74-29,696

BATAKIS, Nikos A., 1943-
NEW SOLUTIONS IN RELATIVISTIC COSMOLOGY.

The City University of New York, Ph.D., 1974
Physics, general

**Xerox University Microfilms**, Ann Arbor, Michigan 48106

# NEW SOLUTIONS IN RELATIVISTIC COSMOLOGY

by

NIKOS BATAKIS

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1974

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Jeffrey M. Cohen

date

Chairman of Examining Committee

5/15/74

date

Marvin H. Mittleman

Executive Officer

Prof. D. Finkelstein

Prof. M. Arons

Prof. H. Soodak

Prof. A. Harvey

Supervisory Committee

The City University of New York

Abstract

NEW SOLUTIONS IN RELATIVISTIC COSMOLOGY

by

Nikos Batakis

Adviser: Professor Jeffrey M. Cohen

A self-contained introduction to Differential Topology is given to facilitate an exposition of (i) the fundamentals of Relativistic Cosmology, (ii) a discussion of symmetries and (iii) the Bianchi classification of spatially homogeneous spacetimes. Three non-stationary cosmological models are presented (Types II, X, V), in which the non-zero vorticity of the cosmological fluid (incoherent matter) is of special interest. Exact solutions are given for simplified cases. A theorem (proposed by Gödel in 1950) on an observable effect of rotating universes is proved and a general discussion is given, based on existing observational data in Relativistic Cosmology.

## Acknowledgement

The present state of knowledge in Physics is, of course, the collective product of many contributions (of varying degrees and forms) and, in that respect, one is indebted to a nearly countless number of individuals.

Here I would like to explicitly express my deep appreciation to my teachers at City University. Professor Jeffrey M. Cohen, to whom I am grateful for his guidance as well as his kindness and understanding, has also given me the benefit of constant consultation and enlightening discussions throughout the development of the present work. I would also like to thank my friends Chris Ftaclas for many valuable discussions, and Kathy Brown for the typing.

TO ERA

TABLE OF CONTENTS

## LIST OF TABLES



## LIST OF FIGURES

## NOTATIONAL AND OTHER CONVENTIONS

In an n-dimensional manifold Greek indices, take the values $1,\ldots,n$, with Einstein's summation convention in effect unless the contrary is specifically stated. In spacetime, the signature of the metric is $(-1,1,1,1)$ with Greek indices running as 0,1,2,3 and Latin as 1,2,3. Bold-faced letters denote tensor quantities, with $\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots$ usually vectors and $\boldsymbol{\sigma}, \boldsymbol{\omega}, \cdots$ 1-forms. The sign of the Riemann and Einstein tensors is fixed by (2.36) and (2.42) respectively. Comma (,) and semicolon (;) denote ordinary and covariant derivatives respectively. In the models of Chapter 5, an effort was made to denote constants with capital letters, while for functions small-case letters were used. The adoption of geometrized units was kept, even in the discussion of data (with minor exceptions), because it was felt that for quantities like, for example, $t = 10^{29}\,cm$, $\rho = 10^{-62}\,cm^{-2}$ the use of conventional units, converting them to $t = 10^{11}$ years, $\rho = 10^{-34}\,g/cm^3$ offered no help for an intuitive digestion of the magnitudes involved. However, for a conversion to conventional units, Table 0.1 might be useful.

The following examples should clarify the enumeration rules for equations,etc.

| | |
|---|---|
| 3.B: | Second section in Chapter 3 |
| (3.76): | Equation(s) 76 in Chapter 3 |
| (37): | Equation(s) 37 in the same chapter in which the reference is made |
| Fig. 2.1: | First figure in Chapter 2 (similarly for tables) |
| NOTE 2: | Footnotes appear collectively at the end (page 114) |

References are specified in the text by the name(s) of the author(s) or editor, followed by the year of the publication.

TABLE 0.1 Conversion to Geometrized Units

| QUANTITY | CONVENTIONAL | GEOMETRIZED |
|---|---|---|
| Length | 1 cm | 1 cm |
| | 1 ly | $.946\times10^{18}$ cm |
| | 1 pc | $3.086\times10^{18}$ cm |
| Time | 1 sec | $2.998\times10^{10}$ cm |
| | 1 year | $0.946\times10^{18}$ cm |
| Mass | 1 g | $7.425\times10^{-29}$ cm |
| Electric Charge | 1 esu=1 $cm^{3/2}\,g^{-1/2}\,sec^{-1}$ | $2.875\times10^{-35}$ cm |
| Temperature | 1 °K | $1.141\times10^{-65}$ cm |
| Energy | 1 erg | $8.262\times10^{-50}$ cm |
| | 1 ev | $1.324\times10^{-61}$ cm |
| Speed of Light | $c=2.998\times10^{10}$ $cm\,sec^{-1}$ | 1 |
| Gravitational Constant | $G=6.673\times10^{-8}$ $cm^{3}\,g^{-1}\,sec^{-2}$ | 1 |
| Plank's Constant | $\hbar=1.055\times10^{-27}$ $cm^{2}\,g\,sec^{-1}$ | $2.612\times10^{-66}$ $cm^{2}$ |
| Boltzmann Constant | $K=1.381\times10^{-16}$ $cm^{2}\,g\,sec^{-2}\,{}^{\circ}K^{-1}$ | 1 |
| Charge Quantum | $e=4.803\times10$ esu | $1.381\times10^{-34}$ cm |
| Proton Rest Mass | $M_p=1.673\times10^{-24}$ g | $1.242\times10^{-52}$ cm |

# I. INTRODUCTION

The Theory of General Relativity, that is Einstein's 1915 Classical Theory (EINSTEIN (1915)) with its subsequent (non-fundamental) mathematical refinement, is the "best" theory available for interpretation of existing observations in Cosmology, in the sense that: (a) it is not contradicted by such observations, (b) conforms with the principle of simplicity. The legibility of criterion (b) has frequently been defended on the grounds of both philosophical preferences and practical convenience.

Before proceeding for a rather rigorous foundation of the so called "Relativistic Cosmologies", it will be useful to give an order-of-magnitude listing of some fundamental quantities together with a few global statements valid in a gross model of our universe, based on the Theory of Relativity, the generally accepted physical theories on matter and radiation and, of course, existing observational data in Cosmology and Astrophysics (see, for example, PEEBLES (1971), ZEL'DOVICH, and NOVIKOV (1964)).

Galaxies, which are generally smaller than $10^{24}$ cm in linear measure, appear to form gravitationally bound clusters with linear dimensions of the order of

$10^{25}$ cm. Measurements on the distribution of those clusters, although not yet quite conclusive, indicate that on a scale of $10^{26}$ cm, and larger, the universe is homogeneous and isotropic. The energy content of the universe has presently a very low average density of the kind which would normally be neglected in almost any other situation encountered in physics. Thus, the rest mass energy density of luminous matter is estimated at

$$\rho_{\text{luminus matter}} \gtrsim \rho_0 = 10^{-59}\ cm^{-2} \qquad (1.1)$$

Of course non-luminous matter, such as intergalactic gas, black holes, etc., is expected to be present, possibly averaging in density up to as much as $10^{3}\rho_0$. Kinetic energy and pressure (other than radiation pressure) contributions presently appear to be of the order of a mere $10^{-6}\rho_0$, hence one is justified to neglect peculiar motions in the rather gross cosmological models. Radiation fields such as the presently observed 2.7 °K isotropic thermal or the anticipated neutrino and gravitational, are estimated to contribute with a density of the order of $10^{-2}\rho_0$. Observed magnetic fields almost certainly do not exceed $10^{-4}\rho_0$. Hubble's "constant" $H$ giving the fractional rate of increase of cosmological distances in our expanding universe is estimated to be of the order

$$H \approx \left[ 1.7 \times 10^{28}\ cm \right]^{-1} \qquad (1.2)$$

Possibly existing inhomogeneities, shear and vorticity (rotation of the universe) must presently be very small according to the above account, but this in no way diminishes their potential importance, especially during the early phases of the cosmic evolution.

Further discussion along these lines will at this point be deferred until the precise definitions of all the quantities and characterizations used above have been given. However, another issue will be briefly mentioned: the question of whether of not the universe is closed. Einstein, who was philosophically inclined towards Mach's view on this subject (MACH (1912), EINSTEIN (1950)), suggested that a global condition excluding open models must complement the (local in character) field equations. Observation is not conclusive on this.

In the following, Chapter 2 presents the fundamental mathematical framework in the form used in the present work. Symmetries, although not fundamental, are technically almost indispensable elements in physical theories, and, therefore, a rigorous definition of spacetime symmetries is given. This also allows a collective presentation of all the various types of derivations used in General Relativity together with the transport laws which they generate. Riemannian Geometry is subsequently introduced with the definition of a metric. A brief review of the techniques of the calculus of generalized exterior forms is also given. The Theory of General Relativity is presented in Chapter 3 in the form of four axioms together with some formulas for later use. A discussion of symmetries in General Relativity follows, with particular attention on the (later exclusively used) spatially homogeneous spacetimes, and a systematic presentation of the various types of the Bianchi classification. Chapter 4 outlines the technique used to obtain an exact solution for a Bianchi Type IX model and also for the three models that follow in Chapter 5: A Type II containing dust and E-M field, which represents a non-isotropic rotating universe with the solution expressed in terms of a linear second order differential equation (equ.(5.29)). And two more models, each representing a dust-filled universe, anisotropic and with rotation different than zero: for the Type IX, the solution is expressed in terms of the second order equation (5.54) while for the Type V, three special cases are studied, model $V_0$ (without rotation), model $V_\epsilon$ (small rotation), and model $V_1$, a (seemingly) peculiar, rotating, profoundly anisotropic and non-stationary universe, for which an exact solution is given. Finally, the proof is provided for a theorem stated by Gödel in 1950, on an observable effect of rotating universes.

# 2. ON DIFFERENTIAL GEOMETRY

## 2.A MANIFOLDS

An n-dimensional differentiable manifold $M$ of class $C^\ell$ without boundary is a structure $\{M, \{U\}, F\}$, where $M$ is a set with elements called points; $\{U\}$ is a collection of open sets $U$ in $M$, each called a neighbourhood of each of its points, such that $M$ is a connected Hausdorff space (NOTE 1); and $F$ is a family of real-valued functions $f$ defined on $M$ such that

(a) If $g$ is a real-valued function on $M$ such that: to every point $p \in M$ there corresponds a neighbourhood $U$ of $p$ and a function $f \in F$ such that $f(p) = g(p)$ for all $p \in U$, then $g \in F$.

(b) If $f_1, \cdots, f_m \in F$ and $g$ is a function $R^m \to R$ of class $C^\ell$ then $g(f_1, \cdots, f_m) \in F$.

(c) To every $p \in M$ there corresponds a $U$ of $p$ and n functions $x^\alpha \in F$ $(\alpha = 1, \dots, n)$ such that (i) the map $\phi : U \ni p \to (x^1(p), \cdots, x^n(p)) \in R^n$ is a homeomorphism of $U$ on a subset of $R^n$ (NOTE 1) (ii) to every $f \in F$ there corresponds a real-valued function (denoted also by) $f$ of class $C^\ell$ defined on $R^n$ such that $f(p) = f(x^1(p), \dots, x^n(p))$.

For the definition of manifolds with boundary, see NOTE 2.

A chart is a pair $(U,\phi)$ as defined in (c) above. A $C^{\ell}$ atlas is a collection of charts $\{(U_k,\phi_k)\}$ such that: $M = \bigcup_k U_k$ and also, if for the charts $(U,\phi_x)$, $(V,\phi_y)$ it is: $U \cap V \neq \emptyset$, the map $\phi_x \circ \phi_y^{-1}$ is a $C^{\ell}$ map of the following sets in $R^n$, $\phi_y(U \cap V) \to \phi_x(U \cap V)$. The set of functions $\{x^{\alpha}(p)\}$ are the local coordinates of $p$. If Part (c) of the above definition can be satisfied for $U = M$, the coordinates are called global. The map $\phi_x \circ \phi_y^{-1}$ is a transformation of (local) coordinates. Fig. 1 gives a pictorial view for some of the above definitions. Starting from one atlas, one can define another atlas as being compatible to the first if their union is an atlas. The collection of all atlases compatible with the initial atlas is the complete atlas of the manifold.

A manifold is orientable if there is an atlas $\{(U_k,\phi_k)\}$ in the complete atlas such that the Jacobian of the transformation $\phi_k \circ \phi_{\ell}^{-1}$ is positive for all non-empty $U_k \cap U_{\ell}$. An atlas is locally finite if every $p \in M$ has an open neighborhood which intersects only a finite number of the sets $U_k$.

$M$ is paracompact if for every atlas $\{(U_k,\phi_k)\}$ there exists a locally finite atlas $\{(V_{\ell},\Psi_{\ell})\}$ with each $V_{\ell}$ contained in some $U_k$.

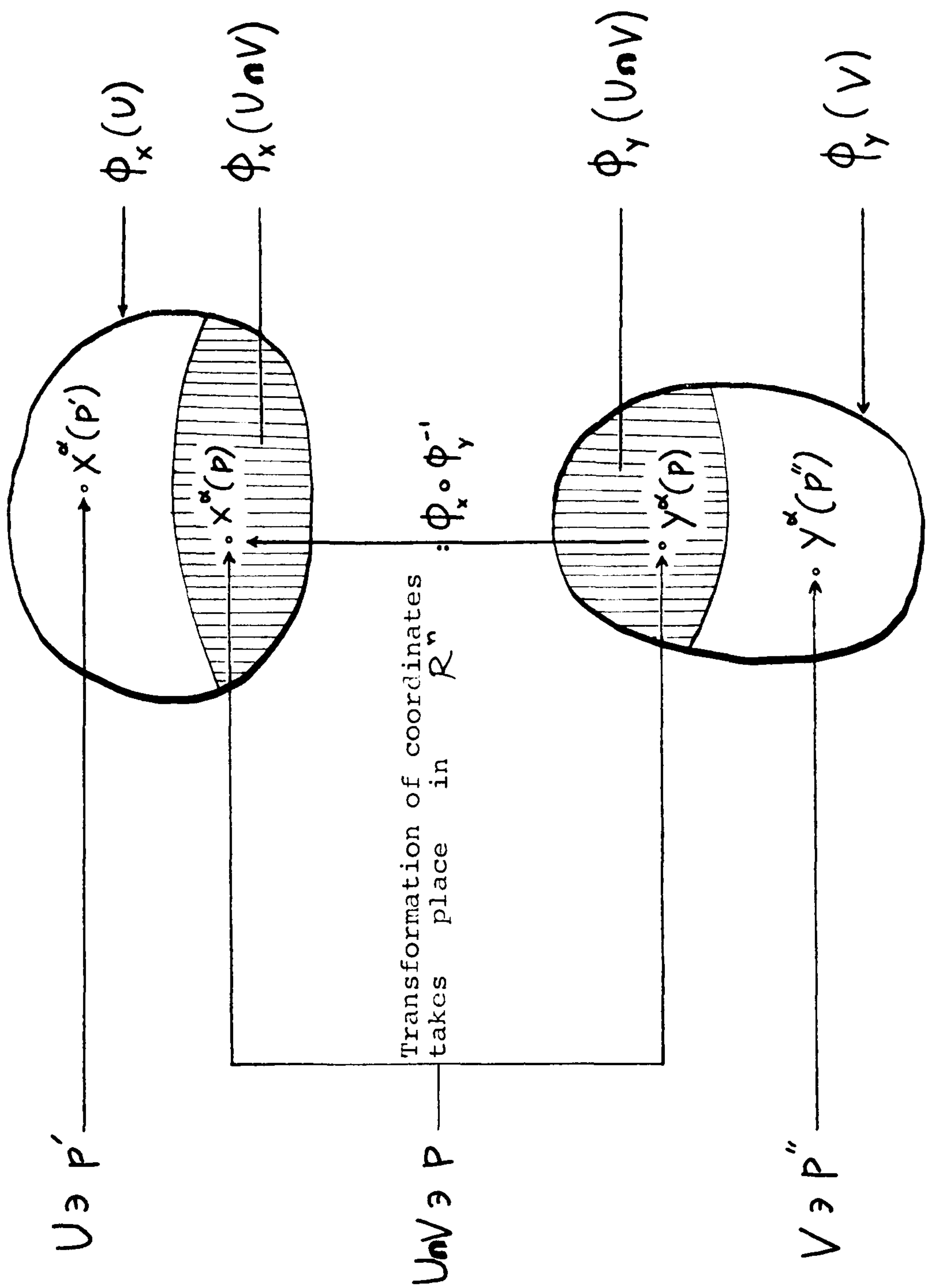



FIGURE 2.1: Coordinatization of a manifold and coordinate transformations.

The following holds on paracompact $C^\ell$ manifolds: for every locally finite atlas $\{(U_\kappa, \phi_\kappa)\}$ there exists a set of $C^\ell$ functions $g_\kappa$ on $M$, with $0 \le g_\kappa \le 1$ and with the support of each $g_\kappa$ contained in $U_\kappa$ such that

$$\sum_\kappa g_\kappa(p) = 1 \qquad (2.1)$$

at every $p \in M$. For this (Partition of Unity Theorem) and other topics developed in this chapter, see, for example, HELGASON (1962), KOBAYASHI and NOMIZU (1963).

If $f \in F$ is not constant on any neighbourhood of any $p \in U$, the set of points in the (open) subset $U$ defined by $f(p) = c$ will form an $n-1$ dimensional manifold (hypersurface) for each $c$ in $f(U)$, and $f$ will be said to be a hypersurface forming function in $U$. Otherwise $f(p) = c$ will define a closed $n$-dimensional subset of $U$ for values $c$ of $f(U)$. By intersecting $r$ hypersurfaces, one obtains, in general, an $n-r$ dimensional submanifold.

A $C^0$ curve in $M$ is a $C^0$ map of a real interval into $M$ :

$$\lambda(t) : \quad R \ni t \longrightarrow p(t) \in M \qquad (2.2)$$

One can define the tangent vector to the curve $\lambda$ at the point $p_0 = p(t_0)$ as the operator

$$\left(\frac{dp}{dt}\right)_\lambda \Big|_{t_0} = \left(\frac{\partial}{\partial t}\right)_\lambda \Big|_{t_0}$$

which maps every function $f \in F$ at $P_0$ to the real number $\left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t_0}$ . Explicitly:

$$\left(\frac{\partial}{\partial t}\right)_\lambda \Big|_{t_0} : F \ni f\Big|_{P_0} \rightarrow \left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t_0} = \lim_{t=t_0} \frac{1}{t-t_0}\Big[f(\lambda(t)) - f(\lambda(t_0))\Big]$$

It can be shown that the tangent vectors at $p \in M$ form an n-dimensional vector space over the reals, the so called tangent vector space $T_p$ at $p \in M$ . For the definition of a vector space and other algebraic structures that will be used later, see NOTE 3.

With $x^\alpha$ local coordinates in a neighbourhood of $p$ one has

$$\left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t_0} = \frac{d x^\alpha(\lambda(t))}{dt}\Big|_{t=t_0} \cdot (\partial_\alpha f)\Big|_{\lambda(t_0)} ,$$

which suggests that the generic vector $\mathbf{u}$ at $p$ can be expressed as $\mathbf{u} = u^\alpha \boldsymbol{\partial}_\alpha \big|_p$ , that is, a linear superposition of the linearly independent tangent vectors $\boldsymbol{\partial}_\alpha$ (see Note 4 for a proof).

The dual of $T_p$, denoted by $T_p^*$, is an n-dimensional vector space over the reals, whose elements, called 1-forms, are the real valued linear functions on $T_p$ :

$$T_p^* \ni \boldsymbol{\omega} : \quad T_p \ni \mathbf{u} \rightarrow \langle \boldsymbol{\omega}, \mathbf{u} \rangle \in R$$

such that

$$\langle \boldsymbol{\omega}, a\mathbf{u} + b\mathbf{v} \rangle = a \langle \boldsymbol{\omega}, \mathbf{u} \rangle + b \langle \boldsymbol{\omega}, \mathbf{v} \rangle$$

for $a, b \in R$, $\boldsymbol{\omega} \in T_p^*$, $\mathbf{u}, \mathbf{v} \in T_p$. See NOTE 3. Given a basis $\{\mathbf{e}_\alpha\}$ in $T_p$ its dual basis $\{\boldsymbol{\omega}^\alpha\}$ in $T_p^*$ is defined by

$$\langle \boldsymbol{\omega}^\alpha, \mathbf{e}_\beta \rangle = \delta^\alpha{}_\beta \tag{2.3}$$

The numbers $\sigma_\alpha$, $v^\alpha$ in the expansions

$$\boldsymbol{\sigma} = \sigma_\alpha \boldsymbol{\omega}^\alpha , \quad \mathbf{v} = v^\alpha \mathbf{e}_\alpha \tag{2.4}$$

are the components of the 1-form $\boldsymbol{\sigma}$ and the tangent vector $\mathbf{v}$ expressed in the (dual) bases $\{\boldsymbol{\omega}^\alpha\}$, $\{\mathbf{e}_\alpha\}$. For every function $f \in F$ and vector $\mathbf{v} \in T_p$, $\mathbf{v}f$ is a real number. Thus, any $f \in F$ can be used to create a 1-form, denoted by $\mathbf{d}f$ according to the definition

$$\langle \mathbf{d}f, \boldsymbol{v} \rangle = \boldsymbol{v} f \qquad (2.5)$$

One can now see that the basis dual to $\{\boldsymbol{\partial}_\alpha\}$ is $\{\mathbf{d}x^\alpha\}$ because,

$$\langle \mathbf{d}x^\alpha, \boldsymbol{\partial}_\beta \rangle = \delta^\alpha{}_\beta \qquad (2.6)$$

Such a basis is called a coordinate basis.

A geometric perspective of 1-forms, analogous to that for the tangent vectors (viewed as the "tangents" to the curve (2.2)) can now be gained if one notices that $f$ = const. for any $f \epsilon F$ with $\mathbf{d}f \neq 0$ defines an n-1 dimensional submanifold of $M$ (a "hypersurface") and

$\langle \mathbf{d}f, \boldsymbol{v} \rangle$ can be interpreted as a measure of "how many" of these hypersurfaces are "pierced through" by $\boldsymbol{v}$. In particular, if $\langle \mathbf{d}f, \boldsymbol{v} \rangle = 0$ the vector $\boldsymbol{v}$ at $p_0$ is "contained" or it is tangent to the hypersurface $f = f(p_0)$ through $p_0$ at $p_0$. The converse of (3) does not hold: for any 1-form $\boldsymbol{\sigma}$, one cannot always find some $f$ such that $\mathbf{d}f = \boldsymbol{\sigma}$. Geometrically this corresponds to the fact that sets of planes, one set at each point $p \epsilon M$, do not necessarily mesh to form integral surfaces $f$ = const.

From the cartesian product

$$\Pi_r^s = \underset{1}{T_p^*} \times \ldots \times \underset{r}{T_p^*} \times \underset{1}{T_p} \times \ldots \times \underset{s}{T_p}$$

one can now define as tensors of type $\binom{r}{s}$ at $p$ the linear real-valued functions on $\Pi_r^s$:

$$\mathbf{G} : \Pi_r^s \ni (\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^r, \mathbf{v}_1, \ldots \mathbf{v}_s) \rightarrow \mathbf{G}(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^r, \mathbf{v}_1, \ldots, \mathbf{v}_s) \in R$$

with

$$\mathbf{G}(\ldots, a\mathbf{A} + b\mathbf{B}, \ldots) = a\mathbf{G}(\ldots, \mathbf{A}, \ldots) + b\mathbf{G}(\ldots, \mathbf{B}, \ldots)$$

$k^{th}$ position

where $a, b \in R$ and $\mathbf{A}, \mathbf{B}$ 1-forms or tangent vectors. The resulting n -dimensional vector space over the reals is denoted by

$$T_s^r(p) = \underset{1}{T_p} \otimes \cdots \otimes \underset{r}{T_p} \otimes \underset{1}{T_p^*} \otimes \cdots \otimes \underset{s}{T_p^*} \tag{2.7}$$

In particular, $T_0^1(p) = T_p$ , $T_1^0(p) = T_p^*$ and vectors and 1-forms are type $\binom{1}{0}$ and $\binom{0}{1}$ tensors, respectively (see NOTE 6).

