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by

lan

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Abstract

CHARACTERIZATION, ESTIMATION, AND PREDICTION
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by

Gary A. Gordon

Advisor: Professor Se Jeung Oh

The mathematical basis for utilizing radar observations to predict the position and configuration of a large collection of dynamically independent scatterers, unresolvable by an observing radar, is discussed. A continuous density function in dynamic state space is introduced to characterize the scatterer ensemble. Additional aspects of the scatterer ensemble model are discussed thoroughly.

A concise state representation for the scatterer ensemble is obtained by associating with it an oriented abstract object, whose output terminal variables correspond to certain aggregate properties of the ensemble. These aggregate properties are the first moments, and second central moments, of the projection of the defined density function onto the position subspace of dynamic state space. The dynamics of the scatterer ensemble are considered, and an input-output-state relation is obtained for the associated oriented abstract object. This relation is shown to have the response separation property, establishing the first moments and second central moments of the density in dynamic state space as state variables for the abstract object. Reducibility of the derived state representation is examined.

A special case of practical importance, in which the scatterers are known a priori to coincide at some instant, is considered. For such an "impulsively dispensed" scatterer ensemble, a significant reduction in the state representation is achieved. In addition, a characterization of impulsively dispensed scatterer ensembles by properties in the space of scatterer dispensing velocities is introduced.

The output terminal variables of the oriented abstract object correspond to observable properties of the scatterer ensemble. Estimation of these properties, on the basis of data obtained by the radar, is considered. Radar models and the

forms of the received voltages are discussed. Then, estimators are introduced and new results regarding estimator bias are obtained.

Prediction of the position and configuration of the scatterer ensemble is reduced to the problem of estimating the state of a time-varying linear system on the basis of a set of noisy observations of the system output. The position and configuration are characterized by the first moments and second central moments of the density in the position subspace. Prediction algorithms are discussed briefly. Also, a method is presented for the estimation of the dispensing time of an impulsively dispensed scatterer ensemble. An example illustrating the application of the formulated estimation and prediction methods is presented, using a Monte Carlo simulation of the radar echoes received from a one-dimensional, impulsively dispensed, scatterer ensemble.

CHAPTER 1

INTRODUCTION

In the study of a system consisting of an ensemble of dynamic points it is often desirable, or indeed necessary, to characterize the dynamical behavior of the geometrical configuration and spatial location of the dynamic points by the behavior of certain aggregate properties of the ensemble. This situation arises, for example, in cases where the dimensionality of the problem is too high for treatment in the ordinary way; where the individual points are not resolvable by the observer, or where the details of individual point motions are not of interest.

A problem of this type, treated by means of an original approach in this dissertation, arises in the radar tracking of a dense ensemble of dynamically independent scatterers. In this thesis, a finite state representation is derived for an ensemble consisting of an arbitrarily large number of scatterers. Formulation of the representation begins with the characterization of the point ensemble by a density function in dynamic state space, and the projection of this density on the position subspace. The first and second moments of the projected density characterize the spatial position and configuration of the ensemble, and are observable. These moments are defined as the output terminal variables of an oriented abstract object associated with the dynamic scatterer ensemble. Input-output-state relations are derived for the abstract object and are shown to have the response separation property, establishing the first and second moments of the density in dynamic state space as state variables for the abstract object. Reducibility of the derived state representation is examined. A special case of practical importance, in which singularity of the density in dynamic state space permits significant reduction of the state representation, is presented.

Additional results specialized to the radar tracking problem are presented in the thesis. It is emphasized, however, that the aspects of the thesis described above have potential application in a broad class of problems involving ensembles of dynamic points.

1.1 Statement of the radar tracking problem

Theoretical problems concerning the radar tracking of point targets have been investigated extensively in recent years. Lately, the radar tracking of cloud-like targets has become of interest in various areas of scientific research (e.g., atmospheric clouds, West-Ford radio research dipole cloud,[31] ion clouds for upper-atmospheric research, etc.).

The research reported in this thesis is concerned with an ensemble of scatterers, each of which is traveling on an independent trajectory. The ensemble is observed by a radar/data processor system at the instants t_i, $i = 1, \ldots, N$. It is necessary for the radar/data processor system to predict the spatial location of the ensemble at time t_{N+1}. The separation between adjacent scatterers is too small for the scatterers to be resolved by the observing radar. Thus, the radar/data processor system must treat the ensemble as a single cloud-like target. Furthermore, the intervals $t_{i+1} - t_i$, $i = 1, \ldots, N$ may be sufficiently large to permit significant deformation of the ensemble configuration during each interval.

Two modes of radar observation are considered, in our study, for the collection of data on the target. The first mode of observation considered is by means of a radar of the monopulse type, where the observation consists of a single transmitted pulse. It is necessary for the scatterer ensemble to be contained within the main lobe of the antenna radiation pattern. Thus, the width of this lobe must be wider than the largest angle subtended at the radar by the scatterer ensemble. To emphasize this requirement, this type of observation is referred to as a wide-beam monopulse radar observation.

The alternate mode of observation considered utilizes a radar whose beam-width is small in comparison to the angular dimensions of the scatterer ensemble. In this case, each observation consists of a set of transmitted pulses. As the pulses are transmitted, the radar is scanned in angle so as to cover the entire solid angle subtended at the radar by the scatterer ensemble.

The prediction of the location of the scatterer ensemble is used to determine the required beam placement and width for a wide-beam monopulse observation, or the required scan pattern for a narrow-beam observation. The realization of this observation-prediction-observation cycle constitutes the tracking problem.

Observation of the scatterer ensemble requires some knowledge of its configuration (i.e , size, shape, and orientation) as well as its position. In addition, information on the cloud configuration permits economical observation. A wide-beam radar can use a reduced beam-width, which results in increased antenna gain and difference channel sensitivity. In contrast, if no information about the cloud configuration were available, the radar would, by necessity, always use the widest obtainable beam-width. A radar with a dish antenna can vary its beam width, within limits, by focusing and defocusing, while a phased-array radar can vary its beam-width by appropriately adjusting the array elements. Similarly, a narrow-beam radar can use available configuration information to reduce the solid angle which is scanned, and thus require fewer transmitted pulses for the observation.

It is the task of the data processor to utilize information obtained at the observation instants t_i, $i=1,\ldots,N$ to predict the location and configuration of the scatterer ensemble at time t_{N+1} and, ultimately, to use this prediction to prepare the radar for an observation at that time. The questions addressed in this thesis concern the nature of the information which is to be obtained by the radar/data processor at each observation, and the manner in which this information is to be used to predict, in some optimum sense, the position and configuration of the ensemble.

1.2 The approach to the tracking problem

In the problem under consideration, the radar/data processor must treat the observed target ensemble as a single cloud-like object. Consistent with this constraint, we formulate a novel approach to the tracking problem, in which the target is considered to be associated with a density function in dynamic state space. The information which the data processor is assumed to extract from the raw data (video voltage samples) acquired by the radar at each observation instant consists of estimates of the centroid and second central moment (SCM) matrix of the projection, on the position subspace, of the density function associated with the scatterer ensemble at that instant. The centroid is the vector of first moments of the projected density, while the SCM matrix is the matrix of second moments and joint second moments taken about the centroid. The SCM matrix is analogous to the covariance matrix in multi-variate statistics.

The elements of the centroid and SCM matrix can be estimated on the basis of raw radar data acquired with a single observation of the scatterer ensemble by a wide-beam monopulse radar or narrow-beam radar scan. These estimates are subject to random errors, resulting from the presence of the usual sources of noise, and in addition from certain inherent properties of a scatterer ensemble, such as the time variations of the radar cross-section of the individual scatterers and the variations of the relative phases of the received echoes from the scatterers.

On the basis of the estimates of the centroid and SCM matrix derived from the data obtained at the observation instants t_i, $i = 1, \ldots, N$, it is assumed that the data processor attempts to predict the centroid and SCM matrix at time t_{N+1}. This approach is taken since :

(1) it is found that these quantities can be predicted on the basis of the information i.e., estimates of the centroid and SCM matrix) obtained at the observation instants

(2) the predicted quantities characterize the position and configuration of the scatterer ensemble at time t_{N+1}.

The characterization of the position and configuration of the scatterer ensemble by the centroid and SCM matrix is analogous to the use of the first moments and second central moments to characterize the concentration of probability density in multivariate statistics. The centroid represents an average scatterer position, while the SCM matrix specifies the mean-squared dimensions of the scatterer ensemble.

The outline of the scatterer ensemble cannot be predicted on the basis of the information derived at the observation instants. The outline could be defined as the continuous surface enclosing the smallest volume but containing all the scatterers. The absence of an outline prediction will not hamper the ability of the radar/data processor system to observe the scatterer ensemble at the instant of prediction since:

(1) The required mean square dimensions of the beam shape or scan coverage for observation can be related directly to the predicted mean square angular dimensions of the scatterer ensemble.

(2) The beam-width or scan coverage used will of necessity be somewhat larger than cloud dimensions to allow for errors in the predictions. In the case of a wide-beam monopulse observation, the beam-width must be somewhat wider than cloud dimensions to reduce certain biases in the estimates of the centroid and SCM matrix caused by variation in the sum-beam pattern.

(3) Beam patterns do not drop abruptly to zero, but attenuate continuously with angular displacement from beam center. Since the beam does not have a distinct edge, there is no need for a distinct cloud outline prediction.

(4) In tracking the ensemble, it is not generally essential that all scatterers be observed. Observation of the bulk of the scatterers is sufficient.

Although knowledge of the outline of the scatterer ensemble is not necessary for observation, it may be desirable to define a canonical bounding surface for the scatterer ensemble as part of a procedure for the determination of beam pointing and width for wide-beam observation, or the determination of the scan pattern for narrow-beam observation. The bounding surface would contain the bulk of the scatterers, but not necessarily all of them.

The procedure by which the canonical bounding surface is defined in terms of the predicted centroid and SCM matrix can be called the <u>bounding policy.</u> One reasonable bounding policy makes use of the eigenvectors and eigenvalues of the SCM matrix to define an ellipsoidal bounding surface[1].

The ellipsoid is centered at the centroid, has principal semi-axes collinear with the eigenvectors and semiaxis magnitudes proportional to the square root of the corresponding eigenvalues. If the eigenvalues are denoted by λ_i, $i=1,2,3$, and K is the proportionality constant, the semiaxis magnitudes are given by $K\sqrt{\lambda_i}$. If $K=\sqrt{5}$, the ellipsoid defined is analogous to the ellipsoid of concentration[2], which is used to indicate the concentration of probability density in multivariate statistics.

An alternate bounding policy defines a rectangular parallelepiped centered at the centroid, with edges colinear with the eigenvectors of the SCM matrix and edge lengths given by $2K\sqrt{\lambda_i}$, where again the λ_i are the eigenvalues and K is some constant. It follows from the Chebeshev inequality that the fraction of the density which is excluded by the surface is bounded by $3/K^2$. However, in the case of this policy, as well as in the case of the bounding policy which uses an ellipsoidal surface, a practical value for K is best determined by considering the class of cloud configurations which are expected to occur and selecting a value of K empirically which gives satisfactory performance. Finally, it is reemphasized that, as indicated above, use of a bounding policy is not a necessary part of a tracking procedure. Procedures for cloud bounding, and for the determination of the required beam placement and width, or scan pattern, from the predictions are system-oriented considerations, and are not considered further in the thesis. The particular aspects of the tracking problem treated in this thesis are discussed in the following section.

1.3 Results of the research

Corresponding to the approach to the tracking problem which was discussed in the last section, two formal objectives arise:

(1) given radar data resulting from an observation, estimate the centroid and SCM matrix of the scatterer ensemble at the instant of observation;

(2) given the set of estimates corresponding to observation instants t_i, $i = 1, \ldots, N$, which contain random errors, optimally estimate the centroid and SCM matrix at time t_{N+1}.

The emphasis of the thesis is directed towards formulating a realization of the second objective. Since, in the context of tracking, the instant t_{N+1} is later than the observation instants, the estimate sought in the second objective is generally a prediction. Thus, the second objective is referred to as the <u>prediction problem</u>, to distinguish it from the first objective, which is referred to as the <u>estimation problem</u>.

The above estimation and prediction problems fall into the class of problem generally referred to as <u>optimization problems</u> [3]. An optimization problem can be decomposed as consisting of

(1) defining the goal to be achieved

(2) obtaining as much information as possible on any present conditions relevant to the goal

(3) identification of all factors in the environment, including physical laws and constraints, which effect our ability to achieve the goal.

(4) determination of the best policy for using the obtained information and environmental factors to achieve the goal.

Determination of the best policy, or solution of the optimization problem, requires:

(1) adequate definition of the problem in physical terms

(2) translation of the physical problem description into mathematical terms

(3) solution of the mathematical problem

In discussing the prediction problem in the thesis, the main effort is expended on the first two aspects of the solution; that is, on the development of an adequate physical and mathematical formulation of the problem. The end result of this formulation is the reduction of the prediction problem to that of estimating the state of a time-varying linear system on the basis of noisy observations of the system output. Since the state estimation problem has been studied extensively[3-10], only a brief discussion is presented to apply the existing results in state estimation theory to the problem at hand.

In the present study, the formulation begins by considering a set of points which is dense in a region of dynamic state space. With each point, a scatterer is associated whose radar cross-section is a random variable. A continuous density function in dynamic state space is introduced, as well as the projection of the density on subspaces of dynamic state space. This density function represents the distribution in state space of the expected values of the radar cross-section associated with the scatterers. In the formulation, each particular scatterer ensemble is specified completely by its density function.

The first moments and second central moments of the density in dynamic state space are introduced, and the time transformation of these quantities is considered. It is shown that unique transformations can be obtained, provided that a certain stationarity assumption is made regarding the statistics of the scatterer radar cross-sections, and provided that the point mass dynamics are linear or linearized about a reference trajectory. Finally, the formulation culminates in the association of an abstract object with the scatterer ensemble, whose state variables are these first moments and second central moments and whose output variables are the elements of the centroid and SCM matrix of the projection of the density on the position subspace. The latter variables are exactly the quantities estimated, at each observation instant, on the basis of the radar data.

Identification of state variables for the scatterer ensemble, which requires thorough mathematical formulation and analysis of the physical situation, constitutes the crucial aspect in the solution of the prediction problem. Treatment of the prediction problem within the context of formal concepts of system theory[11] permits us to make the notion of state for the dynamic scatterer ensemble precise, and to treat in a rigorous manner questions regarding equivalent states and system reduction.

Thus, input-output-state relations for the abstract object associated with the scatterer ensemble are derived and shown to have the response separation property. State and observation equations are presented, as well as the relation between the state variables and the quantities to be predicted. The latter quantities are the same as those observed, the centroid and SCM matrix of the projection of the density on the position subspace. In addition, some general results regarding the possibility of system reduction are derived.

A special case of practical importance (e.g., the West-Ford dipole cloud) is that of impulsively dispensed scatterer ensembles. An impulsively dispensed scatterer ensemble is one which can be considered to be dispensed at an instant from a single point in space. In this special case the order of the system representation can be reduced significantly from that required in the general case. Input-output-state relations, state equations, etc., are derived as in the general case, in the discussion below of impulsively dispensed scatterer ensembles.

The treatment of the prediction problem, described above, constitutes the primary original contribution of this research. Additional results obtained include a characterization of impulsively dispensed scatterer ensembles in the space of scatterer dispensing velocities, and the derivation of the relationship between properties of the scatterer ensemble in dispensing velocity space and corresponding properties in the position subspace at some instant after dispensing. A method of estimating the dispensing time of an impulsively dispensed scatterer ensemble, on the basis of the observed values of the SCM matrix, is presented.

Estimators which use wide-beam monopulse or narrow-beam scan radar data to estimate the centroid and SCM matrix are discussed, and new results regarding estimator bias are obtained.

1.4 Related work in the literature

Ensembles of unresolvable radar scatterers have been considered in the literature both in connection with targets consisting of many dynamically independent point scatterers and targets consisting of a single large object with many scattering centers. Delano[12], Peters and Weimer[13], and Ross and Bechtel[14] considered targets of the latter type, and were concerned primarily with the effects of angular glint in null tracking radars. That is, the phenomena studied consisted of the motion of the apparent null direction, or balance point, which is the direction along which the radar must be pointed to obtain vanishing angle error signals in the radar. The angle pointing error, defined as the difference between the balance point and the true radar centroid, was introduced by Peters and Weimer[13], and certain bounds on the pointing error were discussed by them. Ross and Bechtel[14] computed the angle pointing error as a function of target aspect angle for the special case of a finite, perfectly conducting, right-circular cylinder.

Jacobs[15], Aiken[16], and Rummler[17] concerned themselves with targets consisting of many dynamically independent point scatterers. The problem considered was that of estimating the angular position of the centroid, and also the angular squared width, or second central moment. The estimators studied operate on data obtained by a wide-beam monopulse radar. Expressions are obtained for the expected value and variance of the angular centroid estimators and for the expected value of the angular squared width estimators.

Helms[18] extended the results of Jacobs[15] and Aiken[16] and introduced estimators for the centroid and second central moment matrix of the scatterer ensemble. The estimates are based on data acquired by a wide-beam monopulse radar whose range resolution is fine compared to the extent in range of the scatterer ensemble. These estimators are discussed in greater detail in Chapter 5, below. Hardin[19] discussed procedures for deriving the centroid and second central moment matrix from data obtained by means of a narrow-beam radar scan. In Chapter 5 of this thesis, estimators of the centroid and second central moment matrix for use with data acquired by a somewhat idealized narrow-beam scan are introduced.

Recently, the prediction of the location of a scatterer ensemble on the basis of data acquired at a set of observation instants was discussed by Schweppe and Knudsen[20,21]. In their approach, cloud observation is assumed to consist of the determination of a cloud outline contour in position and possibly doppler. The cloud state is represented by a surface in dynamic state space. In contrast, in the present approach, observation results in estimates of the cloud centroid and SCM matrix. The cloud state is represented by a finite number of state variables (at most 27) which obey certain linear state equations.

In the present approach, cloud observation may be made by means of a wide-beam monopulse radar as well as a narrow-beam radar. On the other hand, an outline contour measurement, essential in the Schweppe and Knudsen approach, requires the use of a narrow-beam radar. For distant or small clouds, which subtend a small angle at the radar, obtaining sufficient angular resolution for a contour measurement may be impossible in practice.

Of somewhat tangential interest is work reported by Lawson and Uhlenbeck[22], Wong, Reed and Kaprielian[23], Childers and Reed[24], Kelly and Lerner[25], and Kerr[26] which deals with the statistical properties of the radar echo from a scatterer ensemble. These statistical properties are the probability density functions of the instantaneous received voltage and power, as well as the autocorrelation functions and power spectra. Some of the results are used below in Chapter 6. The autocorrelation functions are of interest since they are relevant to the determination of a minimum time interval between radar observations at a particular wavelength, such that the estimates of centroid and SCM matrix corresponding to the first observation are practically uncorrelated with those corresponding to the second.

Several of the references cited above[12-19] present work applicable to null tracking of scatterer ensembles. In null tracking, only information corresponding to observations immediately preceding the instant at which beam pointing is to be directed is used in determining the beam pointing direction. This information is only a small fraction of the total available information. The null tracking concept differs essentially from the observation-prediction-observation approach to tracking considered in this thesis. In the present approach, information obtained from all preceding observations is used in directing beam pointing. Furthermore, the present work considers the prediction of the target configuration, as well as its position.

It is emphasized that, among the references cited above, only the work of Schweppe and Knudsen[20,21] considers the prediction of properties of a dynamic scatterer ensemble. The present approach to the prediction problem differs essentially from that of Schweppe and Knudsen, as discussed above. The nature of our approach to and formulation of the prediction problem enables us to take advantage of the formalism of abstract system theory in making the analysis precise and in deriving new results.

1.5 Organization of the dissertation

Essential concepts and definitions are introduced in Chapter 2. These form the basis of our characterization of a scatterer ensemble. A piece-wise continuous density function in dynamic state space associated with the scatterer ensemble is defined. The physical conditions which permit the modelling of a scatterer ensemble in this manner are discussed. Additional aspects of the model, including random scatterer radar cross-section and random phase of the echo received from each scatterer are introduced and justified. Finally, projections of the density on lower dimensional subspaces of dynamic state space, and moments of the density and its projections are discussed.

Chapter 3 includes the identification of state variables for the scatterer ensemble, and the derivation of the state equations. The chapter begins with the representation of the scatterer dynamics, and a discussion of linear dynamics and linearization of nonlinear dynamics. Then, the time transformation of the density in dynamic state space and of its first moments and second central moments are obtained. A crucial assumption regarding the stationarity of certain statistics associated with the scatterer radar cross-sections is discussed in detail. The chapter culminates in the formal identification of the scatterer ensemble with an abstract object or system with certain output and state variables. The input-output-state relations of the system are derived, and it is demonstrated that they have the response separation property[27]. Finally, the possibility of system reduction is investigated.

Chapter 4 deals with the important special case of impulsively dispensed clouds. The transformation with time of the density in the position subspace, as well as of its first moments and second central moments are derived. State variables are identified, input-output-state relations of the system are derived, and questions concerning system reduction are investigated. The characterization of an impulsively dispensed cloud by a density in the space of scatterer dispensing velocities is discussed.

In Chapter 5, estimation of the centroid and SCM matrix on the basis of radar data obtained during a single observation is discussed. Mathematical models of observation by a wide-beam monopulse radar and narrow-beam radar scan are formulated. The estimators are defined and new results concerning estimator bias are obtained.

Chapter 6 contains a description of a method of generalized least squares estimation and its application to the prediction problem. In addition, a method is presented for estimating the dispensing time of an impulsively dispensed scatterer ensemble, based on a set of estimates of the SCM matrix. Finally, an example is presented which illustrates the application of the results obtained in the thesis. The example includes a Monte-Carlo simulation of the radar echoes received from a one-dimensional impulsively dispensed scatterer ensemble, and application of estimators for the centroid and second central moment, as well as use of the prediction algorithms. The effect on predictions of the use of the wrong dispensing time in the system model, and the use of the algorithms formulated for estimation of the dispensing time, are illustrated.

The conclusions and a summary of the contributions of the thesis are presented in Chapter 7. Also presented there are suggestions for extensions of the results obtained and other related work which might be persued.

Generalized least squares prediction algorithms, including a recursive formulation, are discussed in an appendix.

CHAPTER II

SCATTERER ENSEMBLE CHARACTERIZATION

In this chapter, a characterization of the scatterer ensemble by a density function in dynamic state space is introduced. The applicability of this characterization is discussed. Additional aspects of the scatterer ensemble model are introduced and certain assumptions are justified. Projections of the density function on lower dimensional subspaces of dynamic state space, and moments of the density function and its projections are introduced. This enables us to state the estimation and prediction problems mathematically and provides part of the framework for the formulation of the solutions.

2.1 Definition of the density function

Consider a set of points, denoted $\mathscr{I}$, which is dense in a region $\mathscr{V}$ of dynamic state space at time t. Associated with each point $i \in \mathscr{I}$ is a scatterer with radar cross-section $\sigma_i^2(t)$, considered to be a random variable. Denote†

$$s_i^2(t) = \mathcal{E}\{\sigma_i^2(t)\} \quad . \tag{2.1-1}$$

Let $\overline{p}$ denote the dynamic state vector, consisting of position and velocity, and let $\Delta\tau_p$ represent a volume element (and its volume) in dynamic state space at $\overline{p}$. We define

$$\rho_p(\overline{p},t) = \lim_{\Delta\tau_p \to 0} \frac{\sum_I s_i^2(t)}{\Delta\tau_p} \tag{2.1-2}$$

† $\mathcal{E}\{\cdot\}$ represents the expectation operator.

where the summation is over the scatterers contained in the element $\Delta \tau_p$. In the ensuing discussion, a point $i \in \mathscr{S}$ will be referred to as a "scatterer", and the set $\mathscr{S}$ is the "scatterer ensemble".

We assume that $\rho_p(\overline{p},t)$ is defined (i.e., the limit in (2.1-2) exists) and finite for all $\overline{p} \in \mathscr{V}$, and that $\rho_p(\overline{p},t)$ is at least a piecewise continuous function of $\overline{p}$. Then the Riemann integral

$$T(t) = \int_{\mathscr{V}} \rho_p(\overline{p},t)\, d\tau_p \tag{2.1-3}$$

exists. The region $\mathscr{V}$ is of course bounded, though not necessarily connected. The quantity T(t) will be referred to as the total radar cross-section of the scatterer ensemble at time t. Finally, we define

$$f_p(\overline{p},t) = \rho_p(\overline{p},t)/T(t) \quad . \tag{2.1-4}$$

The normalized density function $f_p(\overline{p},t)$, in our formulation, is used to characterize the distribution of the scatterer ensemble in dynamic state space. Substituting (2.1-4) in (2.1-3), we get

$$\int_{\mathscr{V}} f_p(\overline{p},t)\, d\tau_p = 1 \quad . \tag{2.1-5}$$

2.2 Applicability of the characterization; additional aspects of the scatterer ensemble model

(a) The scatterer ensemble as a dense set.

The theory presented in this thesis is intended to apply to actual situations in which a large, but finite, number of scatterers is observed by a radar. The distance and range rate difference between neighboring scatterers in state space is small compared to the resolution of the observing radar. That is, individual scatterers cannot be resolved. However, the scatterer separations are large enough for multiple scattering of the transmitted wave to be negligible. Each scattering object is considered to be an independent point scatterer.

Furthermore, the value of the complex video voltage (or voltages, if the radar is of the monopulse type) appearing at the receiver output at an instant corresponding to a range within the range interval containing the scatterer ensemble is generally determined by the echoes from a large number of scatterers. It is assumed that the beam intersects the ensemble at that range. Defining an interrogation cell loosely as a region of space whose dimensions are indicated by the radar's range resolution, doppler resolution, and beam width, the above conditions can be restated: an interrogation cell which intersects the ensemble generally contains many scatterers.

Neglecting the multiple scattering effect is not necessarily inconsistent with the assumption of small scatterer separation. Multiple scattering can be neglected if scatterer separations are larger than about a wavelength. Since the wavelength is generally orders of magnitude less than the radar resolution, scatterer separations may well be large compared to the first, while at the same time, small compared to the second.

The set $\mathcal{S}$ of scatterers, as defined in Section 2.1, can be considered to arise from a physical situation by letting the number of scatterers become infinite, while the expected radar cross-sections of the scatterers go to zero. The use of a continuous density to characterize the distribution of scatterers is analogous, to some extent, to the use of a continuous density to describe the distribution of mass or charge, in modelling phenomena where the discrete nature of mass and charge can be ignored. Such cases generally arise when the observing apparatus in the phenomena interact with a large number of particles simultaneously.

There is an essential difference, however, between these phenomena and those that we are concerned with. The mass or charge of each particle in an ensemble is a constant, while we have assumed that the radar cross section of each scatterer is a random variable. The function $\rho_p(\overline{p}, t)$ which has been introduced to describe the scatterer distribution is deterministic, since it is defined in terms of the expected values of scatterer radar cross-section. However, the outcome of a radar observation will, of course, depend on the particular values that the scatterers' radar cross-sections attain at the instant of observation. Thus, the density is not an observable quantity in the way that a mass or charge density is observable.

(b) <u>Random scatterer radar cross-section</u>.

The use of a random variable for scatterer radar cross-section is motivated by the desire to include cases where the cross-sections of the scatterers change with time - as when, for example, the scatterers are non-symmetrical tumbling objects. Scatterer rotation is not included in the dynamic description of the scatterers. In fact, since the dynamic state vector $\overline{p}$ consists only of position and velocity, the scatterers appear dynamically, in the formulation, as point masses. Possible motions between various physical components of a scatterer are similarly ignored. Thus, it is logically consistent to introduce a random variable for scatterer radar cross-section, where the value attained by the random variable depends on the unknown scatterer orientation, and possibly the internal component configuration, at the instant. The ensemble of possible orientations and internal configurations for a scatterer, the corresponding values of radar cross-section, and the associated probabilities, determine the distribution function for $\sigma^2(t)$.

The random variable $\sigma^2(t)$ associated with a scatterer must be considered, in some contexts, to be determined by a stochastic process, which can be denoted $\sigma^2(\cdot)$. That is, the random variable $\sigma^2(t)$ arises by considering the value attained, at time t, by the random process $\sigma^2(\cdot)$ associated with the scatterer. The possible need for this point of view is evident when we consider that $\sigma^2(t_1)$ and $\sigma^2(t_2)$ for a scatterer are not generally independent. The ensemble of possible orientations and internal configurations at t_1 and t_2 are related, since presumably the transformation of orientation and internal configuration with time obeys some deterministic law. This subtlety does not enter into the formulation presented in this thesis, but can become important when one investigates the correlation between data obtained as a result of observation of the scatterer ensemble at two different instants.

(c) Random received scatterer echo phase

A second major difference exists between the phenomena associated with a scatterer ensemble and the analogous ensembles of mass or charge particles. In producing phenomena, the masses or charges enter linearly, as scalars, in the appropriate mathematical expressions (i.e., physical laws). Thus, for example, in the case of a static point charge ensemble, the electric field at position $\bar{q}'$ is given by

$$\bar{E}(\bar{q}') = \frac{1}{4\pi\epsilon_o} \sum_i \frac{Q_i(\bar{q}' - \bar{q}_i)}{|\bar{q}' - \bar{q}_i|^3} \tag{2.2-1}$$

where $\bar{q}_i$ and Q_i are the position and charge of particle i, and the summation is over the charges which produce a field at $\bar{q}'$. Similarly, the gravitational potential $P(\bar{q}')$, due to a point mass ensemble, is given by

$$P(\bar{q}') = -G \sum_i \frac{m_i}{|\bar{q}' - \bar{q}_i|} \tag{2.2-2}$$

where m_i is the mass of particle i.

When a continuous charge density $\rho_1(\bar{q})$ is introduced to represent the charge ensemble, defined by

$$\rho_1(\bar{q}) = \lim_{\Delta\tau_q \to 0} \frac{\sum_i Q_i}{\Delta\tau_q} \tag{2.2-3}$$

where $\Delta\tau_q$ is a volume element in position space containing $\bar{q}$, and the summation is over the particles in the element $\Delta\tau_q$, Eq. (2.2-1) becomes

$$\bar{E}(\bar{q}') = \frac{1}{4\pi\epsilon_o} \int \frac{\rho_1(\bar{q})[\bar{q}' - \bar{q}_1]}{|\bar{q}' - \bar{q}_i|^3} d\tau_q \tag{2.2-4}$$

Similarly, introducing $\rho_2(\bar{q})$ to represent the mass distribution, (2.2-2) becomes

$$P(\bar{q}') = -G \int \frac{\rho_2(\bar{q})\, d\tau_q}{|\bar{q}' - \bar{q}_i|} \tag{2.2-5}$$

In contrast, in contributing to the value of video voltage at the output of the receiver at some instant, each scatterer radar cross-section enters as the magnitude of a complex number associated with the scatterer. The phase angle of the complex number is determined by the electric phase (relative to some reference) of the carrier associated with the echo received from the scatterer. Expressions for received video voltages, in terms of the scatterer cross-sections and the received echo phases, will be introduced in Chapter 5.

