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ON THE VOLUME OF TUBES

City University of New York

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ON THE VOLUME OF TUBES

by

ALLEN GORIN

A dissertation submitted to the Graduate Faculty
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1979

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Allen Gorin
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The relationship between the volume of a tubular neighborhood of a submanifold and metric invariants of that submanifold have long been of interest. Steiner examined this problem as long ago as 1840 [12]. The problem was discussed for curves by Hotelling [8], in relation to a statistical problem, which motivated H. Weyl to solve the problem for any submanifold of a space of constant curvature. (Actually, he states his results only for Euclidian space and spheres.)

When the ambient manifold is flat, Weyl obtained the surprising result that the volume, as a function of the radius, is a polynomial. Secondly, the coefficients of the polynomial are products of universal constants, depending on dimensions, and intrinsic integral metric invariants of the submanifold. Specifically, these are the integrals of the p^{th} mean curvatures of the submanifold.

Many questions concerning submanifolds can be discussed by restricting one's attention to its tubular neighborhood. Weyl's result suggests restricting one's attention still further to the formula, $V(r)$, for the volume of the tube, and examining the coefficients of its power series expansion. This was done in 1848 for geodesic circles on surfaces by Bertrand and Digret [2].

When the ambient space is symmetric, N. Grossman [7] has obtained order of magnitude estimates for the growth of tubes of large radius, which depend on the rank of the ambient symmetric space. He uses this estimate along with results of Bott to gain information on the Betti numbers of path spaces from the submanifold to a point off it.

In the case of a complex submanifold of complex projective space, Flaherty [6] has given formulae analogous to Weyl's.

When a lower bound is given on the sectional curvatures of the ambient space, V. Dekster [5] has obtained comparison theorems for the volume of tubes which generalize the Rauch Comparison Theorem for the volume of spheres.

The general problem may be stated as follows: Let $M^{(m)}$ and $N^{(m+k)}$ be compact smooth Riemannian manifolds. Let $f: M \rightarrow N$ be a smooth isometric imbedding. Consider the tubular neighborhood of radius r about $f(M)$ in N . This may be identified with the normal disc bundle of M induced by f . Let $V(r)$ be the volume of the tube as a function of r . This volume in general depends on M , N , r and f . The formalisms for solving this problem can be set up by considering the Jacobi fields induced in the normal bundle by variation of the normal geodesics. See Grossman [7] for an exposition of this. Little has been said about the general case, except that the first approximation to $V(r)$ is $V(r) = V_k r^k \text{vol}(M) + O(r^{k+1})$, where V_k is the volume of the unit k -disc. When M is flat, Weyl's results apply, and $V(r)$ is a polynomial. Little is known about the integral invariants which are the coefficients. The first coefficient is a multiple of the volume, as mentioned above. Also by examining the formulae for the last coefficient, it is seen to be the integral of the Lipschitz-Killing curvature of M , and thus the last coefficient is a multiple of the Euler characteristic. The intermediate coefficients are the integrals of Allendoerfer's [1] p^{th} mean curv-

atures. Thorpe [14] discusses the relation between the constancy of these curvatures and the vanishing of certain Pontryagin classes.

For an arbitrary ambient manifold, the integration for the volume of a tube has not yet been done. The purpose of this paper is to make some contributions in this direction. We consider two classes of problems.

First we consider a submanifold $M^{(m)}$ of R^{m+k} , and take E to be a subbundle of its normal bundle, NM . We discuss the volume of the tubular neighborhood of M in E and prove some theorems about it. The first states that the volume is a polynomial if E is totally geodesic, and gives a non-polynomial counterexample in case it is not. The second considers E to be the first osculating space of M , and shows that if the volume agrees with the Weyl polynomial to second order, then it must in fact be that polynomial and M must lie in some lower dimensional subspace of R^{m+k} . We then consider one sided tubes, i.e.: exponentiating only a quadrant of the normal bundle. The integral of the mean curvature of M appears in this volume formula. These theorems are stated in detail in Chapter 1.

The next situation we consider is the case of an immersion rather than an imbedding of one manifold into another. The formalisms used above still apply, but need to be corrected, for near any coincidence point of the immersion there are coincidence points of the exponentiated disc bundle. We obtain approximations to the volume of the multiple points of the immersed tube, and use these approximations plus Weyl's formula to obtain approximations to the

actual volume of the tubular neighborhood when the ambient space is flat. Examining these formulæ, we note that to a certain order of approximation, the volume is intrinsic to the submanifold. Taking a higher order approximation, we note that the next higher order term depends only on the intrinsic metric of the image of the immersion, the conformal structure of the ambient manifold and the codimension. As a corollary, we will note that if the approximation to the actual volume about the immersed submanifold agrees with the intrinsically defined Weyl polynomial, then the immersion must be an imbedding. These formulæ and theorems will be stated and proven in Chapter 2.

CHAPTER 1

Let $M^{(m)}$ be a compact smooth submanifold of R^{m+k} . Choose an ℓ -dimensional subbundle, E , of the normal bundle NM . We will discuss the tubular neighborhood of radius r of M in E . Its volume form will be constructed and given in local coordinates, which in certain situations lends itself to interesting invariant formulations. In the case $\ell=k$, we have $E=NM$, the full tube. This situation was examined in detail by H. Weyl, and was the occasion of the first substantial insight into the general problem.

Theorem 1. (Weyl [16]) The tubular neighborhood of radius r of M in R^{m+k} has volume

$$V(r) = V_{\ell} \sum_{e} \frac{r^{\ell+e}}{(\ell+2)(\ell+4)\dots(\ell+e)} h_e$$

(e even, $0 \leq e \leq m$), where V_{ℓ} is the volume of the unit ℓ -disc and the h_e are integral metric invariants intrinsic to M . Specifically, h_e is the integral of the p^{th} mean curvature of M , with h_0 being the volume.

We will denote this volume by $W(r)$, calling it the ℓ -dimensional Weyl polynomial of M . It is independent of the isometric imbedding for a fixed codimension. Where the number ℓ is clear, we will merely call it the Weyl polynomial of M .

The volume formula is not a polynomial for all subbundles E of NM . We will show:

Theorem 2. If E is a totally geodesic subbundle of NM , then $V(r)$ is a polynomial. It is of the same form as the Weyl polynomial, although the coefficients are not necessarily intrinsic.

Remark. It is not always true that $V(r)$ is a polynomial. As a counterexample, consider M a non-degenerate space curve, with E generated by the first normal vector. It is easily shown by direct calculation that $V(r)$, in this case, is a polynomial if and only if the torsion of the curve vanishes. This converse does not appear to be true for $\dim M \geq 2$.

We will show the following theorems:

Theorem 3. Let E be such that it contains the image of the vector valued second fundamental form of M . The volume of the tube of radius r about M in E is

$$V(r) = r^\ell V_\ell \cdot \text{vol}(M) + \frac{r^{\ell+2}}{\ell+2} [V_\ell \int h_2 dM + V_m^{-1} \int \|\Pi_{E^\perp} \nabla_w e\|^2 dS^{\ell-1} dS^{m-1} dM] + O(r^{\ell+4}),$$

where the integral is over $UTM \ominus UE$, with $e \in UE$, $w \in UTM$, and Π_{E^\perp} denotes projection onto E^\perp .

This formula agrees with Weyl's formula (Theorem 1) when $k=\ell$, (since $E^\perp = \emptyset$). The integral of $\|\Pi_{E^\perp} \nabla e\|^2$ is a measurement of the "twisting" of E . In particular, when M is a non-degenerate curve:

Corollary. Consider a non-degenerate curve γ in R^{1+k} .

Let E be the subbundle of the normal bundle generated by the first normal vector. (E is also called the first osculating space of γ). The volume of the tube about γ in E is

$$V(r) = 2rL + (r^3/6) \int \tau^2 ds + O(r^5),$$

where L is the length of the curve and τ is the torsion of the curve.

Theorem 4. Let $n(1) \cdots n(\ell)$ be a frame field along M . Consider the region spanned by $m + \sum_{p=1}^{\ell} t_p n(p)_m$, for $t_p \geq 0$ and $\sum_{p=1}^{\ell} t_p^2 \leq r^2$.

This is a generalization of a 1-sided tubular neighborhood in case $\ell=1$. The volume of the '1-sided' tube is

$$V^+(r) = r^{\ell} 2^{-\ell} V_{\ell} \cdot \text{vol}(m) + r^{\ell+1} 2^{-\ell} V_{\ell-1} \int H \cdot n \, dM + O(r^{\ell+2}),$$

where V_{ℓ} is the volume of the unit ℓ -disc, 2^{ℓ} is the number of quadrants in R^{ℓ} , $n = \sum_{i=1}^{\ell} n(i)$ and H is the mean curvature vector of M .

Corollary. Let γ be a non-degenerate curve in R^{1+k} with normal vector n . Let E be the subbundle of $N\gamma$ generated by n . The volume of the 1-sided tube about γ in E is

$$V^+(r) = rL + \frac{1}{2} r^2 \int \kappa \, ds + O(r^3),$$

where κ is the curvature of the curve γ , and L is its length.

We remark that Theorems 1 through 4 also hold when M is a submanifold of any flat space.

The next theorem concerns nicely curved submanifolds, which are a generalization of the concept of a non-degenerate curve.

Theorem 5. Let $M^{(m)}$ be a nicely curved submanifold of R^{n+k} . In such a manifold, the range of the second fundamental form has constant dimension ℓ . Denote this subbundle of the normal bundle by $E^{(\ell)}$.

$V(r)$ denotes the volume of the tube in E . The following are equivalent:

- 1) M is contained in a $m+\ell$ dimensional linear subspace of R^{n+k} ,
- 2) E is a totally geodesic subbundle of NM ,
- 3) $V(r)$ is the ℓ -dimensional Weyl polynomial of M ,
- 4) The coefficient of $r^{\ell+2}$ in the power series expansion for $V(r)$

agrees with the corresponding term in the Weyl polynomial.

Remark. This theorem will be based on a generalization of the fact that if the torsion of a space curve vanishes, then it lies in a plane. For M a submanifold of R^{m+k} and E the range of the second fundamental form, we are motivated to define the torsion of the imbedding of M at a point p as

$$\int \|\Pi_E + \nabla_w e\|^2 d\int^{\ell-1} d\int^{m-1},$$

where the domain of integration is for $w \in UTM_p$ and $e \in UE_p$.

Section 1 will develop the volume form for the tube, and in the process we will demonstrate Theorem 2. Section 2 will discuss the integration of this form over the tube and quadrant-tube, which will yield Theorems 3 and 4. Section 3 will prove the equivalences stated in Theorem 5.

We will use some of Weyl's calculations as building blocks for the volume formula. The proof of Theorem 5 will depend on the notions of osculating spaces and formal imbedding numbers, as developed by Allendoerfer.

§1 The volume form

We are given $M^{(m)}$ a submanifold of R^{n+k} and $E^{(\ell)}$ a subbundle of NM . Let (U, x) be a coordinate patch on M . A coordinatization of the tube in E can be constructed as follows:

Choose $\{n(1) \dots n(\ell)\}$ an orthonormal basis section for $E|_U$.

Define, for $t \in R^\ell$,

$$(1.1) \quad Y(u, t) = x(u) + \sum_{p=1}^{\ell} t_p n(p).$$

The tube of radius r is defined by the restriction $\sum t_p^2 \leq r^2$.

We will give the volume form of this coordinatization. Calculations

will be done locally. Since local results will be invariant, the global versions will follow immediately.

It will be convenient to use indices in the following ranges:

$$1 \leq i, j, p \leq \ell, \quad 1 \leq \alpha, \beta \leq m.$$

Let $\{w_\alpha\}$ be an orthonormal basis for $TM|_U$. Y_α and Y_p will denote differentiation of Y with respect to w_α and $\frac{\partial}{\partial t_p}$ respectively. It follows from 1.1 that:

$$(1.2) \quad Y_p = n(p),$$

$$(1.3) \quad Y_\alpha = X_\alpha + \sum_p t_p n_\alpha(p).$$

We express $n_\alpha(p)$ as a linear combination of the orthonormal vectors x_β and $n(q)$, where $1 \leq \beta \leq m$ and $1 \leq q \leq k$. Extend the basis $\{n(p)\}_{p=1, \ell}$ of E to an orthonormal basis $\{n(p)\}_{p=1, k}$ of NM . We can then write

$$(1.4) \quad n_\alpha(p) = \sum_\beta G_\alpha^\beta(p) X_\beta + \sum_{q=1}^k L_{\alpha q}(p) n(q),$$

where $G_\alpha^\beta(p)$ are the coefficients of the second fundamental form in direction $n(p)$, and $L_{\alpha q}(p) = n_\alpha(q) \cdot n(p)$. G is symmetric in α and β and L is skew-symmetric in p and q . Combining 1.3 and 1.4, we obtain

$$(1.5) \quad Y_\alpha = X_\alpha + \sum_\beta [\sum_p t_p G_\alpha^\beta(p)] X_\beta + \sum_{q=1}^k [\sum_p t_p L_{\alpha q}(p)] n(q).$$

Thus, we can write the volume form of E as

$$(1.6) \quad \det \begin{pmatrix} \delta_{\alpha\beta} + \sum_p t_p G_\alpha^\beta(p) & \sum_p t_p L_{\alpha q}(p) \\ \circ & I_\ell \quad \circ \end{pmatrix} dM dt,$$

$$\begin{matrix} 1 \leq \alpha \leq m \\ 1 \leq \beta \leq m \\ 1 \leq q \leq \ell \end{matrix}$$

and $dt = dt_1 \cdots dt_\ell$. Note that the matrix is $m+\ell$ by $m+k$. The interpretation of \det is to consider the row vectors as in the $m+\ell$ dimensional subspace which they span.