For given dual bases $\{\boldsymbol{\omega}^\alpha\}$ , $\{\mathbf{e}_\nu\}$ the real numbers

$$\mathbf{G}(\boldsymbol{\omega}^{\alpha_1}, \ldots, \boldsymbol{\omega}^{\alpha_r}, \mathbf{e}_{\beta_1}, \ldots, \mathbf{e}_{\beta_s}) \equiv G^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \tag{2.8}$$

are the components of $\mathbf{G} \in T^r_s$ in the given basis; using the notation (2.7), one can then write $\mathbf{G}$ as

$$T^r_s \ni \mathbf{G} = G^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \, \mathbf{e}_{\alpha_1} \otimes \cdots \otimes \mathbf{e}_{\alpha_r} \otimes \boldsymbol{\omega}^{\beta_1} \otimes \cdots \otimes \boldsymbol{\omega}^{\beta_s} \tag{2.9}$$

New tensors from given tensor(s) can be defined in a variety of ways. Expressed in component language (although coordinate independent), these definitions are:

(1) Addition and multiplication by real numbers

(2) Tensor product of $\mathbf{G} \in T^r_s$, $\mathbf{G}' \in T^{r'}_{s'}$

$$(\mathbf{G} \otimes \mathbf{G}')^{\alpha_1 \cdots \alpha_{r+r'}}_{\beta_1 \cdots \beta_{s+s'}} = G^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \cdot G'^{\alpha_{r+1} \cdots \alpha_{r+r'}}_{\beta_{s+1} \cdots \beta_{s+s'}} \tag{2.10}$$

(3) Contraction (summation over) $C^A_B$ of the $A^{th}$ contravariant with $B^{th}$ convariant index. Examples:

$$(C^1_1 \mathbf{G})^{\beta\gamma \cdots}_{\mu\nu \cdots} = \delta^\lambda_\alpha \, G^{\alpha\beta\gamma \cdots}_{\lambda\mu\nu \cdots} \tag{2.11}$$

and $$C^1_1 (\boldsymbol{\sigma} \otimes \mathbf{v}) = \langle \boldsymbol{\sigma}, \mathbf{v} \rangle = \sigma_\alpha v^\alpha$$

(4) Transpose: trivially obtained by changing the position of two indices. Example:

$$\left(G(2 \rightleftarrows 3)\right)_{\alpha\beta\gamma\delta\ldots} = G_{\alpha\gamma\beta\delta\ldots}$$

(5) Symmetrization:

$$\left(G^{symm}\right)_{\alpha_1\cdots\alpha_r} = \frac{1}{r!}\sum_{(\alpha_{p_1}\cdots\alpha_{p_r})} G_{\alpha_1\cdots\alpha_r} \equiv G_{(\alpha_1\cdots\alpha_r)} \quad (2.12)$$

(6) Antisymmetrization:

$$\left(G^{skew}\right)_{\alpha_1\cdots\alpha_r} = \frac{1}{r!}\sum_{(\alpha_{p_1}\cdots\alpha_{p_r})} \begin{bmatrix}\alpha_1\cdots\alpha_r\\ \alpha_{p_1}\cdots\alpha_{p_r}\end{bmatrix} G_{\alpha_1\cdots\alpha_r} \equiv G_{[\alpha_1\cdots\alpha_r]} \quad (2.13)$$

where $\begin{bmatrix}\alpha_1\cdots\alpha_r\\ \alpha_{p_1}\cdots\alpha_{p_r}\end{bmatrix}$ is 0, +1, or -1 according to the permutation relationship of the two sets of indices.

A $C^\ell$ tensor field $G$ of type $\binom{r}{s}$ on a set $V \subset M$ is an assignment of an element of $T^r_s(p)$ to each point $p \in V$ such that the components of $G$ with respect to any coordinate basis defined on any open subset of $V$ are $C^\ell$ functions.-

$T^r_s(M)$ is a module over the ring $F$ (see NOTE 3), or an infinite dimensional vector space over the reals. The direct sum

$$\mathcal{T}(M) = \sum_{all\ r,s} T^r_s(M) \quad (2.14)$$

with the product $\otimes$ defined in (10) forms an algebra over the reals, the so called tensor algebra, of $M$.

In the following, the word "field" is occasionally deleted for brevity.

## 2.C MAPPINGS

Of special importance in studying the topology of a manifold are its well-behaved point transformations, that is, the one-one $C^\ell$ differentiable maps of $M$ onto itself (see, for example, ELLIS (1965)). Each such map is completely determined by a map $\Phi$ on $F$ $(\Phi : F \ni f \rightarrow \Phi f \in F)$ that satisfies (2.5):

$$\left\{\begin{array}{l} \Phi \text{ is non-degenerate} \\ (\text{maps hypersurface forming } f \text{ on hypersurface forming } \\ \Phi f) \\ \Phi \text{ is functional} \\ (\Phi f(x^\alpha(p)) = f(\Phi x^\alpha(p))) \end{array}\right. \qquad (2.15)$$

Every group of maps acting on $F$ defines a group $G$ of transformations $\Phi$ on $M$. $G$ is effective if, whenever $\Phi(p) = p$ for all $p \in M$, $\Phi$ is the identify. Points $p, q$ are equivalent under $G$ if there is a $\Phi \in G$ such that $\Phi(p) = q$. Invariant variety $V$ of $M$ is a subspace of $M$ such that all points equivalent to each $p \in V$ lie in $V$. Minimum invariant variety at $p$ is the subspace of $M$ of least dimensions, containing points equivalent to $p$, that is an invariant variety. $G$ is transitive on a subspace $U$ of $M$ if each two points of $U$ are equivalent to each other under

$G$ (thus $G$ is transitive on each of the minimum invariant varieties it defines), and it is simply transitive if dim. of $U$ = order of the group--otherwise it can be multiple transitive or intransitive. The group of stability of $p \in M$ is the subgroup of $G$ leaving $p$ invariant.

To define maps on manifolds in greater generality, one may proceed as follows.

The Cartesian product $M \times M'$ of manifolds $M, M'$ is a manifold with structure naturally defined by the manifold structures of $M, M'$. For $p \in M, p' \in M'$, $(x^{\alpha}, x^{\alpha'})$ are the coordinates of the point $(p, p') \in M \times M'$ **in the** neighbourhood $U \times U'$ where $U$ is a neighbourhood of $p$ with local coordinates $x^{\alpha}$ as similarly for $p'$.

A map $\phi: M \to M'$ is a $C^{\rho}$ map if, for any local coordinate systems in $M$ and $M'$, the coordinates of the image point $\phi(p) \in M'$ are $C^{\rho}$ functions of the coordinates of $p \in M$. More rigorously, $\phi$ is defined as a $C^{\prime}$ differentiable curve in $M \times M'$. Such a $\phi$ gives rise to the maps

$$\phi^{*} : T^{0}_{s}(\phi(p)) \ni G \to \phi^{*} G \in T^{0}_{s}(p) \quad (2.16)$$

$$\phi_* : T^r_0(p) \ni \mathbf{G} \rightarrow \phi_* \mathbf{G} \in T^r_0(\phi(p)) \qquad (2.17)$$

Using the above equations and (3), one can now show

$$\phi^*(\mathbf{d}f) = \mathbf{d}(\phi^* f) \qquad (2.18)$$

The dimensionality of $\phi_*(T_p)$ is called the rank $r$ of $\phi$ at $p \in M$. $\phi$ is injective at $p$ if $r=n$ (and, therefore $n \leq n'$) and surjective at $p$ if $r=n'$ ($n' \leq n$). $\phi$ can be further characterized as: proper (if: $\phi^{-1}(K)$ is compact (see NOTE 7) for any compact $K \subset M'$), an immersion (: $\phi$ injective at every $p \in M$) an imbedding (: $\phi$ proper one-one immersion). $\phi(M)$ is an n-dimensional immersed (imbedded) submanifold in $M'$ if $\phi$ is an immersion (imbedding). A $C^\ell$ diffeomorphism is a one-one $C^\ell$ map $\phi$ such that $\phi^{-1}$ is also $C^\ell$. $\phi$ is a local diffeomorphism near $p$ if $\phi_*$ is an isomorphism $T_p \rightarrow T_{\phi(p)}$; in that case $\phi_*$ is both injective and surjective at $p$ and $(\phi^{-1})^*$ maps $T^*_p$ to $T^*_{\phi(p)}$. One can therefore define a map $\Phi$

$$\Phi : T^r_s(p) \ni \mathbf{G} \rightarrow \Phi \mathbf{G} \in T^r_s(\phi(p)) \qquad (2.19)$$

such that

$$\mathbf{G}(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^s, \mathbf{v}_1, \dots, \mathbf{v}_r)\Big|_p = (\Phi \mathbf{G})\left((\phi^{-1})^* \boldsymbol{\sigma}^1, \dots, \phi_* \mathbf{v}_r\right)\Big|_{\phi(p)}$$

for any values of r,s. This map preserves all algebraic relations, that is, it is an isomorphism of the tensor algebra $\mathcal{T}(M)$ ; it also preserves the coordinate independent operations such as symmetrization, contraction, etc. When $M' = M$, $\Phi$ is an automorphism. Such automorphisms are special cases ("dragging along"--see below) of the so called ACT-maps $\Phi$ of the tensor algebra. They are defined by the following A, C, T propositions:

$\underline{\underline{A}}$ $\Phi$ is an Automorphism of the tensor algebra (see NOTE 3)

$\underline{\underline{C}}$ $\Phi$ commutes with contractions

$$\left( \Phi \left( C^A_B \mathbf{G} \right) = C^A_B \left( \Phi \mathbf{G} \right) \right)$$

$\underline{\underline{T}}$ $\Phi$ preserves Tensor Type

$$\left( T^r_s(M) \ni \mathbf{G} \to \Phi \mathbf{G} \in T^r_s(M) \right)$$

It can be seen from the following equations that $\Phi$ can be determined completely if its action on $T^0_0$ (that is, on $F$ , which is equivalent to a point tranformation on $M$) and $T_0^1$ is known:

$$\Phi\langle\sigma, v\rangle = \langle\Phi\sigma, \Phi v\rangle \tag{2.20}$$

$$\Phi C = (\Phi C^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s})(\Phi e_{\alpha_1}) \otimes \cdots \otimes (\Phi \omega^{\beta_s}) \tag{2.21}$$

A derivation of a group of ACT-maps is defined, if $C \in T^r_s$ and $I$ the group identify, by

$$DC = \lim_{\Phi \to I} (C - \Phi C) \tag{2.22}$$

Maps like $D$, called ACT-derivations of $\mathcal{T}(M)$ are characterized by the following A-derivation, C, T propositions:

A-derivation  $D$ is a linear map on $\mathcal{T}(M)$, but not an automorphism because it is a derivation.

$$D(A \otimes B) = (DA) \otimes B + A \otimes (DB) \tag{2.23}$$

C  Commutes with Contractions

T  Preserves Tensor Type

One can now see that if $D, D_1, \cdots$ are ACT-derivations and $a, a, \cdots \in R$, then $aD$ and $a_1D_1 + a_2D_2$ are also ACT-derivations. Defining the "product" $[D_1, D_2] = D_1 D_2 - D_2 D_1$ one then sees that all ACT-derivations form a Lie algebra $\mathbf{A}$ over the reals: the "product" is antisymmetric and (identically) satisfies Jacobi's identity, (2.24):

$$[D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0$$

Using standard Lie group techniques, one can now show that an r-dimensional sub-algebra $\mathfrak{g}$ of $\mathbf{A}$ generates an r-parameter Lie group $G_r$ of ACT-maps. Thus,

$$G_r = \{ e^{-D} , D \in \mathfrak{g} \} \tag{2.25}$$

where,

$$e^{-D} = \sum_{n=0}^{\infty} \frac{1}{n!} (-D)^n \tag{2.26}$$

is the ACT-map generated by the element $D$ of the Lie (sub)algebra $\mathfrak{g}$ of ACT-derivations. The algebra $\mathfrak{g}$ is known if a basis $D_i$ $i = 1, \dots, r$ is given plus the structure constants $C_{ij}{}^k$ defining all the commutators ("products") in $\mathfrak{g}$ :

$$[D_i , D_j] = C_{ij}{}^k D_k \tag{2.27}$$

As was the case with ACT-maps, an ACT-derivation is known if its effect on $T^0_0 (\equiv F)$ and $T^1_0 (\equiv T(M))$ is known.

When acting on functions, $D$ can be represented by a vector field $\boldsymbol{v}$. Its effect on vectors can be described by a set of functions $D_\alpha{}^\beta$ for a given basis $\{\mathbf{e}_\alpha\}$ :

$$D(\mathbf{e}_\alpha) = D_\alpha{}^\beta \, \mathbf{e}_\beta \qquad (2.28)$$

This, combined with

$$Df = \boldsymbol{v} f \qquad (2.29)$$

can determine the action of $D$ on any tensor. For example,

$$D(\boldsymbol{\omega}^\alpha) = -D_\beta{}^\alpha \, \boldsymbol{\omega}^\beta \qquad (2.30)$$

A subalgebra $\mathfrak{g}$ of $A$ is an invariance subalgebra on the set of tensors $\{\mathbf{T}\}$ if $D\mathbf{T}=0$ for every $D \in \mathfrak{g}$ and every $\mathbf{T} \in \{\mathbf{T}\}$. The corresponding Lie group of ACT-maps is then an invariance group of $\{\mathbf{T}\}$ (NOTE .).

Table 2.1 summarizes three particularly important examples of ACT-derivations and the ACT maps they

TABLE 2.1 ACT -Maps and -Derivations

| ACT -Derivation $D$ | | | ACT -Map $\Phi$ | | |
|---|---|---|---|---|---|
| Name | Action of $D$ on: Functions | Vectors | Action of $\Phi$ on: Functions | Vectors | Name |
| Tensor Derivation $\Lambda$ | $\Lambda f = 0$ | $\Lambda \mathbf{e}_\mu = c\,\mathbf{e}$ <br> $[\Lambda \boldsymbol{\omega}^\mu = -c\,\boldsymbol{\omega}]$ | $\Phi f = f$ | $\Phi \mathbf{e}_\mu = e^{-c}\mathbf{e}_\mu$ | Conformal Mapping |
| | $\Lambda f = 0$ | $\Lambda \mathbf{e}_\mu = L^\lambda{}_\mu \mathbf{e}$ <br> $[\Lambda \boldsymbol{\omega}^\mu = -L^\mu{}_\lambda \boldsymbol{\omega}]$ | $\Phi f = f$ | $\Phi \mathbf{e}_\mu = (e^{-\Lambda})^\alpha{}_\mu \mathbf{e}_\alpha$ | Rotation in $T(M)$ (Spacetime: Lorentz Transformation) |
| Lie Derivation $L_{\mathbf{v}}$ | $L_{\mathbf{v}} f = \mathbf{v} f$ | $L_{\mathbf{v}} \mathbf{u} = [\mathbf{v}, \mathbf{u}]$ <br> $L_\alpha \mathbf{e}_\beta = C_{\alpha\beta}{}^\gamma \mathbf{e}_\gamma$ <br> $[L_\alpha \boldsymbol{\omega}^\beta = -C_{\alpha\gamma}{}^\beta \boldsymbol{\omega}^\gamma]$ | See Transformation (19) Considered for $M' = M$ | | Dragging along |
| Covariant Derivation $\nabla_{\mathbf{v}}$ | $\nabla_{\mathbf{v}} f = \mathbf{v} f$ | $\nabla_\gamma \mathbf{e}_\beta = \Gamma^\alpha{}_{\beta\gamma} \mathbf{e}_\alpha$ <br> $[\nabla_\gamma \boldsymbol{\omega}^\alpha = -\Gamma^\alpha{}_{\beta\gamma} \boldsymbol{\omega}^\beta]$ <br> NOTE: $\nabla_\alpha \equiv \nabla_{\mathbf{e}_\alpha}$ | See Following Remarks | | Parallel Displacement |

generate. Each is defined through its action on functions and vectors (its action on 1-forms is also given for convenience) according to equations (28), (29), (30). A few explanatory remarks follow.

In the definition of tensor derivations, the (real) number $c$ is the so called conformal factor.

For dragging along, it is $\Phi[\mathbf{u},\mathbf{v}]=[\Phi\mathbf{u},\Phi\mathbf{v}]$ where $\mathbf{u},\mathbf{v}$ are the vector fields corresponding to any two members of the algebra of Lie derivations through the representation: $D\rightarrow\mathbf{u}$ if and only if $Df=\mathbf{u}f,\ f\in F$ . In other words, any dragging along $\Phi$ , gives an automorphism of the Lie algebra of vector fields (defined with the introduction of the same "product" as for [illegible]-derivations, that is, the commutator).

For the case of parallel transfer, the affine connection (or affinity) $\nabla$ obeys the rule

$$\nabla_{f\mathbf{u}+g\mathbf{v}} = f\nabla_{\mathbf{u}} + g\nabla_{\mathbf{v}} \tag{2.31}$$

for every $f,g\in F,\ \mathbf{u},\mathbf{v}\in T^1_0$. The so called rotation coefficients $\Gamma^{\lambda}{}_{\mu\nu}$ are connected with the structure constants $C_{\lambda\mu}{}^{\nu}$ if the torsion tensor is known, which is defined as

$$T(u,v) = \nabla_u v - \nabla_v u - [u,v] \tag{2.32}$$

The integral curves of a differentiable vector field $v$ are defined (locally) as the curves to which $v$ is tangent. These curves lie on the surfaces $f = \text{constant}$ where the $f$'s are solutions to $vf = 0$. In general, the vector field $v$ is parallely transferred along the integral curves of $u$ if

$$\nabla_u v \equiv \dot{v} = 0 \tag{2.33}$$

In particular, if (the so called acceleration) $\dot{u}$ vanishes

$$\dot{u} \equiv \nabla_u u = 0 \tag{2.34}$$

the integral curves of $u$ are called geodesics. The above equation also defines, to within a linear transformation, the parametrization $x^\alpha = x^\alpha(\lambda)$ of the geodesic with $\lambda$ called the affine parameter.

One may regard $\nabla$ as a tensor operator, mapping tensors

$$\nabla: \quad \mathbf{G}(\cdots) \longrightarrow \nabla \mathbf{G}(\cdots;\cdot) \tag{2.35}$$

or, in component language,

$$(\nabla \mathbf{G})^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_{s+1}} = G^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s ; \beta_{s+1}} \tag{2.36}$$

Note that the simple comma is used for

$$\mathbf{e}_\alpha f = f_{,\alpha} \tag{2.37}$$

$\nabla^2 = \nabla\nabla$ is also an operator such that

$$(\nabla^2 G)(\boldsymbol{\omega}^{\alpha_1}, \ldots, \mathbf{e}_{\beta_s}; \mathbf{e}_\lambda; \mathbf{e}_\mu) = G^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s ; \lambda ; \mu} \tag{2.38}$$

and one finds, in this notation (NOTE 10):

$$(\nabla^2 G)(;\mathbf{u};\mathbf{v}) = \nabla_{\mathbf{v}} \nabla_{\mathbf{u}} G - \nabla_{\nabla_{\mathbf{v}} \mathbf{u}} G \tag{2.39}$$

Just as the covariant derivation $\nabla_{\mathbf{u}}$ can be defined through the operator $\nabla$ once the direction $\mathbf{u}$ is given, one can define the tensor derivation $\mathbf{R}(\mathbf{u},\mathbf{v})$ from $\nabla^2$ if the two directions $\mathbf{u}, \mathbf{v}$ are given, as

$$\mathbf{R}(\mathbf{u},\mathbf{v}) = [\nabla_{\mathbf{u}}, \nabla_{\mathbf{v}}] - \nabla_{[\mathbf{u},\mathbf{v}]} \tag{2.40}$$

This operator, called curvature, is linear in $u, v$ and therefore representable by a $\binom{1}{3}$ tensor, the Riemann tensor, with components

$$\mathbf{R}(\boldsymbol{\omega}^\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma, \mathbf{e}_\delta) = R^\alpha{}_{\beta\gamma\delta} = \langle \boldsymbol{\omega}^\alpha, \mathbf{R}(\mathbf{e}_\gamma, \mathbf{e}_\delta)\mathbf{e}_\beta \rangle \tag{2.41}$$

$$= \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\mu} C_{\gamma\delta}{}^\mu$$

The Riemann tensor obeys the (first and second set of the) Bianchi identities:

FIRST SET: $$R^\lambda{}_{[\mu\nu\rho]} = 0 \tag{2.42}$$

SECOND SET: $$R^\lambda{}_{\mu[\nu\rho;\sigma]} = 0$$

It is used next, for the definition or description of: geodesic deviation (FIG. 2.2), parallel transport of a vector along a closed curve (FIG. 2.3), Exponential map and normal Riemannian coordinates (FIG. 2.4).

