-)^n\rho_F(m)$ and $x_{pm}{}^n=\exp\left[\frac{-n}{T}\sqrt{p^2+m^2}\right]<1$.

The momentum integral can be expressed in terms of modified Hankel functions:

$$\int_0^\infty dp\;p^2\exp\left[\frac{-n}{T}\sqrt{p^2+m^2}\right] = -m^3\,\frac{d}{dy}\left[\frac{K_1(y)}{y}\right] = \frac{m^3}{y}\,K_2(y),$$

where $y \equiv \frac{nm}{T}$, yielding the result:

$$\ln\left[1+Z(V_o,T)\right] = \frac{V_o T}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}\rho(m;n)m^2\, K_2\left(\frac{nm}{T}\right)dm. \qquad (\text{A}.18)$$

From this expression for the partition function, we can derive a mathematical statement of the bootstrap condition, and we can infer the behavior of the mass spectrum. We now have:

$$Z(V_o,T)=\exp\left[\frac{V_o T}{2\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\int_0^{\infty}\rho(m;n)m^2\, K_2\left(\frac{nm}{T}\right)dm\right] - 1. \qquad (\text{A}.19)$$

But according to statistical mechanics, the partition function may also be written as:

$$Z(V_o,T) = \int_0^{\infty}\sigma(E)\,\exp\left(\frac{-E}{T}\right)dE,$$

where $\sigma(E)\,dE$ counts the number of states of the thermodynamic system between E and E + dE. This "thermodynamic system" is the fireball being considered, which itself must consist of fireballs according to the bootstrap condition, when $m\to\infty$. The latter fact is expressed by writing Z as (A.19). Thus:

$$Z(T) = \exp\left[\frac{V_o T}{2\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\int_0^{\infty}K_2\left(\frac{nm}{T}\right)\rho(m;n)m^2dm\right] - 1$$

$$\equiv \int_0^{\infty}\exp\left(\frac{-E}{T}\right)\sigma(E)\,dE. \qquad (\text{A}.20)$$

Hagedorn[13] shows further, for $m \to \infty$, that,

$$Z(T) = \exp\left[\int_0^\infty m^{\frac{3}{2}} \rho(m) \exp\left(\frac{-m}{T}\right) dm\right] \equiv \int_0^\infty \sigma(m) \exp\left(\frac{-m}{T}\right) dm \qquad (A.21)$$

where in the rest system of the fireball, $m = E$. He then infers from (A.21) the mathematical statement of the bootstrap:

$$\ln \rho(m) \longrightarrow \ln \sigma(m) \quad \text{for } m \to \infty. \qquad (A.22)$$

From (A.21) and (A.22), Hagedorn shows that the mass spectrum must appear as:

$$\rho(m) \longrightarrow a m^{\frac{-5}{2}} \exp\left(\frac{m}{T_o}\right) \qquad (A.23)$$

where "a" and T_o are constants. This spectrum is assumed to disagree with experiment at high m only, because it includes the many resonances not yet discovered. In the next section, we will concern ourselves with the physical implications of the constant T_o.

3) The Upper Limit of the Temperature.

The experimentally observed particles yield the mass spectrum $\rho(m)$ when the particles' distributions are smoothed by Gauss functions. The lower end of the spectrum is fitted by a function with the required behavior (A.23), and the function is extrapolated to $m \to \infty$. In this way, the parameter T_o may be fixed at (160 ± 10) MeV.[13,14]

Consider the physical meaning of T_o. According to (A.21), the partition function does not exist if $T > T_o$, since:

$$Z(T) = \exp\left[\int_0^{\infty} m^{-1}\ a\ \exp\left(\frac{-m}{T} + \frac{m}{T_o}\right) dm\right] . \tag{A.24}$$

Thus, T_o is an upper limit for the temperature T, and characterizes the statistical equilibrium of the hadronic fireball. At low energies, raising the energy of the fireball will increase the temperature T. But near $T=T_o$, particle creation becomes so violent that T no longer rises; all additional energy input is used to emit particles. In this sense, T_o represents a "boiling point" for all hadronic matter. While a fireball might exist in a state where $T > T_o$, it will very rapidly emit particles to bring it below T_o, and restore statistical equlibrium (the lack of equlibrium above T_o is seen explicitly in (A.24) by the fact that $Z \to \infty$). The temperature T_o represents a "phase transition" to a state of violent particle emission.

Naturally, one would expect this phase transition to manifest itself in the transverse momentum distribution. Indeed, in the next section, we see that its role is important.

4) Transverse Momentum Distribution.

Consider the partition function again, this time without subtracting the ground state:

$$Z = \sum_{(\nu)} \prod_{\alpha k} x_{\alpha k}^{\nu_{\alpha k}} \tag{A.25}$$

where $k \equiv \rho = 0,1,\ldots,$ for bosons and $k \equiv \varphi = 0$ or 1 for fermions. The average occupation numbers $\bar{\nu}_{\alpha k}$ may be calculated from Z in the usual manner:

$$\bar{\nu}_{\alpha k} = x_{\alpha k} \frac{\partial \ln Z}{\partial x_{\alpha k}} = \left[\exp\left(\frac{\sqrt{p_\alpha^2 + m_k^2}}{T} \right) \mp 1 \right]^{-1} \quad \text{for} \begin{cases} \text{bosons.} \\ \text{fermions.} \end{cases} \tag{A.26}$$

If we wish the momentum distribution of a particular particle, we introduce a delta function at its mass, integrate over the mass, and multiply by the density of states $(V_0/2\pi^2)\, dp$, where $\hbar \equiv 1$. We integrate over p_z, the longitudinal momentum, to obtain the transverse momentum distribution, for reasons discussed in Section 1:

$$w_{\substack{B\\F}}(p_\perp)dp_\perp = \text{const}\cdot p_\perp\, dp_\perp \int_0^\infty dp_z \left[\exp\left(\frac{\sqrt{p_z^2 + \mu^2}}{T} \right) \mp 1 \right]^{-1} \tag{A.27}$$

where $\mu^2 \equiv p_\perp^2 + m^2$.

In the previous section, we observed the experimental result $T_0 \approx 160$ MeV, which is of the order of m_π, the pion mass. Thus, if $\mu \gg m_\pi$, the ∓ 1 in (A.27) for bosons and fermions becomes irrelevant. For $T \to T_o$, we have:

$$w(p_\perp) \approx \text{const}\cdot p_\perp \int_{\mu}^{\infty} dx \, \exp\left[-\frac{1}{T_o}\sqrt{x^2+\mu^2}\right]$$

$$= \text{const}\cdot p_\perp \cdot \sqrt{p_\perp^2 + m^2} \cdot K_1\left(\frac{\sqrt{p_\perp^2+m^2}}{T_o}\right). \qquad (\text{A.}28)$$

Since we are assuming that the argument of K_1 is large:

$$w(p_\perp) \approx \text{const}\cdot p_\perp \cdot \sqrt{T_o\sqrt{p_\perp^2+m^2}} \, \exp\left[\frac{-\sqrt{p_\perp^2+m^2}}{T_o}\right]$$

$$\to c\cdot p_\perp^{\frac{3}{2}} \exp\left(\frac{-p_\perp}{T_o}\right) \qquad (\text{A.}29)$$

for $p_\perp \gg T_o$ and $p_\perp \gg m$. Thus, the transverse momentum distribution (if $p_\perp \gg m_\pi$) is of the Boltzmann type when T is near its limit T_o.

5) Average Multiplicity.

If the average occupation number (A.26) is integrated over all mass m_k and all momenta p_α , we obtain the average multiplicity of particles N(E) corresponding to the temperature T, which in turn corresponds to the energy E (note that

$\bar{E}(T) = -\frac{d\ln Z(1/T)}{d(1/T)}$).

Hagedorn[13] finds that N(E) obeys a Poisson distribution, with $\bar{N} \sim \ln E$. Note that this refers to the average multiplicity of first daughters of the initial fireball. According to our assumptions, this is a good approximation to the average multiplicity of final particles.

The logarithmic dependence of $\bar{N}$ on E is certainly consistent with experimental results.[1] But at very high energies, the two-body correlation function is positive[1], so that the multiplicity distribution is not actually Poisson. The products are not statistically independent as the model assumes. Consequently, the model has fallen into disrepute in recent years.

In conclusion, to summarize the basic features of the statistical thermodynamic model:

a) The model predicts that the hadronic mass spectrum rises exponentially with m, a fact which is verified in the lower-mass region, where most particles are assumed to be known.

b) Hadronic matter has a highest temperature T_0, near which additional energy input is used almost entirely to create new particles.

c) A transverse momentum distribution of the Boltzmann type is predicted, and is in good agreement with experiment.[13]

d) While the average multiplicity of products agrees with experiment, the model predicts a Poisson distribution for N(E). Experiment at high energies is inconsistent with this prediction (See Part II, Article 5, for a further discussion of this point).

We will now focus our attention on several models which are more closely related to the multiperipheral cluster model of Part II.

3. CHEW - PIGNOTTI MULTIPERIPHERAL BOOTSTRAP MODEL

At this point, we take the liberty of assuming the reader to be familiar with the "simple" Amati-Fubini-Stanghellini (AFS) multiperipheral model. For a detailed examination of AFS with the modification of clustering, he is referred to Part II.

In the AFS model, the "fundamental" amplitudes $A^R(s_o)$ which are used to construct the rungs of the multiperipheral ladder are not rigidly specified. That is, the theorist may select a reasonable form for them. As an example, AFS took the case of simple φ^3 theory, where $A^R(s_o) \sim \delta(s_o-\mu^2)$. In Part II, we use Feynman Rules and unitarity to calculate $A^R_m(s_o)$ for clusters of m particles on the rungs. In the present article, we will review a model in which a much stronger constraint is placed on these "resonant" amplitudes.

Chew and Pignotti combined Regge theory, bootstrap theory, and the multiperipheral model to produce what is now known as the Chew-Pignotti Model (CP).[15] The CP model supposes that the input interaction is the same as the output interaction, a consistency condition. Since the value of the multiperipheral model lies in its ability to produce an output amplitude A(s,t) for elastic scattering which is consistent with Regge theory, the CP model demands that $A^R(s_o)$ be specified by Regge theory. The sides of the elastic ladder are no longer pions. They are now Regge trajectories. They may be meson trajectories or Pomeranchuk trajectories (the latter being characterized by $\alpha = 1$). The inelastic chain for such a model is shown in Figure 2. The two types of internal vertices are in Figure 3. We now proceed with the calculation of cross sections according to the technique of Chew and Pignotti.[15]

The differential cross section for particles a, b, to interact and form n mesons is:

$$d\sigma_n^{ab} = \frac{1}{v_{rel}} \left|A_n^{ab}\right|^2 d\Phi_n$$

where A_n^{ab} is the amplitude for $ab \longrightarrow abn$, $d\Phi_n$ is the differential phase space, and where

$$v_{rel} = \frac{\sqrt{(p_a \cdot p_b)^2 - p_a^2 p_b^2}}{m_a m_b} .$$

In the center of mass system of the incoming particles, we have:

$$v_{rel} = \frac{\sqrt{\left(\frac{s-m_a^2-m_b^2}{2}\right)^2 - m_a^2 m_b^2}}{m_a m_b} = \sqrt{\left(\frac{s-m_a^2-m_b^2}{2m_a m_b}\right)^2 - 1}$$

where $s \equiv (p_a + p_b)^2$. If we define

$$\cosh \eta_o \equiv \frac{(s - m_a^2 - m_b^2)}{2m_a m_b}, \text{ we have:}$$

$$d\sigma_n^{ab} = \frac{1}{\sinh \eta_o} \left|A_n^{ab}\right|^2 d\Phi_n \tag{A.30}$$

In terms of the final particle momenta p_j, $d\Phi_n$ is:

$$d\Phi_n = d^4p_a\, \delta(p_a{}^2-m_a{}^2)d^4p_1\, \delta(p_1{}^2-m_1{}^2)\ldots d^4p_b\, \delta(p_b{}^2-m_b{}^2)$$

$$\times\ \delta^4\left(\sum_1^n p_i + p_a + p_b - p_a' - p_b'\right).$$

We wish to rewrite this in terms of Toller variables.[16] The $3n+2$ independent variables now consist of $n+1$ momentum transfers Q_{ij}, $n+1$ variables which we call ζ_{ij}, and n variables called ω_i. The Q_{ij} are shown in Figure 4. We define $i\zeta_{ij}$ to be the analytic continuation of the angle in the rest system of Q_{ij} between p_i and p_j. To understand ω_i, go to the rest frame of p_i, where the spatial components of $Q_{i,i+1}$ and $Q_{i-1,i}$ are parallel. Add the momenta to the left of p_i, and separately add those which are to the right. The difference between the angles which these two sums make with the direction of the spatial components of $Q_{i,i+1}$ and $Q_{i-1,i}$ is defined as ω_i. In terms of these variables, the phase space becomes:

$$d\Phi_n = \cosh q_a \cosh q_b \prod_{j=1}^{n} \sinh q_j\, d\,\omega_j \prod_{i=1}^{n+1} dt_i\, d\cosh \zeta_{i,i+1}$$

$$\times\ \frac{\delta(\cosh\eta - \cosh\eta_o)}{\sinh\eta_o} \qquad (A.31)$$

where $\cosh q_j \equiv \dfrac{\mu^2 - t_j - t_{j'}}{2(-t_j)^{\frac{1}{2}}(-t_{j'})^{\frac{1}{2}}}$,

with $t_j \equiv (Q_{j,j+1})^2$, $\cosh q_a = 1-(t_1/4m_a{}^2)$, $\cosh q_b = 1-(t_{n+1}/4m_b{}^2)$, and where, when each ζ_i is large,

$$\cosh\eta \approx \cosh q_a \cosh q_b \prod_{j=1}^{n} (\cosh q_j + \cos\omega_j) \prod_{i=1}^{n+1} \cosh\xi_i . \quad (A.32)$$

We define three more quantities now:

$$\lambda_i \equiv \sqrt{(\cosh q_{i-1} + \cos\omega_{i-1})(\cosh q_i + \cos\omega_i)}$$

$$\exp(x_i) \equiv \lambda_i \cosh\xi_i$$

$$\exp(X_o) \equiv \frac{\cosh\eta}{\sqrt{\cosh q_a \cosh q_b}} \quad (A.33)$$

In terms of these new functions, (A.32) takes on the simple form:

$$X_o = \sum_{i=1}^{n+1} x_i \quad (A.34)$$

since $\prod_{i=1}^{n+1} (\cosh q_{i-1} + \cos\omega_{i-1})(\cosh q_i + \cos\omega_i)$

$= \cosh q_a \cosh q_b \prod_{i=1}^{n} (\cosh q_i + \cos\omega_i)^2$,

where $q_o \equiv q_a$, $q_{n+1} \equiv q_b$, and $\omega_o \equiv \omega_a \equiv \omega_{n+1} \equiv \omega_b = 0$.
Our differential phase space (A.31) is now:

$$d\Phi_n = \frac{1}{\sinh\eta_o} \prod_{j=1}^{n} \frac{\sinh q_j \, d\omega_j}{\cosh q_j + \cos\omega_j} \prod_{i=1}^{n+1} dt_i dx_i \, \delta(X_o - \sum_i x_i) . \quad (A.35)$$

Now that (A.35) gives the phase space in (A.30), we need only write A_n^{ab} and integrate (A.30) to find σ_n^{ab}. This is where the multiperipheral assumption (and the Regge input) enters the theory. We assume the form:

$$A_n^{ab} \sim F_a(t_1)F_b(t_{n+1}) \prod_{j=1}^{n} F_j(t_j, t_{j+1}, \omega_j) \prod_{i=1}^{n+1} (\cosh \xi_i)^{\alpha_i(t_i)} \tag{A.36}$$

where F is large only for small t's and we note here that

$$s_i = t_{i-1} + t_{i+1} + 2(-t_{i-1})^{\frac{1}{2}}(-t_{i+1})^{\frac{1}{2}}$$

$$\times \ (\sinh q_{i-1} \sinh q_i \cosh \xi_i + \cosh q_{i-1} \cosh q_i). \tag{A.37}$$

These expressions for $d\Phi_n$ and A_n^{ab} in (A.30) yield:

$$d\sigma_n^{ab} = \frac{1}{\sinh \eta_0} \left| F_a(t_1)F_b(t_{n+1}) \prod_{j=1}^{n} F_j(t_j, t_{j+1}, \omega_j) \right.$$

$$\times \left. \prod_{i=1}^{n+1} (\cosh \xi_i)^{\alpha_i(t_i)} \right|^2 \frac{1}{\sinh \eta_0}$$

$$\times \prod_{j=1}^{n} \frac{\sinh q_j \, d\omega_j}{\cosh q_j + \cos \omega_j} \prod_{i=1}^{n+1} dt_i dx_i \, \delta(X_0 - \sum_i x_i)$$

Integrating over the $d\omega$'s:

$$d\sigma_n^{ab} = \exp(-2X_0)\, \delta(X_0 - \sum_i x_i) f_a^2(t_1) f_b^2(t_{n+1})$$

$$\times \prod_{j=1}^{n} f_j^2(t_j, t_{j+1}) \prod_{i=1}^{n+1} dt_i dx_i \exp(2\alpha_i x_i) \tag{A.38}$$

where f_j^2 is the product of $|F_j(t_j, t_{j+1}, \omega_j)|^2$ and known functions of t_j, t_{j+1}, and ω_j, integrated over ω_j.

At this point, Chew and Pignotti make an assumption which greatly simplifies the calculation of σ_n^{ab}. This assumption is:

$$\int dt_1 dt_2 \ldots dt_{n+1}\, f_a^2(t_1) f_1^{\,2}(t_1,t_2)\ldots f_n^{\,2}(t_n,t_{n+1}) f_b^{\,2}(t_n)$$

$$\approx G_{ax}^{\,2}(g_M)^{2i}(g_P)^{2(n-i)}G_{by}^{\,2} \qquad \text{(A.39)}$$

where x and y may be M (meson trajectory) or P (Pomeron trajectory) and i is the number of M-M internal vertices in the chain (See Figure 3). This assumption is realized if the t-integrations go from $-\infty$ to 0, if

$$f_i(t_i,t_{i+1}) = g_i f(t_i) f(t_{i+1}),$$

$$f_a(t_1) = G_{ax} f(t_1),$$

$$f_b(t_{n+1}) = G_{by} f(t_{n+1}),$$

f(t) being the same function in all cases, and if we replace $\alpha_i(t_i)$ by α_P or α_M.

As an example, consider Figure 5, where three pions are produced, with exchanges of Regge trajectories M, M, P, and M.

Equations (A.38) and (A.39) indicate the result:

$$d\sigma_3^{aMMPMb} \approx G_{aM}{}^2 G_{bM}{}^2 g_P{}^4 g_M{}^2 \exp\left[2(\alpha_M x_1+\alpha_M x_2+\alpha_P x_3+\alpha_M x_4 - X_o)\right]$$

$$\times\ dx_1 dx_2 dx_3 dx_4\ \delta(-X_o+x_1+x_2+x_3+x_4) \qquad (A.40)$$

Now that we have arrived at an expression for the cross section for ab→abn, we can find the total cross section by summing over n and the possible trajectories. First consider only diagrams with meson trajectories. The n-meson production cross section for such diagrams is:

$$d\sigma_n^{aM\ldots Mb} = G_{aM}{}^2 G_{bM}{}^2 (g_M{}^2)^n \exp\left[2(\alpha_M - 1)X_o\right]$$

$$\times\ dx_1 \ldots dx_{n+1}\ \delta(x_1 + \ldots + x_{n+1} - X_o) \qquad (A.41)$$

The x-integration is

$$\int_o^{X_o} dx_1 \int_o^{x_1} dx_2 \ldots \int_o^{x_n} dx_{n+1}\ \delta(x_1+\ldots+x_{n+1}-X_o) = \frac{(X_o)^n}{n!}\ ,$$

so that we have:

$$\sigma_n^{aM\ldots Mb} = G_{aM}{}^2 G_{bM}{}^2\ \frac{(g_M{}^2 X_o)^n}{n!}\ \exp\left[(2\alpha_M - 2)X_o\right], \qquad (A.42)$$

a Poisson distribution in n. Summing over n yields

$$\sum_n \frac{(g_M{}^2 X_o)^n}{n!} = \exp\,(g_M{}^2 X_o)$$

(recall that the Taylor series for e^{λ} is $\sum_n \frac{\lambda^n}{n!}$), so that, with the definition $2\alpha_M + g_M{}^2 \equiv 2\alpha_{M'}$, we find:

$$\sigma_{tot}^{aM\ldots Mb} = G_{aM}{}^2\, G_{bM}{}^2\, \exp\left[(2\alpha_{M'} - 2)X_o\right] . \qquad (A.43)$$

The Froissart bound[17] requires that $\alpha_{M'} \leq 1$, so:

$$g_M{}^2 \leq 2(1 - \alpha_M) \qquad (A.44)$$

where this condition arises out of unitarity.

Before we examine diagrams in which the Pomeranchuk trajectory is also present, note that the average multiplicity from the pure meson trajectory diagrams may be read directly from the Poisson distribution (A.42), since

$$\langle n \rangle = \sum_n n\, \frac{\lambda^n e^{-\lambda}}{n!} = \sum_n \frac{\lambda^n e^{-\lambda}}{(n-1)!} = \lambda \sum_n \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} = \lambda\ ,$$

so that:

$$\langle n \rangle^{aM\ldots Mb} = g_M{}^2 X_o . \qquad (A.45)$$

In the case of nucleon-nucleon scattering, m_a and m_b are much larger than the average momentum transfer, and the definitions following (A.31) reveal that $\cosh q_a \approx \cosh q_b \approx 1$.

Furthermore, (A.33) then provides $\exp(X_o) = \cosh\eta$, and the delta function in (A.31) yields $\exp(X_o) = \cosh \eta_o$. But

$$\cosh \eta_o \equiv \frac{s - m_a^2 - m_b^2}{2m_a m_b} ,$$

so $X_o \sim \ln(s)$ and (A.45) becomes:

$$\langle n \rangle^{aM\ldots Mb} \sim \ln(s). \tag{A.46}$$

This is the same prediction as the simple AFS model, and also the same as our cluster model in Part II. We must keep in mind that diagrams with Pomeranchuk trajectories are being ignored at this point. Note, too, that because we have a Poisson distribution (A.42), the two-body correlation function $f_2 = \langle n(n-1)\rangle - \langle n \rangle^2$ is zero if we only include the meson trajectories:

$$\langle n^2 \rangle = \sum_n \frac{n^2 \lambda^n e^{-\lambda}}{n!} = \sum_n \frac{n \lambda^n e^{-\lambda}}{(n-1)!} = \lambda \sum_n \frac{n \lambda^{n-1} e^{-\lambda}}{(n-1)!}$$

$$= \lambda \sum_n \frac{(n-1)\lambda^{n-1} e^{-\lambda}}{(n-1)!} + \lambda = \lambda^2 + \lambda ,$$

$$\text{so } f_2 = \langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle = \lambda^2 + \lambda - \lambda^2 \quad - \lambda = 0.$$

This is a different result, both from the simple AFS model ($f_2 \sim -\ln s$) and our cluster model of Part II ($f_2 \sim +\ln s$). The Pomeron is thus necessary if we believe that f_2 depends on s, and if we want to rescue the CP model. We have found in (A.43), however, that any group of adjacent meson trajectories may be replaced by a single trajectory given by $\alpha_{M'} = \alpha_M + \frac{1}{2}g_M^{\,2}$.

We may now write the total cross section as:

$$\sigma_{tot}^{ab} = \sigma_{tot}^{aP\ldots Pb} + \sigma_{tot}^{aP\ldots M'b}$$

$$+ \sigma_{tot}^{aM'\ldots Pb} + \sigma_{tot}^{aM'\ldots M'b} \qquad \text{(A.47)}$$

where it is understood that M' and P trajectories alternate along the chain. The optical theorem tells us that $\sigma_{tot}^{ab} \propto \frac{1}{s}$ X imaginary forward elastic ampliude $ab \rightarrow ab$. Chew and Pignotti assume that this amplitude is dominated by the Pomeranchuk trajectory $\alpha_P(0) = 1$, so that $\sigma_{tot}^{ab} \propto \frac{1}{s} A \sim \frac{1}{s}\, s^{\alpha_P} \sim 1$. That is, σ_{tot}^{ab} should approach a nonvanishing constant at high energy.

Consider the contributions to σ_{tot}^{ab} given by (A.47). In the first term, consider $\sigma_N^{aP\ldots Pb}$, where N is defined as the number of meson trajectory "clusters" M':

$$d\,\sigma_N^{aP\ldots Pb} = G_{aP}^{\,2}G_{bP}^{\,2}(g_P^{\,2})^{2N}\Big[\exp\big[2(\alpha_P x_1 + \alpha_{M'} x_2 + \alpha_P x_3 + \ldots$$

$$+ \alpha_P x_{2N+1} - X_o)\big]\Big] \text{ X } dx_1 dx_2 dx_3 \ldots dx_{2N+1}\, \delta(X_o - \sum_{i=1}^{2N+1} x_i).$$

Integrate over x_{2N+1} using the delta function, so that:

$$\alpha_P x_{2N+1} = \alpha_P(X_o - x_1 - x_2 - x_3 - \ldots - x_{2N}) \text{ and thus:}$$

$$d\sigma_N^{aP\ldots Pb} = G_{aP}{}^2 G_{bP}{}^2 (g_P{}^2)^{2N}$$

$$X\left[\exp\left[2(\alpha_{M'}x_2-\alpha_P x_2+\alpha_{M'}x_4-\alpha_P x_4+-\ldots+\alpha_{M'}x_{2N}-\alpha_P x_{2N}+\alpha_P X_o-X_o)\right]\right]$$

$$X\ dx_1 dx_2 dx_3 \ldots dx_{2N}$$

or more clearly:

$$d\sigma_N^{aP\ldots Pb} = G_{aP}{}^2 G_{bP}{}^2 (g_P{}^2)^{2N}\left[\exp\left[2(\alpha_P - 1)X_o\right]\right]$$

$$X\left[\exp\left[-2(\alpha_P-\alpha_{M'})(x_2+x_4+\ldots+x_{2N})\right]\right]dx_1 dx_2 \ldots dx_{2N}.$$

Change variables:

$z \equiv x_2 + x_4 + \ldots + x_{2N}$

$x_i' \equiv x_i$ for $i = 1,2,3,\ldots,2N-1$.

That is, replace the variable x_{2N} by $z \equiv x_2+x_4+\ldots+x_{2N}$.

Integrate over all x_i', but not over z immediately:

$$\sigma_N^{aP\ldots Pb} = G_{aP}{}^2 G_{bP}{}^2 (g_P{}^2)^{2N}\left[\exp\left[2(\alpha_P - 1)X_o\right]\right]$$

$$X\int_0^{X_o} dz\left[\exp\left[-2(\alpha_P-\alpha_{M'})z\right]\right]dx_1'\ dx_2'\ldots\ dx_{2N-1}'\ .$$

Since we have temporarilly fixed the subsum z, the integrals over odd-indexed x_i' may be done separately from those over even-indexed x_i':

$$\int_0^{X_o - z} dx_1' \int_0^{x_1'} dx_3' \cdots \int_0^{x_{2N-3}'} dx_{2N-1}' = \frac{(X_o - z)^N}{N!} \quad \text{and}$$

$$\int_0^{z} dx_2' \int_0^{x_2'} dx_4' \cdots \int_0^{x_{2N-4}'} dx_{2N-2}' = \frac{z^{N-1}}{(N-1)!} .$$

We are led to the result:

$$\sigma_N^{aP\ldots Pb} = G_{aP}{}^2 G_{bP}{}^2 (g_P{}^2)^{2N} \left[\exp\left[2(\alpha_P - 1)X_o\right] \right]$$
$$X \int_0^{X_o} dz \; \frac{z^{N-1}}{(N-1)!} \; \frac{(X_o - z)^N}{N!} \; \exp\left[-2(\alpha_P - \alpha_{M'})z \right] \qquad (A.48)$$

and, of course, $\sigma_{tot}^{aP\ldots Pb} = \sum_N \sigma_N^{aP\ldots Pb}$. When $\alpha_{M'}$ is near α_P, an approximate closed form for this sum may be obtained:

$$\sigma_{tot}^{aP\ldots Pb} = G_{aP}{}^2 G_{bP}{}^2 \left[\exp\left[(\alpha_P + \alpha_{M'} - 2)X_o\right] \right] \cosh(g_P{}^2 X_o) \qquad (A.49)$$

Similarly, we may look at the other three contributions to (A.47). First, for aP...M'b:

$$\sigma_N^{aP\ldots M'b} = G_{aP}{}^2 G_{bM}{}^2 (g_P{}^2)^{2N+1} \left[\exp\left[2(\alpha_P - 1)X_o\right] \right]$$
$$X \int_0^{X_o} dz \; \frac{z^N}{N!} \; \frac{(X_o - z)^N}{N!} \left[\exp\left[-2(\alpha_P - \alpha_{M'})z \right] \right] \qquad (A.50)$$

where N is the number of M' trajectory clusters, not including the end M'. The same result occurs for aM'...Pb. If

$\alpha_{M'}$ is close to α_P:

$$\sigma_{tot}^{aP\ldots M'b} = G_{aP}^2 G_{bM}^2 \Big[\exp[(\alpha_P + \alpha_{M'} - 2)X_o]\Big] \sinh(g_P^2 X_o). \quad (A.51)$$

Finally, the last contribution to (A.47) is:

$$\sigma_{tot}^{aM'\ldots M'b} = G_{aM}^2 G_{bM}^2 \Big[\exp[(\alpha_P + \alpha_{M'} - 2)X_o]\Big] \cosh(g_P^2 X_o) \quad (A.52)$$

so that (A.47) now reads:

$$\sigma_{tot}^{ab} = (G_{aP}^2 G_{bP}^2 + G_{aM}^2 G_{bM}^2)\cosh(g_P^2 X_o)$$

$$+ \; (G_{aP}^2 G_{bM}^2 + G_{aM}^2 G_{bP}^2)\sinh(g_P^2 X_o)$$

$$\times \; \exp[(\alpha_P + \alpha_{M'} - 2)X_o]. \quad (A.53)$$

At high energy (and thus, high X_o), this is:

$$\sigma_{tot}^{ab} = (G_{aP}^2 G_{bP}^2 + G_{aM}^2 G_{bM}^2 + G_{aP}^2 G_{bM}^2 + G_{aM}^2 G_{bP}^2)$$

$$\times \; \exp[(g_P^2 + \alpha_{M'} + \alpha_P - 2)X_o]$$

so that, since we desire that σ_{tot}^{ab} constant at high energy, we demand:

$$g_P^2 = 2 - \alpha_P - \alpha_{M'} \; . \quad (A.54)$$

For elastic scattering, we can write:

$$d\,\sigma_{el}^{ab} = G_{aP}^{2}G_{bP}^{2}\left[\exp[2(\alpha_P x_1 - X_o)]\right]\; dx_1\;\; \delta(-X_o + x_1)$$

so that we can integrate over x_1, yielding:

$$\sigma_{el}^{ab} = G_{aP}^{2}G_{bP}^{2}\;\exp[2(\alpha_P - 1)X_o] \tag{A.55}$$

allowing the rewriting of (A.52) as:

$$\sigma_{tot}^{ab} = \sigma_{el}^{ab}\left[(1+\gamma_a\gamma_b)\cosh(g_P^{2}X_o)+(\gamma_a+\gamma_b)\sinh(g_P^{2}X_o)\right]$$

$$\times\;\;\exp[(\alpha_{M'} - \alpha_P)X_o] \tag{A.56}$$

where $\gamma_c \equiv G_{cM}^{2}/G_{cP}^{2}$.