Weyl considered the case $\ell=k$, when E is the full normal bundle of M in R^{m+k} . In this situation, I_ℓ fills the lower right hand block of 1.6, thus the volume form becomes, in Weyl's case:

$$(1.7) \quad \det[I + \sum_p t_p G_\alpha^\beta(p)] dM dt.$$

This can be evaluated, and expressed as a polynomial in the t_p . The coefficients of the polynomial are invariant polynomials of the matrices $G(p)$. The volume of the tube of radius r is found by integrating the polynomial over M and the region $\sum t_p^2 \leq r^2$. Weyl does this, and obtains the following formula, given in Theorem 1:

$$V(r) = V_\ell \sum_e \frac{r^{\ell+e}}{(m+2) \cdots (m+e)} h_e,$$

where the sum is for e even, $0 \leq e \leq m$. V_ℓ is the volume of the unit ℓ -disc, and the h_e are intrinsic metric invariants of the submanifold M . Note that e must be even. The terms corresponding to odd e vanish. Weyl shows this using the fact that an integral $\int t_1^{e(1)} \cdots t_\ell^{e(\ell)} dS^{\ell-1}$ vanishes unless all the $e(i)$ are even. This will be used later.

The block matrix 1.6 can be evaluated by the same technique if $L_{\alpha q}(p) = 0$ for $q \geq \ell+1$, $p \leq \ell$ and for all α . In this case, the subspace of definition of the determinant is clear, and the volume form is again the form of 1.7. The volume will hence again be a

polynomial in r . The condition $L_{\alpha q}(p) = 0$ can be restated as $n_{\alpha}(p) \cdot n(q) = 0$, or that $\Pi_{E^{\perp}} \nabla_w e = 0$ for all $w \in TM$ and $e \in E$. We call such a subbundle E totally geodesic. Recall:

Definition. Let E be a subbundle of a vector bundle over M with connection ∇ . E is a totally geodesic subbundle if $\nabla_w e \in E$ for all $e \in E$ and $w \in E^{\perp} TM$.

Theorem 2 is now proven.

Remark. Although when E is totally geodesic, we obtain a polynomial $V(r)$, the coefficients will only be intrinsic to M if E contains the range of the second fundamental form of M . This follows from Weyl's original proof.

When E is not totally geodesic, we can still obtain an approximation to $V(r)$. We do this by using the standard definition of the volume form as $\sqrt{\det g} \, du \, dt$, where g is the metric tensor expressed in local coordinates. Since we are using an orthonormal frame as a basis for TM , rather than the basis associated to a coordinatization of M , the volume form is actually $\sqrt{\det g} \, dM \, dt$. This form will be integrated over M and over the sphere (or quadrant) of radius r in each fiber.

Referring to equations 1.2 and 1.5, and recalling that the X_i are orthonormal, we obtain the components of g as:

For $1 \leq \alpha, \eta \leq m$,

$$(1.8) \quad g_{\alpha\eta} = Y_{\alpha} \cdot Y_{\eta} = \delta_{\alpha\eta} + 2 \sum_{p=1}^{\frac{q}{2}} t_p G_{\alpha}^{\eta}(p) \\ + \sum_{\beta} \left[\sum_i t_i G_{\alpha}^{\beta}(i) \right] \cdot \left[\sum_j t_j G_{\eta}^{\beta}(j) \right] \\ + \sum_{q=1}^k \left[\sum_i t_i L_{\alpha q}(i) \right] \cdot \left[\sum_j t_j L_{\eta q}(j) \right].$$

For $1 \leq \alpha \leq m$ and $1 \leq a \leq \ell$,

$$(1.9) \quad g_{\alpha, (a+m)} = Y_{\alpha} \cdot n(a) = \sum_i t_i L_{\alpha a}(i) .$$

For $1 \leq a, b \leq \ell$,

$$(1.10) \quad g_{(a+m), (b+m)} = \delta_{ab} .$$

Thus, the matrix g is a second degree polynomial in the variables $\{t_i\}$:

$$(1.11) \quad g(t) = I + \sum_i A_i t_i + \sum_{i,j} t_i t_j B_{ij} ,$$

where

$$(1.12) \quad A_i = \begin{pmatrix} 2G_{\alpha}^{\eta}(i) & L_{\alpha p}(i) \\ L_{\alpha p}(i)^{tr} & 0 \end{pmatrix} \quad \begin{matrix} 1 \leq \alpha, \eta \leq m \\ 1 \leq p \leq \ell \end{matrix}$$

$$(1.13) \quad B_{ij} = \begin{pmatrix} \sum_{\beta} G_{\alpha}^{\beta}(i) G_{\eta}^{\beta}(j) + \sum_{q=1}^k L_{\alpha q}(i) L_{\alpha q}(j) & 0 \\ 0 & 0 \end{pmatrix}$$

To find the volume form, we need to calculate $\sqrt{\det g(t)}$. We will first approximate $\det g(t)$ with a Taylor series, then its square root. It will not be necessary for this application to calculate the coefficients of mixed terms, $t_i t_j$, because if $i \neq j$, then $\int t_i t_j dS^{\ell}$ vanishes.

Let $f(t) = \det g(t)$. Note $f(0) = 1$.

Lemma. The first few derivatives of f , at $t = 0$, ($t_1 = t_2 = \dots = t_{\ell} = 0$), are expressed in terms of the A_i and the B_{ij} as:

$$(1.14) \quad f_1(0) = \text{tr} A_i ,$$

$$(1.15) \quad f_{11}(o) = 2[\text{tr } B_{11} + \sigma_2 A_1] ,$$

where tr is the trace and σ_2 is the classical second order orthogonally invariant polynomial in the coefficients of a matrix.

Proof. To evaluate the above derivatives it suffices to restrict f to $\{t | t_a = 0 \text{ for } a \neq 1\}$.

$$\begin{aligned} f(t) &= \det [I + t_1 A_1 + t_1^2 B_{11}] = \det [I + t_1 (A_1 + t_1 B_{11})], \\ &= 1 + t_1 \text{tr}(A_1 + t_1 B_{11}) + t_1^2 \sigma_2 (A_1 + t_1 B_{11}) + O(t_1^3), \\ &= 1 + t_1 \text{tr } A_1 + t_1^2 (\text{tr } B_{11} + \sigma_2 A_1) + O(t_1^3), \end{aligned}$$

which proves the lemma.

Lemma. For $h(t) = \sqrt{f(t)} = \sqrt{\det g(t)}$,

$$(1.16) \quad h(o) = 1 ,$$

$$(1.17) \quad h_1(o) = \frac{1}{2} \text{tr } A_1 ,$$

$$(1.18) \quad h_{11}(o) = \text{tr } B_{11} + \sigma_2 A_1 - \frac{1}{4} (\text{tr } A_1)^2$$

Proof. This follows from the previous lemma by the chain rule.

Letting η_{ij} be the coefficient of $t_i t_j$ in the Taylor series for $h(t)$, we have shown that the coefficient of $dM dt$ in the volume is

$$\begin{aligned} (1.19) \quad &1 + \frac{1}{2} \sum_i t_i \text{tr } A_1 + \frac{1}{2} \sum_i t_i^2 [\text{tr } B_{11} + \sigma_2 A_1 - \frac{1}{4} \text{tr}^2 A_1] \\ &+ \sum_{i \neq j} \eta_{ij} t_i t_j + O(t^3) . \end{aligned}$$

By $O(t^3)$, we mean $O(\sum t_i t_j t_k)$.

We will now calculate the coefficients in terms of the $G_\alpha^\beta(p)$ and the $L_{\alpha p}(q)$.

Lemma. The coefficients in equation 1.19 are given by

$$(1.20) \quad \text{tr} A_i = 2 \sum_{\alpha} G_{\alpha}^{\alpha}(i) ,$$

$$(1.21) \quad \text{tr} B_{ii} = \sum_{\alpha, \beta} G_{\alpha}^{\beta}(i)^2 + \sum_{\alpha} \sum_{q=1}^k L_{\alpha q}(i)^2 ,$$

$$(1.22) \quad \sigma_2 A_i = 4\sigma_2 [G(i)] - \sum_{\alpha} \sum_{q=1}^{\ell} L_{\alpha q}(i)^2 ,$$

$$(1.23) \quad - \frac{1}{2} \text{tr}^2 A_i = - \sum_{\alpha, \beta} G_{\beta}^{\alpha}(i) .$$

Proof. 1.20 and 1.21 follow immediately from equations 1.12 and 1.13. 1.23 follows from 1.20. To show 1.22, we must recall a few facts about the invariant function σ_2 .

The first is that for an arbitrary square matrix, $M = (m_{ij})$, $\sigma_2 M = \sum_{i < j} m_{ii} m_{jj} - m_{ij} m_{ji}$. If the sum is over all i and j , a factor of $\frac{1}{2}$ is introduced.

When the matrix M is symmetric and has the special form $M = \begin{pmatrix} N & H \\ H^{\text{tr}} & O \end{pmatrix}$, it follows directly from the formula for σ_2 that $\sigma_2 M = \sigma_2 N - \frac{1}{2} \sum_{i,j} h_{ij}^2$. Noting that A_i indeed has this special form, (see 1.12), and that $\sigma_2(2N) = 4\sigma_2 N$, 1.22 then follows. The lemma is proven.

The coefficient of t_1^2 in 1.19 is thus

$$\begin{aligned} & \frac{1}{2} \sum_{\alpha, \beta} G_{\alpha}^{\beta}(i)^2 + \frac{1}{2} \sum_{q=1}^k \sum_{\alpha} L_{\alpha q}(i)^2 - \frac{1}{2} \sum_{\alpha, \beta} G_{\alpha}^{\alpha}(i) G_{\beta}^{\beta}(i) \\ & + 2\sigma_2 [G(i)] - \frac{1}{2} \sum_{q=1}^{\ell} \sum_{\alpha} L_{\alpha q}(i)^2 , \end{aligned}$$

which simplified to, (recalling the definition of σ_2)

$$(1.24) \quad \sigma_2 [G(i)] + \frac{1}{2} \sum_{q>l} \sum_{\alpha} L_{\alpha q}(i)^2.$$

We have now shown, combining this lemma and 1.19, that the coefficient of $dMdt$ in the volume form is

$$(1.25) \quad 1 + \sum_{i,\alpha} t_i G_{\alpha}^{\alpha}(i) + \sum_{i<j} \eta_{ij} t_i t_j \\ + \sum_i t_i^2 \left\{ \sigma_2 [G(i)] + \frac{1}{2} \sum_{q>l} \sum_{\alpha} L_{\alpha q}(i)^2 \right\} + O(t^3).$$

Note that the summation of the $L_{\alpha q}(i)$ is only for $q=l+1$ to k , which represents the orthogonal complement of the subbundle E in NM .

§2 The volume

We will now discuss the integration of the volume form over the tube and quadrant of radius r . First, note that when the volume form, 1.25, is integrated over the fiber, (the t variable), we obtain certain integrals as follows:

Lemma. Let B be the ball in t -space, (the fiber), of radius r . Let Q be the quadrant described by $\{t \in B \mid t_i \geq 0 \text{ for all } i\}$. The integrals of t_i and $t_i t_j$ over B and Q are independent of i and j by symmetry and are given by the following formulæ:

$$(2.1) \int_B t_i^2 dt = r^{\ell+2} V_\ell / (\ell+2),$$

$$(2.2) \int_B t_i t_j dt = 0, \text{ for } i \neq j,$$

$$(2.3) \int_B t_i dt = 0,$$

$$(2.4) \int_Q t_i dt = r^{\ell+1} V_{\ell-1} 2^{-\ell}.$$

Proof. Equations 2.1 through 2.3 are immediate consequences of formula 12 in Weyl's paper. Equation 2.4 follows by straightforward integration.

We can now integrate the volume form, 1.25, over the fiber.

Proposition 2.5. The volume of the tube, of radius r , about M in the space E is

$$(2.5) \begin{aligned} V(r) = & r^\ell V_\ell \int dM \\ & + \frac{r^{\ell+2} V_\ell}{(\ell+2)} \int \left\{ \sum_i \sigma_2 [G(1)] + \frac{1}{2} \sum_{q>\ell} \sum_\alpha L_{\alpha q} (1)^2 \right\} dM \\ & + O(r^{\ell+4}). \end{aligned}$$

Proof. Integrate 1.25 over the fiber using Fubini's Theorem and equations 2.1 through 2.3. Note $r^\ell v_\ell = \int_B 1 \cdot dt$. Also, $\int_B 0(t^3) dt = O(r^{\ell+4})$. It is this rather than $O(r^{\ell+3})$ because $\int_B t_i t_j t_k dt = 0$.

We now discuss the geometric significance of the integrands in 2.5. First, we show that the function $\sum_{i,\alpha} \sum_{q>\ell} L_{\alpha q}(i)^2$ is actually invariant and measures the torsion of the subbundle E . We can call this torsion because if it vanishes, then $L_{\alpha q}(i) = 0$ for all i, q and α , which was seen in section 1 to be the definition of E being a totally geodesic subbundle of the normal bundle.

Proposition 2.6. We can express the above function in invariant notation:

$$(2.6) \quad \int_B \sum_{i,\alpha} \sum_{q>\ell} t_i^2 L_{\alpha q}(i)^2 dt = \frac{r^{\ell+2}}{V_m(\ell+2)} \int \| \Pi_{E^\perp} \nabla_w e \|^2 dS^{\ell-1} dS^{m-1},$$

where the integration is over $(w, e) \in UTM \otimes UE$.