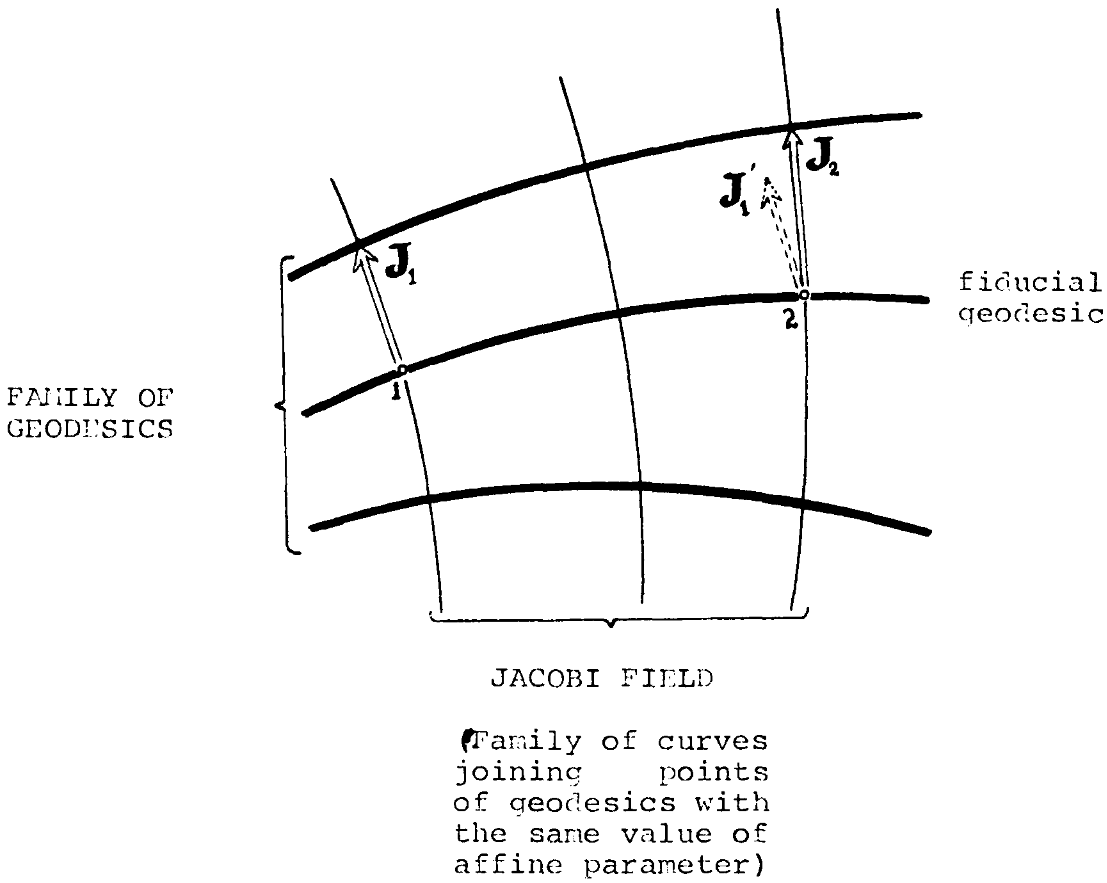



FIGURE 2.2: Geodesic Deviation: $\mathbf{J}_1'$ is the vector $\mathbf{J}_1$ after parallel transfer on the fiducial geodesic from 1 to 2. In general it is different ("deviates") from $\mathbf{J}_1$ already residing at 2, which (like $\mathbf{J}_1$) is defined through the equation of geodesic deviation: $\nabla_u \nabla_u \mathbf{J} + \mathbf{R}(u, \mathbf{J})u = 0$ a special case of the deviation equation (3.53)

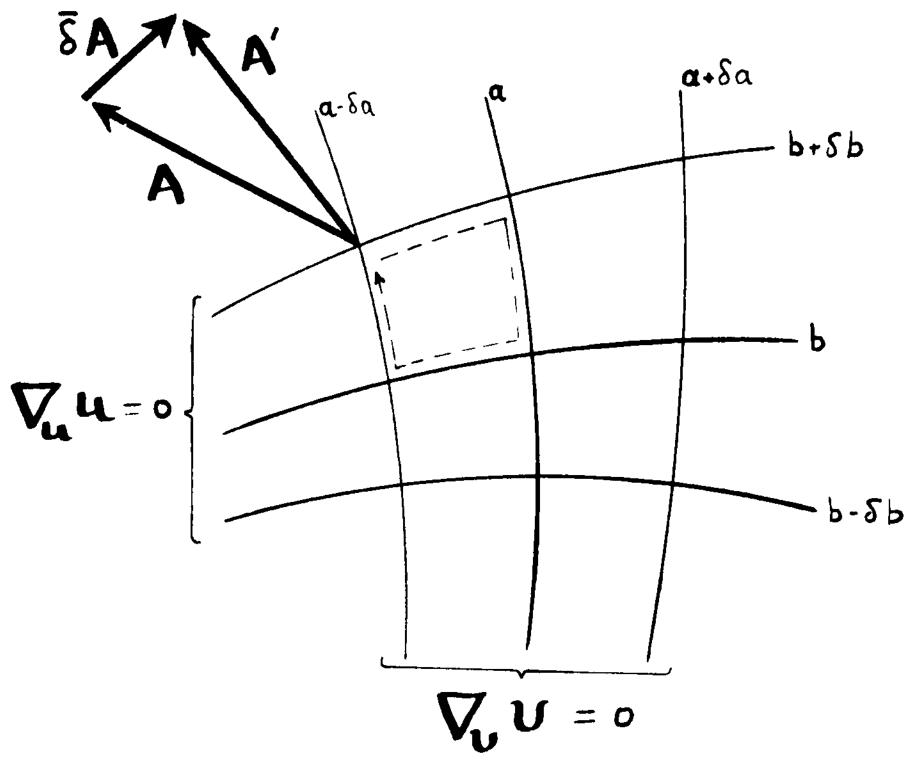



FIGURE 2.3: Parallel Transport Around a Loop:
The numbers $a, a+\delta a, \ldots$ are the values of the affine parameters on the two families of geodesics. The quadrilateral defined by $\mathbf{u}, \mathbf{v}$ is closed because it is assumed that $[\mathbf{u}, \mathbf{v}] = 0$. $\mathbf{A}'$ is the position of $\mathbf{A}$ after parallel transport around the circuit. Then, with $\delta\mathbf{A} = \bar{\delta}\mathbf{A} / (\delta a)(\delta b)$,

$$\delta \mathbf{A} = -\mathbf{R}(\mathbf{u}, \mathbf{v})\,\mathbf{A}$$

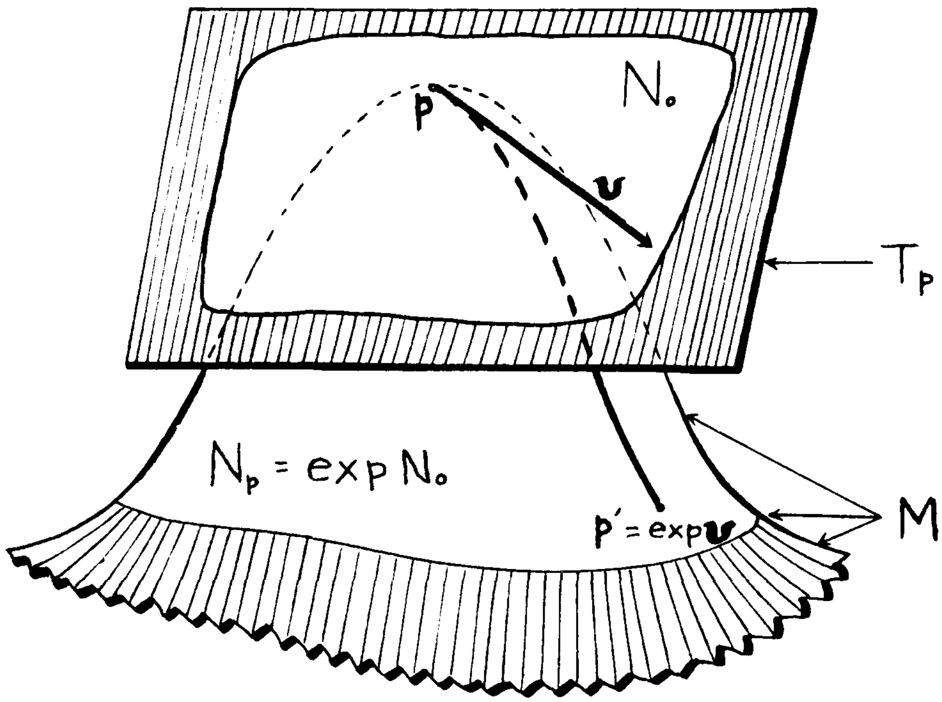



FIGURE 2.4: Exponential Mapping and Normal Coordinates: Exponential map: $T_p \ni \mathbf{v} \to \exp(\mathbf{v}) \in M$. $N_o$ is an open neighbourhood in $T_p$ mapped into $N_p \in M$ with the exp map so that the point $p' \in M$ into which $\mathbf{v}$ is mapped lies on the geodesic $\nabla_{\mathbf{v}}\mathbf{v} = 0$ with affine parameter values $\lambda = 0$ at $p$ and $\lambda = 1$ at $p'$ In the convex normal neighbourhood $N_p$ of $p$ normal coordinates of $p'$ are defined as $x^\alpha = x^\alpha(p')$ where $\mathbf{v} = x^\alpha \mathbf{e}_\alpha$ Then, at $p$, $\mathbf{e}_\alpha = \partial_\alpha$ and $\Gamma^\lambda{}_{\mu\nu} = 0$ (see NOTE 11).

A metric $\mathbf{g}$ is a symmetric tensor in $T^0_2$ with components $\mathbf{g}(\mathbf{e}_\alpha,\mathbf{e}_\beta) \equiv \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta}$, non-degenerate (: $\mathbf{g}(\mathbf{u},\mathbf{v}) = 0$ for given $\mathbf{u}$ and all $\mathbf{v}$ implies $\mathbf{u} = 0$), with "inverse" (also denoted by) $\mathbf{g} \in T^2_0$ with components $g^{\alpha\beta}$ such that

$$g_{\alpha\lambda}\, g^{\lambda\beta} = \delta_\alpha{}^\beta \tag{2.43}$$

The metric is used to define: lengths of vectors ( $u^2 = \mathbf{g}(\mathbf{u},\mathbf{u})$ ), the "cosine of the angle" between them ( $\mathbf{g}(\mathbf{u},\mathbf{v})/|\mathbf{g}(\mathbf{u},\mathbf{u})\,\mathbf{g}(\mathbf{v},\mathbf{v})|^{1/2}$ ), the length $\ell$ of a curve between the points with values $t_1, t_2$ of the curve parameter $t$ ( $\ell = \int_{t_1}^{t_2} |\mathbf{g}(\partial_t,\partial_t)|^{1/2}\, dt$ ), and the "lowering and rising" of indices through the maps in $\mathcal{T}(M)$ naturally defined by $\mathbf{g}$ (in applications, tensors thus mapped to each other, e.g. $T^{\alpha\beta}$, $T_\alpha{}^\beta$ etc., will be considered as representing the same entity, $\mathbf{T}$ ).

With proper choice of basis vectors, $g_{\mu\nu}$ can have ±1 (zeros would indicate a degenerate metric) as its diagonal components at one point with no off-diagonal terms. The resulting sequence of signs is the signature of $\mathbf{g}$. A signature with mixed signs is called pseudo-Eucledian and gives rise to pseudo-Riemannian spaces. Riemann spaces have Eucledian signature at every point.

The central theorem of Riemannian geometry guarrantees the existance of a unique torsion-free affinity in a (psuedo) Riemannian manifold, such that

$$\nabla_{\mathbf{u}}\mathbf{g} = 0 \quad \text{for every} \quad \mathbf{u} \in T^1_0 \qquad (2.44)$$

The affinity is then symmetric in the sense that the rotation coefficients $\Gamma^{\lambda}{}_{\mu\nu}$ are symmetric in $\mu\nu$, in holonomic coordinates. The Riemann tensor has in this case the additional symmetry $R_{\lambda\mu\nu\rho} = R_{[\lambda\mu]\nu\rho}$ which, combined with the ones resulting from its definition leaves $\frac{1}{12}n^2(n^2-1)$ algebraically independent components.

The (symmetric) Ricci tensor, the curvature scalar, the Einstein tensor (symmetric, zero divergence) and the (conformally invariant) Weyl tensor are, in that order, defined as:

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} \qquad (2.45)$$

?

$$R = R^{\mu}{}_{\mu} \qquad (2.46)$$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \qquad (2.47)$$

$$W^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}_{[\gamma} R^{\beta]}_{\delta]} + \frac{1}{3}\delta^{[\alpha}_{[\gamma}\delta^{\beta]}_{\delta]}R \qquad (2.48)$$

A group of ACT maps generated by an r-dimensional algebra of Lie derivations that leave the metric invariant is called a group of isometries. The r independent vector fields $\xi_{\kappa}$ in $T^{1}_{0}(M)$ corresponding to r independent derivations in the Lie algebra are called killing vectors, and the equation expressing the invariance of the metric

$$L_{\xi} g = 0 \qquad (2.49)$$

the killing equation. It can also be written as

$$\xi(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot[\xi,\mathbf{v}] + \mathbf{v}\cdot[\xi,\mathbf{u}] \qquad (2.50)$$

where $\mathbf{u},\mathbf{v}$ are any two vector fields.

Thus each $\xi$ generates a group of maps of $M$ onto itself called the group of motions of the space and every point $p \in M$ "moves" under the action of this group, along the, so called, orbit of $p$. If the orbit of a point $p$ is $p$ itself, it is called a fixed point of ?

the group. If the orbit of every $p \epsilon M$ is a geodesic, $\xi$ is called a geodesic killing vector and the isometry it generates a translation.

## 2.E GENERALIZED EXTERIOR CALCULUS

Consider the tensor $\mathbf{G} \in T^{r}_{s+q}$ , antisymmetric in its last $q$ covariant indices (NOTE 12). Then, with the exterior (or wedge) product " $\wedge$ " between any two 1-forms defined by $\boldsymbol{\sigma} \wedge \boldsymbol{\omega} = \boldsymbol{\sigma} \otimes \boldsymbol{\omega} - \boldsymbol{\omega} \otimes \boldsymbol{\sigma}$, $\mathbf{G}$ can be written as

$$\mathbf{G} = \frac{1}{q!}\, \mathbf{G}_{\lambda_1 \cdots \lambda_q}\, \boldsymbol{\omega}^{\lambda_1} \wedge \ldots \wedge \boldsymbol{\omega}^{\lambda_q} \qquad (2.51)$$

where

$$\mathbf{G}_{\lambda_1 \cdots \lambda_q} = G^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_r \lambda_1 \cdots \lambda_q}\, \mathbf{e}_{\alpha_1} \otimes \cdots \otimes \mathbf{e}_{\alpha_r} \otimes \boldsymbol{\omega}^{\beta_1} \otimes \ldots \otimes \boldsymbol{\omega}^{\beta_s}$$

and, in this form, called an $\binom{r}{s}$ -tensor-valued $q$-form. In this terminology, a function is a scalar-valued 0-form, a vector is a vector-valued 0-form, an ordinary 1-form is a scalar-valued 1-form, a $\binom{1}{1}$ tensor a vector-valued 1-form and so on.

In a manifold with affinity $\boldsymbol{\nabla}$ , one can further define the (generalized) exterior derivative operator $\mathbf{d}$ . This, in addition to being a derivation (but not an ACT derivation), obeys the following rules:

(a) $\mathbf{dG}$ is a $\binom{r}{s}$-tensor-valued $(q+1)$ -form $\mathbf{G}$ as above.

(b) For any tensor-valued 0-form $S$, $dS = \nabla S$

(c) For any scalar-valued 1-form $\omega$, Cartan's structural can be taken as the definition of $d\omega$. With $T$ the torsion tensor,

$$d\omega^{\alpha} = -\omega^{\alpha}_{\ \lambda} \wedge \omega^{\lambda} + \frac{1}{2} T^{\alpha}_{\ \lambda\mu} \omega^{\lambda} \wedge \omega^{\mu} \tag{2.52}$$

where

$$\omega^{\lambda}_{\ \mu} = \Gamma^{\lambda}_{\ \mu\nu} \omega^{\nu} \tag{2.53}$$

the connection forms, with

$$d\omega^{\lambda}_{\ \mu} = -\omega^{\lambda}_{\ \alpha} \wedge \omega^{\alpha}_{\ \mu} + \frac{1}{2} R^{\lambda}_{\ \mu\alpha\beta} \omega^{\alpha} \wedge \omega^{\beta} \tag{2.54}$$

(d) For the exterior product of any tensor-valued $q$-form $G$ with an ordinary (that is, scalar-valued) $p$-form $\sigma$

$$d(G \wedge \sigma) = (dG) \wedge \sigma + (-1)^{q} G \wedge d\sigma \tag{2.55}$$

Although a metric is not needed for the definition of integrals over n-dimensional submanifolds of $M$ (see NOTE 13), more general definitions can be given with the use of the so called canonical n-form on $M$ (or Levi-Civita tensor) $\epsilon$, in terms of which one also defines the duality operation on forms. The components of $\boldsymbol{\epsilon}$ in an orientable manifold, and with respect to any basis with positive orientation, are

$$\epsilon_{\alpha_1 \cdots \alpha_n} = n! \, |g|^{\frac{1}{2}} \, \delta^{1}_{[\alpha_1} \cdots \delta^{n}_{\alpha_n]} \tag{2.56}$$
$$\epsilon^{\alpha_1 \cdots \alpha_n} = (-1)^{\frac{n-s}{2}} \, n! \, |g|^{-\frac{1}{2}} \, \delta^{[\alpha_1}_{1} \cdots \delta^{\alpha_n]}_{n}$$

so that

$$\epsilon_{\alpha_1 \cdots \alpha_n} \epsilon^{\beta_1 \cdots \beta_n} = (-1)^{\frac{n-s}{2}} \, n! \, \delta^{\beta_1}_{[\alpha_1} \cdots \delta^{\beta_n}_{\alpha_n]} \tag{2.57}$$

where $g$ is the determinant of the matrix of the components of $\mathbf{g}$ and $s$ is the signature. The dual of a q-form is an (n-q)-form; thus the dual of $\mathbf{G}$ in (51) is defined as

$$^*\mathbf{G} = \frac{1}{q!} \frac{1}{(n-q)!} \mathbf{G}^{\lambda_1 \cdots \lambda_q} \epsilon_{\lambda_1 \cdots \lambda_q \lambda_{q+1} \cdots \lambda_n} \boldsymbol{\omega}^{\lambda_{q+1}} \wedge \cdots \wedge \boldsymbol{\omega}^{\lambda_n} \tag{2.58}$$

In a quite analogous manner, one can define the exterior product of vectors as $\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \equiv \mathbf{e}_\alpha \otimes \mathbf{e}_\beta - \mathbf{e}_\beta \otimes \mathbf{e}_\alpha$, thus creating bivectors, etc., or more generally tensor-valued $q$-vectors. The duality operation will, in this case, be denoted by $\bigstar$.

The volume of an open set $U$ in $M$ is defined as (see also NOTE 13)

$$\int_U \epsilon \equiv \int_{\phi(U)} |g|^{1/2} dx^1 \cdots dx^n \qquad (2.59)$$

where $\phi(U)$ is the region in $R^n$ onto which $U$ is mapped under the coordination $X^\alpha$. In the same sense

$$\int_U f\epsilon \equiv \int_{\phi(U)} f |g|^{1/2} dx^1 \ldots dx^n \qquad (2.60)$$

is the integral of $f$ over $U$. The integral of $f$ over the manifold $M$, is given by

$$\int_M f\epsilon = \sum_k \int_{\phi_k(U_k)} g_k |g|^{\frac{1}{2}} f\, dx^1 \ldots dx^n \qquad (2.61)$$

where $g_k$ is a partition of the unity of the atlas $\{(U_k, \phi_k)\}$. One can therefore define, in the above fashion, the integral of any $q$-form over any orientable $q$-dimensional submanifold of $M$. Useful in this context is the generalized Stokes Theorem which states for a $q$-form $\sigma$ and any orientable ($q$ + 1)-dimensional (sub)manifold $\Omega$ with boundary $\partial\Omega$ one has

$$\int_\Omega d\sigma = \int_{\partial\Omega} \sigma \qquad (2.62)$$

## 3. GENERAL RELATIVISTIC COSMOLOGY OUTLINED

### 3.A AXIOMATIC FOUNDATION OF GENERAL RELATIVITY

The postulates, on which the Classical Theory of General Relativity can be founded, will be given first and explanatory remarks will follow together with some basic formulas for later use.

SPACE TIME (Einstein's "Continuum"): All events in the universe can be considered as points of a connected, four-dimensional, Hausdorf, $C^\infty$ differentiable manifold, with a pseudo - Riemannian metric defined on it with Lorentz signature, and with free test particles moving on geodesics, determined by the associated symmetric metric affinity.

CAUSALITY: For any two points (events) in any convex, normal neighbourhood, a signal can be sent between them only if they can be joined by a differentiable curve, lying entirely in that neighbourhood, whose tangent vector is everywhere non-zero and non-spacelike.

LOCAL CONSERVATION OF ENERGY AND MOMENTUM: There exists a symmetric tensor, the energy-momentum tensor, which depends on the tensor fields by which the material content of the space is described, their covariant derivatives and the metric, and which vanishes in an open neighbourhood if and only if all fields vanish there. Its divergence is identically zero.

FIELD EQUATIONS: Einstein's equations hold, that is,

$$\mathbf{G} + \Lambda \mathbf{g} = 8\pi \mathbf{T}$$

where $\mathbf{G}$ is Einstein's tensor, $\mathbf{g}$ the metric, $\mathbf{T}$ the energy momentum tensor and $\Lambda$ the cosmological constant (hereafter taken equal to zero).

For a systematic exposition of the theories of Relativity and Gravitation, see, for example: TRAUTMAN, PIRANI and BONDI (1965); ANDERSON (1967); HAWKING and ELLIS (1972); MISNER, THORNE, and WHEELER (1973).

The above postulates must be supplemented by a prescription on how to determine $\mathbf{T}$, which can be done if a Lagrangian is known.

The physical concept of an event arises from our (supposedly objective) ability to assign relative spatial and temporal separations to localized occurences in nature. The assumption that events form a "continuum" is supported by our intuitive conception of space and time and the principle of simplicity. Lorentz signature is chosen because special relativity holds in sufficiently small regions of spacetime, as experiments show.

A vector field $\mathbf{u}$ will be timelike, spacelike or null at a given point $p$, according to whether $\mathbf{u}\cdot\mathbf{u}$ is, in that order, negative positive, or zero at $p$.

The motion of small free particles on geodesics follows from the (experimentally established--see, for example, DICKE (1964)) equality between the inertial and passive gravitational mass. That all matter fields enter the field equations through their stress-energy tensor (only), expresses, on the other hand, the equality between passive and active gravitational mass (KREUZER (1968)).

The momentum $\mathbf{p}$ of a particle with rest mass $m$ is always timelike. An observer with velocity $\mathbf{u}$ will attribute to this particle:

rest mass $$m = (-\mathbf{p}\cdot\mathbf{p})^{1/2} \tag{3.1}$$

energy $$E = -\mathbf{p}\cdot\mathbf{u} \tag{3.2}$$

magnitude of 3-momentum $$|\vec{p}| = \left[(\mathbf{p}\cdot\mathbf{u})^2 + \mathbf{p}\cdot\mathbf{p}\right]^{1/2} \tag{3.3}$$

magnitude of 3-velocity $\quad |\vec{V}| = -|\vec{P}|\,(\mathbf{P}\cdot\mathbf{u})^{-1}$ (3.4)

The red (blue)-shift of a signal received by observer "2" and emitted by "1" is given by $\mathbf{z}$ :

$$1+z = \frac{\lambda_2}{\lambda_1} = \frac{\omega_1}{\omega_2} = \frac{E_1}{E_2} = \frac{\mathbf{u}_1\cdot\mathbf{K}(1)}{\mathbf{u}_2\cdot\mathbf{K}(2)} \tag{3.5}$$

where $\mathbf{u}_2$ is the velocity of observer "2", $\mathbf{K}$(1) the momentum of the photon when emitted at "1" and so on. The calculation of the curvature

$$\mathbf{R} = \mathbf{e}_\lambda \otimes \boldsymbol{\omega}^\mu \otimes \mathbf{R}^\lambda{}_\mu \tag{3.6}$$

is based on the computation of the curvature 2-forms:

(same as (2.54))

$$\mathbf{R}^\lambda{}_\mu \equiv \frac{1}{2} R^\lambda{}_{\mu\alpha\beta}\, \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta = \mathbf{d}\boldsymbol{\omega}^\lambda{}_\mu + \boldsymbol{\omega}^\lambda{}_\alpha \wedge \boldsymbol{\omega}^\alpha{}_\mu$$

As one finds from (2.52)

$$\mathbf{d}\boldsymbol{\omega}^\lambda = -\boldsymbol{\omega}^\lambda{}_\mu \wedge \boldsymbol{\omega}^\mu \tag{3.7}$$

and, using (2.44) or, equivalently

$$\mathbf{d}(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{d}\mathbf{u})\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{d}\mathbf{v} \tag{3.8}$$

one has

$$\mathbf{d} g_{\mu\nu} = \boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu} \tag{3.9}$$

One has therefore exactly enough equations to calculate the connection forms and consequently $\mathbf{R}^{\lambda}{}_{\mu}$ . Alternatively, one can use (2.53) to obtaine the $\Gamma$'s and the components of the Riemann tensor, given by (2.41), here simplified to

$$\begin{aligned} R^{\lambda}{}_{\mu\nu\rho} = {} & \Gamma^{\lambda}{}_{\mu\rho,\nu} - \Gamma^{\lambda}{}_{\mu\nu,\rho} + \Gamma^{\lambda}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\rho} \\ & - \Gamma^{\lambda}{}_{\alpha\rho}\Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\lambda}{}_{\alpha\nu}\Gamma^{\alpha}{}_{\mu\rho} - \Gamma^{\lambda}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\rho\nu} \end{aligned} \tag{3.10}$$

where the fact of zero torsion $\mathbf{T}$ :

$$C_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\mu\nu} \tag{3.11}$$

was used. The "inverse" of (11), as valid in general, is given below for completeness:

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} + C_{\lambda\mu\nu} + C_{\lambda\nu\mu} - C_{\mu\nu\lambda}\right) \tag{3.12}$$

For any function $f$ , if and only if $\mathbf{T} = 0$

$$\mathbf{d}^2 f = 0 \tag{3.13}$$

Also, applying for $\mathbf{S} = \mathbf{d}\boldsymbol{v}$ the equation

$$\langle \mathbf{d}\mathbf{S}, \mathbf{u} \wedge \mathbf{w} \rangle = \nabla_{\mathbf{u}} \langle \mathbf{S}, \mathbf{w} \rangle - \nabla_{\mathbf{w}} \langle \mathbf{S}, \mathbf{u} \rangle - \langle \mathbf{S}, [\mathbf{u}, \mathbf{w}] \rangle \tag{3.14}$$

valid when $\mathbf{T} = 0$ for any tensor valued 1-form $\mathbf{S}$ , vectors $\mathbf{u}, \mathbf{w}$ , one has

$$\overset{2}{\mathbf{d}}\boldsymbol{v} = \mathbf{R}\boldsymbol{v} \tag{3.15}$$

This equation can also be derived directly as follows. From

$$\mathbf{R}(\mathbf{u}, \mathbf{w})\boldsymbol{v} = \nabla_{\mathbf{u}} \nabla_{\mathbf{w}} \boldsymbol{v} - \nabla_{\mathbf{w}} \nabla_{\mathbf{u}} \boldsymbol{v} - \nabla_{[\mathbf{u}, \mathbf{w}]} \boldsymbol{v}$$

(equ. (2.40))

one finds, using $\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} = [\mathbf{u}, \mathbf{w}]$ (zero torsion) and (2.39):

$$\overset{2}{\nabla}\boldsymbol{v}(;\mathbf{w};\mathbf{u}) - \overset{2}{\nabla}\boldsymbol{v}(;\mathbf{u};\mathbf{w}) = \mathbf{R}(\mathbf{u}, \mathbf{w})\boldsymbol{v} \tag{3.16}$$

and in component notation

$$v^{\alpha}{}_{;\nu;\mu} - v^{\alpha}{}_{;\mu;\nu} = R^{\alpha}{}_{\lambda\mu\nu} v^{\lambda} \tag{3.17}$$

The Bianchi identities are

$$\mathbf{d}\mathbf{R} = 0 \quad , \quad \mathbf{d}^{*}\mathbf{G} = 0 \tag{3.18}$$

where

$$\mathbf{G} = C^1_1 (\star \mathbf{R}^\star) \qquad (3.19)$$

is the Einstein tensor written in the presently used notation.