In the previous section, the density $\rho_p(\bar{p},t)$ was defined in terms of the scatterer radar cross-sections only, without consideration of the echo phases. Thus, the phenomena (the video voltages) cannot be expressed in terms of the density $\rho_p(\bar{p},t)$ in a way analogous to (2.2-4) and (2.2-5).

In summary, the received video voltages cannot be expressed in terms of $\rho_p(\bar{p},t)$ due to the randomness of scatterer cross-section, and the dependence of the voltages on the received echo phases. However, in Chapter 5 we will show that certain significant statistics of the voltages can be expressed in terms of $\rho_p(\bar{p},t)$. It will be necessary to use the assumption there that the received echo phases for the scatterers are independent random variables, each uniformly distributed in angle. This assumption is justified in the following discussion.

The received echo phase goes through a complete cycle when the range of a scatterer changes by a distance equal to half the radar wavelength. The radar wavelength is generally much less than the range resolution of the radar.

Furthermore, the scatterers are assumed to be dynamically independent. Thus, considering a set of scatterers in an interrogation cell, the echoes from which determine the receiver output voltages at some instant, it is not unreasonable to suppose that the corresponding set of echo phases are uniformly distributed in angle. Based on these arguments, the consideration of the received echo phase for each scatterer to be a random variable, uniformly distributed in angle, is natural. Certainly, we are justified in assuming that the echo phases for any two scatterers are independent.

Let us not overlook, however, the fact that the position of each scatterer is specified, in our formulation, by its dynamic state vector $\overline{p}$. Thus, this vector determines the range, and hence the phase of the received echo voltages for each scatterer, at least to within a constant which is the same for all scatters. (Obviously, only the relative phases of the received scattered echoes are of significance in any case.) The dynamic state vector of a scatterer is not introduced as a random variable in the formulation. If it were, the function $\rho_p(\overline{p}, t)$ would not be deterministic as defined in (2.1-2), since we could only speak of the probability of a scatterer being present in the element $\Delta\tau_p$.

A theory can be formulated in which the dynamic state vectors for the scatterers are considered to be random variables. In place of (2.1-2) we would define

$$\rho_p(\overline{p}, t) = \lim_{\Delta\tau_p \to 0} \frac{\mathcal{E}\{\sum_i s_i^2\}}{\Delta\tau_p} \tag{2.2-6}$$

where the expectation is taken over the ensemble of possible scatterer configurations, and the summation is over the scatterers in the element $\Delta\tau_p$ for each configuration. However, this approach obscures the basic motivation behind the theory. That is, our interest is in estimating (from radar data) and predicting properties of the particular scatterer ensemble which is being observed. The properties estimated and predicted are the ones intended to characterize the actual configuration of the scatterers being observed, not the average configuration of a statistical ensemble of scatterer configurations which could have been observed.

Thus the dynamic state vector of each scatterer is introduced in the theory as a deterministic, though unknown quantity. However, the dynamic state of a scatterer cannot be observed, since the scatterers are unresolvable. Furthermore, the analysis does not consider estimation and prediction of individual scatterer dynamic states, but only of overall properties of the scatterer ensemble. Finally, the effect on these properties of changes in scatterer positions by amounts of the order of the radar wavelength is negligble. Under these conditions, the "contradiction" between considering the dynamic state to be an unknown deterministic quantity and the received echo phase to be a random variable is surely only a formal one.

Considering the received echo phases to be random variables then, the dynamic independence of the scatterers, and the smallness of the wavelength compared to the range resolution justify the assumption that the relative echo phase between any two scatterers is uniformly distributed in angle. This, of course, implies that each scatterer echo phase, relative to the arbitrary reference, is uniformly distributed in angle. In contrast, if the range resolution were much smaller than a wavelength, the scatterers in an interrogation cell would all have nearly the same echo phase.

It remains to point out that certain remarks made in the discussion of random scatterer cross-section apply here as well. That is, the relative phase between two scatterers is a stochastic process whose value at an instant defines a random variable such as we have been considering. The random variables corresponding to two different times are not independent, since scatterer motion is deterministic. These considerations become significant when one investigates the correlation between data obtained as a result of observation of the scatterer ensemble at two different instants.

(d) Piecewise continuity of the density function.

The assumption that $\rho_p(\overline{p},t)$, defined by (2.1-2), is a piecewise continuous function of $\overline{p}$ is not without significance with regard to the applicability of the introduced scatterer ensemble characterization to actual physical situations. One can inquire as to what properties of a scatterer ensemble make the use of a piecewise continuous density model appropriate. The answer involves the notion of a volume element which is small compared to the dimensions of the scatterer ensemble but which, if placed within the region occupied by the ensemble, generally contains many scatterers. If one imagines the small volume element to be translated throughout the region, the value of the finite sum $\sum_i s_i^2$ over the scatterers in the volume element varies smoothly, in the sense of a piecewise continuous function. That is, abrupt changes in this empirical scatterer density are permitted, but only those of the type present in a piecewise continuous function.

Thus, the physical scatterer ensemble modelled cannot contain isolated scatterers with very large expected radar cross-section. For, if such a scatterer were present, the sum of expected radar cross-sections over a volume element containing the scatterer would be much larger than for neighboring volume elements. This would indicate the necessity of the use of an impulse in the density function introduced to characterize the scatterer ensemble.

The formulation could be generalized to include the presence of impulses in the density function. Much of the analysis and results would be unchanged. However, integrals such as (2.1-3) could not be considered to be Riemann integrals. Furthermore, the presence of isolated large scatterers would make a crucial assumption of the part of the formulation concerned with prediction somewhat forced. This will be discussed further in Chapter 3.

Note that by increasing the dimensions of the small volume element, we reduce the fraction of total expected radar cross-section in the element containing the large scatterer which is associated with this scatterer. Thus, the difference in total expected radar cross-section between this volume element and neighboring volume elements decreases. This illustrates that there is a relationship between permissible values of isolated large scatterers or similarly, permissible local variations in scatterer size and/or average separation, and the minimum dimensions of a volume element which can be considered small in a given physical situation.

2.3 Projections and moments of the density $f_p(\bar{p},t)$

The density $f_p(\bar{p},t)$ cannot be measured by means of a single observation of the observing radar. At best, the radar can only estimate the projection of this density, or properties of the projection, on a four dimensional subspace corresponding to position and doppler. We restrict ourselves to the case where the radar attempts only to estimate properties (i.e., certain moments) of the projection of the density $f_p(\bar{p},t)$ on the three dimensional subspace corresponding to position. The use of doppler measurements is not considered.

(a) Projections of $f_p(\bar{p},t)$ on linear subspaces.

The projection of the density $f_p(\bar{p},t)$ on a linear subspace is easily defined. Let $\bar{a}_i$, $i = 1, \ldots, r$, be an orthonormal basis for the linear subspace and let $\bar{a}_i$, $i = r+1, \ldots, 6$ be chosen so that the $\bar{a}_i$ are an orthonormal basis for dynamic state space. Denote by p_i the components of the dynamic state vector†, i.e.

$$p_i = \bar{p}^T \bar{a}_i \quad . \tag{2.3-1}$$

Finally, define the vectors

$$\bar{p}' = [p_1, \ldots, p_r]^T \tag{2.3-2}$$

$$\bar{p}'' = [p_{r+1}, \ldots, p_6]^T \quad . \tag{2.3-3}$$

Each vector $\bar{p}'$ specifies a point in the linear subspace. The projection

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† Coordinate representations of vectors are considered, throughout the thesis to be matrices with a single column. The superscript T is used to denote transposition.

of $f_p(\bar{p},t)$ on the subspace is defined by†

$$f_{p'}(\bar{p}',t) = \int f_p(\bar{p},t)\, dp_{r+1} \ldots dp_s \tag{2.3-4}$$

$$= \int f_p(\bar{p},t)\, d\tau_{p''} \ , \tag{2.3-5}$$

where $\bar{p}$ in the equations is given by

$$\bar{p} = [\bar{p}' \vdots \bar{p}'']^T \ . \tag{2.3-6}$$

The density $f_{p'}(\bar{p}',t)$ is at least a piecewise continuous function of $\bar{p}'$ and

$$\int f_{p'}(\bar{p}',t) d\tau_{p'} = \int f_p(\bar{p},t) d\tau_{p''}\, d\tau_{p'} = 1 \ . \tag{2.3-7}$$

This density characterizes the distribution of the scatterer ensemble in the subspace of the vectors $\bar{p}'$ in the same way that $f_p(\bar{p},t)$ characterizes the distribution in dynamic state space. That is

$$f_{p'}(\bar{p}',t) = \frac{1}{T(t)} \lim_{\Delta\tau_{p'}\to 0} \frac{\sum_i s_i^2}{\Delta\tau_{p'}} \tag{2.3-8}$$

where T(t) is the total radar cross-section of the scatterer ensemble, and the summation is over the scatterers contained in the element $\Delta\tau_{p'}$.

(b) The position subspace.

Of particular interest is the projection of $f_p(\bar{p},t)$ on the position subspace. We denote the position vector by $\bar{q}$ and the velocity vector by $\bar{u}$, and partition the

† We use the notation $f_s(\cdot\,,t)$ or, if the time dependence is irrelevent, $f_s(\cdot)$ to denote the projection of $f_p(\bar{p},t)$ on the space of vectors (in this case s) indicated by the subscript. Similarly, $d\tau_s$ denotes the volume element in the space.

dynamic state vector $\overline{p}$ according to

$$\overline{p} = [\overline{q} : \overline{u}]^T \quad . \tag{2.3-9}$$

Then

$$f_q(\overline{q},t) = \int f_p(\overline{p},t)\, d\tau_u \quad , \tag{2.3-10}$$

where $\overline{p}$ in (2.3-10) is given by (2.3-9).

The density $f_q(\overline{q},t)$ characterizes the distribution of the scatterer ensemble in position space at time t, in the sense of Eq. (2.3-8).

(c) Moments of $f_p(\overline{p},t)$ and $f_q(\overline{q},t)$.

In Chapter 5, the estimation of moments of $f_q(\overline{q},t)$ from data acquired by a wide-beam monopulse radar or narrow-beam radar will be considered. The moments discussed are the elements of the centroid $\overline{\eta}_q(t)$ and second central moment (SCM) matrix $\Phi_q(t)$ defined by

$$\overline{\eta}_q(t) = \int \overline{q} f_q(\overline{q},t)\, d\tau_q \tag{2.3-11}$$

$$\Phi_q(t) = \int [\overline{q}-\overline{\eta}_q(t)][\overline{q}-\overline{\eta}_q(t)]^T f_q(\overline{q},t) d\tau_q \tag{2.3-12}$$

In addition to being the quantities estimated from radar data at each instant of observation, the centroid and SCM matrix are the quantities whose prediction at time t_{N+1}, based on observations at instants t_i, $i = 1, \ldots, N$, is used to characterize the location in space of the scatterer ensemble at t_{N+1}.

The centroid vector represents an average location of the scatter ensemble. The SCM matrix characterizes the extent of the scatterer ensemble in all spatial directions. To be specific, consider the SCM of the density $f_q(\overline{q},t)$ in the direction of an arbitrary unit vector $\bar{a}$. Denote the SCM by $\mu_q^2(\bar{a},t)$. Then

$$\mu_q^2(\bar{a},t) \equiv \int \{\bar{a}^T[\bar{q}-\bar{\eta}_q(t)]\}^2 f_q(\bar{q},t)\, d\tau_q$$

$$= \int \{\bar{a}^T[\bar{q}-\bar{\eta}_q(t)][\bar{q}-\bar{\eta}_q(t)]^T\bar{a}\} f_q(\bar{q},t) d\tau_q$$

$$= \bar{a}^T\{\int[\bar{q}-\bar{\eta}_q(t)][\bar{q}-\bar{\eta}_q(t)]^T f_q(\bar{q},t) d\tau_q\}\bar{a}$$

$$= \bar{a}^T \Phi_q(t)\bar{a} \quad . \tag{2.3-13}$$

Thus, the SCM matrix determines the SCM of the density in all spatial directions.

The SCM matrix is symmetric, since

$$\Phi_q^T(t) = \{\int[\bar{q}-\bar{\eta}_q(t)][\bar{q}-\bar{\eta}_q(t)]^T f_q(\bar{q},t) d\tau_q\}^T$$

$$= \int \{[\bar{q}-\bar{\eta}_q(t)][\bar{q}-\bar{\eta}_q(t)]^T\}^T f_q(\bar{q},t) d\tau_q$$

$$= \int [\bar{q}-\bar{\eta}_q(t)][\bar{q}-\bar{\eta}_q(t)]^T f_q(\bar{q},t)\, d\tau_q$$

$$= \Phi_q(t) \tag{2.3-14}$$

Thus, only six of the elements of $\Phi_q(t)$ are distinct, and the number of estimated and predicted parameters, including the elements of the centroid vector, is nine in total.

Although the centroid and SCM matrix are the quantities observed (i.e., estimated from radar data) and predicted, it will be seen in Chapter 3 that the elements of $\bar{\eta}_q(t)$ and $\Phi_q(t)$ are not sufficient, in general, to serve as a set of state variables for the scatterer ensemble. However, it will be demonstrated there that the elements of the quantities $\bar{\eta}_p(t)$, $\Phi_p(t)$, defined by

$$\bar{\eta}_p(t) = \int \bar{p} f_p(\bar{p}, t)\, d\tau_p \tag{2.3-15}$$

$$\Phi_p(t) = \int [\bar{p} - \bar{\eta}_p(t)][\bar{p} - \bar{\eta}_p(t)]^T f_p(\bar{p}, t)\, d\tau_p \tag{2.3-16}$$

can, under general conditions, be used as such a set. It will be shown that these state variables satisfy certain time-varying linear equations.

The quantities $\bar{\eta}_p(t)$ and $\Phi_p(t)$ do not have an obvious geometrical interpretation, as in the case of $\bar{\eta}_q(t)$ and $\Phi_q(t)$. However, if the dynamic state vector $\bar{p}$ is assumed to be partitioned as in (2.3-9) some geometrical meaning can be associated with the quantities. First consider $\bar{\eta}_p(t)$: substituting (2.3-9) into (2.3-15) the equations

$$\bar{\eta}_p(t) = \int [\bar{q} \,\vdots\, \bar{u}]^T f_p(\bar{p}, t)\, d\tau_p \tag{2.3-17}$$

$$= \begin{bmatrix} \int \bar{q} f_p(\bar{p}, t)\, d\tau_p \\ \hdashline \int \bar{u} f_p(\bar{p}, t)\, d\tau_p \end{bmatrix} \tag{2.3-18}$$

are obtained. It is to be understood that $\bar{p}$ in (2.3-17) and (2.3-18) is given by (2.3-9). But in (2.3-18), the upper integral can be reduced, using the definition (2.3-10), as shown:

$$\begin{aligned} \int \bar{q} f_p(\bar{p}, t)\, d\tau_p &= \int \bar{q} f_p(\bar{p}, t)\, d\tau_u\, d\tau_q \\ &= \int \bar{q} f_q(\bar{q}, t)\, d\tau_q \\ &= \bar{\eta}_q(t) \end{aligned} \tag{2.3-19}$$

Similarly, for the lower integral in (2.3-18), the equations

$$\int \bar{u}\, f_p(\bar{p},t)\, d\tau_p = \int \bar{u}\, f_p(\bar{p},t)\, d\tau_q\, d\tau_u$$

$$= \int \bar{u}\, f_u(\bar{u},t)\, d\tau_u \qquad (2.3\text{-}20)$$

are obtained, where $f_u(\bar{u},t)$ is the projection of $f_p(\bar{p},t)$ on the velocity subspace. Denoting

$$\bar{\eta}_u(t) = \int \bar{u}\, f_u(\bar{u}, t)\, d\tau_u \qquad (2.3\text{-}21)$$

Eq. (2.3-18) reduces to

$$\bar{\eta}_p(t) = [\bar{\eta}_q(t) \,\vdots\, \bar{\eta}_u(t)]^T \; . \qquad (2.3\text{-}22)$$

Just as $\bar{\eta}_q(t)$ represents an average scatterer position, $\bar{\eta}_u(t)$ represents an average scatter velocity.

Now consider $\Phi_p(t)$, proceeding as above. The partitions (2.3-9) and (2.3-22), applied to the matrix $[\bar{p}-\bar{\eta}_p(t)]\,[\bar{p}-\bar{\eta}_p(t)]^T$, result in

$$[\bar{p}-\bar{\eta}_p(t)]\,[\bar{p}-\bar{\eta}_p(t)]^T = \begin{bmatrix} \bar{q}-\bar{\eta}_q(t) \\ \hline \bar{u}-\bar{\eta}_u(t) \end{bmatrix} \begin{bmatrix} \bar{q}-\bar{\eta}_q(t) \\ \hline \bar{u}-\bar{\eta}_u(t) \end{bmatrix}^T \qquad (2.3\text{-}23)$$

Furthermore, partitioning Φ_p according to

$$\Phi_p = \left[\begin{array}{c|c} \Phi_{11} & \Phi_{12} \\ \hline \Phi_{21} & \Phi_{22} \end{array}\right] \qquad (2.3\text{-}24)$$

and substituting (2.3-23) and (2.3-24) into (2.3-16) the equation

$$\Phi_{11}(t) = \int [\bar{q}-\bar{\eta}_q(t)]\,[\bar{q}-\bar{\eta}_q(t)]^T f_p(\bar{p}, t)\, d\tau_p \qquad (2.3\text{-}25)$$

results. Using (2.3-10) as before, (2.3-25) reduces to

$$\Phi_{11}(t) = \int [\bar{q}-\bar{\eta}_q(t)]\,[\bar{q}-\bar{\eta}_q(t)]^T f_q(\bar{q}, t)\, d\tau_q$$

$$= \Phi_q(t) \qquad (2.3\text{-}26)$$

Similarly

$$\Phi_{22}(t) = \int [\bar{u} - \bar{\eta}_u(t)] [\bar{u} - \bar{\eta}_u(t)]^T f_p(\bar{p}, t)\, d\tau_p$$

$$= \int [\bar{u} - \bar{\eta}_u(t)] [\bar{u} - \bar{\eta}_u(t)]^T f_u(\bar{u}, t)\, d\tau_u \qquad (2.3\text{-}27)$$

The quantity expressed by (2.3-27), which can be denoted $\Phi_u(t)$, is analogous to $\Phi_q(t)$. Specifically, $\Phi_u(t)$ indicates the velocity spread of the scatterer ensemble in all spatial directions. That is, if $\bar{a}$ is an arbitrary unit vector, the SCM of $f_u(\bar{u}, t)$ in the direction of $\bar{a}$, denoted $\mu_u(\bar{a}, t)$, is given by

$$\mu_u(\bar{a}, t) = \int \{ \bar{a}^T [\bar{u} - \bar{\eta}_u(t)] \}^2 f_u(\bar{u}, t) d\tau_u$$

$$= \bar{a}^T \Phi_u(t) \bar{a} \qquad (2.3\text{-}28)$$

The derivation of (2.3-28) follows the steps leading to (2.3-13).

Thus, in the partition (2.3-24) of Φ_p, two of the submatrices, Φ_{11} and Φ_{22} have been identified as the matrices Φ_q and Φ_u respectively, with corresponding geometrical interpretations, provided that $\bar{p}$ is partitioned according to (2.3-9). The submatrices Φ_{12} and Φ_{21} are given, proceeding as before, by

$$\Phi_{12}(t) = \int [\bar{q} - \bar{\eta}_q(t)] [\bar{u} - \bar{\eta}_u(t)]^T f_p(\bar{p}, t) d\tau_p \qquad (2.3\text{-}29)$$

$$\Phi_{21}(t) = \int [\bar{u} - \bar{\eta}_u(t)] [\bar{q} - \bar{\eta}_q(t)]^T f_p(\bar{p}, t) d\tau_p \qquad (2.3\text{-}30)$$

$$= \Phi_{12}^T(t) \qquad (2.3\text{-}31)$$

These matrices contain joint central moments in position and velocity coordinates. No significant geometrical interpretation of these submatrices has been identified.

Introducing the [3 × 6] matrix M, given by

$$M = [\, I \vdots 0 \,] \tag{2.3-32}$$

the relationships between the vectors $\overline{\eta}_q(t)$ and $\overline{\eta}_p(t)$, and between the matrices $\Phi_q(t)$ and $\Phi_p(t)$ can be expressed as

$$\overline{\eta}_q(t) = M\, \overline{\eta}_p(t) \tag{2.3-33}$$

$$\Phi_q(t) = M\, \Phi_p(t)\, M^T \tag{2.3-34}$$

These expressions will be used in the following chapters.

2.4 The statement of the estimation and prediction problems

In terms of the new characterization of scatterer ensembles introduced in the previous sections, we can now state our problem in a formal manner as follows:

(1) given radar data at some instant t_i, estimate the quantities $\overline{\eta}_q(t_i)$, $\Phi_q(t_i)$.

(2) given observations $\hat{\overline{\eta}}_q(t_i)$, $\hat{\Phi}_q(t_i)$, $i = 1, \ldots, N$, predict $\overline{\eta}_q(t_{N+1})$, $\Phi_q(t_{N+1})$, in accordance with state equations derived for $\overline{\eta}_p(t)$, $\Phi_p(t)$. The observations $\hat{\overline{\eta}}_q(t_i)$, $\hat{\Phi}_q(t_i)$ (resulting from application of appropriate estimators to the radar data) contain observation errors due to random scatterer cross-section and received echo phase, as well as the usual observation noise.

These problems are treated in the ensuing chapters. In the second problem, note that the relationship between $\overline{\eta}_q(t)$, $\Phi_q(t)$ and $\overline{\eta}_p(t)$, $\Phi_p(t)$ discussed in sub-section 2.3(c) is crucial. There must, of course, be a known relation between the observed quantities and the state variables of a system in order for prediction to be possible. In the system under study, the observed quantities $\overline{\eta}_q(t)$ and $\Phi_q(t)$ are submatrices of the quantities $\overline{\eta}_p(t)$ and $\Phi_p(t)$, whose elements constitute the state variables of the system, and the required relations are given by (2.3-33) and (2.3-34).

CHAPTER III

STATE VARIABLE REPRESENTATION OF A DYNAMIC SCATTERER ENSEMBLE

In this chapter, the representation of the point-mass dynamics of the scatterers, and related discussions on linear dynamics and linearization of non-linear dynamics are presented. Then the transformation with time of $\bar{\eta}_p(t)$ and $\Phi_p(t)$ is investigated, and it is shown that a unique transformation can be defined under certain general conditions. A crucial "stationarity assumption", regarding certain statistics associated with the scatterer radar cross-sections, is discussed in detail. Then , the scatterer ensemble is formally identified as an abstract object or system, with the elements of $\bar{\eta}_q(t)$ and $\Phi_q(t)$ the output variables, and the elements of $\bar{\eta}_p(t)$ and $\Phi_p(t)$ the state variables. Input-output-state relations for the system are derived and it is demonstrated that they have the response separation property. Finally, the reducibility of the derived state representation of the system is investigated.

3.1 Point-mass dynamics of the scatterers

(a) General point-mass dynamics

It is assumed that the scatterers share a common point-mass dynamics. The laws governing the dynamics are expressed here in an equation of the form

$$\bar{p}(t_i) = \bar{g}(\bar{p}(t_j), t_j, t_i) \quad . \tag{3.1-1}$$

The function $\bar{g}$ specifies the dynamic state $\bar{p}(t_i)$ of a point-mass at the instant t_i, given the dynamic state $\bar{p}(t_j)$ at the instant t_j. The equation (3.1-1) can be considered to result from the integration of differential equations of motion, if the dynamic laws are given in such a form, as is often the case. The function $\bar{g}$ is assumed to be continuous in its arguments. Furthermore, it is assumed that

$$\bar{p}' = \bar{g}(\bar{p}, t_j, t_i) \tag{3.1-2}$$

implies

$$\bar{p} = \bar{g}(\bar{p}', t_i, t_j) \tag{3.1-3}$$

for every $\bar{p}$, t_j and t_i.

The latter assumption implies that $\bar{g}(\cdot, t_j, t_i)$ specifies a one to one (reversible) mapping of dynamic state vectors at time t_j onto the corresponding state vectors at time t_i. In fact, as seen from (3.1-2) and (3.1-3), the inverse mapping is given by $\bar{g}(\cdot, t_i, t_j)$. Finally, since $\bar{g}(\bar{p}, t_j, t_i)$ can be considered to be defined for all dynamic state vectors $\bar{p}$, it is said that (3.1-2) defines a continuous one to one mapping of dynamic state space at time t_j onto dynamic state space at time t_i.

(b) Linear point-mass dynamics.

Of special interest is the case where $\bar{g}$ in (3.1-1) is linear in the variables

$\overline{p}(t_j)$, or the case where, to an acceptable approximation, the function $\overline{g}$ can be linearized about a reference trajectory. Let $\overline{p}_1(t)$ denote the reference trajectory and define

$$\Delta_1\overline{p}(t) = \overline{p}(t) - \overline{p}_1(t) \quad . \tag{3.1-4}$$

The dynamics can then be expressed in the form

$$\Delta_1\overline{p}(t) = G(t, t_o)\,\Delta_1\overline{p}(t_o) \tag{3.1-5}$$

where $G(t, t_o)$ is a $[6 \times 6]$ matrix. This equation specifies the dynamic state of a point mass at time t, relative to the dynamic state of the reference trajectory at time t, given the corresponding quantity at time t_o. Substituting (3.1-4) into (3.1-5) results in the equation

$$\overline{p}(t) = \overline{p}_1(t) + G(t, t_o)\,[\overline{p}(t_o) - \overline{p}_1(t_o)] \tag{3.1-6}$$

which is of the form of Eq. (3.1-1). As a consequence of the assumptions imposed on the continuity and reversibility of a function (3.1-1) which represents the point-mass dynamics, the matrix $G(t, t_o)$ in (3.1-6) is necessarily a continuous function of t and t_o (i.e., each element is a continuous function of t and t_o) and is non-singular for all t and t_o.

As an example of a case where $\overline{g}$ in (3.1-1) is linear in the variables $\overline{p}(t_j)$, consider dynamics specified by the equation

$$\overline{q}(t) = \overline{q}(t_o) + \overline{u}(t_o)\,[t - t_o] - \tfrac{1}{2}\overline{a}[t - t_o]^2 \tag{3.1-7}$$

representing motion in a field of uniform acceleration $\overline{a}$. Differentiating (3.1-7) with respect to t, the expression for the velocity

$$\overline{u}(t) = \overline{u}(t_o) - \overline{a}[t - t_o] \tag{3.1-8}$$

is obtained. For the reference trajectory, the equations

$$\overline{q}_1(t) = \overline{q}_1(t_o) + \overline{u}_1(t_o)[t - t_o] - \tfrac{1}{2}\overline{a}[t - t_o]^2 \tag{3.1-9}$$

$$\overline{u}_1(t) = \overline{u}_1(t_o) - \overline{a}[t - t_o] \tag{3.1-10}$$

hold. Subtracting (3.1-9) from (3.1-7) and (3.1-10) from (3.1-8) results in

$$\overline{q}(t) - \overline{q}_1(t) = \overline{q}(t_o) - \overline{q}_1(t_o) + [\overline{u}(t_o) - \overline{u}_1(t_o)][t - t_o] \tag{3.1-11}$$

$$\overline{u}(t) - \overline{u}_1(t) = \overline{u}(t_o) - \overline{u}_1(t_o) \ . \tag{3.1-12}$$

Finally, assuming the partition

$$\overline{p} = [\overline{q}, \overline{u}]^T \tag{3.1-13}$$

for the dynamic state vector, comparison of (3.1-5) with (3.1-11) and (3.1-12) yields

$$G(t, t_o) = \left[\begin{array}{c|c} I & [t - t_o]I \\ \hline 0 & I \end{array}\right] \tag{3.1-14}$$

where I is the identity matrix of order three.

Note that the matrix $G(t, t_o)$ in (3.1-14) is independent of the reference trajectory $\overline{p}_1(t)$. In fact, it is true in general that, if the point-mass dynamics are given exactly by (3.1-5) for some reference trajectory $\overline{p}_1(t)$, the elements of $G(t, t_o)$ are independent of the reference trajectory chosen. To show this, let $\overline{p}_2(t)$ be an arbitrary trajectory and define

$$\Delta_2\overline{p}(t) = \overline{p}(t) - \overline{p}_2(t) \tag{3.1-15}$$

Then, using (3.1-4) and (3.1-5)

$$\Delta_2\overline{p}(t) = \overline{p}(t) - \overline{p}_2(t)$$

$$= \overline{p}(t) - \overline{p}_1(t) - [\overline{p}_2(t) - \overline{p}_1(t)]$$

$$= G(t, t_o)[\overline{p}(t_o) - \overline{p}_1(t_o)] - G(t, t_o)[\overline{p}_2(t_o) - \overline{p}_1(t_o)]$$

$$= G(t, t_o)[\overline{p}(t_o) - \overline{p}_2(t_o)]$$

$$= G(t, t_o)\Delta_2\overline{p}(t_o) \qquad (3.1\text{-}16)$$

Thus $G(t, t_o)$ is independent of the reference trajectory. The implication of this result is that, if a linearization of non-linear dynamics is accurate, the matrix $G(t, t_o)$ will be a weak function of the reference trajectory.

(c) Linearization of nonlinear dynamics

If $\overline{g}$ in (3.1-1) is not linear in the variables $\overline{p}(t_j)$, an equation of the form (3.1-5) can still be used to describe an approximation to the point-mass dynamics. The approximation can be represented as the equations of first variation with respect to a reference trajectory, $\overline{p}_1(t)$, by expanding the function $\overline{g}$ about the point $(\overline{p}_1(t_o), t_o, t)$ in Taylor series, retaining terms up to first order in $\Delta_1\overline{p}(t_o)$ only. This operation yields

$$\overline{g}(\overline{p}(t_o), t_o, t) = \overline{g}(\overline{p}_1(t_o) + \Delta_1\overline{p}(t_o), t_o, t)$$

$$\approx \overline{g}(\overline{p}_1(t_o), t_o, t) + [G_{mn}]\Delta_1\overline{p}(t) \qquad (3,1\text{-}17)$$

where

$$G_{mn} = \frac{\partial g_m}{\partial p_n} \qquad m, n = 1, \ldots, 6 \qquad (3.1\text{-}18)$$

In (3.1-18), g_m and p_n are components of $\overline{g}(\overline{p}, t_o, t)$ and $\overline{p}$, respectively, and the partial derivatives are evaluated at the point $(\overline{p}_1(t_o), t_o, t)$. Finally, from (3.1-1) and (3,1-17), we obtain

$$\Delta_1 \overline{p}(t) = \overline{p}(t) - \overline{p}_1(t)$$
$$= \overline{g}(\overline{p}(t_o), t_o, t) - \overline{g}(\overline{p}_1(t_o), t_o, t)$$
$$\approx [G_{mn}] \Delta_1 \overline{p}(t) \qquad (3.1\text{-}19)$$

which is an equation of the form (3.1-5), with

$$G(t, t_o) = [G_{mn}] \ . \qquad (3.1\text{-}20)$$

In this case, note that the elements of $G(t, t_o)$ are not independent of the reference trajectory $\overline{p}_1(t)$, as was shown to be true in the case of linear dynamics. The dependence on the reference trajectory arises from the requirement that (3.1-18) be evaluated at $(\overline{p}_1(t_o), t_o, t)$.