Proof. Let $n_1 \cdots n_\ell$ be the already chosen orthonormal basis section for E , and f_1, \dots, f_m be a local orthonormal frame on M . We proceed via a sequence of lemmas.

$$\begin{aligned} \text{Lemma 2.7.} \quad & \int_B \sum_{\alpha,i} \sum_{q>\ell} t_i^2 L_{\alpha q}(i)^2 dt \\ &= r^{\ell+2}/(\ell+2) \int \sum_{\alpha,i} \sum_{q>\ell} t_i^2 L_{\alpha q}(i)^2 dS^{\ell-1} \end{aligned}$$

Proof. Change to spherical coordinates and note that the integrand is quadratic.

$$\begin{aligned} \text{Lemma 2.8.} \quad & \int \sum_{\alpha,i} \sum_{q>\ell} t_i^2 L_{\alpha q}(i)^2 dS^{\ell-1} \\ &= \int \sum_{\alpha,q} \left[\sum_i t_i n_\alpha(i) \cdot n(q) \right]^2 dS^{\ell-1} \end{aligned}$$

Proof. The integrands are equal modulo a function which is a linear combination of terms $t_i t_j$, for $i \neq j$. This function vanishes when integrated over $S^{\ell-1}$.

$$\begin{aligned} \text{Lemma 2.9. } \int \sum_{\alpha, q} [\sum_i t_i n_\alpha(i) \cdot n(q)]^2 dS^{\ell-1} \\ = \int \sum_{\alpha} \|\Pi_{E^+} e_\alpha\|^2 dS^{\ell-1} \end{aligned}$$

Proof. Set $e = \sum_i t_i n(i)$ and then note that the summation is for $q = \ell+1$ to k . Then note that $\{n(q)\}_{q > \ell}$ is an orthonormal basis for E^+ .

$$\begin{aligned} \text{Lemma 2.10. } V_m \int \sum_{\alpha} \|\Pi_{E^+} e_\alpha\|^2 dS^{\ell-1} \\ = \int \|\Pi_{E^+} \nabla_w e\|^2 dS^{\ell-1} dS^{m-1} \end{aligned}$$

Proof. Let $\{f_\alpha\}$ be the orthonormal basis section we have been using for TM. Let $w = \sum_{\alpha} s_{\alpha} f_{\alpha} \in \text{UTM}$. Note that

$$\begin{aligned} (2.10a) \quad \|\Pi_{E^+} \nabla_w e\|^2 &= \|\sum_{\alpha} s_{\alpha} \Pi_{E^+} e_{\alpha}\|^2 = \\ &= \sum_{\alpha} s_{\alpha}^2 \|\Pi_{E^+} e_{\alpha}\|^2 + m(s), \end{aligned}$$

where $m(s)$ is a linear combination of mixed terms, $s_i s_j$, which vanish when integrated over S^{m-1} . When the right hand side of 2.10a is integrated over S^{m-1} , a factor of $\int s_i^2 dS^{m-1}$ is introduced. This is equal to V_m , the volume of the unit m -disc. That is a consequence of formula 12 in Weyl's paper.

Proposition 2.6 follows immediately from the above lemmas.

The other integrand in the volume formula, 2.5, is $\sum_i \sigma_2 [G(i)]$.

In the situation where E contains the image of the second fundamental form, it follows directly from Weyl's discussion, (section 4), that $\sum_1 \sigma_2 [G(1)]$ is actually intrinsic to M . Using the fact that the ambient space is Euclidian, he shows that this object is an invariant polynomial in the curvature tensor of M . Specifically, upon examination, it turns out to be scalar curvature. We use his notation and call it h_2 .

We are now in position to prove Theorem 3. Combine proposition 2.5 with 2.6 and the above discussion. It follows that the volume of the tubular neighborhood, radius r , about M in E is given locally, and thus globally, by:

$$\begin{aligned} V(r) &= r^\ell V_\ell \cdot \text{vol}(M) \\ &+ \frac{r^{\ell+2}}{\ell+2} [V_\ell \int h_2 dM + V_m^{-1} \int \| \Pi_{E^\perp} \nabla_w e \|^2 dS^{\ell-1} dS^{m-1} dM] \\ &+ O(r^{\ell+4}). \end{aligned}$$

Theorem 3 is now proven.

Remark. The deviation of this formula, in the coefficient of $r^{\ell+2}$, from Weyl's formula is the square of the L^2 norm of the torsion of E , which measures the non-geodecity of E as a sub-bundle of NM .

We now prove Theorem 4, which is concerned with the volume of a quadrant tube about M in E . As before, let $n(1) \cdots n(\ell)$ be a local orthonormal partial frame field of NM . This time, we consider the region, Q , parameterized by $m + \sum_{i=1}^{\ell} t_i n(i)$, for $t_i \geq 0$ and $\sum_1 t_i^2 \leq r^2$. Referring to formula 1.25, we can approximate

the volume form by

$$(2.11) \quad [1 + \sum_{i,\alpha} t_i G_\alpha^\alpha(i) + O(t^2)] dM dt.$$

Integrating over the fiber (the t variable), and referring to 2.4 for $\int_Q t_i dt$, we obtain

$$r^\ell 2^{-\ell} V_\ell \int dM + r^{\ell+1} 2^{-\ell} V_{\ell-1} \int \sum_{i,\alpha} G_\alpha^\alpha(i) dM + O(r^{\ell+2}).$$

Noting that $\sum_\alpha G_\alpha^\alpha(i) = \sum_\alpha G_\alpha^\alpha \cdot n(i) = H \cdot n(i)$, where $H = \text{tr } G$ is the mean curvature vector, we then have that the volume is

$$r^\ell 2^{-\ell} V_\ell \cdot \text{vol}(M) + r^{\ell+1} 2^{-\ell} V_{\ell-1} \int [H \cdot \sum_i n(i)] dM + O(r^{\ell+2}).$$

This proves Theorem 4.

§3 Theorem 5

We now discuss and prove Theorem 5. To define the concept of a nicely curved submanifold, we must first define the osculating spaces of the submanifold. We follow the exposition of Spivak, volume 4, Chapter 7.

Let $M^{(m)}$ be a submanifold of R^{m+k} , and ∇ the covariant derivative on R^{m+k} . If x_1 and x_2 are vector fields on M , $\nabla_x Y$ is the normal component of $\nabla_x Y$, known as the vector valued second fundamental form. The range of the second fundamental form, together with TM , comprises the span of $\nabla_{x_1} x_2$ for $x_1, x_2 \in TM$.

Let E_p be the range of the second fundamental form at M_p .

We introduce some notation: $\nabla(x, Y) = \nabla_x Y$, $\nabla(x, Y, Z) = \nabla_x (\nabla_y Z)$, etc.

Definition 3.1. The k^{th} osculating space of M at a point p is the span, at p , of:

$$x_1, \nabla(x_1, x_2), \dots, \nabla(x_1, x_2, \dots, x_k), \text{ for all } x_i \in TM.$$

Denote this $Osc(k, M_p)$.

Note $Osc(1, M_p) = TM_p$ and $Osc(2, M_p) = TM_p \oplus E_p$. We sometimes suppress the M and write $Osc(k)_p$.

Definition 3.2. M is called nically curved if $\dim Osc(k)_p$ is the same at all points $p \in M$.

We now consider only nicely curved submanifolds, and we then have a nested sequence of vector bundles $Osc(k)$, with $Osc(k) \subseteq Osc(k+1)$.

Since $Osc(k) \subseteq NM$ for all k , the sequence must stabilize.

We note that if $Osc(k) = Osc(k+1)$, then $Osc(k+1) = Osc(k+2) = \dots$. Therefore, there exists a minimum ℓ such that $Osc(\ell) = Osc(\ell+1)$. The number, $\dim Osc(\ell, M)$, is denoted $\#(M)$, and is called the formal imbedding number of M . Spivak supplies the following proposition:

Proposition 3.3. Let M be a nicely curved submanifold of R^{m+k} . Then, M is contained in a $\#(M)$ -dimensional linear subspace of R^{m+k} .

We now prove Theorem 5. We restate it:

Theorem 5. Let $E^{(\ell)}$ be the range of the second fundamental form of a nicely curved submanifold $M(m)$ of R^{m+k} . Let $V(r)$ denote the volume of the tubular neighborhood of radius r about M in E . The following are equivalent:

- 1) $V(r)$ is the Weyl polynomial of M ,
- 2) $V(r)$ agrees with the Weyl polynomial to order $r^{\ell+2}$
- 3) E is a totally geodesic subbundle of NM ,
- 4) E is contained in a $m+\ell$ dimensional linear subspace of R^{m+k} ,

Proof. We remember that E is totally geodesic if E is closed under the induced connection on the normal bundle of M , i.e.: Let ∇_V^+ be the induced connection on NM . E is totally geodesic if $\nabla_V^+ e \in E$ for all $e \in E$ and $v \in TM$. A restatement of this is that $\Pi_{E^\perp} \nabla_V^+ e = 0$ for all e and V . Since this is the torsion of M , whose integral is the difference between the Weyl coefficient of $r^{\ell+2}$ and the coefficient of the volume formula (by Theorem 3), we see that (2) and (3) are equivalent. We then recall that $Osc(1, M) = E \otimes TM$, and that $Osc(2, M)$ is

$\{\nabla_x Y \mid x \in TM, Y \in \text{Osc}(1, M)\}$, where ∇ is the connection on R^{m+k} . Assume E to be totally geodesic. Then $\nabla_x e \in E \cap \text{Osc}(1)$ for all $e \in E$. Also, $\nabla_x Y \in E \oplus TM$ for all $Y \in TM$ since the normal component of $\nabla_x Y$ is the vector valued second fundamental form $II(x, Y)$ which is contained in E . Therefore, we have shown that if E is totally geodesic, then $\text{Osc}(1, M) = \text{Osc}(2, M)$. By Proposition 3.3, M is contained in a linear subspace of R^{m+k} of dimension $\#(M) = \dim \text{Osc}(1, M) = \dim (E \oplus TM) = m+l$. Thus we have shown that (3) implies (4). By Weyl's Theorem, (4) implies (1). Trivially, (1) implies (2). We have then shown the following chain of implications:

$$(1) \longrightarrow (2) \longleftarrow (3) \longrightarrow (4) \longrightarrow (1),$$

which proves Theorem 5.

CHAPTER 2

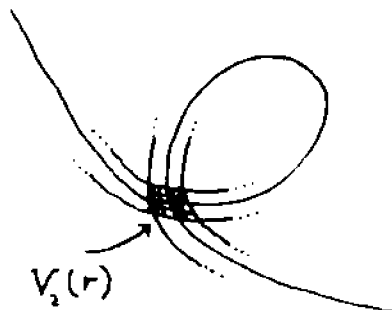
The results so far discussed are for the volumes of tubes about imbedded submanifolds. The problem was attacked by first finding the volume form of the tube and then integrating it over the submanifold. All calculations were done in the normal bundle, with the induced metric.

We now consider the problem of finding the volume of a tubular neighborhood of an immersed submanifold. The formalisms for the imbedded case can be used, however the calculated volume will count certain regions of the tube with multiplicity. If $f: X \rightarrow Y$ is an immersion, then near any ℓ -fold coincidence point of f the tube about $f(X)$ will also have ℓ -fold multiplicity. The problem then becomes to calculate the correction terms to the volume formula obtained from the imbedded case. Specifically, if $V_1(r)$ is the volume of the tube in the normal bundle with the induced metric, and if $V_j(r)$ is the volume of the j -fold coincidence points of the tube of radius r about $f(X)$ in Y , then the actual volume of the tube, as a function of r , will be

$$\begin{aligned} \text{Volume} &= V_1(r) - V_2(r) + 2V_3(r) - \dots + (-1)^{\ell-1} (\ell-1) V_\ell(r), \\ &= V_1(r) + \sum_{j=2}^{\ell} (-1)^{j-1} (j-1) V_j(r) \end{aligned}$$

where ℓ is the maximum coincidence of f .

For example, in the case of an immersed circle in R^2 , generically there are only double points. $V_1(r)$ is given by Weyl's formula as $2rL$, where L is the length of the curve. $V_2(r)$ is the area of the double points of the band:



Thus, the actual area is $2rL - V_2(r)$.

We might hope that $V_2(r)$ would be a 'nice' function, possibly even a polynomial. This is not usually the case. $V_2(r)$ would be a polynomial, for small r , if the curve was linear near each intersection point. Even in the simple case where the curve segments are linear and circular, $V_2(r)$ is transcendental as a function of r . (See examples.) We can, however, obtain an approximation to these volumes in the generic case of a self-transversal immersion. The main result of this chapter will be:

Theorem 1. Let $f: X^{(n)} \rightarrow Y^{(n+k)}$ be a self-transversal isometric immersion between smooth, compact Riemannian manifolds. Consider the tubular neighborhood of radius r about $f(X)$ in Y . For small r , the volume of the ℓ -fold points, $V_\ell(r)$, of this neighborhood is given by the formula:

$$V_\ell(r) = r^{k\ell} (V_k)^\ell \cdot \int |\det^{-1} n| \, dV + O(r^{k\ell+1}),$$

where V_k is the volume of the unit k -disc, dV is the volume form of the ℓ -fold coincidence submanifold of f , and n is a matrix defined along the coincidence set dependent only on the conformal structure of M along that set.

This theorem enables us to obtain an approximation to the volume of the tube in the Euclidian case. It follows from Theorem 1 and Weyl's Theorem that:

Theorem 2. Let $f: X(n) \rightarrow R^{n+k}$ be a self-transversal immersion. For small r , the volume of the tube of radius r about $f(X)$ in R^{n+k} is given by the formula

$$\sum_{0 \leq e \leq k} H_e r^{k+e} - (V_k)^2 r^{2k} \int |\det^{-1} n| dV + O(r^{2k+1}),$$

where the H_e are the coefficients of the ℓ -dimensional Weyl polynomial of X , (as described in Chapter 1, Theorem 1), and the correction term is the volume of the double points of the tube as given in the previous theorem.

Remark. The volume of the tube about $f(X)$ in R^{n+k} can be approximated by a Taylor series in r . We note that the above theorem tells us that:

- 1) If we approximate the volume to order r^{2k-1} , the approximation depends only on the intrinsic metric of X and the codimension of the immersion. (Compare Weyl's Theorem.)
- 2) If we approximate the volume to order r^{2k} , the volume depends only on the above factors, the intrinsic metric of $f(X)$ and the conformal structure of Y along the submanifold of double points of f .