The duals of various forms can be expresse in terms of the following "volume" elements in $M$ :

4-VOLUME $$\mathbf{d}^4\Omega = \frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\,\boldsymbol{\omega}^\alpha\wedge\boldsymbol{\omega}^\beta\wedge\boldsymbol{\omega}^\gamma\wedge\boldsymbol{\omega}^\delta \qquad (3.20)$$

HYPERSURFACE $$\mathbf{d}^3\Sigma_\mu = \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}\,\boldsymbol{\omega}^\alpha\wedge\boldsymbol{\omega}^\beta\wedge\boldsymbol{\omega}^\gamma \qquad (3.21)$$

2-SURFACE $$\mathbf{d}^2 S_{\lambda\mu} = \frac{1}{2!}\epsilon_{\lambda\mu\alpha\beta}\,\boldsymbol{\omega}^\alpha\wedge\boldsymbol{\omega}^\beta \qquad (3.22)$$

LINE $$\mathbf{d}\mathcal{S}_{\lambda\mu\nu} = \epsilon_{\lambda\mu\nu\alpha}\,\boldsymbol{\omega}^\alpha \qquad (3.23)$$

Thus:

| FOR | DUAL IS | |
|---|---|---|
| $f \in F$ | $\star f = f\,\mathbf{d}^4\Omega$ | (3.24) |
| $\mathbf{J} = J_\mu\,\boldsymbol{\omega}^\mu$ | $\star\mathbf{J} = J^\mu\,\mathbf{d}^3\Sigma_\mu$ | (3.25) |
| $\mathbf{F} = \frac{1}{2!}F_{\mu\nu}\,\boldsymbol{\omega}^\mu\wedge\boldsymbol{\omega}^\nu$ | $\star\mathbf{F} = \frac{1}{2!}F^{\mu\nu}\,\mathbf{d}^2 S_{\mu\nu}$ | (3.26) |
| $\mathbf{K} = \frac{1}{3!}K_{\lambda\mu\nu}\,\boldsymbol{\omega}^\lambda\wedge\boldsymbol{\omega}^\mu\wedge\boldsymbol{\omega}^\nu$ | $\star\mathbf{K} = \frac{1}{3!}K^{\lambda\mu\nu}\,\mathbf{d}\mathcal{S}_{\lambda\mu\nu}$ | (3.27) |

From (2.56), one obtains:

$$\epsilon_{\alpha_0\alpha_1\alpha_2\alpha_3}\,\epsilon^{\beta_0\beta_1\beta_2\beta_3} = -(4!)(0!)\,\delta^{\beta_0}_{[\alpha_0}\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}\delta^{\beta_3}_{\alpha_3]} \tag{3.28}$$

$$\epsilon_{\alpha_0\alpha_1\alpha_2\rho}\,\epsilon^{\beta_0\beta_1\beta_2\rho} = -(3!)(1!)\,\delta^{\beta_0}_{[\alpha_0}\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2]} \tag{3.29}$$

$$\epsilon_{\alpha_0\alpha_1\nu\rho}\,\epsilon^{\beta_0\beta_1\nu\rho} = -(2!)(2!)\,\delta^{\beta_0}_{[\alpha_0}\delta^{\beta_1}_{\alpha_1]} \tag{3.30}$$

$$\epsilon_{\alpha_0\mu\nu\rho}\,\epsilon^{\beta_0\mu\nu\rho} = -(1!)(3!)\,\delta^{\beta_0}_{\alpha_0} \tag{3.31}$$

$$\epsilon_{\lambda\mu\nu\rho}\,\epsilon^{\lambda\mu\nu\rho} = -(0!)(4!) \tag{3.32}$$

and using these expressions with $\nabla\epsilon = 0$ and the equation

$$(\mathbf{d}\,\boldsymbol{\sigma})_{\alpha_1\cdots\alpha_{q+1}} = (-1)^q\,\sigma_{[\alpha_1\cdots\alpha_q\,;\,\alpha_{q+1}]} \tag{3.33}$$

valid for any ordinary $q$-form $\boldsymbol{\sigma}$ when $\mathbf{T} = 0$, one obtains:

$$\mathbf{d}(\text{any volume element in (67)}) = 0 \tag{3.34}$$

$$\mathbf{d}\,{}^*\mathbf{J} = J^{\mu}{}_{;\mu}\,\mathbf{d}^4\Omega \tag{3.35}$$

$$\mathbf{d}\,{}^*\boldsymbol{F} = F^{\lambda\mu}{}_{;\mu}\,\mathbf{d}^3\Sigma_\lambda \tag{3.36}$$

and so on. Also, for $\mathbf{G} = \mathbf{e}_\mu G^\mu{}_\nu\,\boldsymbol{\omega}^\nu$

$$\mathbf{d}\,{}^*\mathbf{G} = \mathbf{e}_\mu\,G^{\mu\nu}{}_{;\nu}\,\mathbf{d}^4\Omega \tag{3.37}$$

with similar expressions for other forms.

## 3.B THE DESCRIPTION OF A RELATIVISTIC FLUID

Anticipating the specialization to spatially homogeneous cosmological models (see next section and Chapter 5), a discussion of hypersurfaces will follow.

A hypersurface in $M$ is the image of a 3-dimensional manifold $\mathcal{S}$ under the imbedding

$$\phi : \quad \mathcal{S} \longrightarrow M \tag{3.38}$$

$\phi(\mathcal{S})$ can be locally described by $f = 0$, where $f$ is a properly chosen hypersurface-forming function on $M$; if both $\mathcal{S}$ and $M$ are orientable, the nowhere zero 1-form

$$\boldsymbol{u} = (\text{const})\, \mathbf{d}f \tag{3.39}$$

will characterize portions of $\phi(\mathcal{S})$ as timelike, spacelike or null according to whether $\boldsymbol{u}$ is, in that order, spacelike, timelike or null. The induced metric $\phi^{*}\mathbf{g}$ on $\mathcal{S}$ (the star $*$ has nothing to do with the duality operation on forms--see section 2.C for notation), called the first fundamental form, can be used to define the projection operator $\mathbf{h}$

$$\mathbf{h} = (\phi^*)^{-1}\,\phi^*\,\mathbf{g} \tag{3.40}$$

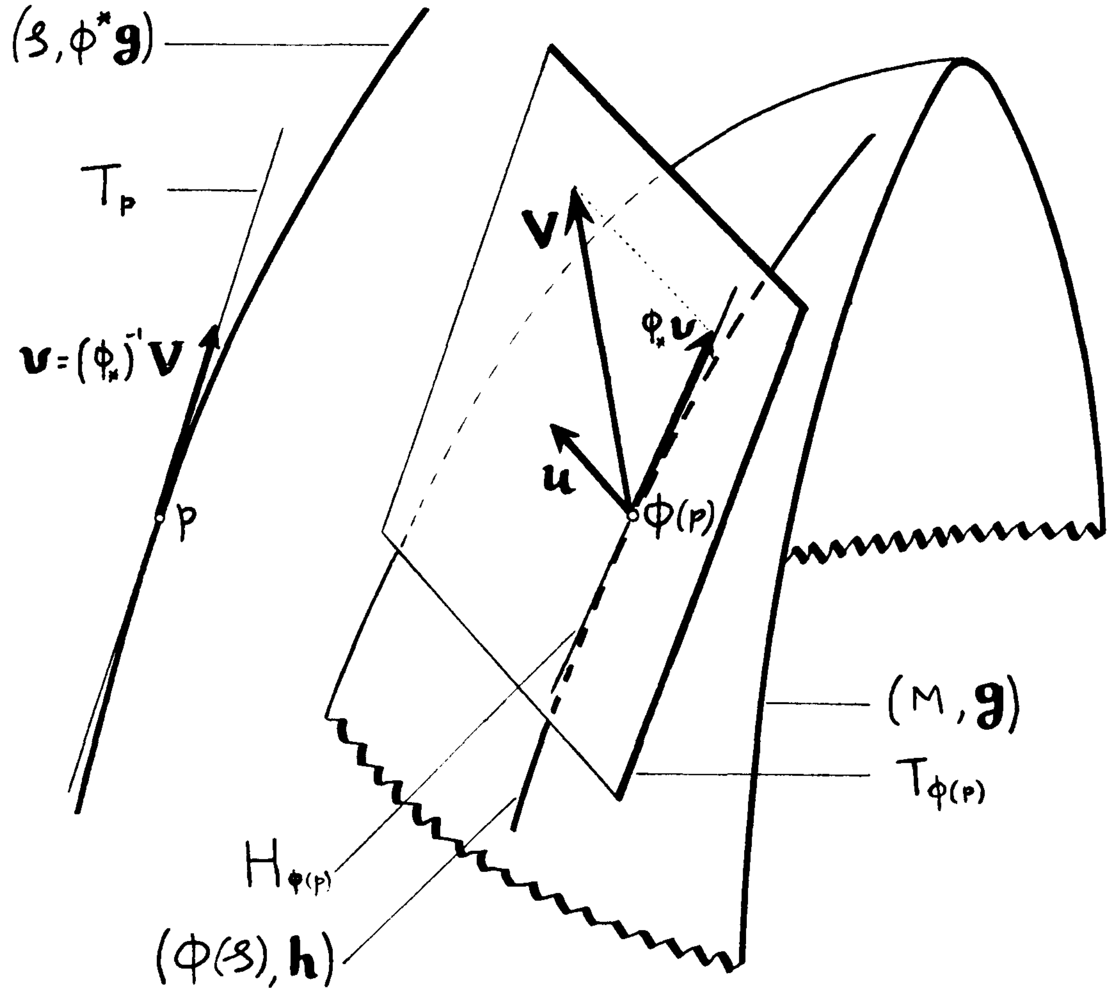
which projects every vector $\mathbf{v} \in T_{\phi(p)}$ , $p \in \mathcal{S}$ into its part lying in the subspace $\phi_*(T_p) = H_{\phi(p)}$ of $T_{\phi(p)}$. See Figure 3.1. If $\mathbf{u}$ is timelike, with $\mathbf{u}\cdot\mathbf{u} = -1$, then

$$\mathbf{h} = \mathbf{g} + \mathbf{u} \otimes \mathbf{u} \tag{3.41}$$

Consider now a congruence of timelike curves in $M$, which can be taken as respresenting the histories of particles or the flow lines of a fluid. A congruence is a family of curves, one through each point of $M$. In a closed compact 4-dimensional submanifold $D$ of $M$ the congruence can be represented by the diffeomorphism

$$[\tau_0, \tau_1] \times \mathcal{S} \longrightarrow D$$

where $[\tau_0, \tau_1]$ a closed interval of the real line and $\mathcal{S}$ a 3-dimensional manifold with boundary. The everywhere timelike vector tangent to the curves can be normalized so that $\mathbf{u}\cdot\mathbf{u} = -1$ and one can now introduce a set of quantities which can describe the kinematical behaviour of the fluid.




$(M, \mathbf{g})$ : Original manifold and metric

$(\mathcal{S}, \phi^* \mathbf{g})$ : Hypersurface and induced metric ("first fundamental form" on $\mathcal{S}$ )

$(\phi(\mathcal{S}), \mathbf{h})$ : Image in $M$ and induced metric

FIGURE 3.1: Imbedding a hypersurface in a manifold.

ACCELERATION $\quad \mathbf{a} = \nabla_{\mathbf{u}} \mathbf{u} = \dot{\mathbf{u}}$ (3.42)

VORTICITY

$$\omega_{\alpha\beta} = h_\alpha{}^\lambda h_\beta{}^\mu u_{[\lambda;\mu]} \tag{3.43}$$

$$\Omega^\alpha = \tfrac{1}{2} \epsilon^{\alpha\lambda\mu\nu} u_\lambda \omega_{\mu\nu} \tag{3.44}$$

$$= \tfrac{1}{2} \epsilon^{\alpha\lambda\mu\nu} u_\lambda u_{\mu;\nu}$$

or,

$$\mathbf{\Omega} = * \tfrac{1}{2} \mathbf{u} \wedge \mathbf{d}\mathbf{u} \tag{3.45}$$

EXPAN SION

$$\theta_{\alpha\beta} = h_\alpha{}^\lambda h_\beta{}^\mu u_{(\lambda;\mu)} \tag{3.46}$$

SHEAR

$$\sigma_{\alpha\beta} = \theta_{\alpha\beta} - \tfrac{1}{3} \theta h_{\alpha\beta} \tag{3.47}$$

VOLUME (OR ISOTROPIC) EXPANSION

$$\theta = h^{\alpha\beta} \theta_{\alpha\beta} = h^{\alpha\beta} u_{\alpha;\beta} = u^\alpha{}_{;\alpha} \tag{3.48}$$

One can now verify the "decomposition" of $\nabla \mathbf{u}$:

$$u_{\alpha;\beta} = - \dot{u}_\alpha u_\beta + \tfrac{1}{3} \theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{3.49}$$

In order to discuss the physical significance of these quantities, one may introduce the Fermi derivative $F_{\mathbf{u}} \mathbf{v}$ of a vector field $\mathbf{v}$ along the congruence:

$$F_{\mathbf{u}} \mathbf{v} = \nabla_{\mathbf{u}} \mathbf{v} - \mathbf{g}(\dot{\mathbf{u}}, \mathbf{v}) \mathbf{u} + \mathbf{g}(\mathbf{u}, \mathbf{v}) \dot{\mathbf{u}} \tag{3.50}$$

If $F_{\mathbf{u}}\mathbf{v} = 0$, $\mathbf{v}$ is said to be Fermi-Walker transported along the congruence. The physical significance of this transport law is that it describes a propagation free of any rotation with its induced inertial forces. Thus, for example, a gyroscope is Fermi-Walker transported along its world line. The lengths and angles between Fermi-Walker propagated vectors are preserved. For vectors in $H_p$, $F_{\mathbf{u}} = \nabla_{\mathbf{u}}$ and if the congruence is geodesic $(\dot{\mathbf{u}} = 0)$, then $F_{\mathbf{u}} = \nabla_{\mathbf{u}}$ for all vectors.

Consider now a family of curves, called a Jacobi field with tangent vector $\mathbf{J}$, defined in terms of the congruence so that

$$L_{\mathbf{u}}\mathbf{J} = 0 \qquad \text{(components: } \dot{J}^{\alpha} = u^{\alpha}{}_{;\beta}J^{\beta}\text{)} \qquad (3.51)$$

One may interpret $\mathbf{J}$ as the "connecting" vector between nearby curves of the congruence. As $\mathbf{J}$ is defined modulo a component parallel to $\mathbf{u}$, one can work with only the projection of $\mathbf{J}$ into $H_p$ at every point $p$. Using for basis a set of orthonormal vectors $\{\mathbf{e}_\alpha\}$, Fermi-Walker transported along the congruence and with $\mathbf{e}_0 = \mathbf{u}$, one obtains from (51)

$$\frac{d}{d\tau} J^{l} = u^{l}{}_{;m} J^{m} \tag{3.52}$$

$$\frac{d^2}{d\tau^2} J^{l} = \left(-R^{l}{}_{omo} + u^{l}{}_{;m} + \dot{u}^{l}\dot{u}_{m}\right) J^{m} \tag{3.53}$$

The second is the so called deviation equation which for a geodesic congruence becomes the equation of geodesic deviation (see figure 2.2).

The solutions of (52) can be expressed as

$$J^{l}(\tau) = A^{l}{}_{m}(\tau)\, J^{m}(\tau_{o}) \tag{3.54}$$

where the matrix $A^{l}{}_{m}$, a function of $\tau$, determines the evolution of $\mathbf{J}$ from its value $\mathbf{J}(\tau_o)$ at an initial position $p_o$. At the initial position then, $A$ is the unit matrix, representable there by a small spherical element of the fluid, according to the rules of the familiar representation of a 3 x 3 matrix by an elipsoid. At the general position $A$ can be written as

$$A^{l}{}_{m} = O^{l}{}_{n} S^{n}{}_{m} \tag{3.55}$$

where $S$ is symmetric and $O$ orthogonal with positive determinant. Now an initially spherical volume of the fluid will in general experience a rotation with respect to a Fermi-Walker transported basis and a (positive or negative) expansion, which can be decomposed, as usual, to a volume (that is, isotropic) expansion and a shear. The matrix $O$ will describe the rotation $\omega_{\ell m}$ and matrix $S$ the expansion $\theta_{\ell m}$ of the fluid, with the determinant of $S$ giving the volume of the initially spherical element of the fluid. Explicitly, one finds:

$$
\begin{aligned}
2\,\omega_{\ell m} &= (A^{-1})^{n}{}_{m}\left(\frac{d}{d\tau}A\right)_{\ell n} - (A^{-1})^{n}{}_{\ell}\left(\frac{d}{d\tau}\right)_{nm} \\
2\,\theta_{\ell m} &= (A^{-1})^{n}{}_{m}\left(\frac{d}{d\tau}A\right)_{\ell m} + (A^{-1})^{n}{}_{\ell}\left(\frac{d}{d\tau}A\right)_{nm} \qquad (3.56) \\
\theta &= \frac{d}{d\tau}\ln(\det A)
\end{aligned}
$$

and, using (52), (53)

$$\frac{d}{d\tau}A^{\ell}{}_{m} = u^{\ell}{}_{;n}\,A^{n}{}_{m} \qquad (3.57)$$

$$\frac{d^{2}}{d\tau^{2}}A^{\ell}{}_{m} = \left(-R^{\ell}{}_{ono} + u^{\ell}{}_{;n} + \dot{u}^{\ell}\dot{u}_{n}\right)A^{n}{}_{m} \qquad (3.58)$$

From (57) applied to the initial point $P_o$ (where $A^{\ell}{}_{m} = \delta^{\ell}{}_{m}$) and the expressions (56) one can now justify the definitions (43), (46), (47), (48) and the terminology used. Equation (58) determines the evolution of the kinematical quantities if the Riemann tensor is known, which, in general, is accomplished by solving the field equations.

Assuming the fluid to be perfect, its equations of motion will be functions of $\rho$ (baryon number density) and $s$ (entropy per baryon):

$$p = p(\rho, s) \qquad T = T(\rho, s) \tag{3.59}$$

From the laws of:

baryon number conservation $\nabla \cdot (\rho \mathbf{u}) = 0$ (3.60)

1st Law of Thermodynamics $d\mathcal{E} = \mu d\rho + \rho ds$ (3.61)

with $\mathcal{E}$ the total energy density, $\mu = \frac{\mathcal{E} - p}{\rho}$ the chemical potential per baryon, one obtains for the pressure and temperature:

$$p = \rho \left(\frac{\partial \mathcal{E}}{\partial \rho}\right)_s - \mathcal{E} \tag{3.62}$$

$$T = \frac{1}{\rho}\left(\frac{\partial \mathcal{E}}{\partial s}\right)_\rho \tag{3.63}$$

and the following consistency relation (Maxwell relation)

$$\left(\frac{\partial p}{\partial s}\right)_{\rho} = \rho^{2}\left(\frac{\partial T}{\partial \rho}\right)_{s} \qquad (3.64)$$

The second law of thermodynamics $\frac{ds}{d\tau} \geqslant 0$ holds with the equality sign for an adiabatic flow and, in that case,

$$\mathbf{T} = (\mathcal{E} + p)\,\mathbf{u}\otimes\mathbf{u} + p\,\mathbf{g} \qquad (3.65)$$

is the energy momentum tensor for the fluid (Lagrangian $= -2\mathcal{E}$).

The component of $\nabla\cdot\mathbf{T} = 0$ along $\mathbf{u}$ reproduces, as expected, the first law (61), while its component on the subspace $H$ gives the

Euler equation $$(\mathcal{E} + p)\,\dot{\mathbf{u}} = -(\mathbf{d}p + \dot{p}\,\mathbf{u}) \qquad (3.66)$$

Noting that for a comoving element of the fluid, with volume $V$ one has

$$\frac{d}{d\tau}\ln V = \nabla\cdot\mathbf{u} \qquad (3.67)$$

one can rewrite $\mathbf{u}\cdot(\nabla\cdot\mathbf{T}) = 0$ (first law of thermodynamics) as

$$\frac{d}{d\tau}(\mathcal{E}V) + p\frac{d}{d\tau}V = 0 \qquad (3.68)$$

If the fluid is electrically charged, there will be a contribution to the energy momentum tensor from the E-M field. The physically observable quantities that describe an E-M field are the components of the (antisymmetric) Faraday tensor

$$\mathbf{F} = \frac{1}{2} F_{\alpha\beta}\, \boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}^{\beta} \qquad (3.69)$$

with

$$F_{i0} = E_i$$

$$F_{ij} = H_k \quad (i,j,k \text{ cyclically } 1,2,3)$$

or, the Maxwell tensor, which is defined as the dual of $\mathbf{F}$. In terms of these tensors, one may form the invariants

$$\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} = |\vec{H}|^2 - |\vec{E}|^2 \qquad (3.70)$$

$$\frac{1}{4} F_{\alpha\beta}\, {}^{*}F^{\alpha\beta} = \vec{H}\cdot\vec{E} \qquad (3.71)$$

The electromagnetic potential $\mathbf{A}$, a 1-form in terms of which $\mathbf{F}$ may be expressed as

$$\mathbf{F} = \mathbf{d}\mathbf{A} \tag{3.72}$$

is measurable only after a gauge has been chosen for it, one such being the usual Larentz gauge

$$\mathbf{d}\,{}^*\mathbf{A} = 0 \tag{3.73}$$

The field equations (Maxwell's equations) are

$$\mathbf{d}\mathbf{F} = 0 \qquad \mathbf{d}\,{}^*\mathbf{F} = 4\pi\,{}^*\mathbf{J} \tag{3.74}$$

where $\mathbf{J}$ is the current 1-form for the electric charge, defined as $\rho\mathbf{u}$, with $\rho$ the charge density for a comoving observer. The energy momentum tensor for the E-M field alone is (NOTE 14)

$$4\pi T_{\mu\nu} = -F_{\mu}{}^{\alpha} F_{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{3.75}$$

The equations of motion (equation for the Lorentz force) will result from $\nabla \cdot \mathbf{T} = 0$ if the massive (of the mass) contribution of the "bare" fluid is included in the energy momentum tensor.