In some cases of non-linear dynamics, it is possible that closed form expressions for the elements of $G(t, t_o)$ can be obtained. A partial illustration is contained in the work by Frick [28] in which the elements of the submatrix G_3 in the partition

$$G(t, t_o) = \left[\begin{array}{c|c} G_1 & G_3 \\ \hline G_2 & G_4 \end{array}\right] \qquad (3.1\text{-}21)$$

are evaluated. The dynamics are assumed to be specified by Kepler's laws and perturbations in velocity of a reference trajectory are considered.

Equation (3.1-18) makes evident a numerical method for evaluating the elements G_{mn}. In the method, the values of the function $\overline{g}(\overline{p}, t_o, t)$ are derived (by numerical integration of appropriate differential equations, for example) for $\overline{p} = \overline{p}_1(t_o)$ and for $\overline{p} = \overline{p}_1(t_o) + \Delta p_n \overline{a}_n$, $n = 1, \ldots, 6$, where the $\overline{a}_n$ are an orthonormal basis for dynamic state space and the Δp_n are small perturbations in the elements of the dynamic state vector $\overline{p}_1(t_o)$. Let $\Delta \overline{g}_n = \overline{g}(\overline{p}_1(t_o) + \Delta p_n \overline{a}_n, t_o, t) - \overline{g}(\overline{p}_1(t_o), t_o, t)$. Then

$$(\Delta \overline{g}_n)_m \approx \frac{\partial g_m}{\partial p_n} \Delta p_n \qquad m, n = 1, \ldots, 6 \qquad (3.1\text{-}22)$$

where $(\Delta \overline{g}_n)_m$ denotes component m of the vector $\Delta \overline{g}_n$. This numerical approach is especially efficient when the function $\overline{g}(\overline{p}, t_o, t)$ can be evaluated without integration, as is the case, for example, when the dynamics are assumed to obey Kepler's laws.

(d) Transformation of the dynamic state vector

The dynamic state vector of a point mass at instant t_i (or t) is given in terms of the dynamic state vector at instant t_j (or t_o) by (3.1-1) in general, or (3.1-4) and (3.1-5) if (3.1-1) is linear in $\overline{p}(t_j)$. In both these cases, the instants t_i and t_j, or t and t_o, are arbitrary. In contrast, when the description of the point mass dynamics, (3.1-4) and (3.1-5), results from a linearization of the dynamics, the instant t_o has a specific interpretation. It is the instant at which the perturbations to the reference trajectory are introduced. That is, in evaluating (3.1-18), we regard the value of t_o to be held fixed, with the elements G_{mn} evaluated as a function of t.

The motivation for considering t_o to be fixed arises from the method by which the elements G_{mn} are computed numerically. It is advantageous to select a value of t_o, and for perturbations $\Delta p_n \overline{a}_n$, n = 1, ..., 6 of the reference trajectory at time t_o, integrate to find the perturbed trajectories as functions of time t. Then, the coefficients are given by (3.1-22).

For any two times t_i and t_j, the transformation of the dynamic state vector from t_j to t_i is given by

$$\Delta_1\overline{p}(t_i) = G(t_i, t_o)\, \Delta_1\overline{p}(t_o)$$

$$= G(t_i, t_o)\, G^{-1}(t_j, t_o)\, \Delta_1\overline{p}(t_j)\ . \tag{3.1-23}$$

using (3.1-5). In (3.1-23), $G(t_i, t_o)$, $G(t_j, t_o)$ are computed as part of the numerical integration of the reference trajectory and the trajectories resulting from the perturbations $\Delta p_n\, \overline{a}_n$, $n = 1, \ldots, 6$ at time t_o.

Defining the matrix $F_p(t_i, t_j)$ by

$$F_p(t_i, t_j) = G(t_i, t_o)\, G^{-1}(t_j, t_o) \quad , \tag{3.1-24}$$

(3.1-23) can be represented as

$$\Delta_1\overline{p}(t_i) = F_p(t_i, t_j)\, \Delta_1\overline{p}(t_j) \quad . \tag{3.1-25}$$

It is clear that $F_p(t_i, t_j)$ is the dynamic state transition matrix. The dependence of the value of the matrix $F_p(t_i, t_j)$ on the value t_o used in linearization, as well as the dependence on the reference trajectory $\overline{p}_1(t)$ used, is not denoted, although this dependence exists.

If the dynamics are linear, and thus given exactly by (3.1-5), the matrix $F_p(t_i, t_j)$ defined by (3.1-24) is independent of t_o. To show this, consider that the equation

$$\Delta_1\overline{p}(t) = G(t, t_o')\, \Delta_1\overline{p}(t_o') \tag{3.1-26}$$

implies that

$$\Delta_1\overline{p}(t_i) = G(t_i, t_o')\, G^{-1}(t_j, t_o')\, \Delta_1\overline{p}(t_j) \tag{3.1-27}$$

and that, in the case of linear dynamics, (3.1-23) and (3.1-27) hold exactly for all values of $\Delta_1\overline{p}(t_j)$. This assures us that

$$G(t_i, t_o') \, G^{-1}(t_j, t_o') = G(t_i, t_o) \, G^{-1}(t_j, t_o) \tag{3.1-28}$$

which completes the proof.

This result emphasizes that a dependence of $F_p(t_i, t_j)$ on t_o arises as a consequence of non-linearity in the dynamic equations. If a linearization of non-linear dynamics is accurate, the dependence of $F_p(t_i, t_j)$ on t_o should be weak.

Finally, we note some well known properties of transition matrices, such as $F_p(t_i, t_j)$, for future reference. The first property is expressed by the relation

$$F_p^{-1}(t_j, t_i) = F_p(t_i, t_j) \tag{3.1-29}$$

and states that the transition matrices which transform the dynamic state vector between two instants are mutual inverses. The second property is that of transistivity, and is represented by the expression

$$F_p(t_i, t_j) = F_p(t_i, t) \, F_p(t, t_j) \quad . \tag{3.1-30}$$

Derivation of these properties is omitted.

3.2 Transformation of $\overline{\eta}_p(t)$ and $\Phi_p(t)$

In Chapter 2, the density function $\rho_p(\overline{p}, t)$ was introduced as the basic characterization of a scatterer ensemble. In the present analysis, two scatterer ensembles are considered to be identical if and only if the associated density functions are identical. On the other hand, in this chapter the elements of the vector of first moments $\overline{\eta}_p(t)$ of the density function, and the elements of matrix of second central moments $\Phi_p(t)$ are identified as state variables for a scatterer ensemble. It is obvious that it is possible for two non-identical scatterer ensembles to have the same state at an instant.

Scatterer ensembles for which $\overline{\eta}_p(t)$ and $\Phi_p(t)$ are equal at some instant are said to be equivalent at that instant. In order for it to be possible to use $\overline{\eta}_p(t)$ and $\Phi_p(t)$ to specify the state of the scatterer ensemble, it must be true that any two scatterer ensembles that are equivalent at an instant are equivalent at all instants, since the state of the scatterer ensemble at an instant must uniquely determine the state at any other instant. This requires that the transformation with time of $\overline{\eta}_p(t)$ and $\Phi_p(t)$ is independent of the density function (i.e., the same for all equivalent scatterer ensembles).

This section considers the transformation with time of the density function $\rho_p(\overline{p}, t)$ and its moments $\overline{\eta}_p(t)$ and $\Phi_p(t)$. Conditions under which the transformations of $\overline{\eta}_p(t)$ and $\Phi_p(t)$ are independent of $\rho_p(\overline{p}, t)$ are identified.

(a) Transformation of $\rho_p(\overline{p}, t)$ with time.

In order to derive the transformations of the moments $\overline{\eta}_p(t)$ and $\Phi_p(t)$ with time, it is necessary to consider first the time variation of the density

$\rho_p(\bar{p}, t)$. Later, it will be seen that, under certain conditions, the transformation of $\bar{\eta}_p(t)$ and $\Phi_p(t)$ does not depend on $\rho_p(\bar{p}, t)$.

Since $\rho_p(\bar{p}, t)$, defined by (2.1-2), is a deterministic function, and since the transformation with time of the dynamic state vector of each scatterer in the ensemble obeys a deterministic law, it is possible that $\rho_p(\bar{p}, t)$ transforms with time in some well-behaved way. In contrast, due to the random scatterer cross-sections and echo phases, measurements of properties of the ensemble which is described by $\rho_p(\bar{p}, t)$ will vary randomly from instant to instant.

The function $\rho_p(\bar{p}, t)$ is a piecewise continuous function of $\bar{p}$, but as yet nothing has been said about its properties with regard to the variable t. Referring back to the definition (2.1-2), it is evident that the behavior of $\rho_p(\bar{p}, t)$ with time depends on the properties of the random processes $\sigma_i^2(t)$, and in particular on the functions $s_i^2(t)$. If no restrictions are placed on the behavior of the functions $s_i^2(t)$, then clearly the transformation with time of $\rho_p(\bar{p}, t)$ will be undefined. Rather than restricting the behavior of the $s_i^2(t)$ directly, the following assumption is made which is fundamental to the present analysis, and will be referred to as the stationarity assumption. It is assumed that for the set of scatterers, denoted $\mathcal{a}$, contained in any element $d\tau_p$ of dynamic state space at some instant, the quantity $\mathcal{E}\sum_{i\in\mathcal{a}} \sigma_i^2(t) = \sum_{i\in\mathcal{a}} s_i^2(t)$ is time invariant. The significance of the stationarity assumption with regard to the application of our analytical approach to actual physical situations will be discussed in Section 3.3 below. In the present section, however, the transformation of the density function $\rho_p(\bar{p}, t)$ with time will be derived under the stationarity assumption.

Suppose that the point-mass dynamics are represented by (3.1-1). The mapping

$$\bar{p} = \bar{g}(\bar{p}', t_j, t_i) \tag{3.2-1}$$

transforms points in dynamic state space at time t_j onto the corresponding points at time t_i. Consider a volume element $d\tau_p'$ at point $\bar{p}'$ in dynamic state space at time t_j. Under the transformation (3.2-1), the volume element $d\tau_p'$ maps onto a volume element $d\tau_p$ at point $\bar{p}$ in dynamic state space, where $\bar{p}$ and $\bar{p}'$ are related by (3.2-1), and the volume of the elements are related by

$$d\tau_p = |J|\, d\tau_p' \quad . \tag{3.2-2}$$

The quantity $|J|$ is the Jacobean of the transformation (3.2-1), defined by

$$|J| = |\det[\, j_{mn}]| \tag{3.2-3}$$

where

$$j_{mn} = \frac{\partial g_m}{\partial p_n} \quad . \tag{3.2-4}$$

In (3.2-4), g_m is the m^{th} component of the vector function $\bar{g}$ and p_n is the n^{th} component of the dynamic state vector. The partial derivative in (3.2-4) is evaluated at the point $(\bar{p}', t_j, t_i)$.

The volume elements $d\tau_p'$ and $d\tau_p$ contain the same set of scatterers, which is denoted $\mathcal{A}$. By the stationarity assumption, we have

$$\sum_{i \in \mathcal{A}} s_i^2(t_i) = \sum_{i \in \mathcal{A}} s_i^2(t_j) \tag{3.2-5}$$

But the definition (2.1-2) implies

$$\sum_{i \in \mathcal{A}} s_i^2(t_j) = \rho_p(\bar{p}', t_j)\, d\tau_p' \tag{3.2-6}$$

$$\sum_{i \in \mathcal{A}} s_i^2(t_i) = \rho_p(\bar{p}, t_i)\, d\tau_p \qquad (3.2\text{-}7)$$

so that

$$\rho_p(\bar{p}, t_i)\, d\tau_p = \rho_p(\bar{p}', t_j)\, d\tau_{p'} \qquad (3.2\text{-}8)$$

Finally, using (3.2-2), the transformation

$$\rho_p(\bar{p}, t_i) = \frac{\rho_p(\bar{p}', t_j)}{|J|} \qquad (3.2\text{-}9)$$

is obtained. It should be noted that $|J|$ is a function of $\bar{p}'$, since in (3.2-4) the partial derivative is evaluated at $(\bar{p}', t_j, t_i)$. Also note that $\bar{p}$ and $\bar{p}'$ in (3.2-9) are related by (3.2-1).

Denoting the region occupied by the scatterer ensemble at time t_j by $\mathcal{V}'$, and the region occupied at time t_i by $\mathcal{V}$, the transformation (3.2-1) defines a one-to-one mapping of $\mathcal{V}'$ onto $\mathcal{V}$. The total radar cross-section of the scatterer ensemble at times t_j and t_i are given by

$$T(t_j) = \int_{\mathcal{V}'} \rho_p(\bar{p}', t_j)\, d\tau_p' \qquad (3.2\text{-}10)$$

$$T(t_i) = \int_{\mathcal{V}} \rho_p(\bar{p}, t_i)\, d\tau_p \qquad (3.2\text{-}11)$$

respectively. Then, as a consequence of (3.2-8), the equality

$$T(t_i) = T(t_j) \qquad (3.2\text{-}12)$$

is obtained. Thus, the total radar cross-section of the scatterer ensemble is time invariant. Finally, using the definition (2.1-4), and the transformations (3.2-9) and (3.2-12), the following relations are obtained:

$$f_p(\bar{p}', t_j) = \rho_p(\bar{p}', t_j)/T(t_j) \tag{3.2-13}$$

$$f_p(\bar{p}, t_i) = \rho_p(\bar{p}, t_i)/T(t_i) \tag{3.2-14}$$

$$= \frac{\rho_p(\bar{p}', t_j)}{|J|\ T(t_i)}$$

$$= \frac{f_p(\bar{p}', t_j)}{|J|} \frac{T(t_j)}{T(t_i)}$$

$$f_p(\bar{p}, t_i) = \frac{f_p(\bar{p}', t_j)}{|J|} \tag{3.2-15}$$

In (3.2-15), the vectors $\bar{p}$ and $\bar{p}'$ are related by (3.2-1), and $|J|$ is given by (3.2-3) and (3.2-4). Equation (3.2-15) defines the transformation with time of $f_p(\bar{p}, t)$.

Finally, consider the case of linear or linearized dynamics, with the time transformation of the dynamic state vector given by (3.1-25). The mapping (3.2-1) is replaced by

$$\bar{p} = \bar{p}_I(t_i) + F_p(t_i, t_j)\ [\bar{p}' - \bar{p}_I(t_j)] \tag{3.2-16}$$

which relates points in dynamic state space at time t_i to corresponding points at time t_j. Equations (3.2-2), (3.2-8), (3.2-9), (3.2-12), and (3.2-15), defining the time transformation of quantities of interest, remain unchanged. The Jacobian $|J|$ in this case is given by

$$|J| = |\det F_p(t_i, t_j)| \tag{3.2-17}$$

applying (3.2-3) and (3.2-4) to the right hand side of (3.2-16).

(b) Transformation of $\overline{\eta}_p(t)$ with time.

In this subsection, an expression for the vector $\overline{\eta}_p(t_i)$ is derived in terms of the density $f_p(\overline{p}, t_j)$, using the general expression (3.1-1) to represent the dynamics. Then, it is shown that in the case of linear, or linearized dynamics, a transformation between $\overline{\eta}_p(t_i)$ and $\overline{\eta}_p(t_j)$ can be defined in terms of the transition matrix $F_p(t_i, t_j)$, and that the transformation is independent of the density function $f_p(\overline{p}, t)$.

Substituting t_i and t_j for t in (2.3-15), and introducing the dummy variable $\overline{p}'$ the equations

$$\overline{\eta}_p(t_i) = \int \overline{p} f_p(\overline{p}, t_i)\, d\tau_p \tag{3.2-18}$$

$$\overline{\eta}_p(t_j) = \int \overline{p}' f_p(\overline{p}', t_j)\, d\tau'_p \tag{3.2-19}$$

are obtained. Assuming the dynamics are given by (3.1-1), and using the stationarity assumption, the mapping (3.2-1) results in Eqs. (3.2-8), (3.2-9), (3,2-12), and (3.2-15). Equations (3.2-8), (3.2-12) and the definition (2.1-4) imply that

$$f_p(\overline{p}, t_i)\, d\tau_p = f_p(\overline{p}', t_j)\, d\tau'_p \quad . \tag{3.2-20}$$

Now, let us transform the space of integration in (3.2-18) according to the mapping (3.2-1). But (3.2-1) is precisely the mapping which led to the equality (3.2-20), so that the equation

$$\overline{\eta}_p(t_i) = \int \overline{g}(\overline{p}', t_j, t_i)\, f_p(\overline{p}', t_j)\, d\tau'_p \tag{3.2-21}$$

results. This equation specifies the vector $\overline{\eta}_p(t_i)$ in terms of the density $f_p(\overline{p}', t_j)$ at another instant. However, in general the integral in (3.2-21) will not be simply related to the centroid $\overline{\eta}_p(t_j)$. In addition, since an infinite number of functions $f_p(\overline{p}', t_j)$ exist for which (3.2-19) will yield the same result, while the results given by (3.2-21) for this set of functions will not necessarily be unique, it is not to be expected that a transformation exists between the vectors $\overline{\eta}_p(t_i)$ and $\overline{\eta}_p(t_j)$. That is, $\overline{\eta}_p(t_j)$ does not uniquely determine $\overline{\eta}_p(t_i)$.

However, consider the results which are obtained when the dynamics are given by (3.1-25). Again the stationarity assumption is invoked, but the mapping (3.2-1) is replaced by (3.2-16). Again (3.2-20) is valid, and when the space of integration in (3.2-18) is transformed according to the mapping (3.2-16), the equation

$$\overline{\eta}_p(t_i) = \int \{\overline{p}_1(t_i) + F_p(t_i, t_j)[\overline{p}' - \overline{p}_1(t_j)]\} f_p(\overline{p}', t_j)\, d\tau_p'$$

results. With the use of (2.1-5), this equation is simplified as follows:

$$\begin{aligned}\overline{\eta}_p(t_i) &= \overline{p}_1(t_i) + \int F_p(t_i, t_j)[\overline{p}' - \overline{p}_1(t_j)] f_p(\overline{p}', t_j)\, d\tau_p' \\ &= \overline{p}_1(t_i) + F_p(t_i, t_j)\left[\int \overline{p}' f_p(\overline{p}', t_j)\, d\tau_p' - \overline{p}_1(t_j)\right] \\ &= \overline{p}_1(t_i) + F_p(t_i, t_j)[\overline{\eta}_p(t_j) - \overline{p}_1(t_j)] \qquad (3.2\text{-}22)\end{aligned}$$

Equation (3.2-22) defines a unique transformation between the vectors $\overline{\eta}_p(t_i)$ and $\overline{\eta}_p(t_j)$. It is to be noted that the assumption of linear dynamics, as well as the stationarity assumption, was necessary for the existance of the transformation. Finally, introducing the quantity

$$\Delta_1\overline{\eta}_p(t) = \overline{\eta}_p(t) - \overline{p}_1(t) \quad , \qquad (3.2\text{-}23)$$

representing the displacement of the vector $\overline{\eta}_p(t)$ from the reference trajectory, (3.2-22) becomes

$$\Delta_1 \overline{\eta}_p(t_i) = F_p(t_i, t_j)\, \Delta_1 \overline{\eta}_p(t_j) \quad . \tag{3.2-24}$$

Note the similarity between (3.2-24) and (3.1-25). The vector $\overline{\eta}_p(t)$ transforms with time in the same way as does the dynamic state vector. Partitioning

$$\overline{\eta}_p(t) = [\overline{\eta}_q(t) \vdots \overline{\eta}_u(t)]^T \tag{3.2-25}$$

as discussed in Subsection 2.3(c), it is seen that the centroid $\overline{\eta}_q(t)$ of the ensemble follows a point mass trajectory, and that the instantaneous velocity of this trajectory is $\overline{\eta}_u(t)$. In the previous discussion of Subsection 2.3(c), the quantities $\overline{\eta}_q(t)$ and $\overline{\eta}_u(t)$ were identified simply as the "average" scatterer position and velocity, respectively. Here it is apparent that

$$\frac{d}{dt}\, \overline{\eta}_q(t) = \overline{\eta}_u(t) \quad , \tag{3.2-26}$$

since this equation is valid for the position and velocity of a point mass.

(c) Transformation of $\Phi_p(t)$ with time.

Substituting t_i and t_j for t in (2.3-16), and introducing the dummy variable $\overline{p}'$, the equations

$$\Phi_p(t_i) = \int [\overline{p} - \overline{\eta}_p(t_i)][\overline{p} - \overline{\eta}_p(t_i)]^T f_p(\overline{p}, t_i)\, d\tau_p \tag{3.2-27}$$

$$\Phi_p(t_j) = \int [\overline{p}' - \overline{\eta}_p(t_j)][\overline{p}' - \overline{\eta}_p(t_j)]^T f_p(\overline{p}', t_j)\, d\tau_p' \tag{3.2-28}$$

are obtained. It is possible to derive an expression for $\Phi_p(t_i)$ in terms of the density $f_p(\overline{p}, t_j)$, as shown for $\overline{\eta}_p(t_i)$ in the previous subsection. As before, in the case of dynamics expressed by (3.1-1), there is in general no unique transformation between the matrices $\Phi_p(t_i)$ and $\Phi_p(t_j)$. Let us then consider the case of dynamics expressed by (3.1-25).

Now, the space of integration in (3.2-27) is to be transformed according to the mapping (3.2-16). Subtracting the vector $\overline{\eta}_p(t_i)$ from both sides of (3.2-16) results in

$$\overline{p}-\overline{\eta}_p(t_i) = \overline{p}_1(t_i) - \overline{\eta}_p(t_i) + F_p(t_i, t_j)[\overline{p}' - \overline{p}_1(t_j)] \quad . \tag{3.2-29}$$

Then, applying the equality (3.2-22), the equation

$$\begin{aligned} \overline{p}-\overline{\eta}_p(t_i) &= F_p(t_i, t_j)[\overline{p}_1(t_j) - \overline{\eta}_p(t_j)] + F_p(t_i, t_j)[\overline{p}' - \overline{p}_1(t_j)] \\ &= F_p(t_i, t_j)[\overline{p}' - \overline{\eta}_p(t_j)] \end{aligned} \tag{3.2-30}$$

is obtained. The mapping defined by (3.2-30) is equivalent to that defined by (3.2-16). Thus, with the use of (3.2-30) and (3.2-20), the transformation of the space of integration in (3.2-27) gives

$$\begin{aligned} \Phi_p(t_i) &= \int F_p(t_i, t_j)[\overline{p}' - \overline{\eta}_p(t_j)][\overline{p}' - \overline{\eta}_p(t_j)]^T F_p{}^T(t_i, t_j) f_p(\overline{p}', t_j) d\tau_p' \\ &= F_p(t_i, t_j)\{\int [\overline{p}' - \overline{\eta}_p(t_j)][\overline{p}' - \overline{\eta}_p(t_j)]^T f_p(\overline{p}', t_j) d\tau_p'\} \; F_p{}^T(t_i, t_j) \\ &= F_p(t_i, t_j)\Phi_p(t_j) F_p{}^T(t_i, t_j) \end{aligned} \tag{3.2-31}$$

Equation (3.2-31) defines a unique transformation between the matrices $\Phi_p(t_i)$ and $\Phi_p(t_j)$. As in the case of the transformation of the vector $\overline{\eta}_p(t)$, the existance of a unique transformation for $\Phi_p(t)$ requires the use of linear dynamics, as well as the stationarity assumption.

3.3 Physical implications of the stationarity assumption

The stationarity assumption has been invoked in connection with the derivation of the time transformation of the properties of the scatterer ensemble which are of interest. This assumption requires that for the set of scatterers, denoted $\mathcal{Q}$, contained in any element $d\tau_p$ of dynamic state space at some instant, the quantity $\mathcal{E}\sum_{i\in\mathcal{Q}} \sigma_i^2(t) = \sum_{i\in\mathcal{Q}} s_i^2(t)$ is time invariant. Note that this assumption does not require that $\mathcal{E}\{\sigma_i^2(t)\}$ is time invariant for each scatterer. However, $\mathcal{E}\{\sigma_i^2(t)\}$ time invariant for all scatterers implies, of course, that $\mathcal{E}\sum_i \sigma_i^2(t)$ is time invariant.

The statement of the stationarity assumption in terms of the sets of scatterers contained in volume elements $d\tau_p$, rather than in terms of the individual scatterers makes the formulation applicable to a broader class of actual physical situations. As an example, consider an ensemble of unsymmetrical tumbling scatterers in orbit above the atmosphere. Each scatterer rotates about an axis whose orientation remains fixed in inertial space as the scatterer traverses its trajectory. As time progresses, however, the aspect of each rotation axis with respect to the radar changes. Thus, the ensemble of possible orientations (relative to the radar) changes with time for each scatterer, and the quantity $\mathcal{E}\{\sigma_i^2\}$ will not be time invarient.

However, consider a large set of these scatterers, and suppose that there is no preferred orientation for the rotation axes of the scatterers. For this set, one might expect that the quantity $\mathcal{E}\sum_i \sigma_i^2(t)$ is nearly time-invarient, and that the quantity becomes less variable with time as the number of scatterers in the set increases.

To formalize these notions, let us consider the following sequence of experiments. In the N^{th} experiment, the orientation of the rotation axis of each of N identical scatterers is selected independently from a distribution in which all orientations are equally likely. Let $\mathcal{Z}$ denote the ensemble of possible combinations of N axis orientations, and $\mathcal{Z} = Z$ represent the set of axis orientations realized in the experiment. Then $\mathcal{E}\{\sigma_i^2(t)\,|\,\mathcal{Z}= Z\}$ is the expected value of radar cross-section of scatterer i, taken over the ensemble of possible orientations for the scatterer corresponding to the axis orientations specified by $\mathcal{Z} = Z$, each such possible scatterer orientation being equally likely.

Now, consider the random variable $\sum_i \sigma_i^2(t)$. Its conditional expectation is given by

$$\mathcal{E}\{\sum_i \sigma_i^2(t)\,|\,\mathcal{Z}= Z\} = \sum_i \mathcal{E}\{\sigma_i^2(t)\,|\,\mathcal{Z}= Z\} \qquad (3.3\text{-}1)$$

It is this expectation which is to be considered to appear in the definition (2.1-2) of the density function, since the orientations of the rotation axes of the scatterers remain fixed in inertial space (at their values $\mathcal{Z} = Z$) as time progresses, and the variation of scatterer orientation with time was a justification for considering $\sigma_i^2(t)$ to be a random variable.

The quantity $\mathcal{E}\{\sum_i \sigma_i^2(t)\,|\,\mathcal{Z}= Z\}$ changes with time as the radar aspect changes. As an indication of the possible variations which the quantity might undergo, let us consider the variations which the quantity experiences as Z is varied over the ensemble $\mathcal{Z}$, with the radar aspect considered to be fixed. Obviously the variations which the quantity undergoes as Z is varied over $\mathcal{Z}$ are greater than those that occur when the radar aspect is changed, since the ensemble of relative configurations (i.e., axis orientations relative to the radar aspect) attained in the latter case is a subset of those attained in the former case.

As a measure of the variation of $\mathcal{E}\{\sum_i \sigma_i^2(t) | \mathcal{Z} = Z\}$ as Z is varied over $\mathcal{Z}$, let us use the variance of the random variable $\mathcal{E}\{\sum_i \sigma_i^2(t) | \mathcal{Z}\}$. We show below that (in the sequence of experiments) as N approaches infinity, this variance goes to zero. This result is interpreted to indicate that $\mathcal{E}\{\sum_i \sigma_i^2(t) | \mathcal{Z} = Z\}$ does not vary appreciably with changing radar aspect, when the number of scatterers is large.

To prove this assertion, we invoke the identity[29]

$$\operatorname{var}\{Y\} = \mathcal{E}\{\operatorname{var}\{Y | X\}\} + \operatorname{var}\{\mathcal{E}\{Y|X\}\} \tag{3.3-2}$$

where X and Y are random variables. Since the first term in the right hand side of (3.3-2) must be positive, we have the inequality

$$\operatorname{var}\{Y\} \geq \operatorname{var}\{\mathcal{E}\{Y|X\}\} \quad . \tag{3.3-3}$$

Substituting $Y = \sum_i \sigma_i^2(t)$, $X = \mathcal{Z}$ in (3.3-3) results in

$$\operatorname{var}\{\sum_i \sigma_i^2(t)\} \geq \operatorname{var}\{\mathcal{E}\{\sum_i \sigma_i^2(t) | \mathcal{Z}\}\} \tag{3.3-4}$$

It is shown that, with assumptions introduced below, the left hand side of (3.3-4) approaches zero as N approaches infinity.

As the number of scatterers in the set increases, it is assumed that the size of the scatterers decrease so that the total scattering cross-section of the set stays the same. In particular, assume that

$$\mathcal{E}\{\sigma_i^2(t)\} = T/N \quad \text{for all i} \tag{3.3-5}$$

where T is some constant. The expectation in (3.3-5) is taken over all scatterer orientations, each being equally likely. Thus for any N, we have

$$\mathcal{E}\{\sum_i \sigma_i^2(t)\} = \sum_i \mathcal{E}\{\sigma_i^2(t)\} = T \tag{3.3-6}$$

It is also assumed that the shape of the probability density function for the random variable $\sigma_i^2(t)$ is similar for all N. That is, if $p_{N_1}(\sigma)$ and $p_{N_2}(\sigma)$ are the densities for two values of N,

$$p_{N_1}(\sigma) = K\, p_{N_2}(K\sigma) \tag{3.3-7}$$

Then

$$\mathcal{E}_{N_1}\{\sigma\} = \frac{1}{K}\mathcal{E}_{N_2}\{\sigma\} \tag{3.3-8}$$

$$\mathrm{var}_{N_1}\{\sigma\} = \frac{1}{K^2}\mathrm{var}_{N_2}\{\sigma\} \tag{3.3-9}$$

where $\mathcal{E}_{N_1}\{\sigma\}$ and $\mathcal{E}_{N_2}\{\sigma\}$ are the means of $p_{N_1}(\sigma)$ and $p_{N_2}(\sigma)$, and $\mathrm{var}_{N_1}(\sigma)$ and $\mathrm{var}_{N_2}(\sigma)$ are the variances of $p_{N_1}(\sigma)$ and $p_{N_2}(\sigma)$. From (3.3-5) and (3.3-8), we obtain

$$K = N_1/N_2 \tag{3.3-10}$$

and assuming that

$$\mathrm{var}_1\{\sigma\} = V \tag{3.3-11}$$

where V is some constant, the equation

$$\mathrm{var}_N\{\sigma\} = \frac{V}{N^2} \tag{3.3-12}$$

results from (3.3-9) and (3.3-10) with $N_2 = 1$, and $N_1 = N$.