As an example, Chew and Pignotti examine the case $g_P^{2}=0$, $\alpha_{M'}=\alpha_P=1$ (so that, from the definition following (A.42), $g_M^{2} = 2(1-\alpha_M)$). Our equations (A.55), (A.56) and (A.42) yield the results:

$$\sigma_{el}^{ab} = G_{aP}^{2}G_{bP}^{2}$$

$$\sigma_{tot}^{ab} = \sigma_{el}^{ab}\;(1 + \gamma_a\gamma_b)$$

$$\sigma_{n}^{ab} = \gamma_a\gamma_b\;\;\sigma_{el}^{ab}\,\frac{(g_M^{2}X_o)^{n}}{n!}\;\;\exp(-g_M^{2}X_o). \tag{A.57}$$

Again, from the argument preceding (A.46), we have $X_o \sim \ln(s)$, and for this version of the CP model:

$$\langle N \rangle \sim X_o \sim \ln(s). \tag{A.58}$$

But again, since (A.57) is Poisson, $f_2 = 0$. Apparently, we need $g_P{}^2 \neq 0$ if we want a nonzero two-body correlation function in this model, so that terms of the forms (A.49), (A.50) and (A.52) must be added to the right side of (A.57). Furthermore, experimental evidence for diffractive dissociation ($N = 0$, aPM'b) indicates $g_P{}^2 \approx 0.02$ according to Chew and Pignotti. (At moderate lab energies, and $\alpha_{m'} \approx \alpha_P$, (A.50) provides $\sigma_{N=0}^{aPM'b} = \sigma_{el}^{ab}\,\gamma_b(g_P{}^2 X_o)$).

In the next **chapter**, we will investigate another modification of multiperipheral models, which has more promise for fitting experimental correlation functions than the CP model. Indeed, we will use the CP model as an example of a multiperipheral model which can itself be modified according to the prescription of the next article.

4. THE ABSORPTIVE MULTIPERIPHERAL MODEL

The absorptive multiperipheral model (AMPM) is the only model of those discussed thus far which yields the correlation function behavior:

$$f_m \sim (\ln s)^m \qquad (A.59)$$

If the Koba-Nielsen-Olesen[5] (KNO) scaling factors, $\langle n^q \rangle / \langle n \rangle^q$, experimentally rise with q, then the behavior (A.59) is warranted. In the simple multiperipheral model and the cluster model of Part II, $f_m \sim \ln(s)$ and the KNO factors are not only independent of s, but they are also independent of q. However, the sign of the coefficient of $\ln(s)$ in f_2 differs between the simple AFS model and our cluster model (See Part II). The special behavior of (A.59) merits our review of AMPM.

Absorption was first applied to the peripheral (one-meson exchange) model by Gottfried and Jackson[18], and then applied to the multiperipheral model by Finkelsten and Zachariasen[19]. The Finkelstein and Zachariasen model (FZ) resembles the CP model in that it is a bootstrap approach. An elastic amplitude is obtained, which in the high s limit is a self-consistent solution at finite and zero momentum transfers t. Furthermore, the total cross section does not fall at high s in the FZ model.

Caneschi and Schwimmer have calculated the partial cross sections, average multiplicity and correlation functions for the FZ model[20]. We will reiterate their calculations here, as they are of interest as a comparison with our other models.

To introduce absorption, the impact parameter representation is invoked, and the Pomeron ($\alpha = 1$) is given the representation $i\Theta(B_o(Y) - B)$, where $Y \equiv \ln(s)$, B is the impact parameter, and $B_o = R_oY$; R_o is a constant which we set at one. Absorption is used to prevent violation of the Froissart bound by the output amplitude.[17] The partial cross section for producing N particles when the incident particles differ in rapidities by $Y = \ln(s)$ and they have impact parameter B is:

$$\sigma_N(Y,B) = S(Y,B)M_N(Y,B), \tag{A.60}$$

where

$$M_N(Y,B)=g^N\int\prod_{i=1}^{N-1}\left[d^2b_i dy_i\, \Theta(y_i{}^2-b_i{}^2)\right]\delta(Y-\sum_{i=1}^{N-1}y_i)$$
$$\times\ \delta(B-\sum_{i=1}^{N-1}b_i) \tag{A.61}$$

and S(Y,B) is the absorption factor, $1 + iT(Y,B)$. y_i and b_i are the rapidity and impact parameter differences between the i^{th} and $i+1^{st}$ particles in the chain and $\sqrt{g}$ is the coupling constant.

Caneschi and Schwimmer now apply Laplace transforms in Y and B to (A.61), yielding:

$$M_N(X,C) \equiv g^N \int d^2B\ dY\ e^{XY} e^{BC} M_N(Y,B) = g^N (X^2-C^2)^{-\frac{3}{2}(N-1)} \Theta(X^2-C^2). \tag{A.62}$$

The main contribution to the inverse transform occurs at

$$X_{max} = \frac{3(N-1)Y}{Y^2-B^2}, \quad C_{max} = \frac{3(N-1)B}{Y^2-B^2}. \tag{A.63}$$

The method of steepest descent about X_{max}, C_{max} provides:

$$M_N(Y,B) = g^N \frac{(\sqrt{Y^2 - B^2})^{3N-6}}{(3N-5)!} \Theta(Y^2 - B^2). \tag{A.64}$$

A self-consistent solution is obtained with:

$$T(Y,B) = i\Theta(Y^2-B^2) - 3i\Theta(Y^2-B^2) g^{-\frac{5}{3}} \sqrt{Y^2-B^2} \exp\left[-g^{\frac{1}{3}} \sqrt{(Y^2-B^2)}\right]. \tag{A.65}$$

Combining (A.60) with (A.64) and (A.65) thus gives us the partial cross section:

$$\sigma_N^{FZ}(Y,B) = \frac{3(g^{\frac{1}{3}}\sqrt{Y^2-B^2})^{3N-5}}{(3N-5)!} \left[\exp\left(-g^{\frac{1}{3}}\sqrt{Y^2-B^2}\right)\right] \Theta(Y^2-B^2). \tag{A.66}$$

Equation (A.66) can be integrated over B to obtain the dependence of the partial cross section on N and Y:

$$\sigma_N(Y) = \frac{6g^{-\frac{2}{3}}}{(3N-5)!} \gamma(3N-3,\ g^{\frac{1}{3}}Y), \tag{A.67}$$

where γ is the incomplete gamma function[21]:

$$\gamma(3N-3,\ g^{\frac{1}{3}}Y) \equiv \int_0^{g^{\frac{1}{3}}Y} e^{-t}\, t^{3N-4}\, dt. \tag{A.68}$$

The result (A.67) was used by Caneschi and Schwimmer to calculate the average multiplicity and correlation functions using a generating function defined by Mueller[3]:

$$S(Y,Z) \equiv \sum_N z^N \sigma_N(Y). \tag{A.69}$$

Note that:

$$F_m \equiv \langle N(N-1)\ldots(N-m+1)\rangle = \frac{1}{S(Y,z)} \frac{\partial^m S}{\partial z^m}\Bigg|_{z=1} \tag{A.70}$$

and the correlation functions f_i are given by[3]:

$$F_m = m! \sum_{n_i} \delta\left(m - \sum_i n_i\right) \prod_{i=1}^{\infty} \left(\frac{f_i}{i!}\right)^{n_i - n_{i+1}} \frac{1}{(n_i - n_{i+1})!} \tag{A.71}$$

For example, the two-body correlation function is:

$$f_2 = F_2 - f_1{}^2 = F_2 - F_1{}^2 = \langle N(N-1)\rangle - \langle N\rangle^2 . \tag{A.72}$$

The FZ generating function is obtained by substituting (A.67) into (A.69):

$$\begin{aligned} S(Y,z) &= \sum_N z^N \frac{6g^{-\frac{2}{3}}}{(3N-5)!} \gamma(3N-3,\ g^{\frac{1}{3}}Y) \\ &= \frac{2g^{-\frac{2}{3}} z^{-\frac{5}{3}}}{(z^{\frac{1}{3}}-1)^2} \left[\left[g^{\frac{1}{3}}(z^{\frac{1}{3}}-1)Y-1\right]\exp\left[(z^{\frac{1}{3}}-1)g^{\frac{1}{3}}Y\right] + 1\right]. \end{aligned} \tag{A.73}$$

The leading terms of F_m, using (A.73) in (A.70), are:

$$F_m \sim \frac{2}{m+2}\left[\frac{1}{3}\, g^{\frac{1}{3}} Y\right]^m . \qquad (A.74)$$

Thus, the leading terms of the correlation functions for $m \geq 2$ are:

$$f_m \sim F_m - (F_1)^m \sim \frac{2}{m+2}\left(\frac{1}{3} g^{\frac{1}{3}} Y\right)^m - \left(\frac{2}{9} g^{\frac{1}{3}} Y\right)^m \sim \left[\frac{2}{m+2} - \left(\frac{2}{3}\right)^m\right]\left(\frac{1}{3} g^{\frac{1}{3}} Y\right)^m \qquad (A.75)$$

where $f_1 = \langle N \rangle = \frac{2}{9} g^{\frac{1}{3}} Y$. In the CP model, $F_m = (F_1)^m$ and the leading terms cancel. In fact, all terms except the first order term in Y cancel. But in the FZ model, this cancellation does not occur, and $f_m \sim (\ln s)^m$, as claimed earlier. In particular, the two-body correlation function is:

$$f_2 \sim \left(\frac{1}{2} - \frac{4}{9}\right)\left(\frac{1}{3} g^{\frac{1}{3}} Y\right)^2 \sim \frac{1}{162}\, g^{\frac{2}{3}} (\ln s)^2 . \qquad (A.76)$$

Since the coefficients of $(\ln s)^m$ in F_m and $(F_1)^m$ are unequal,

$$\frac{F_m}{(F_1)^m} = \frac{\langle n^m \rangle}{\langle n \rangle^m} \neq 1 .$$

At high energies, we have:

$$\frac{\langle n^m \rangle}{\langle n \rangle^m} = \frac{2}{m+2}\, \frac{\left(\frac{1}{3} g^{\frac{1}{3}} Y\right)^m}{\left(\frac{2}{9} g^{\frac{1}{3}} Y\right)^m} = \frac{2}{m+2}\left(\frac{3}{2}\right)^m \geq 1 . \qquad (A.77)$$

Note that (A.77) is independent of s, that it equals one when m=1, and that it increases as m increases. Contrast this to the CP model, where $(\langle n^m\rangle / \langle n\rangle^m) = 1$ for all m.

We will see in Part II, that although the two-body correlation function in our multiperipheral cluster model goes as ln(s), we can nevertheless drive f_2 positive by including clusters of sufficient size, and we can fit both f_2 and $f_1 = \langle n\rangle$ by including clusters of sufficient variety.

Before beginning our study of the cluster model, we will review one more model of related interest, the Mueller analysis of inclusive processes.

5. MUELLER ANALYSIS OF INCLUSIVE PROCESSES.

The following chapter stands apart from the rest in two respects. Firstly, we will examine a type of distribution largely neglected so far in this paper, except for some discussion in the context of the diffractive fragmentation model. Secondly, we will review a method of analysis which lends itself not to one particular model, but rather to many models.

Inclusive processes are those in which some of the products are not observed experimentally, so that the reaction may be written:

$A + B \rightarrow C_1 + C_2 + \ldots + C_n$ + anything,

where $C_1, C_2, \ldots, C_n$ are observed products. For simplicity, we will consider the one-particle inclusive process:

$A + B \rightarrow C$ + anything.

Much work has been done on these processes in specific models (See, for example, Ref. 20, pp. 41-43). We choose to follow the more general analysis developed by Mueller.[22]

Consider first the experimental observations regarding one-particle inclusive processes. Let the momenta of the incident particles A and B be p_1 and p_2, respectively, and let that of the observed product C be q. If ω is the energy of C, we define the one-particle inclusive distribution as:

$$\rho_{AB}{}^{C} \equiv \omega \frac{d\sigma}{d^3q} = \frac{1}{2s} \sum_{n=2}^{\infty} \sum_{k} \int \prod_{i=1}^{n} \frac{d^3q_i}{(2\pi)^3 2\omega_i} \omega_k \delta^3(q_k - q)$$

$$\times\ \delta^4(p_1 + p_2 - \sum_j^n q_j) |T(p_1 + p_2 \rightarrow q_1 + \ldots + q_n)|^2 ,$$

(A.78)

where $s \equiv (p_1+p_2)^2$, the sum over k is taken only over particles of type C, and the sum over n includes a sum over all particle types and spins. We also define $t \equiv (p_1-q)^2$, $u \equiv (p_2-q)^2$ and $M^2 \equiv s+t+u-m_A{}^2-m_B{}^2-m_C{}^2$. Of the variables s, t, u, and M^2, any three may be chosen as the independent variables on which $\rho_{AB}{}^{C}$ depends. Let q_L and q_T denote longitudinal and transverse components of momenta (parallel and perpendicular to $\vec{p}_1-\vec{p}_2$, respectively). We also define the rapidity variable y as:

$$y \equiv \ln \frac{\omega + q_L}{m_T} = \tfrac{1}{2} \ln \frac{\omega + q_L}{\omega - q_L} \qquad \text{(A.79)}$$

where $m_T{}^2 \equiv m_C{}^2 + q_T{}^2$ so that $\omega^2-q_L{}^2=m_C{}^2+q_T{}^2=m_T{}^2$. Note now that:

$$\sinh y = \frac{e^y-e^{-y}}{2} = \tfrac{1}{2}\left[\sqrt{\frac{\omega+q_L}{\omega-q_L}} - \sqrt{\frac{\omega-q_L}{\omega+q_L}}\right] = \frac{\omega+q_L-(\omega-q_L)}{2\sqrt{\omega^2-q_L{}^2}}$$

$$= \frac{q_L}{\sqrt{\omega^2-q_L{}^2}} = \frac{q_L}{\sqrt{q_T{}^2+m_C{}^2}} = \frac{q_L}{m_T} , \qquad \text{(A.80)}$$

and we also have:

$$\cosh y = \frac{e^{y}+e^{-y}}{2} = \frac{1}{2}\left[\sqrt{\frac{\omega + q_L}{\omega - q_L}} + \sqrt{\frac{\omega - q_L}{\omega + q_L}}\right]$$

$$= \frac{(\omega + q_L) + (\omega - q_L)}{2\sqrt{\omega^2 - q_L^{2}}} = \frac{\omega}{m_T} \quad , \qquad (A.81)$$

and finally:

$$dq_L = m_T \cosh y \; dy = \omega \, dy, \quad dy = dq_L/\omega \, . \qquad (A.82)$$

The range of y is $Y \equiv \ln(s/m_T^{2})$, since $\omega_{cm} = \pm \sqrt{s}/2$ at the limits.

We are now prepared to discuss some experimental observations. Firstly, $\rho_{AB}^{\;C}$ decreases exponentially with $|\vec{q}_T| \equiv q_T$. Furthermore, it is a function only of q_T and the scaling parameter $x \equiv 2q_L^{cm}/\sqrt{s}$. At any fixed q^{lab}, $\rho_{AB}^{\;C}$ approaches a limit as $s \to \infty$. This last result is known as limiting fragmentation.

There are several other experimental results worth mentioning before we study the Mueller analysis of $\rho_{AB}^{\;C}$. In the region where $(\frac{Y}{2} + y)$ is finite and $t \sim s$ (so that $x < 0$, the target fragmentation region), $\rho_{AB}^{\;C}$ is independent of the projectile A except for the normalization proportional to the total cross section $\sigma_T(AB)$. In the region where $x \approx 0$, y finite (central region), there is a plateau in $\rho_{AB}^{\;C}$, and the production of particles and antiparticles is equal. In all regions, most products are pions. We will discuss the three regions of y in greater detail shortly (See Figure 6).

While the discussion which follows was originated by Mueller[22], its details have been extracted from two reviews.[23,24] The core of Mueller's treatment consists of a generalization of the optical theorem. Recall that the usual optical theorem relates the imaginary part of the forward elastic amplitude to the total cross section. The generalized Mueller version relates the one particle inclusive distribution $\rho_{AB}^{\;C}$ to the imaginary part of the forward three-body elastic amplitude for the reaction $A\bar{C}B \rightarrow A\bar{C}B$:

$$\rho_{AB}^{\;C} = \frac{1}{s} A_{A\bar{C}B\rightarrow A\bar{C}B}\Big|_{\text{forward}} \qquad (A.83)$$

where the imaginary part is understood (See Figure 7). Mueller combines (A.83) and Regge formalism to analyze $\rho_{AB}^{\;C}$ in the light of the experimental picture described above. We will first examine the rapidity regions as promised earlier.

The projectile fragmentation region has finite $(\frac{Y}{2}-y) = \ln\frac{s}{m_T^2} - y$. Now $y = \sinh^{-1}\frac{q_L}{m_T} = \sinh^{-1}\frac{x\sqrt{s}}{2m_T}$, using the experimental result of scaling. At high energies s, y is large and $\frac{1}{2}e^y \approx \frac{x\sqrt{s}}{2m_T}$, so that $y = \ln\frac{x\sqrt{s}}{m_T}$ and

$$\frac{Y}{2}-y = \ln\frac{s}{m_T^2} - \ln\frac{x\sqrt{s}}{m_T} = \ln\frac{sm_T}{m_T^2 x\sqrt{s}} = \ln\frac{\sqrt{s}}{xm_T} \propto -\ln x.$$

Thus, $x>0$ in this region (remember that $|x|<1$). Consider t and u in the projectile fragmentation region:

$$t = (p_1-q)^2 \approx m_A^2+m_C^2-\frac{2m_A^2}{\sqrt{s}}m_T\sinh y - m_T\sqrt{s}\,e^{-y}$$

$$= m_A^2+m_C^2-\frac{2m_A^2}{\sqrt{s}}m_T\left(\frac{x\sqrt{s}}{m_T}\right)-m_T\sqrt{s}\left(\frac{m_T}{x\sqrt{s}}\right) = m_A^2+m_C^2-2m_A^2x-\frac{m_T^2}{x}, \qquad (A.84)$$

$$u \approx m_B{}^2 + m_C{}^2 - \frac{2m_B{}^2}{\sqrt{s}} m_T \sinh y - m_T\sqrt{s}\, e^y$$

$$= m_B{}^2 + m_C{}^2 - \frac{2m_B{}^2}{\sqrt{s}} m_T\left(\frac{x\sqrt{s}}{m_T}\right) - m_T\sqrt{s}\left(\frac{x\sqrt{s}}{m_T}\right)$$

$$= m_B{}^2 + m_C{}^2 - 2m_B{}^2 x - xs \ . \qquad (A.85)$$

Thus, in the projectile fragmentation region, t is finite and $u \sim -s$. That is, the momentum transfer between the target (B) and the product (C) grows with the energy. Finally, consider the behavior of M^2 in this region:

$$M^2 = (P-q)^2 = P^2+q^2-2P\cdot q = s+m_C{}^2-2\sqrt{s}\,\omega = s+m_C{}^2-2\sqrt{s}\,m_T\cosh y \qquad (A.86)$$

where $P \equiv p_1+p_2$. If $(\frac{Y}{2} - y)$ is finite, $\frac{Y}{2} \sim y$, $\cosh y \sim \cosh\frac{Y}{2} \sim \sqrt{s}$, and $M^2 \sim s$. In fact, since $\frac{t-u}{s} = x$ when the masses are negligible compared to the variables s, t, u, we have:

$$M^2 = s + t + u = s(1 - x) + 2t \ . \qquad (A.87)$$

Throughout most of the projectile fragmentation region, $M^2 \sim s$. But if we are at the edge of the distribution, $y \to \frac{Y}{2}$, $x \to 1$ and $\frac{M^2}{s} \to 0$. This edge will be of great interest later.

Now we turn to the central region, where y itself is finite. Using the first lines of (A.84) and (A.85) in this region, we find that t and u go as $\sqrt{s}$, and in particular:

$$tu \approx m_T{}^2 s \quad \text{and} \quad M^2 \sim s. \qquad (A.88)$$

Finally, the target fragmentation region is characterized by finite $(\frac{Y}{2} + y)$, and the roles of t and u are reversed from their behavior in the projectile fragmentation region. Thus, u is fixed and $t \sim s$, while $M^2 = s(1 + x) + 2u$. Again, $M^2 \sim s$ except in the edge region, where $\frac{M^2}{s} \to 0$, $y \to \frac{-Y}{2}$, and $x \to -1$.

We are now prepared to calculate the one-particle inclusive distributions. Consider the three-body elastic amplitude of (A.83) and Figure 7. In the projectile fragmentation region, where t is fixed and s and $u \to \infty$, Regge theory demands that a Reggeon be exchanged between B and $\bar{C}$ (See Figure 8). Since this Reggeon has the vacuum quantum numbers (in the crossed channel, Reggeon $\to B\bar{B}$), it is a Pomeron. The amplitude for Figure 8 is:

$$A_{A\bar{C}B \to A\bar{C}B}\Big|_{\text{forward}} \longrightarrow \beta_{AC}(q_T, \tfrac{Y}{2}-y)\ s^{\alpha(0)} \beta_B \ . \qquad (A.89)$$

where we could have chosen M^2 or $(-u)$ instead of s, and where β_B is the B-Pomeron coupling and β_{AC} is the coupling of the Pomeron to the combination (AC). From (A.83), we observe:

$$\rho_{AB}{}^{C} = \beta_{AC}(q_T, \tfrac{Y}{2}-y)\, \beta_B\ s^{\alpha(0)-1}. \qquad (A.90)$$

Pomeron factorization implies that this β_B is the β_B of the total cross section:

$$\sigma_{AB} = \beta_A \beta_B\ s^{\alpha(0)-1} \ , \qquad (A.91)$$

so that we find:

$$\frac{1}{\sigma_{AB}} \rho_{AB}^{\ C} = \frac{\beta_{AC}(q_T, \frac{Y}{2} - y)}{\beta_A} , \qquad (A.92)$$

which is independent of the target B. Since $\alpha(0) = 1$, (A.90) reveals that scaling is obtained. The reader is reminded of the experimental results of scaling, limiting fragmentation, and the fact that $\rho_{AB}^{\ C}/\sigma_{AB}$ is independent of the target in the projectile fragmentation region. These results are thus predicted by the theory.

We will now examine the same three-body amplitude $A_{A\bar{C}B \rightarrow A\bar{C}B}$ in the central region, where t and u are large, so that two Reggeons are introduced, one between A and $\bar{C}$, the other between $\bar{C}$ and B, both being Pomerons (See Figure 9). The amplitude is thus:

$$A_{A\bar{C}B \rightarrow A\bar{C}B}\Big|_{\text{forward}} \rightarrow \beta_A(-t)^{\alpha(0)}\beta_C(-u)^{\alpha(0)}\beta_B \ . \qquad (A.93)$$

Now (A.88) says that $tu = (q_T^{\ 2} + m_C^{\ 2})s$, so that (A.93) becomes:

$$\rho_{AB}^{\ C} \rightarrow s^{\alpha(0)-1} \beta_A \beta_B \beta_C(q_T) \qquad (A.94)$$

and (A.91) allows us to write

$$\frac{1}{\sigma_{AB}} \rho_{AB}^{\ C} \rightarrow \beta_C(q_T). \qquad (A.95)$$

Since the trajectories are Pomerons, $\alpha(0) = 1$ and (A.94) exhibits scaling. Factorization (A.91) implies that $\rho_{AB}{}^{C}/\sigma_{AB}$ is independent of A and B (A.95); i.e., in the central region, the distribution is unaffected by the nature of the incident particles, it depends only on the type of product and the product's q_T. This is the central region of Figure 6, governed by $\beta_C(q_T)$, independent of y. The projectile freagmentation region of the figure is given by (A.92) and the target fragmentation region is given by:

$$\frac{\beta_{BC}(q_T, \frac{Y}{2} + y)}{\beta_B} .$$

Thus, Regge theory, with its factorization and Pomeron trajectory, reproduces the important experimental results.

We must now consider the special regions which we have called the "edges" of the rapidity distributions, the fragmentation regions where $\frac{M^2}{s} \to 0$, $x \to \pm 1$, $y \to \pm\frac{Y}{2}$, with t or u fixed. We can take this limit in two ways; either s and $M^2 \to \infty$ and then $\frac{M^2}{s} \to 0$, or $s \to \infty$ with M^2 fixed, then $M^2 \to \infty$. In both cases, t or u is fixed.

Consider first taking $s \to \infty$ with M^2 fixed and t fixed (the edge of the projectile fragmentation region). The inclusive amplitude $A + B \to C$ + anything is then essentially $A + B \to C + X$ where "X" has mass M. For $A\bar{C} \to \bar{B}X$, $\cos\theta_{cm} \propto s/M^2$, so Regge theory provides the result:

$$\frac{d\,\sigma_{AB}^{C}}{dt\;dM^2} \rightarrow \frac{1}{16\pi s^2}\left|\sum_i \beta_{AC}^{\;i}(t)\left(\frac{s}{M^2}\right)^{\alpha_i(t)}\beta_B^{\;i}(t,M^2)\left[\frac{-1+\tau_i e^{-i\pi\alpha_i}}{\sin\pi\alpha_i}\right]\right|^2$$

(A.96)

where τ_i are the trajectory signatures, $\alpha_i(t)$ are the trajectories in the $A\bar{C}$ channel, $\beta_{AC}^{\;i}(t)$ their couplings to the AC structure, and $\beta_B^{\;i}(t, M^2)$ their couplings to B and X (See Figure 10). Now we let M^2 become infinite. Note that $|\beta_B^{\;i}(t, M^2)|^2$ is proportional to the "total cross section" for $\alpha_i B \rightarrow X$ at total center of mass energy M^2. As $M^2 \rightarrow \infty$, the "elastic amplitude" related to this "cross section" must be dominated by Pomeron exchange ($\alpha_i B \rightarrow \alpha_i B$):

$$|\beta_B^{\;i}(t,M^2)|^2 \longrightarrow \beta_{RRP}^{\;i}(t)\;(M^2)^{\alpha(0)}\beta_B\;. \tag{A.97}$$

The new quantity $\beta_{RRP}(t)$ is the Pomeron coupling to the Reggeon $\alpha_i(t)$, called the triple Regge vertex for Pomeron-Reggeon-Reggeon (See Figure 11). Noting that

$$\rho = \omega\,\frac{d\sigma}{d^3q} = \frac{s}{\pi}\,\frac{d\,\sigma}{dt\;dM^2}\;,$$

we can write, using (A.96) and (A.97):

$$\frac{\rho_{AB}^{C}}{\beta_A\,\beta_B} \rightarrow \frac{1}{16\pi^2}\;\frac{|\beta_{AC}(t)|^2}{\beta_A}\left(\frac{s}{M^2}\right)^{2\alpha_R(t)-1}(M^2)^{\alpha(0)-1}\,\beta_{RRP}(t).$$

(A.98)

The other approach to the limit, where we take s, $M^2 \rightarrow \infty$, and then $(M^2/s) \rightarrow 0$, yields:

$$\frac{\rho_{AB}^{\ C}}{\beta_A \ \beta_B} \rightarrow \frac{\beta_{AC}(q_T, \ \frac{Y}{2} - y)}{\beta_A} \ s^{\alpha(0)-1} \tag{A.99}$$

from (A.90), where $y \longrightarrow (Y/2)$. Now $\alpha(0) = 1$, and in our limit, (A.87) tells us that $(M^2/s) = 1 - x$. Equating (A.98) and (A.99):

$$\beta_{AC}(q_T, \ \tfrac{Y}{2}-y) \sim (1-x)^{1-2\alpha_R(t)} \ . \tag{A.100}$$

In the target fragmentation region, u is fixed instead of t, and this analysis results in:

$$\beta_{BC}(q_T, \ \tfrac{Y}{2}+y) \sim (1+x)^{1-2\alpha_R(u)} \ . \tag{A.101}$$

We may remark at this point that the triple Regge vertex becomes a triple Pomeron vertex for a reaction such as $AB \rightarrow AX$, where the Reggeon in the crossed channel $A\bar{A}$ has the vacuum quantum numbers. We can now show that if the Pomeron trajectory is flat at one ($\alpha(t) \equiv 1$), that $\beta_{PPP}(t)=0$ for all t. We will make use of the sum rule:

$$\langle n \rangle \ \sigma_{AB} = \sum_C \int \frac{d\sigma_{AB}^{\ C}}{dt \ dM^2} \ dt \ dM^2 \tag{A.102}$$

where $\langle n \rangle$ is the average multiplicity of final particles of type C. A lower bound is obtained by taking only C=A in the sum, and integrating only over the triple Regge region, where t is small and M^2 is large. Then the integrand is obtained from (A.98), with $\rho = \frac{s}{\pi} \frac{d\sigma}{dt\, dM^2}$ and R=P, C=A:

$$\langle n \rangle \, \sigma_{AB} \geq \int_{M^2_{min}}^{s} dM^2 \int_{t_{max}}^{0} dt \, \frac{1}{16\pi s^2} \left(\frac{s}{M^2}\right)^{2\alpha(t)}$$

$$\times \, (M^2)^{\alpha(0)} \, |\beta_A(t)|^2 \, \beta_{PPP}(t) \, \beta_B \, . \qquad (A.103)$$

If $\alpha(t) \cong 1$, the integral over M^2 goes as $\int^{s} \frac{dM^2}{M^2} \sim \ln s$, and we find:

$$\langle n \rangle \, \sigma_{AB} \geq \ln(s). \qquad (A.104)$$

We will also use the momentum conservation sum rule:

$$P_\mu \, \sigma_{AB} = \sum_C \int dt \int dM^2 \, \frac{d\sigma^C_{AB}}{dt\, dM^2} \, q_\mu \quad . \qquad (A.105)$$

Again selecting the lower bound with C=A and the triple Regge region, and selecting the sum of the energy and longitudinal momentum components:

$$\sigma_{AB} \geq \int_{M^2_{min}}^{s} dM^2 \int_{t_{max}}^{0} dt \;\; \frac{d\sigma^A_{AB}}{dt\, dM^2} \left(1 - \frac{M^2}{s}\right). \qquad (A.106)$$

Here, $\alpha(t) \cong 1$ implies (using (A.98) again):

$$\sigma_{AB} \geq \ln(s). \qquad (A.107)$$

This is inconsistent. The assumption $\alpha(t) \cong 1$ implies that

σ_{AB} is independent of s, and here it also implies (A.107). Kinematically, since $\langle n \rangle \sim 1$ when $x \geq 0.5$, (A.104) also leads to the result (A.107). We must assume that if the Pomeron trajectory is flat at one, then $\beta_{PPP}(t) = 0$ for all t.

Experimental data suggests that $\alpha(t)$ is rather flat, but that β_{PPP} is nonzero.[24] The inevitable prediction is (A.107), but it is still to early to commit ourselves to a nonconstant total cross section at high energies.

APPENDIX B. $\langle n \rangle$ AND f_2 IN THE NOVA MODEL.

If $\sigma_n \propto n^{-2}$, then

$$\sigma_{tot} = \sum_{n}^{\frac{1}{2}\sqrt{s}} n^{-2} = 1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{s/4} .$$

At high s, σ_{tot} is relatively independent of s.