## 3.C ON SYMMETRIC SPACES AND THE BIANCHI CLASSIFICATION

The $G_1$ of ACT maps $\Phi$ generated by a Lie derivation $L_{\xi}$ is an affine transformation if it commutes with parallel displacement:

$$\Phi(\nabla_{\mathbf{u}}\mathbf{v}) = \nabla_{\Phi\mathbf{u}}(\Phi\mathbf{v}) \tag{3.76}$$

For zero torsion and noting that (2.39) is also valid for Lie derivations, the above can be written as:

$$0 = \lim_{\Phi\to 1}\left[\nabla_{\mathbf{u}}\mathbf{v} - \Phi^{-1}(\nabla_{\Phi\mathbf{u}}\Phi\mathbf{v})\right]$$

$$= \lim_{\Phi\to 1}\left[(\nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\Phi\mathbf{u}}\mathbf{v}) + (\nabla_{\Phi\mathbf{u}}\mathbf{v} - \nabla_{\Phi\mathbf{u}}\Phi\mathbf{v}) + (\nabla_{\Phi\mathbf{u}}\Phi\mathbf{v} - \Phi^{-1}(\nabla_{\Phi\mathbf{u}}\Phi\mathbf{v}))\right]$$

$$= \nabla_{L_{\xi}\mathbf{u}}\mathbf{v} + \nabla_{\mathbf{u}}(L_{\xi}\mathbf{v}) - L_{\xi}(\nabla_{\mathbf{u}}\mathbf{v})$$

hence

$$\nabla_{\mathbf{u}}(L_{\xi}\mathbf{v}) = (L_{\xi}(\nabla\mathbf{v}))(\,;\mathbf{u}) \tag{3.77}$$

and similarly, for any tensor $\mathbf{G}$

$$\nabla_{\mathbf{u}}(L_{\xi}\mathbf{G}) = (L_{\xi}(\nabla\mathbf{G}))(\,;\mathbf{u}) \tag{3.78}$$

As a by-product of the above derivation, one obtains using (2.40)

$$\xi^{\alpha}{}_{;\lambda;\mu} = - R^{\alpha}{}_{\lambda\beta\mu}\,\xi^{\beta} \qquad (3.79)$$

The integrability conditions for (76)$\Longleftrightarrow$(77) are

$$L_{\xi} T = 0 \qquad , \qquad L_{\xi} R = 0 \qquad (3.80)$$

for the torsion and curvature tensors.

An isometry generated by $L_{\xi}$ is also an affine transformation in spacetime so, from $L_{\xi}\mathbf{g} = 0$ and (78),

$$\xi_{(\alpha;\beta)} = 0 \qquad (3.81)$$

if and only if $\xi$ is a killing vector. It also follows from (77) that for a killing vector $\xi$

$$\xi(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot L_{\xi}\mathbf{v} + \mathbf{v}\cdot L_{\xi}\mathbf{u}$$

or, with $\mathbf{u} = \mathbf{e}_{\alpha}$ , $\mathbf{v} = \mathbf{e}_{\beta}$ , $\xi = \partial_t$

$$\partial_t g_{\alpha\beta} = C_{\alpha\beta t} + C_{\beta\alpha t}$$

which shows that in a coordinate basis (where $C_{(\alpha\beta)t} = 0$ )

$g_{\alpha\beta}$ is independent of the coordinate corresponding to the killing vector.

From (81), one finds that for any $\mathbf{u}$,

$$(\xi_\alpha u^\alpha)_{;\beta}\, u^\beta = \xi_\alpha u^\alpha{}_{;\beta}\, u^\beta$$

hence, $\xi_\alpha u^\alpha$ is constant along an affinely scaled geodesic with tangent $\mathbf{u}$. Also, $\xi_\alpha \xi^\alpha$ is constant along the trajectories of $\boldsymbol{\xi}$, as follows from $(\xi_\alpha \xi^\alpha)_{;\beta}\, \xi^\beta = 0$ and if the isometry is a translation $\xi_\alpha \xi^\alpha$ is constant everywhere. The last statement holds because in $\xi_{\alpha;\beta}\xi^\beta = \lambda \xi_\alpha$, $\lambda = 0$ as a consequence of (81), and one obtains then $(\xi_\alpha \xi^\alpha)_{;\beta} = 0$.

Spacetime will have the highest possible symmetry if it admits the maximum allowed number of killing vectors. This number is equal to the maximum dimensionality of the invariance Lie algebra of the metric, that is 1/2n(n+1). Thus the flat spacetime of Special Relativity (Minkowski space) possesses 10 killing vector fields.

It will now be shown that for every killing vector $\boldsymbol{\xi}$ there corresponds a conservation law for the vector $\mathbf{P} = \boldsymbol{\xi}\cdot\mathbf{T}$ (component of the energy-momentum flux density along the direction of $\xi$ ). From (2.50) and $\nabla \cdot \mathbf{T} = 0$ , one finds that $\nabla \cdot \mathbf{P} = 0$, which shows that the flux of $\mathbf{P}$ over the closed hypersurface $\partial D$ of any compact 4-dimentional region $D$ is zero:

$$\int_{\partial D} P^{\mu} \mathbf{d}\Sigma_{\mu} = \int_{\partial D} {}^{*}\mathbf{P} = \int_{D} \mathbf{d}\, {}^{*}\mathbf{P} = \int_{D} (\nabla \cdot \mathbf{P}) \mathbf{d}^{4}\Omega = 0 \qquad (3.82)$$

In flat spacetime and Cartesian coordinates, there exist four killing vectors $\mathbf{P}_{\alpha} = \partial_{\alpha}$ generating translations and establishing conservation for energy and momentum and six others, $\mathbf{M}_{\alpha\beta} = g_{\mu[\alpha} x^{\mu} \partial_{\beta]}$ generating six rotations and giving the conservation law for angular momentum.

Spacetime will, in general, admit no killing vectors but the lack of any conservation laws (although one realizes that such laws are not fundamental or necessary elements of a physical theory) is apparent only in a global sense, since one can always, using normal coordinates, make negligible the flux for linear and angular momentum over a closed hypersurface by considering a sufficiently small region of spacetime.

Assume now that a spacetime is spatially homogeneous, that is, it admits a $G_3$ group of motions simply transitive on spacelike hypersurfaces. These, so called, surfaces of homogeneity define a unique, normal to them and future directed vector field (or 1-form) $\mathbf{n}$, with

$$\mathbf{n}\cdot\mathbf{n} = -1 \quad , \quad \nabla_{\mathbf{n}}\mathbf{n} = 0 \quad , \quad \mathbf{n}\wedge d\mathbf{n} = 0 \qquad (3.83)$$

Because it now follows that $d\mathbf{n} = 0$, there exists a function $t$, with $\mathbf{n} = -dt \neq 0$ such that the equation for each hypersurface $\mathcal{S}$ will be $t$ = const. and a parametrization $\mathcal{S}(\bar{t})$ can be made, where $\bar{t} = \bar{t}(t)$ with $\frac{d\bar{t}}{dt} \neq 0$. The metric in $\mathcal{S}(t)$ will be $\mathbf{g} + \mathbf{n}\otimes\mathbf{n}$ and the general discussion of 3.B is directly applicable.

Introduce now an orthonormal basis $\{\mathbf{e}_\alpha\}$ such that $\mathbf{e}_0 = \mathbf{n}$. Using the equations: $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, $(\nabla_{\mathbf{u}}\mathbf{e}_{(\alpha})\cdot\mathbf{e}_{\beta)} = 0$ (following from (9) and (2.53)), (11) and (12), one can compute the structure constants $C_{\alpha\beta}{}^{\gamma}$ (see, for example, ELLIS and MACCALLUM (1969)). They are

$$C_{\ell m}{}^{n} = \epsilon_{\ell m r} S^{rn} + a_{[\ell}\delta^{n}_{m]} \qquad (3.84)$$

where,

$$a_\ell = C_{\ell m}{}^{m}$$

$$S^{rn} = \frac{1}{2} C_{\ell m}{}^{[r} \epsilon^{n]\ell m}$$

and

$$C_{\alpha\beta 0} = 0$$

$$C_{\ell 0 m} = n_{\ell ; m} + \mathbf{e} \cdot (\nabla_{\mathbf{n}} \mathbf{e}_m) \tag{3.85}$$

The last equation can be rewritten as

$$C_{\ell 0 m} = n_{\ell ; m} + \epsilon_{\ell r m} \Omega^{r} \tag{3.86}$$

where

$$\Omega^{\alpha} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} n_{\beta} \mathbf{e}_{\gamma} \cdot (\nabla_{\mathbf{n}} \mathbf{e}_{\delta})$$

is the rotation of the basis vectors with respect to Fermi Walker transported basis. If $\xi_\ell$ are the three killing vectors generating the $G_3$, one can choose the basis (BESIDES $\mathbf{e}_0 = \mathbf{n}$) so that $[\mathbf{e}_\ell, \xi_m] = 0$.

Applying the Jacobi identity (2.24) for $\mathbf{e}_n, \xi_\ell, \xi_m$ one sees that $C_{\ell m}{}^{n}$ are functions of time only (NOTE 15). Further, one can calculate the time derivatives of the structure constants by applying (2.24) to $\mathbf{e}_0, \mathbf{e}_\ell, \mathbf{e}_m$ and finally for $\mathbf{e}_\ell, \mathbf{e}_m, \mathbf{e}_n$ to obtain a constraint among the $C$'s; this constraint written in terms of $S_{\ell m}$ and $a_\ell$, in terms of which the $C$'s are expressed in (84), is

$$S_{\ell m} a^m = 0 \tag{3.87}$$

Performing an orthogonal (only time dependent) rotation of the basis $\{e_m\}$ one can give $S_{\ell m}$ and $a_\ell$ the canonical form

$$\begin{aligned} S_{\ell m} &= \mathrm{diag}\,(s_1, s_2, s_3) \\ a_\ell &= (a\,\delta^0_{s_1}, 0, 0) \end{aligned} \tag{3.88}$$

into which the constraint (87) has been absorbed, since $\delta^0_{s_1} = 0$ (if $s_1 \neq 0$), and $\delta^0_0 = 1$.

On any chosen hypersurface $\mathcal{S}$ the basis for the killing vectors can be chosen so that

$$\tilde{C}_{\ell m}{}^n \overset{*}{=} C_{\ell m}{}^n \tag{3.89}$$

where

$$[\xi_\ell, \xi_m] = \tilde{C}_{\ell m}{}^n \xi_n \tag{3.90}$$

and the $*$ in (89) indicates that the identification is valid on a specified $\mathcal{S}$ since $C_{\ell m}{}^n$ are functions of time while $\tilde{C}_{\ell m}{}^n$ constants (NOTE 16). The expression (84) will therefore also hold for the $\tilde{C}$'s but with the $S^{rm}$, $a^{\ell}$ constants. Using the canonical form (88), one has

$$\tilde{C}_{\ell m}{}^{n} = \epsilon_{\ell m}{}^{n} \tilde{S}_{n}\,, \quad \tilde{C}_{12}{}^{2} = \tilde{C}_{13}{}^{3} = \tilde{a}$$
$$\tilde{C}_{2m}{}^{m} = \tilde{C}_{3m}{}^{m} = 0 \qquad \text{(NO SUMS)} \qquad (3.91)$$

One is free to renumber the $\xi$'s , reverse their directions and rescale them. Table 3.1 gives all the possible independent combinations for $\tilde{S}_{\ell}$, $\tilde{a}$ together with the Bianchi type of the group of motions tht the corresponding killing vectors generate.

Types $\mathrm{VI}_h$ and $\mathrm{VII}_h$ are actually one-parameter families of groups; $h = \frac{a^2}{S_2 S_3}$ was chosen for the parametrization because it is an invariant as one can show by calculating the time derivatives of $S_{\ell m}$ and $a_{\ell}$ . Therefore, it is always:

$$h = \frac{a^2}{S_2 S_3} = \frac{\tilde{a}^2}{\tilde{S}_2 \tilde{S}_3} \qquad (3.92)$$

When used to determine the $\tilde{C}$'s, the values for $\tilde{S}_{\ell}$, $\tilde{a}$ can be chosen as ±1 (when not zero) unless none of $\tilde{a}, \tilde{S}_2, \tilde{S}_3$ is zero, in which case $\tilde{S}_2$ , $\tilde{S}_3$ can be ±1 and $\tilde{a}$ will be given by the invariant $h$ from (92).

TABLE 3.1 Classifying the Possible Structure Constants

| $\tilde{S}_1$ | $\tilde{S}_2$ | $\tilde{S}_3$ | $\tilde{a}$ | Bianchi Type |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | I |
| + | 0 | 0 | 0 | II |
| + | + | 0 | 0 | VII ($\equiv$ $VII_0$) |
| + | − | 0 | 0 | VI ($\equiv$ $VI_0$) |
| + | + | + | 0 | IX |
| + | + | − | 0 | VIII |
| 0 | 0 | 0 | + | V |
| 0 | 0 | + | + | IV |
| 0 | + | + | + | $VII_h$ |
| 0 | + | − | + | $VI_h$ with ($VI_{-1} \equiv III$) |

TABLE 3.2 The Bianchi Classification of Homogeneous Spaces

| Bianchi Type | $d\sigma^1$ | $d\sigma^2$ | $d\sigma^3$ |
|---|---|---|---|
| I | $0$ | $0$ | $0$ |
| II | $\sigma^2\wedge\sigma^3$ | $0$ | $0$ |
| III | $\sigma^1\wedge\sigma^2$ | $0$ | $0$ |
| IV | $0$ | $\sigma^1\wedge\sigma^2$ | $\sigma^1\wedge(\sigma^2+\sigma^3)$ |
| V | $0$ | $\sigma^1\wedge\sigma^2$ | $\sigma^1\wedge\sigma^3$ |
| VI | $\sigma^2\wedge\sigma^3$ | $\sigma^1\wedge\sigma^3$ | $0$ |
| $\mathrm{VI}_h$ | $0$ | $a\sigma^1\wedge\sigma^2+\sigma^3\wedge\sigma^1$ | $-\sigma^1\wedge\sigma^2-a\sigma^3\wedge\sigma^1$ |
| VII | $\sigma^2\wedge\sigma^3$ | $\sigma^3\wedge\sigma^1$ | $0$ |
| $\mathrm{VII}_h$ | $0$ | $a\sigma^1\wedge\sigma^2+\sigma^3\wedge\sigma^1$ | $\sigma^1\wedge\sigma^2-a\sigma^3\wedge\sigma^1$ |
| VIII | $\sigma^2\wedge\sigma^3$ | $\sigma^3\wedge\sigma^1$ | $-\sigma^1\wedge\sigma^2$ |
| IX | $\sigma^2\wedge\sigma^3$ | $\sigma^3\wedge\sigma^1$ | $\sigma^1\wedge\sigma^2$ |

From (2.53), (7), (11) one has for a basis $\{\omega^\lambda\}$

$$d\omega^\nu = -\frac{1}{2} C_{\lambda\mu}{}^\nu \omega^\lambda \wedge \omega^\mu \tag{3.93}$$

This equation with the values supplied by (91) was used to construct Table 3.2, which gives the differential relations (93) for the 1-forms $\sigma^\ell$ dual to the killing vectors $\xi_\ell$ of the various Bianchi types.

Type III is derived from $VI_h$ for $h=-1$; notice however that the $d\sigma^\ell$ given above were obtained from the original ones (which were: $d\sigma^1 = 0$, $d\sigma^2 = -d\sigma^3 = \sigma^1 \wedge (\sigma^2 - \sigma^3)$ ) by a linear transformation of the basis. The Types VI and VII can be recovered from $VI_h$ and $VII_h$ when $h = 0$; in these last two types, the value of $\tilde{a}$ is given by (92) for $\tilde{s}_1 = -1$ and $\tilde{s}_2 = +1$.

# 4. EXACT SOLUTIONS

## 4.A OUTLINE OF A TECHNIQUE

An exact solution of Einstein's field equations, representing a cosmological model, is a spacetime $(M, g)$ in which the components of the metric are known functions of the coordinates and the material content of the model is described by appropriate geometric quantities, usually scalar vector or tensor fields, whose components are known functions of the coordinates and such that the energy momentum tensor defined by them satisfy Einstein's field equations. For the complete understanding of a model, the above information,being local as a result of the use of the machinery of differential geometry, must be supplemented by a study of its global topological characteristics (see, for example, HAWKING and ELLIS (1973)).

When the restriction of isotropy is relaxed for models complying with the Cosmological Principle, that is, for spatially homogeneous spacetimes, the most straightforward search for an exact solution would start by writing the metric as

$$\mathbf{g} = -\mathbf{d}t \otimes \mathbf{d}t + g_{\ell m}\, \boldsymbol{\sigma}^{\ell} \otimes \boldsymbol{\sigma}^{m} \tag{4.1}$$

with the $g_{\ell m}$ unknown functions of the time $t$, and the $\boldsymbol{\sigma}$ 's chosen from one of the Bianchi types described in 3.C and listed in TABLE 3.2.

In terms of a properly chosen orthonormal basis

$$\boldsymbol{\omega}^{0} = \mathbf{d}t\,, \qquad \boldsymbol{\omega}^{\ell} = A^{\ell}{}_{m}\, \boldsymbol{\sigma}^{m} \tag{4.2}$$

with the $A$'s functions of $t$ (only), the metric will be

$$\mathbf{g} = -\boldsymbol{\omega}^{0} \otimes \boldsymbol{\omega}^{0} + \boldsymbol{\omega}^{1} \otimes \boldsymbol{\omega}^{1} + \boldsymbol{\omega}^{2} \otimes \boldsymbol{\omega}^{2} + \boldsymbol{\omega}^{3} \otimes \boldsymbol{\omega}^{3} \tag{4.3}$$

Then, from

$$\mathbf{d}\boldsymbol{\omega}^{\lambda} = \Gamma^{\lambda}{}_{\mu\nu}\, \boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu} \tag{4.4}$$

which follows from (3.7) and (2.53), one can uniquely determine the rotation coefficients if one uses, in addition to (4), the symmetry

$$\Gamma_{(\lambda\mu)\nu} = 0 \tag{4.5}$$

which follows from (2), (2.53) and (3.9). Consequently, one can obtain expressions for the components of both the Riemann and Ricci tensors from (3.10) and its $C^1_2$ contraction:

$$R_{\mu\rho} = \Gamma^{\alpha}{}_{\mu\rho,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\rho} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\rho} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\rho\alpha} \qquad (4.6)$$

To write down the field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \qquad (4.7)$$

or,

$$R_{\mu\nu} = 8\pi T_{\mu\nu} - 4\pi T g_{\mu\nu} \qquad (4.8)$$

(where $R$ and $T$ are $C^1_1$ contractions of $C^1_2\mathbf{R}$ and $\mathbf{T}$), one must know $\mathbf{T}$, which presents no difficulties ordinarily.

Thus, for the case of dust, for example, $\mathbf{T}$ is given by (3.65) reduced to

$$\mathbf{T} = \varepsilon\, \mathbf{u} \otimes \mathbf{u} \qquad (4.9)$$

for an E-M field by (3.75) and so on.

The field equations must be supplemented by equations which the material content of spacetime must obey, that is, conservation (e.g. continuity) equations, field equations and equations of motion. One example in each category respectively is given below:

Baryon number conservation $$\nabla \cdot (\rho \mathbf{u}) = 0 \tag{4.10}$$

Maxwell's equations $$\mathbf{d}\mathbf{F} = 0 \ , \ \mathbf{d}{}^*\mathbf{F} = 4\pi\, {}^*\mathbf{J} \tag{4.11}$$

Spatial component of (vanishing) divergence of T $$\mathbf{h}(\nabla \cdot \mathbf{T}) = 0 \tag{4.12}$$

Of course, not all of these are independent equations, but it is usually useful to consider even those among them that follow from Einstein's field equation (e.g. (12) above). The solution of the resulting system of differential equations for $g_{\ell m'}$ if one exists, can sometimes be expedited if one makes the "right" transformation $t \to T$ :

$$\mathbf{d}T = f(t)\,\mathbf{d}t \tag{4.13}$$

The existence of a solution, for at least some finite time, can be secured if one shows that the initial value equations

$$G_{0\mu} = 8\pi T_{0\mu} \tag{4.14}$$

can be satisfied on an "initial" spacelike hypersurface of homogeneity (Cauchy problem--see, for example Y. Bruhat's article in WITTEN (1962)).

## 4. B A BIANCHI TYPE IX MODEL

Using mainly the technique outlined above, an exact solution for a Bianchi Type IX universe was obtained.

This model represents a closed homogeneous anisotropic world filled with an E-M field and a massless scalar field. It reduces to Taub-NUT-M spaces (TAUB (1951); NEWMAN, TAMBURINO and UNTI (1963); MISNER (1963)) when it is empty, it is a generalization of Brill's E-M universe (BRILL 1964) when the scalar field vanishes and, finally, represents the scalar-tensor theory (see, for example, DICKE (1964)) analog of these spacetimes. For details, see BATAKIS and COHEN (1972).