Finally,

$$\mathrm{var}\{\sum_i \sigma_i^2(t)\} = \sum_{i=1}^{N} \mathrm{var}_N\{\sigma_i^2(t)\}$$

$$= \frac{V}{N} \tag{3.3-13}$$

which approaches zero as N approaches infinity. Referring back to (3.3-4), it is seen that for infinite N, the random variable $\mathcal{E}\{\sum_i \sigma_i^2(t)|\mathcal{Z}\}$ is equal, with probability one, to its expected value, given by $\mathcal{E}\{\sum_i \sigma_i^2(t)\} = T$. This, of course, implies that the quantity $\mathcal{E}\{\sum_i \sigma_i^2(t)|\mathcal{Z} = Z\}$ is equal, with probability one, to T

This completes the proof of the assertions which were made. Note that the approach used makes use of the limiting concept adopted throughout, whereby the set of scatterers in a volume element is allowed to approach infinity, while the sizes of the scatterers approach zero, such that the total radar cross-section of scatterers in the volume element remains constant.

Finally, it is noted that the formulation could be generalized, at the expense of added complexity, to the case where the scatterers are not all identical. The crucial concept is that as the number of scatterers in an element approaches infinity, the size of all the scatterers approach zero. If a finite scatterer is present, for which $\mathcal{E}\{\sigma_i^2(t)\}$ is time-varying, then obviously $\mathcal{E}\sum_i \sigma_i^2(t)$ will not be time-invariant. Such a scatterer would correspond to an impulse in the density function $\rho_p(\bar{p}, t)$. Thus, if impulses were present in $\rho_p(\bar{p}, t)$, it would have to be assumed that the associated scatterers were either symmetric, or moved so as to always present the same aspect to the radar, in order for the stationarity assumption to hold. The forced nature of these required assumptions is among the reasons that the presence of impulses in $\rho_p(\bar{p}, t)$ is excluded from the present formulation.

3.4 The scatterer ensemble as a formal system

The transformation equations

$$\bar{\eta}_p(t_i) = \bar{p}_1(t_i) + F_p(t_i, t_j)\,[\bar{\eta}_p(t_j) - \bar{p}_1(t_j)] \qquad (3.2\text{-}22)$$

$$\Phi_p(t_i) = F_p(t_i, t_j)\,\Phi_p(t_j)\,F_p^{\,T}(t_i, t_j) \qquad (3.2\text{-}31)$$

derived for $\bar{\eta}_p(t)$ and $\Phi_p(t)$ suggest that these quantities can be used to specify the state of the scatterer ensemble. This notion is made more precise here by means of the formal concepts of system theory as introduced by Zadeh and Desoer[11]. The primary objectives of the present section are the development of the concept of the scatterer ensemble as an oriented abstract object, and the identification of a set of state variables for the object which satisfy the consistency conditions. In addition, the formalism introduced here will enable us to examine certain questions concerning system reduction in the following section, and in Chapter 4.

(a) Definition of the abstract object.

The scatterer ensemble is a physical object with which we associate an abstract object whose terminal variables are considered to be the elements of the centroid vector $\bar{\eta}_q(t)$ and the independent elements of the SCM matrix $\Phi_q(t)$. These terminal variables are the quantities estimated by the data processor, at each instant of observation, on the basis of the data acquired by the radar. Thus, these variables are considered to be output variables of the abstract object. The abstract object has no input terminal variables, so the input segment space for the object can be assumed to consist of a single time function, and the input-output relation to consist of this single time function paired with each of the possible output functions $\bar{\eta}_q(t)$, $\Phi_q(t)$. If an input-output-state relation can be written for the object, then it is a member of the class of objects known as sources. The single time function in the input space is introduced only to place objects with no inputs into the framework of general oriented abstract objects, with which input-output relations are associated. The input time function need not appear explicitly (though it may) in the input-output-state relation.

Since the number of output variables in our present discussion is nine (i.e., three elements of $\bar{\eta}_q(t)$ and six independent elements of $\Phi_q(t)$), the output space, defined as the range of the output at an instant t, can be considered to be a subset of the space $\mathcal{R}^9$ of all real nine-tuples. This range is independent of t. The output segment space consists of the functions $\bar{\eta}_q(t)$, $\Phi_q(t)$ corresponding to the set of all possible scatterer ensembles which satisfy certain assumptions made in the preceding formulation (i.e., characterization by a piece-wise continuous density function $\rho_p(\bar{p}, t)$, the stationarity assumption, and common scatterer point-mass dynamics). The complete set of output functions is generated by considering all possible densities $f_p(\bar{p}, t_o)$ at some instant t_o, and the output functions corresponding to each density.

The problem of defining a state for the ensemble is stated loosely as that of parametrizing the output segment space of the object such that a label (the state of the object at time t_o) is associated uniquely with each possible output segment $\{\bar{\eta}_q(t), \Phi_q(t), t > t_o\}$ for every t_o. (Since our abstract object has no input terminal variables and a degenerate input segment space of a single function, mention of dependence on the input will be omitted.) However, in order to qualify for the state of the object, the parametrization must satisfy the set of four consistency conditions. Or alternatively, if an input-output-state relation which has the response separation property can be obtained for the object, the parametrization introduced by the input-output-state relation qualifies as the state of the object[27].

(b) Identification of state variables

Here, we derive input-output-state relations for the abstract object and show that they have the response separation property. The moments $\overline{\eta}_p(t)$ and $\Phi_p(t)$ of the density function $f_p(\overline{p}, t)$ are thereby identified as state variables for the scatterer ensemble. We begin by invoking the stationarity assumption, and assuming that the dynamics are given by (3.1-25); that is, the dynamics are either linear or linearized about a reference trajectory. It is maintained that, with these assumptions, the state of the scatterer ensemble at time t_o is given by the six components of the vector $\overline{\eta}_p(t_o)$ together with the 21 independent elements of the matrix $\Phi_p(t_o)$.

Combining Eqs. (2.3-33) and (3.2-22) yields

$$\overline{\eta}_q(t) = M\{\overline{p}_1(t) + F_p(t, t_o)\,[\overline{\eta}_p(t_o) - \overline{p}_1(t_o)]\} \tag{3.4-1}$$

while combining Eqs. (2.3-34) with (3.2-31) yields

$$\Phi_q(t) = M F_p(t, t_o)\,\Phi_p(t_o)\,F_p^{\,T}(t, t_o)\,M^T \tag{3.4-2}$$

where the matrix M is given by (2.3-32).

It is shown below that (3.4-1) and (3.4-2) constitute input-output-state relations for the abstract object under study, with the response separation property. That is, for all τ such that

$$t_o < \tau \leq t \tag{3.4-3}$$

we have

$$\overline{\eta}_q(t) = M\{\overline{p}_1(t) + F_p(t, \tau)[\overline{\eta}_p(\tau) - \overline{p}_1(\tau)]\} \tag{3.4-4}$$

$$\Phi_q(t) = M F_p(t, \tau)\Phi_p(\tau) F_p^T(t, \tau) M^T \tag{3.4-5}$$

where $\overline{\eta}_p(\tau)$ and $\Phi_p(\tau)$ depend only on $\overline{\eta}_p(t_o)$ and $\Phi_p(t_o)$.

To prove the assertion, we substitute $t_i = \tau$ and $t_j = t_o$ in (3.2-22) and (3.2-31), obtaining

$$\overline{\eta}_p(\tau) = \overline{p}_1(\tau) + F_p(\tau, t_o)[\overline{\eta}_p(t_o) - \overline{p}_1(t_o)] \tag{3.4-6}$$

$$\Phi_p(\tau) = F_p(\tau, t_o)\Phi_p(t_o) F_p^T(\tau, t_o) \tag{3.4-7}$$

Solving (3.4-6) for $[\overline{\eta}_p(t_o) - \overline{p}_1(t_o)]$ and substituting in (3.4-1) gives

$$\overline{\eta}_q(t) = M\{\overline{p}_1(t) + F_p(t, t_o) F_p^{-1}(\tau, t_o)[\overline{\eta}_p(\tau) - \overline{p}_1(\tau)]\} \tag{3.4-8}$$

while solving (3.4-7) for $\Phi_p(t_o)$ and substituting in (3.4-2) gives

$$\Phi_q(t) = M F_p(t, t_o) F_p^{-1}(\tau, t_o)\Phi_p(\tau) F_p^{-1T}(\tau, t_o) F_p^T(t, t_o) M^T \tag{3.4-9}$$

However, from (3.1-29) and (3.1-30),

$$\begin{aligned} F_p(t, t_o) F_p^{-1}(\tau, t_o) &= F_p(t, t_o) F_p(t_o, \tau) \\ &= F_p(t, \tau) \end{aligned} \tag{3.4-10}$$

which when substituted in (3.4-8) and (3.4-9) gives Eqs. (3.4-4) and (3.4-5), which were to be obtained. Finally, it is evident from (3.4-6) and (3.4-7) that

$\overline{\eta}_p(\tau)$ and $\Phi_p(\tau)$ depend only on $\overline{\eta}_p(t_o)$ and $\Phi_p(t_o)$, so the proof is complete.

Thus, with the stationarity assumption, and linear dynamics, the state of the scatterer ensemble at time t_o is given by the elements of $\overline{\eta}_p(t_o)$ together with the independent elements of $\Phi_p(t_o)$. The state equations are given by (3.2-22) and (3.2-31). Note that (3.4-1) and (3.4-2) are linear in the state variables. Thus, the scatterer ensemble, considered to be an abstract object as discussed above, constitutes a linear time-varying system. There are 27 state variables in all, and the state space (not to be confused with dynamic state space) is a subspace of the space of all real ordered 27-tuples, $\Re^{27}$.

Since the abstract object under consideration has no inputs, linearity is equivalent to zero-input linearity. Examining (3.4-1), it is apparent that to adhere strictly to the formal definition of zero input linearity [30], the state of the object at time t_o should be considered to contain the elements of the vector $\Delta_1\overline{\eta}_p(t_o) = \overline{\eta}_p(t_o) - \overline{p}_1(t_o)$ rather than $\overline{\eta}_p(t_o)$, and the output terminal variables should be considered to include $\Delta_1\overline{\eta}_q(t) = \overline{\eta}_q(t) - \overline{q}_1(t)$, rather than $\overline{\eta}_q(t)$. The input-output-state relation (3.4-1) is replaced by the equation

$$\Delta_1\overline{\eta}_q = M F_p(t, t_o)\, \Delta\overline{\eta}_p(t_o) \tag{3.4-11}$$

which, since the trajectory $\overline{p}_1(t)$ is known, is completely equivalent‡.

‡ However, for convenience, and since no confusion can result, elements of $\overline{\eta}_p(t_o)$ and $\overline{\eta}_q(t)$ will still be referred to as state variables and output variables.

3.5 Reducibility of the system representation

An input-output-state relation for the scatterer ensemble, with the state specified by 27 state variables, has been established. The question arises as to whether the system, as represented, is in reduced form. That is, whether it is possible to represent the state of the scatterer ensemble by fewer than 27 state variables, so that the output functions $\overline{\eta}_q(t)$ and $\Phi_q(t)$ are still uniquely determined for all $t > t_o$ by the specification of the state at any instant t_o. To investigate the reducibility of the system representation, it is necessary to discuss first the nature of the state space in some detail.

(a) The nature of the state space.

It is obvious that for any vector $\overline{\eta}_p(t_o)$, it is possible to find some density function $f_p(\overline{p}, t_o)$ for which $\overline{\eta}_p(t_o)$ is given by (2.3-15), i.e.

$$\overline{\eta}_p(t_o) = \int \overline{p}\, f_p(\overline{p}, t_o)\, d\tau_p \quad . \tag{3.5-1}$$

Thus, the space of state variables corresponding to the elements of the vector $\overline{\eta}_p(t_o)$ is the space of all real six-tuples, $\Re^6$. On the other hand, it is not true that the space of state variables corresponding to the independent elements of the matrix $\Phi_p(t_o)$ is the space of all real 21-tuples. This follows from the restriction that for every density function $f_p(\overline{p}, t_o)$, the matrix $\Phi_p(t_o)$ given by (2.3-16), i.e.

$$\Phi_p(t_o) = \int [\overline{p} - \overline{\eta}_p(t_o)]\, [\overline{p} - \overline{\eta}_p(t_o)]^T\, f_p(\overline{p}, t_o)\, d\tau_p \tag{3.5-2}$$

must be positive semi-definite. To prove that $\Phi_p(t_o)$ necessarily has this property, let $\overline{a}_p$ be an arbitrary vector in dynamic state space, and define

$\mu_p^2(\bar{a}_p, t_o)$ by

$$\mu_p^2(\bar{a}_p, t_o) = \int \{\bar{a}_p^T [\bar{p} - \bar{\eta}_p(t_o)]\}^2 f_p(\bar{p}, t_o)\, d\tau_p \quad . \tag{3.5-3}$$

Since the integrand in (4.2-57) is positive for all $\bar{p}$,

$$\mu_p^2(\bar{a}_p, t_o) \geq 0 \quad . \tag{3.5-4}$$

But by successive manipulation, and the use of (3.5-2),

$$\mu_p^2(\bar{a}_p, t_o) = \int \bar{a}_p^T [\bar{p} - \bar{\eta}_p(t_o)][\bar{p} - \bar{\eta}_p(t_o)]^T \bar{a}_p f_p(\bar{p}, t_o)\, d\tau_p$$

$$= \bar{a}_p^T \{ \int [\bar{p} - \bar{\eta}_p(t_o)][\bar{p} - \bar{\eta}_p(t_o)]^T f_p(\bar{p}, t_o)\, d\tau_p \} \bar{a}_p$$

$$= \bar{a}_p^T \Phi_p(t_o)\, \bar{a}_p \quad . \tag{3.5-5}$$

Equation (3.5-4) implies that the quadratic form in (3.5-5) is non-negative, and since $\bar{a}_p$ is arbitrary, $\Phi_p(t_o)$ is positive semi-definite. Thus, the space of the state variables contained in the matrix $\Phi_p(t_o)$ is confined to those points in $\mathfrak{R}^{21}$ corresponding to matrices $\Phi_p(t_o)$ which are positive semi-definite.

(b) Reducibility of $\bar{\eta}_p(t_o)$.

It is shown in this subsection that it is not possible to use a proper subset of the elements of $\bar{\eta}_p(t_o)$ in a state specification. If it were possible to specify the state by a proper subset of the elements of $\bar{\eta}_p(t_o)$ together with $\Phi_p(t_o)$, the output functions $\bar{\eta}_q(t)$ and $\Phi_q(t)$ would have to be independent of the remaining elements of $\bar{\eta}_p(t_o)$, since these remaining elements are not determined by the set used in the state specification. The possibility of specifying the state in the above manner is precluded by the following argument.

Consider two distinct vectors $\overline{\eta}_p(t_o)$ and $\overline{\eta}_p'(t_o)$. We show that these distinct vectors necessarily give rise to distinct output functions $\overline{\eta}_q(t)$ and $\overline{\eta}_q'(t)$. To prove this assertion, assume two distinct vectors, $\overline{\eta}_p(t_o)$ and $\overline{\eta}_p'(t_o)$ specify the output functions $\overline{\eta}_q(t)$ and $\overline{\eta}_q'(t)$ according to (3.4-1), and that these output functions are identical for $t > t_o$. Then,

$$\overline{\eta}_q(t) = M\{\overline{p}_1(t) + F_p(t, t_o)[\overline{\eta}_p(t_o) - \overline{p}_1(t_o)]\} \tag{3.5-6}$$

$$\overline{\eta}_q'(t) = M\{\overline{p}_1(t) + F_p(t, t_o)[\overline{\eta}_p'(t_o) - \overline{p}_1(t_o)]\} \tag{3.5-7}$$

and by assumption,

$$\overline{\eta}_q(t) = \overline{\eta}_q'(t) \qquad t > t_o \quad . \tag{3.5-8}$$

Subtracting (3.5-7) from (3.5-6) and using (3.5-8) results in

$$M F_p(t, t_o)[\overline{\eta}_p(t_o) - \overline{\eta}_p'(t_o)] = 0 \qquad t > t_o \quad . \tag{3.5-9}$$

The vector $\overline{\eta}_p(t_o) - \overline{\eta}_p'(t_o)$ is some non-zero vector in $\Re^6$, which can be denoted $\Delta_1\overline{p}(t_o)$; i.e.

$$\Delta_1\overline{p}(t_o) \equiv \overline{\eta}_p(t_o) - \overline{\eta}_p'(t_o) \tag{3.5-10}$$

so that (3.5-9) becomes

$$M F_p(t, t_o)\,\Delta_1\overline{p}(t_o) = 0 \qquad t > t_o \quad . \tag{3.5-11}$$

Now consider the trajectory determined by

$$\overline{p}(t_o) = \overline{p}_1(t_o) + \Delta_1\overline{p}(t_o) \tag{3.5-12}$$

where $\overline{p}_1(t)$ is the reference trajectory, and $\Delta_1\overline{p}(t_o)$ is defined by (3.5-10). For this trajectory

$$\overline{p}(t) = \overline{p}_1(t) + \Delta_1\overline{p}(t) \tag{3.5-13}$$

where from (3.1-25)

$$\Delta_1\overline{p}(t) = F_p(t, t_o)\Delta_1\overline{p}(t_o) \quad . \tag{3.5-14}$$

Finally, partitioning $\Delta_1\overline{p}(t)$ according to

$$\Delta_1\overline{p} = [\Delta_1\overline{q} \;\vdots\; \Delta_1\overline{u}]^T \tag{3.5-15}$$

and using (2.3-32), the equation

$$\Delta_1\overline{q}(t) = M F_p(t, t_o)\Delta_1\overline{p}(t_o) \tag{3.5-16}$$

results. But (3.5-11) implies that

$$\Delta_1\overline{q}(t) = 0 \qquad t > t_o \tag{3.5-17}$$

which indicates that the trajectory $\overline{p}(t)$ coincides with the reference trajectory. This leads us to conclude that

$$\Delta_1\overline{p}(t_o) = \overline{\eta}_p(t_o) - \overline{\eta}_p{}'(t_o) = 0 \quad , \tag{3.5-18}$$

which contradicts the assumption that $\overline{\eta}_p(t_o)$ and $\overline{\eta}_p{}'(t_o)$ are distinct. Thus, all six elements of $\overline{\eta}_p(t_o)$ are required in the specification of the state of the system.

(c) Reducibility of $\Phi_p(t_o)$.

In this subsection, we show that no principal minor of $\Phi_p(t_o)$ (e.g., such as $\Phi_q(t_o)$) can be used in the specification of the state of the scatterer ensemble at time t_o. Since the remaining elements of $\Phi_p(t_o)$ are not determined by the elements of any principal minor, the output functions $\overline{\eta}_q(t)$ and $\Phi_q(t)$ would have to be independent of each of the remaining elements, in order for it to be possible to specify the state by a principal minor.

That this is not possible is implied by the following somewhat stronger result, proved below: the output functions are not uniquely determined by

$\overline{\eta}_p(t_o)$ together with any proper subset of the elements of $\Phi_p(t_o)$ formed by deleting the elements of a principal submatrix of $\Phi_p(t_o)$. To prove this assertion, assume that the output function $\Phi_q(t)$ is uniquely determined by such a proper subset. Then the output function must be independent of the values of the elements in the deleted principal minor. We contradict this by demonstrating that it is always possible to find two scatterer ensembles for which the matrices $\Phi_p(t_o)$ and $\Phi_p'(t_o)$ differ only in the deleted elements, but for which the output functions are necessarily distinct.

Let $\Phi_p(t_o)$ be any positive semi-definite matrix, and let a $[6 \times 6]$ matrix A be formed by setting to zero all elements not contained in the deleted principal minor and choosing the remaining elements of A so that it is positive semi-definite, but not the zero matrix. Define

$$\Phi_p'(t_o) = \Phi_p(t_o) + A \tag{3.5-19}$$

The matrix $\Phi_p'(t_o)$ is positive semi-definite, since it is the sum of two positive semi-definite matrices. Also, $\Phi_p'(t_o)$ differs from $\Phi_p(t_o)$ only in a subset of the deleted elements. Both $\Phi_p(t_o)$ and $\Phi_p'(t_o)$ can be associated with scatterer ensembles since, for any positive semi-definite matrix $\Phi_p(t_o)$, some density function $f_p(\overline{p}, t_o)$ can be found for which (3.5-2) gives $\Phi_p(t_o)$.

By hypothesis, the output functions corresponding to $\Phi_p(t_o)$ and $\Phi_p'(t_o)$ are identical. Using (3.4-2), the functions $\Phi_q(t)$ and $\Phi_q'(t)$ defined by

$$\Phi_q(t) = M F_p(t, t_o)\, \Phi_p(t_o)\, F_p^T(t, t_o)\, M^T \tag{3.5-20}$$

$$\Phi_q'(t) = M F_p(t, t_o)\, \Phi_p'(t_o)\, F_p^T(t, t_o)\, M^T \tag{3.5-21}$$

are identical for $t > t_o$. Thus, subtracting (3.5-21) from (3.5-20) gives, for $t > t_o$

$$M F_p(t, t_o) [\Phi_p(t_o) - \Phi_p'(t_o)] F_p^T(t, t_o) M^T = 0 \qquad (3.5\text{-}22)$$

But then, from (3.5-19) we have

$$M F_p(t, t_o) A F_p^T(t, t_o) M^T = 0 \qquad (3.5\text{-}23)$$

The matrix A is positive semi-definite, and thus some density function $f_p(\bar{p}, t_o)$ can be found for which $\Phi_p(t_o)$ given by (3.5-2) equals the matrix A. Let $f_p'''(\bar{p}, t_o)$ denote any such density and $\Phi_p'''(t_o) = A$ the corresponding matix. Then (3.5-23) can be rewritten

$$M F_p(t, t_o) \Phi_p'''(t_o) F_p^T(t, t_o) M^T = 0 \qquad (3.5\text{-}24)$$

and (3.4-2) and (3.5-24) imply that

$$\Phi_q'''(t) = 0 \qquad t > t_o \qquad (3.5\text{-}25)$$

where $\Phi_q'''(t)$ is the SCM matrix of the projection of $f_p'''(\bar{p}, t)$ on the position subspace.

Consequently, the density function $f_q'''(\bar{q}, t)$ is zero for all $t > t_o$, except at one point. Thus the trajectories of all scatterers in any ensemble corresponding to $f_p'''(\bar{p}, t)$ have the same trajectory, which implies that

$$\Phi_p'''(\bar{p}, t) = 0 \qquad \text{all } t \qquad (3.5\text{-}26)$$

In particular

$$\Phi_p'''(p, t_o) = A = 0 \qquad (3.5\text{-}27)$$

which contradicts our requirement that A not be the zero matrix. Thus it is not possible to delete the elements of any principal submatrix of $\Phi_p(t_o)$, and uniquely specify the output function $\Phi_q(t)$ by the remaining elements.

It should be pointed out that the results of this subsection apply for all point-mass dynamics. If particular point-mass dynamics are specified, the state representation can be, at least in principle, examined for reducibility with respect to the elements of $\Phi_p(t_o)$. In practice, this examination may be extremely difficult, as the transition matrix for the elements of $\Phi_p(t)$ will rarely be given in closed form.

3.6 State representation when dynamics are non-linear

State variables have been identified for the scatterer ensemble, using the stationarity assumption and linear dynamics. The present section includes a brief discussion of the representation of the state of the scatterer ensemble in the case of non-linear dynamics.

In this case, the elements of the quantities $\bar{\eta}_p(t_o)$ and $\Phi_p(t_o)$ cannot be used to specify the state of the scatterer ensemble (considered to be an abstract object as discussed above) at time t_o. In Subsections 3.2(b) and 3.2(c) above, it was pointed out that in general $\bar{\eta}_p(t)$ and $\Phi_p(t)$ are not uniquely determined by $\bar{\eta}_p(t_o)$ and $\Phi_p(t_o)$. That is, it is possible that two different densities $f_p^{(1)}(\bar{p}, t_o)$ and $f_p^{(2)}(\bar{p}, t_o)$ may exist for which $\bar{\eta}_p^{(1)}(t_o)$ and $\bar{\eta}_p^{(2)}(t_o)$ are identical and $\Phi_p^{(1)}(t_o)$ and $\Phi_p^{(2)}(t_o)$ are identical, and yet $\bar{\eta}_p^{(1)}(t)$ or $\Phi_p^{(1)}(t)$ is different from $\bar{\eta}_p^{(2)}(t)$ or $\Phi_p^{(2)}(t)$. Thus, the output functions $\bar{\eta}_q(t)$ and $\Phi_q(t)$, which are sub-matrices of $\bar{\eta}_p(t)$ and $\Phi_p(t)$, are not uniquely determined by $\bar{\eta}_p(t_o)$ and $\Phi_p(t_o)$, and consequently the use of $\bar{\eta}_p(t_o)$ and $\Phi_p(t_o)$ to specify the state of the scatterer ensemble would not satisfy the first self-consistency condition[11] for the state representation.

It is easily seen that the state of the ensemble can be specified by the density function $f_p(\bar{p}, t_o)$. The density function at any time t is given in terms of $f_p(\bar{p}, t_o)$ by (3.2-15), and the output functions $\bar{\eta}_q(t)$ and $\Phi_q(t)$ are given by (2.3-15) and (2.3-16), followed by (2.3-33) and (2.3-34). However, in this case the dimension of state space is infinite and a state estimation approach to the prediction problem such as is outlined in Section 2.4 is not applicable.

CHAPTER IV

IMPULSIVELY DISPENSED SCATTERER ENSEMBLES

In this chapter, due to its practical importance, the special case of impulsively dispensed scatterer ensembles is discussed. An impulsively dispensed scatterer ensemble is comprised of scatterers which have been dispensed simultaneously from some mechanism, which is referred to as the dispenser. Artificially produced clouds of scatterers are likely to be formed in this manner. A good example is the West-Ford dipole cloud[31]. Also, it may be possible in some cases to consider a scatterer ensemble to be impulsively dispensed regardless of the particular physical process which caused the formation of the scatterer ensemble.

The primary significance of impulsively dispensed clouds is the fact that if it is known (to the radar/data processor system) that a cloud is impulsively dispensed, and the time of scatterer dispensing is known, the number of state variables necessary to represent the cloud state is significantly reduced, in comparison to the number necessary in the general case discussed in the preceding chapter. Since the time required for operation of a state estimation filter for a linear time-varying system, and the detrimental effects of round off error in numerical computations, increase rapidly with the order of the system (i.e., the number of state variables), the reduction of the system order presents a great practical advantage.

4.1 The transformation of scatterer position

In the present formulation, the scatterers in an impulsively dispensed cloud are considered to be dispensed, at a single instant, from a single point in space. The instant is referred to as the dispensing instant. In practice, a cloud can be considered to be impulsively dispensed if the duration of scatterer dispensing is much less than the time interval between dispensing and cloud observation, and if the distances (at the dispensing instant) between various centers in a dispensing complex are much smaller than the cloud dimensions at instants of observation.

Denote the trajectory of the dispenser (which is considered to be a point-mass) by $\overline{p}_d(t)$, and the dispensing instant by t_d. The condition that the scatterers are all dispensed from the dispenser at time t_d is represented by the equation

$$\overline{p}_i(t_d) = \overline{p}_d(t_d) + [0 \vdots \overline{w}_i]^T \qquad (4.1\text{-}1)$$

where the dynamic state vector is assumed to be partitioned according to (3.1-13), as usual. Here $\overline{p}_i(t_d)$ and $\overline{p}_d(t_d)$ are respectively the dynamic state vector of the i^{th} scatterer in the ensemble, and the dispenser, at the dispensing instant. Note that the position of each scatterer coincides with that of the dispenser at the dispensing instant. The difference in velocity between the i^{th} scatterer and the dispenser, at the dispensing instant, is considered to result from the operation of the dispensing mechanism. This velocity difference, denoted $\overline{w}_i$, is referred to as the dispensing velocity of the i^{th} scatterer.

As a first step towards deriving the transformation with time of scatterer position, an equation is obtained which relates the position of a scatterer, at some time t (after dispensing) to its dispensing velocity. One of the results of the present chapter is the derivation of transformation equations for $\overline{\eta}_q(t)$ and $\Phi_q(t)$. It will be seen that transformation equations which do not depend on the density $f_p(\overline{p},t)$ can be obtained for an impulsively dispensed cloud, with the use of linear dynamics and the stationarity assumption. Since, as we have seen previously, such transformations of $\overline{\eta}_p(t)$ and $\Phi_p(t)$ cannot be obtained in the case of nonlinear dynamics, and since $\overline{\eta}_q(t)$ and $\Phi_q(t)$ are submatrices of $\overline{\eta}_p(t)$ and $\Phi_p(t)$ respectively, it is evident that transformations for $\overline{\eta}_q(t)$ and $\Phi_q(t)$ cannot be obtained in the case of non-linear dynamics. Consequently, in the ensuing discussion, it will be assumed that the point-mass dynamics are described by (3.1-25).

Substituting

$$\overline{p}_1(t) = \overline{p}_d(t) \tag{4.1-2}$$

in the definition (3.1-4) for $\Delta_1\overline{p}(t)$, and substituting (3.1-4) in (3.1-25), we obtain

$$\overline{p}(t_i) - \overline{p}_d(t_i) = F_p(t_i,t_j)\,[\overline{p}(t_j) - \overline{p}_d(t_j)] \tag{4.1-3}$$

Equation (4.1-2) implies that the dispenser trajectory is being used for the reference trajectory. Substituting $t_i = t$ and $t_j = t_d$ in (4.1-3) yields

$$\overline{p}(t) - \overline{p}_d(t) = F_p(t,t_d)[\overline{p}(t_d) - \overline{p}_d(t_d)] \tag{4.1-4}$$

and using (4.1-1), with the subscript i dropped, to substitute for $\overline{p}(t_d) - \overline{p}_d(t_d)$ in (4.1-4) results in the equation

$$\bar{p}(t) - \bar{p}_d(t) = F_p(t, t_d) [0 \vdots \bar{w}]^T . \tag{4.1-5}$$

Finally, partitioning the matrix F_p into four [3×3] matrices according to

$$F_p = \left[\begin{array}{c|c} F_1 & F_3 \\ \hline F_2 & F_4 \end{array}\right] \tag{4.1-6}$$

and substituting (4.1-6) into (4.1-5) the equations

$$\bar{q}(t) - \bar{q}_d(t) = F_3(t, t_d) \bar{w} \tag{4.1-7}$$

$$\bar{u}(t) - \bar{u}_d(t) = F_4(t, t_d) \bar{w} \tag{4.1-8}$$

result. Equation (4.1-7) relates the position of a scatterer at time t (after dispensing) to its dispensing velocity.

The density $f_p(\bar{p}, t_d)$ is singular, since all the scatterers are at the same position. It is assumed that the projection of this density on the velocity subspace (i.e., $f_u(\bar{u}, t_d)$) is non-singular. Furthermore, it is assumed that, at any instant of observation t, the projection of the density $f_p(\bar{p}, t)$ on the position subspace (i.e., $f_q(\bar{q}, t)$) is non-singular. The latter assumption implies that at time t, the scatterers cannot be contained in a plane. This further implies that the matrix $F_3(t, t_d)$ is non-singular, for if it were, (4.1-7) would map the space of vectors $\bar{w}$ onto a planar subspace of the position subspace.

