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STACKS, CO-STACKS AND AXIOMATIC HOMOLOGY THEORY

by

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INTRODUCTION

The word "stack" was first used by Spanier and Whitehead in [10] as a translation of "faiceau" (sheaf). There they introduced the notions of covariant stacks and contravariant stacks to provide local coefficients for homology and co-homology of CW-complexes in order to study the S-theory. Later, Spanier [9] modified the definition of a stack as a functor from an abstract simplicial complex, regarded as a category, to the category of groups to study the generalized theory of higher order operations. There the meaning of stacks was changed from "topological sheaves" to "simplicial sheaves" and they worked very well. This thesis constitutes an attempt to explain why they are there and where they belong.

First, we generalize the notion of covariant stacks and contravariant stacks as follows: Consider a simplicial set (i.e. a semi-simplicial complex) as a category by regarding its simplexes as objects of the category and the incidence maps restricted to simplexes as morphisms. Call this a simplicial category, then a prestack is defined as a contravariant functor on a simplicial category and a co-prestack as a covariant functor. Stacks and co-stacks are those functors which transform the restrictions of degeneracy operators to isomorphisms. In this way, we obtain "combinatorial sheaves and co-sheaves".

Then we develop the theory in a way which is in some sense parallel to the theory of sheaves. It turns out that stacks and co-stacks belong to a homology theory on a generalized category of the category of pairs of simplicial sets, namely the category of K-pairs (simplicial pairs over a fixed simplicial set K). They provide coefficients for the theory.

It will be proved that the theory (called homology theory over K) to which they belong is unique. This explains why they worked so well. The following words are copied from Eilenberg and Steenrod [2, pp. x–xi] to support this point of view.

> The great gain of an axiomatic treatment lies in the simplification obtained in proofs of theorems. Proofs based directly on the axioms are usually simple and conceptual. It is no longer necessary for a proof to be burdened with the heavy machinery used to define the homology[1]. Nor is one faced at the end of a proof by the question, does the proof still hold if another homology theory replaces the one used? When a homology theory has been shown to satisfy the axioms, the machinery of its construction may be dropped.

The construction of the homology theory over K is also stimulated by the following question: Associated to a category $\mathcal{C}$ and a fixed object K in $\mathcal{C}$ there is a category $\mathcal{C}_K$ of objects over K . If K is a terminal object of $\mathcal{C}$, then $\mathcal{C}_K$ is just $\mathcal{C}$. Suppose that there is a homology theory on $\mathcal{C}$. Can this theory be generalized to a theory on $\mathcal{C}_K$? The answer of this question might be generally "yes". At least, it is true when $\mathcal{C}$ is the category of simplicial pairs.

This thesis consists of five chapters. Chapter I introduces categorical machineries and the notion of simplicial categories. The main body of the treatment begins in chapter II, where the notion of

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(1) The word group in the original statement is omitted because the homology over K has more general values.

co-prestacks, co-stacks, relative co-stacks, and their duals are introduced. A relative co-stack on a simplicial set X modulo a simplicial subset X' with values in a co-complete abelian category $Ⓐ$ is a co-stack $p_o A$ with supports in $X-X'$.

To each simplicial map $f:X \to Y$, regarding as a functor between simplicial categories, construct a pair of adjoint functors $f_{\#} \dashv f^{\#}:Ⓐ^X \to Ⓐ^Y$ between the category of co-prestacks on X and the category of co-prestacks on Y both with values in $Ⓐ$. The pull-back functor $f^{\#}$ is defined by composition with the functor f .

These functors play the major technical role of the development. In particular, if $x^{\delta}:\Delta^n \to X$ denotes the unique simplicial map determined by the correspondence $\delta^n \to x$, $x \in X_n$, the functor $x_{\#}$ and $x^{\#}$ are used to supply projectives and injectives.

If $Ⓐ$ is a co-complete AB4 abelian category with a fixed projective generator P , then both $f_{\#}$ and $f^{\#}$ are exact, the co-stack P^n on Δ^n with constant value P is a projective generator of $Ⓐ^{\Delta^n}$, and $Ⓐ^X$ is a co-complete AB4 abelian category with a projective generator $U = \amalg_{x \in X} x_{\#}P^n$. Thus we can do homological algebra by projective resolutions.

Some other major results of chapter II are summarized as follows: The category $ⓃⒶ^X$ of co-stacks is a reflective and co-reflective Serre subcategory of $Ⓐ^X$. The reflector N_* is exact; the co-reflector N^* is co-kernel preserving (i.e. right exact). Let $i:X' \to X$ be the inclusion map of the simplicial subset X' of X , then $i^{\#}i_{\#} = 1$ and $Ⓐ^{X'}$ is isomorphic to a reflective Serre subcategory of $Ⓐ^X$. The reflector is exact. The category $Ⓐ^{(X,X')}$

of relative co-prestacks is a co-reflective Serre subcategory of $Ⓐ^X$. The co-reflector p_o is exact. If A is a co-stack in $Ⓐ^X$, i.e. $A \in (NA)^X$, then p_oA and all $f^{\#}A$ are co-stacks.

Chapters III and IV concern the homology theeory. The value category Ⓐ used is a co-complete AB4 abelian category with a fixed projective generator P as has been mentioned before. The homology and co-homology are discussed simultaneously. Precisely, the co-homology part is completely omitted because of the duality.