# 5. THREE ROTATING UNIVERSES

## 5.A A BIANCHI TYPE II MODEL WITH DUST AND E-M FIELD

Starting with

$$
\begin{aligned}
g = & -dt\otimes dt + a^2(\sigma^1\otimes\sigma^1 + \lambda^2\,\sigma^2\otimes\sigma^2 + \\
& (\nu^2\lambda^2+\mu^2)\sigma^3\otimes\sigma^3 + \nu\lambda^2\sigma^2\otimes\sigma^3 + \nu\lambda^2\sigma^3\otimes\sigma^2)
\end{aligned}
\tag{5.1}
$$

the diagonal form (4.2) is obtained with the tetrad

$$
\begin{aligned}
&\omega^0 = dt && \omega^1 = a\sigma^1 \\
&\omega^2 = \lambda a(\sigma^2+\nu\sigma^3) && \omega^3 = \mu a\sigma^3
\end{aligned}
\tag{5.2}
$$

Assuming that $d\sigma^1 = \sigma^2\wedge\sigma^3$, $d\sigma^2 = d\sigma^3 = 0$ (Bianchi type II), one derives

$$
\begin{aligned}
d\omega^0 &= 0 \\
d\omega^1 &= \frac{a_{,0}}{a}\,\omega^0\wedge\omega^1 + \frac{1}{\lambda\mu a}\,\omega^2\wedge\omega^3 \\
d\omega^2 &= \frac{(\lambda a)_{,0}}{\lambda a}\,\omega^0\wedge\omega^2 + \frac{\lambda\,\nu_{,0}}{\mu\,\nu}\,\omega^0\wedge\omega^3 \\
d\omega^3 &= \frac{(\mu a)_{,0}}{\mu a}\,\omega^0\wedge\omega^3
\end{aligned}
\tag{5.3}
$$

Equations (4.4), (4.5) now give the rotation coefficients and those different than zero are

$$\Gamma_{101} = \frac{a_{,0}}{a} \qquad \Gamma_{202} = \frac{(\lambda a)_{,0}}{\lambda a} \qquad \Gamma_{303} = \frac{(\mu a)_{,0}}{\mu a}$$

$$\Gamma_{203} = \Gamma_{320} = \Gamma_{302} = \frac{\lambda \nu_{,0}}{2\mu} \tag{5.4}$$

$$\Gamma_{123} = -\Gamma_{231} = \Gamma_{312} = \frac{1}{2\lambda\mu a}$$

The components of the Ricci tensor follow from (4.6)

$$\begin{aligned}
-R_{00} &= \left(\frac{(\lambda\mu a^3)_{,0}}{\lambda\mu a^3}\right)_{,0} + \left(\frac{a_{,0}}{a}\right)^2 + \left(\frac{(\lambda a)_{,0}}{\lambda a}\right)^2 + \left(\frac{(\mu a)_{,0}}{\mu a}\right)^2 + \frac{1}{2}\left(\frac{\lambda \nu_{,0}}{\mu}\right)^2 \\
R_{11} &= \left(\frac{a_{,0}}{a}\right)_{,0} + \frac{(\lambda\mu a^3)_{,0}}{\lambda\mu a^3}\frac{a_{,0}}{a} + \frac{1}{2}(2\lambda\mu a)^{-2} \\
R_{22} &= \left(\frac{(\lambda a)_{,0}}{\lambda a}\right)_{,0} + \frac{(\lambda\mu a^3)_{,0}}{\lambda\mu a^3}\frac{(\lambda a)_{,0}}{\lambda a} - \frac{1}{2}\left(\frac{\lambda \nu_{,0}}{\mu}\right)^2 - \frac{1}{2}(2\lambda\mu a)^{-2} \\
R_{33} &= \left(\frac{(\mu a)_{,0}}{\mu a}\right)_{,0} + \frac{(\lambda\mu a^3)_{,0}}{\lambda\mu a^3}\frac{(\mu a)_{,0}}{\mu a} + \frac{1}{2}\left(\frac{\lambda \nu_{,0}}{\mu}\right)^2 - \frac{1}{2}(2\lambda\mu a)^{-2} \\
R_{23} &= \left(\frac{\lambda \nu_{,0}}{2\mu}\right)_{,0} - \frac{(\lambda^2 a^3)_{,0}}{\lambda^2 a^3}\frac{\lambda \nu_{,0}}{2\mu} \\
R_{01} &= R_{02} = R_{03} = R_{12} = R_{31} = 0
\end{aligned} \tag{5.5}$$

A special feature of this model is the non-zero vorticity of its contents: a pressureless fluid with density $\rho$ and velocity

$$\mathbf{u} = -u^0 \boldsymbol{\omega}^0 + u^1 \boldsymbol{\omega}^1 \qquad (\mathbf{u}\cdot\mathbf{u} = -1) \tag{5.6}$$

will have rotation, according to (3.45),

$$\boldsymbol{\Omega} = \frac{u^1}{2\lambda\mu a}\left(u^1 \boldsymbol{\omega}^0 - u^0 \boldsymbol{\omega}^1\right) \tag{5.7}$$

Equation $\nabla \cdot \mathbf{T} = 0$, written as

$$T^{\alpha 0}{}_{,0} + \Gamma^{\alpha}{}_{\mu\nu} T^{\mu\nu} + \Gamma^{\mu}{}_{\nu\mu} T^{\alpha\nu} = 0 \tag{5.8}$$

and applied to the energy momentum tensor of the dust with non-zero components

$$\begin{aligned} 8\pi T^{00} &= \rho (u^0)^2, \quad 8\pi T^{11} = \rho (u^1)^2 \\ 8\pi T^{01} &= \rho u^0 u^1 \end{aligned} \tag{5.9}$$

gives, after some algebra and two integrations ($M, N$ are constants)

$$\lambda\mu a^3 \rho u^0 = M \tag{5.10}$$

$$u^1 a = N \tag{5.11}$$

The magnitude of the rotation vector (7) can now be written as

$$|\Omega| = \frac{u^1}{2\lambda\mu a} = \frac{N}{2\lambda\mu a^2} \tag{5.12}$$

An E-M field with

$$\begin{aligned} F &= -e_l\, \omega^0 \wedge \omega^l + \tfrac{1}{2} \epsilon^l{}_{mn} h_l\, \omega^m \wedge \omega^n \\ {}^*F &= h_l\, \omega^0 \wedge \omega^l + \tfrac{1}{2} \epsilon^l{}_{mn} e_l\, \omega^m \wedge \omega^n \end{aligned} \tag{5.13}$$

will have, according to (3.75), energy momentum tensor

$$4\pi T_{\mu\nu} = \begin{array}{c|c|c|c}
\frac{1}{2}(h^2+e^2) & & & \\
\hline
h_2e_3-h_3e_2 & \frac{1}{2}(-h_1^2+h_2^2+h_3^2-e_1^2+e_2^2+e_3^2) & & \\
\hline
h_3e_1-h_1e_3 & -h_1h_2-e_1e_2 & \frac{1}{2}(h_1^2-h_2^2+h_3^2+e_1^2-e_2^2+e_3^2) & \\
\hline
h_1e_2-h_2e_1 & -h_3h_1-e_3e_1 & -h_2h_3-e_2e_3 & \frac{1}{2}(h_1^2+h_2^2-h_3^2+e_1^2+e_2^2-e_3^2)
\end{array} \tag{5.14}$$

where,

$$h^2 = \sum h_\ell^2 \quad , \quad e^2 = \sum e_\ell^2$$

If one writes down Maxwell's equations (4.11) with $h_1 = e_1 = 0$ and also $\mathbf{J} = 0$, the resulting system of differential equations can be integrated; the solution, after this was done, was found to be ( $H_i$ , $E_i$ are constants):

$$h_2 = \frac{H_2 + H_3\nu}{\mu a^2} \qquad h_3 = \frac{H_3}{\lambda a^2}$$
$$e_2 = \frac{E_2 + E_3\nu}{\mu a^2} \qquad e_3 = \frac{E_3}{\lambda a^2} \tag{5.15}$$

With the simplifying assumptions

$$E_2 = -H_3 \quad , \quad E_3 = H_2 \, , \quad H^2 \equiv H_2^2 + H_3^2 \tag{5.16}$$

$$\lambda = \mu \tag{5.17}$$

one obtains from (14) the following non-zero components for the energy momentum tensor of the E-M field

$$\left|\begin{array}{l} 8\pi T_{00} = 8\pi T_{11} = \dfrac{H^2(2+\nu^2)}{\lambda^2 a^4} \\ 8\pi T_{22} = 8\pi T_{33} = \dfrac{H^2\nu^2}{\lambda^2 a^4} \\ 8\pi T_{01} = \dfrac{2H^2}{\lambda^2 a^4} \\ 8\pi T_{23} = -\dfrac{2H^2\nu}{\lambda^2 a^4} \end{array}\right. \tag{5.18}$$

The field equations are now

$$R_{00} = R_{11} = \frac{H^2(2+\nu^2)}{\lambda^2 a^4} + \rho(u^0)^2 - \frac{1}{2}\rho \tag{5.19}$$

$$R_{22} = -\frac{H^2\nu^2}{\lambda^2 a^4} + \frac{1}{2}\rho \tag{5.20}$$

$$R_{33} = \frac{H^2\nu^2}{\lambda^2 a^4} + \frac{1}{2}\rho \tag{5.21}$$

$$R_{23} = -\frac{2H^2\nu}{\lambda^2 a^4} \tag{5.22}$$

where $R_{\mu\nu}$ is given by (5) with $\lambda = \mu$. Because $R_{01} = 0$, one also has the relation

$$2H^2 = MN \tag{5.23}$$

From $R_{33} - R_{11} = \frac{2H^2\nu}{\lambda^2 a^4}$ and (22), one finds with one integration ( $P$ is a constant)

$$\nu_{,0} = \frac{\sqrt{2}HP}{\nu^2\lambda^2 a^3} \tag{5.24}$$

$$\nu^3\lambda a = P \tag{5.25}$$

Using (24) in the form of (4.13), that is,

$$d\nu = \frac{\sqrt{2}H\nu}{\lambda a^2}\, dt \tag{5.26}$$

and also (25), one can introduce $\nu$ as the independent variable in the equation $R_{11} = R_{11}$. When this is done, and with a prime denoting differentiation with respect to $\nu$ , one obtains the following linear second order equation

$$(\ln a^4)'' + \frac{4}{\nu}(\ln a^4)' + \frac{a^4\nu^4}{2H^2P^2} + 1 = 0 \tag{5.27}$$

Therefore, if $y(\nu)$ is a solution of

$$y'' + \frac{4}{\nu}y' + \nu^4 e^y + 1 = 0 \tag{5.28}$$

one has

$$a = \left[ 2 H^2 P^2 e^{y(\nu)} \right]^{\frac{1}{4}} \qquad (5.29)$$

with similar expressions for the other quantities $\lambda, \rho, u^0, u^1$, obtained from (10), (11), (25). This formally completes the solution, with $\nu$ considered a new time coordinate and with (26) giving the corresponding transformation.

## 5.B A DUST FILLED BIANCHI TYPE IX UNIVERSE

For a Bianchi type IX spacetime ($\mathbf{d}\boldsymbol{\sigma}^{\ell} = \boldsymbol{\sigma}^{m} \wedge \boldsymbol{\sigma}^{n}$, $(\ell, m, n)$ cyclically (1,2,3)-- section 3.C), the metric

$$\mathbf{g} = -\mathbf{d}t \otimes \mathbf{d}t + a^2 \left[ \boldsymbol{\sigma}^1 \otimes \boldsymbol{\sigma}^1 + \lambda^2 \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^2 + (\mu^2 + \nu^2 \lambda^2) \boldsymbol{\sigma}^3 \otimes \boldsymbol{\sigma}^3 + \nu \lambda^2 \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^3 + \nu \lambda^2 \boldsymbol{\sigma}^3 \otimes \boldsymbol{\sigma}^2 \right] \tag{5.30}$$

is diagonal for the orthonormal basis

$$\boldsymbol{\omega}^0 = \mathbf{d}t \qquad \boldsymbol{\omega}^1 = a \boldsymbol{\sigma}^1$$
$$\boldsymbol{\omega}^2 = \lambda a (\boldsymbol{\sigma}^2 + \nu \boldsymbol{\sigma}^3) \qquad \boldsymbol{\omega}^3 = \mu a \boldsymbol{\sigma}^3 \tag{5.31}$$

with

$$\mathbf{d}\boldsymbol{\omega}^0 = 0$$

$$\mathbf{d}\boldsymbol{\omega}^1 = \frac{a_{,0}}{a} \boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 + \frac{1}{\lambda \mu a} \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3$$

$$\mathbf{d}\boldsymbol{\omega}^2 = \frac{(\lambda a)_{,0}}{\lambda a} \boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^2 + \frac{\lambda \nu_{,0}}{\mu} \boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^3 + \frac{\nu}{a} \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 + \frac{(1+\nu^2)\lambda}{\mu a} \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^1 \tag{5.32}$$

$$\mathbf{d}\boldsymbol{\omega}^3 = \frac{(\mu a)_{,0}}{\mu a} \boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^3 + \frac{\mu}{\lambda a} \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 + \frac{\nu}{a} \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^1$$

As done before, the non-zero rotation coefficients are

$$
\begin{aligned}
&\Gamma_{101} = \frac{a_{,0}}{a} \qquad \Gamma_{202} = \frac{(\lambda a)_{,0}}{\lambda a} \qquad \Gamma_{303} = \frac{(\mu a)_{,0}}{\mu a} \\
&\Gamma_{302} = \Gamma_{203} = \Gamma_{320} = \frac{\lambda \nu_{,0}}{2\mu} \\
&\Gamma_{212} = -\Gamma_{313} = \frac{\nu}{a} \\
&\Gamma_{\substack{312\\231\\123}} = \frac{\substack{+\\-\\+}\, 1 \;\; \substack{+\\+\\-}\, \mu^2 \;\; \substack{-\\+\\+}\, (1+\nu^2)\lambda^2}{2\lambda\mu a}
\end{aligned}
\tag{5.33}
$$

The components of the Ricci tensor are given by (34 - 40) and the $G_{00}$ component of the Einstein tensor for $\lambda = \mu$ by (41).

The matter content of this spacetime will be described by the following non-zero components of the energy momentum tensor

$$
\begin{aligned}
8\pi T_{00} &= \rho (u^0)^2 \\
8\pi T_{11} &= \rho (u^1)^2 \qquad \text{WITH:} \quad 8\pi T = -\rho \\
8\pi T_{01} &= -\rho u^0 u^1
\end{aligned}
\tag{5.42}
$$

where $\rho$ is energy density and

$$
\mathbf{u} = -u^0 \boldsymbol{\omega}^0 + u^1 \boldsymbol{\omega}^1 \quad (\mathbf{u}\cdot\mathbf{u} = -1) \tag{5.43}
$$

With some algebra and two integrations, it now follows

$$-R_{00} = (\ln \lambda\mu a^3)_{,0,0} + ((\ln a)_{,0})^2 + ((\ln \lambda a)_{,0})^2 + ((\ln \mu a)_{,0})^2 + 2\left(\frac{\lambda}{2\mu}\nu_{,0}\right)^2 \tag{5.34}$$

$$R_{11} = (\ln a)_{,0,0} + (\ln \lambda\mu a^3)_{,0}(\ln a)_{,0} - 2\left(\frac{\nu}{a}\right)^2 + \frac{1-[(1+\nu^2)\lambda^2-\mu^2]^2}{2(\lambda\mu a)^2} \tag{5.35}$$

$$R_{22} = (\ln \lambda a)_{,0,0} + (\ln \lambda\mu a^3)_{,0}(\ln \lambda a)_{,0} - 2\left(\frac{\lambda}{2\mu}\nu_{,0}\right)^2 + \frac{(1+\nu^2)^2\lambda^4-(\mu^2-1)^2}{2(\lambda\mu a)^2} \tag{5.36}$$

$$R_{33} = (\ln \mu a)_{,0,0} + (\ln \lambda\mu a^3)_{,0}(\ln \mu a)_{,0} + 2\left(\frac{\lambda}{2\mu}\nu_{,0}\right)^2 + \frac{\mu^4-[(1+\nu^2)\lambda^2-1]^2}{2(\lambda\mu a)^2} \tag{5.37}$$

$$R_{01} = \frac{\nu}{a}\left(\ln\frac{\mu}{\lambda}\right)_{,0} + \frac{(1+\nu^2)\lambda^2-\mu^2}{2\mu^2 a}\nu_{,0} \tag{5.38}$$

$$R_{23} = \left(\frac{\lambda}{2\mu}\nu_{,0}\right)_{,0} + (\ln \lambda^2 a^3)_{,0}\left(\frac{\lambda}{2\mu}\nu_{,0}\right) + \frac{(1+\nu^2)\lambda^2+\mu^2-1}{\lambda\mu a^2}\nu \tag{5.39}$$

$$R_{02} = R_{03} = R_{12} = R_{31} = 0 \tag{5.40}$$

$$G_{00} \overset{[\lambda=\mu]}{=} 3((\ln a)_{,0})^2 + 4(\ln\lambda)_{,0}(\ln a)_{,0} + ((\ln\lambda)_{,0})^2 - \frac{1}{4}(\nu_{,0})^2 + \frac{4\lambda^2+2\lambda^2\nu^2-4\lambda^4\nu^2-\lambda^4\nu^4-1}{4\lambda^4 a^2} \tag{5.41}$$

from $\nabla \cdot \mathbf{T} = 0$ ( $M, N$ are constants)

$$\lambda \mu a^3 \rho u^0 = M \tag{5.44}$$

$$u^1 a = N \tag{5.45}$$

Again, using the definition (3.45), one sees that the rotation vector is not zero

$$\boldsymbol{\Omega} = \frac{u^1}{2\lambda\mu a}\left(u^1 \boldsymbol{\omega}^1 - u^0 \boldsymbol{\omega}^0\right) \tag{5.46}$$

with magnitude

$$|\Omega| = \frac{|u^1|}{2\lambda\mu a} = \frac{|N|}{2\lambda\mu a^2} \tag{5.47}$$

The field equations (4.7), that is,

$$R_{\mu\nu} = 8\pi T_{\mu\nu} - 4\pi T$$

now show that $R_{22} = R_{33}$ , which, assuming from now on

$$\lambda = \mu \tag{5.48}$$

reads

$$\lambda a \nu_{,0} = -\nu\left[(2+\nu^2)\lambda^2 - 1\right]^{1/2} \tag{5.49}$$

Also, from $\quad R_{01} = -\rho u^0 u^1$

$$\lambda^2 a^3 \nu^2 \nu_{,0} = -2MN \tag{5.50}$$

Combining this with (59), one obtains

$$\nu^3\left[(2+\nu^2)\lambda^2 - 1\right]^{\frac{1}{2}} \lambda a^2 = 2MN \tag{5.51}$$

It can be checked now that $R_{23}$ given by (39) is identically zero.

Using (50) in the form (4.13), that is,

$$d\nu = -\frac{2MN}{\nu^2\lambda^2 a^3}\, dt \tag{5.52}$$

one can introduce $\nu$ as the independent variable in

$$(\ln \lambda^2 a^4)_{,0,0} + 6\left((\ln a)_{,0}\right)^2 + 6(\ln\lambda)_{,0}(\ln a)_{,0} + 2\left((\ln\lambda)_{,0}\right)^2 + \frac{1}{2}(\nu_{,0})^2 + \frac{1-4\nu^2\lambda^4-\nu^4\lambda^4}{2\lambda^4 a^2} = 0$$

which follows by subtracting the 00 from the 11 field equation. The result, with a prime denoting differentiation with respect to $\nu$, was found to be

$$(\ln \lambda^2 a^4)'' + \left(\ln \frac{2MN}{\nu^2\lambda^2 a^3}\right)'(\ln \lambda^2 a^4)' +$$

$$+ 6((\ln a)')^2 + 2((\ln \lambda)')^2 + 6(\ln \lambda)'(\ln a)' + \tag{5.53}$$

$$+ \frac{(1 - 4\nu^2\lambda^4 - \nu^4\lambda^4)\nu^4 a^4}{2(2MN)^2} + \frac{1}{2} = 0$$

Eliminating $a$ between (51), (53), a second order differential equation for $\lambda$ with respect to $\nu$ is derived. The result of this calculation was:

$$\left[(-4\nu^6 - 16\nu^4 - 16\nu^2)\lambda^5 + (4\nu^4 + 8\nu^2)\lambda^3\right] \lambda'' +$$

$$\left[(4\nu^6 + 16\nu^4 + 16\nu^2)\lambda^4 + \nu^2\right] (\lambda')^2 +$$

$$\left[(-8\nu^5 - 24\nu^3 - 16\nu)\lambda^5 + (12\nu^3 - 4\nu)\lambda^3 + 6\nu\lambda\right] \lambda' + \tag{5.54}$$

$$\left[(-2\nu^4 + 4\nu^2 + 36)\lambda^6 + (-\nu^4 - 4\nu^2 - 36)\lambda^4 + (2\nu^2 + 11)\lambda^2 - 1\right] = 0$$

One could, instead of (53), use the $G_{oo}$ equation to derive a first order differential equation which, however, was found to be more complicated than (54).

If $\lambda$ is known as a function of $\nu$, the rest of the variables $a, \rho, u^0, u'$ will follow from (43), (44), (45), (48) and (51). This completes the calculation formally, since $\nu$ can be considered the new time coordinate, with (52) expressing the corresponding transformation.

## 5.C THREE DUST-FILLED BIANCHI TYPE V MODELS

In a Bianchi type V spacetime ($\mathbf{d}\sigma^1 = 0$ , $\mathbf{d}\sigma^{2(3)} = \sigma^1 \wedge \sigma^{2(3)}$), the metric

$$\mathbf{g} = -\mathbf{d}t \otimes \mathbf{d}t + a^2 \Big( \sigma^1 \otimes \sigma^1 + \nu\, \sigma^1 \otimes \sigma^2 + \nu\, \sigma^2 \otimes \sigma^1 + (\lambda^2 + \nu^2)\, \sigma^2 \otimes \sigma^2 + \mu^2\, \sigma^3 \otimes \sigma^3 \Big) \tag{5.55}$$

assumes the diagonal form (4.2) for the basis

$$\begin{aligned} \omega^0 &= \mathbf{d}t & \omega^2 &= \lambda a\, \sigma^2 \\ \omega^1 &= a(\sigma^1 + \nu \sigma^2) & \omega^3 &= \mu a\, \sigma^3 \end{aligned} \tag{5.56}$$

with:

$$\left| \begin{aligned} \mathbf{d}\omega^0 &= 0 \\ \mathbf{d}\omega^1 &= \frac{a_{,0}}{a}\, \omega^0 \wedge \omega^1 + \frac{\nu_{,0}}{\lambda}\, \omega^0 \wedge \omega^2 + \frac{\nu}{\lambda a}\, \omega^1 \wedge \omega^2 \\ \mathbf{d}\omega^2 &= \frac{(\lambda a)_{,0}}{(\lambda a)}\, \omega^0 \wedge \omega^2 + \frac{1}{a}\, \omega^1 \wedge \omega^2 \\ \mathbf{d}\omega^3 &= \frac{(\mu a)_{,0}}{(\mu a)}\, \omega^0 \wedge \omega^3 + \frac{1}{a}\, \omega^1 \wedge \omega^3 + \frac{\nu}{\lambda a}\, \omega^3 \wedge \omega^2 \end{aligned} \right. \tag{5.57}$$

The (other than zero) rotation coefficients are

$$
\left|
\begin{array}{l}
\Gamma_{101} = \dfrac{a_{,0}}{a} \qquad \Gamma_{202} = \dfrac{(\lambda a)_{,0}}{\lambda a} \qquad \Gamma_{303} = \dfrac{(\mu a)_{,0}}{\mu a} \\
\Gamma_{212} = \Gamma_{313} = \dfrac{1}{a} \qquad \Gamma_{121} = \Gamma_{323} = -\dfrac{\nu}{\lambda a} \\
\Gamma_{102} = \Gamma_{201} = \Gamma_{210} = \dfrac{\nu_{,0}}{2\lambda}
\end{array}
\right.
\tag{5.58}
$$

The following components of the Riemann tensor are indicative of the anisotropy of the homogeneous hypersurfaces:

$$
\begin{aligned}
R_{1212} &= \frac{a_{,0}}{a}\frac{(\lambda a)_{,0}}{\lambda a} - \frac{1}{a^2} - \left(\frac{\nu_{,0}}{2\lambda}\right)^2 - \left(\frac{\nu}{\lambda a}\right)^2 \\
R_{3131} &= \frac{(\mu a)_{,0}}{\mu a}\frac{a_{,0}}{a} - \frac{1}{a^2} \\
R_{2323} &= \frac{(\lambda a)_{,0}}{\lambda a}\frac{(\mu a)_{,0}}{\mu a} - \frac{1}{a^2} - \left(\frac{\nu}{\lambda a}\right)^2
\end{aligned}
\tag{5.59}
$$

The components of the Ricci tensor are

$$
-R_{00} = (\ln \lambda\mu a^3)_{,0,0} + ((\ln a)_{,0})^2 + ((\ln \lambda a)_{,0})^2 + ((\ln \mu a)_{,0})^2 + 2(\nu_{,0}/2\lambda)^2 \tag{5.60}
$$

$$
R_{11} = (\ln a)_{,0,0} + (\ln \lambda\mu a^3)_{,0}(\ln a)_{,0} - \frac{2(\lambda^2+\nu^2)}{\lambda^2 a^2} \tag{5.61}
$$

$$
R_{22} = (\ln \lambda a)_{,0,0} + (\ln \lambda\mu a^3)_{,0}(\ln \lambda a)_{,0} + 2\left(\frac{\nu_{,0}}{2\lambda}\right)^2 - \frac{2(\lambda^2+\nu^2)}{\lambda^2 a^2} \tag{5.62}
$$

$$
R_{33} = (\ln \mu a)_{,0,0} + (\ln \lambda\mu a^3)_{,0}(\ln \mu a)_{,0} - \frac{2(\lambda^2+\nu^2)}{\lambda^2 a^2} \tag{5.63}
$$

$$
-R_{01} = \frac{1}{a}(\ln \lambda\mu)_{,0} + \frac{3\nu\nu_{,0}}{2\lambda^2 a} \tag{5.64}
$$

$$-R_{02} = \frac{\nu}{\lambda a}\left(\ln \frac{\lambda^2}{\mu}\right)_{,0} - \frac{3\,\nu_{,0}}{2\lambda a} \tag{5.65}$$

$$R_{12} = \left(\frac{\nu_{,0}}{2\lambda}\right)_{,0} + \left(\ln \mu a^3\right)_{,0}\frac{\nu_{,0}}{2\lambda} \tag{5.66}$$

$$R_{03} = R_{31} = R_{23} = 0 \tag{5.67}$$

The pressureless perfect fluid content of the space is described by the energy momentum tensor

$$8\pi \mathbf{T} = \rho\, \mathbf{u} \otimes \mathbf{u} \tag{5.68}$$

where,

$$\mathbf{u} = -u^0 \boldsymbol{\omega}^0 + u^1 \boldsymbol{\omega}^1 + u^2 \boldsymbol{\omega}^2 \qquad (\mathbf{u}\cdot\mathbf{u} = -1) \tag{5.69}$$

and the components of $\boldsymbol{\nabla}\cdot\mathbf{T} = 0$ (equ. (8)) are

$$\begin{aligned}
&\left(\rho(u^0)^2\right)_{,0} + \rho(u^1)^2\frac{a_{,0}}{a} + \rho(u^2)^2\frac{(\lambda a)_{,0}}{\lambda a} + \rho u^1 u^2\frac{\nu_{,0}}{\lambda} + \rho(u^0)^2 X = 0\\
&\left(\rho u^0 u^1\right)_{,0} + \rho u^0 u^1\frac{a_{,0}}{a} - \rho(u^2)^2\frac{1}{a} - \rho u^1 u^2\frac{\nu}{\lambda a} + \rho u^0 u^1 X = 0\\
&\left(\rho u^0 u^2\right)_{,0} + \rho u^0 u^2\frac{(\lambda a)_{,0}}{\lambda a} + \rho u^1 u^2\frac{1}{a} + \rho u^0 u^1\frac{\nu_{,0}}{\lambda} + \rho(u^1)^2\frac{\nu}{\lambda a} + \rho u^0 u^2 X = 0
\end{aligned} \tag{5.70}$$

where,

$$X \equiv \frac{(\lambda \mu a^3)_{,0}}{\lambda \mu a^3} + \frac{2u^1}{a u^0} - \frac{2\nu u^2}{\lambda a u^0} \tag{5.71}$$

According to (3.45), the rotation of the fluid is not identically zero:

$$2\Omega = u^1 u^2 \left( \frac{u^0}{a u^1} + \frac{\nu u^0}{\lambda a u^2} + \left(\ln \frac{u^1 u^2}{\lambda}\right)_{,0} + \frac{u^1}{\lambda u^2} \nu_{,0} \right) \omega^3 \qquad (5.72)$$

Search for solutions to the field equations (4.7) will be carried for the following three special cases, models $V_0$, $V_\varepsilon$, $V_1$.