Now for the average multiplicity, we have:

$$\langle n \rangle = \frac{\sum_n n\sigma_n}{\sigma_{tot}} \propto \sum_n \frac{1}{n} \rightarrow \int_1^{\frac{1}{2}\sqrt{s}} \frac{dn}{n} \propto \ln(s) \qquad (B.1)$$

and for the Mueller two body correlation function:

$$f_2 = \langle n(n-1) \rangle - \langle n \rangle^2 = \langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle$$
$$\propto \langle n^2 \rangle - (\ln s)^2$$

at high s. But $\langle n^2 \rangle$ is simply:

$$\langle n^2 \rangle = \frac{\sum_n n^2\sigma_n}{\sigma_{tot}} \propto \sum_n^{\frac{1}{2}\sqrt{s}} 1 \propto \sqrt{s}$$

so that, finally:

$$f_2 \propto \sqrt{s} \qquad (B.2)$$

as was to be proven. Higher order correlation functions may be examined in a similar manner.

APPENDIX C. ALTERNATE DERIVATION OF THE CORRELATION FUNCTIONS IN THE MULTIPERIPHERAL CLUSTER MODEL.

Let m represent the number of particles in the cluster. The average multiplicity of particles is then:

$$\langle n \rangle = \sum_m m \langle n_m \rangle \tag{C.1}$$

where, as before, $\langle n_m \rangle$ is the average number of clusters of type m in the inelastic chain. That is:

$$\langle n_m \rangle \equiv \frac{1}{A(s,-\mu^2)} \sum_{n_1 n_2 \ldots} n_m A^s_{n_1 n_2 \ldots}(s,-\mu^2). \tag{C.2}$$

Recall our equation for the elastic diagram (Figure 13a)):

$$A_{n_1 n_2 \ldots}(A_1^R, A_2^R, \ldots) = \int \ldots \int \frac{d^4 q_1 \ldots d^4 q_{N-1}}{(8\pi^4)^{N-1}}$$
$$\times \frac{A^R_{m_1}(s_o) A^R_{m_2}(s_1) \ldots A^R_{m_N}(s_{N-1})}{(q_1{}^2-\mu^2)^2 \ldots (q_{N-1}{}^2-\mu^2)^2}$$

where N is the total number of clusters (not generally equal to η, the total particle multiplicity) and we have suppressed the arguments $p, p', m_1, m_2, \ldots, m_N$, on the left in order to emphasize the functional dependence on the resonant amplitudes A^R_m. Note that n_1 of the m_j equal one, n_2 of the m_j equal two, etc.

Now vary A^R_1, holding A^R_m for $m \neq 1$ fixed:

$$A_{n_1 n_2 \ldots}(A_1^R+\delta A_1^R, A_2^R, \ldots) = \int \ldots \int \frac{d^4q_1 \ldots d^4q_{N-1}}{(8\pi^4)^{N-1}}$$

$$\times \frac{\overbrace{(A_1^R+\delta A_1^R)(A_1^R+\delta A_1^R)\ldots(A_1^R+\delta A_1^R)}^{n_1 \text{ factors}}\overbrace{A_2^R A_2^R \ldots A_2^R}^{n_2 \text{ factors}}\overbrace{A_3^R A_3^R \ldots A_3^R}^{n_3 \text{ factors}}\ldots}{(q_1{}^2-\mu^2)^2(q_2{}^2-\mu^2)^2\ldots(q_{N-1}{}^2-\mu^2)^2}$$

where $\delta A_1^R(s_j) \equiv \lambda\delta(s_j' - s_j)$, and we have:

$$\delta A_{n_1 n_2 \ldots} = A_{n_1 n_2 \ldots}(A_1^R+\delta A_1^R, A_2^R, A_3^R, \ldots) - A_{n_1 n_2 \ldots}(A_1^R, A_2^R, A_3^R, \ldots)$$

so that:

$$\frac{\delta A_{n_1 n_2 \ldots}}{\delta A_1^R(s_o)} \equiv \lim_{\lambda \to o} \frac{\delta A_{n_1 n_2 \ldots}}{\lambda} = \int \ldots \int \frac{d^4q_1 \ldots d^4q_{N-1}}{(8\pi^4)^{N-1}}$$

$$\times \frac{\overbrace{A_1^R A_1^R \ldots A_1^R}^{n_1-1 \text{ factors}}\overbrace{A_2^R A_2^R \ldots A_2^R}^{n_2}\overbrace{A_3^R A_3^R \ldots A_3^R}^{n_3}\ldots\delta(s_j'-s_j)}{(q_1{}^2-\mu^2)^2\ldots(q_2{}^2-\mu^2)^2}$$

+ all other permutations of the missing $A_1^R(s_j)$ replaced by $\delta(s_j'-s_j)$; a total of n_1 such terms,

where $s_j'=s_o$ for all j. By multiplying by $A_1^R(s_o)$ and integrating over s_o, we replace the missing factor in $A_{n_1 n_2 \ldots}$ in each of the n_1 terms, obtaining:

$$\int ds_o A_1^R(s_o) \frac{\delta A_{n_1 n_2 \ldots}}{\delta A_1^R(s_o)} = n_1 A_{n_1 n_2 \ldots}$$

Similarly, by varying $A_m^R(s_o)$ for any specific m:

$$\int ds_o \; A_m^R(s_o) \frac{\delta A_{n_1 n_2 \ldots}}{\delta A_m^R(s_o)} = n_m A_{n_1 n_2 \ldots} \quad .$$

We sum over permutations of the m_j on both sides, with the n_j fixed:

$$\int ds_o \; A_m^R(s_o) \frac{\delta A^s_{n_1 n_2 \ldots}}{\delta A_m^R(s_o)} = n_m A^s_{n_1 n_2 \ldots} \quad .$$

Finally, we sum over the n_j to obtain the average number of clusters of each type m:

$$\langle n_m \rangle = \frac{1}{A} \sum_{n_1 n_2 \ldots} n_m A^s_{n_1 n_2 \ldots} = \frac{1}{A} \sum_{n_1 n_2 \ldots} \int ds_o A_m^R(s_o) \frac{\delta A^s_{n_1 n_2 \ldots}}{\delta A_m^R(s_o)}$$

$$= \frac{1}{A} \int ds_o \; A_m^R(s_o) \frac{\delta A}{\delta A_m^R(s_o)} \quad .$$

Thus, the average multiplicity of particles is:

$$\langle n \rangle = \sum_m \frac{m}{A(s,-\mu^2)} \int ds_o \; A_m^R(s_o) \frac{\delta A(s,-\mu^2)}{\delta A_m^R(s_o)} \quad .$$

It is time to make use of the energy dependence of $A(s,-\mu^2)$ (equation (2.5)). We obtain:

$$\langle n \rangle = \sum_m m \int ds_o \; A_m^R(s_o) \frac{\delta \ln A(s,-\mu^2)}{\delta A_m^R(s_o)}$$

$$= \sum_m m \int ds_o \; A_m^R(s_o) \left[\frac{\delta \alpha}{\delta A_m^R(s_o)} \ln(s) + \frac{\delta \ln \gamma_\alpha(-\mu^2)}{\delta A_m^R(s_o)} \right]$$

$$= \left[\sum_m m \int ds_o \; A_m^R(s_o) \frac{\delta \alpha}{\delta A_m^R(s_o)} \right] \ln(s) + \text{constant.}$$

We will calculate the functional derivative of α with respect to $A_m^R(s_o)$ by using the functional dependence exhibited in the eigenvalue equation (2.9). Differentiate (2.9) implicitly with respect to $A_{m'}^R(s_o)$:

$$\int \frac{ds_o}{16\pi^3} \, \delta A_{m'}^R(s_o) \int_o^1 dx \, \frac{x^\alpha(1-x)}{s_o x+\mu^2(1-x)^2}$$
$$+ \int \frac{ds_o}{16\pi^3} \sum_m A_m^R(s_o) \int_o^1 dx \, \frac{\delta(x^\alpha)(1-x)}{s_o x+\mu^2(1-x)^2} = 0$$

where

$$\delta(x^\alpha) = \delta(e^{\ln x^\alpha}) = \delta\left[(e^\alpha)^{\ln x}\right] = (\ln x)(e^\alpha)^{(\ln x-1)} e^\alpha \delta\alpha$$

$$= (\ln x) x^\alpha \delta\alpha \, .$$

Recall that

$$\delta A_{m'}^R(s_o) = \lambda\delta(s_o'-s_o) \text{ and } \frac{\delta\alpha}{\delta A_{m'}^R(s_o)} \equiv \lim_{\lambda\to o} \frac{\delta\alpha}{\lambda} \, .$$

Thus, our differential eigenvalue equation becomes:

$$\lambda\int_o^1 dx \frac{x^\alpha(1-x)}{s_o' x+\mu^2(1-x)^2} + (\delta\alpha)\int ds_o \sum_m A_m^R(s_o) \int_o^1 dx (\ln x) \frac{x^\alpha(1-x)}{s_o x+\mu^2(1-x)^2}$$
$$=0.$$

Dividing by λ and taking $\lambda\to 0$, we obtain:

$$\frac{\delta\alpha}{\delta A^R_{m'}(s_o')} = -\left[\int ds_o \sum_m A^R_m(s_o)\int_o^1 dx\ (\ln x)\ \frac{x^\alpha(1-x)}{s_o x+\mu^2(1-x)^2}\right]^{-1}$$
$$\times \int_o^1 dx\ \frac{x^\alpha(1-x)}{s_o' x+\mu^2(1-x)^2} \quad . \qquad (C.3)$$

Substitute this into our equation for the average multiplicity of particles to obtain:

$$\langle n\rangle = -\left[\int ds_o \sum_m A^R_m(s_o)\int_o^1 dx\ (\ln x)\ \frac{x^\alpha(1-x)}{s_o x+\mu^2(1-x)^2}\right]^{-1}$$
$$\times\left[\sum_{m'} m'\int ds_o'\ A^R_{m'}(s_o')\int_o^1 dx\ \frac{x^\alpha(1-x)}{s_o' x+\mu^2(1-x)^2}\right]\ln(s)$$
$$+ \text{ constant} \qquad (C.4)$$

which is identical to (2.18).

Now consider the two body correlation function:

$$f_2 = \langle n(n-1)\rangle - \langle n\rangle^2$$

$$= \langle(\sum_m mn_m)(\sum_m mn_m-1)\rangle - (\sum_m m\langle n_m\rangle)^2$$

$$= \sum_m\left[m^2(\Delta n_m)^2 - m\langle n_m\rangle\right] + 2\sum_{\substack{mm'\\ m\neq m'}} mm'(\langle n_m n_{m'}\rangle - \langle n_m\rangle\langle n_{m'}\rangle) \quad (C.5)$$

where $\Delta n_m \equiv \sqrt{\langle n_m{}^2\rangle - \langle n_m\rangle^2}$ and where

$$\langle n_m{}^2\rangle \equiv \frac{1}{A}\sum_{n_1 n_2\ldots} n_m{}^2\ A^s_{n_1 n_2\ldots} \quad .$$

By taking

$$\frac{\delta A_{n_1 n_2 \dots}(s,-\mu^2)}{\delta A_m^R(s_o)}$$

and again varying $A_m^R(s_o)$, we arrive at:

$$\langle n_m(n_m-1)\rangle = \frac{1}{A(s,-\mu^2)}\iint ds_o ds_o' \; A_m^R(s_o)A_m^R(s_o')\frac{\delta^2 A(s,-\mu^2)}{\delta A_m^R(s_o)\delta A_m^R(s_o')} \; .$$

If we had taken our first functional derivative and varied $A_{m'}^R(s_o)$ for $m'\neq m$, we would have obtained:

$$\langle n_m n_{m'}\rangle = \frac{1}{A(s,-\mu^2)}\iint ds_o ds_o' \; A_m^R(s_o)A_{m'}^R(s_o')\frac{\delta^2 A(s,-\mu^2)}{\delta A_m^R(s_o)\delta A_{m'}^R(s_o')} \; .$$

We need only plug these results for $\langle n_m{}^2\rangle$ and $\langle n_m n_{m'}\rangle$ into our equation for f_2 to find:

$$f_2 = \sum_m m(m-1)\langle n_m\rangle - m^2\langle n_m\rangle^2 - 2\sum_{\substack{mm' \\ m\neq m'}} mm'\langle n_m\rangle\langle n_{m'}\rangle$$
$$+ \sum_{mm'} \frac{mm'}{A(s,-\mu^2)}\iint ds_o ds_o' \; A_m^R(s_o)A_{m'}^R(s_o')\frac{\delta^2 A(s,-\mu^2)}{\delta A_m^R(s_o)\delta A_{m'}^R(s_o')}$$

where

$$\langle n_m\rangle = \frac{1}{A(s,-\mu^2)}\int ds_o \; A_m^R(s_o) \; \frac{\delta A(s,-\mu^2)}{\delta A_m^R(s_o)}.$$

Note now that

$$\frac{\partial^2 \ln f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{1}{f}\frac{\partial f}{\partial y}\right) = -\frac{1}{f^2}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{1}{f}\frac{\partial^2 f}{\partial x\partial y}$$

so that

$$\frac{1}{f}\,\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial \ln f}{\partial x}\,\frac{\partial \ln f}{\partial y} + \frac{\partial^2 \ln f}{\partial x\partial y}\,.$$

In terms of our function $A(s,-\mu^2)$ and its functional derivatives:

$$\frac{1}{A}\,\frac{\delta^2 A}{\delta A^R_m \delta A^R_{m'}} = \frac{\delta \ln A}{\delta A^R_m}\,\frac{\delta \ln A}{\delta A^R_{m'}} + \frac{\delta^2 \ln A}{\delta A^R_m \delta A^R_{m'}}$$

where the arguments have been suppressed for clarity. Making use of this result and the energy dependence of $A(s,-\mu^2)$ (equation (2.5)), f_2 becomes, after cancellation of terms:

$$f_2 = \left[\sum_m m(m-1)\int ds_o A^R_m(s_o)\frac{\delta\alpha}{\delta A^R_m(s_o)} + \sum_{mm'} mm'\iint ds_o ds_o' A^R_m(s_o) A^R_{m'}(s_o') \times \frac{\delta^2\alpha}{\delta A^R_m(s_o)\delta A^R_{m'}(s_o')}\right]\ln(s) + \text{constant.}$$

To evaluate f_2 further, we take the first functional derivative of α with respect to $A^R_m(s_o)$, as calculated for $\langle n\rangle$, and differentiate again. The previous result for the first derivative can be written in the form:

$$\int_o^1 dx\,\frac{x^{\alpha}(1-x)}{s_o'x+\mu^2(1-x)^2} + \left(\frac{\delta\alpha}{\delta A^R_{m'}(s_o')}\right)\int ds_o \sum_m A^R_m(s_o)\int_o^1 dx\frac{(\ln x)x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}$$

$$= 0.$$

The suggested second differentiation thus yields:

$$(\delta\alpha)\int_0^1 dx\,\frac{(\ln x)x^{\alpha}(1-x)}{s_o'x+\mu^2(1-x)^2}$$

$$+\delta\left(\frac{\delta\alpha}{\delta A^R_{m'}(s_o')}\right)\int ds_o\sum_m A^R_m(s_o)\int_0^1 dx\,\frac{(\ln x)x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}$$

$$+\left(\frac{\delta\alpha}{\delta A^R_{m'}(s_o')}\right)\int ds_o\,\delta A^R_{m''}(s_o)\int_0^1 dx\,\frac{(\ln x)x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}$$

$$+\left(\frac{\delta\alpha}{\delta A^R_{m'}(s_o')}\right)(\delta\alpha)\int ds_o\sum_m A^R_m(s_o)\int_0^1 dx\,\frac{(\ln x)^2 x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2} = 0.$$

Note again that $A^R_{m''}(s_o) = \lambda'\delta(s_o''-s_o)$ and also note that

$$\frac{\delta\alpha}{\delta A^R_{m''}(s_o'')} \equiv \lim_{\lambda'\to 0}\frac{\delta\alpha}{\lambda'}\,, \text{ with}$$

$$\frac{\delta^2\alpha}{\delta A^R_{m'}(s_o')\delta A^R_{m''}(s_o'')} \equiv \lim_{\lambda'\to 0}\frac{1}{\lambda'}\,\delta\left(\frac{\delta\alpha}{\delta A^R_{m'}(s_o')}\right). \quad \text{Thus:}$$

$$\frac{\delta^2\alpha}{\delta A^R_{m'}(s_o')\delta A^R_{m''}(s_o'')} = -\left[\int ds_o\sum_m A^R_m(s_o)\int_0^1 dx\,\frac{(\ln x)x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\right]^{-1}$$

$$\times\left\{\left[\int ds_o\sum_m A^R_m(s_o)\int_0^1 dx\,\frac{(\ln x)^2x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\right]\frac{\delta\alpha}{\delta A^R_{m'}(s_o')}\,\frac{\delta\alpha}{\delta A^R_{m''}(s_o'')}\right.$$

$$\left.+\int_0^1 dx\frac{(\ln x)x^{\alpha}(1-x)}{s_o'x+\mu^2(1-x)^2}\,\frac{\delta\alpha}{\delta A^R_{m''}(s_o'')}+\int_0^1 dx\frac{(\ln x)x^{\alpha}(1-x)}{s_o''x+\mu^2(1-x)^2}\,\frac{\delta\alpha}{\delta A^R_{m'}(s_o')}\right\}$$

(C.6)

Substituting this and our previous result for the first derivative into our equation for f_2, we find:

$$f_2 = \left[\frac{1}{\phi_o^{(1)}}\right]\Bigg\{ -\sum_{m'} m'(m'-1)\int ds_o'\; A^R_{m'}(s_o') \int_o^1 dx\, \frac{x^{\alpha}(1-x)}{s_o'x+\mu^2(1-x)^2}$$

$$- \sum_{m'm''} m'm'' \iint ds_o' ds_o''\; A^R_{m'}(s_o')\; A^R_{m''}(s_o'') \left(\frac{1}{\phi_o^{(1)}}\right)$$

$$X\Bigg[\left(\int ds_o \sum_m A^R_m(s_o) \int_o^1 dx \frac{(\ln x)^2 x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\right)\left(\int_o^1 dx \frac{x^{\alpha}(1-x)}{s_o' x+\mu^2(1-x)^2}\right)$$

$$X\left(\int_o^1 dx \frac{x^{\alpha}(1-x)}{s_o'' x+\mu^2(1-x)^2}\right)\left(\frac{1}{\phi_o^{(1)}}\right)$$

$$+\; 2\left(\int_o^1 dx \frac{(\ln x)x^{\alpha}(1-x)}{s_o' x+\mu^2(1-x)^2}\right)\int_o^1 dx \frac{x^{\alpha}(1-x)}{s_o'' x+\mu^2(1-x)^2}\Bigg]\Bigg\} \ln(s)$$

$$+ \text{ constant} \qquad \text{(C.7)}$$

where the reader is reminded that

$$\phi_o^{(1)} = \int ds_o \sum_m A^R_m(s_o) \int_o^1 dx\, \frac{(\ln x)x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2} \;.$$

Note that (C.7) is the same result as (2.19).

Finally, we examine f_3:

$$f_3 = \langle n(n-1)(n-2)\rangle - 3\langle n\rangle f_2 - \langle n\rangle^3 \;. \qquad \text{(C.8)}$$

Using the fact that $f_2 = \langle n(n-1)\rangle - \langle n\rangle^2$, we obtain:

$$f_3 = \langle n^3\rangle - 3\langle n^2\rangle\langle n\rangle - 3\langle n^2\rangle + 2\langle n\rangle^3 + 3\langle n\rangle^2 + 2\langle n\rangle$$

where $n = \sum_m m n_m$. Thus, f_3 becomes:

$$f_3 = \sum_m \Big[m^3(\langle n_m^3\rangle - 3\langle n_m^2\rangle\langle n_m\rangle + 2\langle n_m\rangle^3) - 3m^2(\langle n_m^2\rangle - \langle n_m\rangle^2) + 2m\langle n_m\rangle \Big]$$

$$+ \sum_{\substack{mm' \\ m\neq m'}} \Big\{ \tfrac{3}{2} mm'^2 \Big[\langle n_m n_{m'}\rangle^2 - \langle n_m\rangle\langle n_{m'}^2\rangle - 2\langle n_{m'}\rangle(\langle n_m n_{m'}\rangle - \langle n_m\rangle\langle n_{m'}\rangle)$$

$$- 3mm'(\langle n_m n_{m'}\rangle - \langle n_m\rangle\langle n_{m'}\rangle) \Big] \Big\}$$

$$+ \sum_{\substack{mm'm'' \\ \text{all unequal}}} 6mm'm'' \Big[\frac{\langle n_m n_{m'} n_{m''}\rangle}{6} - \frac{\langle n_m n_{m'}\rangle\langle n_{m''}\rangle}{2} + 2\frac{\langle n_m\rangle\langle n_{m'}\rangle\langle n_{m''}\rangle}{6} \Big]$$

(C.9)

We have already calculated $\langle n_m\rangle$, $\langle n_m^2\rangle$, and $\langle n_m n_{m'}\rangle$. To obtain the cubic averages, we need:

$$\langle n_m(n_m-1)(n_m-2)\rangle = \frac{1}{A(s,-\mu^2)} \sum_{n_1 n_2 \ldots} n_m(n_m-1)(n_m-2)A^S_{n_1 n_2 \ldots}$$

$$= \frac{1}{A(s,-\mu^2)} \iiint ds_o\, ds_o'\, ds_o''\, A^R_m(s_o) A^R_m(s_o') A^R_m(s_o'') \frac{\delta^3 A(s,-\mu^2)}{\delta A^R_m(s_o)\,\delta A^R_m(s_o')\,\delta A^R_m(s_o'')}$$

and two other relations:

$$\langle n_m n_{m'}(n_{m'}-1)\rangle = \frac{1}{A(s,-\mu^2)} \iiint ds_o\, ds_o'\, ds_o''\; A^R_m(s_o) A^R_{m'}(s_o') A^R_{m'}(s_o'')$$

$$\times \frac{\delta^3 A(s,-\mu^2)}{\delta A^R_m(s_o)\,\delta A^R_{m'}(s_o')\,\delta A^R_{m'}(s_o'')}$$

$$\langle n_m n_{m'} n_{m''}\rangle = \frac{1}{A(s,-\mu^2)} \iiint ds_o\, ds_o'\, ds_o''\; A^R_m(s_o) A^R_{m'}(s_o') A^R_{m''}(s_o'')$$

$$\times \frac{\delta^3 A(s,-\mu^2)}{\delta A^R_m(s_o)\,\delta A^R_{m'}(s_o')\,\delta A^R_{m''}(s_o'')} .$$

Thus we have:

$$\langle n_m{}^3\rangle = 3\langle n_m(n_m-1)\rangle + \langle n_m\rangle + \frac{1}{A(s,-\mu^2)}\iiint ds_o\, ds_o'\, ds_o''\, A_m^R(s_o) A_m^R(s_o') A_m^R(s_o'')$$

$$\times\ \frac{\delta^3 A(s,-\mu^2)}{\delta A_m^R(s_o)\,\delta A_m^R(s_o')\,\delta A_m^R(s_o'')}$$

and:

$$\langle n_m n_{m'}{}^2\rangle = \langle n_m n_{m'}\rangle + \frac{1}{A(s,-\mu^2)}\iiint ds_o\, ds_o'\, ds_o''\, A_m^R(s_o) A_{m'}^R(s_o') A_{m'}^R(s_o'')$$

$$\times\ \frac{\delta^3 A(s,-\mu^2)}{\delta A_m^R(s_o)\,\delta A_{m'}^R(s_o')\,\delta A_{m'}^R(s_o'')}\ .$$

We can now use these results, together with previous results for $\langle n_m\rangle$, $\langle n_m{}^2\rangle$, $\langle n_m n_{m'}\rangle$, to write f_3 in terms of functional derivatives of $A(s,-\mu^2)$. Suppress the arguments of $A(s,-\mu^2)$ and $A_m^R(s_o)$ for clarity, and make use of the identity:

$$\frac{\partial^3 \ln f}{\partial x\partial y\partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \ln f}{\partial y}\frac{\partial \ln f}{\partial z} + \frac{\partial^2 f}{\partial y\partial z}\right]$$

$$= -\frac{\partial^2 \ln f}{\partial x\partial y}\frac{\partial \ln f}{\partial z} - \frac{\partial \ln f}{\partial y}\frac{\partial^2 \ln f}{\partial x\partial z} - \frac{1}{f}\frac{\partial f}{\partial x}\left(\frac{1}{f}\frac{\partial^2 f}{\partial y\partial z}\right) + \frac{1}{f}\frac{\partial^3 f}{\partial x\partial y\partial z}$$

$$= -\frac{\partial^2 \ln f}{\partial x\partial y}\frac{\partial \ln f}{\partial z} - \frac{\partial \ln f}{\partial y}\frac{\partial^2 \ln f}{\partial x\partial z} - \frac{1}{f}\frac{\partial f}{\partial x}\left(\frac{\partial^2 \ln f}{\partial y\partial z} + \frac{\partial \ln f}{\partial y}\frac{\partial \ln f}{\partial z}\right)$$

$$+ \frac{1}{f}\frac{\partial^3 f}{\partial x\partial y\partial z}$$

$$= -\frac{\partial^2 \ln f}{\partial x\partial y}\frac{\partial \ln f}{\partial z} - \frac{\partial \ln f}{\partial y}\frac{\partial^2 \ln f}{\partial x\partial z} - \frac{\partial \ln f}{\partial x}\frac{\partial^2 \ln f}{\partial y\partial z}$$

$$- \frac{\partial \ln f}{\partial x}\frac{\partial \ln f}{\partial y}\frac{\partial \ln f}{\partial z} + \frac{1}{f}\frac{\partial^3 f}{\partial x\partial y\partial z}\ .$$

In terms of our functional derivatives of $A(s,-\mu^2)$, this identity becomes:

$$\frac{1}{A}\frac{\delta^3 A}{\delta A^R_m \delta A^R_{m'} \delta A^R_{m''}} = \frac{\delta^3 \ln A}{\delta A^R_m \delta A^R_{m'} \delta A^R_{m''}} + \frac{\delta \ln A}{\delta A^R_m}\frac{\delta \ln A}{\delta A^R_{m'}}\frac{\delta \ln A}{\delta A^R_{m''}}$$
$$+ \frac{\delta^2 \ln A}{\delta A^R_m \delta A^R_{m'}}\frac{\delta \ln A}{\delta A^R_{m''}} + \frac{\delta^2 \ln A}{\delta A^R_{m'} \delta A^R_{m''}}\frac{\delta \ln A}{\delta A^R_{m}} + \frac{\delta^2 \ln A}{\delta A^R_{m''} \delta A^R_{m}}\frac{\delta \ln A}{\delta A^R_{m'}} \qquad (C.10)$$

where $A^R_m \equiv A^R_m(s_o)$, $A^R_{m'} \equiv A^R_{m'}(s'_o)$, and $A^R_{m''} \equiv A^R_{m''}(s''_o)$. Together with the identity previously derived for the second derivative, this identity can be used to obtain:

$$f_3 = \sum_m \Bigg[m^3\Bigg(\iiint ds_o ds'_o ds''_o A^R_m A^R_m A^R_m \frac{\delta^3 \ln A}{\delta A^R_m \delta A^R_m \delta A^R_m}$$
$$+3\iint ds_o ds'_o A^R_m A^R_m \frac{\delta^2 \ln A}{\delta A^R_m \delta A^R_m} + \int ds_o A^R_m \frac{\delta \ln A}{\delta A^R_m}\Bigg)$$
$$-3m^2\Bigg(\iint ds_o ds'_o A^R_m A^R_m \frac{\delta^2 \ln A}{\delta A^R_m \delta A^R_m} + \int ds_o A^R_m \frac{\delta \ln A}{\delta A^R_m}\Bigg) + 2m\int ds_o A^R_m \frac{\delta \ln A}{\delta A^R_m}\Bigg]$$
$$+ \sum_{\substack{mm' \\ m=m'}} \Bigg[3mm'^2\Bigg(\iiint ds_o ds'_o ds''_o A^R_m A^R_{m'} A^R_{m'} \frac{\delta^3 \ln A}{\delta A^R_m \delta A^R_{m'} \delta A^R_{m'}}$$
$$+ \iint ds_o ds'_o A^R_m A^R_{m'} \frac{\delta^2 \ln A}{\delta A^R_m \delta A^R_{m'}} - 3mm' \iint ds_o ds'_o A^R_m A^R_{m'} \frac{\delta^2 \ln A}{\delta A^R_m \delta A^R_{m'}}\Bigg]$$
$$+ \sum_{\substack{mm'm'' \\ \text{all unequal}}} mm'm'' \iiint ds_o ds'_o ds''_o A^R_m A^R_{m'} A^R_{m''} \frac{\delta^3 \ln A}{\delta A^R_m \delta A^R_{m'} \delta A^R_{m''}} .$$

We now make use of the energy dependence of our forward absorptive elastic amplitude $A(s,-\mu^2) = s^{\alpha}\,\gamma_{\alpha}(-\mu^2)$ to provide asymptotically:

$$\frac{f_3}{\ln s}=\sum_m\Big[m^3\iiint ds_o ds_o' ds_o'' A_m^R A_m^R A_m^R \frac{\delta^3\alpha}{\delta A_m^R\delta A_m^R\delta A_m^R}$$

$$+3m^2(m-1)\iint ds_o ds_o' A_m^R A_m^R \frac{\delta^2\alpha}{\delta A_m^R\delta A_m^R} + m(m-1)(m-2)\int ds_o A_m^R \frac{\delta\alpha}{\delta A_m^R}\Big]$$

$$+\sum_{\substack{mm'\\ m\neq m'}}\Big[3mm'^2\iiint ds_o ds_o' ds_o'' A_m^R A_{m'}^R A_{m'}^R \frac{\delta^3\alpha}{\delta A_m^R\delta A_{m'}^R\delta A_{m'}^R}$$

$$+\frac{3}{2}mm'(m+m'-2)\iint ds_o ds_o' A_m^R A_{m'}^R \frac{\delta^2\alpha}{\delta A_m^R\delta A_{m'}^R}\Big]$$

$$+\sum_{\substack{mm'm''\\ \text{all unequal}}} mm'm''\iiint ds_o ds_o' ds_o'' A_m^R A_{m'}^R A_{m''}^R \frac{\delta^3\alpha}{\delta A_m^R\delta A_{m'}^R\delta A_{m''}^R}$$

where the arguments s_o, s_o', s_o'', are understood for consecutive A^R **within** each term. We now evaluate the functional derivatives of α with respect to A_m^R by making use of the first iteration of the eigenvalue equation (2.9). We have already calculated the first and second derivatives in our examination of $\langle n\rangle$ and f_2. The third derivative is obtained by leaving the second derivative in the following form and differentiating implicitly again:

$$\Big[\int ds_o\sum_m A_m^R(s_o)\int_0^1 dx\frac{(\ln x)x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\Big]\frac{\delta^2\alpha}{\delta A_m^R(s_o')\delta A_{m'}^R(s_o'')}$$

$$+\Big[\int ds_o\sum_m A_m^R(s_o)\int_0^1 dx\frac{(\ln x)^2x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\Big]\frac{\delta\alpha}{\delta A_m^R(s_o')}\frac{\delta\alpha}{\delta A_{m'}^R(s_o'')}$$

$$+\Big[\int_0^1 dx\frac{(\ln x)x^{\alpha}(1-x)}{s_o' x+\mu^2(1-x)^2}\Big]\frac{\delta\alpha}{\delta A_{m'}^R(s_o'')}+\Big[\int_0^1 dx\frac{(\ln x)x^{\alpha}(1-x)}{s_o'' x+\mu^2(1-x)^2}\Big]\frac{\delta\alpha}{\delta A_m^R(s_o')}=0.$$