For each $A \in Ⓐ^X$, defined a chain complex $\underset{\sim}{C}A \in dⒶ$ by letting $(\underset{\sim}{C}A)_q = \amalg_{x \in X_q} Ax$ with the obvious derivation. Then $\underset{\sim}{C}:Ⓐ^X \to dⒶ$ is an exact additive functor. A simplicial map $f:X \to Y$ induces a chain map $\underset{\sim}{C}f:\underset{\sim}{C}(f^{\#}B) \to \underset{\sim}{C}B$, $B \in Ⓐ^Y$. The homology of a K-pair $(X,X')_\varphi$ with coefficients in a co-stack $A \in (NA)^K$ is defined by chain homology as $\widetilde{H}_*(X,X)_\varphi = \overline{H}\underset{\sim}{C}(p_o\varphi^{\#}A)$. It is proved in chapter III that the homology can be computed by the generalized torsion functor Tor . When Ⓐ is a category of modules, a few overlaps occur here with [6].

In chapter III we also prove a set of propositions which are stated as axioms for homology theory in the beginning of chapter IV.

The exactness, excision, and (strong) additivity axioms are stated and verified as usual. The dimension axiom is quite different from the usual one because we need it for constructing the local coefficients of a given homology theory. The homotopy axiom is stronger than the usual one. It is verified in an unexpected way. The strong form is demanded for the proof of the uniqueness of $\widetilde{H}_*$ in the infinite dimensional case.

For proving the uniqueness theorem, Milnor's proof in [8] does not apply directly to the semi-simplicial case. However, after modifying the homotopy axiom and adding a deformation axiom, his construction works well.

Chapter V may be regarded as a direct application of the theory of stacks and co-stacks. The value category is $\textcircled{Ab}$, the category of abelian groups. The integer group Z plays the role of P . The theorems of chapter IV state that there is a unique homology theory $\widetilde{H}_*$ on the category of simplicial pairs $\textcircled{C}'$ with local coefficients in groups. This is a direct generalization of the usual semi-simplicial homology theory.

The notion of K-modules is introduced. The category of K-modules, $\textcircled{M}_K$, is identified as $\textcircled{NAb}^K$ under the isomorphism between the category $\textcircled{Ab}^K$ and the category of groups over K , $\textcircled{Ab}_K$.

In the last section, we show that when K is finite, the K-module $U^* = N^*(\amalg_{\sigma \in NK} \sigma_\# Z^n)$ is a small projective generator of $\textcircled{M}_K$. Thus $\textcircled{M}_K$ is equivalent to $\textcircled{M}_R$, the category of right R-modules, where R is the ring of endomorphisms of U^* . The structure of the ring R is exhibitted for the case that K is a finite simplicial complex.

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CHAPTER I. PRELIMINARIES

1. CATEGORICAL PRELIMINARIES

Let Ⓒ be a category. The set of morphisms $f:A \to B$ of objects A and B in Ⓒ is denoted by Ⓒ(A,B). The dual category is denoted by $Ⓒ^{o}$.

A category is complete (co-complete) if every pair of morphisms has a difference kernel (co-kernel) and every indexed set of objects a product (co-product). Hence, an abelian category is complete (co-complete) if and only if it is closed under the formation of products (co-products).

An abelian category Ⓐ is called a Grothendieck category if it is co-complete, and the co-limit (i.e. right limit) of a collection of short exact sequences indexed by a directed set is a short exact sequence. This last property is referred to as AB5 in [4]. The following theorem is proved in [4].

Proposition 1.1. A Grothendieck category possessing a generator has enough injectives.

The dual notion of a Grothendieck category is that of a co-Grothendieck category, e.g. the category Ⓐⓑ of abelian groups is Grothendieck, its dual category $Ⓐⓑ^{o}$ of compact abelian groups is co-Grothedieck. The integer group Z is a projective generator of Ⓐⓑ , its dual, the circle group, is an injective co-generator of $Ⓐⓑ^{o}$.

A category is <u>well-powered</u> <u>(co-well-powered)</u> if the family of subobjects (quotient objects) of any object is a set.

<u>Proposition 1.2.</u> (Freyd [3]) An abelian category possessing a generator is well-powered and co-well-powered.

<u>Proof</u>. Let Ⓐ be an abelian category with a generator G . If A is an object in Ⓐ , then a subobject $A' \to A$ is distinguished by the subset Ⓐ (G,A') of the set Ⓐ (G,A) [3, p. 69], and then Ⓐ is well-powered.

Now, for every quotient $A \to Q$ of A , there corresponds a unique kernel $K \to A$ which is a subobject of A , and non-isomorphic quotients determine non-isomorphic kernels. Thus the cardinality of quotients is no more than that of subobjects. Ⓐ is co-well-powered. Q.E.D.

Let Ⓐ be a subcategory of Ⓑ, Ⓐ ⊂ Ⓑ . Ⓐ is a <u>replete subcategory</u> in Ⓑ if for every B in Ⓑ isomorphic to an object A in Ⓐ it is the case that B in Ⓐ .

Given B in Ⓑ , an object B' in Ⓐ is a reflection of B in Ⓐ if there is a morphism $B' \to B$ of Ⓑ such that for any A in Ⓐ and morphism $A \to B$ of Ⓑ , there is a unique morphism $A \to B'$ of Ⓐ which makes the diagram

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow & \\ B' & \longrightarrow & B \end{array}$$

commutative.

The notion of <u>co-reflection</u> is defined dually. If every object in Ⓑ has a reflection (co-reflection) in Ⓐ , we say that Ⓐ is a reflective (co-reflective) subcategory of Ⓑ . The functor N_* which assigns to each object B in Ⓑ a reflection in Ⓐ is called a reflector. A co-reflector N^* is defined dually.

Note. We are interchanging the terms of reflectivity and coreflectivity defined in [3]. The reason is: When a certain object is characterized by a universal mapping property with the unique morphism taking the object as its domain, we call this object a "co-something." e.g. a co-product (formerly called a sum) of a set of objects A_α in a category Ⓐ is an object $\amalg_\alpha A_\alpha$ in Ⓐ together with a set of morphisms $inj_\alpha : A_\alpha \to \amalg_\alpha A_\alpha$, called the system of injections for $\amalg_\alpha A_\alpha$, such that for any object A in Ⓐ and set of morphisms $f_\alpha : A_\alpha \to A$ there is a unique morphism g of Ⓐ 'defined on' $\amalg_\alpha A_\alpha$ such that the diagrams

(1.1)
$$\begin{array}{ccc} A_\alpha & \xrightarrow{inj_\alpha} & \amalg_\alpha A_\alpha \\ & {\scriptstyle f_\alpha}\searrow & \downarrow{\scriptstyle g} \\ & & A \end{array}$$
commute.