An interesting consequence of Theorem 2 is:

Theorem 3. If the volume of the tube of radius r about $f(X)$ in R^{n+k} agrees with the k -dimensional Weyl polynomial of X in the $2k^{\text{th}}$ term of its power series expansion, then f is an imbedding.

This follows from Theorem 2 because in a self-transversal immersion the existence of any coincidence point implies the existence of a double point.

In the case of a circle self-transversally immersed in the plane we will see that:

Corollary 1. Let $f:S^1 \rightarrow R^2$ be a self-transversal immersion. The double points are a finite set of points $\{X_i\}_{i=1,q}$. The area of the double points of the band of radius R near an X_i is, for small R :

$$\int_{-R}^R \int_{-R}^R 1/\sin\theta(t) dt_1 dt_2,$$

where t_1 and t_2 are normal coordinates measuring distance to each curve segment thru X_i , and $\theta(t)$ is the angle between the normals (tangents) of the level curves of the bands.

Corollary 2. If $f:S^1 \rightarrow R^2$ is self-transversal, with double points $\{X_i\}_{i=1,n}$, then the volume of the tube of radius r about the image of S^1 in R^2 is given by the formula:

$$2rL - 4r^2 \sum_{i=1}^n 1/\sin\theta_i + O(r^3),$$

where θ_i is the angle of coincidence of the curve at each X_i and L is the length of the curve.

We will show that for a surface in space:

Corollary 3. If $f:X^{(2)} \rightarrow R^3$ is a self-transversal immersion of a surface in space, then the volume of the tube of radius r about $f(X)$ is

$$2r \cdot \text{Area}(X) - 4r^2 \int |1/\sin\theta| ds + O(r^3),$$

where the integration is over the curves of double points and θ is the angle of self-intersection of the surface at each double point.

We consider again $f: X^{(n)} \rightarrow Y^{(n+k)}$.

Corollary 4. Let $f: X^{(n)} \rightarrow Y^{(n+k)}$ be a self-transversal immersion. If $n+k-\ell k = 0$, then the ℓ -fold points of a self-transversal immersion are a finite set of points. The first approximation to the volume of the ℓ -fold points of the tube depend only on

- 1) the number of coincidence points,
- 2) the conformal structure of M at the coincidence points,
- 3) the codimension k .

We now consider the case of a self-transversal immersion into Euclidian space where the map is linear in the neighborhood of an isolated ℓ -fold point.

Corollary 5. If $f: X^{(n)} \rightarrow \mathbb{R}^{n+k}$, and if $f(X)$ is linear near a particular isolated ℓ -fold point, ($n+k-\ell k = 0$), then for small r the volume of the ℓ -fold points of the tube of radius r , near that point, is the volume of the corresponding parallelogram:

$$r^{k\ell} (V_k)^\ell \det^{-1} n.$$

Remark. This formula, which is exact in the linear case, is the first approximation to the volume in the non-linear case.

(Compare to Theorem 1.)

§1 GENERAL POSITION

Given a self-transversal immersion $f: X \rightarrow Y$, the tangent planes f_*TX are in general position at any coincidence point. In a subsequent section we will construct a coordinate system in a neighborhood of any coincidence point adapted to the problem of finding the volume of the ℓ -fold points of the tubular neighborhood of radius r about $f(X)$ in Y . This coordinate system will be based on the following fact about subspaces in general position:

Proposition. If $\{E_i\}_{i=1, \ell}$ are linearly independent in \mathbb{R}^P , then $\mathbb{R}^P \simeq \left[\bigcap_{i=1}^{\ell} E_i \right] \oplus E_1^\perp \dots \oplus E_\ell^\perp$.

This states that if we consider the position coordinate along the coincidence set, $\bigcap_{i=1}^{\ell} E_i$, and the normal position coordinates with respect to each subspace E_i , they together form a coordinate system in a neighborhood of the coincidence set. This will be generalized to transversal intersections of submanifolds in the third section.

This section will be devoted to listing the basic definitions and facts we will need associated with general position. They are standard and so are stated without proof.

We remark that the definitions and results stated here extend immediately from subspaces of \mathbb{R}^P to continuous subbundles of vector bundles.

We first give three equivalent definitions of general position:

Definition 1.1. A set, $\Sigma = \{E_1, \dots, E_\ell\}$, of subspaces through the origin in \mathbb{R}^P is in general position if for all distinct subsets

$$J \subseteq \{1, \dots, \ell\}: \quad \sum_{i \in J} \text{codim } E_i = \text{codim } \bigcap_{i \in J} E_i.$$

Definition 1.1a. The set Σ is in general position if 1.1 is true for only $J = \{1, \dots, \ell\}$.

Definition 1.1b. The set Σ is in general position if its orthogonal complement, $\{E_1^\perp, \dots, E_\ell^\perp\}$, is a linearly independent set of subspaces of \mathbb{R}^p .

We remark that the property of being in general position is generic. This can be stated precisely by viewing Σ as an element in the ℓ -fold direct sum of Grassmanian manifolds of appropriate type, and stating that the subset of such Σ in general position is open and dense.

A motivating example to keep in mind is the set of coordinate hyperplanes $E_i = \{x \mid x_i = 0\}$ in \mathbb{R}^p . Note that the genericity of general position, along with definition 1.1b, implies that almost all sets of p vectors in \mathbb{R}^p are linearly independent.

We now give two decompositions of the space \mathbb{R}^p with respect to such a Σ .

Proposition 1.2. Let $\Sigma = \{E_1, \dots, E_\ell\}$ be in general position in \mathbb{R}^p . Let $E = \bigcap_{i=1}^{\ell} E_i$. \mathbb{R}^p can then be decomposed as:

$$(1.2) \quad \mathbb{R}^p = E \oplus \bigoplus_{i=1}^{\ell} E_i.$$

Proposition 1.3. Let Σ and E be as above. Let $W_a = \bigcap_{i \neq a} E_i \cap E^\perp$. \mathbb{R}^p can then be written as:

$$\mathbb{R}^p = E \oplus \bigoplus_{i=1}^{\ell} W_i.$$

Proposition 1.4. Let Σ and E be as above. Let $F_a = \bigcap_{i \neq a} E_i$. The set $\Sigma' = \{F_1, \dots, F_\ell\}$ is in general position in E .

Since our reason for examining this topic is to aid us in

studying self-transversal immersions, we will need some information on the objects defined above in the case where the dimensions (codimensions) of the E_i are constant.

Proposition. Let $\Sigma = \{E_1, \dots, E_\ell\}$ be in general position in R^{n+k} . Assume, for all i , that $\dim E_i = n$, (equivalently, $\text{codim } E_i = k$). Let E , F_a and W_a be as previously defined. The dimensions of these objects are independent of the subscript "a" and are given by:

$$(1.5) \quad \dim E = \dim \left(\bigcap_{i=1}^{\ell} E_i \right) = n + k - \ell k,$$

$$(1.6) \quad \dim F_a = \dim \left(\bigcap_{i \neq a} E_i \right) = n + k - (\ell - 1)k,$$

$$(1.7) \quad \dim W_a = \dim (F_a \cap E^{\perp}) = k,$$

$$(1.8) \quad \dim (E_i \cap E^{\perp}) = \ell k - k.$$

§2 The Set of Multiple Points

Let $N^{(n)}$ and $M^{(n+k)}$ be smooth compact Riemannian manifolds. Let $f:N \rightarrow M$ be a smooth immersion, i.e.: Df has rank n everywhere. Assume f is an isometry, i.e.: N has the induced metric.

Definition. A smooth immersion $f:N \rightarrow M$ is called self-transversal or an immersion with normal crossings if at any ℓ -fold point, $\ell \geq 2$, with $m=f(u_1)=\dots=f(u_\ell)$, $\{u_i\}_{i=1,\ell}$ being distinct points in N , the tangent planes $f_*TN_{u_i}$ are in general position in TM_m .

The self-transversal immersions are a natural subset of all immersions to consider because of the following theorem:

Theorem 2.1. (Whitney, Thom, [17]). The set of self-transversal immersions from N to M are open and dense (generic) in the function space of all immersions from N to M .

The set of self-intersections of a self-transversal immersion has a nice structure:

Theorem 2.2. (Whitney, Thom, [17]). Let f be a smooth self-transversal immersion from $N^{(n)}$ to $M^{(n+k)}$. The set of points, in $f(N)$, of multiplicity ℓ is the immersed image in M of a smooth manifold of dimension $n+k-\ell k$.

Corollary. The set of points with multiplicity exactly ℓ is a submanifold of M .

Corollary. $f(N)$ is a stratified set/smooth simplicial complex with strata/skeletons in dimensions $n+k-\ell k$, for values of ℓ from 1 to $[(n+k)/k]$.

Remark. There is a similar theory of self-intersections for topological manifolds. Also discussed are the homology classes in N represented by the inverse image of the k -fold points. References are available in Steenrod's Index under the subject heading 'Coincidences'.

§3 Coordinate Systems in a Neighborhood of a Coincidence Point.

Let $f: N^{(n)} \rightarrow M^{(n+k)}$ be a self-transversal immersion. Let $m \in S_\ell$, the submanifold of strictly ℓ -fold points of f in M . We will generalize the main result from section 1 on choosing coordinates near a coincidence point. The coordinates will consist of a coordinate function along the coincidence submanifold and normal position coordinates with respect to each sheet of the immersion. This coordinate system is the natural one to use in finding the volume of the ℓ -fold points of the tubular neighborhood, radius r , around $f(N)$ in M since the region is simply defined as the set of points whose normal position coordinates to N all have length less than r . We remark that the existence of these coordinates depends strongly on considering only self-transversal immersions. This section will be devoted to first, the construction of this and another interesting coordinate system, second, to the illustration of this construction in the case of a curve in the plane.

We begin with $m \in S_\ell$, the submanifold of M consisting of strictly ℓ -fold points of the immersion f .

From Whitney's Theorem, 2.2, we know that $\dim S_\ell = n+k-\ell k$. Let \tilde{W} be an open neighborhood of m in M such that:

- 1) $\tilde{W} \cap S_{\ell+1} = \emptyset$,
- 2) $f^{-1}(\tilde{W}) = \bigcap_{i=1}^{\ell} V_i$, where the V_i are disjoint coordinate

patches in N .

Let $D = \bigcap_{i=1}^{\ell} f(V_i)$, which is an open neighborhood of m in S_ℓ .

Restrict \tilde{W} and the V_i so that (α, D) is a coordinate neighborhood, with $\alpha: U \rightarrow D$, for $U \subseteq \mathbb{R}^{n+k-\ell k}$.

We now construct, for each sheet of the immersion through m , Fermi coordinates adapted to the coincidence submanifold. These will be based on proposition 1.2, which will then yield a direct sum decomposition of TM_m .

We now apply proposition 1.2. Since f is self-transversal, the set $\{f_*TV_i\}_{i=1,\ell}$ is in general position in TM . We note that $(f_*TV_i)^\perp = f_*NV_i$ and that $TD = \bigcap_{i=1,\ell} f_*TV_i$. Proposition 1.2 can then be rewritten as

$$(3.1) \quad TM|_D \simeq TD \oplus \bigoplus_{i=1,\ell} f_*NV_i.$$

We now construct coordinates for a neighborhood of D within each sheet $f(V_i)$. Let σ be a permutation of $\{1,2,\dots,\ell\}$ with no fixed points. Let $b = \sigma(a)$, so that $\bigcap_{i \neq a} f_*TV_i \cap ND$ is a subbundle of $f_*TV_b \cap ND$. The fiber dimensions are k and $\ell k - k$ respectively, by 1.7 and 1.8. Choose an orthonormal basis section $\{V_{bj}\}_{j=1,k}$ for $\bigcap_{i \neq a} f_*TV_i \cap ND$. Extend this to a basis $\{V_{bj}\}_{j=1,\ell k - 1}$ for $f_*TV_b \cap ND$. The reason we construct the basis in this manner is that the collection $\{V_{bj}\}_{\substack{b=1,\ell \\ j=1,k}}$ will then be a basis for ND . This will be exploited later. We coordinatize a neighborhood of D in $f(V_b)$ using Fermi coordinates:

$$(3.2) \quad \tilde{F}_b(u_b, y_b) = \exp \begin{matrix} f(V_b) \\ \sum_{j=1}^{\ell k - k} y_{bj} V_{bj} \\ \alpha(u_b) \end{matrix}$$

where $y_b = (y_{b1} \dots y_{b,\ell k - k}) \in \mathbb{R}^{\ell k - k}$.

We now construct coordinates for a neighborhood of D in M . Let $\{n_{bj}\}_{j=1,k}$ be an orthonormal basis section for f_*NV_b . Coordinatize a neighborhood of D in M by:

$$(3.3) F_b(u_b, y_b, t_b) = \exp_{\tilde{F}_b}(u_b, y_b) \prod_{j=1}^k t_{bj}^{n_{bj}},$$

where $t_b = (t_{b1} \dots t_{bk}) \in \mathbb{R}^k$.

Let W_b be the coordinate neighborhood associated with F_b , and \tilde{W} as previously defined. Let $W = \bigcap_{i=1}^{\ell} W_i \cap \tilde{W}$. Restrict D and V_i further so that they are contained in W . Let $(U_i, Y_i, T_i): W \rightarrow \mathbb{R}^{n+k}$ be the inverse function to F_i . We will now choose convenient subsets of these functions as coordinates on W .

Proposition 3.4. The function $(U_1, T_1, \dots, T_\ell): W \rightarrow \mathbb{R}^{n+k}$ is a diffeomorphism (coordinate map) in a neighborhood of D in M .

Proof. We write the function in detail, $U_1 = (U_{11} \dots U_{1, n+k-\ell k})$, $T_a = (T_{a1} \dots T_{ak})$. The set D is defined by $T_1 = \dots = T_\ell = 0$. The set $\{\nabla U_{1j}\}_{j=1, n+k-\ell k}$ is a basis for TD , since $U_1|_{T=0}$ is a coordinate map on D . By the construction of the functions T_{aj} , it follows that $\nabla T_{aj}|_{T=0} = n_{aj}$. Since each set $\{n_{aj}\}_{j=1, k}$ is a basis for f_*TV_a , and by 3.1, $TM \cong \bigoplus_{i=1}^{\ell} f_*TV_i$, the gradients of our function are linearly independent along D . By continuity, they are also linearly independent in a neighborhood of D . The proposition then follows from the inverse function theorem.