Finally, solving (4.1-7) for $\bar{w}$ and substituting into (4.1-8) yields

$$\bar{u}(t) - \bar{u}_d(t) = F_4(t, t_d) F_3^{-1}(t, t_d) [\bar{q}(t) - \bar{q}_d(t)] \tag{4.1-9}$$

which states that the velocity of a scatterer at time t is a function of its position. This clearly implies that the density $f_p(\bar{p}, t)$ is singular. It can be said that it is the singularity of the density $f_p(\overline{p}, t)$ for an impulsively dispensed cloud, or equivalently the non-independence of scatterer position and velocity, which permits us to use a fewer number of state variables to represent the cloud (as discussed in section 4.4 below).

Let us now proceed to the derivation of the transformation of scatterer position with time. Substituting t_i and t_j for t in (4.1-7) yields the equations

$$\overline{q}(t_i) - \overline{q}_d(t_i) = F_3(t_i, t_d)\overline{w} \tag{4.1-10}$$

$$\overline{q}(t_j) - \overline{q}_d(t_j) = F_3(t_j, t_d)\overline{w} \tag{4.1-11}$$

Solving (4.1-11) for $\overline{w}$ and substituting into (4.1-10) yields

$$\overline{q}(t_i) - \overline{q}_d(t_i) = F_3(t_i, t_d)\, F_3^{-1}(t_j, t_d)[\overline{q}(t_j) - \overline{q}_d(t_j)] \tag{4.1-12}$$

which is the required transformation. Finally, introducing

$$F_q(t_i, t_j) = F_3(t_i, t_d)F_3^{-1}(t_j, t_d) \tag{4.1-13}$$

Eq. (4.1-12) becomes

$$\overline{q}(t_i) - \overline{q}_d(t_i) = F_q(t_i, t_j)[\overline{q}(t_j) - \overline{q}_d(t_j)] \tag{4.1-14}$$

Although, in deriving (4.1-14), we have used the dispenser trajectory as the reference trajectory, it is not necessary to know the dispenser trajectory to compute $F_q(t_i, t_j)$. The matrix $F_3(t, t_d)$ is a submatrix of $F_p(t, t_d)$. The matrix $G(t, t_o)$ was shown in Subsection 3.1b to be independent of the reference trajectory, provided that the dynamics are exactly linear. This implies that $F_p(t, t_d)$, $F_3(t, t_d)$ and $F_q(t_i, t_j)$ are independent of the reference trajectory. However, note that knowledge of the dispensing time is essential for the computation of $F_q(t_i, t_j)$, as is evident from (4.1-13).

4.2 Transformation of $f_q(\overline{q}, t)$, $\overline{\eta}_q(t)$, and $\Phi_q(t)$ with time

In the previous section, a transformation for scatterer position, relative to the dispenser trajectory, was derived. Invoking the stationarity assumption, it is now possible to derive transformations for $f_q(\overline{q}, t)$, $\overline{\eta}_q(t)$ and $\Phi_q(t)$, using methods similar to those used in Section 3.2 to derive transformations for $f_p(\overline{p}, t)$, $\overline{\eta}_p(t)$ and $\Phi_p(t)$.

(a) Transformation of $f_q(\overline{q}, t)$.

Consider the mapping

$$\overline{q} = \overline{q}_d(t_i) + F_q(t_i, t_j) \left[\overline{q}' - \overline{q}_d(t_j) \right] \tag{4.2-1}$$

which transforms scatterer positions $\overline{q}'$ at time t_j onto the corresponding scatterer positions $\overline{q}$ at time t_i. Under this mapping, a volume element $d\tau_q'$ at position $\overline{q}'$ maps onto the volume element $d\tau_q$ at point $\overline{q}$, where

$$d\tau_q = \left| \det F_q(t_i, t_j) \right| d\tau_q' \tag{4.2-2}$$

Since the volume elements $d\tau_q'$ and $d\tau_q$ contain the same set of scatterers, denoted $\mathcal{A}$, we have

$$\sum_{i \in \mathcal{A}} s_i^2(t_i) = \sum_{i \in \mathcal{A}} s_i^2(t_j) \tag{4.2-3}$$

But, from (2.3-8),

$$\sum_{i \in \mathcal{A}} s_i^2(t_i) = T(t_i)\, f_q(\overline{q}, t_i)\, d\tau_q \tag{4.2-4}$$

$$\sum_{i \in \mathcal{A}} s_i^2(t_j) = T(t_j)\, f_q(\overline{q}', t_j)\, d\tau_q' \tag{4.2-5}$$

It has already been established that the total radar cross-section of the ensemble is time invariant. Thus, with (3.2-12) and the preceding four equations, the transformation

$$f_q(\bar{q}, t_i) = \frac{f_q(\bar{q}', t_j)}{|\det F_q(t_i, t_j)|} \tag{4.2-6}$$

is obtained.

(b) Transformation of $\bar{\eta}_q(t)$.

Let us begin by substituting t_i and t_j for t in (2.3-11), introducing the dummy variable $\bar{q}'$:

$$\bar{\eta}_q(t_i) = \int \bar{q} f_q(\bar{q}, t_i)\, d\tau_q \tag{4.2-7}$$

$$\bar{\eta}_q(t_j) = \int \bar{q}' f_q(\bar{q}', t_j) d\tau_q' \ . \tag{4.2-8}$$

Now, transform the space of integration in (4.2-7) according to the mapping (4.2-1). But from (3.2-12), (4.2-4) and (4.2-5), under this mapping we have

$$f_q(\bar{q}, t_i)\, d\tau_q = f_q(\bar{q}', t_j)\, d\tau_q' \tag{4.2-9}$$

Thus Eq. (4.2-7) gives us

$$\begin{aligned} \bar{\eta}_q(t_i) &= \int \{ \bar{q}_d(t_i) + F_q(t_i, t_j) [\bar{q}' - \bar{q}_d(t_j)]\} f_q(\bar{q}', t_j)\, d\tau_q' \\ &= \bar{q}_d(t_i) + \int F_q(t_i, t_j) [\bar{q}' - \bar{q}_d(t_j)] f_q(\bar{q}', t_j) d\tau_q' \\ &= \bar{q}_d(t_i) + F_q(t_i, t_j) [\bar{\eta}_q(t_j) - \bar{q}_d(t_j)] \end{aligned} \tag{4.2-10}$$

where Eqs. (2.3-7) and (4.2-8) have been used. This is the desired transformation for the centroid vector.

(c) Transformation of $\Phi_q(t)$.

To derive the transformation of the SCM matrix, substitute t_i and t_j in (2.3-12) to obtain

$$\Phi_q(t_i) = \int [\bar{q}-\bar{\eta}_q(t_i)]\,[\bar{q}-\bar{\eta}_q(t_i)]^T f_q(\bar{q}, t_i)\, d\tau_q \qquad (4.2\text{-}11)$$

$$\Phi_q(t_j) = \int [\bar{q}'-\bar{\eta}_q(t_j)][\bar{q}'-\bar{\eta}_q(t_j)]^T f_q(\bar{q}', t_j)\, d\tau_q' \qquad (4.2\text{-}12)$$

The mapping (4.2-1) is equivalent to

$$\begin{aligned}
\bar{q}-\bar{\eta}_q(t_i) &= \bar{q}_d(t_i) - \bar{\eta}_q(t_i) + F_q(t_i, t_j)[\bar{q}' - \bar{q}_d(t_j)] \\
&= F_q(t_i, t_j)\,[\bar{q}_d(t_j) - \bar{\eta}_q(t_j)] + F_q(t_i, t_j)[\bar{q}' - \bar{q}_d(t_j)] \\
&= F_q(t_i, t_j)\,[\bar{q}' - \bar{\eta}_q(t_j)] \qquad (4.2\text{-}13)
\end{aligned}$$

where (4.2-10) has been used. Transforming the space of integration in (4.2-11) according to (4.2-13), and using (4.2-9) yields

$$\begin{aligned}
\Phi_q(t_i) &= \int F_q(t_i, t_j)\,[\bar{q}'-\bar{\eta}_q(t_j)]\,[\bar{q}'-\bar{\eta}_q(t_j)]^T F_q^T(t_i, t_j)\; f_q(\bar{q}', t_j)\,d\tau_q' \\
&= F_q(t_i, t_j)\,\{\int [\bar{q}'-\bar{\eta}_q(t_j)]\,[\bar{q}'-\bar{\eta}_q(t_j)]^T f_q(\bar{q}', t_j)\,d\tau_q'\}\, F_q^T(t_i, t_j) \\
&= F_q(t_i, t_j)\,\Phi_q(t_j)\,F_q^T(t_i, t_j) \qquad (4.2\text{-}14)
\end{aligned}$$

which is the desired transformation for the second central moment matrix.

4.3 Cloud characterization in dispensing velocity space

In this section, a characterization of impulsively dispensed scatterer ensembles by means of certain properties of the ensemble in the space of dispensing velocities, $\overline{w}$, is introduced. The significance of this characterization is that the properties introduced do not depend on the time, as do properties such as $f_q(\overline{q}, t)$, $\overline{\eta}_q(t)$, and $\Phi_q(t)$ which have already been discussed. This makes the properties of the ensemble in dispensing velocity space especially suitable for the specification of the characteristics of an impulsively dispensed ensemble. In addition, since the scatterer dispensing velocities are determined by the operation of the dispensing mechanism, properties of the cloud in dispensing velocity space are direct indications of the operation of the dispensing mechanism.

Let us begin by defining a density function for the ensemble in dispensing velocity space. But first recall the definition of the density functions $f_p(\overline{p}, t)$ and $f_q(\overline{q}, t)$. At any instant at which the scatterer ensemble is observed (or potentially observed) by the radar, the expected value of radar cross-section, $s_i^2(t)$, of each scatterer is defined by virtue of the ensemble of possible interactions (by means of a transmitted pulse) between the radar and the scatterer. The quantities $s_i^2(t)$ and the vectors $\overline{p}_i(t)$ or $\overline{q}_i(t)$ associated with the scatterers then determine the density functions $f_p(\overline{p}, t)$ and $f_q(\overline{q}, t)$. The stationarity assumption, and the dynamics common to the scatterers in the ensemble, determine the transformation with time of $f_p(\overline{p}, t)$ generally, and of $f_q(\overline{q}, t)$ in the case of impulsively dispensed clouds.

In place of defining a density function for the ensemble in dispensing velocity space directly in terms of the expected radar cross-sections $s_i^2(t_d)$, the density is defined by making use of the known transformation (4.1-7) between the dispensing velocity of a scatterer and its position at time t after dispensing. This is done since: (1) the radar may not be considered to be capable of observing the ensemble at the dispensing instant (e.g., dispensing occurs when the dispenser is over the horizon); or (2) since the scatterers are all at the same position at the dispensing instant, the concept of $s_i^2(t_d)$ being defined by virtue of interaction between the radar and scatterer i becomes less appealing (the ensemble of scatterers will constitute a single point target).

Thus, with $f_q(\overline{q}, t)$ the density function in position space of an impulsively dispensed scatterer ensemble at time t, the density function of the ensemble in dispensing velocity space is defined by

$$f_w(\overline{w}) = \left|\det F_3(t, t_d)\right| f_q(\overline{q}, t) \tag{4.3-1}$$

where in (4.3-1), vectors $\overline{w}$ and $\overline{q}$ are related by

$$\overline{q} - \overline{q}_d(t) = F_3(t, t_d)\overline{w} \tag{4.3-2}$$

This definition is satisfactory only if the density $f_w(\overline{w})$ is independent of the time t. That this is true is a consequence of the stationarity assumption, as is shown in the following discussion. Let $f_q(\overline{q}, t_1)$ and $f_q(\overline{q}, t_2)$ be the density functions in position space at times t_1 and t_2 and define

$$f_w(\overline{w}) = \left|\det F_3(t_1, t_d)\right| f_q(\overline{q}, t_1) \tag{4.3-3}$$

where

$$\overline{q} - \overline{q}_d(t_1) = F_3(t_1, t_d)\,\overline{w} \tag{4.3-4}$$

and

$$f_w'(\overline{w}) = \left|\det F_3(t_2, t_d)\right| f_q(\overline{q}', t_2) \tag{4.3-5}$$

where

$$\overline{q}' - \overline{q}_d(t_2) = F_3(t_2, t_d)\,\overline{w} \quad . \tag{4.3-6}$$

Equations (4.3-4) and (4.3-6) imply that $\overline{q}'$ and $\overline{q}$ are related by

$$\begin{aligned} \overline{q} - \overline{q}_d(t_1) &= F_3(t_1, t_d)\, F_3^{-1}(t_2, t_d)\, [\overline{q}' - \overline{q}_d(t_1)] \\ &= F_q(t_1, t_2)\, [\overline{q}' - \overline{q}_d(t_1)] \end{aligned} \tag{4.3-7}$$

which by comparison with (4.2-1) implies that

$$f_q(\overline{q}, t_1) = \frac{f_q(\overline{q}', t_2)}{\left|\det F_q(t_1, t_2)\right|} \tag{4.3-8}$$

using (4.2-6).

Then, using (4.3-3), (4.3-5) and (4.3-8)

$$\begin{aligned} f_w(\overline{w}) &= \left|\det F_3(t_1, t_d)\right| \frac{f_q(\overline{q}', t_2)}{\left|\det F_q(t_1, t_2)\right|} \\ &= \frac{\left|\det F_3(t_1, t_d)\right|}{\left|\det F_3(t_2, t_d)\right|} \; \frac{f_w'(\overline{w})}{\left|\det F_q(t_1, t_2)\right|} \end{aligned} \tag{4.3-9}$$

But

$$F_3(t_1, t_d)\, F_3^{-1}(t_2, t_d) = F_q(t_1, t_2) \tag{4.3-10}$$

which implies that

$$\left|\det F_3(t_1, t_d)\right| \left|\det F_3(t_2, t_d)\right|^{-1} = \left|\det F_q(t_1, t_2)\right| \tag{4.3-11}$$

so that (4.3-9) becomes

$$f_w(\overline{w}) = f_w'(\overline{w}) \tag{4.3-12}$$

which was the required result.

Having defined the density function $f_w(\overline{w})$ for an impulsively dispensed scatterer ensemble, the centroid $\overline{\eta}_w$ and SCM matrix Φ_w are defined as

$$\overline{\eta}_w = \int \overline{w}\, f_w(\overline{w})\, d\tau_w \tag{4.3-13}$$

$$\Phi_w = \int [\overline{w} - \overline{\eta}_w]\, [\overline{w} - \overline{\eta}_w]^T\, f_w(\overline{w})\, d\tau_w \;. \tag{4.3-14}$$

The transformations between $\overline{\eta}_w$ and $\overline{\eta}_q(t)$ and between Φ_w and $\Phi_q(t)$ follow immediately from (4.3-1) and (4.3-2). Under the mapping (4.3-2) a volume element $d\tau_w$ at point $\overline{w}$ maps into the volume element $d\tau_q$ at point $\overline{q}$ where $\overline{q}$ and $\overline{w}$ are related, of course, by (4.3-2) and

$$d\tau_q = |\det F_3(t, t_d)|\, d\tau_w \;. \tag{4.3-15}$$

Thus, with (4.3-1),

$$f_w(\overline{w})\, d\tau_w = f_q(\overline{q}, t)\, d\tau_q \tag{4.3-16}$$

Finally, transforming the space of integration in (4.3-13) according to (4.3-2) results in

$$\begin{aligned} \overline{\eta}_w &= \int F_3^{-1}(t, t_d)\, [\overline{q} - \overline{q}_d(t)]\, f_q(\overline{q}, t)\, d\tau_q \\ &= F_3^{-1}(t, t_d)\, [\overline{\eta}_q(t) - \overline{q}_d(t)] \end{aligned} \tag{4.3-17}$$

which is the desired transformation between $\overline{\eta}_w$ and $\overline{\eta}_q(t)$.

The quantity $\overline{\eta}_w$ is a vector in dispensing velocity space, and comparing (4.3-17) with (4.1-7), it is evident that the centroid $\overline{\eta}_q(t)$ of the scatterer ensemble traverses the point-mass trajectory of a scatterer dispensed from the dispenser, at time t_d, with velocity $\overline{\eta}_w$.

To derive the transformation between Φ_w and $\Phi_q(t)$, note that the mapping (4.3-2) is equivalent to

$$\overline{q}-\overline{\eta}_q(t) = \overline{q}_d(t) - \overline{\eta}_q(t) + F_3(t, t_d)\overline{w}$$

$$= -F_3(t, t_d)\overline{\eta}_w + F_3(t, t_d)\overline{w}$$

$$= F_3(t, t_d)\,[\overline{w}-\overline{\eta}_w] \tag{4.3-18}$$

where (4.3-17) has been used. Thus, transforming the space of integration in (4.3-14) according to (4.3-18) results in

$$\Phi_w = \int F_3^{-1}(t, t_d)[\overline{q}-\overline{\eta}_q(t)]\,[\overline{q}-\overline{\eta}_q(t)]^T\, F_3^{-1T}(t, t_d) f_q(\overline{q}, t)\, d\tau_q$$

$$= F_3^{-1}(t, t_d)\, \Phi_q(t)\, F_3^{-1T}(t, t_d)\ . \tag{4.3-19}$$

The quantities $\overline{\eta}_w$ and Φ_w characterize the scatterer ensemble in dispensing velocity space. The centroid $\overline{\eta}_w$ represents an average dispensing velocity, while the matrix Φ_w determines the spread in dispensing velocities, in the sense of the SCM, in all directions. For an arbitrary unit vector $\overline{a}$, the SCM of the density $f_w(\overline{w})$ in the direction of $\overline{a}$, denoted by $\mu_w^2(\overline{a})$, is given by

$$\mu_w(\overline{a}) \equiv \int \{\, \overline{a}^T\, [\overline{w}-\overline{\eta}_w]\,[\overline{w}-\overline{\eta}_w]^T \overline{a}\} f_w(\overline{w})\, d\tau_w$$

$$= \overline{a}^T\, \Phi_w\, \overline{a}\quad . \tag{4.3-20}$$

Finally, note that the values of $\overline{\eta}_w$ and Φ_w given by (4.3-17) and (4.3-19) are necessarily independent of t, since the density function $f_w(\overline{w})$ defined by (4.3-1) is independent of t.

4.4 State variable characterization of an impulsively dispensed scatterer ensemble

In this section, sets of variables sufficient to specify the state of an impulsively dispensed scatterer ensemble, which is considered to be an abstract object as in Section 3.4 above, are identified. It is found that the number of required state variables is substantially less than is necessary to specify the state of a scatterer ensemble which is not impulsively dispensed. In order for the radar/data processor to take advantage of the potential simplification in the state representation of the scatterer ensemble, it is necessary for the radar/data processor to know a priori that the ensemble is impulsively dispensed, and to know the dispensing time. It is assumed throughout this section that this information is available to the radar/data processor.

If it is known to the radar/data processor that the ensemble is impulsively dispensed, but the dispensing time is not known, it can be estimated on the basis of observations of the second central moment matrix, $\Phi_q(t)$. One procedure for obtaining an estimate uses an iterative least-squares technique. This method will be discussed in Chapter 6.

(a) The dispenser trajectory assumed to be known.

A state representation of impulsively dispensed scatterer ensembles is established by writing input-output-state relations for the associated abstract object, and demonstrating that they have the response separation property. It is assumed here that the dispenser trajectory $\overline{p}_d(t)$ is known, but this restriction will be removed subsequently. The output terminal variables of the abstract object are the elements of $\overline{\eta}_q(t)$ and $\Phi_q(t)$, as in Section 4.3. Substituting $t_i = t$ and $t_j = t_o$

in (4.2-10) and (4.2-14) yields

$$\bar{\eta}_q(t) = \bar{q}_d(t) + F_q(t, t_o)\,[\bar{\eta}_q(t_o) - \bar{q}_d(t_o)] \tag{4.4-1}$$

$$\Phi_q(t) = F_q(t, t_o)\,\Phi_q(t_o)\,F_q^T(t, t_o) \tag{4.4-2}$$

which specifies the output variables at time t as functions of the elements of $\bar{\eta}_q(t_o)$ and $\Phi_q(t_o)$. It is maintained that these elements serve to specify the state of the object at time t_o. That is, (4.4-1) and (4.4-2) are input-output state relations which have the response separation property.

To prove this, it is necessary to show that for all τ satisfying

$$t_o < \tau \leq t \tag{4.4-3}$$

we have

$$\bar{\eta}_q(t) = \bar{q}_d(t) + F_q(t, \tau)\,[\bar{\eta}_q(\tau) - \bar{q}_d(\tau)] \tag{4.4-4}$$

and

$$\Phi_q(t) = F_q(t, \tau)\,\Phi_q(\tau)\,F_q^T(t, \tau) \tag{4.4-5}$$

where $\bar{\eta}_q(\tau)$ and $\Phi_q(\tau)$ depend only on $\bar{\eta}_q(t_o)$ and $\Phi_q(t_o)$. Let us begin by substituting $t_i = \tau$ and $t_j = t_o$ in (4.2-10) and (4.2-14), resulting in

$$\bar{\eta}_q(\tau) = \bar{q}_d(\tau) + F_q(\tau, t_o)\,[\bar{\eta}_q(t_o) - \bar{q}_d(t_o)] \tag{4.4-6}$$

$$\Phi_q(\tau) = F_q(\tau, t_o)\,\Phi_q(t_o)\,F_q^T(\tau, t_o) \tag{4.4-7}$$

Then, solving (4.4-6) for $[\bar{\eta}_q(t_o) - \bar{q}_d(t_o)]$ and substituting into (4.4-1), and solving (4.4-7) for $\Phi_q(t_o)$ and substituting into (4.4-2), the equations

$$\bar{\eta}_q(t) = \bar{q}_d(t) + F_q(t, t_o)\,F_q^{-1}(\tau, t_o)\,[\bar{\eta}_q(\tau) - \bar{q}_d(\tau)] \tag{4.4-8}$$

$$\Phi_q(t) = F_q(t, t_o) F_q^{-1}(\tau, t_o)\,\Phi_q(\tau)\,F_q^{-1T}(\tau, t_o)\,F_q^T(t, t_o) \tag{4.4-9}$$

are obtained. But (4.1-13) implies that

$$F_q(t, t_o)F_q^{-1}(\tau, t_o) = F_3(t, t_d)\, F_3^{-1}(t_o, t_d)\, [F_3(\tau, t_d)F_3^{-1}(t_o, t_d)]^{-1}$$

$$= F_3(t, t_d)\, F_3^{-1}(t_o, t_d)\, F_3(t_o, t_d)\, F_3^{-1}(\tau, t_d)$$

$$= F_3(t, t_d)\, F_3^{-1}(\tau, t_d) = F_q(t, \tau) \qquad (4.4\text{-}10)$$

which, when substituted in (4.4-8) and (4.4-9), give Eqs. (4.4-4) and (4.4-5) which were to be derived. Finally, from (4.4-6) and (4.3-7) it is obvious that $\overline{\eta}_q(\tau)$ and $\Phi_q(\tau)$ depend only on $\overline{\eta}_q(t_o)$ and $\Phi_q(t_o)$, so the proof is complete. Equations (4.4-6) and (4.4-7) are the state equations.

The state variables identified for the impulsively dispensed cloud include the three elements of the centroid vector, $\overline{\eta}_q(t_o)$, and the six independent elements of the SCM matrix, $\Phi_q(t_o)$. The total number of state variables therefore is nine, which is substantially less than the number of state variables, 27, necessary in the general case. The state space, in the case of impulsively dispensed clouds, is a subset of $\mathcal{R}^9$. While the vector $\overline{\eta}_q(t_o)$ is arbitrary, the matrix $\Phi_q(t_o)$ must be positive definite since, in (2.3-13), $\mu_q^2(\overline{a}, t) \geq 0$ for all vectors $\overline{a}$, and since it has been assumed that the scatterers cannot be contained in a plane, which excludes $\mu_q^2(\overline{a}, t) = 0$. It is evident that the state space associated with impulsively dispensed clouds is a subspace of the state space associated with general clouds, since $\overline{\eta}_q(t_o)$ and $\Phi_q(t_o)$ are submatrices of $\overline{\eta}_p(t_o)$ and $\Phi_p(t_o)$ and since the class of impulsively dispensed clouds is included in the class of general clouds. This implies that $\overline{\eta}_p(t_o)$ and $\Phi_p(t_o)$ can be used to specify the state of an impulsively dispensed cloud (as would be necessary if the radar/data processor did not know a priori that a cloud was impulsively dispensed) at a great cost in additional complexity.

It is an immediate consequence of the fact that the output and state variables are identical that the system is in reduced form. If a subset of the nine state variables contained in $\overline{\eta}_q(t_o)$ and $\Phi_q(t_o)$ were specified, the output functions $\overline{\eta}_q(t)$, $\Phi_q(t)$ would not be completely determined for $t \geq t_o$, since they would not even be determined for $t = t_o$.

The system under discussion is linear, since the input-output-state relations (4.4-1) and (4.4-2) are linear in the state variables. Again, to adhere strictly to the definition of zero-input linearity, the state of the object at time t_o could be considered to contain the elements of the vector $\Delta_d \overline{\eta}_q(t_o) \equiv \overline{\eta}_q(t_o) - \overline{q}_d(t_o)$, and the output variables to contain the elements of $\Delta_d \overline{\eta}_q(t) \equiv \overline{\eta}_q(t) - \overline{q}_d(t)$.

In Section 3.5, it was shown that it is not sufficient to use a subset of the vector $\overline{\eta}_p(t_o)$ in the state specification, nor is it possible to delete a principal minor of $\Phi_p(t_o)$ from the state specification. However, in the case of impulsively dispensed clouds, it is possible to use $\overline{\eta}_q(t_o)$ and $\Phi_q(t_o)$ to specify the state. There is no contradiction, since in adding the limitation that the cloud is impulsively dispensed, the abstract object has been changed from what it was previously. Insight is gained into the nature of the change of the abstract object by considering the manner in which the proofs in Section 3.5 regarding system reduction become invalid when the cloud is restricted to be impulsively dispensed.

In the proof that a subset of $\overline{\eta}_p(t_o)$ could not be used in a state specification, the independence of the elements of $\overline{\eta}_p(t_o)$ was evoked (see page 72). These elements are independent in the general case, but not in the case of an impulsively dispensed cloud. This follows from the relation (4.1-9) between the velocity and position of each scatterer in an impulsively dispensed cloud. In fact, considering (4.1-9) to be a mapping from the position subspace onto the velocity subspace, it can be shown that

$$\overline{\eta}_u(t) - \overline{u}_d(t) = F_3(t, t_d)\, F_3^{-1}(t, t_d)\, [\overline{\eta}_q(t) - \overline{q}_d(t)] \qquad (4.4\text{-}11)$$

Similarly, the elements of $\Phi_p(t_o)$, in the case of impulsively dispensed clouds, have inter-relationships among themselves imposed by the relation (4.1-9). Such inter-relationships were tacitly excluded in the proof, in Section 3.5, that a principal submatrix of $\Phi_p(t_o)$ could not be deleted from the state specification. When such inter-relationships exist, the deleted elements may be determined by the elements retained in the state specification, which would exclude the possibility that $\Phi'_p(t_o)$ differs from $\Phi_p(t_o)$.

(b) <u>The dispenser trajectory assumed to be unknown</u>.

Let us now consider the situation where the dispenser trajectory $\overline{p}_d(t)$ is not known. Equation (4.4-2) still constitutes a valid input-output state relation for the object, since none of the quantities in (4.4-2) depend on the dispenser trajectory. (It has been shown that $F_q(t_i, t_j)$ is independent of the reference trajectory.) However, (4.4-1) cannot be considered to specify the output function $\overline{\eta}_q(t)$ in terms of the state $\overline{\eta}_q(t_o)$, which immediately rules out (4.4-1) as an input-output-state relation.

This difficulty can be overcome by including the dynamic state vector,

$\overline{p}_d(t_o)$, of the dispenser in the specification of the state of the object at time t_o. Then, since $\overline{q}_d(t)$ is determined by $\overline{p}_d(t_o)$, Eq. (4.4-1) again yields an input-output-state relation. To make this explicit, let us use the relation

$$\overline{q}_d(t) = M\overline{p}_d(t) = M\{\overline{p}_1(t) + F_p(t, t_o)[\overline{p}_d(t_o) - \overline{p}_1(t_o)] \qquad (4.4\text{-}12)$$

where $\overline{p}_1(t)$ is an arbitrary reference trajectory. Substituting this expression into (4.4-1) gives

$$\overline{\eta}_q(t) = \overline{q}_1(t) + MF_p(t, t_o)[\overline{p}_d(t_o) - \overline{p}_1(t_o)] + F_q(t, t_o)[\overline{\eta}_q(t_o) - \overline{q}_d(t_o)] \quad (4.4\text{-}13)$$

which relates the output function $\overline{\eta}_q(t)$ to the state variables in $\overline{p}_d(t_o)$ and $\overline{\eta}_q(t_o)$. The number of these state variables is nine.

The system, as represented by the input-output-state relations (4.4-13) and (4.4-2) cannot be in reduced form. This is evident from the fact that it is possible to use the input-output-state relation (3 4 1), which applies to impulsively dispensed clouds, in place of (4.4-13). The former equation requires that the state specification include $\overline{\eta}_p(t_o)$, which contains six elements, while the latter equation involves nine state variables. Thus, in the case of an impulsively dispensed cloud with unknown dispenser trajectory, the elements of $\overline{\eta}_p(t_o)$ and $\Phi_q(t_o)$ should be used as the state variables of the object, with the input-output-state relations given by (3.3-14) and (4.4-2).

With this representation, the system is in reduced form. That is, no two states of the system are equivalent. It has already been pointed out that $\Phi_q(t_o) \neq \Phi'_q(t_o)$ implies that the output functions corresponding to the states are not equal, at least at time t_o. Also, with the dispenser trajectory considered to be unknown, the elements of $\overline{\eta}_p(t_o)$ are independent (Eq. (4.4-11) no longer defines a function relating $\overline{\eta}_u(t_o)$ to $\overline{\eta}_q(t_o)$), and the proof, in Section 3.5, that $\overline{\eta}_p(t_o) \neq \overline{\eta}'_p(t_o)$ implies that the output functions corresponding to the states are not equal, applies.

For a direct proof that the system, as represented by (4.4-13) and (4.4-2), is not in reduced form, it is shown that the system has equivalent states. In particular, it is shown that two states $\overline{p}_d(t_o)$, $\overline{\eta}_q(t_o)$, $\Phi_q(t_o)$ and $\overline{p}_e(t_o)$, $\overline{\eta}'_q(t_o)$, $\Phi'_q(t_o)$ are equivalent if

$$\Phi_q(t_o) = \Phi'_q(t_o) \tag{4.4-14}$$

$$\overline{\eta}_q(t_o) = \overline{\eta}'_q(t_o) \tag{4.4-15}$$

$$M F_p(t_d, t_o) \overline{p}_d(t_o) = M F_p(t_d, t_o) \overline{p}_e(t_o) \tag{4.4-16}$$

The last equation states that

$$\overline{q}_d(t_d) = \overline{q}_e(t_d) \; ; \tag{4.4-17}$$

that is: that the trajectories $\overline{p}_d(t)$ and $\overline{p}_e(t)$ corresponding to the dynamic state $\overline{p}_d(t_o)$ and $\overline{p}_e(t_o)$, coincide at the dispensing instant. The trajectories which pass through the point $\overline{q}_d(t_d)$ at time t_d are infinite in number, and our assertion states that the dynamic state vector, $\overline{p}_e(t_o)$, corresponding to any of these trajectories, when incorporated into a state specification, gives rise to the same output functions $\overline{\eta}_q(t)$, $\Phi_q(t)$, as does the dynamic state vector $\overline{p}_d(t_o)$ of the dispenser at time t_o.