Proposition 1.3. Reflections (co-reflections) are unique up to isomorphism.

This follows immediately from the universal mapping property defining them.

Proposition 1.4. Let J : Ⓐ → Ⓑ be the inclusion functor. Ⓐ is a reflective subcategory if and only if J has an adjoint (i.e. right adjoint), and is a co-reflective subcategory if and only if J has a co-adjoint (i.e. left adjoint). The adjoint (co-adjoint) is unique up to isomorphism. It is "the" reflector (co-reflector). The reflector preserves limits (i.e. left limits), and the co-reflector preserves co-limits (i.e. right limits).

Again, these are direct consequences of their definitions.

Proposition 1.5. Let Ⓑ be a complete, co-complete, well-powered, and co-well-powered category and Ⓐ a full subcategory replete in Ⓑ such that Ⓐ is closed under the formation of products and subobjects. Then Ⓐ is a co-reflective subcategory of Ⓑ .

This is a consequence of Freyd's adjoint functor theorem [3, pp. 84-87].

Proposition 1.6. If a functor $F : Ⓐ \to Ⓑ$ has an exact adjoint (co-adjoint), it preserves projectives (injectives).

Proof. Let $G : Ⓑ \to Ⓐ$ be an exact functor which is the adjoint of F , and let $0 \to B' \to B \to B'' \to 0$ be exact in Ⓑ . Then

$$0 \to GB' \to GB \to GB'' \to 0$$

is exact in Ⓐ . If P is a projective in Ⓐ , then

$$0 \to Ⓐ(P,GB') \to Ⓐ(P,GB) \to Ⓐ(P,GB'') \to 0$$

is exact. By adjointness, we have an exact sequence

$$0 \to Ⓑ(FP,B') \to Ⓑ(FP,B) \to Ⓑ(FP,B'') \to 0$$

Hence, FP is a projective in Ⓑ .

Corollary 1.7. Let Ⓐ be an exact co-reflective (reflective) subcategory of an abelian category Ⓑ , then the co-reflector N^* (reflector N_*) preserves projectives (injectives).

We conclude this section with the notion of a Serre subcategory. Let Ⓑ be an abelian category. A full subcategory Ⓐ of Ⓑ is a Serre subcategory if whenever $A' \to A \to A''$ is exact in Ⓑ and A' , A'' are in Ⓐ , it is the case that A is also in Ⓐ. i.e. Ⓐ is an _épaisse_ subcategory in the sense of Grothendieck [4]. Equivalently, a full subcategory Ⓐ ⊂ Ⓑ is a Serre subcategory of Ⓑ if and only if Ⓐ contains the zero object and the objects which are isomorphic to subobjects or quotient objects of an object A in Ⓐ, and also, every extension of two objects A' and A'' in Ⓐ is again in Ⓐ (see [4]).

2. SIMPLICIAL CATEGORIES

Let $X = \{X_q\}$ be a simplicial set (i.e. semi-simplicial complex) with simplexes denoted by $x, x', \ldots,$ and incidence maps by $\mu, \ldots$. X can be regarded as a category, denoted by Ⓧ, with objects all simplexes and morphisms $\mu_x : x \to x'$ for all incidence maps μ such that $\mu x = x'$. This is a small category called the _simplicial category determined by_ X .

If $f : X \to Y$ is a simplicial map, then f commutes with incidence maps and therefore determines a functor $f :$ Ⓧ → Ⓨ. This functor is called a _simplicial functor_. If Ⓒ denotes the category of simplicial sets with morphisms all simplicial maps, then corresponding to it, we have a category of small categories Ⓧ, Ⓨ, ... with morphisms all simplicial functors.

Recall that if Δ^n denotes the simplicial analogue of the standard affine n-simplex, then Δ^n , $n = 0,1,2,\ldots$, are distinguished in Ⓒ by the fact that Δ^n has a unique non-degenerate n-simplex δ^n which has the property that every q-simplex σ of Δ^n can be written uniquely as $\sigma = \sigma^* \delta^n$, where σ^* is the incidence map determined by σ (see, for example, [7, p. 238]).

If x is an n-simplex of X, then there is a unique simplicial map $x^{\delta} : \Delta^{n} \to X$ such that $x^{\delta}(\delta^{n}) = x$. There is a bijection

$$(2.1) \qquad \textcircled{C}(\Delta^{n}, X) \to X_{n}$$

defined by $x^{\delta} \to x$, where $x \in X_{n}$.

In categorical terms, for every $x \in \textcircled{X}$ corresponding to an n-simplex of X, there is a unique simplicial functor $x^{\delta} : \Delta^{n} \to \textcircled{X}$ such that $x^{\delta}(\delta^{n}) = x$. This fact plays an essential role in the next chapter.

For notational convenience, we shall write I for Δ^{1} to stand for any simplicial set arising from a decomposition of the unit interval. δ, 0 and 1 are the only non-degenerate simplexes. Degenerate simplexes over 0 and 1 are again denoted by 0 and 1, respectively.

CHAPTER II. STACKS AND CO-STACKS

3. PRESTACKS AND CO-PRESTACKS

Let X be a simplicial set and let Ⓐ be any category. A co-prestack on X with values in Ⓐ is, by definition, a covariant functor A : Ⓧ → Ⓐ. For two co-prestacks A, A' : Ⓧ → Ⓐ, a natural transformation $f = \{f_x\} : A \to A'$ is called a co-prestack map. The class of all co-prestacks on X with values in Ⓐ form a category (a functor category) $Ⓐ^X$ with morphisms the co-prestack maps.