A similar proposition can be proved using tangential rather than normal coordinates.

Proposition 3.5. The function $(U_1, Y_{11} \dots Y_{1k} \dots Y_{\ell 1} \dots Y_{\ell k}): W \rightarrow \mathbb{R}^{n+k}$ is a diffeomorphism (coordinate map) in a neighborhood of D in M .

Proof. Note that $\nabla Y_{aj}|_{T=0} = V_{aj}$, by 3.2 and 3.3. These were constructed so that for each a , $\{V_{aj}\}_{j=1, k}$ is a basis for

$\bigcap_{i \neq a} f_* TV_i \cap ND$. By proposition 1.3, these are linearly independent and span ND. The proof then follows as in proposition 3.4.

We now illustrate the preceding construction in the case of a curve in the plane. The coincidence set of a self-transversal immersion of a compact curve in R^2 is a finite set of points. Let m be one of these points. Let γ_1 and γ_2 represent arc segments of the 2 "sheets" of the immersion through m . Choosing Fermi coordinates on γ_1 with respect to m is merely choosing arclength as a coordinate, y_1 , with the v_1 of 3.2 being equal to $\frac{\partial}{\partial y_1} \Big|_0$. The point m corresponds to $y_1 = 0$. We now choose n_1 , a normal vector field along γ_1 . The function $F_1(y_1, t_1) = \gamma_1(y_1) + t_1 n_1(y_1)$ is a coordinate map in a neighborhood of m . Compare this to 3.3.

We remark that first, $\frac{\partial F_1}{\partial t_1}(y_1, t_1) = n_1(y_1)$, second that

$\frac{\partial F_1}{\partial y_1}(y_1, t_1) = \dot{\gamma}_1(y_1) + t_1 \dot{n}_1(y_1) = \dot{\gamma}_1(y_1)[1 - k_1(y_1)t_1]$ by Frenet's formula. Thus the components of the metric tensor are (3.6) $g_{11}=1$, $g_{12}=g_{21}=0$, $g_{22}=[1 - k_1(y_1)t_1]^2$. Therefore F_1 is non-singular so long as $t_1 < 1/k_1(y_1)$, which is an elementary result in the Morse Theory of focal points as well.

Proposition 3.4 states that (t_1, t_2) is a coordinatization of a neighborhood of m . The Jacobian of this map is $(\frac{\partial}{\partial t_1} \quad \frac{\partial}{\partial t_2}) =$

$(\nabla T_1 \quad \nabla T_2) = (n_1 \quad n_2)^{-1}$. Thus the area form is

$$(3.7) \quad |\det^{-1}(n_1 \quad n_2)| dt_1 dt_2$$

This area form is independent of the choice of n_1 . An alternate choice of n_1 would have to be of the form $\hat{n}_1 = O_1 n_1$, where O_1 is an

orthonormal transformation of NY_1 . Since NY_1 is one dimensional, $O_1 = \pm 1$. Therefore, $(n'_1 \ n'_2) = \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} (n_1 \ n_2)$. Thus $\det (n'_1 \ n'_2) = \pm 1 \cdot \det (n_1 \ n_2)$, which shows the invariance of the area form.

The area form in this case admits a representation in terms of the angle θ between n_1 and n_2 . This is natural since $n_1 \cdot n_2 = \cos\theta$, and the invariance theory of $GL(n, R)$ states that an invariant polynomial of a matrix must be a polynomial in the $(n_i \cdot n_j)$.

We note

$$\det^2(n_1 \ n_2) = \det(n_1 \cdot n_2) = \det \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} = \sin^2\theta.$$

Thus the area form becomes

$$(3.8) \quad 1/\sin\theta \ dt_1 \ dt_2$$

where $\theta(t_1, t_2)$ is the angle, $\theta \in (0, \pi)$, between $n_1(t)$ and $n_2(t)$.

We note for future reference that

$$(3.9) \quad (n_1 \cdot n_2)^{-1} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sin^2\theta} & \frac{-\cos\theta}{\sin^2\theta} \\ \frac{-\cos\theta}{\sin^2\theta} & \frac{1}{\sin^2\theta} \end{pmatrix}.$$

The next section will construct the volume form in the general case. We will then give some formulae in both the general case and in the special case of a transversally immersed curve in the plane.

§4 The Volume Form

We are given a self-transversal immersion $f:N(n) \rightarrow M(n+k)$, and an ℓ -fold coincidence point m . We have defined in the previous section the coordinate functions near m in M consisting of position coordinates along the coincidence submanifold and normal position vectors with respect to each sheet of the immersion. We will discuss the volume form of this coordinate system in general and in a few restricted cases. The main result is that the volume form, along the coincidence submanifold, has the following form:

$$\det^{-1}(n_{11} \cdots n_{\ell k}) dt dV,$$

where $\{n_{ij}\}_{j=1,k}$ is an orthonormal basis for the normal space of the i^{th} sheet of the immersion and dV is the volume form of the coincidence submanifold. We will note that the function $\det^{-1}(n)$ depends only on the conformal structure at the coincidence points.

For W a neighborhood in M of a coincidence point, the normal coordinate map is $U \times T: W \rightarrow R^{n+k-\ell k} \times R^{\ell k}$, where $T = (T_{11} \cdots T_{\ell k})$.

The volume form is

$$(4.1) \quad \det \left(\frac{\partial}{\partial u} \quad \frac{\partial}{\partial t_{11}} \quad \frac{\partial}{\partial t_{\ell k}} \right) du dt,$$

where $\frac{\partial}{\partial u} = \left(\frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_{n+k-\ell k}} \right)$, $du = du_1 \cdots du_{n+k-\ell k}$ and $dt =$

$dt_1 \cdots dt_{\ell k}$. The wedge product symbol is omitted, as we are interested in unoriented volumes. (Remark: The form in 4.1 is assumed to be positively oriented.)

The level surface defined by $t_{ij}=0$, for all i and j , is the patch D in the coincidence set of the immersion, along which we will explicitly calculate the volume form in terms of the normal

spaces to the immersion and the volume form of the submanifold of ℓ -fold points.

Along D , $\frac{\partial}{\partial u_i} \Big|_D \in TD$ and $\frac{\partial}{\partial t_{aj}} \Big|_D \in ND$. Hence, the determinant

in 4.1 splits, and the volume form, along D , becomes:

$$(4.2) \quad \det\left(\frac{\partial}{\partial u} \Big|_D\right) \det\left(\frac{\partial}{\partial t_{11}} \Big|_D \cdots \frac{\partial}{\partial t_{\ell k}} \Big|_D\right) du dt, \text{ where } \frac{\partial}{\partial u_i} \text{ and } \frac{\partial}{\partial t_{aj}}$$

are viewed as vectors in TD and ND respectively, so that the matrices are square and the operation of determinant makes sense.

Let $dV = \det\left(\frac{\partial}{\partial u} \Big|_D\right) du$, which is the volume form of the submanifold S_ℓ of ℓ -fold points. Hence, the volume form 4.1 can be written as

$$(4.3) \quad \det\left(\frac{\partial}{\partial t_{11}} \Big|_D \cdots \frac{\partial}{\partial t_{\ell k}} \Big|_D\right) dt dV.$$

Remark: It is not necessarily true that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are orthogonal where $t \neq 0$, (ie: off D).

Before continuing, we note a fact about the $\frac{\partial}{\partial t_{ij}}$.

Remark: $\left\{ \frac{\partial}{\partial t_{aj}} \Big|_D \right\}_{j=1, k}$ is a basis for $\bigcap_{i \neq a} f_*TV_i \cap ND$, ie:

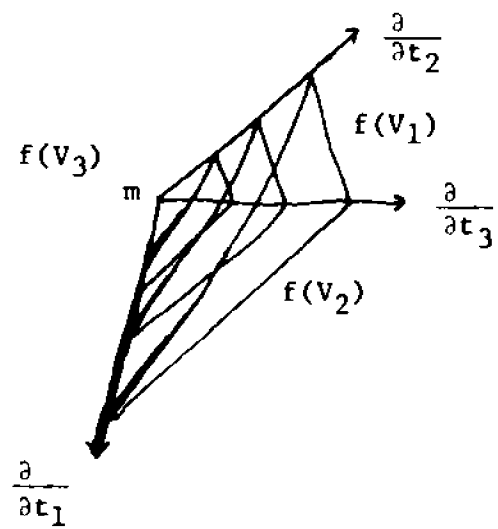
for $T\left(\bigcap_{i \neq a} f(V_i)\right) \cap ND$.

Proof. Using the definition of the coordinate system, from 3.2 and 3.3, we see that $\frac{\partial}{\partial t_{aj}} \Big|_D \in ND$. The vector $\frac{\partial}{\partial t_{aj}}$ is by def-

inition tangent to the level surfaces of the functions T_{bi} , for $b \neq a$. The level surface defined by $\{T_{bi}=0, \text{ for all } i\}$ is simply $f(V_b)$. Thus, for $b \neq a$, $\frac{\partial}{\partial t_{aj}} \in f_*TV_b$, the tangent to the b^{th} sheet of the immersion through D . We then have k linearly independent vectors in $\bigcap_{i \neq a} f_*TV_i \cap ND$, which is k -dimensional by 1.7.

The remark is proven.

In the case of a triple point of an immersed surface in R^3 , the diagram is:



For convenience of notation, we renumber the n_{ij} to $\{n_\alpha\}_{\alpha=1, \dots, \ell k}$. Also, let $n_\alpha = \nabla T_\alpha$ be an extension of the vector fields along the submanifold to a neighborhood of it.

Proposition. The following formulæ are true for the normal coordinate system in a neighborhood of the ℓ -fold point, m :

$$(4.4) \quad \det\left(\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_{\ell k}}\right) = \det^{-1}(n_1 \cdots n_{\ell k}),$$

$$(4.5) \quad \frac{\partial}{\partial t_\alpha} = \sum_{\eta=1}^{\ell k} g^{\eta\alpha} n_\eta,$$

where $(g^{\eta\alpha}) = (n_a \cdot n_b)^{-1}$.

Remark: This proposition will be useful when $t=0$, so that du and dt are orthogonal. Also, in the case of an isolated ℓ -fold point, so that $du=1$, it will yield exactly the volume form in a neighborhood of m .

Proof of the proposition: The normal coordinate map is

$U_x T: W \rightarrow \mathbb{R}^{n+k}$, where W is a neighborhood in M . Let F be the ℓk -dimensional level surface defined by $U = \text{constant}$. $\{\nabla T_{ij}\}$ and $\{\frac{\partial}{\partial t_{ij}}\}$ are both bases for TF . Consider DT restricted to

TF , $DT: TF \rightarrow \mathbb{R}^{\ell k}$. Given $(t_1 \dots t_{\ell k})$ the standard coordinates on $\mathbb{R}^{\ell k}$, and $(\frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_{\ell k}})$ a basis for $TR^{\ell k}$, we can then write DT in

matrix form.

Viewing ∇T_i and n_i as row vectors,

$$(4.5) \quad DT = \begin{pmatrix} \nabla T_1 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} n_1 \\ \vdots \\ \vdots \end{pmatrix},$$

where the vectors are viewed as in TF , so that the matrices are square. By the definition of the vectors $\frac{\partial}{\partial t_\alpha}$ in TF , we have

$$(4.6) \quad \langle DT, \frac{\partial}{\partial t_\alpha} \rangle = e_\alpha = (0 \dots 1_\alpha \dots)^{\text{tr}} \text{ in } TR^{\ell k}. \text{ Componentwise, this states that}$$

$$(4.7) \quad \nabla T_\beta \cdot \frac{\partial}{\partial t_\alpha} = n_\beta \cdot \frac{\partial}{\partial t_\alpha} = \delta_{\alpha\beta}.$$

In matrix notation this becomes

$$(4.8) \quad (n_1 \dots n_{\ell k})^{\text{tr}} \left(\frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_{\ell k}} \right) = I,$$

which proves 4.4.

To prove 4.5, we write $\frac{\partial}{\partial t_\alpha}$ as a linear combination of the n_β (which are a basis for TF):

$$(4.9) \quad \frac{\partial}{\partial t_\alpha} = \sum a_\beta n_\beta.$$

then,

$$(4.10) \quad \delta_{\alpha\beta} = \frac{\partial}{\partial t_\alpha} \cdot n_\beta = \sum a_\beta n_\beta \cdot n_\beta = \sum a_\beta g_{\beta\beta},$$

which can be rewritten as

$$(4.11) \quad e_\alpha = ga, \text{ or}$$

$$(4.12) \quad a = g^{-1} e_\alpha.$$

Substituting 4.12 into 4.9, we obtain the result 4.5. This completes the proof of the proposition.

Proposition 4.13. The volume form of the normal coordinate system, along the submanifold of ℓ -fold points, D , (ie; at $t_{ij}=0$ for all i, j), is

$$(4.13) \quad \det^{-1}(n_{11} \cdots n_{\ell k}) dt dV,$$

where dV is the volume form for D and the n_{ij} are the normal vectors to the immersion.

Proof. Use Proposition 4.4 and equation 4.3.

Proposition 4.14. If the generic ℓ -fold point, m , is isolated (ie. $n+k-\ell k=0$), then the volume form of the normal coordinate map in a neighborhood of m is

$$(4.14) \quad \det^{-1}(n_{11} \cdots n_{\ell k}) dt .$$

Proof. Use Proposition 4.13 and equation 4.1, in the special case $D = \{m\}$.