To prove this assertion, let the output function $\overline{\eta}_q(t)$ be given by (4.4-13) and define $\overline{\eta}'_q(t)$ by

$$\overline{\eta}'_q(t) = \overline{q}_1(t) + M F_p(t, t_o)[\overline{p}_e(t_o) - \overline{p}_1(t_o)] + F_q(t, t_o)[\overline{\eta}_q(t_o) - \overline{q}_e(t_o)] \tag{4.4-18}$$

Obviously, the input-output-relation (4.4-2) need not be considered in the present discussion. Subtracting (4.4-18) from (4.4-13) gives

$$\overline{\eta}_q(t) - \overline{\eta}_q'(t) = M F_p(t, t_o) [\overline{p}_d(t_o) - \overline{p}_e(t_o)] + F_q(t, t_o) [\overline{q}_e(t_o) - \overline{q}_d(t_o)] \, . \tag{4.4-19}$$

Now

$$M F_p(t, t_o) [\overline{p}_d(t_o) - \overline{p}_e(t_o)] = \overline{q}_d(t) - \overline{q}_e(t) \tag{4.4-20}$$

and since trajectory $\overline{p}_e(t)$ is such that the equation

$$\overline{p}_e(t_d) = \overline{p}_d(t_d) + [0 \vdots \overline{w}]^T \tag{4.4-21}$$

holds for some $\overline{w}$, we have

$$F_q(t, t_o) [\overline{q}_e(t_o) - \overline{q}_d(t_o)] = \overline{q}_e(t) - \overline{q}_d(t) \tag{4.4-22}$$

using (4.1-14). Substituting (4.4-20) and (4.4-22) into (4.4-19) gives finally

$$\overline{\eta}_q(t) - \overline{\eta}_q'(t) = 0 \qquad \text{all } t \tag{4.4-23}$$

which proves our assertion.

Thus, all states $\overline{p}_e(t_o)$, $\overline{\eta}_q(t_o)$, $\Phi_q(t_o)$ for which

$$\overline{q}_e(t_d) = M F_p(t_d, t_o) \overline{p}_e(t_o) = \overline{q}_d(t_d) \tag{4.4-24}$$

are equivalent, which indicates that, in the case where the dispenser trajectory is not known, the system as represented by the input-output-state relations (4.4-13) and (4.4-2) is not in reduced form. But another conclusion can be drawn relevant to the case where information about the dispenser trajectory is available. If only the position of the dispenser at the dispensing instant, $\overline{q}_d(t_d)$, is known, any trajectory $\overline{p}_e(t)$ for which (4.4-24) holds can be used in place of $\overline{p}_d(t)$ in the input-output-state relation (4.4-1). That is, any trajectory which coincides with the dispenser trajectory at the dispensing instant can be used in place of the dispenser trajectory. This increases, to some extent, the number of situations in which the minimum number of state variables can be used to represent the state of an impulsively dispensed scatterer ensemble.

CHAPTER V

ESTIMATION OF THE CENTROID AND SECOND CENTRAL MOMENT MATRIX

This chapter discusses the estimation of the centroid, $\overline{\eta}_q(t)$, and second central moment matrix, $\Phi_q(t)$, of a scatterer ensemble based on data acquired by means of a single pulse transmitted by a wide-beam monopulse radar, or by means of an instantaneous scan performed by a narrow-beam radar. Idealized radar models are introduced, the forms of the received voltages are discussed, and estimators of the cloud parameters of interest are formulated. Properties of the estimators are discussed following development of estimators for the wide-beam monopulse and narrow-beam radars.

5.1 Estimators for a wide-beam monopulse radar

(a) Radar model

Consider a wide-beam monopulse radar[32] located at the origin of a coordinate system, $\mathcal{C}$. The basis vectors for $\mathcal{C}$ are denoted $\overline{a}^{(m)}$, $m = 1, 2, 3$ and are oriented in directions of increasing range, elevation, and traverse, respectively. Assume that the complex video voltages received in the sum and difference channels, when a pulse transmitted at $t = 0$ is reflected from a point scatterer, are given by

$$h^{(m)}(t) = \begin{cases} k^{(m)} \alpha^{(m)} \sigma e^{j\psi}/r^2 & 2r \leq ct < 2(r+\Delta R) \\ 0 & \text{otherwise} \end{cases} \qquad (5.1\text{-}1)$$

for $m = 1, 2, 3$, corresponding to the sum, elevation difference, and azimuth difference channels respectively. In (5.1-1), r is the range to the scatterer,

σ^2 is its radar cross-section, and ψ is the phase of the received echo. The $k^{(m)}$ are radar constants, ΔR is the radar range resolution, $\alpha^{(1)} \equiv 1$, and $\alpha^{(m)}$, $m = 2,3$ are direction cosines of the position vector of the scatterer, given by

$$\alpha^{(m)} = \overline{q}^T \,\overline{a}^{(m)}/r \qquad m = 2,3 \tag{5.1-2}$$

where $\overline{q}$ represents the scatterer position in coordinate system $\mathcal{C}$.

The above is a reasonable model of the performance of an amplitude monopulse radar for point targets located near the $\overline{a}^{(1)}$ axis, in the region where the one way voltage sum beam gain is approximately constant, while the difference beam gains are approximately linearly related to angular displacement.

For a scatterer which is located near the $\overline{a}^{(1)}$ axis,

$$(\alpha^{(2)})^2 + (\alpha^{(3)})^2 << 1 \tag{5.1-3}$$

so that

$$\overline{q}^T \,\overline{a}^{(1)}/r = \sqrt{1 - (\alpha^{(2)})^2 - (\alpha^{(3)})^2} \cong 1 \tag{5.1-4}$$

With this approximation

$$\overline{q} \cong r\,\overline{\alpha} \tag{5.1-5}$$

where

$$\overline{\alpha} = [\,\alpha^{(1)}\,,\, \alpha^{(2)}\,,\, \alpha^{(3)}\,]^T \,. \tag{5.1-6}$$

(b) Voltages received by the radar

Assume that the scatterer ensemble $\mathcal{S}$ is contained within the beam. The complex video voltages received in the sum and difference channels, when a pulse is transmitted at $t = 0$, are then given by

$$v_o^{(m)}(t) = \sum_{i_t} k^{(m)} \alpha_i^{(m)} \sigma_i e^{j\psi_i}/r_i^2 \qquad (5.1\text{-}7)$$

where the summation is over scatterers for which

$$ct/2 - \Delta R < r_i \le ct/2 \qquad (5.1\text{-}8)$$

If the voltages $v_o^{(m)}(t)$ are sampled at the instants t_n, $n = 1, 2, \ldots, N$ corresponding to ranges $R_n = ct_n/2$, where

$$R_n = R_o + n\Delta R \qquad n = 1, \ldots, N \qquad (5.1\text{-}9)$$

the voltage samples $v_o^{(m)}(t_n)$ are given by (5.1-7) with the summation over scatterers for which

$$R_{n-1} < r_i \le R_n \qquad (5.1\text{-}10)$$

It is assumed that R_o and N are chosen so that the range interval $(R_o, R_N]$ contains the scatterer ensemble.

Finally, define the calibrated voltage samples $v_n^{(m)}$ by

$$v_n^{(m)} \equiv R_n^2 v_o^{(m)}(t_n)/k^{(m)} = R_n^2 \sum_{i_n} \alpha_i^{(m)} \sigma_i e^{j\psi_i}/r_i^2 \qquad (5.1\text{-}11)$$

with the summation again over scatterers in the interval (5.1-10).

(c) Definition of the estimators

Forming the vectors

$$\overline{v}_n = [v_n^{(1)}, v_n^{(2)}, v_n^{(3)}]^T \qquad n = 1, \ldots, N, \qquad (5.1\text{-}12)$$

the estimators of centroid, second moment matrix, and SCM matrix are defined respectively by

$$\hat{\bar{\eta}}_q = \frac{\mathrm{Re}\sum_n R_n v_n^{(1)} \bar{v}_n^*}{\sum_n v_n^{(1)} v_n^{(1)*}} \tag{5.1-13}$$

$$\hat{\Psi}_q = \frac{\mathrm{Re}\sum_n R_n^2 \bar{v}_n \bar{v}_n^{T*}}{\sum_n v_n^{(1)} v_n^{(1)*}} \tag{5.1-14}$$

$$\hat{\Phi}_q = \hat{\Psi}_q - \hat{\bar{\eta}}_q \hat{\bar{\eta}}_q^T . \tag{5.1-15}$$

Estimators of this form were first proposed by Helms[18], and are related to certain maximum likelihood estimators discussed by Aiken[16]. The m^{th} component of $\hat{\bar{\eta}}_q$ is given by

$$(\hat{\bar{\eta}}_q)_m = \frac{\mathrm{Re}\sum_n R_n v_n^{(1)} v_n^{(m)*}}{\sum_n v_n^{(1)} v_n^{(1)*}} \tag{5.1-16}$$

while element (ℓ, m) of $\hat{\Psi}_q$ is given by

$$(\hat{\Psi}_q)_{\ell,m} = \frac{\mathrm{Re}\sum_n R_n^2 v_n^{(\ell)} v_n^{(m)*}}{\sum_n v_n^{(1)} v_n^{(1)*}} \tag{5.1-17}$$

These estimators will be shown below to have certain desirable properties.

5.2 Estimators for a narrow-beam radar

(a) Radar model

Assume that the complex video voltage received by the radar, when a pulse transmitted at $t = 0$ is reflected from a point scatterer, is given by

$$h^{(m)}(t) = \begin{cases} k\,\sigma\, e^{j\psi}/r^2 & 2r \le ct < 2(r+\Delta R);\ (\theta,\varphi)\in\Delta\,\Omega^{(m)} \\ 0 & \text{otherwise} \end{cases} \qquad (5.2\text{-}1)$$

where k, σ, ψ, r, and ΔR have the same meaning they had in subsection 5.1a. The elements of the ordered pair (θ,φ) are the polar and azimuthal angles of the scatterer, and $\Delta\Omega^{(m)}$ is a solid angle, subtended at the radar, corresponding to beam position m.

(b) Voltages received by the radar

If the scatterer ensemble is observed by the radar, and the received video voltage is sampled and calibrated as were the voltages in subsection 5.1b, the calibrated voltage samples

$$v_n^{(m)} = R_n^{\,2} \sum_{i_{mn}} \sigma_i\, e^{j\psi_i}/r_i^{\,2} \qquad (5.2\text{-}2)$$

are obtained, where the ranges R_n are defined by (5.1-9), and the summation is over scatterers for which (5.1-10) and

$$(\theta_i, \varphi_i) \in \Delta\,\Omega^{(m)} \qquad (5.2\text{-}3)$$

are true.

Again, it is supposed that R_0 and N are chosen so that the range interval $(R_0, R_N]$ contains the scatterer ensemble. In addition, assume that the radar scans (instantaneously) over the entire solid angle containing the cloud. The scan is taken to consist of a set of beam positions corresponding to $m = 1, 2, \ldots,$ and the corresponding solid angles $\Delta\Omega^{(m)}$ are assumed to be non-overlapping.

(c) Definition of the estimators

Let us associate, with each calibrated voltage sample $v_n^{(m)}$, the position vector $\overline{q}_{mn}$ of some point in the region $R_{n-1} < r \leq R_n$, $(\theta, \varphi) \in \Delta\Omega^{(m)}$. The estimators of centroid and second moment matrix are defined respectively by

$$\hat{\overline{\eta}}_q = \frac{\sum_{m,n} \overline{q}_{mn} v_n^{(m)} v_n^{(m)*}}{\sum_{m,n} v_n^{(m)} v_n^{(m)*}} \tag{5.2-4}$$

$$\hat{\Psi}_q = \frac{\sum_{m,n} \overline{q}_{mn} \overline{q}_{mn}^T v_n^{(m)} v_n^{(m)*}}{\sum_{m,n} v_n^{(m)} v_n^{(m)*}} \tag{5.2-5}$$

Properties of these estimators, and of the estimators which use wide-beam monopulse radar data will now be discussed.

5.3 A basic property of the estimators

Each of the estimators (5.1-13), (5.1-14), (5.2-4), and (5.2-5) for the centroid or second moment matrix is in the form of a quotient. It is asserted that, in each case, the ratio of the expected value of the numerator to the expected value of the denominator gives approximately the quantity estimated, provided that the radar resolution ΔR (and $\Delta\Omega^{(m)}$ in the narrow-beam case) is fine compared to the cloud dimensions. This property of the estimators is verified by substituting the expression for the calibrated voltage samples (i.e., either (5.1-11) or (5.2-2)) and taking expected values of numerator and denominator. In taking the expected values, the assumption that the echo phases ψ_i are independent random variables, each uniformly distributed in angle, is essential. This assumption was discussed previously in subsection 2.2(c).

(a) The estimated quantities

The quantities estimated are the centroid $\overline{\eta}_q$, the second moment matrix Ψ_q, and SCM matrix Φ_q, which can be written

$$\overline{\eta}_q = \frac{\int \overline{q}\,\rho_q(\overline{q})\,d\tau_q}{T} \tag{5.3-1}$$

$$\Psi_q = \frac{\int \overline{q}\,\overline{q}^T\,\rho_q(\overline{q})\,d\tau_q}{T} \tag{5.3-2}$$

$$\Phi_q = \frac{\int [\overline{q}-\overline{\eta}_q][\overline{q}-\overline{\eta}_q]^T\,\rho_q(\overline{q})\,d\tau_q}{T} \tag{5.3-3}$$

omitting the notation of the dependence on the time. The relationship $\rho_q(\overline{q}) = T f_q(\overline{q})$ has been used, where T is the total radar cross-section of the scatterer ensemble.

It is shown that the expected values of the denominators of the estimators are equal approximately to T, while the expected values of the numerators of the estimators are equal approximately to the corresponding numerator in the above expressions for the estimated quantities.

(b) Expectation of the denominators of the estimators

First consider the denominators of the estimators (5.1-13) and (5.1-14). Substituting $v_n^{(1)}$ from (5.1-11) and taking the expectation gives

$$\mathcal{E}\sum_n v_n^{(1)} v_n^{(1)*} = \mathcal{E}\sum_n R_n^4 \sum_{i_n, k_n} \alpha_i^{(1)} \alpha_k^{(1)} \sigma_i \sigma_k e^{j(\psi_i - \psi_k)}/r_i^2 r_k^2$$

$$= \sum_n R_n^4 \sum_{i_n} s_i^2/r_i^4 \tag{5.3-4}$$

since $\alpha_i^{(1)} = 1$, all i, and

$$\mathcal{E}\{e^{j(\psi_i - \psi_k)}\} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases} \tag{5.3-5}$$

for independent, uniformly distributed scatterer echo phases.

Now, assuming that $\Delta R << R_{n-1}$ so that for each scatterer in the interval (5.1-10), $r_i/R_n \cong 1$, the equation

$$\mathcal{E}\sum_n v_n^{(1)} v_n^{(1)*} \cong \sum_n \sum_{i_n} s_i^2 = \sum_i s_i^2 \tag{5.3-6}$$

is obtained, with the last summation taken over the entire ensemble. But, by the definition of $\rho_q(\bar{q})$,

$$\sum_i s_i^2 = \int \rho_q(\bar{q})\, d\tau_q = T \tag{5.3-7}$$

which establishes the desired result.

Proceeding in a similar manner, substituting $v_n^{(m)}$ from (5.2-2) into the denominator of (5.2-4) or (5.2-5) and taking the expected value gives

$$\mathcal{E}\sum_{m,n} v_n^{(m)}\, v_n^{(m)*} = \mathcal{E}\sum_{m,n} R_n^4 \sum_{i_{mn}, k_{mn}} \sigma_i\, \sigma_k\, e^{j(\psi_i-\psi_k)}/r_i^2\, r_k^2$$

$$= \sum_{m,n} R_n^4 \sum_{i_{mn}} s_i^2/r_i^4 \qquad (5.3\text{-}8)$$

$$\cong \sum_{m,n} \sum_{i_{m,n}} s_i^2 = \sum_i s_i^2 = T \qquad (5.3\text{-}9)$$

using the arguments used previously.

(c) <u>Expectation of the numerators of the wide-beam monopulse estimators</u>

Substituting $v_n^{(1)}$ and $\overline{v}_n$ from (5.1-11) and (5.1-12) into the numerator of (5.1-13) and taking the expectation yields

$$\mathcal{E}\,\mathrm{Re} \sum_n R_n\, v^{(1)} \overline{v}_n^{\,*} = \mathcal{E}\,\mathrm{Re} \sum_n R_n^5 \sum_{i_n k_n} \alpha_i^{(1)}\, \overline{\alpha}_k\, \sigma_i\, \sigma_k\, e^{j(\psi_i-\psi_k)}/r_i^2\, r_k^2$$

$$= \sum_n R_n^5 \sum_{i_n} \overline{\alpha}_i\, s_i^2/r_i^4 \qquad (5.3\text{-}10)$$

using the definition (5.1-6) and equation (5.3-5).

Now (5.3-10) is equivalent to

$$\mathcal{E}\,\mathrm{Re} \sum_n R_n\, v_n^{(1)} \overline{v}_n^{\,*} = \sum_n R_n^5 \sum_{i_n} r_i\, \overline{\alpha}_i\, s_i^2/r_i^5$$

$$\cong \sum_n R_n^5 \sum_{i_n} \overline{q}_i\, s_i^2/r_i^5 \qquad (5.3\text{-}11)$$

with (5.1-5) used to obtain the approximation. The errors in scatterer positions arising from use of the approximation (5.1-5) are likely to be negligible in comparison to cloud dimensions if the scatterers in the ensemble are not excessively displaced from the $\overline{a}^{(1)}$ axis. The displacements will certainly be small if the ensemble is contained within the beam of a practical monopulse radar. Finally, assume that ΔR is small compared to cloud dimensions, and that both are much smaller than the range to the cloud. Then, the errors in scatterer positions arising from use of the approximation $r_i \cong R_n$ for scatterers in the interval (5.1-10) will be negligible in comparison with cloud dimensions. This approximation gives

$$\mathcal{E}\,\mathrm{Re}\sum_n R_n\, v_n^{(1)}\, \overline{v}_n^{\,*} \cong \sum_n \sum_{i_n} \overline{q}_i\, s_i^2 = \sum_i \overline{q}_i\, s_i^2$$

$$= \int \overline{q}\, \rho_q(\overline{q})\, d\tau_q \qquad (5.3\text{-}12)$$

which is the desired result.

Proceeding in a similar manner, substitution of $\overline{v}_n$ from (5.1-11) and (5.1-12) into the numerator of (5.1-14) and taking the expectation yields

$$\mathrm{Re}\sum_n R_n^2\, \overline{v}_n\, \overline{v}_n^{T*} = \mathrm{Re}\sum_n R_n^6 \sum_{i_n k_n} \overline{\alpha}_i\, \overline{\alpha}_k^T\, \sigma_i\, \sigma_k\, e^{j(\psi_i-\psi_k)}/r_i^2\, r_k^2$$

$$= \sum_n R_n^6 \sum_{i_n} \overline{\alpha}_i\, \overline{\alpha}_k^T\, s_i^2/r_i^4$$

$$= \sum_n R_n^6 \sum_{i_n} r_i^2 \overline{\alpha}_i \overline{\alpha}_k^T\, s_i^2/r_i^6$$

$$\cong \sum_n R_n^6 \sum_{i_n} \overline{q}_i\, \overline{q}_i^T\, s_i^2/r_i^6$$

$$\cong \sum_n \sum_{i_n} \overline{q}_i\, \overline{q}_i^T\, s_i^2 = \sum_i \overline{q}_i\, \overline{q}_i^T\, s_i^2$$

$$= \int \overline{q}\, \overline{q}^T \rho_q(\overline{q})\, d\tau_q \qquad (5.3\text{-}13)$$

which is the desired result. The arguments and approximations used are the same as those used previously.

(d) Expectation of the numerators of the narrow-beam estimators

Substituting $v_n^{(m)}$ from (5.2-2) into the numerator of (5.2-4) and taking the expected value gives

$$\mathcal{E} \sum_{m,n} \overline{q}_{mn} v_n^{(m)} v_n^{(m)*} = \mathcal{E} \sum_{m,n} \overline{q}_{mn} R_n^4 \sum_{i_{mn} k_{mn}} \sigma_i \sigma_k e^{j(\psi_i - \psi_k)} / r_i^2 r_k^2$$

$$= \sum_{m,n} \overline{q}_{mn} R_n^4 \sum_{i_{mn}} s_i^2 / r_i^4 \qquad (5.3\text{-}14)$$

Now assuming that $\Delta R << R_n$, the approximation

$$\mathcal{E} \sum_{m,n} \overline{q}_{mn} v_n^{(m)} v_n^{(m)*} \cong \sum_{m,n} \overline{q}_{mn} \sum_{i_{mn}} s_i^2 \qquad (5.3\text{-}15)$$

is obtained. And finally, assuming that ΔR and $\Delta \Omega^{(m)}$ are small compared to cloud dimensions, the errors in scatterer positions involved in the approximation $\overline{q}_i \cong \overline{q}_{mn}$, for all scatterers in the volume element corresponding to $\overline{q}_{mn}$, will be negligible compared to cloud dimensions. The use of this approximation yields

$$\mathcal{E} \sum_{m,n} \overline{q}_{mn} v_n^{(m)} v_n^{(m)*} \cong \sum_{m,n} \sum_{i_{mn}} \overline{q}_i s_i^2$$

$$= \sum_i \overline{q}_i s_i^2 = \int \overline{q} \rho_q(\overline{q}) d\tau_q \qquad (5.3\text{-}16)$$

which was to be obtained.

Proceeding similarly with the numerator of (5.2-5) gives

$$\mathcal{E} \sum_{m,n} \overline{q}_{mn} \overline{q}_{mn}^T v_n^{(m)} v_n^{(m)*} = \sum_{m,n} \overline{q}_{mn} \overline{q}_{mn}^T R_n^4 \sum_{i_{mn}} s_i^2 / r_i^4 \qquad (5.3\text{-}17)$$

$$\cong \sum_{m,n} \overline{q}_{mn} \overline{q}_{mn}^T \sum_{i_{mn}} s_i^2 \cong \sum_{m,n} \sum_{i_{mn}} \overline{q}_i \overline{q}_i^T s_i^2$$

$$= \sum_i \overline{q}_i \overline{q}_i^T s_i^2 = \int \overline{q} \, \overline{q}^T \rho_q(\overline{q}) \, d\tau_q \qquad (5.3\text{-}18)$$

which was the desired result.

This completes the verification of the property of the estimators which is under discussion.

5.4 Asymptotic bias of the estimators

In the previous section, it was shown that the ratio of the expected values of the numerator and denominator of each of the estimators (5.1-13), (5.1-14), (5.2-4), and (5.2-5) give approximately the quantity estimated, provided that the resolution of the radar is fine compared to cloud dimensions. In the present section, it is shown that as the radar resolution becomes arbitrarily fine, the bias of each estimator approaches zero. It is sufficient to show that the variance of the denominator of each estimator approaches zero, since then the expectation of the estimator is equal to the ratio of the expectations of the numerator and denominator.

First consider the denominators of the estimators (5.1-13) and (5.1-14). Denoting by

$$u_n = v_n^{(1)} v_n^{(1)*} \tag{5.4-1}$$

the power or intensity corresponding to the n^{th} sum channel calibrated voltage sample, the variance of the denominators is given by

$$\operatorname{var} \sum_n u_n = \sum_n \operatorname{var} u_n \ . \tag{5.4-2}$$

But the random variables u_n are exponentially distributed[22], so

$$\operatorname{var} \sum_n u_n = \sum_n \mathcal{E}^2 \{ u_n \} \tag{5.4-3}$$

Now, from (5.3-4) and (5.3-6), $\mathcal{E}\{u_n\}$ is given exactly by

$$\mathcal{E}\{u_n\} = \sum_n R_n^4 \sum_{i_n} s_i^2 / r_i^4 \tag{5.4-4}$$

and approximately by

$$\mathcal{E}\{u_n\} \cong \sum_{i_n} s_i^2 \tag{5.4-5}$$

the approximation in (5.4-5) becoming exact as $\Delta R \to 0$. From (5.4-4) it is evident that $\mathcal{E}\{u_n\} \to 0$ as $\Delta R \to 0$, since $\rho_q(\overline{q})$ is finite everywhere. (This would not be true if $\rho_q(\overline{q})$ were permitted to contain impulses.)

For a given ΔR,

$$\operatorname{var} \sum_n u_n \leq \mathcal{E}\{u_n\}_{max} \sum_n \mathcal{E}\{u_n\} \tag{5.4-6}$$

where $\mathcal{E}\{u_n\}_{max}$ is the maximum value for all n. Using (5.4-5)

$$\sum_n \mathcal{E}\{u_n\} \cong T \tag{5.4-7}$$

the approximation becoming exact as $\Delta R \to 0$. Thus, since $\mathcal{E}\{u_n\}_{max} \to 0$ in (5.4-6) as $\Delta R \to 0$, the assertion that $\operatorname{var} \sum_n u_n \to 0$ is proved.

The corresponding proof for the denominators of the estimators (5.2-4) and (5.2-5) is almost identical to the preceding and will be omitted. The only difference is that the summations over the index n are replaced by double summations over indices m and n. Note that $\Delta\Omega^{(m)} \to 0$ is not a necessary assumption for the proof, since $\Delta R \to 0$ is enough to ensure that $\mathcal{E}\{v_n^{(m)} v_n^{(m)*}\}_{max} \to 0$.

CHAPTER VI

PREDICTION OF CENTROID AND SCM MATRIX

In previous chapters, the scatterer ensemble has been discussed as an abstract object having as outputs the centroid $\overline{\eta}_q(t)$ and SCM matrix $\Phi_q(t)$. Also, observation of the output variables has been related to the physical problem in terms of the estimation of the centroid and SCM matrix on the basis of data acquired by a wide-beam monopulse radar or a narrow-beam radar scan. State variables have been identified for the abstract object and the state equations derived.

In this chapter, it is assumed that a set of observations $\hat{\overline{\eta}}_q(t_i)$, $\hat{\Phi}_q(t_i)$, $i = 1, \ldots, N$ are available. The primary problem considered is the prediction of $\overline{\eta}_q(t_{N+1})$, $\Phi_q(t_{N+1})$. This problem is considered for the case of an impulsively dispersed cloud as well as for the general case. In discussing the case of an impulsively dispensed cloud, it is assumed that the dispenser trajectory is unknown. Estimation of the dispensing time, based on the observations, will be considered.

To demonstrate the application of several main features of the approach to estimation and prediction presented in this and previous chapters, an example, using a one dimensional scatterer ensemble is presented, and the results are discussed.

6.1 Least squares estimation

In the general case, and in the case of impulsively dispensed clouds with known dispensing time, the prediction of $\overline{\eta}_q(t_{N+1})$ and $\Phi_q(t_{N+1})$ involves the prediction of the state of a time-varying linear system based on perturbed observations of the system output. The prediction algorithms considered here are based on a generalized least-squares estimation technique,[33] which is described briefly in this section, and in greater detail in Appendix A.

Suppose a time-varying linear system obeys the state equation

$$\overline{x}(t_i) = L(t_i, t_j)\, \overline{x}(t_j) \tag{6.1-1}$$

where $\overline{x}(t)$ is the state vector for the system and $L(t_i, t_j)$ is the state transition matrix. Let the vector of observed output variables be denoted $\overline{y}(t)$ and system observation be described by the equation

$$\overline{y}(t) = H(t)\, \overline{x}(t) + \overline{n}(t) \tag{6.1-2}$$

where $H(t)$ is a matrix and $\overline{n}(t)$ is a random vector representing observation noise.

Given observations $\overline{y}(t_i)$, $i = 1, \ldots, N$, form the matrix R according to

$$R = \begin{bmatrix} H(t_N) \\ \hline H(t_{N-1})\, L(t_{N-1}, t_N) \\ \hline \vdots \\ \hline H(t_1)\, L(t_1, t_N) \end{bmatrix} \tag{6.1-3}$$

Then the least squares estimate of $\overline{x}(t_N)$, denoted $\overline{x}^*(t_N)$, is given by

$$\overline{x}^*(t_N) = (R^T R)^{-1} R^T Y \tag{6.1-4}$$

where the vector Y is defined by

$$Y = \begin{bmatrix} \bar{y}(t_N) \\ \hdashline \bar{y}(t_{N-1}) \\ \hdashline \vdots \\ \hdashline \bar{y}(t_1) \end{bmatrix} \tag{6.1-5}$$

The least squares prediction of $\bar{x}(t_{N+1})$ is then given by

$$\bar{x}^*(t_{N+1}) = L(t_{N+1}, t_N)\, \bar{x}^*(t_N) \tag{6.1-6}$$

The predicted state vector is optimum in the least squares sense, and if the observation noise vectors $\bar{n}(t_i)$, $i = 1, \ldots, N$ have zero mean, the prediction is unbiased.

6.2 Application to prediction for scatterer ensembles

(a) Prediction in the general case

The state variables consist of the elements of $\Delta_1\overline{\eta}_p(t)$ and the independent elements of $\Phi_p(t)$, and the state equations are given by (3.2-24) and (3.2-31). Observation is described by

$$\Delta_1 \hat{\overline{\eta}}_q(t_i) = M\,\Delta_1\,\overline{\eta}_p(t_i) + \overline{u}_q(t_i) \tag{6.2-1}$$

$$\hat{\Phi}_q(t_i) = M\,\Phi_p(t_i)\,M^T + V_q(t_i) \tag{6.2-2}$$

where $\Delta_1\hat{\overline{\eta}}_q(t_i)$ and $\hat{\Phi}_q(t_i)$ are the observed values of the cloud centroid and SCM matrix (i.e., estimated from radar data), $M = [I \vdots 0]$, $\overline{u}_q(t_i)$ and $V_q(t_i)$ are a random vector and random matrix representing the observation errors which include the effects of random scatterer radar cross-section and echo phase, as well as the usual sources of noise.

The technique described in the previous section can be used to form the least squares estimates of $\Delta_1\overline{\eta}_p(t_{N+1})$ and $\Phi_p(t_{N+1})$, from which the desired predicted quantities can be obtained. These are given by

$$\Delta_1\overline{\eta}_q^{\star}(t_{N+1}) = M\,\Delta_1\,\overline{\eta}_p^{\star}(t_{N+1}) \tag{6.2-3}$$

$$\Phi_q^{\star}(t_{N+1}) = M\Phi_p^{\star}(t_{N+1})\,M^T \tag{6.2-4}$$

Note that the state and observation equations for $\Delta_1\overline{\eta}_p(t)$ and $\Phi_p(t)$ are not coupled. Thus, prediction of $\Delta_1\overline{\eta}_q(t_{N+1})$ and $\Phi_q(t_{N+1})$ can be treated separately. To obtain the least squares prediction for $\Delta_1\overline{\eta}_q(t_{N+1})$, the algorithm of the previous section is applied with $H(t) = M$, $L(t_i, t_j) = F_p(t_i, t_j)$, $\overline{y}(t_i) = \Delta_1\hat{\overline{\eta}}_q(t_i)$, and $\overline{x}(t) = \Delta_1\overline{\eta}_p(t)$. The algorithm yields $\Delta_1\overline{\eta}_p^{\star}(t_{N+1})$ and finally $\Delta_1\overline{\eta}_q^{\star}(t_{N+1})$ is given by (6.2-3).