A "partially dual" notion of a co-prestack is a prestack, which is, by definition, a covariant functor A^+ : $Ⓧ^o$ → Ⓐ. The dual of a co-prestack A : Ⓧ → Ⓐ is a prestack A^o : $Ⓧ^o$ → $Ⓐ^o$. e.g. the dual of a co-prestack of abelian groups is a prestack of compact abelian groups, and this statement is self-dual.

If Ⓐ is an abelian category, it is clear that $Ⓐ^X$ and its duals are also abelian. Kernels, co-kernels, images, co-images, products, and co-products are constructed "pointwise" (cf. [3]). The notions of exactness and limits, co-limits (if they exist in Ⓐ) are also defined pointwise. For example, if $\{A^\alpha\}$ is a direct system of co-prestacks in $Ⓐ^X$ and if, for every x in X, the co-limit $\varinjlim A^\alpha x$ exists in Ⓐ, then the co-limit $\varinjlim A^\alpha$ is the co-prestack defined by $(\varinjlim A^\alpha)x = \varinjlim A^\alpha x$ for all x in X.

Now, let Ⓐ be a co-complete abelian category, and let Ⓧ, Ⓨ be simplicial categories. A simplicial functor f : Ⓧ → Ⓨ

induces a functor $f_{\#} : Ⓐ^X \to Ⓐ^Y$ in the following way.

For each A in $Ⓐ^X$, define a functor $f_{\#}A : Ⓨ \to Ⓐ$ in such a fashion that, on objects $y \in Ⓨ$, $(f_{\#}A)y = \amalg_{fx=y} Ax$ while on morphisms (incidence maps) $\mu_y : y \to y'$ of Ⓨ,

$$(f_{\#}A)\mu_y : \amalg_{fx=y} Ax \to \amalg_{fx=y'} Ax$$

is the sum $\amalg_{(f\mu_x = \mu_y)} A\mu_x$, where μ_x is a morphism (incidence map) of Ⓧ. Since **f**, **A**, and $\amalg$ are functors, $f_{\#}A$ is a well-defined functor on Ⓨ to Ⓐ. Hence corresponding to each A in $Ⓐ^X$, we have a $f_{\#}A$ in $Ⓐ^Y$.

To check that the correspondence $A \to f_{\#}A$ is functorial, let $\varphi = \{\varphi_x\} : A \to A'$ be a co-prestack map in $Ⓐ^X$. Then, for every y in Y, the set $\{\varphi_x | fx=y\}$ of maps determines uniquely a map $\psi_y : \amalg_{fx=y} Ax \to \amalg_{fx=y} A'x$. It is clear that $\psi = \{\psi_y\} : f_{\#}A \to f_{\#}A'$ is a co-prestack map in $Ⓐ^Y$. Let $f_{\#}(\varphi) = \psi$, the equalities $f_{\#}(1_A) = 1_{f_{\#}A}$ and $f_{\#}(\varphi_1\varphi_2) = f_{\#}(\varphi_1)f_{\#}(\varphi_2)$ are obvious.

The functor $f_{\#} : Ⓐ^X \to Ⓐ^Y$ defined above is called the <u>push-out functor</u> induced by f. $f_{\#}A$ is the <u>push-out</u> of A by f.

On the other hand, f induces by composition, a functor $f^{\#} : Ⓐ^Y \to Ⓐ^X$ such that for each $B \in Ⓐ^Y$, $f^{\#}B = Bf$ is a composite of functors. $f^{\#}$ is called the <u>pull-back functor</u> induced by f, and $f^{\#}B$ the <u>pull-back</u> of B by f.

<u>Proposition 3.1</u>. $f^{\#}$ is the adjoint of $f_{\#}$, i.e. there is a natural isomorphism

(3.1) $$Ⓐ^X(A, f^{\#}B) \to Ⓐ^Y(f_{\#}A, B),$$

$A \in Ⓐ^X$, $B \in Ⓐ^Y$.

<u>Proof</u>. Let $\varphi = \{\varphi_x | x \in X\}$ be in $Ⓐ^X(A, f^{\#}B)$. For $y \in Y$

and all $x \in X$ such that $fx = y$, the universal mapping diagram of the co-product $\coprod_x A(x)$,

$$(3.2) \qquad \begin{array}{ccccc} Ax & \xrightarrow{\text{inj}_x} & \coprod_x Ax & = & (f_{\#}A)y \\ & \searrow^{\varphi x} & \downarrow{\scriptstyle \psi y} & & \downarrow{\scriptstyle \psi y} \\ & & (f^{\#}B)x = (Bf)x & = & By \end{array} ,$$

shows that there is an isomorphism from $\textcircled{A}^X(A, f^{\#}B)$ onto $\textcircled{A}^Y(f_{\#}A, B)$ defined by $\varphi \to \psi$ such that

$$\varphi_x = \psi_y \text{ inj } x \ , \ fx = y \ .$$

<u>Corollary 3.2.</u> $f^{\#}$ is exact and $f_{\#}$ is right exact (i.e. co-kernel preserving).

<u>Corollary 3.3.</u> $f_{\#}$ pres^rves projectives.

This follows from proposition 1.6 since $f_{\#}$ has an exact adjoint $f^{\#}$.

<u>Remark</u>. If $\textcircled{A}$ satisfies the axiom AB4 of Grothendieck [4], i.e. a co-product of a family of short exact sequences of $\textcircled{A}$ is again a short exact sequence of $\textcircled{A}$, then $f_{\#}$ is exact. In this case, $f^{\#}$ preserves injectives.