Proposition 4.15. The coefficient of the volume form of the normal coordinate system, $\det^{-1}(n_{11} \cdots n_{\ell k})$, restricted to the coincidence submanifold D , is independent (up to sign) of the choice of the n_{ij} . We can now refer to $\det n$, which depends only on the conformal structure of the ambient space along D .

Proof. It suffices to show for a particular i that if a different orthonormal basis $\{n'_{ij}\}_{j=1,k}$ is chosen for f_*NV_i , then

$\det (n_{11} \cdots n_{\ell k})$ does not change. Without loss of generality, assume $i=1$.

We know that two orthonormal bases for the same subspace are related by an orthonormal transformation: $n'_{1j} = O \cdot n_{1j}$. ND splits into the orthogonal sum

$$ND = f_{\star}NV_1 \oplus E,$$

where E is the orthogonal complement of $f_{\star}NV_1$ in ND . We consider all the n_{1j} as being in ND , for the purpose of defining $\det(n_{11} \cdots n_{\ell k})$. Choose an orthonormal basis for ND containing $(n_{11} \cdots n_{1k})$. The matrix $(n_{11} \cdots n_{\ell k})$ can be written componentwise in this basis as:

$$(4.16) \quad \begin{pmatrix} I & * \\ 0 & \pi_E(n_{21} \cdots n_{\ell k}) \end{pmatrix}$$

The linear transformation O on $f_{\star}NV_1$ can be considered as the restriction of a linear transformation on all of ND , $O \oplus I$, which fixes the orthogonal complement of $f_{\star}NV_1$.

To find $\det (n_{11} \cdots n_{\ell k})$, we apply \det to 4.15, which yields

$$(4.17) \quad \det [\pi_E(n_{21} \cdots n_{\ell k})],$$

viewing the column vectors as elements of E , so that the matrix is square and \det makes sense. Changing the basis of $f_{\star}NV_1$ is equivalent to applying the linear transformation $O \oplus I$ to 4.15, which yields a matrix of the same form, whose determinant is again equal, up to sign, to 4.17. This proves the proposition.

§5 The Volume

We will now derive the formula for the volume of the ℓ -fold points of a tubular neighborhood, with small radius r , about an immersed submanifold. We will also give other formulae for this volume in some restricted cases.

We repeat Theorem 1 for reference:

Theorem 1 Let $f: N(n) \rightarrow M(n+k)$ be a self-transversal isometric immersion. Consider the tubular neighborhood of radius r about $f(X)$ in Y . For small r , the volume of the ℓ -fold points, $V_\ell(r)$, of this neighborhood is given by the formula.

$$V_\ell(r) = r^{k\ell} (V_k)^\ell \int |\det^{-1} n| dV + O(r^{k\ell+1}),$$

where V_k is the volume of the unit k -disc, dV is the volume form of the ℓ -fold coincidence submanifold of f , and n is the matrix of normal vectors defined along the coincidence set, as defined in section 4, dependent only on the conformal structure of M .

We now give the proof of Theorem 1. Note that the set of ℓ -fold points of the tube may have some additional multiplicity. For example, when we immerse a surface in R^3 , the set of double points of the tube is a neighborhood of the curves of double points of the immersion. The tube will thus have multiplicity 3 near each triple point of the immersion. Since we will be expressing the volume as an integral over f_ℓ , the submanifold of strictly ℓ -fold points, our technique counts the regions of the tube of even higher multiplicity more than once. This does not have any effect on the formula in Theorem 1, as it is only an approximation to order $r^{k\ell}$, while the regions counted with multiplicity are

$\ell+1$ -fold points, and thus have volume of order $r^{k(\ell+1)}$.

Recall that f_ℓ is the submanifold of ℓ -fold points of the immersion. Let $N_\epsilon(f_\ell)$ be the normal disc bundle of radius ϵ of f_ℓ . For the rest of this section, the radius of the tube will be referred to as R . For small R , the set of ℓ -fold points of the tubular neighborhood, radius R , of $f(N)$ will be contained in $\exp N_\epsilon(f_\ell)$. Let $m \in f_\ell$, and choose normal coordinates, as defined in §3, in a neighborhood W of m in M . Restrict W such that

- 1) $W \cap f_{\ell+1} = \emptyset$, and
- 2) $W = \exp N_\epsilon D$, where D is a coordinate neighborhood of m in f_ℓ .

The volume of the ℓ -fold points of the tubular neighborhood of $f(N)$, radius R , over the region D in f_ℓ , is

$$(5.2) \quad \int_D \int_B J(u,t) dt_{11} \dots dt_{\ell k} du,$$

where $J(u,t) = \det \left(\frac{\partial}{\partial u} \frac{\partial}{\partial t} \right)$, (cf 4.1), and $B \subset R^{\ell k} = \{(t_{11} \dots t_{\ell k})\}$ is defined by the restrictions $\sum_{j=1}^k t_{aj}^2 \leq R^2$, for $a = 1$ to ℓ .

We change coordinates, replacing each set $\{t_{aj}\}_{j=1,k}$ by (r_a, θ_a) , the corresponding spherical coordinates. Note $\theta_a \in S^{k-1}$. Hence $dt_{a1} \dots dt_{ak} = r_a^{k-1} dr_a d\theta_a$. The domain, B , of r and θ is defined by the set of restrictions $r_a \leq R$, for all a . The formula for volume, 5.2, then becomes

$$(5.3) \quad \int_D \int_{\theta_a \in S^{k-1}} \int_{r_a \leq R} J(u,r,\theta) \bigwedge_{a=1}^{\ell} (r_a^{k-1} dr_a d\theta_a) dv$$

where the integrals are iterated from $a=1$ to ℓ , and J is taken always positive.

Recall from Proposition 4.13 that $J(u, o) = \det n$, where n is the matrix of normal vectors along the coincidence submanifold.

We define an auxiliary function

$$(5.4) \quad F(R, \theta, u) = \int_{r_a \leq R} J(r, \theta, u) \bigwedge_{a=1}^{\ell} (r_a^{k-1} dr_a).$$

The technique we will use is to approximate F to the first order in R , notice the approximation is independent of θ , and then substitute it into 5.3. We suppress the dependence of F on θ and u , and write $F(R)$. Introduce the auxiliary function

$$(5.5) \quad H(R_1, \dots, R_\ell) = \int_{r_a \leq R_a} J(r) \bigwedge_{a=1}^{\ell} (r_a^{k-1} dr_a).$$

The relation between derivatives of F and H is:

$$(5.5) \quad \frac{\partial F}{\partial R}(R) = [(\sum_{i=1}^{\ell} \frac{\partial}{\partial R_i}) H](R, \dots, R),$$

$$(5.6) \quad D^n F = \frac{\partial^n F}{\partial R^n} = [(\sum_{i=1}^{\ell} \frac{\partial}{\partial R_i})^n H](R, \dots, R).$$

Notation. Let $\bar{R} = (R_1 \dots R_\ell)$. Let $\bar{R}=0$ mean $R_1=R_2=\dots=R_\ell=0$. If $I = \{j(1) \dots j(q)\} \subseteq \{1, \dots, \ell\}$, then $E(I)$ will mean the evaluation operator on a function of $(r_1, r_2, \dots, r_\ell)$ which sets $r_a=R_a$ for each $a \in I$. Let $D(I) = \frac{\partial^q}{\partial R_{j(1)} \dots \partial R_{j(q)}}$. We will call any partial derivative simple if it can be written as a single term $\frac{\partial^\alpha}{\partial R_1^{\alpha(1)} \dots \partial R_\ell^{\alpha(\ell)}}$.

Proposition 5.7. The lowest order derivative of H which does not vanish at $\bar{R}=0$ is

$$(5.7) \quad \frac{\partial^{k\ell}}{\partial R_1^k \dots \partial R_\ell^k} H \Big|_{\bar{R}=0} = [(k-1)!]^\ell \cdot \det^{-1} n.$$

This is the only simple derivative of order $k\ell$ which is non-zero at $\bar{R}=0$.

Proof. Any derivative of H is a sum of simple derivatives.

We consider these and list some properties.

If $I \neq \{1, \dots, \ell\}$, then $D(I)H|_{\bar{R}=0} = 0$, since

$$(5.8) \quad [D(I)H](R_1 \dots R_\ell) = \int_{\substack{r_a \leq R_a \\ a \notin I}} \{E(I)[J(r)]\} \left[\prod_{a \in I} R_a^{k-1} \right] \wedge_{a \notin I} (r_a^{k-1} dr_a).$$

Any simple derivative of H can be factored into $[L \circ D(I)]H$, where L is a simple and involves only $\frac{\partial}{\partial R_i}$ with $i \in I$. If $I \neq \{1, \dots, \ell\}$, then $[L \circ D(I)]H|_{\bar{R}=0} = 0$, since L commutes with the integral sign in 5.8. We have thus shown that any simple differential operator which does not contain $\frac{\partial^\ell}{\partial R_1 \dots \partial R_\ell}$ as a factor, when applied to H , vanishes at $\bar{R}=0$.

Consider now simple derivatives of H of the form $L \circ \frac{\partial^\ell H}{\partial R_1 \dots \partial R_\ell}$.

We see, from 5.8, that

$$(5.9) \quad \frac{\partial^\ell H}{\partial R_1 \dots \partial R_\ell} = J(R_1 \dots R_\ell) \prod_{a=1}^{\ell} R_a^{k-1}.$$

The lowest order derivative of 5.9 which will not vanish at $\bar{R}=0$ is

$$(5.10) \quad \frac{\partial^{\ell(k-1)}}{\partial R_1^{k-1} \dots \partial R_\ell^{k-1}} \circ \frac{\partial^\ell H}{\partial R_1 \dots \partial R_\ell} \Big|_{\bar{R}=0} = [(k-1)!]^\ell \cdot J(0 \dots 0).$$

This is the only simple derivative of order $\ell(k-1)$ of 5.9 which will be non-zero at $\bar{R}=0$. Combining 5.9 and 5.10, and recalling Proposition 4.13, the proof of the proposition is complete.

Proposition 5.11. For $n < k\ell$, $D^n F|_{R=0} = 0$. The first nonvanishing derivative of F at $R=0$ is:

$$(5.11) \quad D^{k\ell} F|_{R=0} = \frac{(k\ell)!}{k^\ell} \cdot \frac{1}{\det n}$$

Proof. By 5.6, $D^n F$ is a linear combination, with multinomial coefficients, of derivatives of H of order n . Proposition 5.7 implies the vanishing of $D^n F|_{\bar{R}=0}$ for $n < \ell k$. It also implies, (for transversal intersections), the non-vanishing of $D^{\ell k} F|_{\bar{R}=0}$. It remains to find the correct coefficient. This is equivalent to the following classical combinatorial problem:

Given the polynomial $(\sum_{i=1}^{\ell} a_i)^k$, the coefficient of $a_1^k a_2^k \dots a_{\ell}^k$ is $\frac{(k\ell)!}{(k!)^{\ell}}$. This can be seen as follows: Given $k\ell$ positions, count

the number of ways of rearranging $a_1 \dots a_1 a_2 \dots a_2 \dots a_{\ell} \dots a_{\ell}$. Distinguishing between the repeated variables, there are $(k\ell)!$ ways of doing this. The multiplicity of each rearrangement assuming distinguished variables is $k!$, which is the number of ways of rearranging k copies of a_i . Thus, to count the actual number of rearrangements, we must divide $(k\ell)!$ by $k!$ ℓ -times. This yields the stated result.

Thus, from 5.6 and proposition 5.7,

$$\begin{aligned} D^{\ell k} F|_{\bar{R}=0} &= \frac{(k\ell)!}{(k!)^{\ell}} \cdot [(k-1)!]^{\ell} \cdot \det^{-1} n, \\ &= \frac{(k\ell)!}{k^{\ell}} \det^{-1} n, \end{aligned}$$

which proves the proposition.

Using Taylor's Theorem, we have that for small R ,

$$\begin{aligned}
 (5.12) \quad F(R) &= \frac{1}{(k\ell)!} D^{k\ell} F|_{R=0} \cdot R^{k\ell} + O(R^{k\ell+1}), \\
 &= \frac{1}{k^\ell} \cdot \frac{1}{\det n} \cdot R^{k\ell} + O(R^{k\ell+1})
 \end{aligned}$$

by proposition 5.11. Substituting this into 5.3, the formula for the volume becomes

$$(5.13) \quad \int_D \int_{S^{k-1}} k^{-\ell} |\det^{-1} n| R^{k\ell} + O(R^{k\ell+1}) \, d\theta \, dV.$$

Since the domains of integration, S^{k-1} and f_ℓ , are compact,

$$\int_D \int_{S^{k-1}} O(R^{k\ell+1}) \, d\theta \, dV = O(R^{k\ell+1}).$$

Also, $\det^{-1} n|_D$ is dependent only on u , not θ . Let ω_{k-1} = volume of the unit $k-1$ sphere. 5.13 becomes

$$(5.14) \quad R^{k\ell} (\omega_{k-1}/k)^\ell \int_D |\det^{-1} n| \, dV + O(R^{k\ell+1}).$$

Note that $\omega_{k-1}/k = V_k$ = volume of the unit k -disc, since $V_k = \int_0^1 \omega_{k-1} r^{k-1} \, dr = \omega_{k-1}/k$. Substituting this into 5.14, the

volume formula is as stated in the Theorem,

$$(5.15) \quad (V_k)^\ell R^{k\ell} \int_D |\det^{-1} n| \, dV + O(R^{k\ell+1}).$$

This is the volume of the ℓ -fold points of the tube of radius R within $\exp ND$. Cover f_ℓ except for a set of measure zero by such D 's, and sum. This yields the stated result for volume, counting multiplicities. As was argued in the beginning of this section, the result is therefore true for the actual volume. This proves Theorem 1.