The next differentiation then yields:

$$
\begin{aligned}
&\left[\int ds_o \sum_m A_m^R(s_o) \int_0^1 dx \frac{(\ln x)x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\right] \frac{\delta^3 \alpha}{\delta A_m^R(s_o')\,\delta A_{m'}^R(s_o'')\,\delta A_{m''}^R(s_o''')} \\
&+\left[\int ds_o \sum_m A_m^R(s_o) \int_0^1 dx \frac{(\ln x)^2 x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\right] \frac{\delta \alpha}{\delta A_{m''}^R(s_o''')} \; \frac{\delta^2 \alpha}{\delta A_m^R(s_o')\,\delta A_{m'}^R(s_o'')} \\
&+\left[\int_0^1 dx \frac{(\ln x)x^{\alpha}(1-x)}{s_o''' x+\mu^2(1-x)^2}\right] \frac{\delta^2 \alpha}{\delta A_m^R(s_o')\,\delta A_{m'}^R(s_o'')} \\
&+\left[\int ds_o \sum_m A_m^R(s_o) \int_0^1 dx \frac{(\ln x)^2 x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\right] \left[\frac{\delta^2 \alpha}{\delta A_m^R(s_o')\,\delta A_{m''}^R(s_o''')} \; \frac{\delta \alpha}{\delta A_{m'}^R(s_o'')} \right. \\
&\qquad\qquad \left. + \frac{\delta \alpha}{\delta A_m^R(s_o')} \; \frac{\delta^2 \alpha}{\delta A_{m'}^R(s_o'')\,\delta A_{m''}^R(s_o''')}\right] \\
&+\left[\int ds_o \sum_m A_m^R(s_o) \int_0^1 dx \frac{(\ln x)^3 x^{\alpha}(1-x)}{s_o x+\mu^2(1-x)^2}\right] \frac{\delta \alpha}{\delta A_m^R(s_o')} \; \frac{\delta \alpha}{\delta A_{m'}^R(s_o'')} \; \frac{\delta \alpha}{\delta A_{m''}^R(s_o''')} \\
&+\left[\int_0^1 dx \frac{(\ln x)^2 x^{\alpha}(1-x)}{s_o''' x+\mu^2(1-x)^2}\right] \frac{\delta \alpha}{\delta A_m^R(s_o')} \; \frac{\delta \alpha}{\delta A_{m'}^R(s_o'')} \\
&+\left[\int_0^1 dx \frac{(\ln x)x^{\alpha}(1-x)}{s_o' x+\mu^2(1-x)^2}\right] \frac{\delta^2 \alpha}{\delta A_{m'}^R(s_o'')\,\delta A_{m''}^R(s_o''')} \\
&+\left[\int_0^1 dx \frac{(\ln x)^2 x^{\alpha}(1-x)}{s_o' x+\mu^2(1-x)^2}\right] \frac{\delta \alpha}{\delta A_{m'}^R(s_o'')} \; \frac{\delta \alpha}{\delta A_{m''}^R(s_o''')} \\
&+\left[\int_0^1 dx \frac{(\ln x)x^{\alpha}(1-x)}{s_o'' x+\mu^2(1-x)^2}\right] \frac{\delta^2 \alpha}{\delta A_m^R(s_o')\,\delta A_{m''}^R(s_o''')} \\
&+\left[\int_0^1 dx \frac{(\ln x)^2 x^{\alpha}(1-x)}{s_o'' x+\mu^2(1-x)^2}\right] \frac{\delta \alpha}{\delta A_m^R(s_o')} \; \frac{\delta \alpha}{\delta A_{m''}^R(s_o''')} \\
&= 0.
\end{aligned}
$$

This result, combined with the expressions for the first and second derivatives previously found, and the definitions

(2.14) and (2.15), yield for the three body correlation function, asymptotically:

$$
\begin{aligned}
\frac{f_3}{\ln s} = {} & \sum_m m(m-1)(m-2)\left(\frac{-F^o_m}{\phi^1_o}\right) \\
& + \sum_{mm'} \frac{3mm'}{2}(m+m'-2)\left(\frac{-1}{(\phi^1_o)^2}\right)\left(\frac{\phi^2_o}{\phi^1_o} F^o_m F^o_{m'} - F^1_m F^o_{m'} - F^1_{m'} F^o_m\right) \\
& + \sum_{mm'm''} mm'm''\left(\frac{-1}{(\phi^1_o)^3}\right)\Bigg\{\frac{\phi^2_o}{\phi^1_o}\Bigg[F^o_m\left(\frac{\phi^2_o F^o_{m'} F^o_{m''}}{\phi^1_o} - F^1_{m'} F^o_{m''} - F^1_{m''} F^o_{m'}\right) \\
& \qquad + F^o_{m'}\left(\frac{\phi^2_o F^o_{m''} F^o_m}{\phi^1_o} - F^1_{m''} F^o_m - F^1_m F^o_{m''}\right) \\
& \qquad + F^o_{m''}\left(\frac{\phi^2_o}{\phi^1_o} F^o_m F^o_{m'} - F^1_m F^o_{m'} - F^1_{m'} F^o_m\right)\Bigg] \\
& \quad - F^1_m\left(\frac{\phi^2_o}{\phi^1_o} F^o_{m'} F^o_{m''} - F^1_{m'} F^o_{m''} - F^1_{m''} F^o_{m'}\right) \\
& \quad - F^1_{m''}\left(\frac{\phi^2_o}{\phi^1_o} F^o_m F^o_{m'} - F^1_m F^o_{m'} - F^1_{m'} F^o_m\right) \\
& \quad - \phi^3_o F^o_m F^o_{m'} F^o_{m''} + \phi^1_o\left(F^2_m F^o_{m'} F^o_{m''} + F^2_{m'} F^o_m F^o_{m''} + F^2_{m''} F^o_m F^o_{m'}\right)\Bigg\}
\end{aligned}
$$

which is, in fact, just (2.20) with some terms expanded out.

APPENDIX D. THE SIMPLE MULTIPERIPHERAL ρ-DOMINANCE MODEL.

The invariant amplitude for two pions of momenta p,q, to interact to form a single ρ of momentum k, which decays into two pions of momenta p,q, is:

$$M = \frac{f_\rho^{\,2}(ip^\mu+iq^\mu)(g_{\mu\nu}+k_\mu k_\nu/m_\rho^{\,2})(ip^\nu+iq^\nu)}{s_o-m_\rho^{\,2}+i\epsilon} \tag{D.1}$$

since the Lagrangian density is

$$\mathcal{L} = ig'\rho_\mu \varphi \overleftrightarrow{\partial^\mu} \varphi = ig'\rho_\mu\left[\varphi\partial^\mu\varphi - (\partial^\mu\varphi)\varphi\right],$$

and where $s_o = (p+q)^2$. The $g_{\mu\nu}$ term yields s_o and the $k_\mu k_\nu$ term yields

$$\frac{\left[(p+q)\cdot k\right]^2}{m_\rho^{\,2}} = \frac{s_o^{\,2}}{m_\rho^{\,2}},$$

so that we have:

$$M = \frac{-f_\rho^{\,2}}{s_o-m_\rho^{\,2}+i\epsilon}\left(s_o + \frac{s_o^{\,2}}{m_\rho^{\,2}}\right) \tag{D.2}$$

and the corresponding resonant amplitude is:

$$A_\rho^R(s_o) = \mathrm{Im}(M) = \mathrm{Im}\left[\frac{-f_\rho^{\,2}(s_o-m_\rho^{\,2}-i\epsilon)}{(s_o-m_\rho^{\,2})^2+\epsilon^2}\left(s_o + \frac{s_o^{\,2}}{m_\rho^{\,2}}\right)\right] \tag{D.3}$$

where $\epsilon \to 0$. Therefore, A_ρ^R becomes:

$$A_\rho^R(s_0) = \lim_{\epsilon\to 0} \frac{f_\rho{}^2\epsilon}{(s_0-m_\rho{}^2)^2+\epsilon^2}\left(s_0 + \frac{s_0{}^2}{m_\rho{}^2}\right).$$

This is zero at $s_0 \neq m_\rho{}^2$ and infinite at $s_0 = m_\rho{}^2$. Moreover, note that:

$$\int_{-\infty}^{\infty} ds_0 A_\rho^R(s_0) f(s_0) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} ds_0 \frac{f_\rho{}^2\epsilon(s_0+s_0{}^2/m_\rho{}^2)f(s_0)}{(s_0-m_\rho{}^2)^2+\epsilon^2}.$$

Let $s_0 = y+m_\rho{}^2$ and then $y = \epsilon\tan\theta$:

$$\int_{-\infty}^{\infty} ds_0 A_\rho^R(s_0) f(s_0) = \lim_{\epsilon\to 0} f_\rho{}^2\int_{-\infty}^{\infty}\frac{d(\frac{y}{\epsilon})}{(\frac{y}{\epsilon})^2+1}(2+\frac{y}{m_\rho{}^2})(y+m_\rho{}^2)f(y+m_\rho{}^2)$$

$$= \lim_{\epsilon\to 0} f_\rho{}^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\ (2+\frac{\epsilon}{m_\rho{}^2}\tan\theta)(\epsilon\tan\theta+m_\rho{}^2)f(\epsilon\tan\theta+m_\rho{}^2)$$

$$= f_\rho{}^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\ (2m_\rho{}^2)f(m_\rho{}^2)$$

$$= f_\rho{}^2(2m_\rho{}^2)\pi f(m_\rho{}^2).$$

Thus, $A_\rho^R(s_0) = 2\pi m_\rho{}^2 f_\rho{}^2\delta(s_0-m_\rho{}^2) \equiv m_\rho{}^2 g_\rho{}^2\delta(s_0-m_\rho{}^2)$. The integrations over s_0 in the calculations of $\langle n\rangle$ and the correlation functions are thus trivial:

$$\int ds_0 \frac{A_\rho^R(s_0)}{s_0x+\mu^2(1-x)^2} = \frac{m_\rho{}^2 g_\rho{}^2}{m_\rho{}^2x+\mu^2(1-x)^2}$$

and the integrations over x are again done by numerical analysis, using the IBM 360 computer:

$$F_{\rho}^{(k)} = \frac{m_{\rho}^{2} g_{\rho}^{2}}{16\pi^{3}} \int_{0}^{1} dx \, \frac{x^{\alpha}(1-x)(\ln x)^{k}}{m_{\rho}^{2}x+\mu^{2}(1-x)^{2}} \quad . \tag{D.4}$$

The results yield negative two body correlation function regardless of the value of m_{ρ}. The ρ-dominance modification of the simple multiperipheral model does not provide the desired result of a positive f_2.

APPENDIX E. A POSSIBLE ARGUMENT FOR ADOPTING THE SCALAR MULTIPERIPHERAL MODEL TO PROTON-PROTON SCATTERING.

Ref. 28 provides a possible justification for applying the AFS model, or our cluster version of it, which consists exclusively of scalar particles, to proton-proton scattering. If we define $A^R_{\pi N}$ to be the resonant amplitude for a pion and a proton to interact forming another pion and proton, which again interact to yield a pion-proton pair with the same momenta as the incident pair, then the elastic diagram of Fig. 21 becomes:

$$A^{pp}(pp',m_1\ldots m_N) = \int\ldots\int \frac{d^4q\,d^4q_1\ldots d^4q_{N-1}d^4q'}{(8\pi^4)^{N+1}}$$
$$\times\ \frac{A^R_{\pi N}((p+q)^2)A^R_{m_1}((q-q_1)^2)\ldots A^R_{m_N}((q_{N-1}-q')^2)A^R_{\pi N}((q'-p')^2)}{(q^2-\mu^2)^2(q_1{}^2-\mu^2)^2\ldots(q_{N-1}{}^2-\mu^2)^2(q'^2-\mu^2)^2} \quad (E.1)$$

We define the subsums:

$$A^{pp(s)}(pp',n_1n_2\ldots) = \sum_{\substack{\text{permutations}\\ \text{of } m_1\ldots m_N\\ (\text{fixed } n_1n_2\ldots)}} A^{pp}(pp',m_1\ldots m_N). \quad (E.2)$$

We cannot write a recursion formula for $A^{pp(s)}$, as we are assuming that the protons occur only in the end links of the chain. Yet we can write:

$$A^{pp}(pp',m_1\ldots m_N) = \iint \frac{d^4q\ d^4q'}{(8\pi^4)^2}$$
$$\times\ \frac{A^R_{\pi p}((p+q)^2)\ A(qq',m_1\ldots m_N)\ A^R_{\pi p}((q'-p')^2)}{(q^2-\mu^2)^2(q'^2-\mu^2)^2}$$

which is a prescription for writing A^{pp} from the πp resonant amplitude and the $\pi\pi$ elastic diagram, $A^{\pi\pi}(pp',m_1,\ldots,m_N) \equiv A(pp',m_1\ldots m_N)$. Summing over permutations of $m_1\ldots m_N$, we have:

$$A^{pp(s)}(pp',n_1n_2\ldots) = \iint \frac{d^4q\ d^4q'}{(8\pi^4)^2}$$
$$\times\ \frac{A^R_{\pi p}((p+q)^2)\ A^s(qq',n_1n_2\ldots)\ A^R_{\pi p}((q'-p')^2)}{(q^2-\mu^2)^2(q'^2-\mu^2)^2}\ .$$

Finally, summing over $n_1n_2\ldots$ yields:

$$A^{pp}(pp') = A^R_{pp}(pp') + \iint \frac{d^4q d^4q'}{(8\pi^4)^2}\ \frac{A^R_{\pi p}((p+q)^2)A(qq')A^R_{\pi p}((q'-p')^2)}{(q^2-\mu^2)^2(q'^2-\mu^2)^2}$$

where $A(qq')$ is the total absorptive forward elastic amplitude for $\pi\pi$ scattering, and we have found that this is simply:

$A(pp') \equiv \iint A(s,u)\ \delta[s-(p+p')^2]\ \delta(u+p'^2)\ ds\ du$, and

$A(s,-\mu^2) = s^{\alpha}\ \varphi_{\alpha}(-\mu^2)$.

Now we can also write, for πp scattering (Figure 22):

$$A^{\pi p}(pp',m_1\ldots m_N) = \int \frac{d^4q}{8\pi^4}\ \frac{A^R_{\pi p}((p+q)^2)\ A(qp',m_1\ldots m_N)}{(q^2-\mu^2)^2} \qquad (\mathrm{B}.3)$$

and after summing over permutations of $m_1 \ldots m_N$, and then over $n_1 n_2 \ldots$,

$$A^{\pi p}(pp') = A^R_{\pi p}(pp') + \int \frac{d^4q}{8\pi^4} \frac{A^R_{\pi p}((p+q)^2)\ A(qp')}{(q^2-\mu^2)^2} \quad .$$

If we follow the AFS procedure as used for A(pp'), and apply it to $A^{\pi p}$ and A^{pp}, we obtain:

$$A^{\pi p}(su) = \frac{1}{s}\int ds_o \frac{A^R_{\pi p}(s_o)}{16\pi^3}\int_o^s ds' \int_{u'_{min}}^{\infty} du' \frac{A(s'u')}{(u'+\mu^2)^2} \quad , \tag{E.4}$$

$$A^{pp}(su) = \frac{1}{s}\int ds_o \frac{A^R_{\pi p}(s_o)}{16\pi^3}\int_o^s ds' \int_{u'_{min}}^{\infty} du' \frac{A^{\pi p}(s'u')}{(u'+\mu^2)^2} \quad . \tag{E.5}$$

where $u'_{min} = x(u'+s_o/(1-x))$, and $x = s'/s$. Making use of the fact that $A(s'u') = s^{\alpha}\varphi_{\alpha}(u')$, we find:

$$A^{\pi p}(su) = s^{\alpha}\left[\int \frac{ds_o}{16\pi^3} A^R_{\pi p}(s_o)\int_o^1 dx \int_{u'_{min}}^{\infty} du' \frac{x^{\alpha}\varphi_{\alpha}(u')}{(u'+\mu^2)^2}\right] . \tag{E.6}$$

$$A^{pp}(su) = s^{\alpha}\left[\int \frac{ds_o}{16\pi^3} A^R_{\pi p}(s_o)\int_o^1 dx \int_{u'_{min}}^{\infty} du' \frac{x^{\alpha}\varphi_{\alpha}^{\pi p}(u')}{(u'+\mu^2)^2}\right], \tag{E.7}$$

so that

$$A^{\pi p}(su) = s^{\alpha}\varphi_{\alpha}^{\pi p}(u) \quad \text{and} \quad A^{pp}(su) = s^{\alpha}\varphi_{\alpha}^{pp}(u),$$

and where the first of these results was used in (E.7) to obtain the second result. The pp elastic amplitude has the same energy dependence in this model as the $\pi\pi$ amplitude; note that the exponent α is the same exponent as in the $\pi\pi$ case, since $A(su) = s^{\alpha}\, \varphi_{\alpha}(u)$ was used to obtain $A^{pp}(su) = s^{\alpha}\, \varphi_{\alpha}^{pp}(u)$. Our application is thus justified, if we believe the assumption that the protons are confined to the end links of the chain in the manner prescribed. Note that this assumption is consistent with the experimental result that most of the products of inelastic pp processes are pions.

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TABLE 1

Model	$\langle n \rangle$	f_2	f_n
Diffractive Fragmentation - Nova	$\ln(s)$	$\sqrt{s}$	$(\sqrt{s})^{n-1}$
Statistical Thermodynamic	$\ln(s)$	0	0
Chew-Pignotti Multiperipheral Bootstrap with Meson Trajectories Only	$\ln(s)$	0	0
Absorptive Multiperipheral	$\ln(s)$	$(\ln(s))^2$	$(\ln(s))^n$
Simple AFS Multiperipheral	$\ln(s)$	$-\ln(s)$	$\sim\ln(s)$
Multiperipheral Cluster Model	$\ln(s)$	$+\ln(s)$	$\sim\ln(s)$

TABLE 2

M	$g_m^{\,3}$	$\frac{\langle n\rangle}{\ln s}$	$\frac{f_2}{\ln s}$	$\frac{f_3}{\ln s}$
1	$g_1^{\,2} = 240\pi^2$	1.23	-0.62	0.59
2	$g_1^{\,2} = 60\pi^2 \quad g_2^{\,2} = 123$	1.44	-0.15	-0.34
3	$g_1^{\,2} = g_2^{\,2} = 0 \quad g_3^{\,3} = 0.20$	2.73	1.21	-0.40
3	$g_1^{\,2} = 40\pi^2 \quad g_2^{\,2} = 61.9$ $g_3^{\,2} = 0.093$	2.09	0.524	11.6
4	$g_1^{\,2} = g_2^{\,2} = g_3^{\,2} = 0 \quad g_4^{\,2}=7.7\times10^{-5}$	3.37	3.08	0.414
5	$g_m^{\,2}=0$ for $m<5 \quad g_5^{\,2} = 2.2\times10^{-8}$	4.03	5.67	3.61
6	$g_m^{\,2}=0$ for $m<6 \quad g_6^{\,2} = 5.9\times10^{-12}$	4.70	8.99	---
7	$g_m^{\,2}=0$ for $m<7 \quad g_7^{\,2} = 1.44\times10^{-15}$	5.37	13.0	---
8	$g_m^{\,2}=0$ for $m<8 \quad g_8^{\,2} = 5\times10^{-21}$	5.98	18.8	---
8	$g_1^{\,2}=99.9 \quad g_2^{\,2}=92.7 \quad g_3^{\,2}=11.2\times10^{-4}$ $g_4^{\,2}=10^{-8} \quad g_5^{\,2}=9\times10^{-13}$ $g_6^{\,2}=9\times10^{-17} \quad g_7^{\,2}=10^{-21}$ $g_8^{\,2}=1.4\times10^{-21}$	2.98	7.81	24.4
8	**$g_2^{\,2}=100 \quad g_8^{\,2}=1.41\times10^{-21}$ others=0**	**2.98**	**7.92**	**25.2**
8	**$g_2^{\,2}=97.9 \quad g_8^{\,2}=1.47\times10^{-21}$ others=0**	**3.05**	**8.22**	**25.5**

TABLE 3

$\eta = \sum_m n_m m$	$A^s(n_1 n_2 \ldots n_8)$	Coefficient of $\frac{16\pi^3\mu^2}{s}$ in A^s
15	$A^s(00500000)$	$10^9(\ln s)^3$
16	$A^s(00000002)$	10^7
16	$A^s(16,0000000)$	$10^{-20}(\ln s)^{14}$
16	$A^s(08000000)$	$10^{23}(\ln s)^6$
16	$A^s(10000011)$	$10^3(\ln s)$
16	$A^s(04000001)$	$10^{16}(\ln s)^3$
16	$A^s(00040000)$	$10^3(\ln s)^2$
16	$A^s(06010000)$	$10^{19}(\ln s)^5$
18	$A^s(00600000)$	$10^{11}(\ln s)^4$
18	$A^s(09000000)$	$10^{25}(\ln s)^7$
20	$A^s(0,10,000000)$	$10^{28}(\ln s)^8$
20	$A^s(06000001)$	$10^{25}(\ln s)^5$