4. PROJECTIVES, INJECTIVES, AND GENERATORS IN $\textcircled{A}^X$

Recall that the simplicial categories Δ^n and the simplicial functors x^δ are distinguished in the category of simplicial categories. Let $\textcircled{A}$ be any category. For an object P in $\textcircled{A}$, let P^n denote the constant co-prestack on Δ^n with constant value P .

<u>Lemma 4.1.</u> There is a bijection

$$(4.1) \qquad \Phi : \textcircled{A}^{\Delta^n}(P^n, A) \to \textcircled{A}(P, A\delta^n)$$

<u>Proof.</u> Let

$$\varphi = \{\varphi_\sigma \mid \sigma \epsilon \Delta^n\} : P^n \to A$$

be a natural transformation. The commutative diagram

$$(4.2) \qquad \begin{array}{ccc} P & \xrightarrow{\varphi_{\delta^n}} & A\delta^n \\ & \searrow^{\varphi_\sigma} & \downarrow A\sigma^* \\ & & A\sigma = A(\sigma^*\delta^n) \end{array}$$

shows that φ is completely determined by φ_{δ^n} and vice versa. Hence the correspondence $\varphi \to \varphi_{\delta^n}$ gives rise to a bijection Φ .

<u>Lemma 4.2.</u> If Ⓐ is abelian and P is a projective of Ⓐ , then P^n is a projective of $Ⓐ^{\Delta^n}$.

This follows immediately from lemma 4.1 and the pointwise exactness of co-products.

<u>Theorem 4.3.</u> If Ⓐ is a co-complete abelian category and P is a projective generator of Ⓐ , then $U = \amalg_{x \epsilon X}(x_\# P^n)$ is a projective generator of $Ⓐ^X$, where $x_\# P^n$ is the push-out of P^n by x^δ , and $\dim x = n$.

<u>Proof.</u> P^n is projective by lemma 4.2, therefore $x_\# P^n$ is projective by corollary 3.3. U is a co-product of projectives, U is projective.

To show that U is a generator, we shall use the well-known fact that a projective U is a generator if and only if $Ⓐ^X(U,A)$ is non-trivial for all non-trivial $A \epsilon Ⓐ^X$ [3, p. 68]. Look at

$$(4.3) \qquad \begin{aligned} Ⓐ^X(U,A) &= Ⓐ^X(\amalg x_\# P^n, A) \approx \Pi\, Ⓐ^X(x_\# P^n, A) \\ &\approx \Pi\, Ⓐ^{\Delta^n}(P^n, x^\# A) = \Pi\, Ⓐ^{\Delta^n}(P^n, Ax\ ) \ , \end{aligned}$$

where the co-product and the products are taken over all $x \epsilon X$.

The last term is, by lemma 4.1, isomorphic to

$$\Pi \, Ⓐ(P, Ax^{\delta}(\delta^{n})) = \Pi \, Ⓐ(P, Ax) \; .$$

Hence we have

$$(4.4) \qquad Ⓐ^{X}(U, A) \approx \Pi_{x \in X} Ⓐ(P, Ax) \; .$$

Now, A is non-trivial implies that Ⓐ(P,Ax) is non-trivial for some $x \in X$ since P is a projective generator of Ⓐ. Hence, by 4.4, $Ⓐ^{X}(U,A)$ is non-trivial and thus U is a projective generator. Q.E.D.

This shows that if Ⓐ is a co-complete abelian category with projective generators, then, for any simplicial category Ⓧ, $Ⓐ^{X}$ has projective generators. Hence, in this case, $Ⓐ^{X}$ has enough projectives.

If Ⓐ is a Grothendieck category, then, by pointwise construction of co-limits in $Ⓐ^{X}$, it is easily shown that $Ⓐ^{X}$ is also a Grothendieck category. If, in addition, Ⓐ has a generator, then $Ⓐ^{X}$ is Grothendieck with generator and therefore has enough injectives (proposition 1.1).

5. STACKS AND CO-STACKS; NORMALIZATION

Recall that in a simplicial set X, the face operators d^{i}, or d, and the degeneracy operators s^{j}, or s, are distinguished. They satisfy the usual incidence relations. In particular

$$(5.1) \qquad d^{i}s^{j} = 1 \quad \text{for} \quad i = j \quad \text{or} \quad i = j+1 \; .$$

As a simplicial category, Ⓧ has distinguished morphisms $s_{x} = s \mid x$ and $d_{sx} = d \mid sx$, $x \in X$, such that $d_{sx} \, s_{x} = 1_{x}$.

A co-prestack $A \in Ⓐ^{X}$ is <u>normalized</u> if all $A(s_{x})$ are isomorphisms in Ⓐ. A normalized co-prestack is called a co-stack. Since

(5.2) $$A(d_{sx}s_x) = A(d_{sx})A(s_x) = 1 \ , \ \text{all} \ x \in X \ ,$$

a co-prestack is a co-stack (normalized) if and only if all $A(d_{sx})$ are isomorphisms in Ⓐ. The class of all co-stacks on X with values in Ⓐ form a full subcategory of $Ⓐ^X$. This subcategory is denoted by $(NA)^X$. Obviously, all constant co-prestacks are normalized and therefore in $(NA)^X$.

If a prestack A^+ on X with values in Ⓐ has the property that all $A^+(s_x)$ are isomorphisms in Ⓐ, then it is called a stack (compare with the notion of a stack defined by Spanier [9, p. 512]). Again, a prestack A^+ is a stack if and only if all $A^+(d_{sx})$ are isomorphisms in Ⓐ.

If Ⓐ is abelian, so is $Ⓐ^X$, and we have

Proposition 5.1. $(NA)^X$ is a Serre subcategory of $Ⓐ^X$.