This theorem enables us to obtain an approximation of the volume of the tube in the Euclidian case. It follows from Theorem 1 and Weyl's Theorem that:

Theorem 2: Let $f: X^{(n)} \rightarrow R^{n+k}$ be a self-transversal immersion. For small r , the volume of the tube of radius r about $f(X)$ in R^{n+k} is given by the formula

$$\sum_{0 \leq e \leq k} H_e r^{k+e} - (V_k)^2 r^{2k} \int |\det^{-1} n| dV + O(r^{2k+1}),$$

where the H_e are the coefficients of the ℓ -dimensional Weyl polynomial of X , (as described in Chapter 1, Theorem 1), and the correction term is the volume of the double points of the tube as given in the previous theorem.

In the case of a circle self-transversally immersed in the plane we see that:

Corollary 1. Let $f: S^1 \rightarrow R^2$ be a self-transversal immersion. The double points are a finite set of points $\{X_i\}_{i=1,q}$. The area of the double points of the band of radius R near an X_i is, for small R :

$$(5.16) \quad \int_{-R}^R \int_{-R}^R 1/\sin\theta(t) dt_1 dt_2,$$

where t_1 and t_2 are normal coordinates measuring distance to each curve segment thru X_i , and $\theta(t)$ is the angle between the normals (tangents) of the level curves of the bands.

Proof. Equations 5.2 and 3.8.

Corollary 2. If $f: S^1 \rightarrow R^2$ is self-transversal, with double points $\{X_i\}_{i=1,n}$, then the volume of the tube of radius r about the image of S^1 in R^2 is given by the formula:

$$(5.17) \quad 2rL - 4r^2 \sum_{i=1}^n 1/\sin\theta_i + O(r^3),$$

where θ_i is the angle of coincidence of the curve at each X_i and L is the length of the curve.

Proof. Follows immediately from Theorem 2 or from equation 3.8.

Corollary 3. If $f: X^{(2)} \rightarrow R^3$ is a self-transversal immersion of a surface in space, then the volume of the tube of radius r about $f(X)$ is

$$r \cdot \text{Area}(X) - 4r^2 \int |1/\sin\theta| ds + O(r^3),$$

where the integration is over the curves of double points and θ is the angle of self-intersection of the surface at each double point.

Proof. Follows immediately from Theorem 2.

We consider again $f: X^{(n)} \rightarrow Y^{(n+k)}$.

Corollary 4. Let $f: X^{(n)} \rightarrow Y^{(n+k)}$ be a self-transversal immersion. If $n+k-\ell k = 0$, then the ℓ -fold points of a self-transversal immersion are a finite set of points. The first approximation to the volume of the ℓ -fold points of the tube depend only on

- 1) the number of coincidence points,
- 2) the conformal structure of M at the coincidence points,
- 3) the codimension k .

Proof. Follows immediately from Theorem 1.

We now consider the case of a self-transversal immersion into Euclidian space where the map is linear in the neighborhood of an isolated ℓ -fold point.

Corollary 5. If $f: X^{(n)} \rightarrow R^{n+k}$, and if $f(X)$ is linear near a particular isolated ℓ -fold point, ($n+k-\ell k = 0$), then for small r the volume of the ℓ -fold points of the tube of radius r , near that point, is the volume of the corresponding parallelogram:

$$(5.18) \quad r^{k\ell} (v_k)^\ell \det^{-1} n.$$

Remark: This formula, which is exact in the linear case, is the first approximation to the volume in the non-linear case. (Compare to Theorem 1).

Proof. We use equations 5.3, 5.11 and the fact that the Jacobian will be locally constant.

56 Higher Order Approximations in the Plane

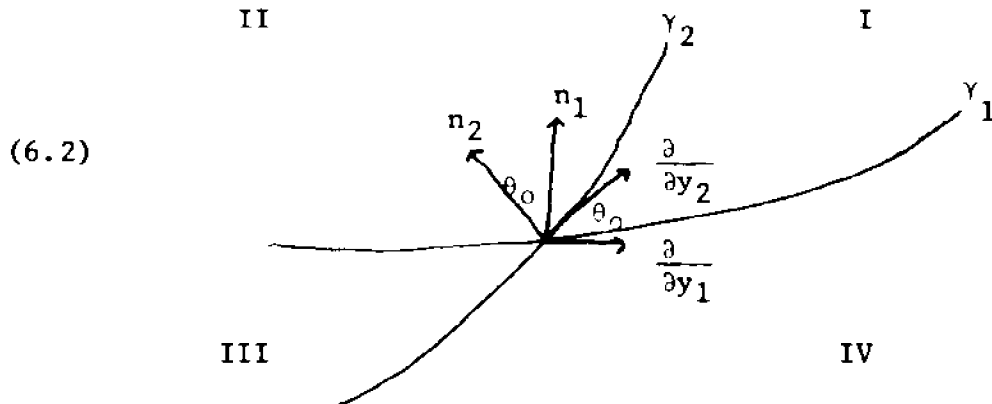
To obtain more accurate approximations to the volume of a tube, we need to better approximate the volume of the l -fold points of a tubular neighborhood of an immersed submanifold. This volume is a function of the radius, the manifold, and the intersection submanifolds. In the case of an imbedded submanifold of R^n , there are only 1-fold points, and the volume, as demonstrated by H. Weyl, is a polynomial in the radius whose coefficients are intrinsic metric invariants of the submanifold. The volume of the points of higher multiplicity does not appear to be so nice, even in the simple case of an intersection between linear and circular curve segments in the plane. (See example.) The only situation in which we know the volume will be a polynomial is in the case of an isolated coincidence point of an immersion into R^n , where the immersion is linear in a neighborhood of the coincidence point. This is shown in corollary to Theorem 1. It also occurs when there is great symmetry, as in examples 2 and 3. We can however, obtain higher order approximations in the particular example of a self-transversal immersion of a curve in the plane. We will comment on how this might also be accomplished in more general situations.

Let $f:S^1 \rightarrow R^2$ be a self-transversal immersion of the circle in the plane. The coincidence set is a finite set of points $\{X_i\}_{i=1,q}$. We will construct a normal coordinate system in a neighborhood of a particular X_i , as described in section 3. The formula for the area of the double points of the band of radius R , near X_i , for small R , is given as a corollary to Theorem 1. We recall it:

$$(6.1) \quad \text{vol}(R) = \int_R^R \int_R^R 1/\sin\theta(t) dt_1 dt_2,$$

where t_i is the distance of a point to the i^{th} sheet (curve segment) of the immersion through the coincidence point, and θ is the angle between the level curves at that point. We now construct this coordinate system in detail. Let m be a particular intersection point, and γ_1 and γ_2 represent the two arc segments of the immersion near m . We make the generic assumption that the curvatures of each segment are non-zero at, and therefore near, m . Choose some orientation for R^2 . Let y_1 be arc length on γ_1 , chosen such that the curvature of $\gamma_1(y_1)$ is positive. This is equivalent to the assertion that (\vec{t}_1, \vec{n}_1) has positive orientation, with $\vec{t}_1 = \dot{\gamma}_1(y_1)$ and $n_1 = \dot{\gamma}_1 / \|\dot{\gamma}_1\|$. We now renumber, if necessary, so that (n_1, n_2) is positively oriented. We then construct, as in section 3, Fermi coordinates near m with respect to each γ_i . Let $F_1(y_1, t_1) = \gamma_1(y_1) + t_1 n_1(y_1)$. Now, let Y_1 and T_1 be the inverse functions to F_1 , defined on a neighborhood of m . By proposition 3.4, (T_1, T_2) is a diffeomorphism from a neighborhood of m to a neighborhood of zero in R^2 . We extend the definition of n_1 to this neighborhood by setting $n_1 = \nabla T_1$. These coordinate systems were discussed in detail in section 3.

We draw the following diagram, and label the quadrants I through IV with respect to the y_1 , as in the Cartesian plane. Note that the diagram is drawn showing the angle between n_1 and n_2 acute. This does not affect the ensuing arguments.



Note that the angle between the level curves, γ_1 and γ_2 , is equal to the angle between the n_1 by elementary geometry.

To calculate higher order approximations to the area formula given in 6.1, we will use Taylor's formula, and therefore will need to be able to calculate the derivatives of the area with respect to R . We will see that these will involve θ_0 , the curvatures and their derivatives at m , and the derivatives of θ with respect to t_1 and t_2 . We will give a formula for θ as a function of y_1 and y_2 . We will then give formulæ for the derivatives $\frac{dy_j}{dt_j}$ in

terms of θ , t and the curvature functions. This would enable us to recursively calculate the higher order derivatives of y with respect to the t_j in terms of θ , t , the derivatives of the curvatures and the lower order derivatives of y with respect to the t_j . Given these formulæ for derivatives of θ with respect to the y_j , and for derivatives of y_j with respect to the t_j , we can calculate

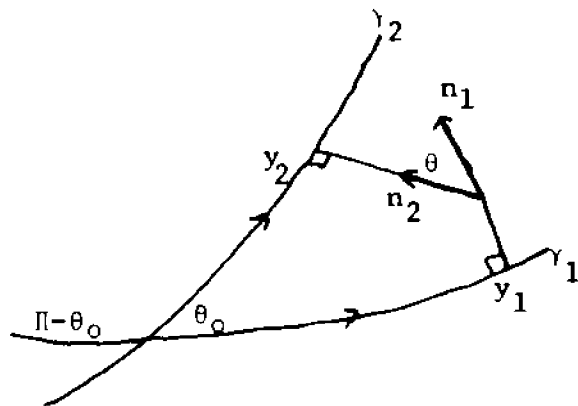
the derivatives of θ with respect to the t_i by the chain rule. We will carry out this calculation to obtain an approximation to the area to order R^3 . First, we give the formula for $\theta(y)$.

Proposition 6.3. Given n_1 and n_2 , $n_i = \nabla T_i$, in a neighborhood of m , and y_i arc length on γ_i , and θ the angle between n_1 and n_2 . Then

$$(6.3) \quad \theta(y_1, y_2) = \theta_0 - \int_0^{y_1} k_1(y_1) dy_1 + \int_0^{y_2} k_2(y_2) dy_2.$$

Proof. We apply the Umlaufsatz in each of the quadrants I through IV. Compare diagram 6.2. The Umlaufsatz states that for a simple closed piecewise C^2 curve in the plane, the integral of the curvature in the counter-clockwise direction plus the sum of the exterior angles is 2π . Let $k_i(y_i)$ be the curvature function for γ_i . It is positive by construction.

In quadrant I:

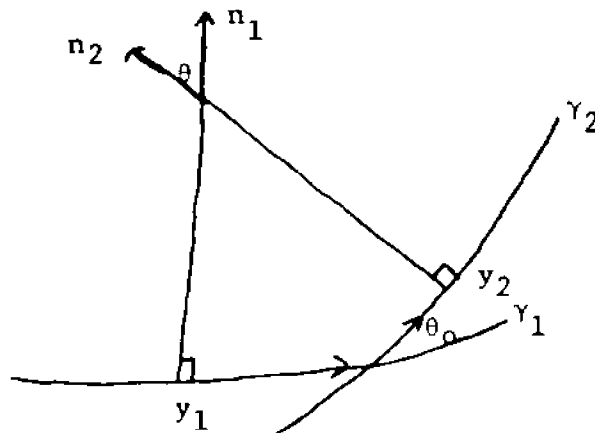


Applying the Umlaufsatz, we have

$$2\pi = \int_0^{y_1} k_1(y_1) dy_1 - \int_0^{y_2} k_2(y_2) dy_2 + \theta + (\pi - \theta_0) + \frac{\pi}{2} + \frac{\pi}{2},$$

which is equivalent to 6.3

In quadrant II, recalling $y_1 \leq 0$ and $y_2 \geq 0$:

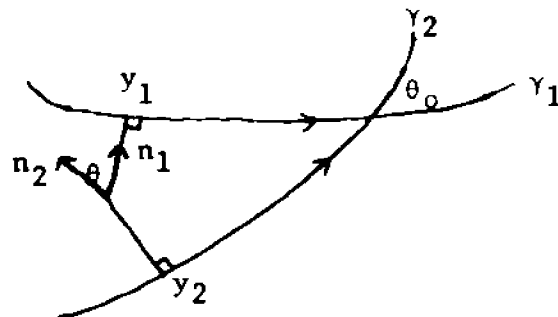


We now have that

$$2\pi = \int_0^{y_2} k_2(y_2) dy_2 + \int_{y_1}^0 k_1(y_1) dy_1 + \theta_0 + (\pi - \theta_0) + \frac{\pi}{2} + \frac{\pi}{2},$$

which is again equivalent to 6.3.

In quadrant III, recalling $y_1 \leq 0$ and $y_2 \leq 0$:

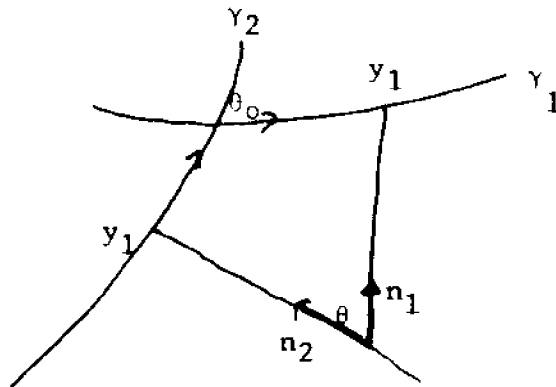


We then have

$$2\pi = \int_{y_2}^0 k_2(y_2) dy_2 - \int_{y_1}^0 k_1(y_1) dy_1 + \theta + (\pi - \theta_0) + \frac{\pi}{2} + \frac{\pi}{2},$$

which is again equivalent to 6.3.

In quadrant IV, recalling $y_1 \geq 0$ and $y_2 \leq 0$:



We then have

$$2\pi = -\int_0^{y_1} k_1(y_1) dy_1 - \int_{y_2}^0 k_2(y_2) dy_2 + \theta_0 + (\pi - \theta) + \frac{\pi}{2} + \frac{\pi}{2},$$

which is once more equivalent to 6.3. This completes the proof of the proposition.