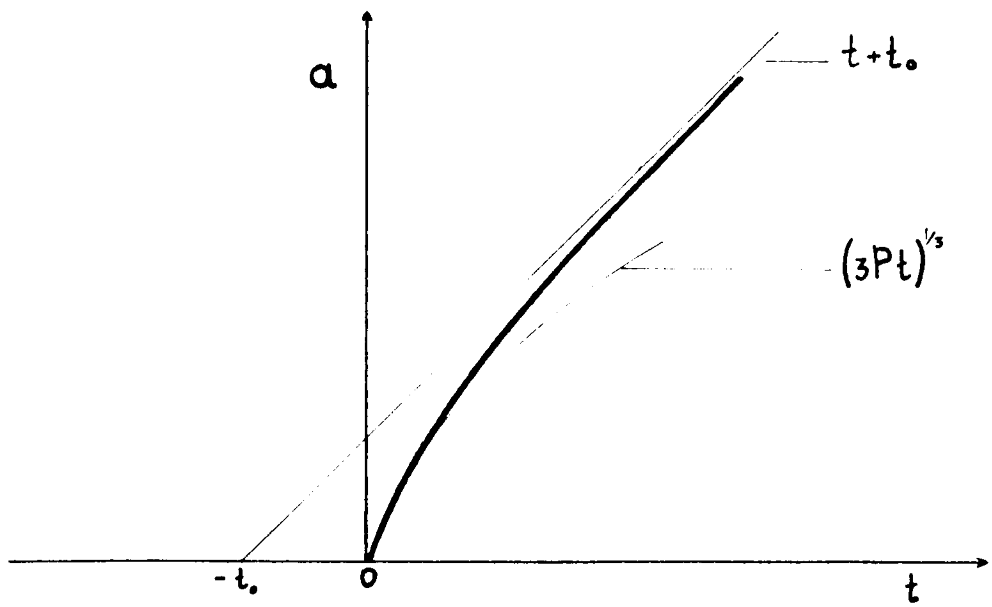
<u>MODEL $V_0$</u> represents a homogeneous, anisotropic, non-rotating universe, filled with dust, and it is obtained for $u^1 = u^2 = \nu = 0$. Then, $u^0 = 1$ and (70) gives ($M$ is a constant)

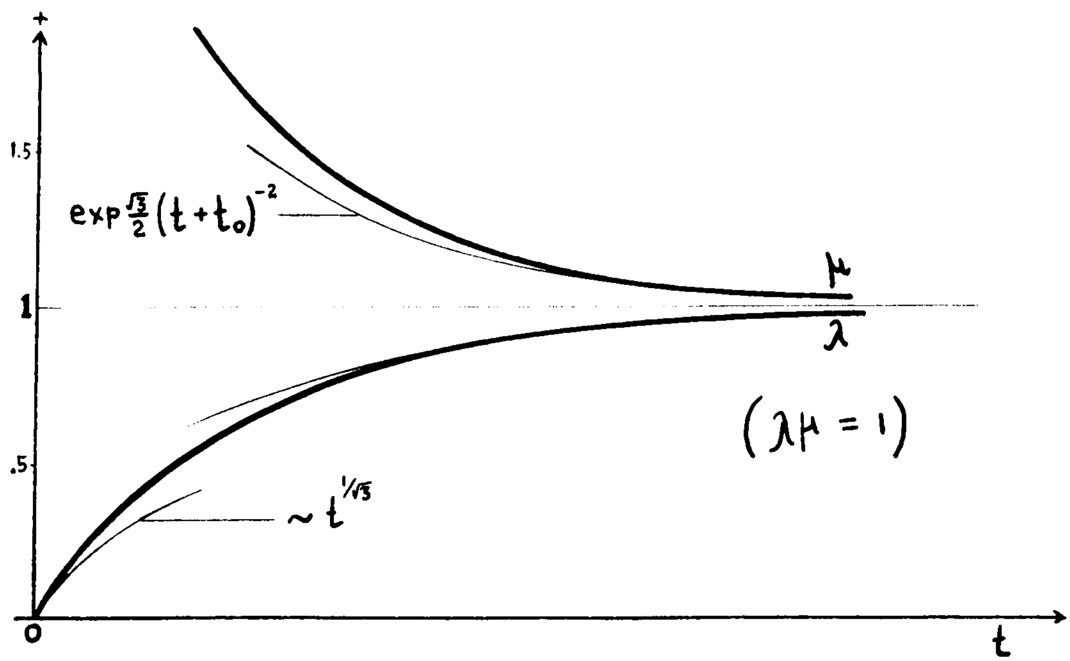
$$\rho_0 a_0^3 = 3M \qquad (5.73)$$

The index $o$ on the quantities $a, \lambda, \mu, \rho$ specializes them for the presently treated case. From $R_{22} = R_{33}$ and, using the condition $\lambda_0 \mu_0 = \text{const}$ which follows from $R_{01} = 0$, one obtains

$$-\frac{\mu_{0,0}}{\mu_0} = \frac{\lambda_{0,0}}{\lambda_0} = \frac{\sqrt{3}\,P}{a_0^3} \qquad (P = \text{const}) \qquad (5.74)$$

To find the time dependence of $a_0$, one can use the $G_{00}$ equation:




$$g = -dt \otimes dt + a^2(\sigma^1 \otimes \sigma^1 + \lambda^2 \sigma^2 \otimes \sigma^2 + \mu^2 \sigma^3 \otimes \sigma^3) \left[ d\sigma^1 = 0 \, , \, d\sigma^{2(3)} = \sigma^1 \wedge \sigma^{2(3)} \right]$$

FIGURES 5.1 and 5.2: $a$ and $\lambda, \mu$ vs. $t$ for Model $V_0$

$$\frac{a_{,0}}{a}\frac{(\lambda a)_{,0}}{\lambda a}+\frac{(\lambda a)_{,0}}{\lambda a}\frac{(\mu a)_{,0}}{\mu a}+\frac{(\mu a)_{,0}}{\mu a}\frac{a_{,0}}{a}-3\,\frac{\lambda^2+\nu^2}{\lambda^2 a^2}=\rho(u^0)^2 \qquad (5.75)$$

which gives

$$dt=[a_0^4+Ma_0^3+P^1]^{-\frac{1}{2}}a_0^2\,da. \qquad (5.76)$$

Figurative graphs for $\lambda_0, \mu_0, a_0$ as functions of time, when $\lambda_0\mu_0 = 1$ are given in figures 5.1 and 5.2; they show that the anisotropy damps out rapidly and the model ultimately develops very nearly like a Friedman universe with negative curvature (see Chapter 6).

MODEL $V_6$ is a perturbed state of the previous model so that a small rotation appears. To first order in the perturbation parameter $\varepsilon$, one has

$$\left|\begin{array}{l} \nu=\varepsilon\nu_1\\ \lambda=\lambda_0+\varepsilon\lambda_1\\ \mu=\mu_0+\varepsilon\mu_1\\ a=a_0+\varepsilon a_1 \end{array}\right. \qquad \left|\begin{array}{l} \rho=\rho_0+\varepsilon\rho_1\\ u^0=1\\ u^1=\varepsilon v^1\\ u^2=\varepsilon v^2 \end{array}\right. \qquad (5.77)$$

To first order in $\varepsilon$ one obtains from (71) after some algebra and two integrations ($C_1, C_2$ are constants):

$$v^1 a = C_1 \qquad (5.78)$$

$$v^2 a = C_2 \tag{5.79}$$

Field equations $R_{12} = \rho u^1 u^2$ and $R_{02} = -\rho u^0 u^2$ give

$$\left(3\frac{\lambda_{0,0}}{\lambda_0}\frac{\nu_1}{\lambda_0 a_0} - 3\frac{\nu_{1,0}}{2\lambda_0 a_0} - \rho_0 v^2\right)\varepsilon + O(\varepsilon^2) = 0$$

$$\left(\left(\frac{\nu_1}{2\lambda_0}\right)_{,0} + (\ln \mu_0 a_0^3)\frac{\nu_{1,0}}{2\lambda_0}\right)\varepsilon + O(\varepsilon^2) = 0$$

and they are both satisfied by the solution

$$\nu_1 = \frac{C_2 M}{\sqrt{3}\,P} - \frac{(\text{const.})}{\mu_0} \tag{5.80}$$

If now one requires that $\lambda \to 1$, $\mu \to 1$, $\nu_1 \to 0$ as $t \to \infty$ (80) gives

$$\nu_1 = \frac{C_2 M}{\sqrt{3}\,P}(1 - \lambda_0) \tag{5.81}$$

The magnitude of the rotation vector (72) is

$$|\Omega| = \frac{C_2 \varepsilon}{2\lambda_0 a_0^2} \tag{5.82}$$

and its direction along $\boldsymbol{\omega}^3$ .

Thus, a solution has been obtained for the lowest order of each quantity in (77).

<u>MODEL V$_1$.</u> An exact solution exists if $u^2 = 0$; (71) gives, in

this case, after some algebra and two integrations, ($C$, $M$ are constants)

$$u'a = C \tag{5.83}$$

$$u^0 \rho \lambda \mu a^3 = 3 M \nu^2 \tag{5.84}$$

$$\frac{\nu_{,0}}{\nu} = - C a^{-1} (a^2 + C^2)^{-\frac{1}{2}} \tag{5.85}$$

From $R_{12} = R_{02} = 0$, one has ($G$, $N$ are constants)

$$\frac{\mu a^3}{\lambda} \nu_{,0} = - \frac{GMC}{N} \tag{5.86}$$

$$\mu^2 \nu^3 = G^3 \lambda^4 \tag{5.87}$$

and using these expressions, one obtains from $R_{01} = -\rho u^0 u'$ after some algebra and one integration ($Q$ is a constant)

$$\lambda \mu = q^3 \tag{5.88}$$

where, $$q^2 = Q - G\nu - N\nu^2 \tag{5.89}$$

the quantities $\lambda, \mu, a$ (and, consequently $u^0, u', \rho$) can be given as functions of $\nu$, which is connected with $t$ through a transformation of the type (4.13), following from (86), (87), and (88):

$$dt = -\frac{NA^3}{MC}\left[1+\left[1+\frac{2C^2}{A^2}q^2\right]^{\frac{1}{2}}\right]^{\frac{3}{2}}\frac{d\nu}{\nu q^2} \tag{5.90}$$

The solution is

$$\lambda^2 = G^{-1}q^2\nu \tag{5.91}$$

$$\mu^2 = G\,q^4\nu^{-1} \tag{5.92}$$

$$a = A\left[1+\left(1+\frac{2C^2}{A^2}q^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}q^{-1} \tag{5.93}$$

where the constants $A, C, G, M, N, Q$ are subject to the restrictions

$$A^2 = \frac{M}{\sqrt{2}\,|N|} \qquad NC > 0$$
$$Q - G\nu - N\nu^2 \geqslant 0 \qquad G\nu > 0 \tag{5.94}$$

An interesting conclusion follows from (94): $\nu$ (and $\mu'$) cannot change sign throughout the entire evolution of this rotating universe, and this combined with (90) shows that $\nu$ is a monotonic function of $t$, constantly decreasing in absolute value. Further statements on this model, which seems to have some very peculiar characteristics, will be reserved until a more elaborate analysis has been completed.

## 5.D AN OBSERVABLE EFFECT OF THE COSMOLOGICAL ROTATION

In the same paper in which he described his rotating model, Gödel also stated (without proof) the following theorem (GODEL (1950)):

If $N_1$, $N_2$ are the numbers of galaxies in the two hemispheres into which a spatial sphere (i.e., one situated in the 3-space orthonogal to $\boldsymbol{u}$ at the point under consideration) of radius $r$, small compared with the world radius $R$, is decomposed by a plane orthogonal to $\boldsymbol{\Omega}$, then

$$\frac{N_1 - N_2}{N_1 + N_2} = \frac{9}{8} |\Omega| r R H \tag{5.95}$$

where $H$ is Hubble's constant.

This theorem essentially connects the "tilt" of a tilted homogeneous model (see, for example, KING and ELLIS (1973)) with the inhomogeneity an observer measures in the fluid's comoving rest space. For a tilted universe with non-zero vorticity, it also allows, in principle, an observational estimate of the magnitude of the rotation by means of galaxy counts.

A proof of the theorem is given below.

In the orthochronous frame naturally defined by the hypersurfaces of homogeneity, one has for the velocity $\mathbf{u}$ and the vector $(\mathbf{AO})$ connecting a given spatial position of the observer at $t = t_0$ and $t = t_0 + \delta t$ (Figure 5.3)

$$\mathbf{u} = -u^0 \boldsymbol{\omega}^0 + u^1 \boldsymbol{\omega}^1 , \qquad (\mathbf{AO}) = (\delta t)\boldsymbol{\omega}^0 \tag{5.96}$$

The position of the generic point $B$ in the fluid's comoving rest space (where the observation takes place) is represented by the spacelike vector $\boldsymbol{\xi}$. The requirement that $\boldsymbol{\xi} \cdot \mathbf{u} = 0$ shows that, with proper orientation of the orthonormal basis $\{\boldsymbol{\omega}^\ell\}$ and $\xi = |\boldsymbol{\xi} \cdot \boldsymbol{\xi}|^{\frac{1}{2}}$,

$$\boldsymbol{\xi} = -u^1 \xi \boldsymbol{\omega}^0 + u^0 \xi \boldsymbol{\omega}^1 \tag{5.97}$$

Noticing that the vector $(\mathbf{AB}) = (\mathbf{AO}) + \boldsymbol{\xi}$ , that is,

$$(\mathbf{AB}) = (\delta t - u^1 \xi)\boldsymbol{\omega}^0 + u^0 \xi \boldsymbol{\omega}^1$$

lies in the hypersurface of homogeneity one obtains

$$\delta t = u^1 \xi \tag{5.98}$$

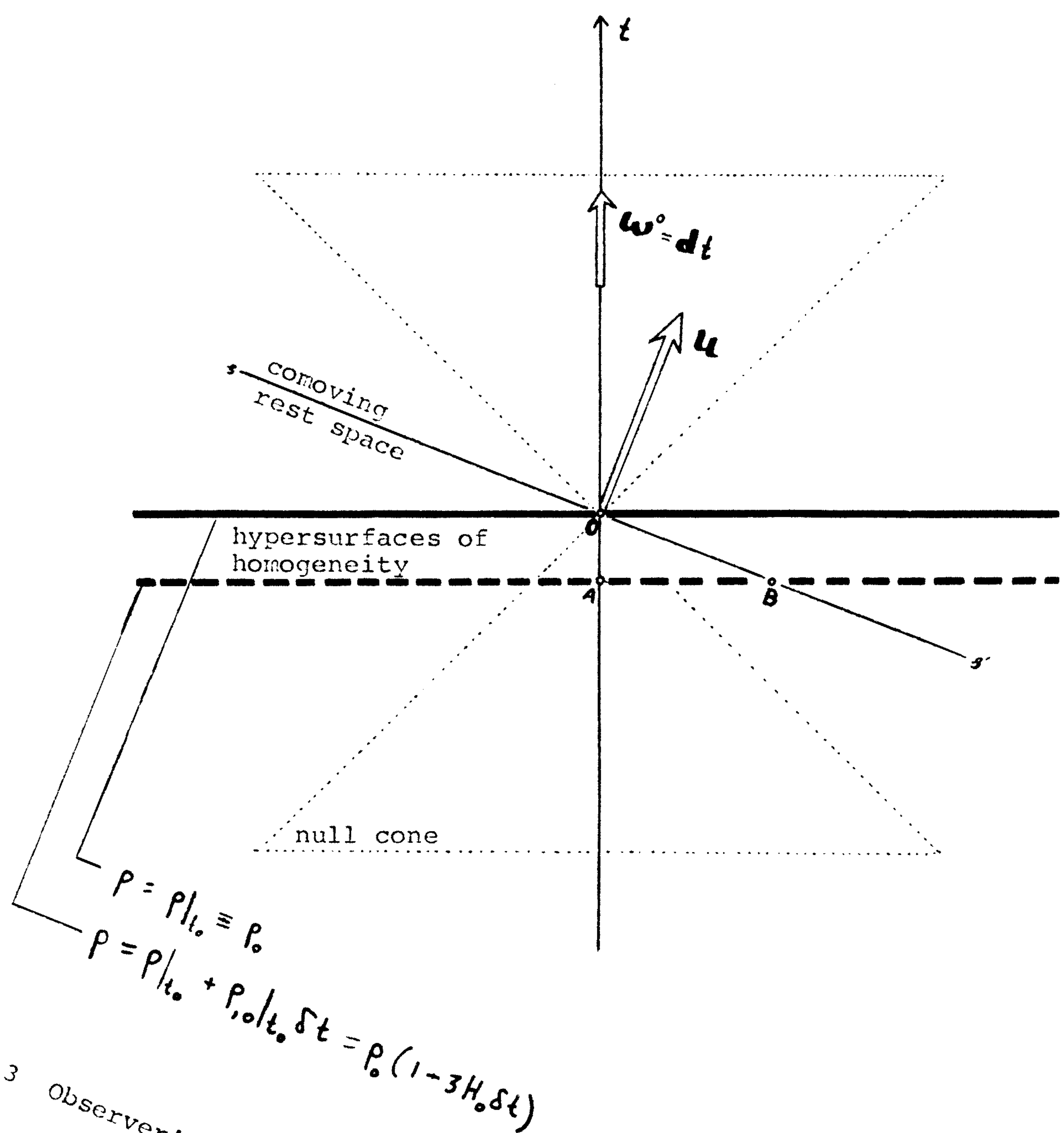



FIGURE 5.3 Observer's neighbourhood in a Tilted Spacetime

The density of the fluid at $B$ is

$$\rho = \rho|_0 + \rho_{,0}|_0 \, \delta t$$

which, using (98) and $\rho_{,0} = -3\frac{R_{,0}}{R}\rho = -3H\rho$ (see, for example equations (6.7), (6.15)), becomes

$$\rho = \rho_0 (1 - 3Hu'\xi)$$

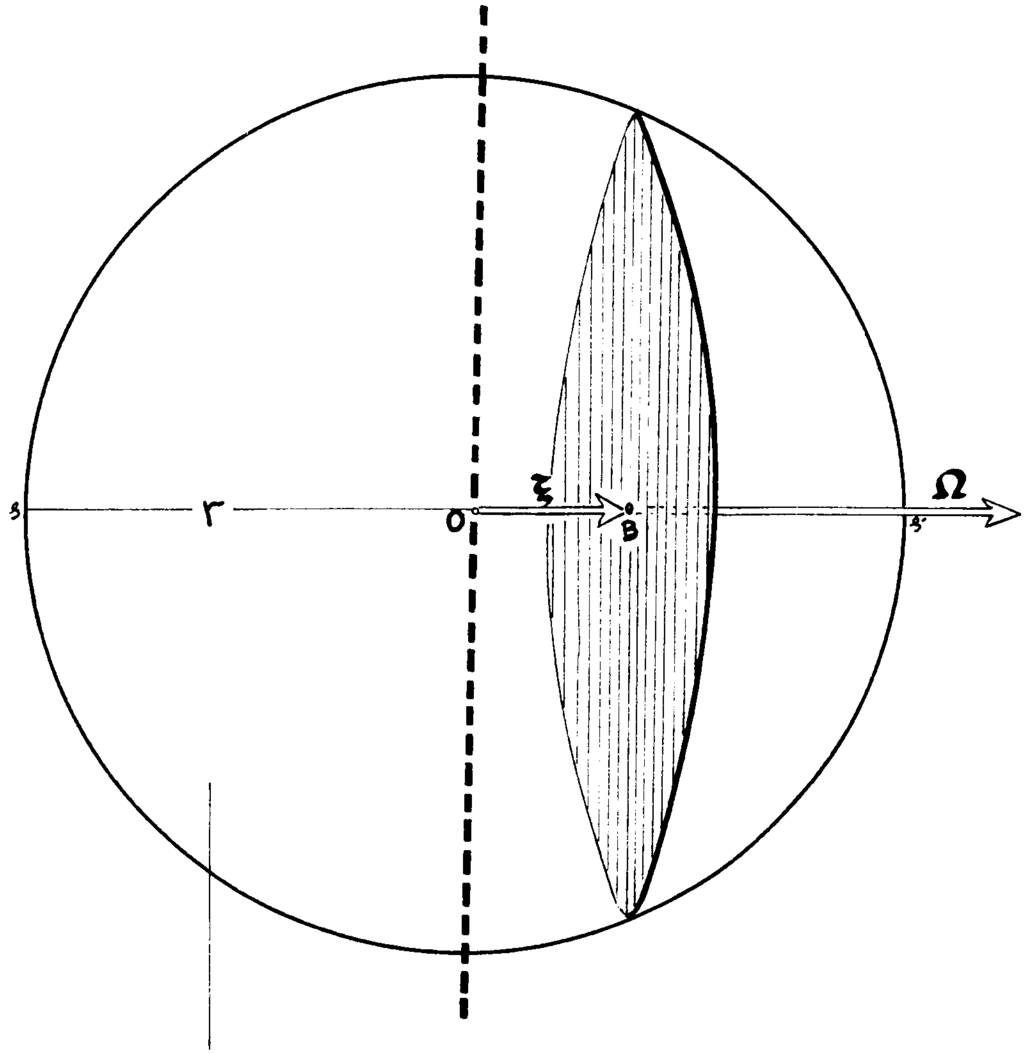
The point $B$ of Figure (5.3) represents actually a circle with radius $\sqrt{r^2-\xi^2}$ in the sphere of radius $r$ in which the measurement takes place; furthermore, this circle as illustrated in figure 5.4 is orthogonal to $\Omega$ because it is orthogonal to $\xi$, which is (anti)parallel to $\Omega$ (equations (97) and (46)). With volume element then $\pi(r^2-\xi^2)d\xi$ and corresponding "number of galaxies"

$$dN = \pi\rho_0 (r^2-\xi^2)(1-3Hu'\xi)d\xi$$

the integrals

$$N_1 = \int_{\xi=-r}^{\xi=0} dN \qquad\qquad N_2 = \int_{\xi=0}^{\xi=r} dN \qquad (5.99)$$

will represent the numbers of galaxies in the two




This half of the
sphere contains $N_1$
number of galaxies.
The other half
contains $N_2$.