Proof. Let A' be a subobject of $A \in (NA)^X$. Then there is a canonical monomorphism $i_x : A'x \to Ax$ in Ⓐ for every $x \in X$. In the commutative diagram

$$\begin{array}{ccc} A'x & \xrightarrow{i_x} & Ax \\ {\scriptstyle A'(d_{sx})}\uparrow & & \uparrow {\scriptstyle A(d_{sx})} \\ A'(sx) & \xrightarrow{i_{sx}} & A(sx) \end{array} \quad ,$$

$i_xA'(d_{sx}) = A(d_{sx})i_{sx}$, and $A(d_{sx})$ is an isomorphism. Hence $i_xA'(d_{sx})$ is a monomorphism and so is $A'(d_{sx})$. But $A'(d_{sx})$ is epic by (5.2). Therefore $A'(d_{sx})$, and so $A'(s_x)$, are isomorphisms. This shows that $A' \in (NA)^X$. Dually, we can show that quotient objects of $A \in (NA)^X$ are again in $(NA)^X$.

Now, let $A' \xrightarrow{\varphi} A \xrightarrow{\psi} A''$ be exact in $\textcircled{A}^X$ with $A', A'' \in \textcircled{NA}^X$. Then $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \psi$, being subobject of A' and quotient object of A'', respectively, are in $\textcircled{NA}^X$. Hence in the commutative diagram

$$\begin{array}{ccccccccc} \operatorname{Ker} \varphi_x & \rightarrow & A'x & \rightarrow & Ax & \rightarrow & A''x & \rightarrow & \operatorname{Coker} \psi_x \\ 1 \downarrow & & 2 \downarrow & & \downarrow A(s_x) & & 3 \downarrow & & 4 \downarrow \\ \operatorname{Ker} \varphi_{sx} & \rightarrow & A'(sx) & \rightarrow & A(sx) & \rightarrow & A''(sx) & \rightarrow & \operatorname{Coker} \psi_{sx} \end{array}$$

the rows are exact in $\textcircled{A}$, the maps 1, 2, 3, 4 are isomorphisms. By the five lemma, $A(s_x)$ is an isomorphism. Thus A is in $\textcircled{NA}^X$. $\textcircled{NA}^X$ is a Serre subcategory of $\textcircled{A}^X$. Q.E.D.

$\textcircled{NA}^X$, being a Serre subcategory, is abelian. If $\textcircled{A}$ is Grothendieck, so are $\textcircled{A}^X$ and $\textcircled{NA}^X$.

In the rest of this chapter, $\textcircled{A}$ will always denote a complete and co-complete abelian category with a fixed projective generator P. $\textcircled{A}^X$ is a complete and co-complete abelian category with a projective generator U as we have shown. By proposition 1.2, $\textcircled{A}^X$ is well-powered and co-well-powered.

Now, $\textcircled{NA}^X$, being a Serre subcategory of $\textcircled{A}^X$, is closed under the formation of subobjects. It is clear that it is a full subcategory replete in $\textcircled{A}^X$ and that it is closed under the formation of products. Hence, by proposition 1.5, we have

<u>Theorem 5.2.</u> $\textcircled{NA}^X$ is a co-reflective subcategory of $\textcircled{A}^X$.

Let $N^* : \textcircled{A}^X \rightarrow \textcircled{NA}^X$ be the co-reflector, then, by proposition 1.4, N^* is the co-adjoint of the inclusion functor $J : \textcircled{NA}^X \rightarrow \textcircled{A}^X$ and N^* preserves co-limits. Also, N^* preserves projectives (corollary 1.7).

The construction of co-reflections can be found in [3].

<u>Proposition 5.3</u>. Let NX be the set of non-degenerate simplexes of X, then $U^* = N^* \coprod_{x \in NX} (x_\# P^n)$ is a projective generator of $\textcircled{\scriptsize NA}^X$.

The proof is similar to that of theorem 4.3.

<u>Theorem 5.4</u>. $\textcircled{\scriptsize NA}^X$ is a reflective subcategory of $\textcircled{\scriptsize A}^X$.

We shall prove this directly by constructing the reflections.

For each co-prestack $A \in \textcircled{\scriptsize A}^X$, let $\bar{A}$ be the co-stack with values $\bar{A}x \approx Ax$ for $x \in NX$ and $\bar{A}(sx) \approx Ax$ for any degeneracy operator s. Then, by (5.2), all $\bar{A}(d)\bar{A}(s) = 1$ on which they defined. Now, define a map $\varphi : \bar{A} \to A$ as follows: For each $x \in NX$, and all degeneracy operators $s, s', \ldots$, let

(5.3)
$$\begin{array}{ccccccc} \bar{A}x & \xrightarrow{\bar{A}(s)} & \bar{A}(sx) & \xrightarrow{A(s')} & \bar{A}(s'sx) & \longrightarrow & \cdots \\ {\scriptstyle\varphi_x}\downarrow & & {\scriptstyle\varphi_{sx}}\downarrow & & {\scriptstyle\varphi_{s'sx}}\downarrow & & \\ Ax & \xrightarrow{A(s)} & A(sx) & \xrightarrow{A(s')} & A(s'sx) & \longrightarrow & \cdots \end{array}$$

be a commutative diagram with

(5.4) $$\varphi_x = 1 \ , \ \varphi_{sx} = A(s)\bar{A}(d) \ , \ \varphi_{s'sx} = \bar{A}(s')\bar{A}(s)\bar{A}(d)A(d'), \ \ldots$$

such that

(5.5) $$\bar{A}(d)\bar{A}(s) = 1 \ , \ \bar{A}(d')\bar{A}(s') = 1 \ , \ \ldots \ .$$