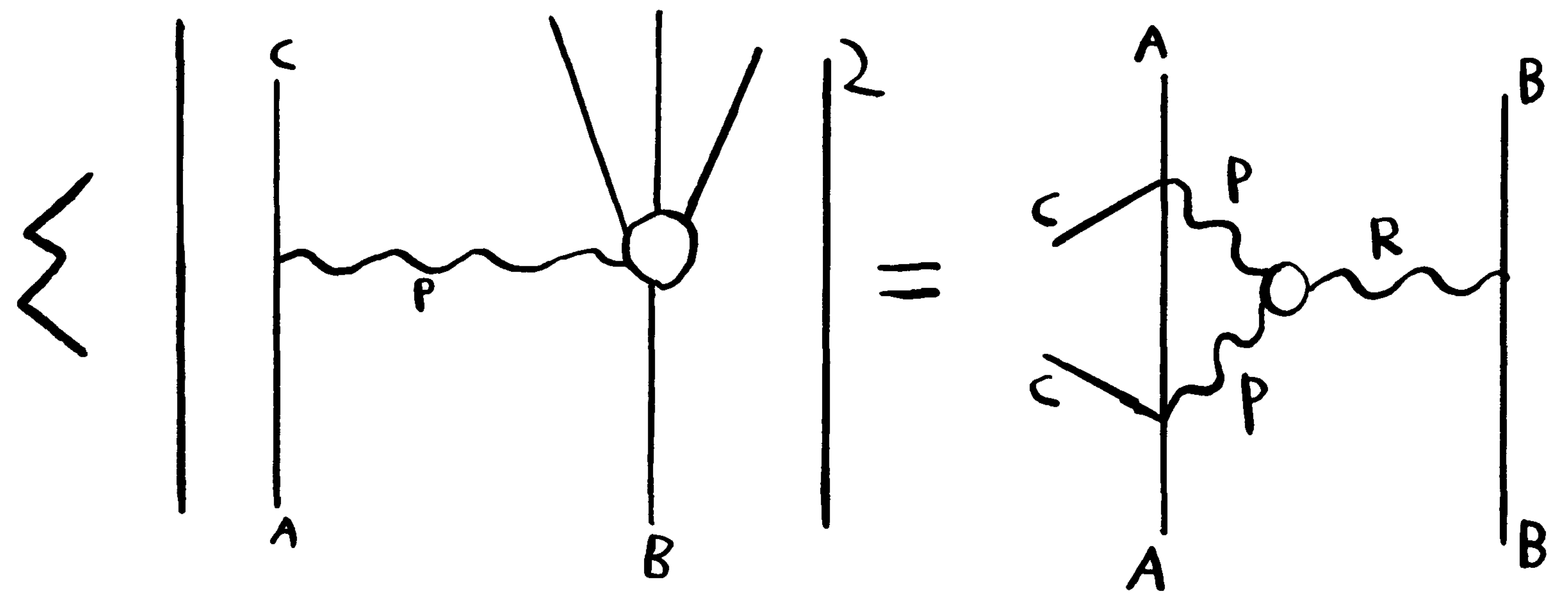


Figure 1

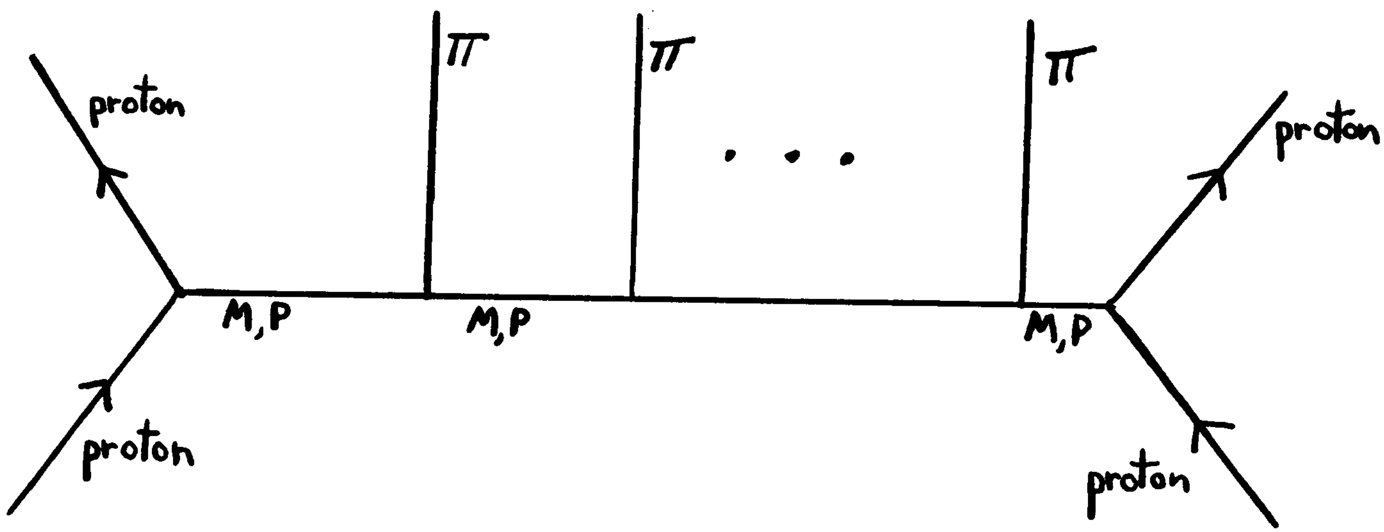


Figure 2

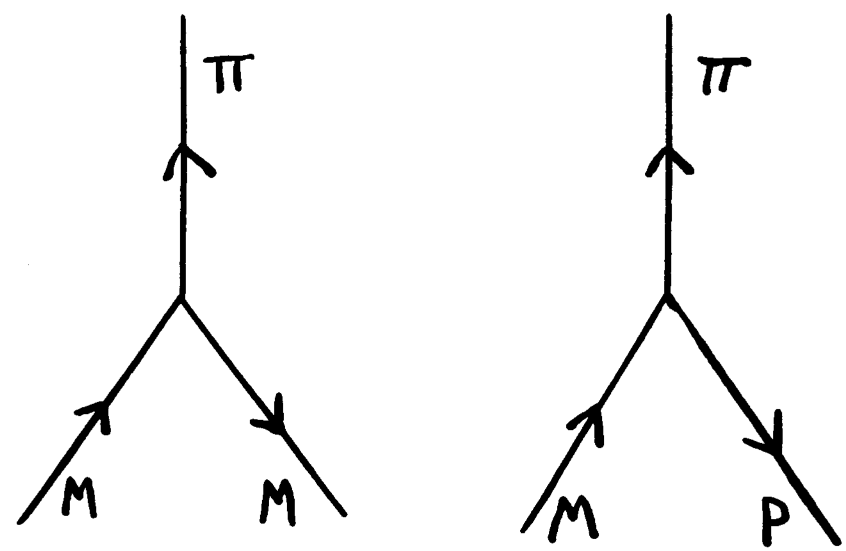


Figure 3

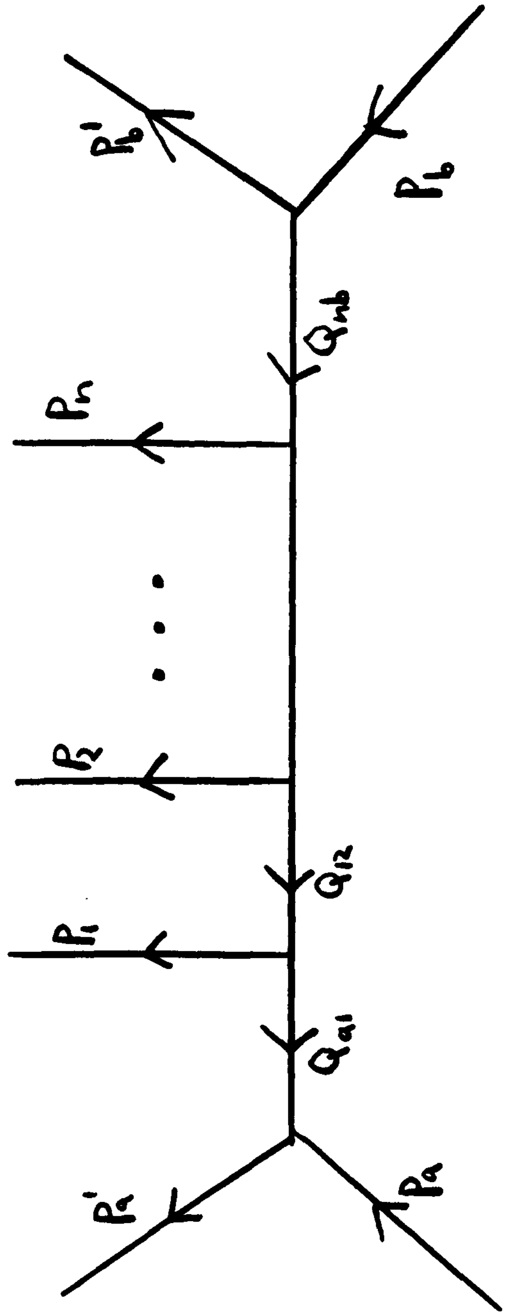


Figure 4

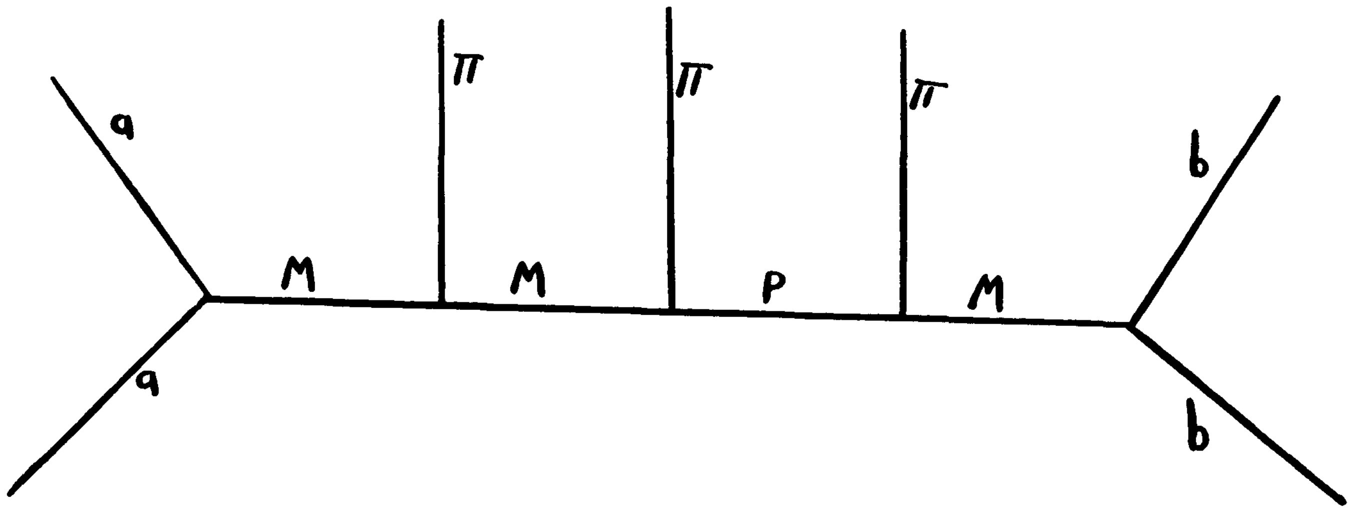


Figure 5

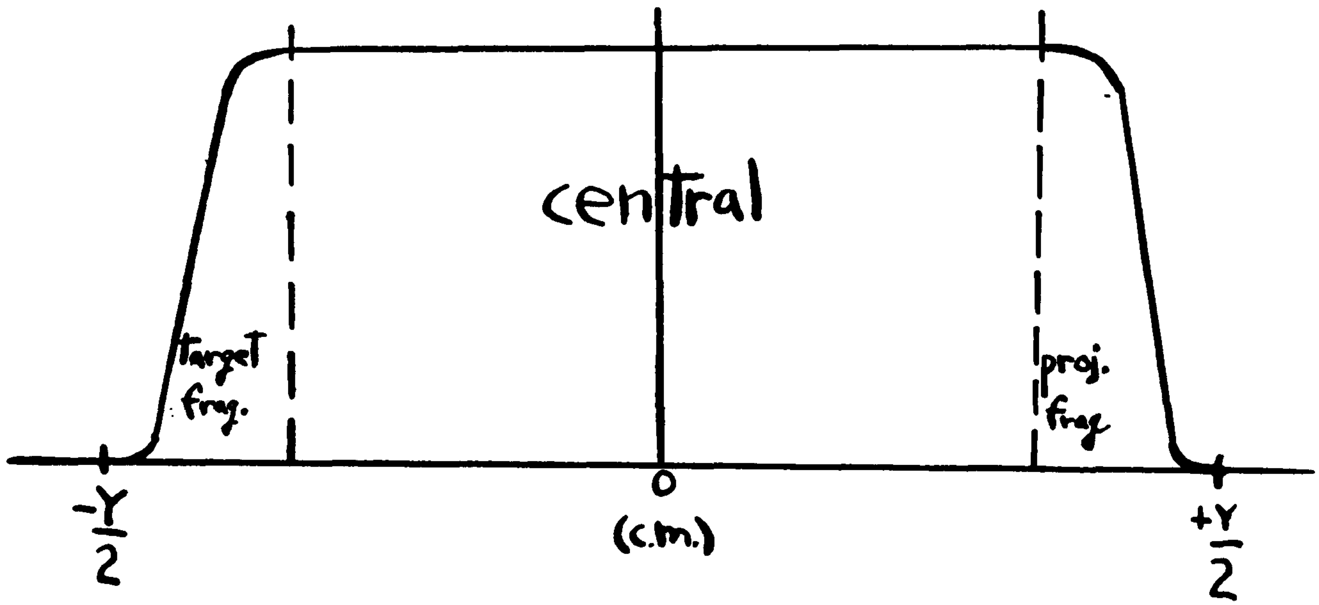


Figure 6

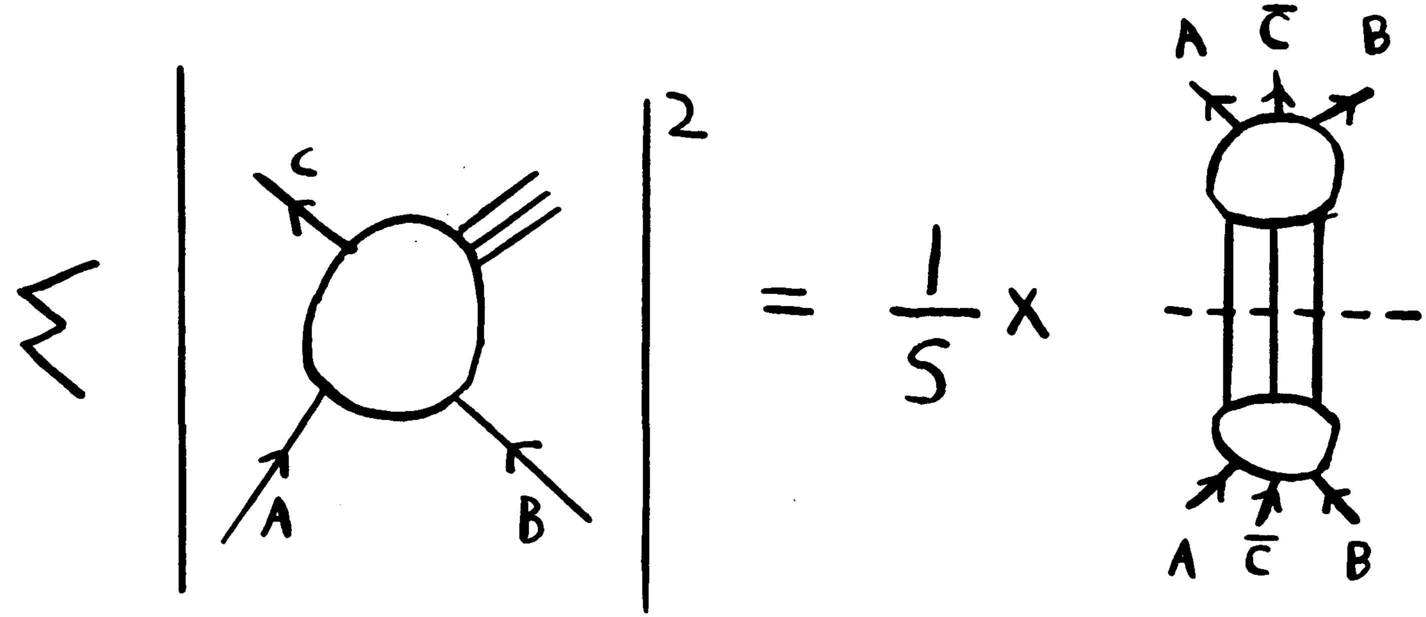


Figure 7

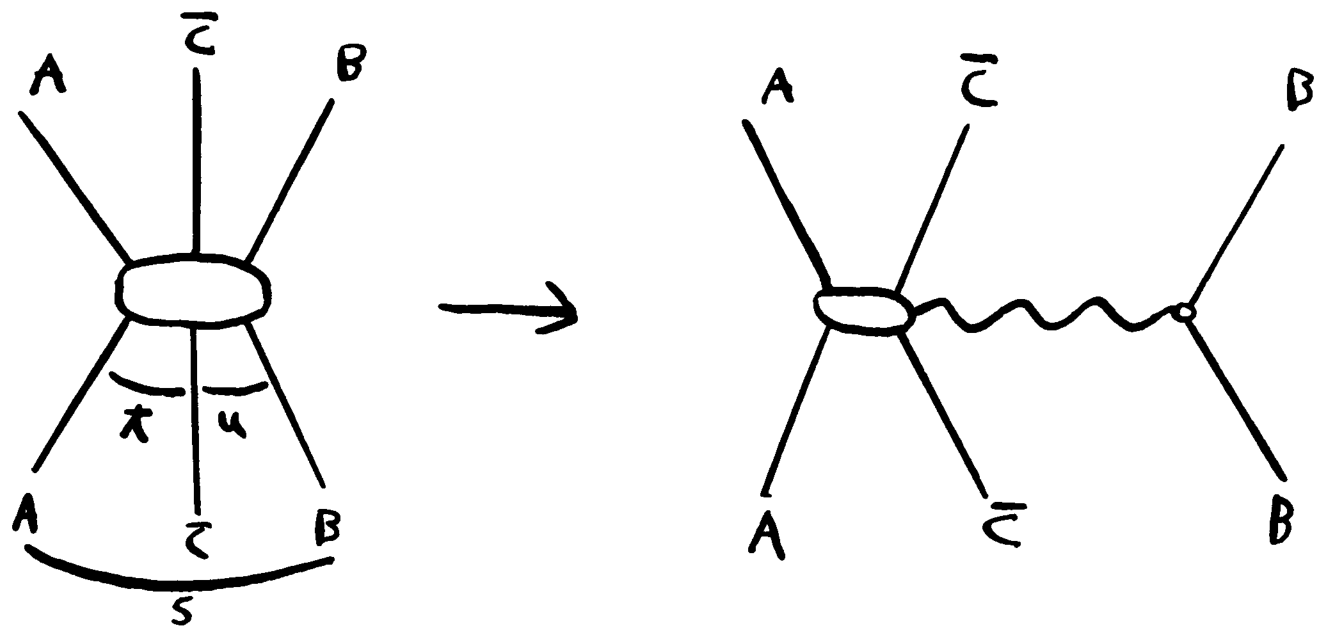


Figure 8

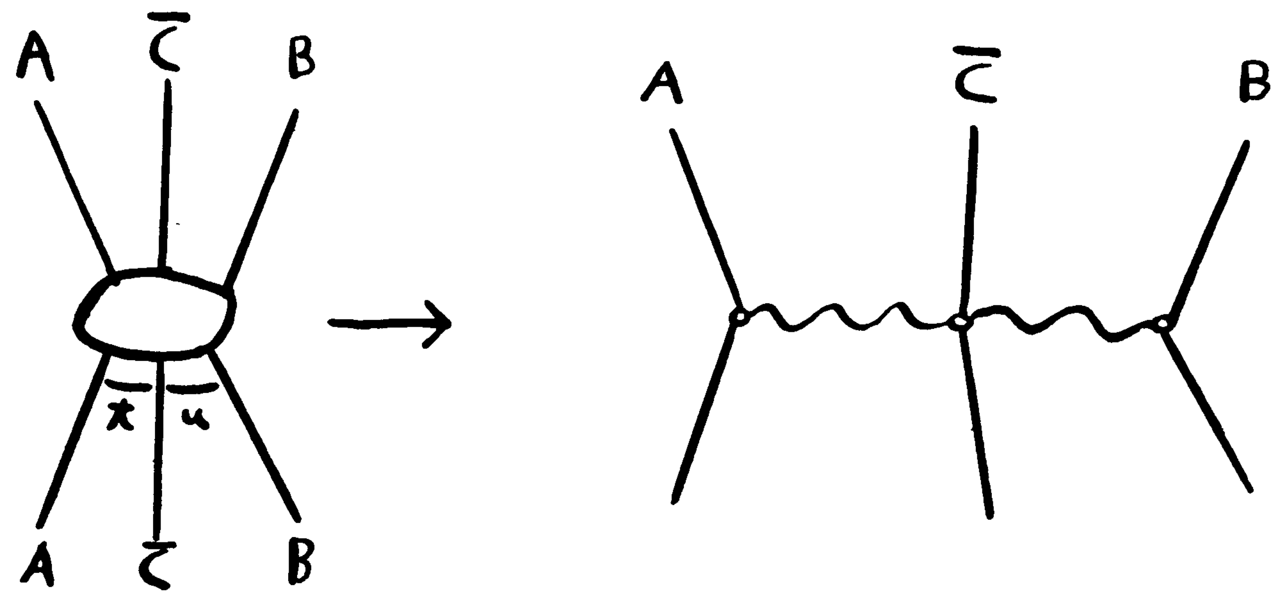


Figure 9

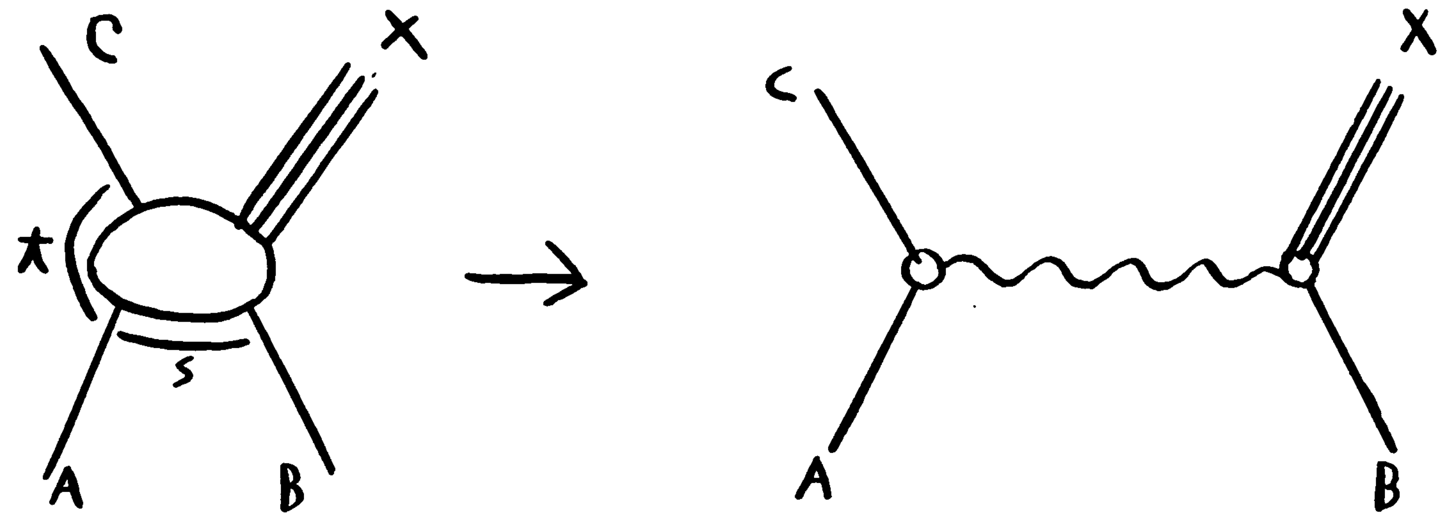


Figure 10

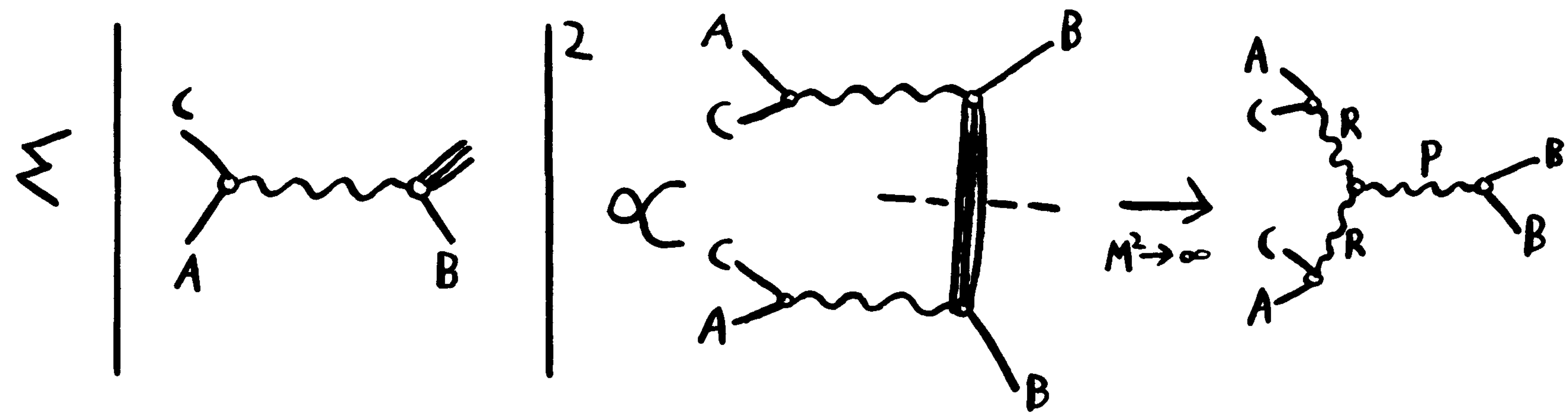


Figure 11

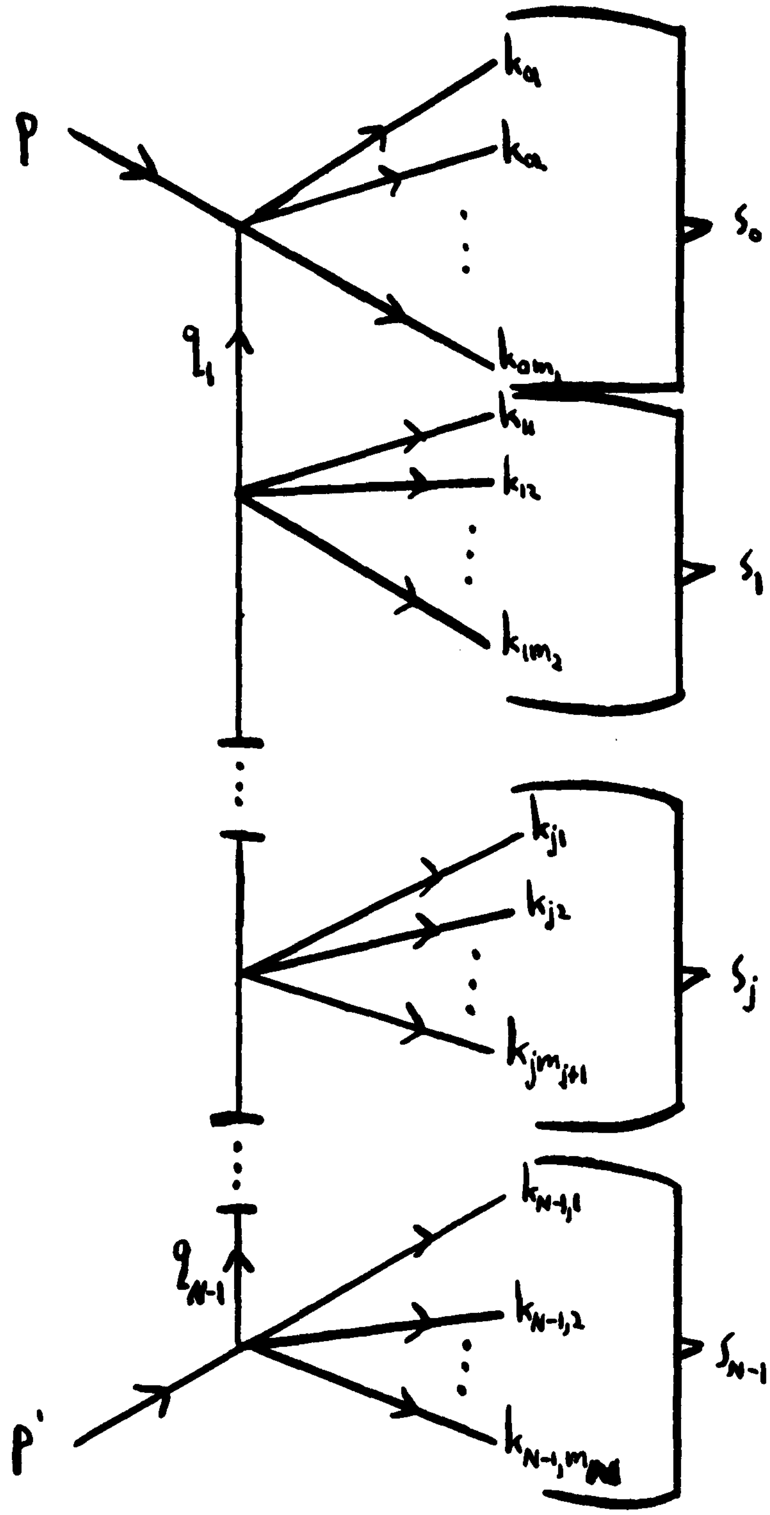


Figure 12 a)

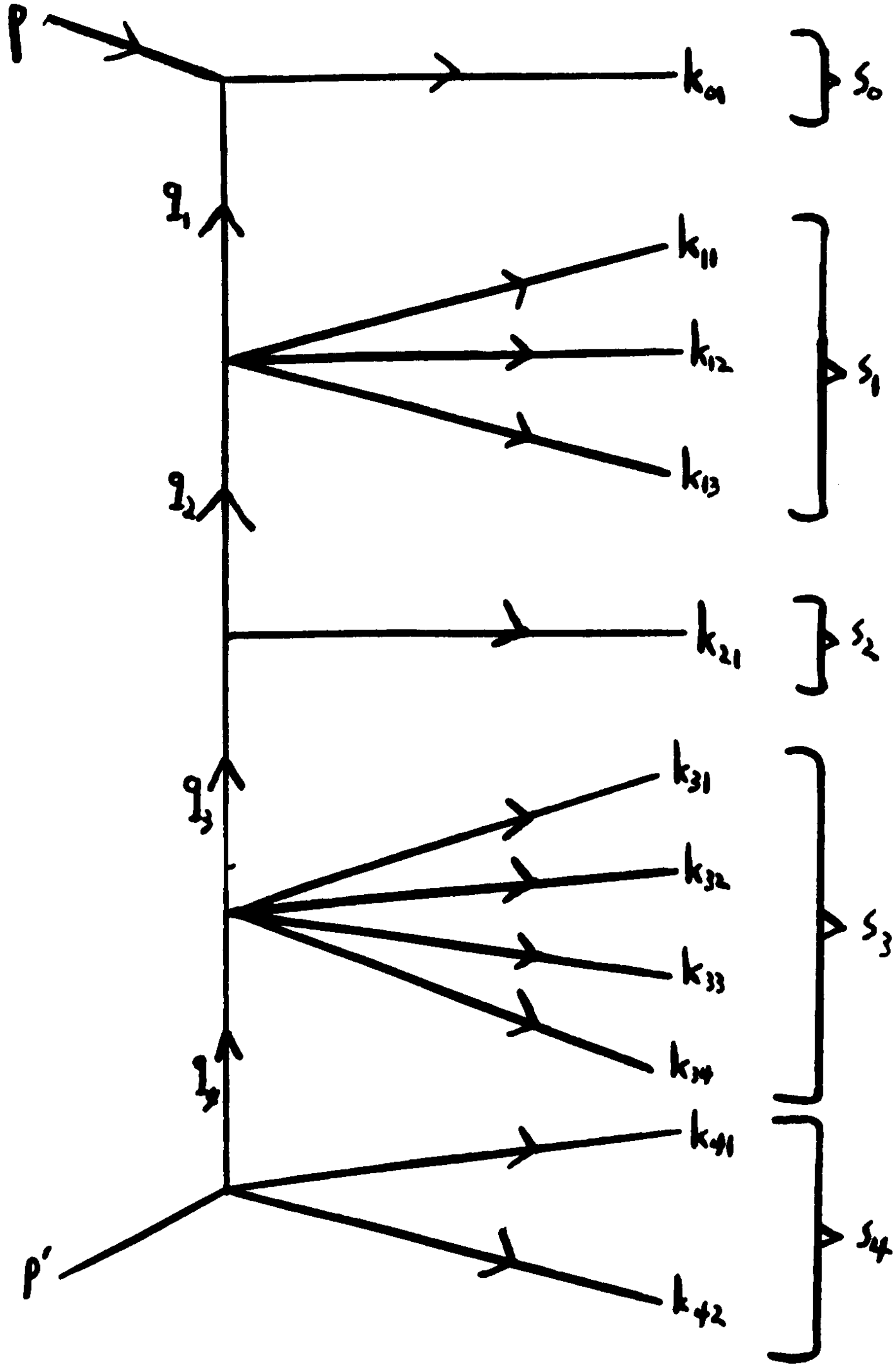
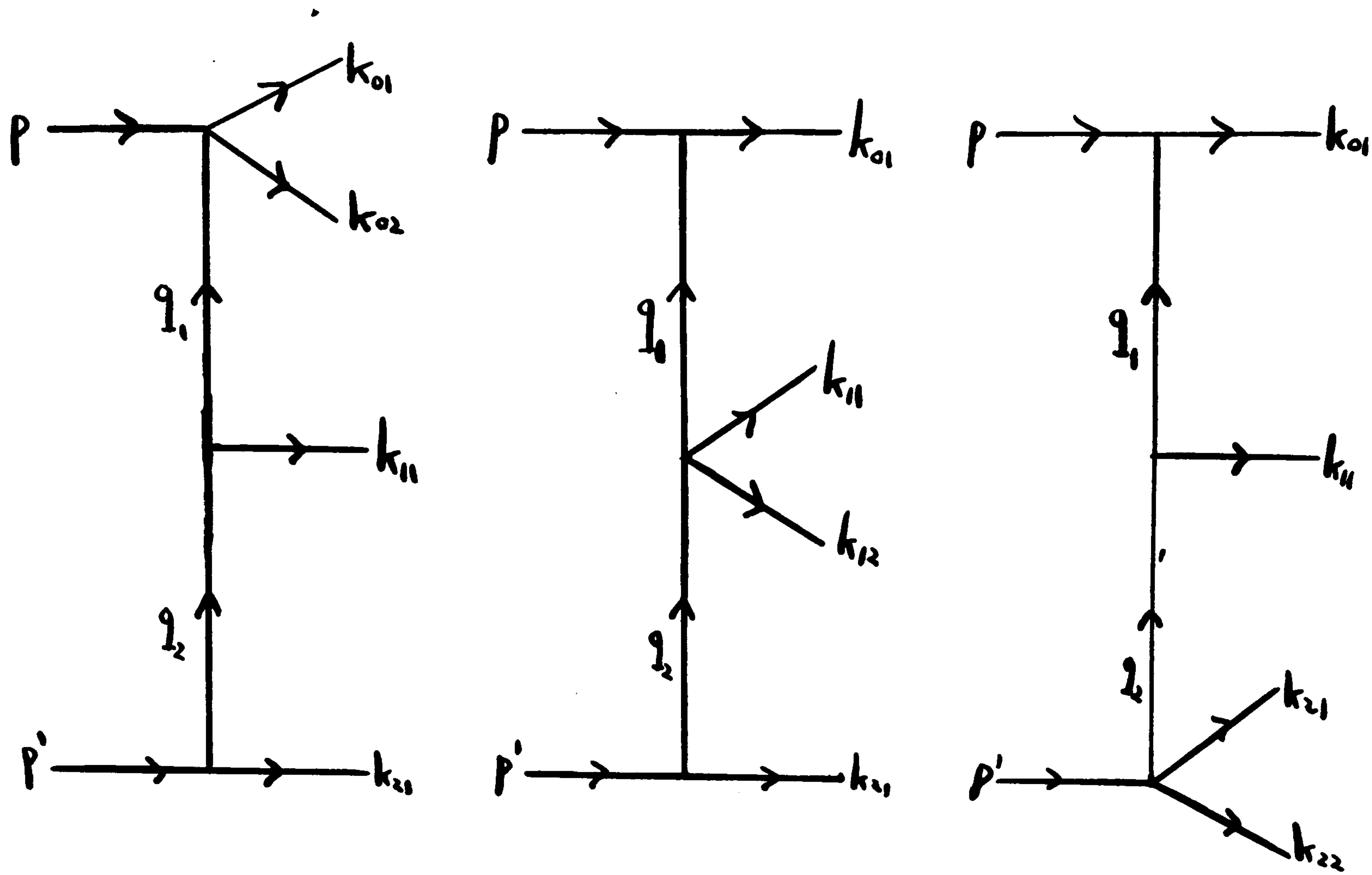

p
k_{01}
s_0
q_1
k_{11}
k_{12}
s_1
k_{13}
q_2
k_{21}
s_2
k_{31}
q_3
k_{32}
s_3
k_{33}
q_4
k_{34}
k_{41}
s_4
p'
k_{42}

Figure 12 b)



p
k_{01}
k_{02}
q_1
k_{11}
q_2
p'
k_{21}
p
k_{01}
q_1
k_{11}
k_{12}
q_2
p'
k_{21}
p
k_{01}
q_1
k_{11}
q_2
k_{21}
p'
k_{22}

Figure 12 c)

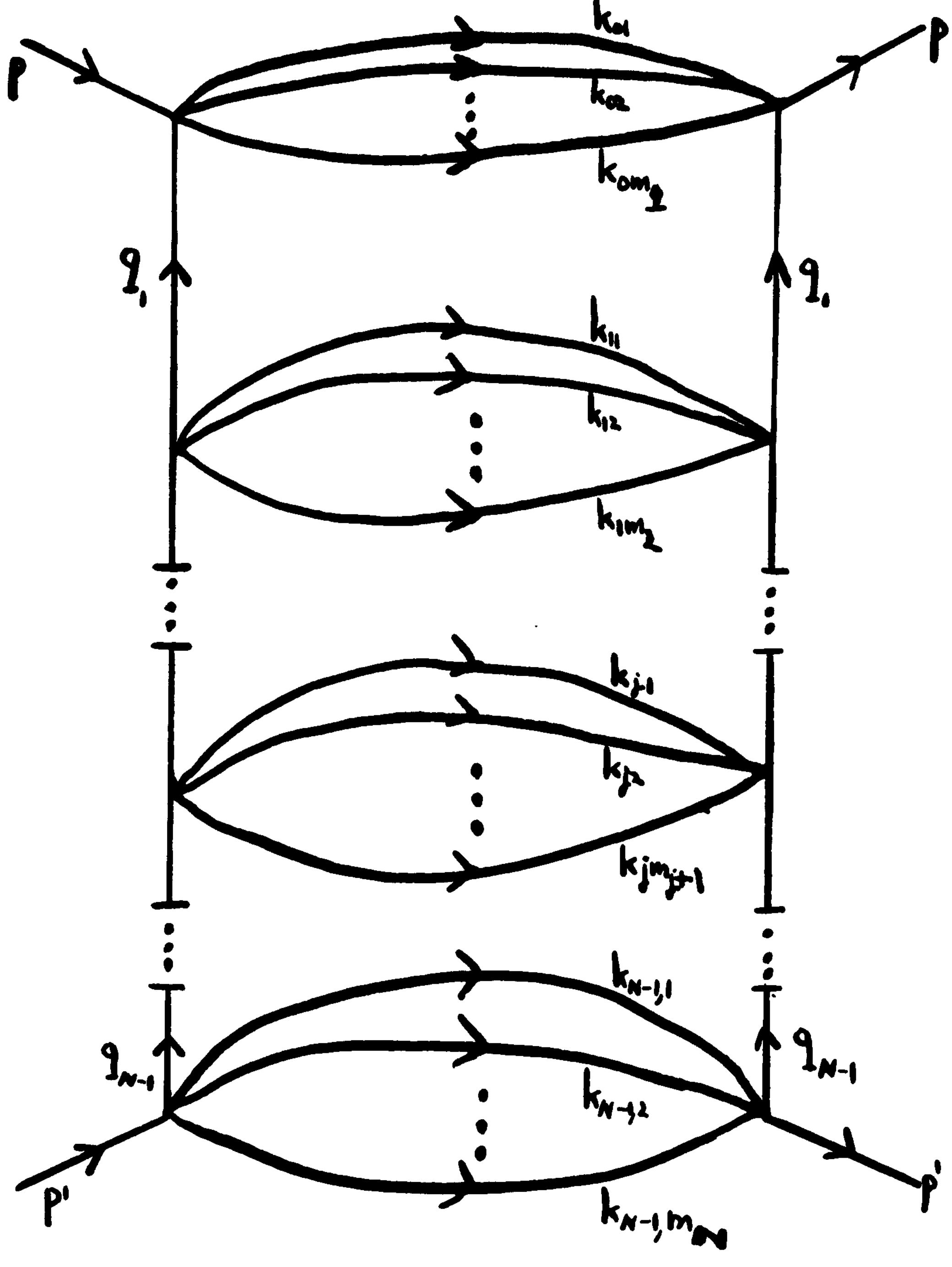

p
p
k_{01}
k_{02}
k_{0m_1}
q_1
q_1
k_{11}
k_{12}
k_{1m_2}
k_{j1}
k_{j2}
$k_{jm_{j+1}}$
$k_{N-1,1}$
q_{N-1}
q_{N-1}
$k_{N-1,2}$
p'
p'
k_{N-1,m_N}