FIGURE 5.4: Sphere in Observer's Rest Space
(Detail of FIG. 5.3)

hemispheres into which the plane orthogonal to $\boldsymbol{\Omega}$ (e.g., circle at $O$ , like the one at $B$ ) divides the sphere with radius $r$ . After the short calculation, one obtains

$$\frac{N_1 - N_2}{N_1 + N_2} = \frac{9}{8} u' r H \tag{5.100}$$

Equation (95) can now be established if one substitutes in (100) the equation $u' = |\Omega| R$ which follows from (47) with $\lambda \approx \mu \approx 1$ (small anistropies) and (6.5).

## 6. DISCUSSION AND CONCLUSIONS

Recalling the remarks made in Chapter I concerning the observed universe, it should be evident now that what is to be understood by "present stage of evolution" or "instant of time" is a (nearly) homogeneous and isotropic spacelike hypersurface, which for "present" includes the observer. If the metric has the form

$$\mathbf{g} = -\mathbf{d}t \otimes \mathbf{d}t + R^2 \gamma_{\ell m} \mathbf{d}x^\ell \otimes \mathbf{d}x^m \tag{6.1}$$

with $R = R(t)$, $\gamma_{\ell m} = \gamma_{\ell m}(x^n)$, the corresponding coordinates are called synchronous because the intrinsic geometry of the hypersurface ($\gamma_{\ell m}$) is "decoupled" from the time except for the overall "expansion parameter" $R$, which is constant on a given hypersurface. This name for $R$ (which is also called the radius of the universe if spherical coordinates are used for $x^\ell$ --see (3)) is given because the distance $\ell$ of any two points in the hypersurface at any given time $t$ is $\ell(t) = \ell(t_0) R(t)$, where $t_0$ a properly chosen "reference" instant. The intrinsic geometry of each hypersurface is characterized by constant curvature ($R$ is not to be confused with the curvature scalar(2.46))

$$^{(3)}R_{\ell mnr} = \frac{k}{R^2}\left(\gamma_{\ell n}\gamma_{mr} - \gamma_{\ell r}\gamma_{mn}\right) \qquad (6.2)$$

where $k$ has been normalized to take the values $-1, 0, +1$ characterizing respectively the open with negative curvature, flat and closed Friedman models. It should be noted here that zero (or negative) curvature does not necessarily imply an open model. The hypersurfaces of the closed Friedman model (which will be discussed as a prototype in the sequel) are 3-spheres and the line element in hyperpolar coordinates is

$$ds^2 = -(dt)^2 + R^2\left[(d\chi)^2 + \sin^2\chi(d\theta)^2 + \sin^2\chi\sin^2\theta(d\phi)^2\right] \qquad (6.3)$$

with the range of the coordinates $\phi, \theta, \chi$ being $[0, 2\pi]$, $[0, \pi]$ and $[0, \pi]$, respectively. Notice that the quantity $R$ in (3) is proportional to the expansion parameter $a$ in (5.30) when $\lambda = \mu = 1$, $\nu = 0$. One way to obtain the exact relation is to compare the volume of the 3-sphere which is $16\pi^2 a^3$ (see, for example, BATAKIS and COHEN (1972)) and also

$$V = R^3 \int_{\chi\theta\phi} \sin\chi \sin\theta \, d\chi \, d\theta \, d\phi = 2\pi^2 R^3 \qquad (6.4)$$

Hence,

$$R = 2a \tag{6.5}$$

It should also be noticed that in the density $\rho$ used in Chapter 5, a factor $8\pi$ was absorbed, therefore,

$$\rho = 8\pi \rho_D \tag{6.6}$$

where $\rho_D$ the density of a dust filled model, satisfying the equation (met with in all the models of Chapter 5)

$$\rho_D R^3 = \text{constant} \tag{6.7}$$

For a universe filled with thermal E-M radiation, the equation analogous to (7) is

$$\rho_{EM} R^4 = \text{const.} \quad , \quad T \sim R^{-1} \tag{6.8}$$

where the law $\rho_{EM} \sim T^4$ for the temperature of the radiation was also used.

The $G_{00}$ field equation

$$3\left(\frac{R_{,0}}{R}\right)^2 + \frac{3k}{R^2} = 8\pi T_{00} \tag{6.9}$$

now gives for:

DUST:
$$R = \frac{1}{2} R^{(D)}_{max} (1 - \cos\eta)$$
$$t = \frac{1}{2} R^{(D)}_{max} (\eta - \sin\eta) \tag{6.10}$$

RADIATION:
$$R = \frac{1}{2} R^{(EM)}_{max} \sin\eta \tag{6.11}$$
$$t = \frac{1}{2} R^{(EM)}_{max} (1 - \cos\eta)$$

where,

$$d\eta = R^{-1} dt \tag{6.12}$$

and the (constants) $R_{max}$, maximum radii of expansion, can be expressed in terms of quantities observed at any instant--e.g. today (index o):

DUST:
$$R^{(D)}_{max} = \frac{8\Pi}{3} R_o^3 \rho_o^{(D)} \tag{6.13}$$

RADIATION:
$$R^{(EM)}_{max} = \frac{8\Pi}{3} R_o^4 \rho_o^{(EM)} \tag{6.14}$$

The so-called "arc parameter time" $\eta$ is connected with the measurable Hubble "constant" (a function of time)

$$H = \frac{R_{,o}}{R} \tag{6.15}$$

or its inverse, the "Hubble time", by

DUST: $$H^{-1} = \frac{1}{2} R^{(D)}_{max} \frac{(1-\cos\eta)^2}{\sin\eta} \tag{6.16}$$

RADIATION: $$H^{-1} = \frac{1}{2} R^{(EM)}_{max} \frac{\sin^2\eta}{\cos\eta} \tag{6.17}$$

To obtain a formula for the red (or blue)-shift for a light ray connecting two observers (comoving with the fluid at the instants of emission and reception) one can, because of the spherical symmetry, write the line element as $-(dt)^2 + R^2 \sum(\sigma^i)^2$ where the $\sigma$'s are such that $\xi^\mu_\ell = \delta^\mu_\ell$ are the components of three killing vectors. Orienting the basis so that light travels along the 'one' direction, one obtains for the components of its momentum: $K_\alpha = (E, RE, 0, 0)$, where the fact $K_\alpha K^\alpha = 0$ was also considered. Using now the definition (3.5) and recalling the relevant remarks that followed equation (3.81), one obtains:

$$1 + z = \frac{[R^{-1} E R]_{em.}}{[R^{-1} E R]_{rec.}} = \frac{[R^{-1} (\mathbf{K}\cdot\xi_1)]_{em.}}{[R^{-1} (\mathbf{K}\cdot\xi_1)]_{rec.}} = \frac{[R^{-1}]_{em.}}{[R^{-1}]_{rec.}}$$

Hence, (6.18)

$$1 + z = \frac{\lambda_{rec.}}{\lambda_{em.}} = \frac{R_{rec.}}{R_{em.}}$$

that is, $z$ is related to the radius $R$ of the universe at the instant of emission of the photon.

Discussion of the events that took place during the first few seconds of the cosmic evolution ($t < \sim 10^{11} cm$) requires extreme extrapolations of the known physical laws. Consequently for $t$ smaller than, say, one second and down to a scale of $t \sim \sqrt{\hbar}$ ($\sim 10^{-33} cm \sim 10^{-43} sec$), present investigations are bound to be rather speculative and perhaps in inevitable need of theoretical developments.

After the universe is a few seconds old ($t \gtrsim 10^{10} cm$), all nucleon—antinucleon pairs recombine as pair production drastically decreases with dropping temperature ($T \lesssim 10^{10}$ °K), all hyperons and mesons decay and, with the density $\rho$ at about $10^{-23} cm^{-2}$, gravitational and neutrino radiation, in thermal equilibrium with matter until then, decouples and, subsequently, cools down according to (8). Such radiation fields are expected to exist today with their spectra enormously redshifted, of course, according to (18).

Premodial element formation occurred during the next phase, which lasts until $t \sim 10^{14} cm$. The high energy baryons slow down sufficiently during this time, to allow the creation of stable particles and nuclei like $He^4$. At this point, a distinct phase in the evolution has been reached: the universe is about 20 minutes old with energy density $\rho \sim 10^{-29} cm^{-2}$ in the form of a hot plasma (at $T \sim 10^9\,{}^\circ K$) consisting of 25% He 75% protons, electrons and photons, plus, of course, the decoupled graviational and neutrino radiations.

Subsequently, due to the prevailing thermal equilibrium, matter and E-M radiation cool down together in this radiation-dominated phase (heat capacity for E-M radiation much greater than for matter), until the temperature drops to $\sim 3000\,{}^\circ K$, at $t \approx 10^{23} cm$ with $\rho \sim 10^{-48} cm^{-2}$; at this point the universe assumes a new outlook: the photons no longer have enough energy to keep the hydrogen atoms ionized, recombination occurs, thermal radiation decouples from matter and cools down in the now matter dominated universe, according to (8); this radiation is the today observed "background", redshifted, according to (18), by a $Z \sim 1000$, indicating that $\frac{R_{TODAY}}{R_{RECOMBIN.}} \approx 1000$.

Shortly thereafter, at $t \approx 10^{26} cm$, with radiation pressure no longer affecting matter, condensation around possibly existing "peaks" in the mass-energy density is for the first time allowed (formation of stars, galaxies, etc.).

Given the present state of observational cosmology, the study of more complex models is necessary not for "fitting to the next decimal" of existing data, but rather for the investigation of fundamental open issues and some apparent difficulties of the Big-Bang theory.

Thus, to explain the observed homogeneity, answer questions concerning galaxy formation and study the conditions near the initial singularity, inhomogeneous models are being studied. Details on this and also on anisotropic models can be found, for example, in ZEL'DOVICH and NOVIKOV (1974). Similarly, besides the need to explain the observed isotropy (or, the possibly existing today small anisotropies), the study of anisotropic models seems fundamental for issues like particle creation or the initial singularity. A related example can be given. During a finite time $\Delta t$ and corresponding arc parameter change $\Delta\eta$, an event at ( $t_o$ , $\chi = o$ , $\theta$ , $\phi$ ) can be influenced by events that lie in the solid angle with hyperpolar coordinates at most equal to $\Delta\eta$. Thus, one sees that there is a "horizon" around any event such that events "beyond" the horizon (that is, events with $\chi > \Delta\eta$) could have no causal effect on it (the event) during $\Delta t$. If the universe was Friedman at all times, one has then to explain why the today observed background (thermal) radiation is isotropic, since every part of it, when it was last scattered at $z \sim 1000$ had been influenced by neighboring portions of the radiation differing by only a few degrees of hyper - polar angle $\chi$. The anisotropic mixmaster universe (MISNER (1969)) shows, among other things, how one can set the ground for dissolving such difficulties.

Although Relativistic Cosmology allows the existence of rotating models, it has been argued that such models possibly violate Mach's principle because rotation of the universe is defined with respect to its inertial frames (the Fermi-Walker propagating frames introduced in section 3.B), which, in turn, according to Mach's principle, are determined by the bulk mass content of the universe ("fixed stars"). This seems to be the case for Gödel's model (GODEL (1950)) and the above version of Mach's principle, but more realistic situations show that no contradiction exists. Thus the Brill-Cohen metric for a thin rotating shell demonstrates, among other issues, a dragging of the inertial frames inside the shell, in accordance with Mach's Principle. For this and examples on finite rotating objects, see COHEN (1967) and (1968), BRILL and COHEN (1966), COHEN and BRILL (1968). A comprehensive listing of models with rotation is given by KRASINSKI (1973).

Rotating models, besides deserving investigation on their own merit, will possibly have an important contribution to the resolution of issues such as the rotation of galaxies, apparent difficulties with conservation of energy in Relativistic Astrophysics, and one of the greatest riddles in Relativistic Cosmology, the Initial Singularity.

FOOTNOTES

1. A set $T$ of arbitrary elements called points is a topological space if to each subset $U \subset T$ there is associated a set $\bar{U}$ (the "closure" of $U$) such that (i) if $U$ contains only one point, then $\bar{U} = U$ (ii) if $U, V \subset T$ then $\overline{U \cup V} = \bar{U} \cup \bar{V}$ (iii) $\bar{\bar{U}} = \bar{U}$. A topological space $T$ can also be defined by giving a "base" (that is a "complete system of neighbourhoods") which is a collection $\{U\}$ of open subsets of $T$ such that every open subset of $T$ is the union of some members of the base and $\emptyset, T \in \{U\}$ A set is closed if $U = \bar{U}$ and it is open if $U \setminus T$ is closed. $T$ is a Hausdorff space if Hausdorff's axiom holds: for every pair of distinct points $a, b \in T$ there exists two distinct open sets, $U_a$, $U_b$ such that $a \in U_a$, $b \in U_b$, $U_a \cap U_b = \emptyset$

2. The definition of a manifold with boundary is obtained if in part (C) property (i) of the definition of a manifold without, "$R^n$" is replaced by "$\frac{1}{2} R^n$", the "lower half of $R^n$, obtainable from $R^n = \{(x^1, \dots, x^n), -\infty < x^i < +\infty, i = 1, \dots, n\}$ if the range of one of the $x^i$, say $X^1$ is restricted to $-\infty < x^1 \leq 0$.

3. An algebraic structure $\{B; \Omega, \times; \cdot, +, *, \dots\}$ is composed of: the "basic" non-empty set $B$; a set of "operators" $\Omega$ together with a map $\times : \Omega \times B \to B$ ("exterior operation"); and "internal operations", that is, maps $\cdot, +, *, \dots$ of $B \to B$ and/or $\Omega \to \Omega$ satisfying various rules of linearity, commutativity, distributivity, associativity, etc., which characterize the individual structure. Examples:

Set $B$ : $\{B; \emptyset; \emptyset\}$ (set of operators empty; no internal operations)

Group $G$ : $\{G; \emptyset; *\}$ ( $*$ is the usual group product)

Ring $r$ : $\{r; \emptyset; + \cdot\}$ ( $+$ is abelian group product; $\cdot$ is linear w.r.t. $+$ )

Field $F$ : $\{F; \emptyset; + *\}$ ( $*$ is group product linear w.r.t. the abelian group product $+$ )

Module $m$ over the ring $r$ : $\{m; r, \times; +, +, \cdot\}$ $\begin{pmatrix} + \text{ makes } m \text{ an abelian group} \\ +,\cdot \text{ make } r \text{ a ring} \\ \times \text{ is linear w.r.t. } + \text{ , etc.} \end{pmatrix}$

Vector space $V$ over the field $F$ : $\{V; F, \times; +, +, *\}$ $\begin{pmatrix} + \text{ makes } V \text{ an abelian group} \\ +,* \text{ make } F \text{ a field} \\ \times \text{ linear, etc.} \end{pmatrix}$

Algebra $A$ over the field $F$ : $\{A; F, \times; +, \cdot, +, *\}$ $\begin{pmatrix} +,\cdot \text{ make } A \text{ a ring} \\ +,* \text{ make } F \text{ a field} \\ \times \text{ linear, etc.} \end{pmatrix}$

An isomorphism $\phi$ between two algebraic structures is a one--one map which preserves their algebraic structure (that is the internal and external operation). Example: $\phi(a * b) = (\phi a) * (\phi b)$

4. If they were not linearly independent, then $v^\alpha \partial_\alpha = 0$ with at least one $v^\alpha \neq 0$ ; but choosing $x^\alpha$ as $f$ one has from the definition of tangent vectors that $v^\beta \partial_\beta x^\alpha = 0$ , or $v^\alpha = 0$ .

5. The dual of $T_p^*$ is isomorphic to $T_p$ and one can therefore identify $(T_p{}^*)^*$ with $T_p$ .

6. Upper indices are also called contravariant, lower covariant.

7. The topological space $M$ is compact if from every finite covering of $M$ by open sets it is possible to select a finite covering. A subset $U \subset M$ is compact if it is compact considered as a topological space. Manifolds are reasonable models of spacetime if they are noncompact because then a line element field is admissible. See, for example, HAWKING and ELLIS (1973).

8. Not to be confused with the "invariant subgroup" of $G$ corresponding to an ideal in $g$ .

9. A similar definition is not possible for $L$ because for Lie derivatives, one has (unlike (2.31))

$$L_v e_\mu = - v^\lambda{}_{,\mu} e_\lambda + C_{\alpha\mu}{}^\lambda v^\alpha e_\lambda$$

10. Although $\nabla G(\ ;u)$ is the same as $\nabla_u G$ , (2.31) demonstrates that $\nabla^2 G(\ ;u;v)$ is not the same as $\nabla_v \nabla_u G$ (apart from the difference of the latter from $\nabla_u \nabla_v G$ ).

11. In the normal coordinates introduced, the geodesic from $P$ for a given vector $v = v^\alpha \partial_\alpha$ is given by $x^\alpha = \lambda v^\alpha$ with $\lambda$ the affine parameter.
$M$ is geodesically complete if exp is defined for any vector in $T_p$ for every $p \in M$ .

12. This choice for the position of the skew indices is only made for convenience in the notation. They could be at the beginning or even scattereed among the other covariant indices.

13. The (coordinate independent) integral of an n-form $\sigma$ over an n-dimensional oriented submanifold $\Omega$ of $M$ is defined as $\int_\Omega \sigma = \int_{\phi(\Omega)} D\,dx^1 \cdots dx^n$ --an integral in $R^n$, where $\phi$ is a map coordinatizing $\Omega$ (with positive orientation) and $\sigma = D\,dx^1 \wedge \cdots \wedge dx^n$ . The quantity $D$ is called a scalar density of weight 1 and transforms, of course, like the (only) components of an n-form: $D = D' \,|J|$ where $|J|$ the Jacobian of the coordinate transformation $x \to x'$ .

14. As follows from the Lagrangian $L = -\frac{1}{8\pi} F_{\alpha\beta} F^{\alpha\beta}$

15. Thus, they generate on their own a group of transformations on $S(t)$ , the so called reciprocal group of $G_3$ .

16. Because the killing vectors form an algebra over the reals, if $\xi$ is a killing vector, $f\xi$ with $f \in F$ will also be a killing vector only if, in general, $f$ is a constant.

## REFERENCES


Anderson, J. L. "Principles of Relativity Physics" Acc. Press New York (1967)

Batakis, N. and Cohen, J.M. "Closed Anisotropic Cosmological Models" Ann. Physics 73 578 (1972)

Brill, D.R. and Cohen, J.M. "Cartan Frames and the General Relativisitic Dirac Equation" J. Math Phys. 7 238 (1965)

Brill, D.R. "Electromagnetic Fields in a Homogeneous Non-isotropic Universe" Phys. Rev. B 133 845 (1964)

Brill, D.R. and Cohen, J.M. "Rotating Masses and their Effect on Inertial Frames" Phys. Rev. B 143 1011 (1966)

Cohen, J.M. in "Relativity Theory and Astrophysics, I. Relativity and Cosmology" ed. by J. Ehlers, A.M.S. Providence, R. I. (1967)

Cohen, J. M. "Friedman Cosmological Models with both Radiation and Matter" Nature 216 249 (1967)

Cohen, J.M. "Angular Momentum and the Kerr Metric" J. Math. Phys. 9 905 (1968)

Cohen, J.M. "Gravitational Collapse of Rotating Bodies" Phys. Rev. 173 1258 (1968)

Cohen, J. M. "Exploding Spheres of Dust" Intern J. Th. Phys. 1 361 (1968)

Cohen, J.M. and Brill, D.R. "Further Examples of 'Machian' Effects of Rotating Bodies in General Relativity" Nuovo Cim. 56B 290 (1968)

Dicke, R.H. "The Theoretical Significance of Experimental Relativity" Gordon and Breach, New YORK (1964)

Einstein, A. "Die Feldgleichungen der Gravitation" Preuss Akad. Wiss. Berlin, Sitzber p.p. 844-847 (1915)

Einstein, A. "The Meaning of Relativity" Princeton Un. Press, Princeton, N.J., (1950) Ellis, G.F.R. "The Geometry of Relativity" lecture notes, Dept. of Appl. Math and Th. Phys., Cambridge (1965)

Ellis, G.F.R. and MacCallum, M.A.H. "A Class of Homogeneous Cosmological Models" Commun. Math Phys. 12 108 (1969)

Gödel, K. "Rotating Universes in General Relativity Theory" Proc. Int. Cong. Math. 1 175 (1950)

Hawking, C.W. and Ellis G.F.R. "The Large Scale Structure of Spacetime" Cambridge Univ. Press, Cambridge (1973)

Helgason, S. "Differential Geometry and Symmetric Spaces" Acc. Press, New York (1962)

King, A.R. and Ellis, G.F.R. "Tilted Homogeneous Cosmological Models" Commun. Math Phys. 31 209 (1973)

Kobayashi, S. and Nomizu, K. "Foundations of Differential Geometry" Interscience, New York (1963)

Krasinski, A. "Some Solutions of the Einstein Field Equations for a Rotating Perfect Fluid" Preprint No. 29, Polish Acad. of Sciences (inst. of Astr.), Warsaw (1973). Also to appear in the J. Math Phys.

Kreuzer, L.B. "Experimental Measurement of the Equivalence of Active and Passive Gravitational Mass" Phys. Rev. 169 1007 (1968)

Mach, E. "Die Mechanik in Ihrer Entwicklung Historisch Kritisch Dargestellt" Brockhaus, Leipzig (1912). English translation by T. J. McCormack as "The Science of Mechanics" Open Court, La Salle Ill. (1960).

Misner, C.W. "The Flatter Regions of Newman, Unti and Tamburino's Generalized Schwarzschild Space" J. Math. Phys. 4 924 (1963)

Misner, C.W. "Mixmaster Universe" Phys. Rev. Lett. 22 1071 (1969)

Misner, C.W., Thorne, K.S., Wheeler, J.A. "Gravitation" Freeman, San Francisco (1973)

Newman, E.T., Tamburino, L. and Unti, T. "Empty Space Generalization of the Schwarzschild Metric" J. Math. Phys. 4 915 (1963)

Peebles, P.J.E. "Physical Cosmology" Princeton Univ. Press Princeton, N.J. (1971)

Taub, A.H. "Empty Space-Times Admitting a Three Parameter Group of Motions" Ann. Math. 53 472 (1951)

Trautman, A., Pirani, F.A.E. and Bondi, H. "Lectures on General Relativity" 1964 Brandei's Summer Lectures, Prentice Hall, Englewood Cliffs, N.J. (1965)

Witten, L., editor, "Gravitation: An Introduction to Current Research" Wiley, New York (1962)

Zel'Dovich, Ya.B. and Novikov, I.D. in "Relativistic Astrophysics" Vol. II "The Universe and Relativity" Univ. of Chicago Press, Chicago, Ill. (1974)