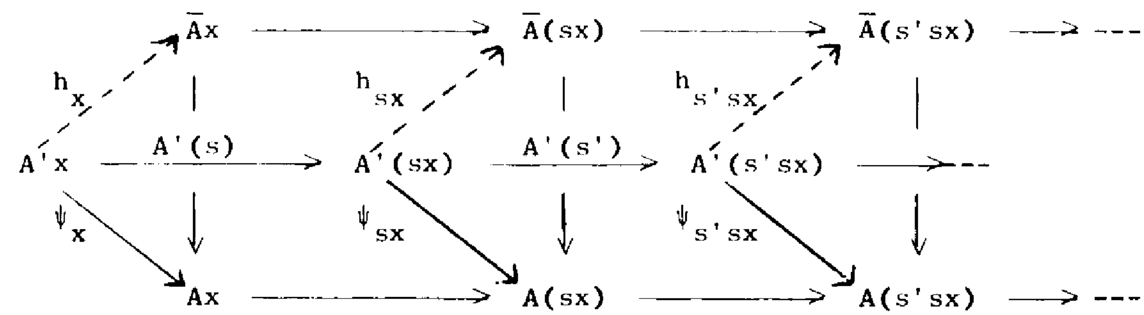
Let $\varphi = \{\varphi_x \mid x \in X\}$. Then φ is a co-prestack map. We shall prove that $\bar{A}$ together with the map φ form a reflection of A.

Let $A' \in \textcircled{\scriptsize NA}^X$ be a co-stack, then all $A'(s_x)$, $A'(d_{sx})$ are isomorphisms and all $A'(d)A'(s) = 1$ on which they defined. Given a co-prestack map $g : A' \to A$,

$$\psi = \{\psi_x : A'x \to Ax \mid x \in X\} \ ,$$

consider, for each $x \in NX$, the "solid" diagram

(5.6)



obtained by adjoining the A'-row to the diagram (5.3). Define maps

(5.7) $$h_x = \psi_x \ , \ h_{sx} = \bar{A}(s)\psi_x A'(d) \ ,$$

$$h_{s'sx} = \bar{A}(s')\bar{A}(s)\psi_x A'(d)A'(d') \ , \ \dots \ ,$$

where $A'(d)A'(s) = 1 \ , \ A'(d')A'(s) = 1 \ , \ \dots$

Then $h = \{h_x \mid x \in X\}$ is a co-stack map and all rectangles of all such solid diagrams are commutative.

Now, combining (5.4) and (5.7), we have

$$\varphi_{sx}h_{sx} = A(s)\bar{A}(d)\bar{A}(s)\psi_x A'(d) = A(s)\psi_x A'(d) = \psi_{sx} \ ,$$

(5.8) $$\varphi_{s'sx}h_{s'sx} = A(s')\varphi_{sx}\bar{A}(d')\bar{A}(s')1_{sx}A'(d') = A(s')\varphi_{sx}h_{sx}A'(d')$$

$$= A(s')\psi_{sx}A'(d') = \psi_{s'sx} \ , \ \dots$$

Hence $\varphi h = \psi$. The uniqueness of h is obvious.

These show that $\bar{A}$ with φ is "the" reflection of A , and thus $Ⓝⓐ^X$ is reflective in $Ⓐ^X$. Q.E.D.

We remark that, in general, the reflection and co-reflection are not isomorphic.

Corollary 5.5. The reflector $N_* : Ⓐ^X \to Ⓝⓐ^X$ is exact.

This follows immediately from the construction of reflections and the pointwise exactness of co-prestacks and co-stacks.

As a direct consequence of this we have (by corollary 1.7)

Corollary 5.6. The inclusion functor J preserves projectives.

Dual to the remark following theorem 5.2, N_* preserves limits and injectives.

From proposition 5.3 and corollary 5.6, we see that $Ⓝⓐ^X$ has enough projectives and these are projective in $Ⓐ^X$. Since J is exact, a projective resolution of $A \in Ⓝⓐ^X$ taken in $Ⓝⓐ^X$ is still a projective solution of A in the big category $Ⓐ^X$.

6. RELATIVE CO-PRESTACKS AND CO-STACKS

Let X' be a simplicial subset of X and let $i: Ⓧ' \to Ⓧ$ be the inclusion map. Then the inclusion functor $i: X' \to X$ is a full embedding. The functor $i_\#: Ⓐ^{X'} \to Ⓐ^X$ maps each $A' \in Ⓐ^{X'}$ onto $i_\# A' = A \in Ⓐ^X$ with <u>support</u> in X', i.e. $Ax = A'x$ if $x \in X'$ and $Ax = 0$ if $x \in X - X'$. It is clear that $i_\#$ is an exact full embedding, and that $i_\# A^{X'}$ is a Serre subcategory of A^X. Identify $A^{X'}$ as $i_\# Ⓐ^{X'}$ and regard $i_\#$ as an inclusion functor.

Now, consider $i^\#: Ⓐ^X \to Ⓐ^{X'}$, the adjoint of $i_\#$. It is clear that $i^\# i_\#$ is the identity functor of $Ⓐ^{X'}$ and so $(i_\# i^\#) i_\# Ⓐ^{X'} = i_\# Ⓐ^{X'}$. $i_\# i^\#$ is an exact reflector of $Ⓐ^X$ onto $i_\# Ⓐ^{X'}$. Hence we have

<u>Proposition 6.1.</u> If X' is a simplicial subset of X, then $Ⓐ^{X'}$ is isomorphic to (and is identified as) a reflective Serre subcategory of $Ⓐ^X$.

In fact, $i_\# i^\# A$, the reflection of $A \in Ⓐ^X$ is a subobject of A with support in X'. Let $p_o A$ denote the quotient $A/i_\# i^\# A$, we have

$$(6.1) \qquad 0 \to i_\# i^\# A \to A \to p_o A \to 0$$

exact in $\textcircled{A}^X$, where $p_o A$ has support in $X - X'$. On the other hand, every co-prestack of $\textcircled{A}^X$ having its support in $X - X'$ is a quotient of some $i_\# i^\# A \to A$ in $\textcircled{A}^X$, i.e. it is of the form $p_o A$. A co-prestack with support in $X - X'$ is called a <u>relative co-prestack</u>. Similarly, a co-stack with support in $X - X'$ is called a <u>relative co-stack</u>.