Figure 13 a)

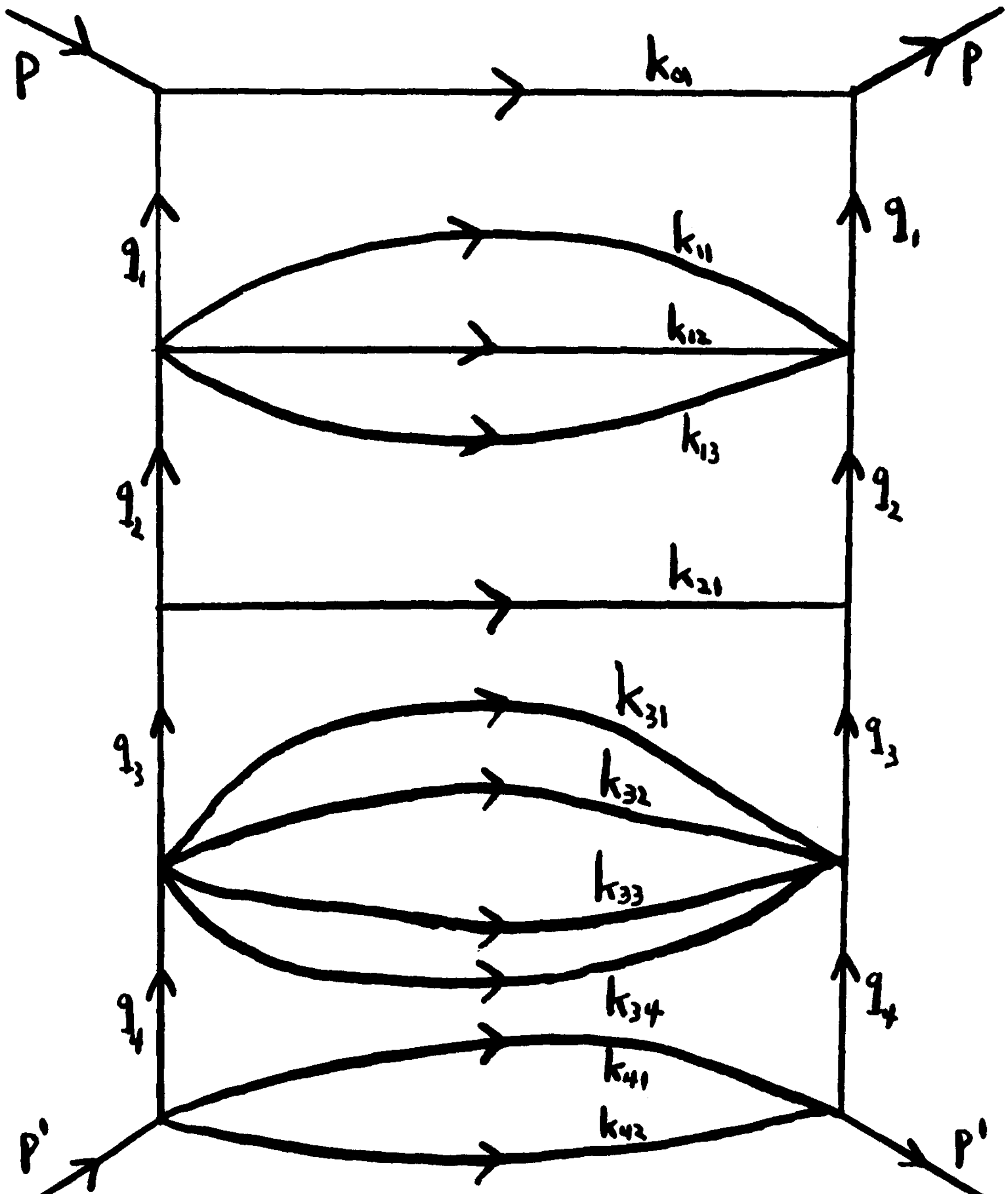
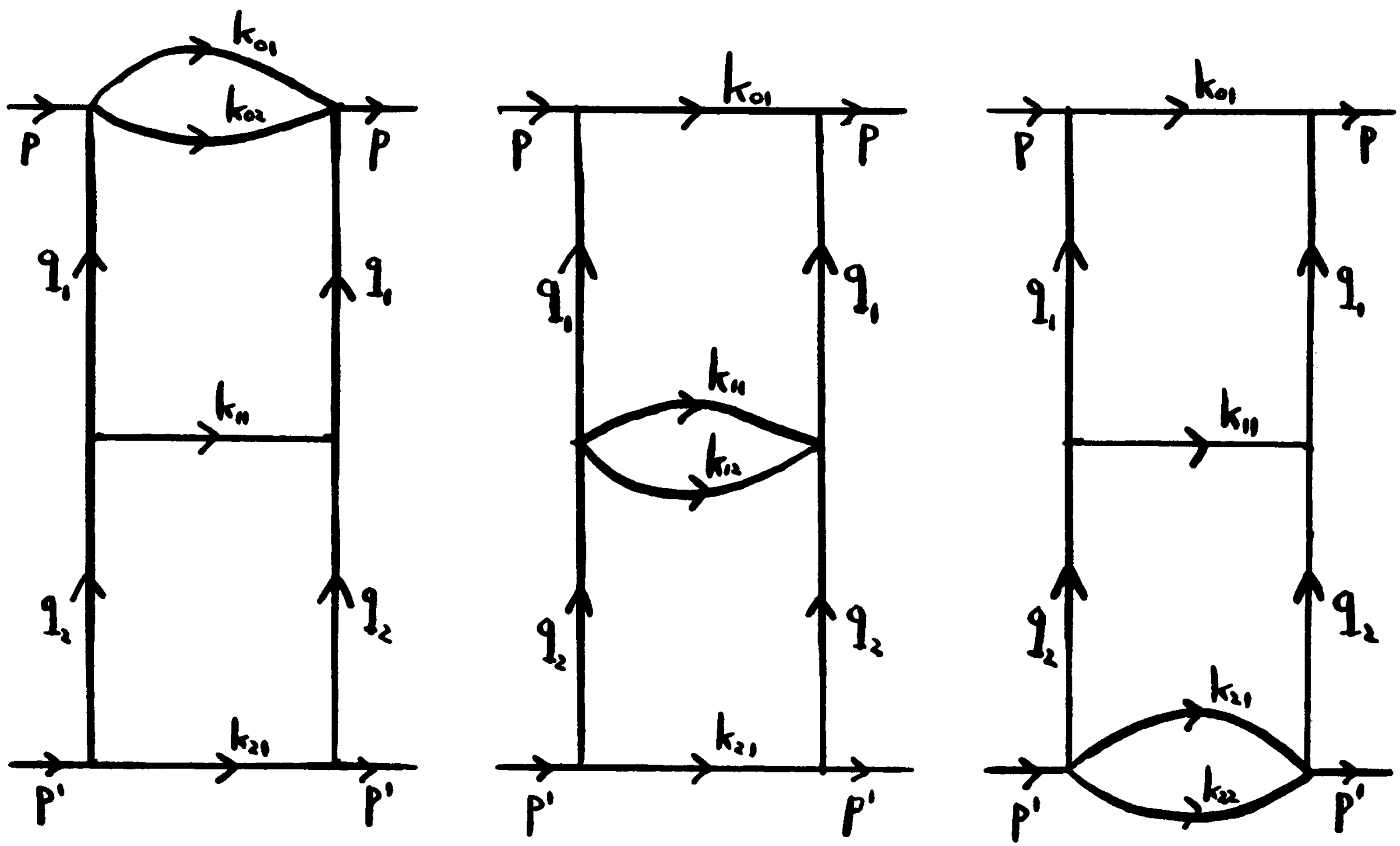

P
k_{01}
P
q_1
k_{11}
q_1
k_{12}
k_{13}
q_2
q_2
k_{21}
k_{31}
q_3
q_3
k_{32}
k_{33}
k_{34}
q_4
q_4
k_{41}
k_{42}
P'
P'

Figure 13 b)



k01
k02
p
p
q1
q1
k11
q2
q2
k21
p'
p'
k01
p
p
q1
q1
k11
k12
q2
q2
k21
p'
p'
k01
p
p
q1
q1
k11
q2
q2
k21
k22
p'
p'

Figure 13 c)

$A^R_m(s_0) = \mathrm{Im}$

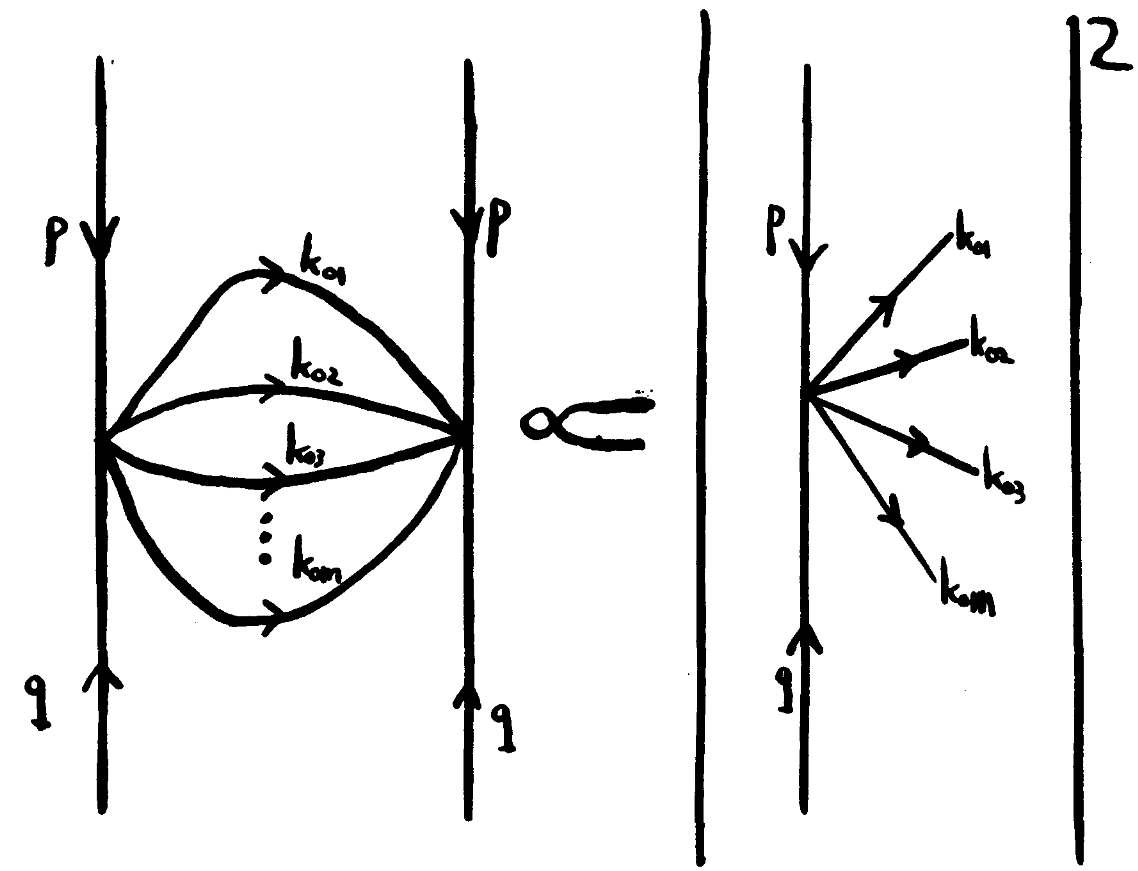


Figure 14

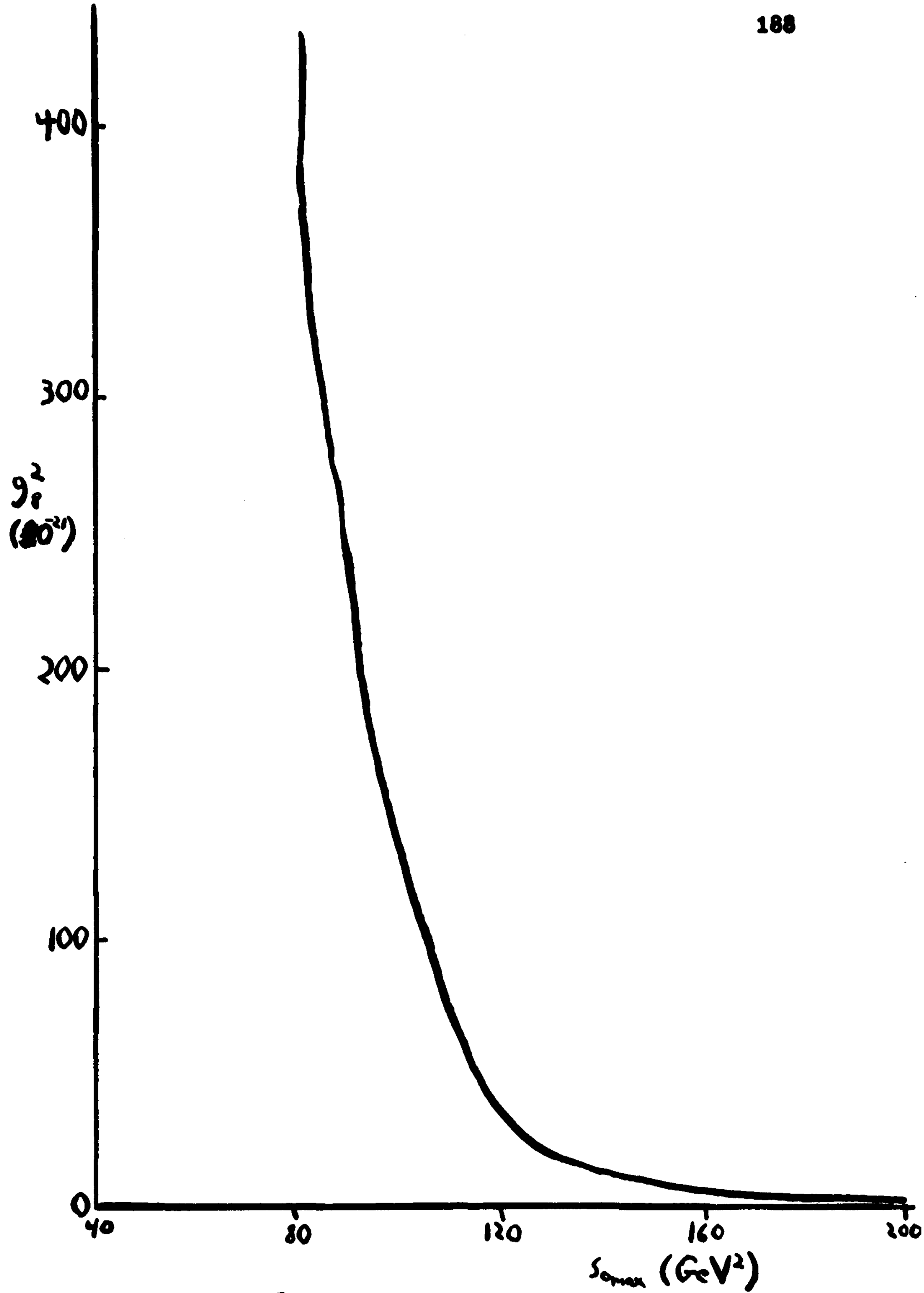
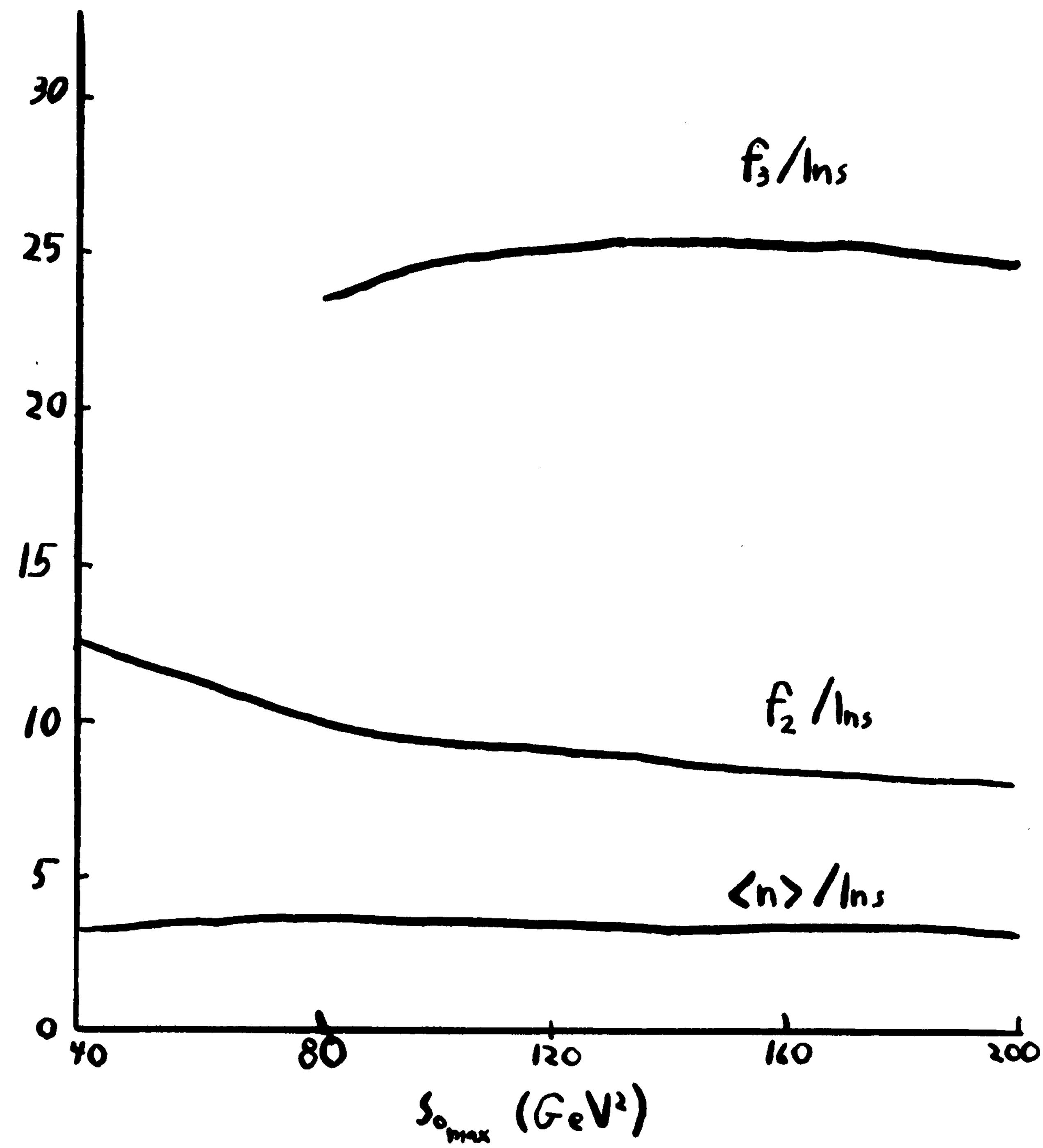


Figure 15a)



30
25
20
15
10
5
0
$f_3/\ln s$
$f_2/\ln s$
$\langle n\rangle/\ln s$
40
80
120
160
200
$s_{o_{max}}$ (GeV^2)

Figure 15 b)

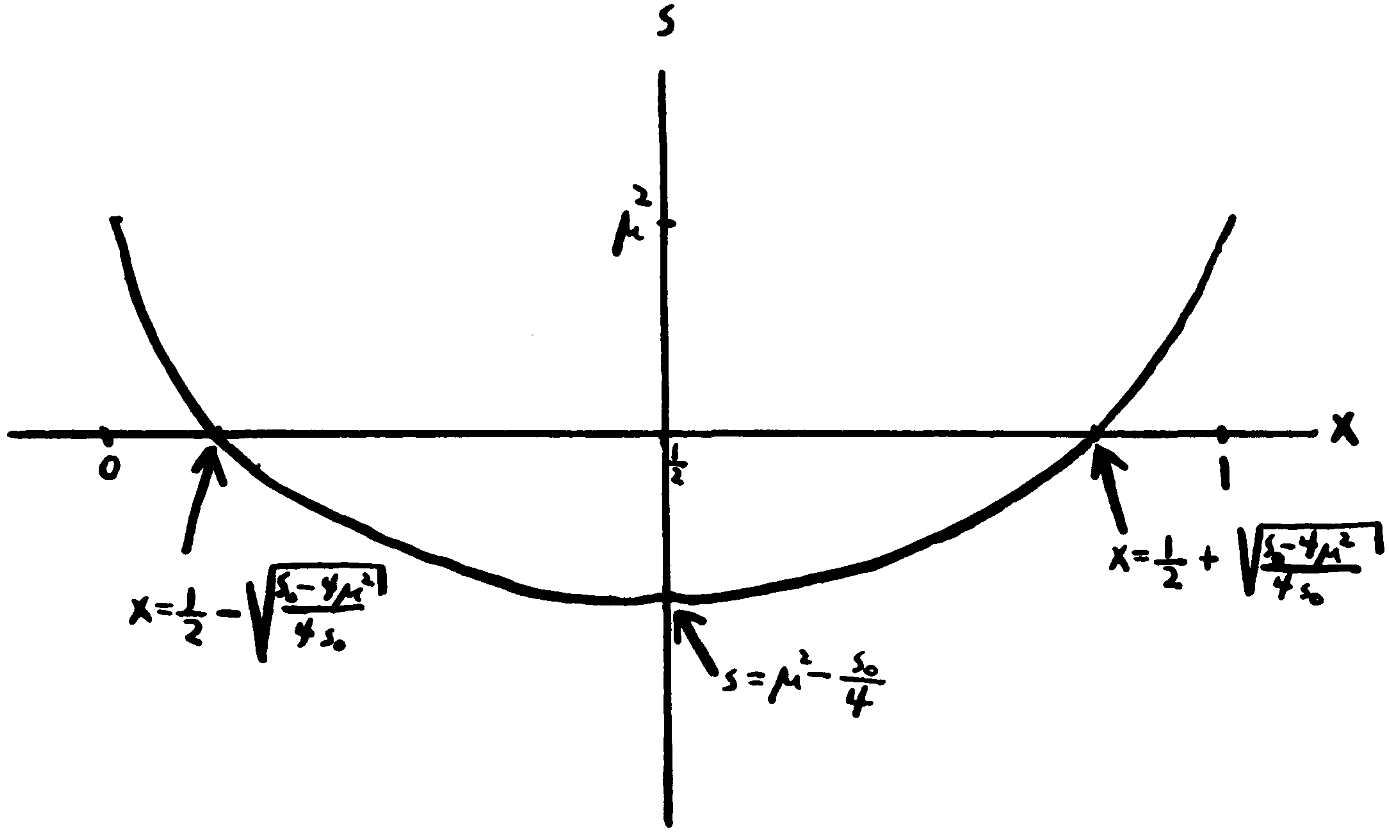

s
μ^2
x
0
$\frac{1}{2}$
1
$x=\frac{1}{2}-\sqrt{\frac{s_0-4\mu^2}{4s_0}}$
$s=\mu^2-\frac{s_0}{4}$
$x=\frac{1}{2}+\sqrt{\frac{s_0-4\mu^2}{4s_0}}$

Figure 16

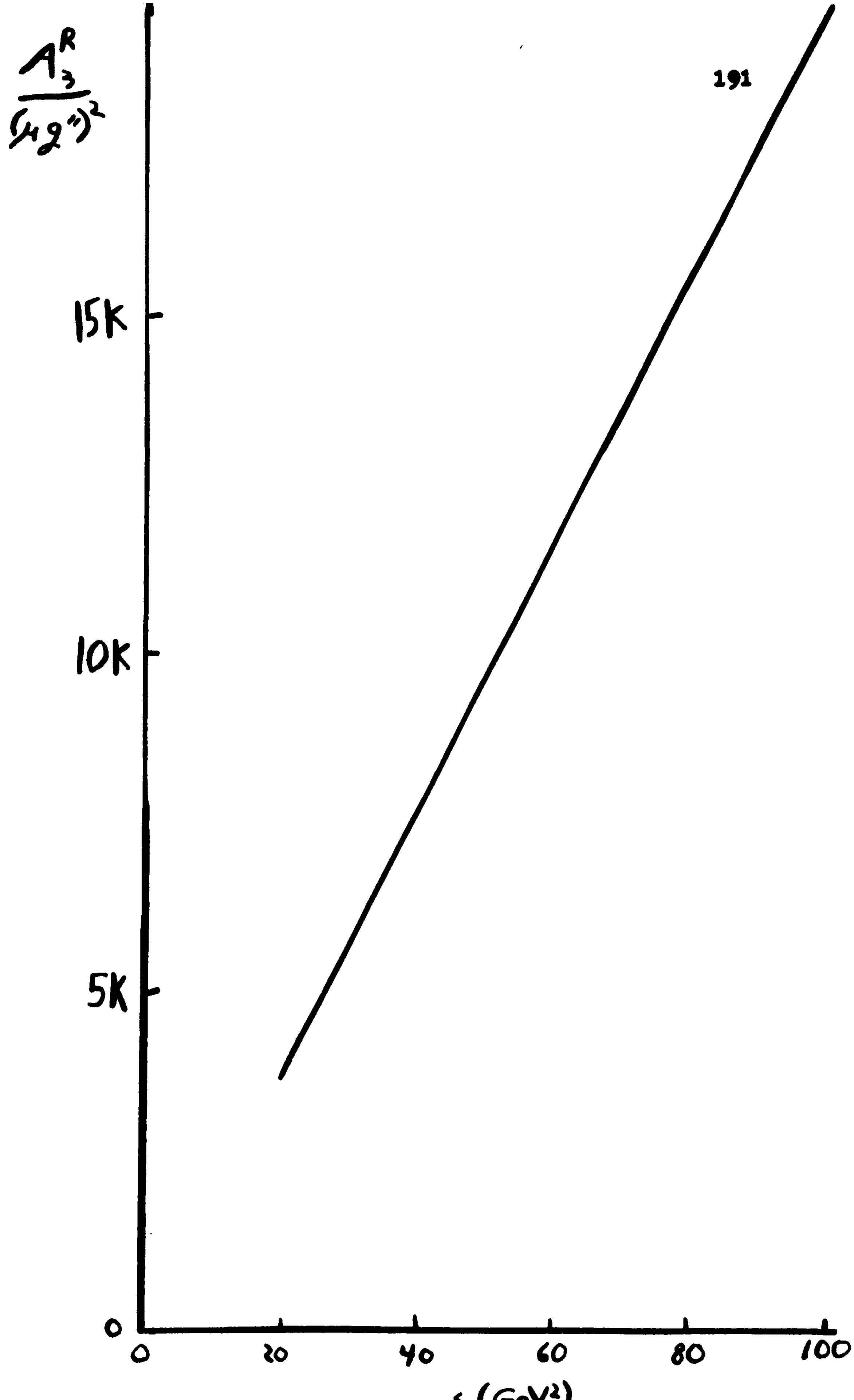


Figure (7a)

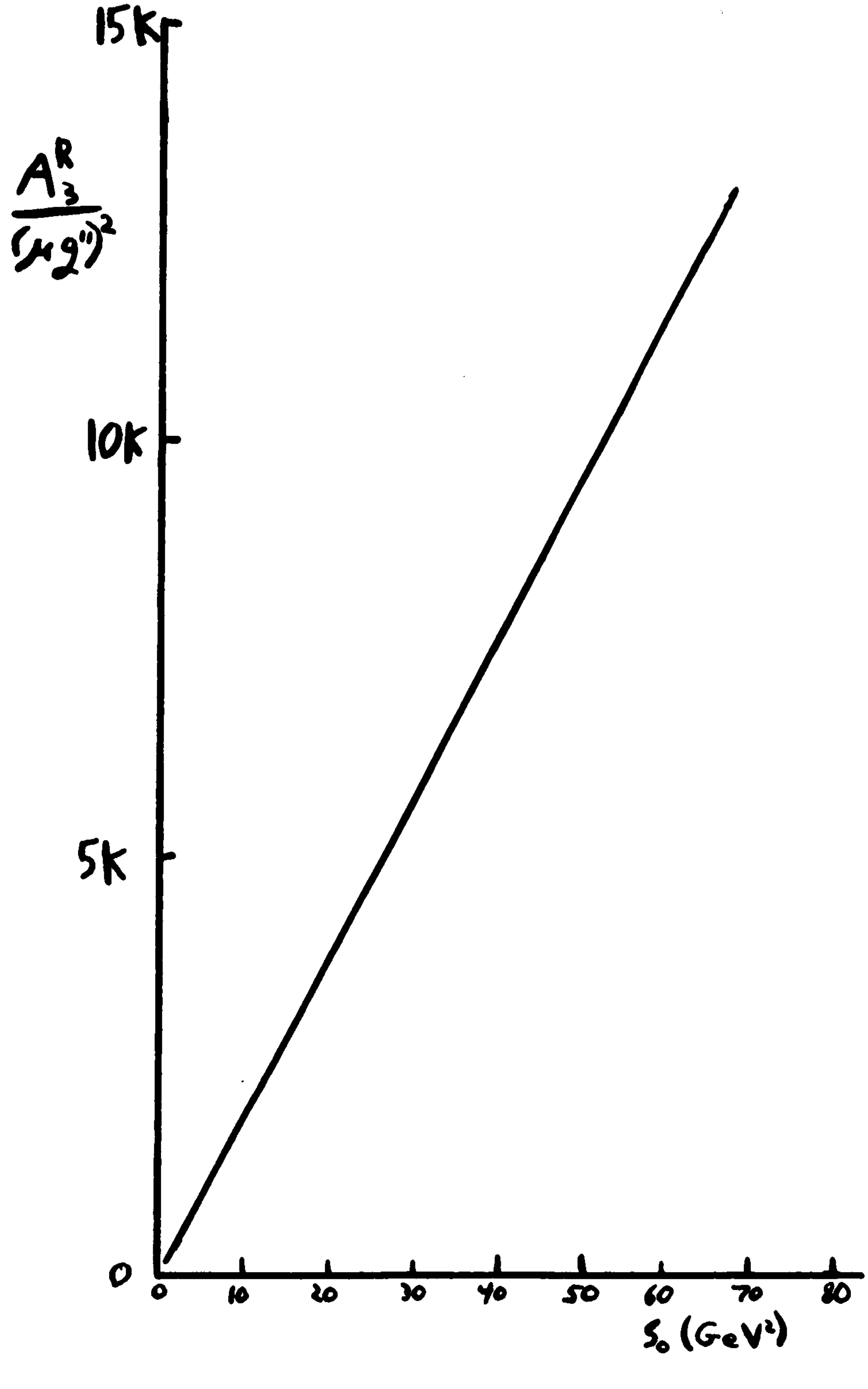


Figure 17b)