Corollary. The derivatives of θ with respect to y are

$$(6.4) \quad \frac{\partial \theta}{\partial y_1} = -k_1(y_1) \quad \frac{\partial \theta}{\partial y_2} = k_2(y_2).$$

We now calculate $\frac{dy_1}{dt_j}$. We have $n_i = \nabla t_i$, which are unit vec-

tors, and ∇y_i , which are perpendicular to the n_i , with norm $1 - k_i(y_i)t_i$. This is a consequence of Frenet's formula. By the

hypothesis that (n_1, n_2) is positively oriented, we have that

$$(6.5) \quad \nabla y_1 \cdot n_2 = -(1 - k_1(y_1)t_1)\sin\theta$$

$$(6.6) \quad \nabla y_2 \cdot n_1 = (1 - k_2(y_2)t_2)\sin\theta,$$

where θ is the angle between n_1 and n_2 . We also have that

$$(6.7) \quad \frac{\partial}{\partial t_j} = g^{1j}n_1 + g^{2j}n_2,$$

where

$$(6.8) \quad (g^{ij}) = (n_i \cdot n_j)^{-1} = \begin{pmatrix} -\frac{\cos\theta}{\sin^2\theta} & \frac{1}{\sin^2\theta} \\ \frac{1}{\sin^2\theta} & -\frac{\cos\theta}{\sin^2\theta} \end{pmatrix}$$

From calculus, we know that

$$(6.9) \quad \frac{\partial y_1}{\partial t_j} = \nabla y_1 \cdot \frac{\partial}{\partial t_j}.$$

Lemma. The derivatives $\frac{\partial y_1}{\partial t_j}$ are

$$(6.10) \quad \frac{\partial y_1}{\partial t_1} = -(1 - k_1(y_1)t_1)/\sin\theta$$

$$(6.11) \quad \frac{\partial y_1}{\partial t_2} = (1 - k_1(y_1)t_1)\cot\theta$$

$$(6.12) \quad \frac{\partial y_2}{\partial t_1} = (1 - k_2(y_2)t_2)\cot\theta$$

$$(6.13) \quad \frac{\partial y_2}{\partial t_2} = -(1 - k_2(y_2)t_2)/\sin\theta$$

Proof. We use 6.5 through 6.9:

$$\frac{\partial y_1}{\partial t_1} = \nabla y_1 \cdot \frac{\partial}{\partial t_1}$$

$$\begin{aligned}
&= \nabla y_1 \cdot (g^{11} n_1 + g^{12} n_2) \\
&= -g^{12} (1 - k_1(y_1)t_1)\sin\theta \\
&= -(1 - k_1(y_1)t_1)/\sin\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y_1}{\partial t_2} &= \nabla y_1 \cdot \frac{\partial}{\partial t_2} \\
&= \nabla y_1 \cdot (g^{12} n_1 + g^{22} n_2) \\
&= -g^{22} (1 - k_1(y_1)t_1)\sin\theta \\
&= + (1 - k_1(y_1)t_1)\cot\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y_2}{\partial t_1} &= \nabla y_2 \cdot \frac{\partial}{\partial t_1} \\
&= \nabla y_2 \cdot (g^{11} n_1 + g^{12} n_2) \\
&= g^{11} (1 - k_2(y_2)t_2)\sin\theta \\
&= -(1 - k_2(y_2)t_2)\cot\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y_2}{\partial t_2} &= \nabla y_2 \cdot \frac{\partial}{\partial t_2} \\
&= \nabla y_2 \cdot (g^{12} n_1 + g^{22} n_2) \\
&= g^{12} (1 - k_2(y_2)t_2)\sin\theta \\
&= (1 - k_2(y_2)t_2)\cot\theta
\end{aligned}$$

This completes the proof of the lemma.

From this lemma, we immediately have

$$(6.14) \quad \frac{\partial y_1}{\partial t_1} \Big|_o = \frac{\partial y_2}{\partial t_2} \Big|_o = -1/\sin\theta_o$$

$$(6.15) \quad \frac{\partial y_1}{\partial t_2} \Big|_o = \frac{\partial y_2}{\partial t_1} \Big|_o = \cot\theta_o \quad .$$

We can now calculate $\frac{\partial \theta}{\partial t_1}$, which by the chain rule is equal to

$\frac{\partial \theta}{\partial y_1} \frac{\partial y_1}{\partial t_1} + \frac{\partial \theta}{\partial y_2} \frac{\partial y_2}{\partial t_1}$. We calculate, using 6.4 and 6.10 through 6.13,

$$(6.16) \quad \begin{aligned} \frac{\partial \theta}{\partial t_1} &= \frac{\partial \theta}{\partial y_1} \frac{\partial y_1}{\partial t_1} + \frac{\partial \theta}{\partial y_2} \frac{\partial y_2}{\partial t_1} \\ &= k_1(y_1)(1 - k_1(y_1)t_1)/\sin\theta \\ &\quad + k_2(y_2)(1 - k_2(y_2)t_2)\cot\theta. \end{aligned}$$

$$(6.17) \quad \begin{aligned} \frac{\partial \theta}{\partial t_2} &= \frac{\partial \theta}{\partial y_1} \frac{\partial y_1}{\partial t_2} + \frac{\partial \theta}{\partial y_2} \frac{\partial y_2}{\partial t_2} \\ &= -k_1(y_1)(1 - k_1(y_1)t_1)\cot\theta - k_2(y_2)(1 - k_2(y_2)t_2)/(\sin\theta). \end{aligned}$$

We note for reference that

$$(6.18) \quad \begin{aligned} \frac{\partial \theta}{\partial t_1} + \frac{\partial \theta}{\partial t_2} &= [k_1(y_1)(1 - k_1(y_1)t_1) - k_2(y_2)(1 - k_2(y_2)t_2)] \\ &\quad \cdot [1/\sin\theta - \cot\theta] \end{aligned}$$

Examining formulæ 6.10 through 6.13, which describe $\frac{dy_1}{dt_j}$, and formulæ 6.16 and 6.17 which describe $\frac{\partial \theta}{\partial t_1}$, we see that the derivatives of order q with respect to t are expressible as an algebraic function of $\sin\theta$, $\cos\theta$, t and the derivatives $k_1^{(1)}$ through $k_1^{(q-1)}$.

Repeating 6.1, the area of the double points of the band of radius R around the segments γ_1 and γ_2 is, for small R :

$$A(R) = \int_{-R}^R \int_{-R}^R 1/\sin\theta(t_1, t_2) dt_1 dt_2.$$

We remember from calculus that the derivative of a function $\phi(R) = \int_{-R}^R \phi(s) ds$ is $\phi'(R) = \phi(R) + \phi(-R)$. We can then calculate

the derivatives of $A(R)$:

$$(6.19) \quad \dot{A}(R) = \int_{-R}^R 1/\sin\theta(t_1, R) + 1/\sin\theta(t_1, -R) dt, + \\ \int_{-R}^R 1/\sin\theta(R, t_2) + 1/\sin\theta(-R, t_2) dt_2.$$

Therefore, $\dot{A}(0) = 0$

$$(6.20) \quad \ddot{A}(R) = 2 \sum_{\alpha=1}^2 \sum_{\beta=1}^2 1/\sin\theta((-1)^\alpha R, (-1)^\beta R) \\ + \sum_{\beta=1}^2 \int_{-R}^R -(\cos\theta/\sin^2\theta) \frac{\partial\theta}{\partial t_2} (t_1, (-1)^\beta R) dt, \\ + \sum_{\alpha=1}^2 \int_{-R}^R -(\cos\theta/\sin^2\theta) \frac{\partial\theta}{\partial t_1} ((-1)^\alpha R, t_2) dt_2.$$

Therefore, $\ddot{A}(0) = 8/\sin\theta_0$, which agrees with the approximation formula for the area given in the corollary to Theorem 1.

$$(6.21) \quad \ddot{A}(R) = 3 \sum_{\alpha=1}^2 \sum_{\beta=1}^2 (-\cos\theta/\sin^2\theta) \left(\frac{\partial\theta}{\partial t_1} + \frac{\partial\theta}{\partial t_1} \right) (-R)^\alpha, (-R)^\beta \\ + \sum_{\beta=1}^2 \int_{-R}^R \frac{\partial}{\partial R} \left[(-\cos\theta/\sin^2\theta) \frac{\partial\theta}{\partial t_2} (t_1, (-1)^\beta R) \right] dt_1 \\ + \sum_{\alpha=1}^2 \int_{-R}^R \frac{\partial}{\partial R} [*] dt.$$

Therefore $\ddot{A}(0) = (-12 \cos\theta_0/\sin^2\theta_0) \left(\frac{\partial\theta}{\partial t_1} \Big|_0 + \frac{\partial\theta}{\partial t_2} \Big|_0 \right)$.

From 6.18, we have that $\frac{\partial\theta}{\partial t_1} + \frac{\partial\theta}{\partial t_2}$ at $t_1 = t_2 = 0$ is $(k_1(0) - k_2(0))$.

$(\csc\theta_0 - \cot\theta_0)$, which equals $(k_{10} - k_{20}) (1 - \cos\theta_0)/\sin\theta_0$. Therefore

$$(6.22) \quad \ddot{A}(0) = \frac{-12\cos\theta_0}{\sin^2\theta_0} \frac{1 - \cos\theta_0}{\sin\theta_0} (k_{10} - k_{20})$$

$$= \frac{-12\cos\theta_0}{(1+\cos\theta_0)\sin\theta_0} (k_{10} - k_{20}).$$

Using the derivatives of A at $R = 0$ and Taylor's Theorem, we have that

$$(6.23) \quad A(R) = (4/\sin\theta_0)R^2 - 2(k_{10} - k_{20})\cot\theta_0/(1 + \cos\theta_0)R^3 + O(R^4)$$

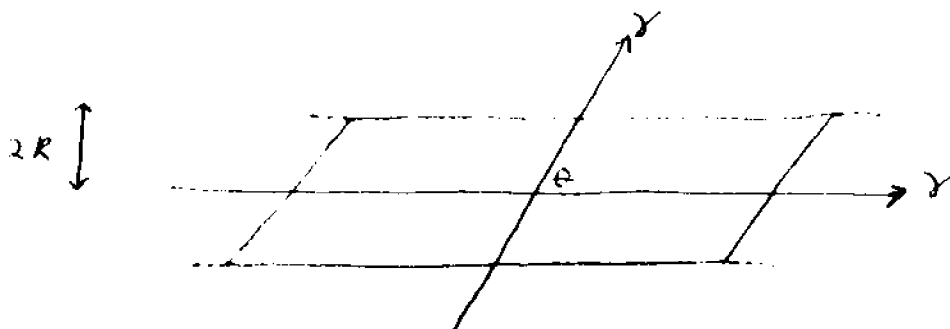
For higher order approximations, we could calculate higher derivatives of A at $R = 0$. These would involve trigonometric functions of θ_0 and the derivatives of θ with respect to t . We can easily see from 6.21 and our comments on $\frac{\partial\theta}{\partial t_1}$ that $A^{(q)}(0)$ is an algebraic function of $\sin\theta_0$, $\cos\theta_0$ and derivatives of the curvatures at the coincidence point of order less than $q-3$.

To carry out this procedure in a more general setting, such as an isolated coincidence point of order ℓ of a self-transversal immersion $f:N^{(n)} \rightarrow M^{(n+k)}$ we need a higher dimensional analogue to the Umlaufsatz. It must apply to curved polyhedra of codimension 1 in M . A possible candidate is a generalization of the Gauss-Bonnet formula for polyhedra. (Reference: Milnor's note on Euler characteristic and measure.)

Examples

The area of double points of tubular neighborhoods of self-transversal immersions are listed in a few special cases. The calculations are straightforward calculus and trigonometry, so are omitted.

Example 1. If a curve intersects itself in the plane and is linear in a neighborhood of a coincidence point:



The area of the double points of the band of radius R is

$$(1) \quad 4R^2/\sin\theta$$

Example 2. In the situation above, if one segment is linear and the other circular of radius ρ the area is

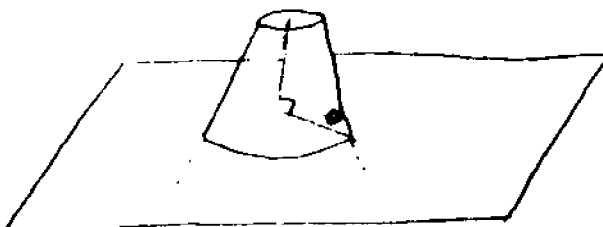
$$(2) \quad \frac{1}{2} \sum_{\alpha=1}^2 \sum_{\beta=1}^2 \left\{ [R + (-1)^\alpha \cos\theta] \sqrt{(\rho + (-1)^\beta R) - (R + (-1)^\alpha \rho \cos\theta)^2} + [\rho + (-1)^\beta R] \arcsin \frac{R + (-1)^\alpha \rho \cos\theta}{\rho + (-1)^\beta R} \right\}$$

This formula, although precise, gives little insight into its relation with the geometry of the curve. Applying the results of

Chapter 2, section 6, the area of the double points is:

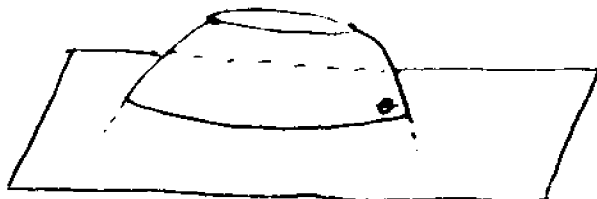
$$(2a) \quad 4R^2/\sin\theta + (2R^3/\rho) \cot\theta/(1 + \cos\theta) + O(R^4).$$

Example 3. Let a surface be immersed in R^3 , and let the curve of double points and the surface near those coincidence points look like a right circular cone and a plane:



The volume of the double points, for small R , is $4LR^2/\sin\theta$, where L is the length of the circle of intersection and θ the angle of intersection.

Example 4. As in example 3, except consider a sphere of radius ρ rather than a right circular cone. The volume is the same, $4LR^2/\sin\theta$.



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