To obtain the least squares prediction for $\Phi_q(t_{N+1})$, it is necessary first to form a vector, say $\overline{\phi}_p(t)$, from the independent elements of $\Phi_p(t)$. Then (3.2-31) yields the state equation

$$\overline{\phi}_p(t_i) = L_p(t_i, t_j)\, \overline{\phi}_p(t_j) \qquad (6.2\text{-}5)$$

where the elements of the transition matrix $L_p(t_i, t_j)$ are given by sums of products of elements of the dynamic state transition matrix $F_p(t_i, t_j)$. Similarly, a vector $\overline{\phi}_q(t)$ is formed from the independent elements of the matrix $\Phi_q(t)$. It can be assumed that the elements of $\overline{\phi}_q(t)$ and $\overline{\phi}_p(t)$ are arranged so that

$$\overline{\phi}_p(t) = [\,\overline{\phi}_q(t) \,\vdots\, \overline{r}(t)\,]^T \qquad (6.2\text{-}6)$$

where the vector $\overline{r}(t)$ represents the elements in $\overline{\phi}_p(t)$ which are not in $\overline{\phi}_q(t)$. Then

$$\overline{\phi}_q(t) = M'\overline{\phi}_p(t) = [\, I \,\vdots\, 0\,]^T\, \overline{\phi}_p(t) \qquad (6.2\text{-}7)$$

where M' is $[21 \times 6]$.

The algorithm of the previous section can now be applied with $H(t) = M'$, $L(t_i, t_j) = L_p(t_i, t_j)$, $\overline{y}(t_i) = \hat{\overline{\phi}}_q(t_i)$, and $\overline{x}(t) = \overline{\phi}_p(t)$. The algorithm yields $\overline{\phi}_p^*(t_{N+1})$ and $\overline{\phi}_q^*(t_{N+1})$ is given by

$$\overline{\phi}_q^*(t_{N+1}) = M'\, \overline{\phi}_p^*(t_{N+1}). \qquad (6.2\text{-}8)$$

(b) Prediction for impulsively dispensed clouds

We assume that the dispenser trajectory is not known. The state variables then consist of the elements of $\Delta_1 \overline{\eta}_p(t)$ and the indpendent elements of $\Phi_q(t)$, and the state equations are given by (3.2-24) and (4.2-14). Observation is described by (6.2-1) and

$$\hat{\Phi}_q(t_i) = \Phi_q(t_i) + V_q(t_i) \tag{6.2-9}$$

where $V_q(t_i)$ is a random matrix, as before.

The prediction of $\overline{\eta}_q(t_{N+1})$ is obtained as in the previous subsection. To obtain the least squares prediction for $\Phi_q(t_{N+1})$, consider that (4.2-14) yields the state equation

$$\overline{\phi}_q(t_i) = L_q(t_i, t_j)\,\overline{\phi}_q(t_j) \tag{6.2-10}$$

where $\overline{\phi}_q(t)$ is the vector of independent elements of $\Phi_q(t)$ introduced in the previous subsection, and the elements of the transition matrix $L_q(t_i, t_j)$ are given by sums of products of elements of the matrix $F_q(t_i, t_j)$.

Now the algorithm of Section 6.1 can be applied with $H(t) = I$, $L(t_i, t_j) = L_q(t_i, t_j)$, $\overline{y}(t_i) = \hat{\overline{\phi}}_q(t_i)$, and $\overline{x}(t) = \overline{\phi}_q(t)$. The algorithm yields $\overline{\phi}_q^{\,*}(t_{N+1})$.

6.3 Estimation of the dispensing time

Assume that it is known a priori that a cloud is impulsively dispensed, but the dispensing time is not known. A method is discussed here which uses a set of measurements $\hat{\Phi}_q(t_i)$, $i = 1, \ldots, N$ to estimate the dispensing time, t_d. The method presented is iterative, and the estimate t_d^* obtained is optimum in the least squares sense.

To define the optimum estimate, let $t_d^{(k)}$ be an estimate of the dispensing time, and let $\Phi_q^{(k)}(t_N)$ be the least squares estimate of $\Phi_q(t_N)$ obtained by using the technique outlined in the previous sections, with $t_d^{(k)}$ used as the dispensing time. Let $\Phi_q^{(k)}(t_i)$ be defined as

$$\Phi_q^{(k)}(t_i) = F_q^{(k)}(t_i, t_N)\, \Phi_q^{(k)}(t_N)\, F_q^{(k)T}(t_i, t_N) \tag{6.3-1}$$

where $F_q^{(k)}(t_i, t_N)$ is given by (4.1-13) with $t_d = t_d^{(k)}$. Let $\overline{\phi}_q(t)$ be defined as above, with obvious interpretations for $\hat{\overline{\phi}}_q(t_i)$ and $\overline{\phi}_q^{(k)}(t_i)$, and define

$$e^{(k)} = \sum_{i=1}^{N} [\hat{\overline{\phi}}_q(t_i) - \overline{\phi}_q^{(k)}(t_i)]^T [\hat{\overline{\phi}}_q(t_i) - \overline{\phi}^{(k)}(t_i)] \tag{6.3-2}$$

which is the sum of the squared residuals. The value of $t_d^{(k)}$ which minimizes $e^{(k)}$ is the optimum estimate, t_d^*, of the dispensing time.

Obviously $e^{(k)}$ is a function of $t_d^{(k)}$, and it is possible to minimize $e^{(k)}$ by finding a value of $t_d^{(k)}$ for which $\frac{d e^{(k)}}{d t_d^{(k)}}$ vanishes. One iterative numerical technique uses approximate values of $\frac{de^{(k)}}{dt_d^{(k)}}$ and $\frac{d^2 e^{(k)}}{d t_d^{(k)2}}$ computed as follows.

Expanding $e^{(k)}(t^{(k)} + h)$ in Taylor's series

$$e^{(k)}(t^{(k)} + h) = e^{(k)}(t^{(k)}) + \left.\frac{d e^{(k)}}{d t^{(k)}}\right|_{t^{(k)}} h + \frac{1}{2} \left.\frac{d^2 e^{(k)}}{d t^{(k)2}}\right|_{t^{(k)}} h^2 + \ldots \tag{6.3-3}$$

where h is a time increment small in comparison to the desired precision in the estimate t_d^*. Also

$$e^{(k)}(t^{(k)}-h) = e^{(k)}(t^{(k)}) - \left.\frac{d\,e^{(k)}}{d\,t^{(k)}}\right|_{t^{(k)}} h + \frac{1}{2}\left.\frac{d^2 e^{(k)}}{d\,t^{(k)2}}\right|_{t^{(k)}} h^2 + \ldots \quad (6.3\text{-}4)$$

Adding and subtracting yields

$$\left.\frac{d^2 e^{(k)}}{d\,t^{(k)2}}\right|_{t^{(k)}} \cong \frac{e^{(k)}(t^{(k)}+h) + e^{(k)}(t^{(k)}-h) - 2\,e^{(k)}(t^{(k)})}{h^2} \quad (6.3\text{-}5)$$

$$\frac{d\,e^{(k)}}{d\,t^{(k)}} \cong \frac{e^{(k)}(t^{(k)}+h) - e^{(k)}(t^{(k)}-h)}{2\,h} \quad (6.3\text{-}6)$$

The iterative technique, based on the Newton-Raphson method[34], chooses $t_d^{(k+1)}$ as

$$t_d^{(k+1)} = t_d^{(k)} - \frac{\{d\,e^{(k)}/dt^{(k)}\}\big|_{t^{(k)}}}{\{d^2 e^{(k)}/d\,t^{(k)2}\}\big|_{t^{(k)}}} \quad (6.3\text{-}7)$$

Iteration proceeds until the desired precision in the estimate t_d^* is obtained. An initial estimate of t_d can be obtained, for example, by fitting a low-order polynomial (by least squares) to the set of data points corresponding to the largest eigenvalues of $\hat{\Phi}_q(t_i)$, $i = 1, \ldots, N$ and choosing for $t_d^{(O)}$ the intercept of the polynomial with the t_d axis. This is justified by the observation that at $t = t_d$, the eigenvalues of $\Phi_q(t)$ are zero.

6.4 One dimensional example

To illustrate the application of the analyses presented in the thesis, an example which involves a computer simulation of a scatterer ensemble is discussed. For simplicity, the simulated scatterer ensemble is one-dimensional. An impulsively dispensed cloud is simulated, but prediction algorithms for general clouds as well as those for impulsively dispensed clouds are applied. The estimation of the dispensing time, and the effect of the use, in obtaining predictions of an incorrect dispensing time are illustrated.

(a) Radar and target models in the simulation

Let us assume dynamics given by

$$q(t) = q(t_o) + u(t_o)[t - t_o] - \tfrac{1}{2} a [t - t_o]^2 \qquad (6.4\text{-}1)$$

where a is a constant, and simulate an impulsively dispensed cloud whose density function in dispensing velocity space is given by

$$f_w(w) = \begin{cases} \dfrac{1}{2\, w_{max}} & |w| \le w_{max} \\ 0 & |w| > w_{max} \end{cases} \qquad (6.4\text{-}2)$$

Then, using a reference trajectory $p(t)$, the dynamics (6.4-1) can be represented by (3.1-5) with

$$G(t, t_o) = \begin{bmatrix} 1 & t-t_o \\ 0 & 1 \end{bmatrix} \qquad (6.4\text{-}3)$$

(See the example involving (3.1-7) - (3.1-14).)

Equations (3.1-24), (4.1-6), and (4.1-13) imply

$$F_p(t_i, t_j) = \begin{bmatrix} 1 & t_i - t_j \\ 0 & 1 \end{bmatrix} \tag{6.4-4}$$

$$F_3(t, t_d) = t - t_d \tag{6.4-5}$$

$$F_q(t_i, t_j) = (t_i - t_d)/(t_j - t_d) \tag{6.4-6}$$

while (4.3-1) and (4.3-2) yield

$$f_q(q,t) = \begin{cases} \frac{1}{2} w_{max}[t - t_d] & |q - q_d(t)| \le w_{max}[t - t_d] \\ 0 & |q - q_d(t)| > w_{max}[t - t_d] \end{cases} \tag{6.4-7}$$

for the density in position space.

The radar is located at $q = 0$, transmits pulses at the instants $t_1 + i\Delta t$, $i = 1, 2, \ldots,$ and uses the (calibrated) received voltage samples to form the estimates

$$\hat{\eta}_q = \frac{\sum_n R_n v_n v_n^*}{\sum_n v_n v_n^*} = \frac{\sum_n R_n v_n^2}{\sum_n v_n^2} \tag{6.4-8}$$

$$\hat{\Phi}_q = \frac{\sum_n [R_n - \hat{\eta}_q(t)]^2 v_n^2}{\sum_n v_n^2} \tag{6.4-9}$$

(The formulation in Section 5.1 can be considered to apply here, with the identifications $\hat{\eta}_q = (\hat{\bar{\eta}}_q)_1$, $\hat{\Phi}_q = (\hat{\Phi}_q)_{1,1}$, and $v_n = v_n^{(1)}$, the symbol on the right being the one used in Section 5.1.) The voltages are given, in terms of the scatterer parameters by

$$v_n = R_n^2 \sum_{i_n} \sigma_i e^{j\psi_i}/r_i^2 \tag{6.4-10}$$

with the summation over scatterers for which $R_{n-1} < r_i \leq R_n$.

In the simulation, the random variables v_n^2 are generated using Monte-Carlo techniques. The random variable v_n^2 is exponentially distributed,[22] with expectation

$$\mathcal{E}\{v_n^2\} = \mathcal{E}\{R_n^4 \sum_{i_n, k_n} \sigma_i \sigma_k e^{j(\psi_i - \psi_k)} / r_i^2 r_k^2\}$$

$$= R_n^4 \sum_{i_n} s_i^2 / r_i^4$$

$$= R_n^4 \int_{R_{n-1}}^{R_n} \frac{f_q(q,t)}{q^4} dq \qquad (6.4\text{-}11)$$

with $f_q(q,t)$ in (6.4-11) given by (6.4-7).

(b) Estimation and prediction results

In the simulation, a set of numbers $\{v_n^2\}$ is generated corresponding to each instant of observation, t_i. Each number v_n^2 is drawn from an exponential distribution with the mean given by (6.4-11). Estimates $\hat{\eta}_q$ and $\hat{\Phi}_q$ are formed from each set $\{v_n^2\}$ according to (6.4-8) and (6.4-9). Finally, using estimates $\hat{\eta}_q(t_i)$, $\hat{\Phi}_q(t_i)$, $i = 1, \ldots, N$ the algorithms discussed above are used to predict $\eta_q(t_{N+1})$, $\Phi_q(t_{N+1})$. The convergence of the predictions $\eta_q^*(t_{N+1})$ and $\Phi_q^*(t_{N+1})$ to the correct values, with increasing N, is then studied. The values $a = 10$, $t_d = 0$, $q_d(t_d) = 2 \times 10^5$, $u_d(t_d) = 3 \times 10^3$, $t_1 = 200$, $\Delta t = 3$, $w_{max} = 100$, and $\Delta R = 100$ are used in the simulation.

The normalized estimation (i.e., observation) errors $e_\eta(t_i)$, $e_\Phi(t_i)$ defined by

$$e_{\eta}(t_i) = [\hat{\eta}_q(t_i) - \eta_q(t_i)]/2\, w_{max}[t_i - t_d] \tag{6.4-12}$$

$$e_{\Phi}(t_i) = [\hat{\Phi}_q(t_i) - \Phi_q(t_i)]/\Phi_q(t_i) \tag{6.4-13}$$

where, from (6.4-7)

$$\eta_q(t) = q_d(t) \tag{6.4-14}$$

$$\Phi_q(t) = 4\, w_{max}^2 [t - t_d]^2/12 \tag{6.4-15}$$

are plotted in Figures 1 and 2. Also shown in the figures are the prediction residuals $r_{\eta}(t_{N+1})$ and $r_{\Phi}(t_{N+1})$, defined by

$$r_{\eta}(t_{N+1}) = [\eta_q^*(t_{N+1}) - \eta_q(t_{N+1})]/2\, w_{max}[t_{N+1} - t_d] \tag{6.4-16}$$

$$r_{\Phi}(t_{N+1}) = [\Phi_q^*(t_{N+1}) - \Phi_q(t_{N+1})]/\Phi_q(t_{N+1}) \tag{6.4-17}$$

where $\eta_q^*(t_{N+1})$ and $\Phi_q^*(t_{N+1})$ are the least squares predictions, based on the observations $\hat{\eta}_q(t_i)$, $\hat{\Phi}_q(t_i)$, $i = 1, \ldots, N$, obtained using the previously discussed methods.

Predictions for $\eta_q(t_{N+1})$ are obtained using the general formulation only, since it is not assumed that the dispenser trajectory is known, while predictions for $\Phi_q(t_{N+1})$ are obtained with the general formulation and the formulation for impulsively dispensed clouds. Convergence (with increasing N) of the prediction residuals, relative to the observation errors, is evident in all cases. In the prediction of $\Phi_q(t_{N+1})$, use of the formulation for impulsively dispensed clouds leads to markedly more rapid convergence.

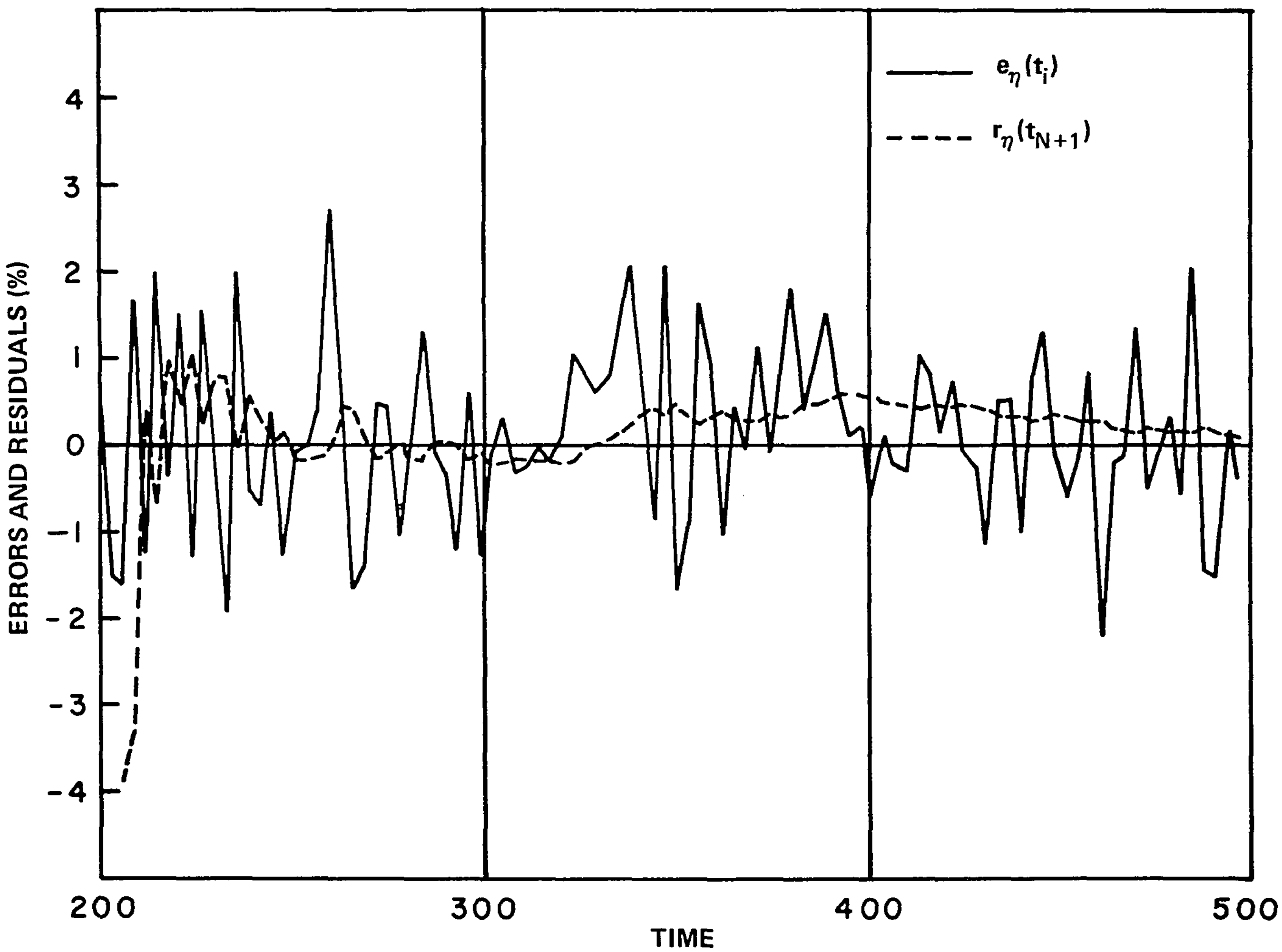

Figure 1. Centroid observation errors, $e_\eta(t_i)$, and prediction residuals, $r_\eta(t_{N+1})$.

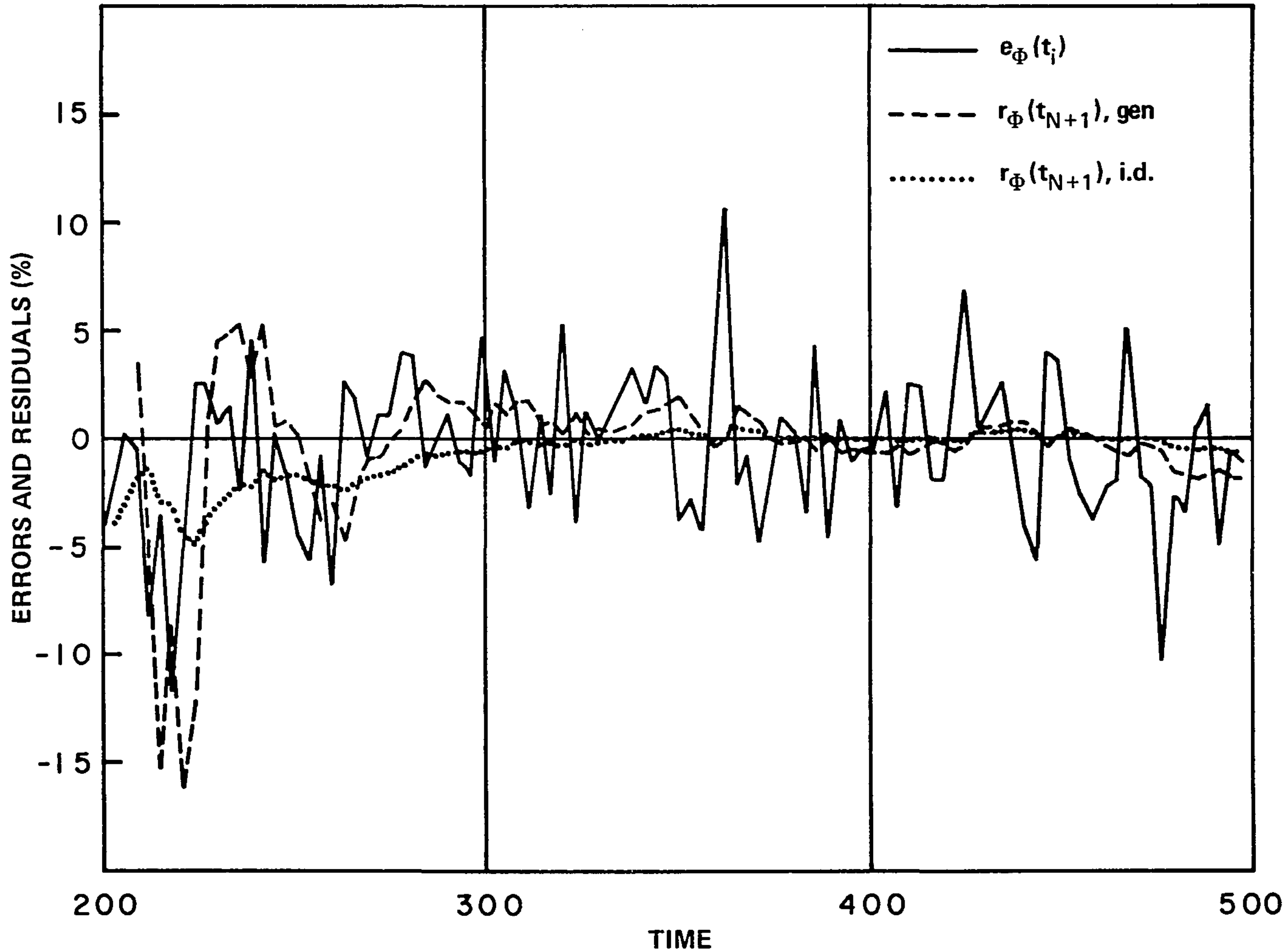

Figure 2. SCM observation errors, $e_\Phi(t_i)$, and prediction residuals, $r_\Phi(t_{N+1})$, using general formulation (gen) and impulsively dispensed formulation (i.d.).

(c) <u>Effect of using incorrect dispensing time; estimation of dispensing time</u>.

Use of an incorrect value of the dispensing time, t_d, in computing the matrix $F_q(t_i, t_j)$ in the state equation (4.2-14) can lead to bias in the prediction of $\Phi_q(t_{N+1})$. To qualitatively illustrate this effect, the prediction algorithm (corresponding to the formulation for impulsively dispensed clouds) used in the example to obtain $\Phi_q^*(t_{N+1})$ is reapplied, this time using incorrect values for the dispensing time.

The correct value of the dispensing time is $t_d = 0$. In Figure 3, normalized prediction residuals are shown which result when the correct value for dispensing time is used in applying the prediction algorithm (this curve also appears in Figure 2), and when the incorrect values $t_d = +25$ and $t_d = -25$ are used in applying the prediction algorithm. Note that the use of $t_d = +25$ leads to a positive bias in the predictions, while the use of $t_d = -25$ leads to a negative bias in the predictions.

When the prediction is based on a system model which includes a value of the dispensing time which is smaller than (i.e., previous to) the actual dispensing time, the cloud appears (based on the observations $\hat{\Phi}(t_i)$, $i = 1, \ldots, N$) to be growing more slowly than it actually is. (It has had more time to attain its observed size.) Thus, the predicted cloud size will tend to be smaller than the actual cloud size. That is, the prediction will be negatively biased. Similarly, use in the system model of a dispensing time which is larger than (i.e., after) the actual dispensing time, leads to a faster apparent cloud growth, and consequently a positive bias in the prediction.

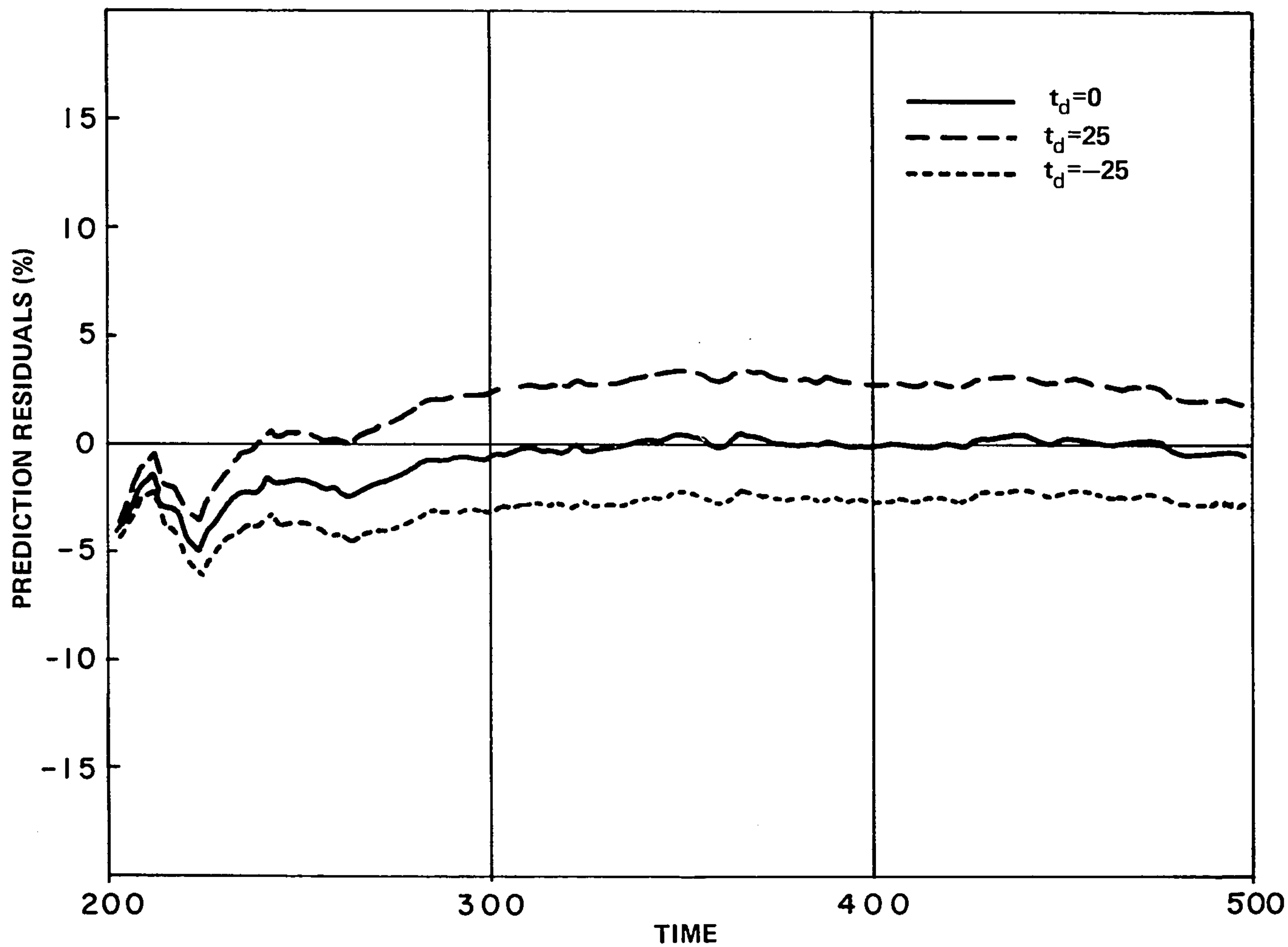

Figure 3. SCM prediction residuals $r_{\Phi}(t_{N+1})$, with correct and incorrect values of the dispensing time, t_d, in the system model.

Since use of the incorrect dispensing time leads to bias in the prediction of $\Phi_q(t_{N+1})$, the accuracy with which the dispensing time can be estimated from the observations (i.e., assuming that it is not known a priori) is of importance. The use of observations to estimate the dispensing time is illustrated with the simulated observations of the example. The observations $\hat{\Phi}_q(t_i)$, $i = 1, \ldots, N$ yield an optimum estimate t_d^* of the dispensing time, obtained as discussed in Section 6.3. As N increases and the data base expands, the accuracy of the estimate t_d^* is expected to increase. In Figure 4, the estimate of dispensing time is plotted as a function of t_N, the time of the last observation used in obtaining the estimate.

Initially, the errors in the estimated dispensing time, which result from the errors in the observations, are substantial. However, as the data base expands, the estimates quickly converge to the correct value of the dispensing time (which is zero). For each value of N, the initial estimate $t_d^{(o)}$ was obtained by fitting a linear curve to the two points defined by $(t_1, \sqrt{\hat{\Phi}_q(t_1)})$ and $(t_N, \sqrt{\hat{\Phi}_q(t_N)})$ and finding the intercept on the time axis. The iteration converged to within a small fraction of a second within three iterations in all cases.