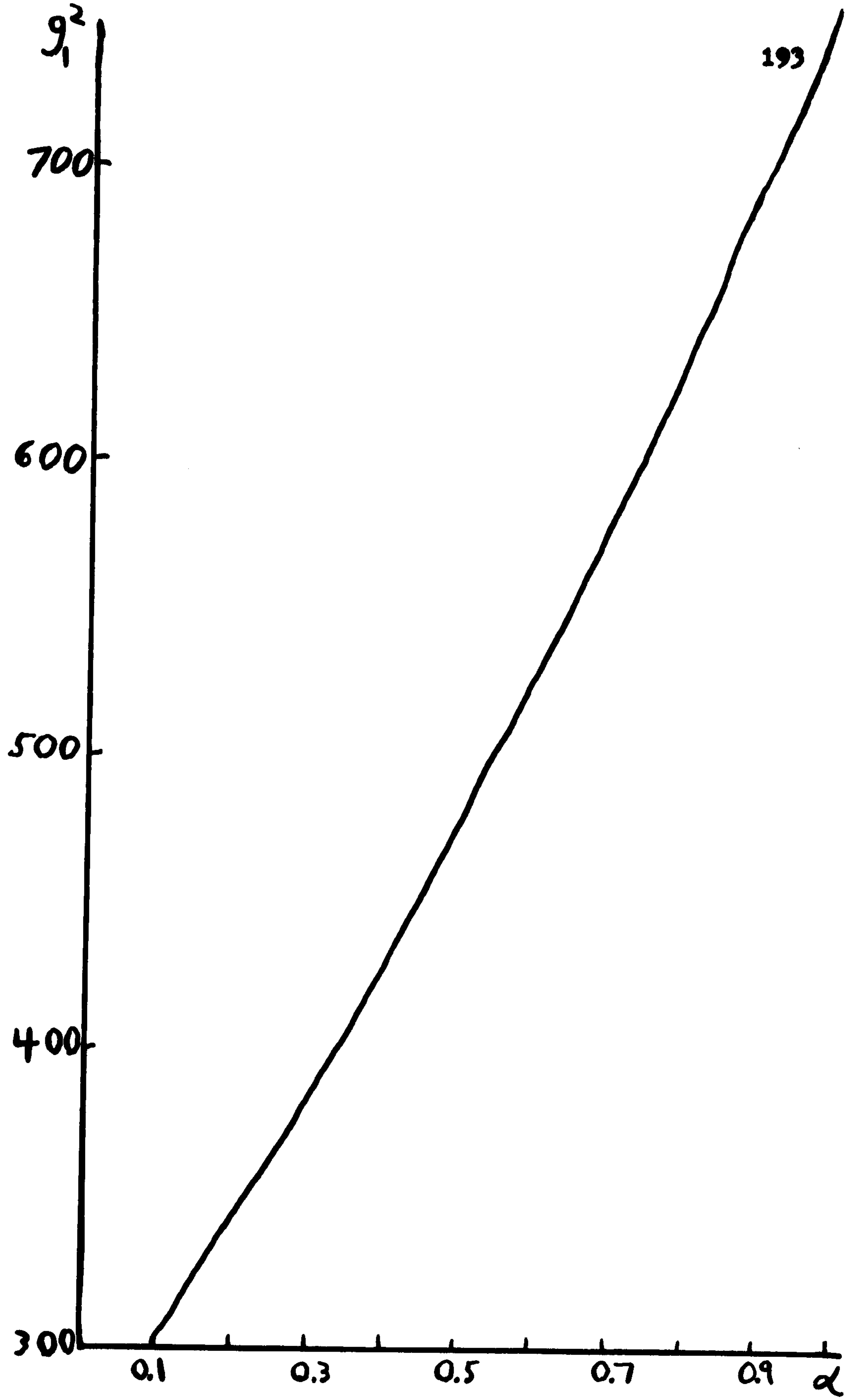


Figure 18a)

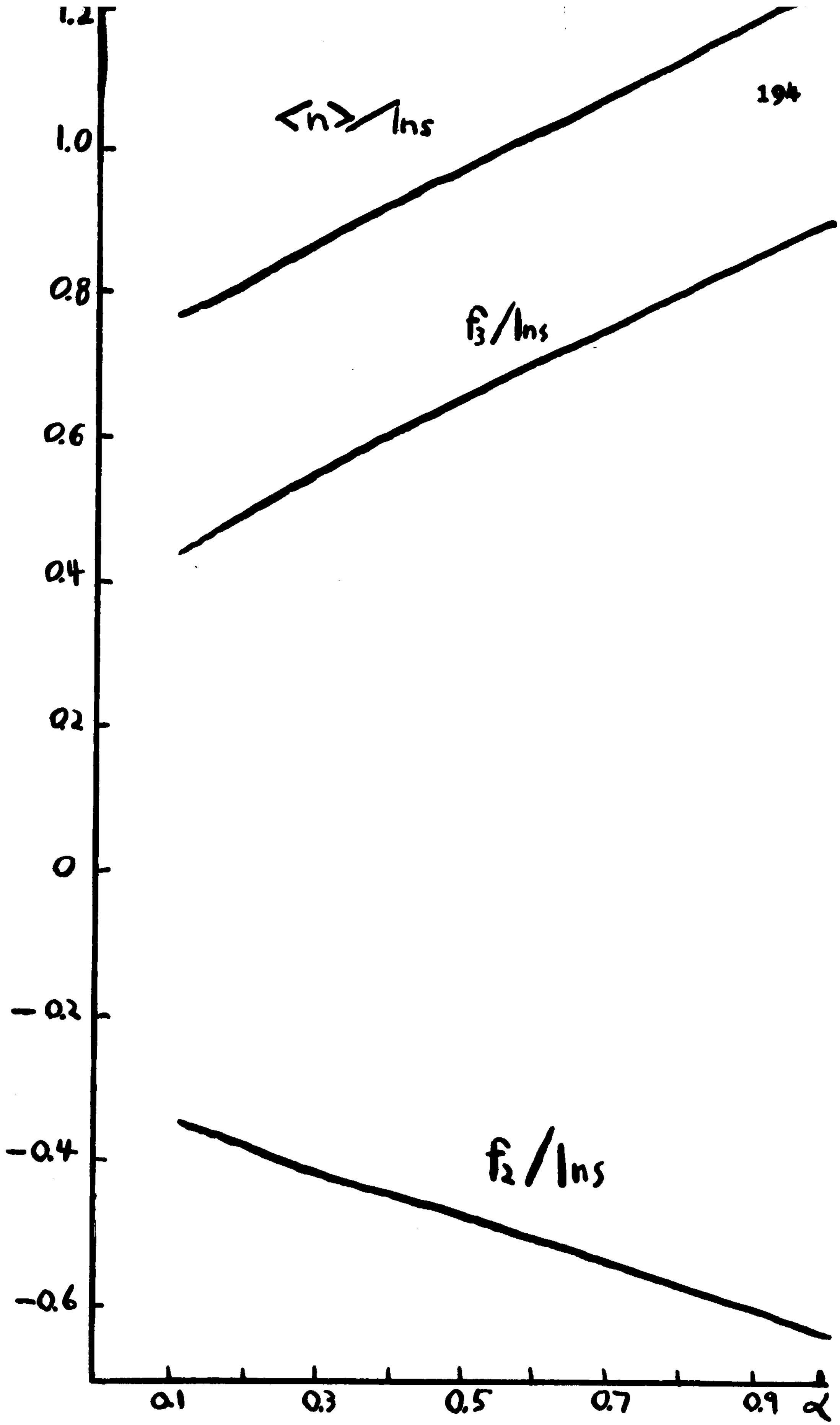


Figure 18 b)

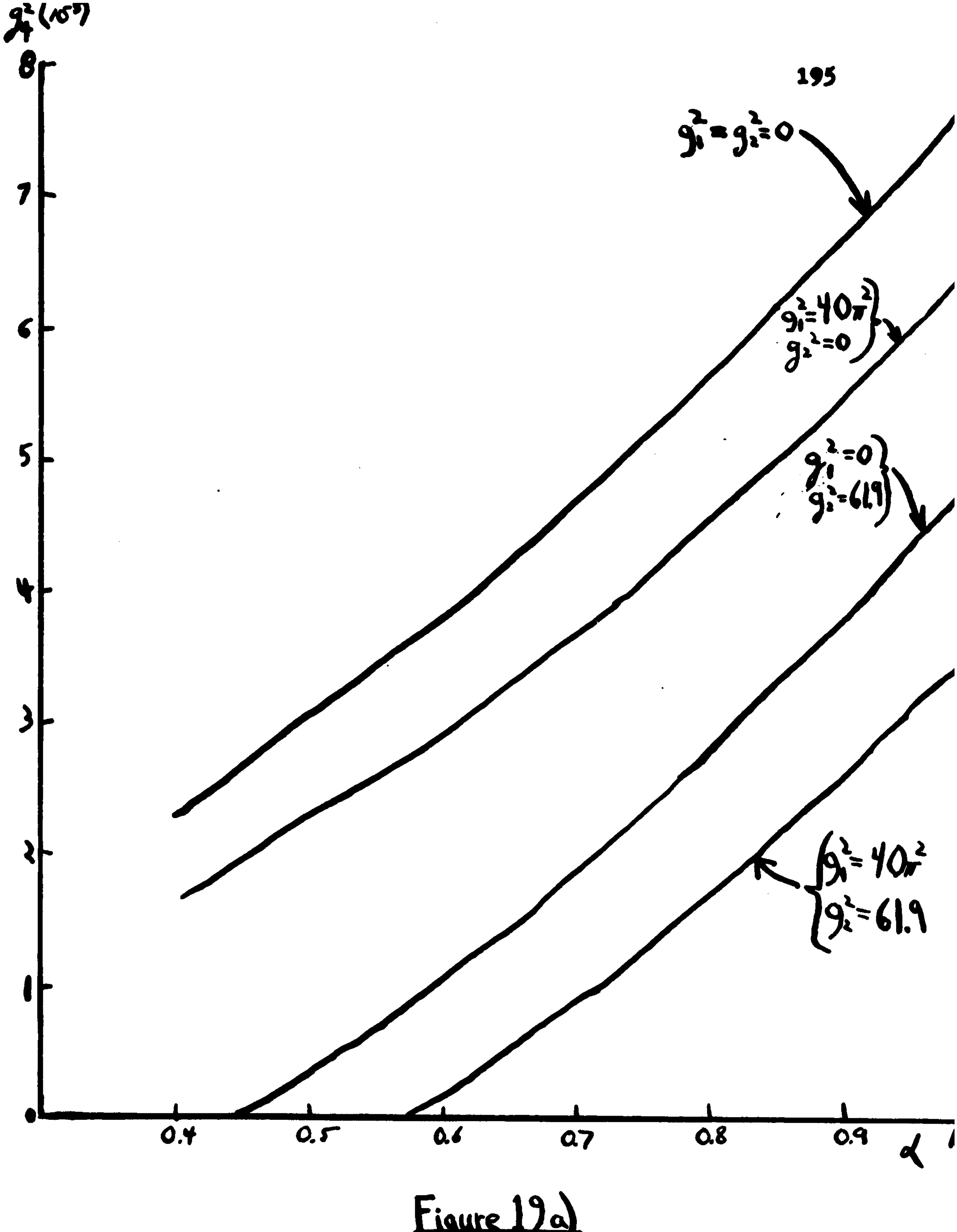

$g_4^2\,(10^3)$
8
7
6
5
4
3
2
1
0
0.4
0.5
0.6
0.7
0.8
0.9
α
$g_1^2 = g_2^2 = 0$
$g_1^2 = 40\pi^2$, $g_2^2 = 0$
$g_1^2 = 0$, $g_2^2 = 61.9$
$g_1^2 = 40\pi^2$, $g_2^2 = 61.9$

Figure 19a)

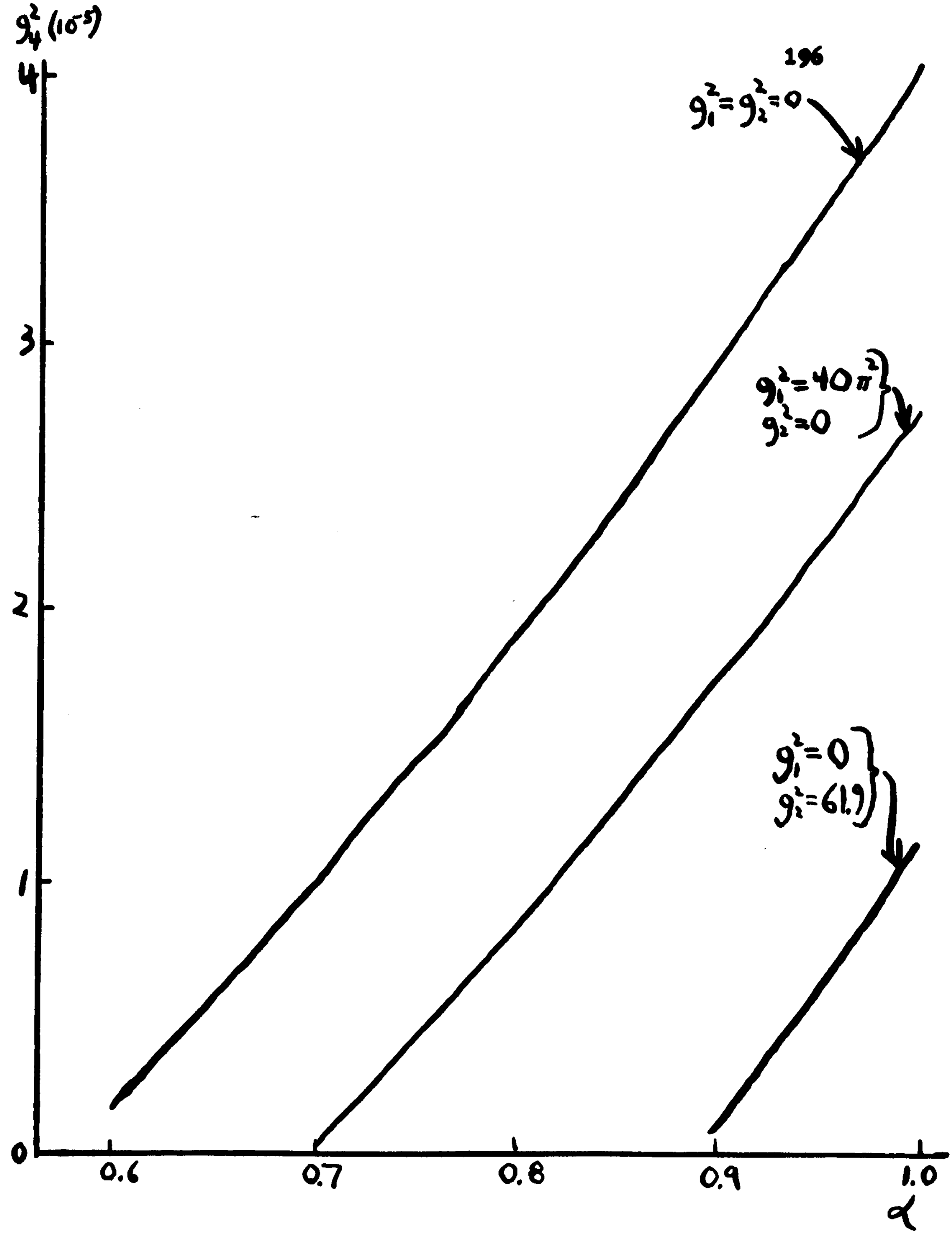
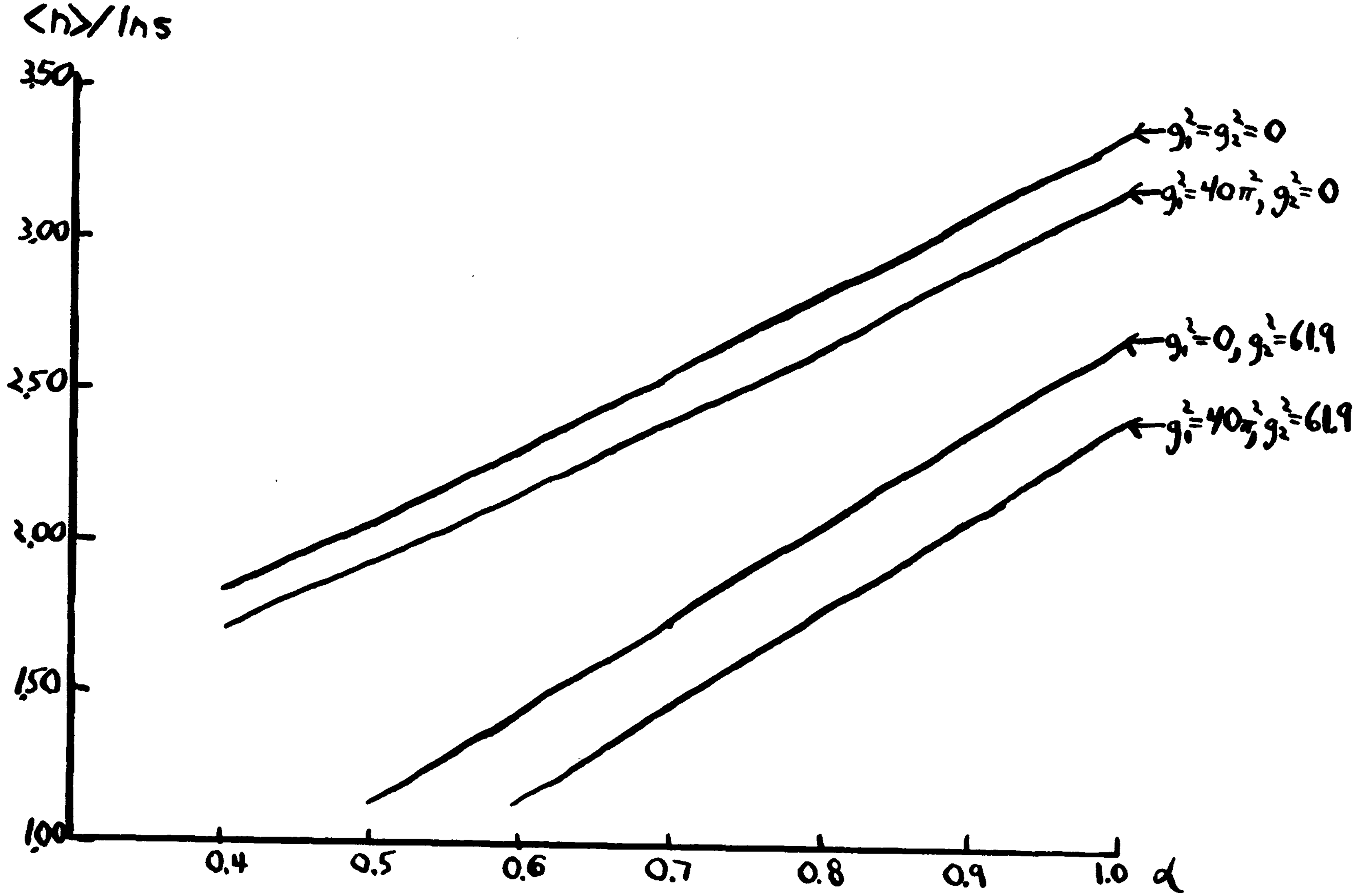

$g_4^2\,(10^{-5})$
4
3
2
1
0
0.6
0.7
0.8
0.9
1.0
α
$g_1^2 = g_2^2 = 0$
$g_1^2 = 40\pi^2$
$g_2^2 = 0$
$g_1^2 = 0$
$g_2^2 = 61.9$

Figure 19 b)



<n>/ln s
3.50
3.00
2.50
2.00
1.50
1.00
0.4
0.5
0.6
0.7
0.8
0.9
1.0
α
$g_1^2=g_2^2=0$
$g_1^2=40\pi^2,\ g_2^2=0$
$g_1^2=0,\ g_2^2=61.9$
$g_1^2=40\pi^2,\ g_2^2=61.9$

Figure 19 c)

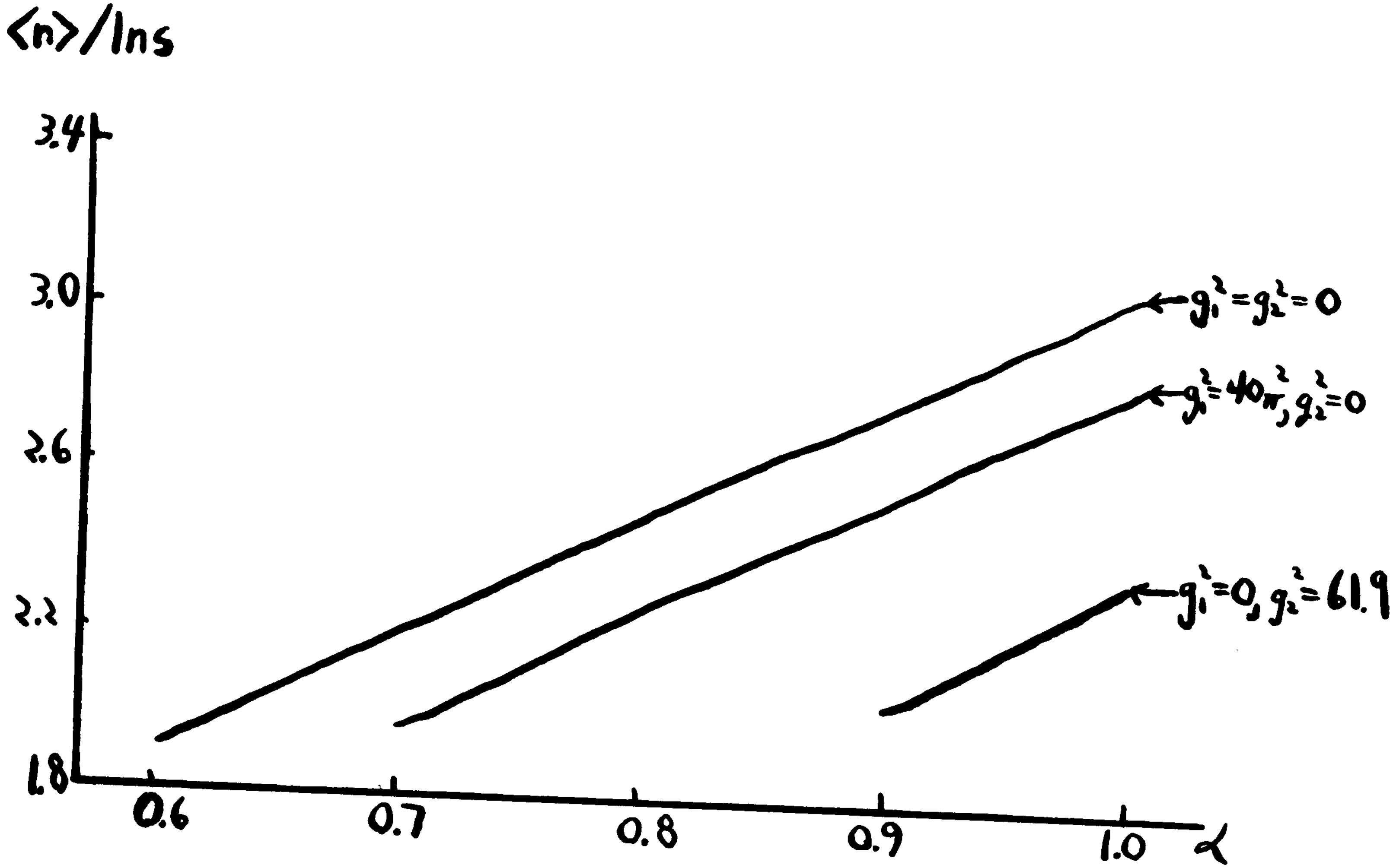

$\langle n \rangle / \ln s$
3.4
3.0
2.6
2.2
1.8
0.6
0.7
0.8
0.9
1.0
α
$\tilde{g}_1^2 = \tilde{g}_2^2 = 0$
$\tilde{g}_1^2 = 40\pi^2, \tilde{g}_2^2 = 0$
$\tilde{g}_1^2 = 0, \tilde{g}_2^2 = 61.9$

Figure 19 d)

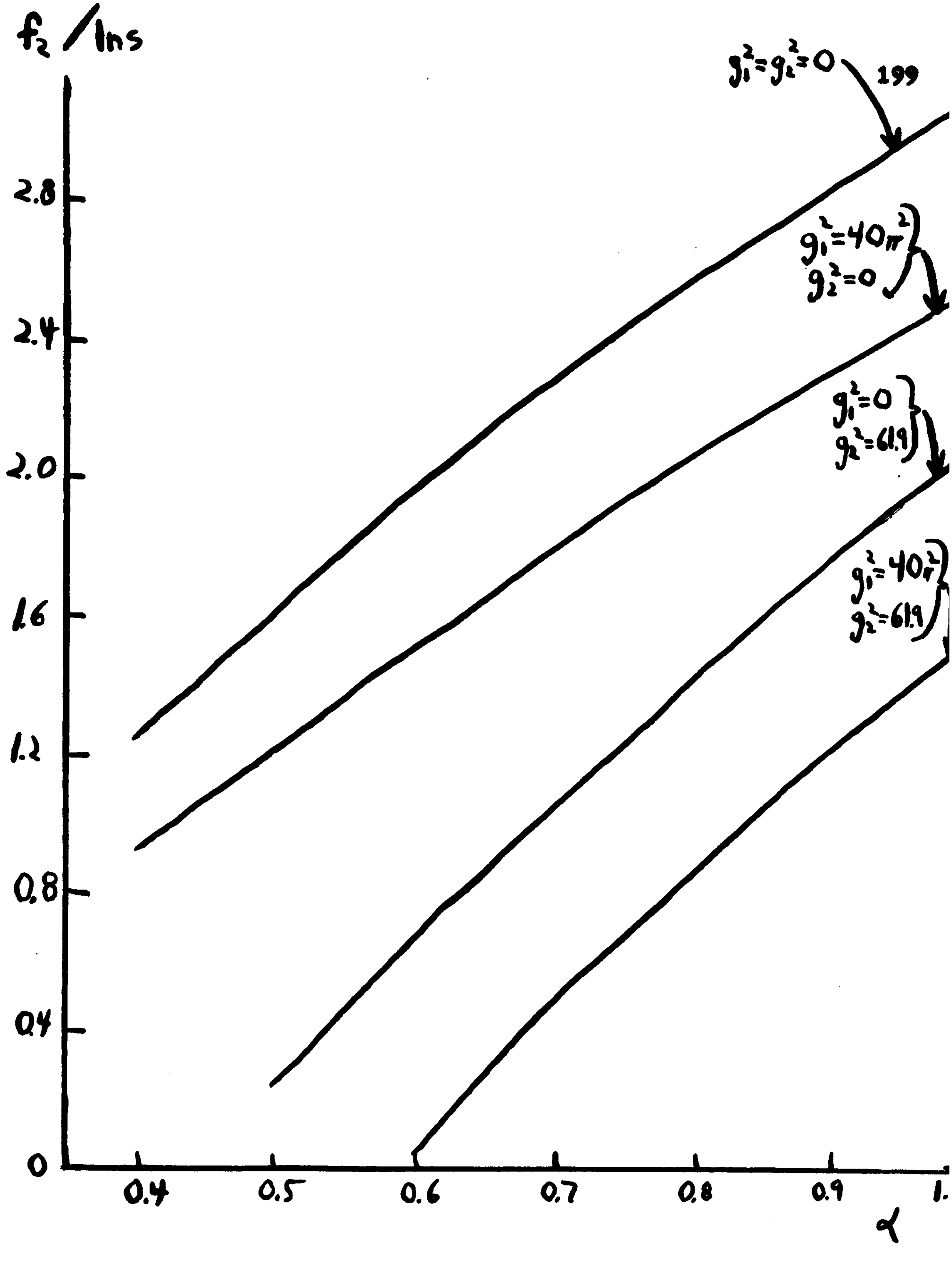

f_2 / lns
$g_1^2 = g_2^2 = 0$
$g_1^2 = 40\pi^2$
$g_2^2 = 0$
$g_1^2 = 0$
$g_2^2 = 61.9$
$g_1^2 = 40\pi^2$
$g_2^2 = 61.9$
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0
0.4
0.5
0.6
0.7
0.8
0.9
1.
α

Figure 19e)

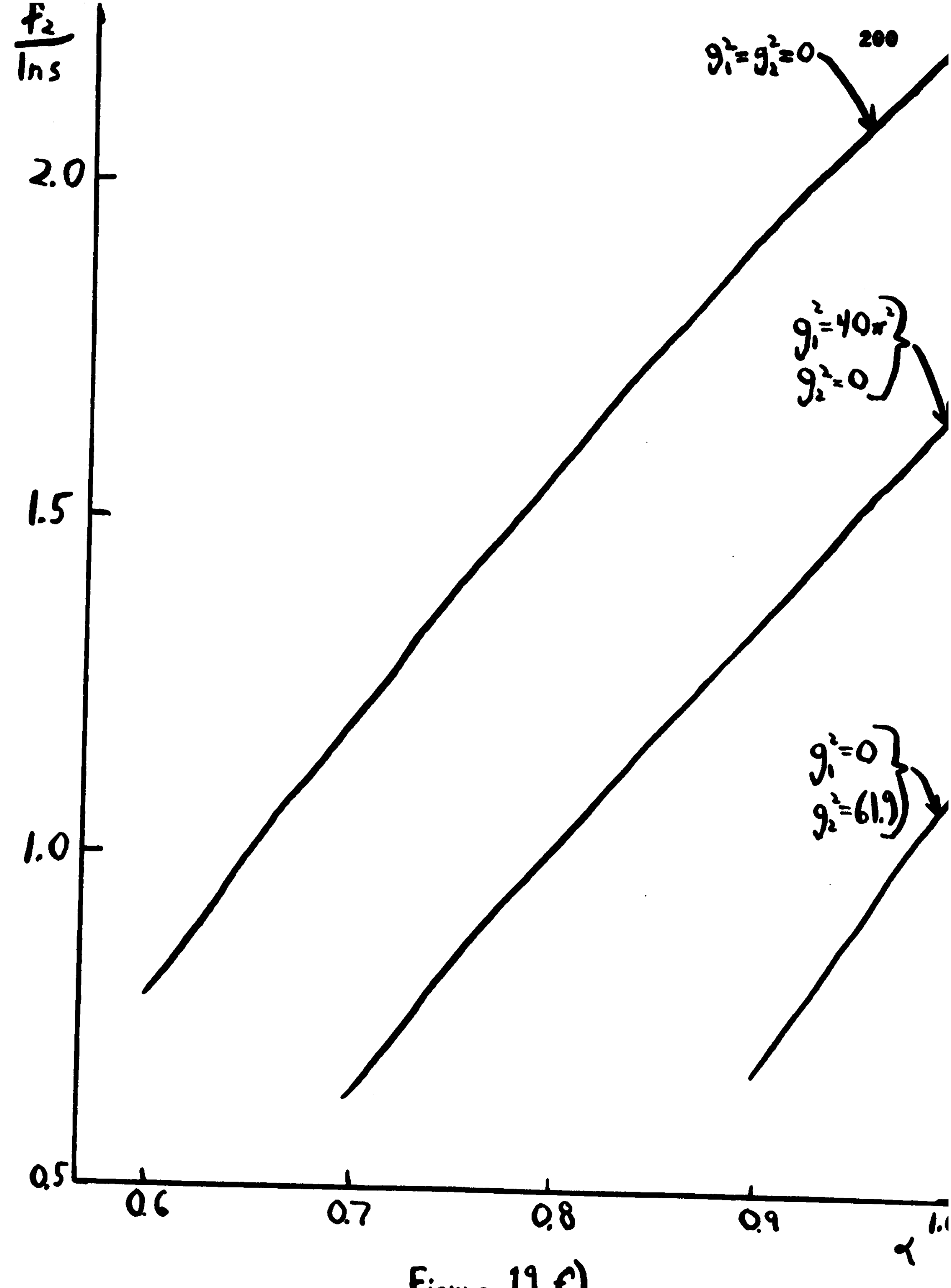

$f_2 / \ln s$
2.0
1.5
1.0
0.5
0.6
0.7
0.8
0.9
1.0
α
$g_1^2 = g_2^2 = 0$
$g_1^2 = 40\pi^2$, $g_2^2 = 0$
$g_1^2 = 0$, $g_2^2 = (61.9)$

Figure 19 f)

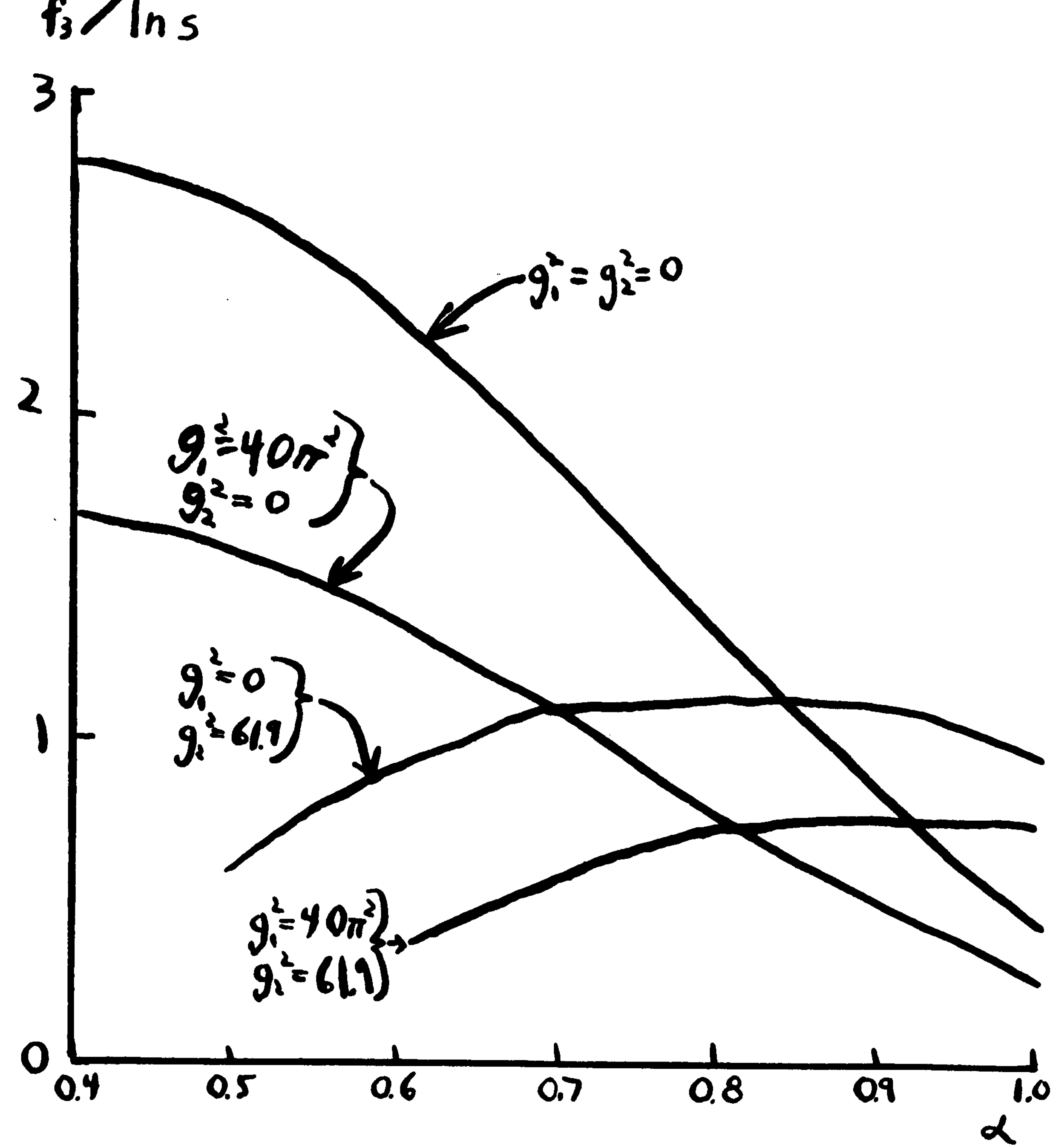
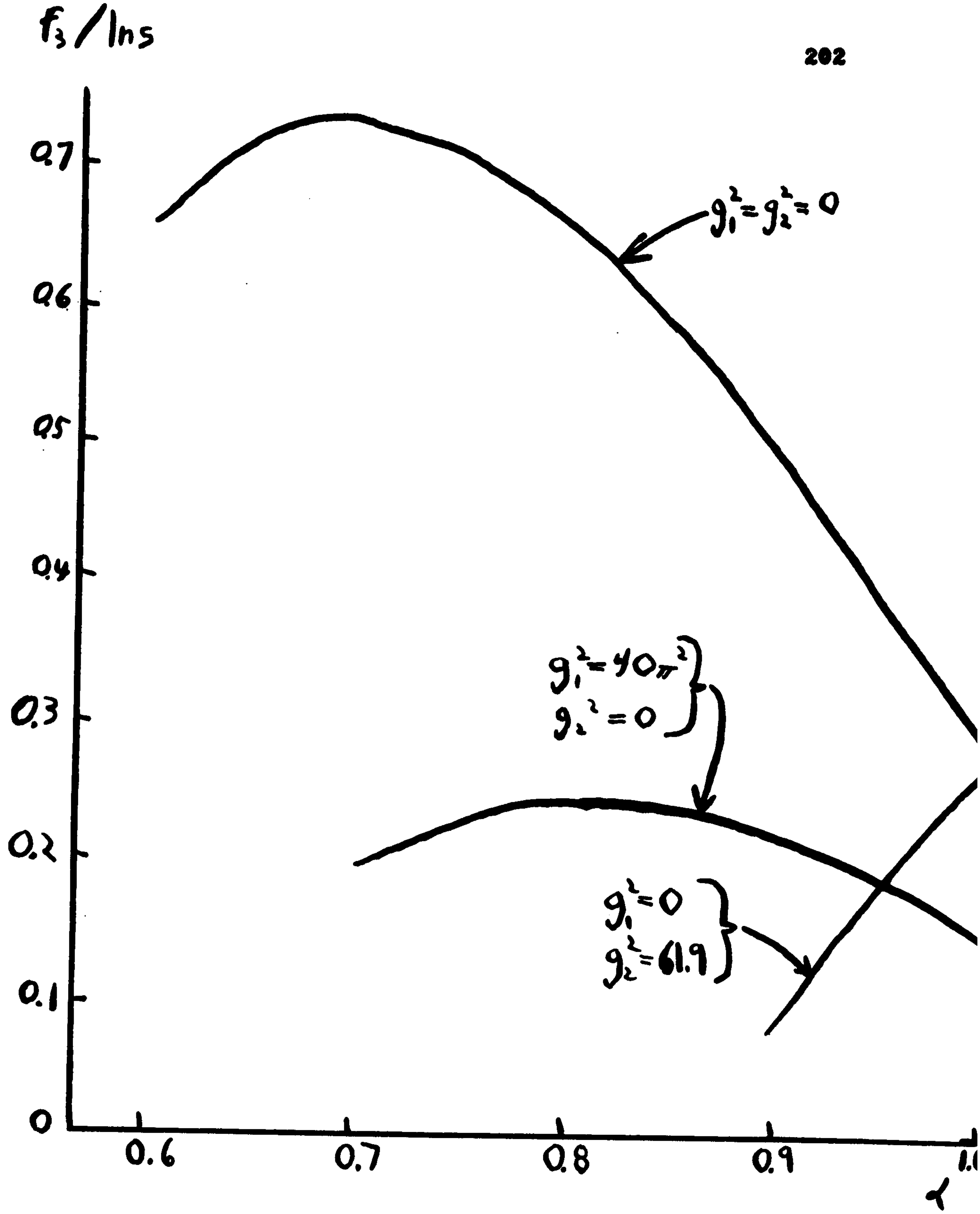

$f_3 / \ln s$
3
2
1
0
0.4
0.5
0.6
0.7
0.8
0.9
1.0
α
$g_1^2 = g_2^2 = 0$
$g_1^2 = 40\pi^2$
$g_2^2 = 0$
$g_1^2 = 0$
$g_2^2 = 61.9$
$g_1^2 = 40\pi^2$
$g_2^2 = 61.9$

Figure 19 g)



$f_3/\ln s$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.6
0.7
0.8
0.9
1.0
α
$g_1^2 = g_2^2 = 0$
$g_1^2 = 40\pi^2$
$g_2^2 = 0$
$g_1^2 = 0$
$g_2^2 = 61.9$

Figure 19 h)

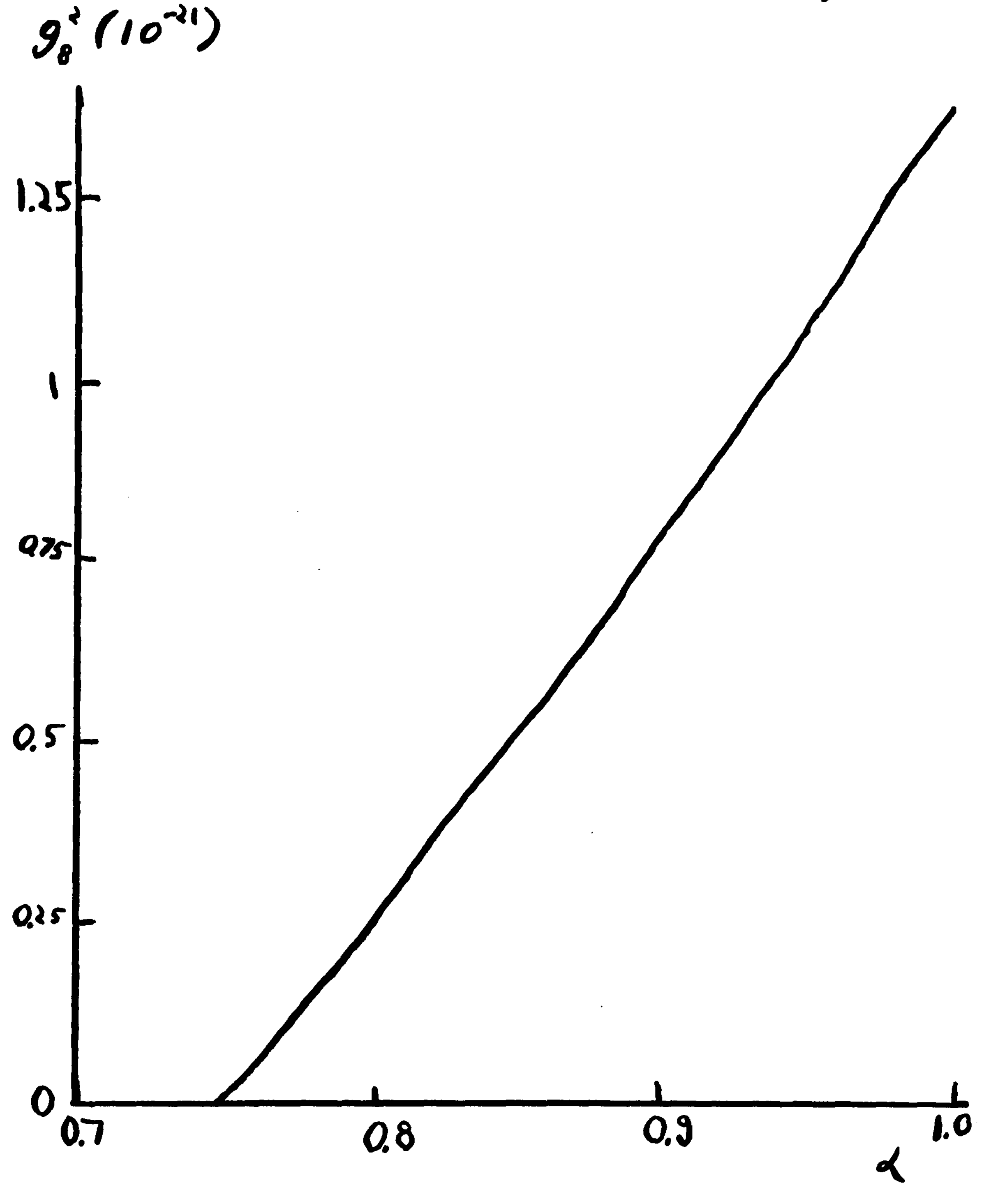


Figure 20 a)

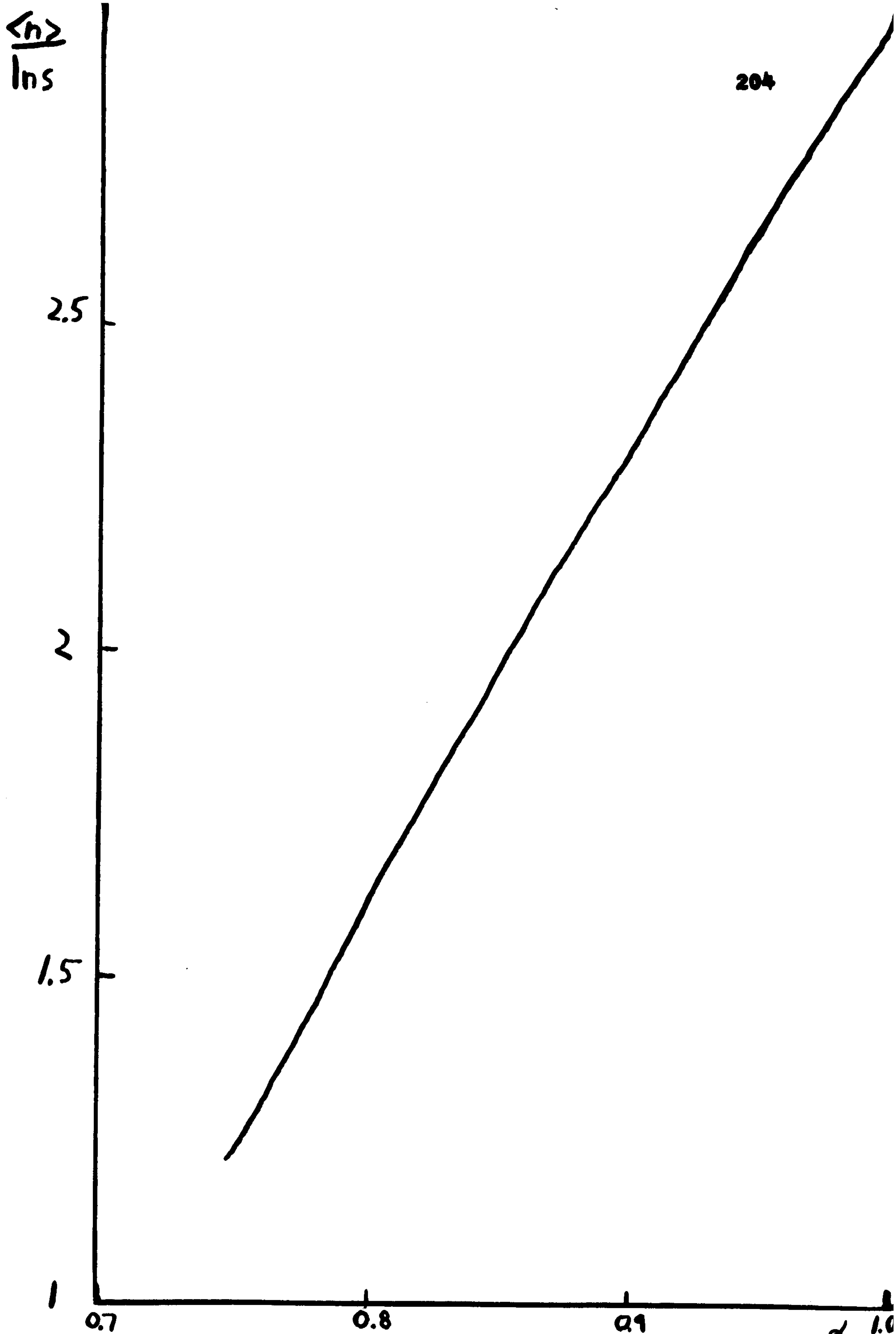


Figure 20 b)

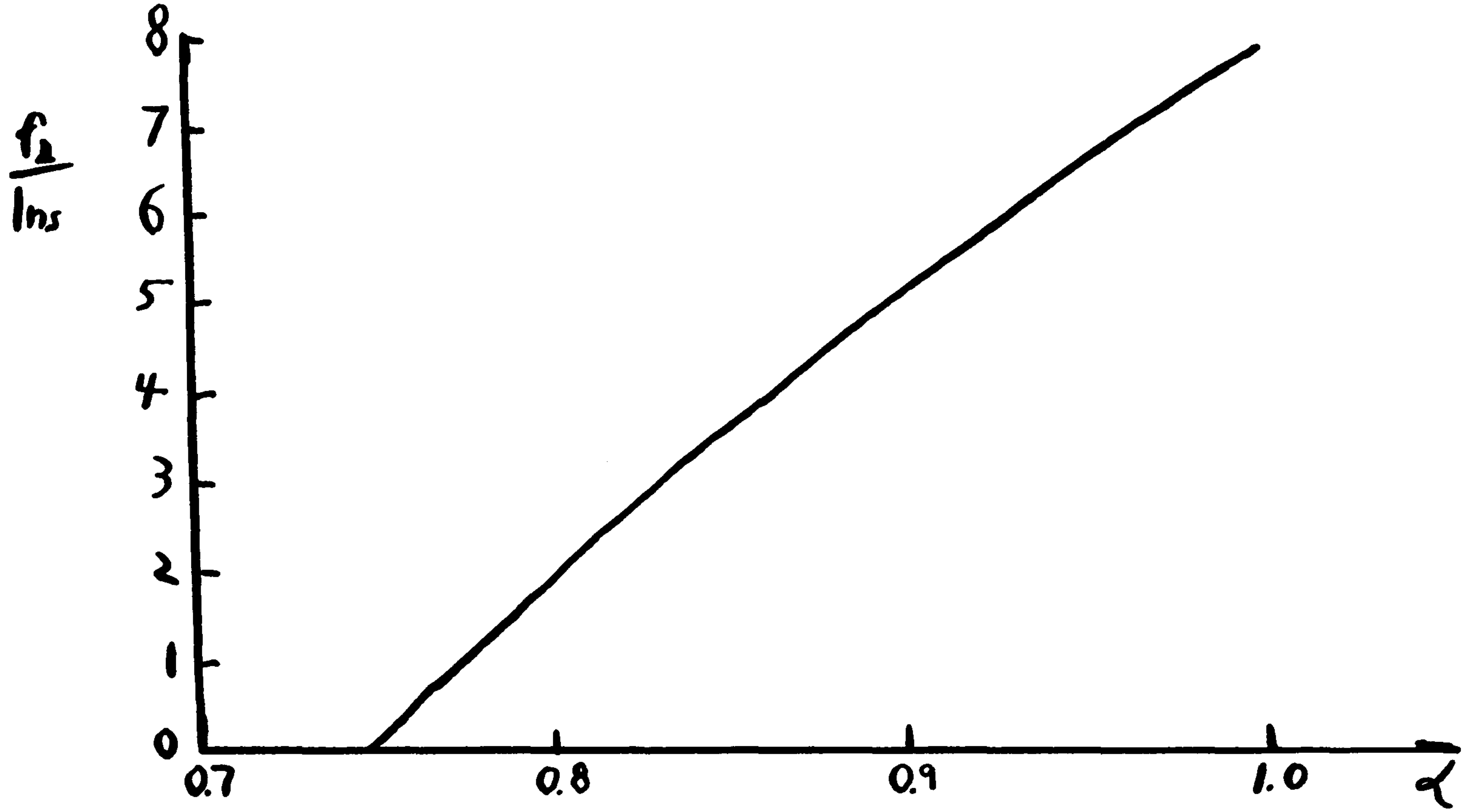


Figure 20 c)

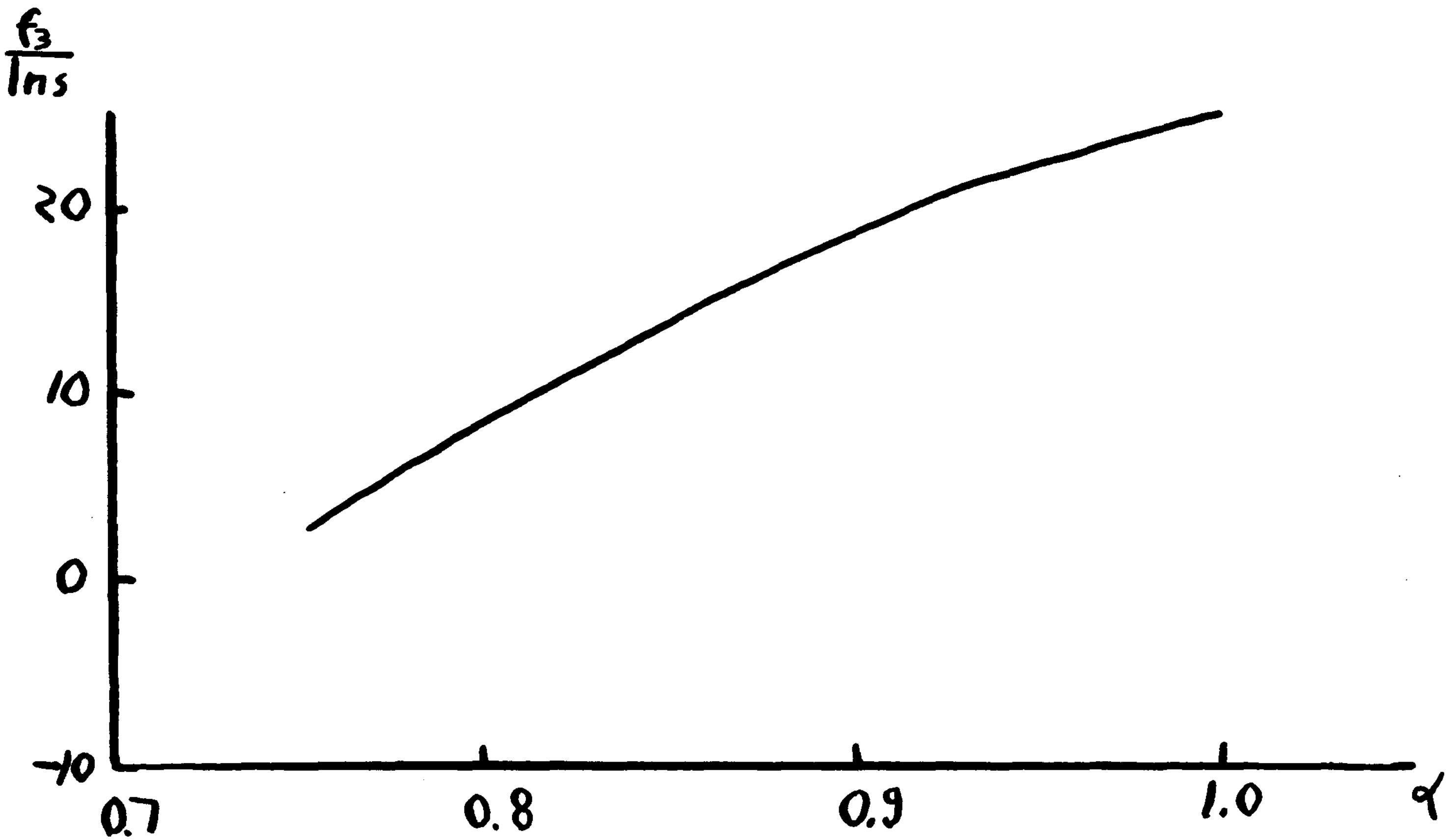


Figure 20 d)

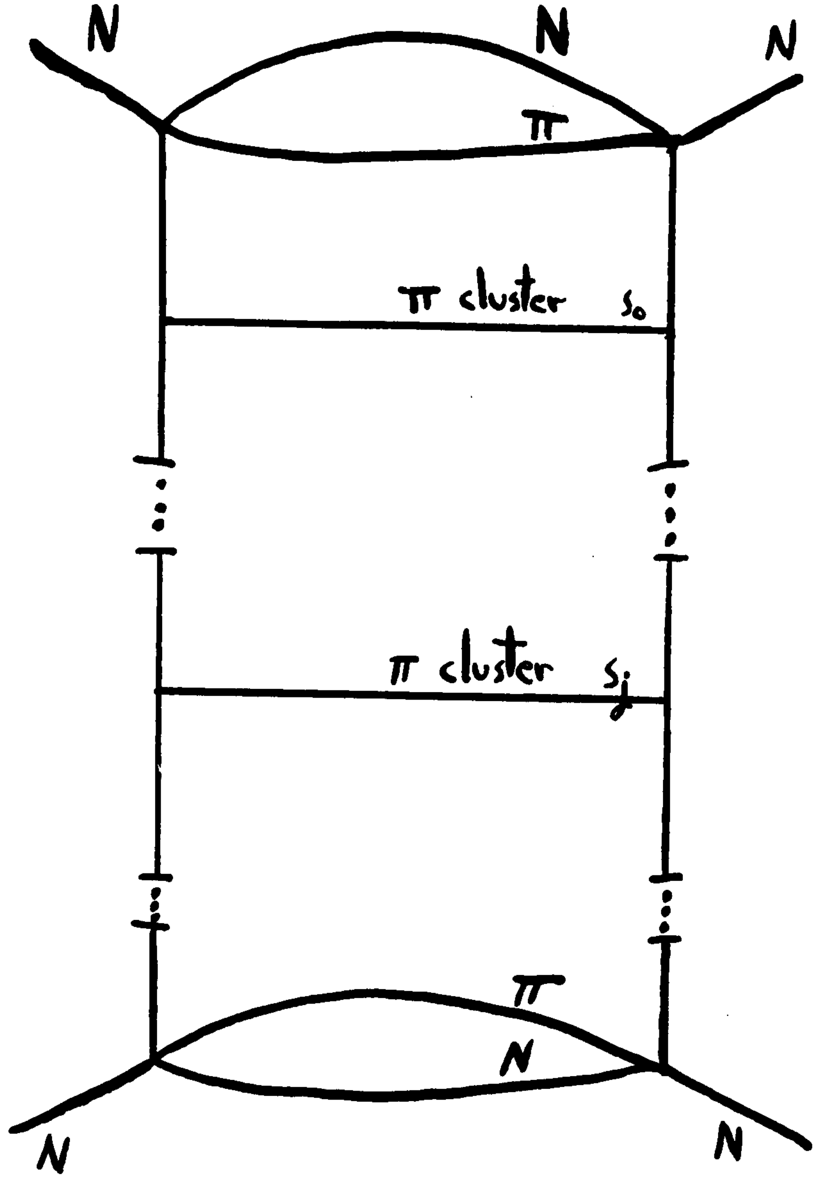


Figure 21

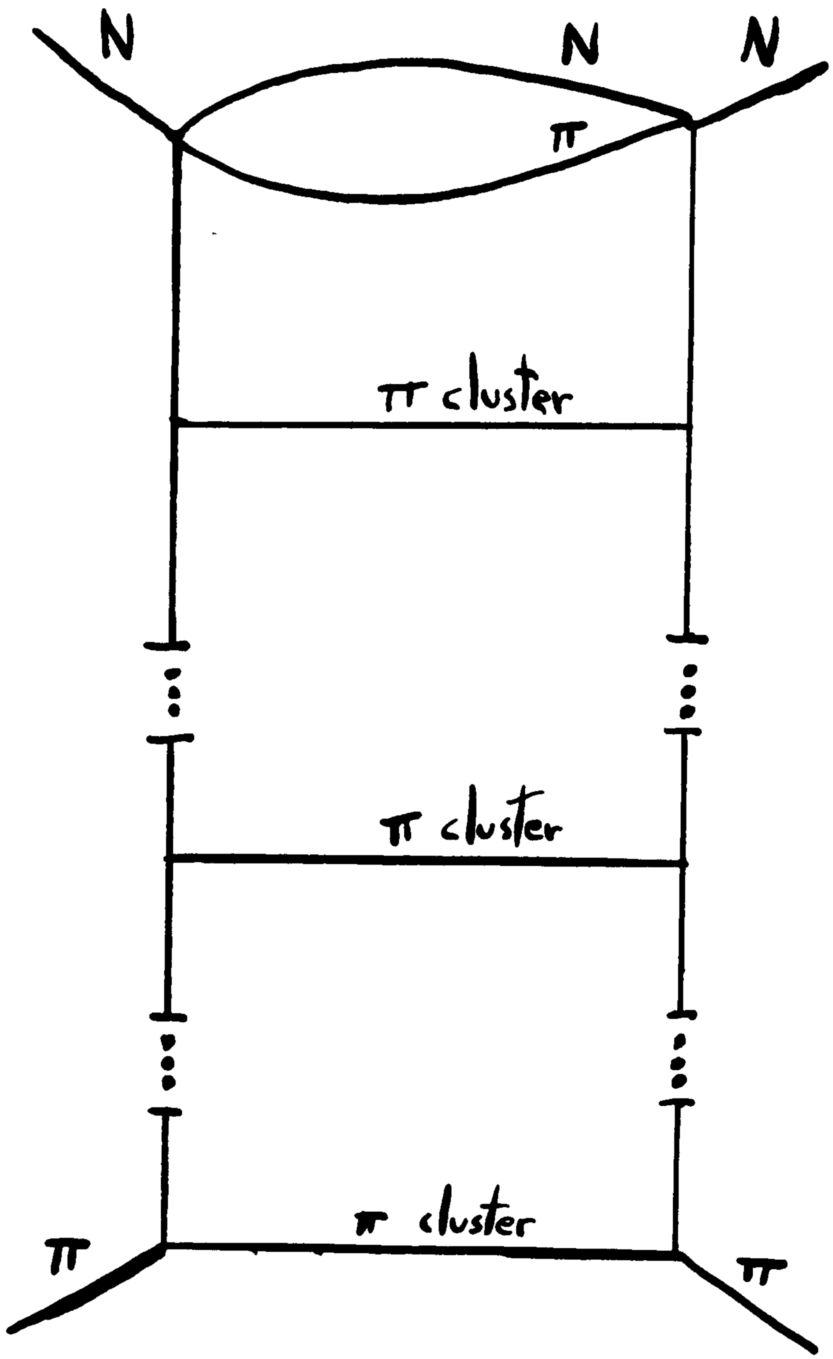


Figure 22