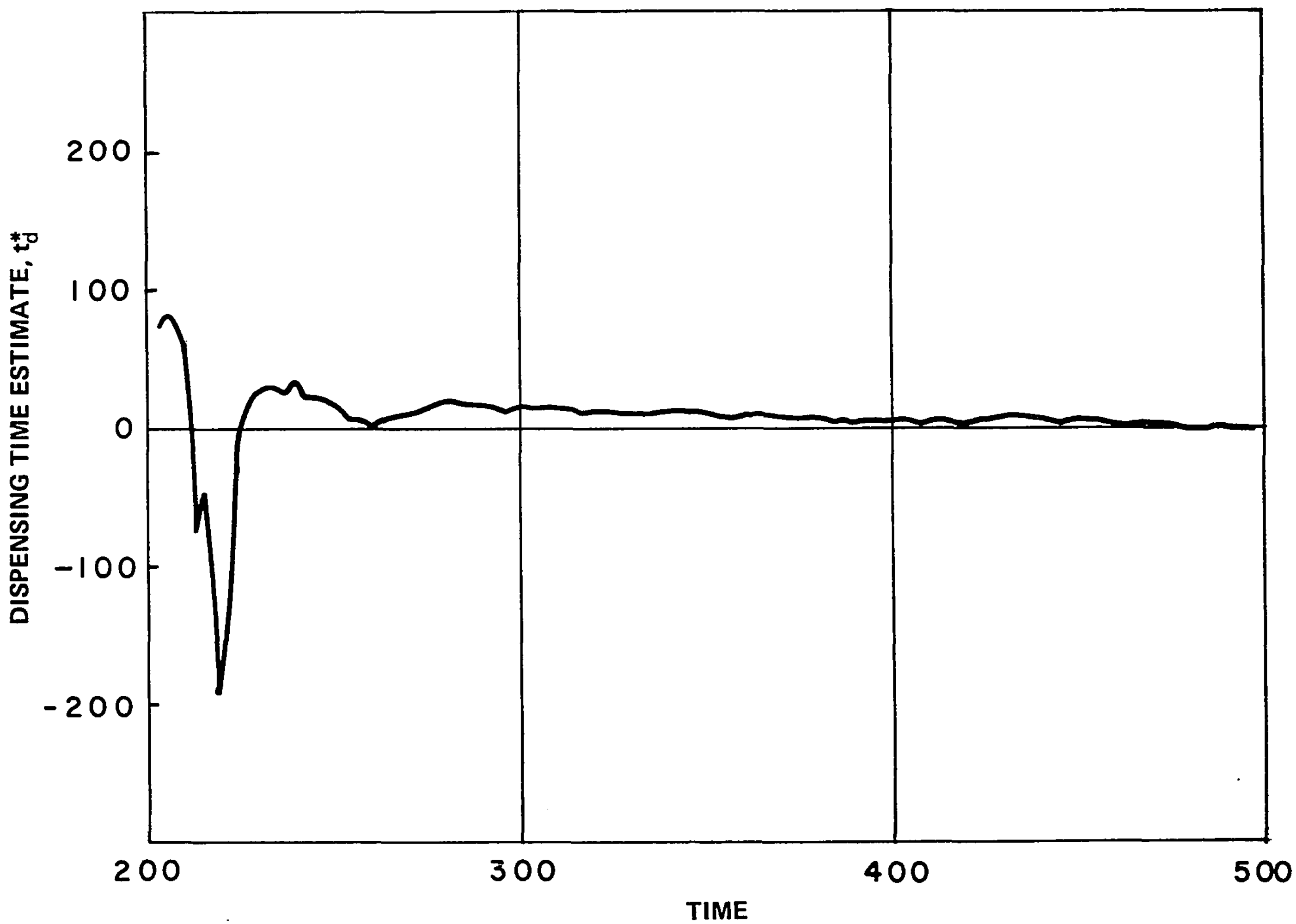

Figure 4. Convergence of dispensing time estimate, t_d^*, to correct value as data base expands.

CHAPTER 7

CONCLUSIONS

In this final chapter, a summary of the contributions of the research reported in this thesis is presented. In addition, some suggestions are made for possible extensions of the present work, and other related efforts which might be pursued in future research.

7.1 Contributions of the research

A new approach has been presented for the radar tracking of a dense ensemble of dynamically independent scatterers. The essence of the approach consists of an observation - prediction - observation schema, in which the quantities predicted are certain aggregate properties of the ensemble. These aggregate properties are defined by formulating a precise mathematical model of the scatterer ensemble. This model makes use of a dense set of points to represent the scatterers in dynamic state space, and a density function, $\rho_p(\bar{p}, t)$, to represent the distribution, in dynamic state space, of scatterer radar cross-section at time t. Projections of the density $\rho_p(\bar{p}, t)$ on lower dimensional sub-spaces of dynamic state space are introduced, as well as moments of $\rho_p(\bar{p}, t)$ and its projections. The first moments and second central moments of the projection of $\rho_p(\bar{p}, t)$ on the position subspace, referred to as the <u>centroid</u> and <u>second central moment (SCM) matrix</u>, are the predicted aggregate properties of the ensemble.

In order to make the scatterer ensemble model as general as possible, the scatterer radar cross-sections and the phases of the received scatterer echoes are considered to be random variables. Prediction of the centroid and SCM matrix requires definition of the density function $\rho_p(\bar{p}, t)$ in terms of the expected values of the scatterer radar cross-section, and the introduction of the stationarity assumption. The discovery of the tracking approach, formulation of a model of the scatterer ensemble which permits the theoretical realization of this approach, and the thorough examination of the applicability of the model to actual physical situations, constitute a major contribution of this research.

Another major contribution of the research is the formulation and analysis of the prediction problem for the case of an arbitrary scatterer ensemble, and for the special case of an impulsively dispensed scatterer ensemble. Time transformation of the density $\rho_p(\bar{p}, t)$ and of its first moments and second central moments are derived, and it is shown that the time transformation of the moments are unique provided that the point-mass dynamics are linear, or linearized about a reference trajectory. The notion of state for the scatterer ensemble is then introduced in a precise manner by the association of an abstract oriented object with the scatterer ensemble, whose output terminal variables are the elements of the centroid and SCM matrix. Input-output-state relations are derived for the abstract object and shown to have the response separation property, establishing the first moments and second central moments of $\rho_p(\bar{p}, t)$ as the state variables. By these means, prediction of the centroid and SCM matrix of the scatterer ensemble is reduced to the problem of predicting the state of a time-varying linear system, based on noisy observations of the system output. Some results concerning reduction of the order of the system representation are derived.

In the case of impulsively dispensed scatterer ensembles, it is found that the order of the system representation can be reduced significantly, relative to the order of the system in the general case. This reduction is seen to result from the singularity of the density $\rho_p(\bar{p},t)$. It is shown that, if the dispenser trajectory is known, the elements of the centroid and SCM matrix themselves suffice for the state representation. On the other hand, if the dispenser trajectory is unknown, the centroid is replaced by the first moments of $\rho_p(\bar{p},t)$ in the state representation. In addition, it is shown, based on an argument establishing the equivalence of states, that knowledge of the dispenser position at the instant of dispensing is sufficient to allow use of the minimum order state representation.

Additional results of the research include a characterization of impulsively dispensed scatterer ensembles in the space of scatterer dispensing velocities, and derivation of the relationship between properties of the ensemble in dispensing velocity space and corresponding properties in the position subspace at an instant after dispensing. A method for the estimation of the dispensing time of an impulsively dispensed scatterer ensemble, on the basis of the observed values of the SCM matrix, is obtained. Observation of the scatterer ensemble by a wide-beam monopulse radar and by a narrow-beam radar scan is examined in detail. New results concerning the bias of estimators of the centroid and SCM matrix are presented.

7.2 Suggested extensions and related work

The theoretical foundations of the adopted approach for the radar tracking of dense scatterer ensembles are discussed comprehensively in this thesis. The research does, however, pose several questions which are of interest primarily with regard to the possibility of obtaining improved tracking performance, of coping with the non-ideal performance of actual radar systems, as opposed to the idealized performance assumed in this research, and of compensating for the effects of model deviations from the physical world.

(a) Effects of non-linear dynamics

Linearization of the point mass dynamics, relative to a reference trajectory, is an essential element in the formulation of the state representation for the scatterer ensemble. Deviation of the point-mass dynamics from linearity, in the physical world, gives rise to system modelling errors which can cause eventual divergence in the prediction algorithms. A somewhat similar problem arises in point target tracking, when linearized dynamics are used in the system model. However, the effects of dynamic non-linearity may be more serious in the case of scatterer ensemble tracking than in the case of point target tracking, since linearization plays a more fundamental role in the formulation of the system model in the former case than in the latter. This is clear from the fact that, in the case of ensemble tracking, the linearity assumption is necessary for the state of the scatterer ensemble to be defined at all, while in point target tracking, linearization merely serves to simplify the form of the state equations.

Possible methods of diminishing the modelling errors, such as relinearization and iterative differential correction, should be studied in the context of the scatterer ensemble tracking problem. The nature of the effect of the modelling errors on the performance of the prediction algorithms is of interest. Modification of the prediction algorithms, for control of divergence, is also an area for future study. Modifications which might be investigated include, for example, the introduction of input noise to the system model, and the use of fading memory.

(b) Radar model refinements

The radar models introduced in Chapter 5 were obviously idealized. The non-ideal performance of real radar systems will affect the quality of the estimates of the centroid and SCM matrix which are obtained on the basis of data acquired at an observation instant. As an example, the variation of the sum-beam pattern will introduce bias in the estimates obtained by a monopulse radar with the estimators discussed in Chapter 5. The magnitude of this bias, and the possibility of its reduction through modification of the estimators, is of interest. In addition, the effect of radar noise on the bias of the estimators requires further investigation.

(c) Definition of beam-width or scan pattern requirements

The beam-width necessary for a wide-beam monopulse observation, or the scan pattern necessary for a narrow-beam observation are determined on the basis of the predicted SCM matrix. However, several other factors, such as uncertainty in the predicted centroid and SCM matrix, reduction of estimate bias, increase in antenna gain (wide-beam observation) or decrease in the number of pulse transmissions (narrow-beam case), must enter into the considerations. Definition of the various tradeoffs, leading to some optimum schema for the beam-width or scan pattern determination is of practical importance. In this context, the study of cloud bounding policies, referred to in the introduction, is likely to be useful.

(d) Second-order statistics of the estimators

In Chapter 5, we have discussed the first order statistics of the estimators of the centroid and SCM matrix. There are certain benefits to be derived from study of the second order statistics of these estimators. One object of such a study would be the derivation of the covariance matrix of the estimated parameters, initially in terms of the basic properties of the scatterer ensemble (i.e., the density function in the position subspace), and ultimately in terms of observable scatterer ensemble properties, such as the centroid, SCM matrix (and possibly higher moments), and the signal to noise ratio. Such relations would be useful, for example, in the determination of the required observation rate to achieve a desired prediction accuracy. Also, availability of the second order statistics in this form makes possible, without recourse to examination of the residuals, the use of prediction algorithms satisfying the minimum variance criterion, which is often somewhat preferable to least squares prediction.

Other second order statistics of interest are the time correlations of the estimates. Specifically, the time interval within which the estimates become uncorrelated indicates the rate at which observation at a particular radar wavelength yields independent estimates. Since observation at a higher rate is wasteful of radar resources, these correlation times are of practical importance.

APPENDIX: GENERALIZED LEAST SQUARES ESTIMATION

This appendix discusses the generalized least squares algorithm stated and applied in Chapter 6. The least squares criterion is stated in terms of the description of a general time-varying linear system. Then the estimation algorithm presented in Chapter 6 is derived and discussed. In addition, two recursive forms of the least squares algorithm are obtained. Finally, extensions to weighted least squares and unbiased minimum variance estimation are discussed. Further discussion of these subjects can be found in various references[6,8,9].

(a) The least squares criterion

Consider a system described by the state equation

$$\overline{x}_i = L_{i,j}\,\overline{x}_j \tag{A-1}$$

where $\overline{x}_i = \overline{x}(t_i)$ is the state vector, and $L_{i,j} = L(t_i, t_j)$ is the transition matrix. Observation is described by

$$\overline{y}_i = H_i\,\overline{x}_i + \overline{n}_i \tag{A-2}$$

where $\overline{y}_i = \overline{y}(t_i)$ is the vector of observed variables, $H_i = H(t_i)$ is the observation matrix, and $\overline{n}_i = \overline{n}(t_i)$ is a random error term.

Assume that observations $\overline{y}_i$, $i = 1, \ldots, N$ are available, and that the state $\overline{x}_N$ is to be estimated. Denote the least squares estimate of $\overline{x}_N$, based on the observations $\overline{y}_i$, $i = 1, \ldots, N$, by $\overline{x}^*_{N,N}$. In terms of this estimate, the observation residuals are defined to be $\overline{y}_i - H_i L_{i,N}\overline{x}^*_{N,N}$, $i = 1, \ldots, N$. The least squares estimate, $\overline{x}^*_{N,N}$, minimizes, by definition, the cost criterion

$$e = \sum_{i=1}^{N} (\bar{y}_i - H_i L_{i,N} \bar{x}^*_{N,N})^T (\bar{y}_i - H_i L_{i,N} \bar{x}^*_{N,N}) \tag{A-3}$$

To simplify the expression for the cost criterion, the quantities

$$R_N = \begin{bmatrix} H_N \\ \hdashline H_{N-1} L_{N-1,N} \\ \hdashline \vdots \\ \hdashline H_1 L_{1,N} \end{bmatrix} \tag{A-4}$$

$$Y_N = \begin{bmatrix} \bar{y}_N \\ \hdashline \vdots \\ \hdashline \bar{y}_1 \end{bmatrix} \tag{A-5}$$

are introduced. In terms of these quantities,

$$e = [Y_N - R_N \bar{x}^*_{N,N}]^T [Y_N - R_N \bar{x}^*_{N,N}] \tag{A-6}$$

(b) Solution for least squares estimate

It is necessary to find the vector $\bar{x}^*_{N,N}$ which minimizes the cost e in (A-6). Two methods of solution will be presented. First, consider that (A-6) is equivalent to the expression†

$$e = [(R^T R)^{\frac{1}{2}} \bar{x}^*_{N,N} - (R^T R)^{-\frac{1}{2}} R^T Y]^T [(R^T R)^{\frac{1}{2}} \bar{x}^*_{N,N} - (R^T R)^{-\frac{1}{2}} R^T Y]$$
$$- Y^T R (R^T R)^{-1} R^T Y + Y^T Y \tag{A-7}$$

as can be verified by expanding both expressions. It is assumed that the matrix

† The subscripts N on matrices R and Y are deleted.

R_N has full column rank, so that $R_N^T R_N$ is positive definite, and the matrix $(R_N^T R_N)^{\frac{1}{2}}$, such that $[(R_N^T R_N)^{\frac{1}{2}}]^2 = R_N^T R_N$, exists and is non-singular.

Of the terms in (A-7), only the first depends on $\bar{x}^*_{N,N}$. Since this term must be non-negative, the minimum value of e will be obtained if the first term vanishes. That is, if

$$(R_N^T R_N)^{\frac{1}{2}} \bar{x}^*_{N,N} - (R_N^T R_N)^{-\frac{1}{2}} R_N^T Y_N = 0 \quad . \tag{A-8}$$

This requires that

$$\bar{x}^*_{N,N} = (R_N^T R_N)^{-1} R_N^T Y_N \tag{A-9}$$

which defines the least squares estimate for $\bar{x}_N$ based on observations $\bar{y}_i$, $i = 1, \ldots, N$.

As a second method of solution for $\bar{x}^*_{N,N}$, we differentiate (A-6) with respect to the elements of $\bar{x}^*_{N,N}$. Denoting by $[\frac{\partial e}{\partial \bar{x}^*_{N,N}}]$ the vector of derivatives of e with respect to elements of $\bar{x}^*_{N,N}$, we have

$$\left[\frac{\partial e}{\partial \bar{x}^*_{N,N}}\right] = 2 R_N^T R_N \bar{x}^*_{N,N} - 2 R_N^T Y_N \tag{A-10}$$

A necessary condition for $\bar{x}^*_{N,N}$ to minimize e is that the derivative, given by (A-10), vanish. This condition yields the least squares estimate (A-9). To show that this value of $\bar{x}^*_{N,N}$ does give a minimum, we evaluate the matrix of second derivatives of e with respect to the elements of $\bar{x}^*_{N,N}$, denoted by $[\frac{\partial^2 e}{\partial \bar{x}^{*2}_{N,N}}]$. Differentiating (A-10), we obtain

$$\left[\frac{\partial^2 e}{\partial \bar{x}^{*2}_{N,N}}\right] = 2 R_N^T R_N \quad . \tag{A-11}$$

Since this matrix is positive definite, (A-9) gives the minimum value for e. Thus $\overline{x}^*_{N,N}$, defined by (A-9), is indeed the least squares estimate.

(c) Least squares estimate corresponding to an arbitrary instant.

Here it is shown that the least squares estimate $\overline{x}^*_{N+1,N}$ of $\overline{x}_{N+1}$, based on the observations $\overline{y}_i$, $i = 1, \ldots, N$, is given by

$$\overline{x}^*_{N+1,N} = L_{N+1,N}\, \overline{x}^*_{N,N} \quad . \tag{A-12}$$

It should be noted that t_{N+1} need not be later than t_N in general, nor need it be distinct from the instants t_i, $i = 1, \ldots, N$.

By definition, $\overline{x}^*_{N+1,N}$ minimizes

$$e' = [\, Y_N - R'_N\, \overline{x}^*_{N+1,N}]^T\, [Y_N - R'_N\, \overline{x}^*_{N+1,N}] \tag{A-13}$$

where Y_N is given by (A-5) and R'_N is defined as

$$R'_N = \begin{bmatrix} H_N\, L_{N,N+1} \\ \hline H_{N-1}\, L_{N-1,N+1} \\ \hline \vdots \\ \hline H_1\, L_{1,N+1} \end{bmatrix} \tag{A-14}$$

But, from the previous proofs, $\overline{x}^*_{N+1,N}$ is given by

$$\overline{x}^*_{N+1,N} = (R'^{\,T}_N\, R'_N)^{-1}\, R'^{\,T}_N\, Y_N \; . \tag{A-15}$$

To derive (A-12), note that R_N and R'_N, given by (A-4) and (A-14), are related by the expression

$$R'_N = R_N\, L_{N,N+1} \tag{A-16}$$

which, when substituted into (A-15) yields

$$
\begin{aligned}
\bar{x}^{*}_{N+1,N} &= \left(L^{T}_{N,N+1}\, R^{T}_{N}\, R_{N}\, L_{N,N+1}\right)^{-1} L^{T}_{N,N+1}\, R^{T}_{N}\, Y_{N} \\
&= L^{-1}_{N,N+1}\, R^{-1}_{N}\, R^{-1T}_{N}\, L^{-1T}_{N,N+1}\, L^{T}_{N,N+1}\, R^{T}_{N}\, Y_{N} \\
&= L_{N+1,N}\left(R^{T}_{N}\, R_{N}\right)^{-1} R^{T}_{N}\, Y_{N} \\
&= L_{N+1,N}\, \bar{x}^{*}_{N,N}\ .
\end{aligned}
$$

This completes the proof that $\bar{x}^{*}_{N+1,N}$, given by (A-12), is the least squares estimate of $\bar{x}_{N+1}$, based on the observations $\bar{y}_i$, $i = 1, \ldots, N$.

(d) <u>Bayes recursive algorithm</u>

It is desirable to be able to form the least squares estimate $\bar{x}^{*}_{N+1,N+1}$, when a new observation vector $\bar{y}_{N+1}$ becomes available, in terms of the previously obtained estimate $\bar{x}^{*}_{N,N}$ and the new observation vector. This and the next section discuss recursive algorithms which permit estimation in this manner.

Assuming that the observation vectors $\bar{y}_i$, $i = 1, \ldots, N+1$ are available, form the quantities

$$
R_{N+1} = \begin{bmatrix} H_{N+1} \\ \hdashline H_N\, L_{N,\,N+1} \\ \hdashline \vdots \\ \hdashline H_1\, L_{1,N+1} \end{bmatrix} = \begin{bmatrix} H_{N+1} \\ \hdashline R_N\, L_{N,N+1} \end{bmatrix} \qquad \text{(A-17)}
$$

$$Y_{N+1} = \begin{bmatrix} \bar{y}_{N+1} \\ \hdashline \vdots \\ \hdashline \bar{y}_1 \end{bmatrix} = \begin{bmatrix} \bar{y}_{N+1} \\ \hdashline Y_N \end{bmatrix} \tag{A-18}$$

Then the least squares estimate $\bar{x}^*_{N+1,N+1}$ is given by

$$\bar{x}^*_{N+1,N+1} = (R^T_{N+1} R_{N+1})^{-1} R^T_{N+1} Y_{N+1} \tag{A-19}$$

To derive a recursive formulation, substitute the rightmost expressions of (A-17) and (A-18) into (A-19). This yields

$$\bar{x}^*_{N+1,N+1} = (H^T_{N+1} H_{N+1} + L^T_{N,N+1} R^T_N R_N L_{N,N+1})^{-1} (H^T_{N+1} \bar{y}_{N+1} + L^T_{N,N+1} R^T_N Y_N) . \tag{A-20}$$

Denoting

$$P_{N+1} = (H^T_{N+1} H_{N+1} + L^T_{N,N+1} R^T_N R_N L_{N,N+1})^{-1} \tag{A-21}$$

and adding and subtracting $H^T_{N+1} H_{N+1} \bar{x}^*_{N+1,N}$ within the second brackets in (A-20) gives

$$\begin{aligned} \bar{x}^*_{N+1,N+1} &= P_{N+1} (L^T_{N,N+1} R^T_N Y_N + H^T_{N+1} H_{N+1} \bar{x}^*_{N+1,N} \\ &\qquad + H^T_{N+1} \bar{y}_{N+1} - H^T_{N+1} H_{N+1} \bar{x}^*_{N+1,N}) \\ &= P_{N+1} (H^T_{N+1} H_{N+1} + L^T_{N,N+1} R^T_N R_N L_{N,N+1}) \bar{x}^*_{N+1,N} \\ &\qquad + P_{N+1} H^T_{N+1} (\bar{y}_{N+1} - H_{N+1} \bar{x}^*_{N+1,N}) \end{aligned} \tag{A-22}$$

In obtaining (A-22), the relations (A-9) and (A-12) have been used. Finally, (A-21) implies that

$$\bar{x}^{*}_{N+1,N+1} = \bar{x}^{*}_{N+1,N} + P_{N+1} H^{T}_{N+1} (\bar{y}_{N+1} - H_{N+1}\bar{x}^{*}_{N+1,N}) \tag{A-23}$$

which is the desired recursive formula.

Recursive computation of the matrix P_{N+1} defined by (A-21) is possible. In obtaining (A-20), it was seen that

$$(R^{T}_{N+1} R_{N+1})^{-1} = P_{N+1} \quad . \tag{A-24}$$

Then, changing indices and substituting into (A-21) gives

$$P_{N+1} = (H^{T}_{N+1} H_{N+1} + L^{T}_{N,N+1} P^{-1}_{N} L_{N,N+1})^{-1} \ . \tag{A-25}$$

To initialize the above recursive least squares algorithm, we must find an index N for which R_N has full column rank. Then $R^{T}_{N} R_{N}$ is positive definite, and P_N and P_{N+1} can be computed from (A-24) and (A-25), since the matrix in parenthesis will be non-singular in each case. Note that once P_N is positive definite for some N, it will be positive definite for all higher indices, since R_N will have full column rank. An initial estimate of $\bar{x}^{*}_{N,N}$ is computed with (A-9). Subsequently, (A-12), (A-23), and (A-25) are used recursively to obtain the estimates.

(e) Kalman recursive algorithm

Note that (A-25) requires two matrix inversions for computation, and that the inverted matrices have the order equal to the number of state variables. A recursive algorithm which requires only a single matrix inversion is obtained by applying the matrix inversion lemma to (A-25). This gives us the expression

$$P_{N+1} = M_{N+1} - M_{N+1} H_{N+1}^T (H_{N+1} M_{N+1} H_{N+1}^T + I)^{-1} H_{N+1} M_{N+1} \tag{A-26}$$

where, for brevity, we denote

$$M_{N+1} = L_{N+1,N} P_N L_{N+1,N}^T \tag{A-27}$$

In (A-26), the order of the matrix $(H_{N+1} M_{N+1} H_{N+1}^T + I)$ is the same as the number of observed variables. The number of observed variables may be substantially less than the number of state variables, which is an additional advantage of the present formulation over that discussed previously.

The reduction of the number of inverted matrices from two to one, and the decrease in the order of the matrices results in computational savings. It cannot be assumed, however, that the computation (A-26) will be less sensitive than the computation (A-25) to errors due to the limitations on the precision of the computing machine, since the subtraction in (A-26) leads to a loss in numerical accuracy.

(f) Some properties of the least squares estimate

If the random error terms $\bar{n}_i$, $i = 1, \ldots, N$ in (A-2) have zero means, the estimate $\bar{x}^*_{N,N}$ defined by (A-9) is unbiased. To show this, define

$$E_N = \begin{bmatrix} \bar{n}_N \\ --- \\ \vdots \\ --- \\ \bar{n}_1 \end{bmatrix} \tag{A-28}$$

so that relation (A-2) for $i = 1, \ldots, N$ can be expressed as

$$Y_N = R_N \bar{x}_N + E_N \quad . \tag{A-29}$$

Substituting this expression for Y_N into (A-9) and taking the expectation yields

$$\mathcal{E}\{\bar{x}^*_{N,N}\} = (R_N^T R_N)^{-1} R_N^T (R_N \, \mathcal{E}\{\bar{x}_N\} + \mathcal{E}\{E_N\}) = \mathcal{E}\{\bar{x}_N\} = \bar{x}_N \qquad \text{(A-30)}$$

since $\mathcal{E}\{E_N\} = \bar{0}$ by hypothesis. Thus, the estimate is unbiased.

The second order statistics of the least squares estimate $\bar{x}^*_{N,N}$ can be expressed in terms of the second order statistics of the random error terms $\bar{n}_i$, $i = 1, \ldots, N$, as follows. Denote by

$$V_N = \mathcal{E}\{E_N E_N^T\} \qquad \text{(A-31)}$$

the co variance matrix of the random errors $\bar{n}_i$, $i = 1, \ldots, N$, and by

$$S_N = \mathcal{E}\{\bar{x}^*_{N,N} \bar{x}^{*T}_{N,N}\} - \bar{x}_N \bar{x}_N^T \qquad \text{(A-32)}$$

the covariance matrix of the state estimate. Then, using (A-9) and (A-29), we obtain

$$S_N = \mathcal{E}\{[\bar{x}_N + (R_N^T R_N)^{-1} R_N^T E_N][\bar{x}_N + (R_N^T R_N)^{-1} R_N^T E_N]^T\} - \bar{x}_N \bar{x}_N^T$$

$$= \mathcal{E}\{(R_N^T R_N)^{-1} R_N^T E_N E_N^T R_N (R_N^T R_N)^{-1}\}$$

$$+ \mathcal{E}\{\bar{x}_N E_N^T R_N (R_N^T R_N)^{-1}\} + \mathcal{E}\{(R_N^T R_N)^{-1} R_N^T E_N \bar{x}_N^T\} \qquad \text{(A-33)}$$

$$= (R_N^T R_N)^{-1} R_N^T V_N R_N (R_N^T R_N)^{-1} \quad . \qquad \text{(A-34)}$$

The second two terms in (A-33) vanish, since $\mathcal{E}\{E_N\} = \bar{0}$ by hypothesis. Equation (A-34) relates the covariance matrix, S_N, of the state estimate to the covariance matrix, V_N, of the observation errors.

(g) Weighted least squares estimate

Some of the observations $\overline{y}_i$, $i = 1, \ldots, N$ may be considered to be more reliable than others. Instead of minimizing the cost criterion (A-6), the more general cost criterion

$$e = [Y_N - R_N \overline{x}^*_{N,N}]^T Q_N [Y_N - R_N \overline{x}^*_{N,N}] \tag{A-35}$$

can be employed, where Q_N is a positive definite weighting matrix, which weighs more heavily the residuals corresponding to the more reliable observations. The vector $\overline{x}^*_{N,N}$ which minimizes (A-35) is defined as the weighted least squares estimate of $\overline{x}_N$.

The methods of section (b) above used to minimize (A-6) can be used here to minimize (A-35), yielding

$$\overline{x}^*_{N,N} = (R_N^T Q_N R_N)^{-1} R_N^T Q_N Y_N \tag{A-36}$$

for the weighted least squares estimate . It is easily shown that (A-12) holds for the weighted least squares estimates, and that the estimates are unbiased. The estimate covariance matrix S_N is now found to be given by

$$S_N = (R_N^T Q_N R_N)^{-1} R_N^T Q_N V_N Q_N R_N (R_N^T Q_N R_N)^{-1} . \tag{A-37}$$

The algorithm for weighted least squares estimation can be recursively formulated, using the methods used previously for recursive formulation of the least squares estimation algorithm. The weighted least squares estimate is found to be given by

$$\bar{x}^{*}_{N+1,N+1} = \bar{x}^{*}_{N+1,N} + P_{N+1} H^{T}_{N+1} Q_{(N+1)} (\bar{y}_{N+1} - H_{N+1} \bar{x}^{*}_{N+1,N}) \tag{A-38}$$

where P_{N+1} is given, in terms of P_N, by

$$P_{N+1} = (H^{T}_{N+1} Q_{(N+1)} H_{N+1} + L^{T}_{N,N+1} P^{-1}_{N} L_{N,N+1})^{-1} \tag{A-39}$$

in the Bayes formulation. In deriving these equations, it is assumed that the weighting matrix Q_{N+1} has the form

$$Q_{N+1} = \left[\begin{array}{c|c|c|c} Q_{(1)} & & & \bar{0} \\ \hline & Q_{(2)} & & \\ & & \ddots & \\ \hline \bar{0} & & & Q_{(N+1)} \end{array}\right] = \left[\begin{array}{c|c} Q_N & \bar{0} \\ \hline \bar{0} & Q_{(N+1)} \end{array}\right] \tag{A-40}$$

where the square matrices $Q_{(i)}$, $i = 1, \ldots, N+1$ have, as their order, the number of observed variables.

As before, the algorithm is initialized at an index value N for which R_N has full column rank. Then, to initialize the algorithm, $\bar{x}^{*}_{N,N}$ is computed from (A-36) and P_N is given by

$$P_N = (R^{T}_{N} Q_N R_N)^{-1} \ . \tag{A-41}$$

The Kalman recursive formulation is obtained from (A-39), with the use of the inversion lemma, giving

$$P_{N+1} = M_{N+1} - M_{N+1} H^{T}_{N+1} (H_{N+1} M_{N+1} H^{T}_{N+1} + Q^{-1}_{(N+1)})^{-1} H_{N+1} M_{N+1} \tag{A-42}$$

where M_{N+1} is given by (A-27).

One choice for the weighting matrix Q_N is the inverse of the covariance matrix V_N of the observation errors, defined by (A-31). If V_N is known, and Q_N is set equal to V_N^{-1}, then it can be shown[9] that the estimate given by (A-36) is the minimum variance unbiased estimate. That is, the estimate $\bar{x}^*_{N,N}$ will minimize simultaneously each of the diagonal elements of S_N, the state estimate covariance matrix. The matrix S_N, in this case, is given by

$$S_N = R_N^{-1} V_N R_N^{-1T} \tag{A-43}$$

as obtained by substituting $Q_N = V_N^{-1}$ into (A-37). If the observation errors are jointly gaussian, the minimum variance unbiased estimate of x_N is also the maximum likelihood estimate[9].

Since they are algebriacally equivalent to (A-36), the recursive formulations described by (A-38) together with either (A-39) or (A-42) also yield the minimum variance unbiased estimate of the state, provided that $Q_N = V_N^{-1}$. Note that since Q_{N+1} must be of the form (A-40), the second order statistics of the observation errors must satisfy $\mathcal{E}\{\bar{n}_i \bar{n}_j^T\} = 0$ for $\bar{n}_i \neq \bar{n}_j$, in order for the recursive formulations to yield the minimum variance unbiased estimate. That is, the observation errors corresponding to two different instants must be uncorrelated.

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