Let $\textcircled{A}^{(X,X')}$ denote the full subcategory of $\textcircled{A}^X$ consisting of all relative co-prestacks. It is clear that $\textcircled{A}^{(X,X')}$ is also a Serre subcategory of $\textcircled{A}^X$. Moreover,

(6.2) $$\textcircled{A}^X(A,A'') = \textcircled{A}^{(X,X')}(p_o A,A'')$$

for any $A \in \textcircled{A}^X$ and $A'' \in \textcircled{A}^{(X,X')}$. We have

<u>Proposition 6.2.</u> p_o is an exact co-reflector of $\textcircled{A}^X$ onto its Serre subcategory $\textcircled{A}^{(X,X')}$.

Similar statements are true for $\textcircled{NA}^{X'}$, $\textcircled{NA}^X$, and $\textcircled{NA}^{(X,X')}$.

As a consequence of proposition 6.1 and 6.2, p_o and $i_\#$ preserve projective resolutions.

CHAPTER III. HOMOLOGY OF SIMPLICIAL PAIRS

Throughout this chapter, Ⓐ will denote a co-complete abelian category with the property AB4 and having a fixed projective generator P . Therefore $Ⓐ^X$ is co-complete, abelian, AB4 , and has a projective generator $U = \amalg_{x \in X} x_{\#}P^n$ (theorem 4.3).

A pair of simplicial sets (X,X') such that X' is a simplicial subset of X is called a simplicial pair. The class of all simplicial pairs form a category, denoted by Ⓒ' , with morphisms simplicial maps $f:(X,X') \to (Y,Y')$ such that $f(X') \subset Y'$. We write X for (X,∅) , where ∅ is the vacuous set.

7. CHAIN COMPLEXES, HOMOLOGY

For each $A \in Ⓐ^X$, let $CA = \{C_q\}$ be an object of the graded category $Ⓐ^\infty$ of Ⓐ such that $C_q = \amalg_{x \in X_q} Ax$ with the system of injections denoted by $\{inj_x \mid x \in X_q\}$. For every $q > 0$, let $d_q : C_q \to C_{q-1}$ denote the unique morphism satisfying the equality

(7.1) $$d_q \; inj_x = \sum_{i=0}^{q} (-1)^i inj_{d^i x} A(d^i_x) \; , \; x \in X_q \; .$$

Then $d_q d_{q+1} = 0$ and therefore (CA,d) , simply written CA , is a chain complex.

Observe that the map $A \to CA$ defines an exact additive functor $C:Ⓐ^X \to dⒶ$, the category of chain complexes (CA,d) and thus $\bar{H}C$ is a homology functor on $Ⓐ^X$ to $Ⓐ^\infty$, where $\bar{H}$ is the usual homology functor on dⒶ to $Ⓐ^\infty$ (cf. [5]).

Let (X,X') be a simplicial pair. The (relative) homology of (X,X') with coefficients in $A \in \mathbb{A}^X$, denoted by $H_*(X,X';A)$, is defined as

(7.2) $$H_*(X,X';A) = \overline{H}\underset{\sim}{C}(p_o A) .$$

In particular, $H_*(X;A) = \overline{H}\underset{\sim}{C}A$ is the (absolute) homology of X. Observe that if $f:(X,X') \to (Y,Y')$ is a simplicial map, then, by the definition of $f_\#$, we have

$$\underset{\sim}{C}f_\# A = \underset{\sim}{C}A \quad \text{and} \quad \underset{\sim}{C}p_o f_\# A = \underset{\sim}{C}p_o A .$$

Hence

(7.3) $$H_*(X,X';A) = H_*(Y,Y';\ f_\# A)$$

for any $A \in \mathbb{A}^X$.

Now, let $f(X,X') \to (Y,Y')$ be a simplicial map and let B be in $\mathbb{A}^Y$. Then the chain complex $\underset{\sim}{C}(f^\# B)$ of $f^\# B \in \mathbb{A}^X$ has

(7.4) $$\underset{\sim}{C}_q = \amalg_{x \in X_q} (f^\# B)x = \amalg_{x \in X_q} Bf(x) .$$

f induces a chain map

$$\underset{\sim}{C}f = \{f_q\} : \underset{\sim}{C}(f^\# B) \to \underset{\sim}{C}B$$

defined by the equalities

(7.5) $$f_q \, inj_x = inj_{fx} , \ x \in X_q , \ q = 0,1,2 , \ldots$$

To check that $f_q d_{q+1} = d_{q+1} f_{q+1}$ is just a routine computation. This shows that f induces a morphism

(7.6) $$\overline{H}\underset{\sim}{C}f = f_*:H_*(X,X';f^\# B) \to H_*(Y,Y';B)$$

which is called the morphism induced by f.

Proposition 7.1. For any $A \in \mathbb{A}^X$, there is a homology sequence (i.e. an exact triangle for H_*) for each pair (X,X'):

(7.7) $$\ldots \to H_q(X';i^\# A) \xrightarrow{i_*} H_q(X;A) \xrightarrow{j_*} H_q(X,X';A) \xrightarrow{\partial} H_{q-1}(X';i^\# A) \to \ldots ,$$

where i and j are inclusion maps. ∂ is called a boundary operator.

<u>Proof</u>. Since $0 \to i_{\#}i^{\#}A \to A \to p_{o}A \to 0$ is exact in $\textcircled{A}^{X}$, $0 \to \underset{\sim}{C}(i_{\#}i^{\#}A) \to \underset{\sim}{C}A \to \underset{\sim}{C}(p_{o}A) \to 0$ is exact in $d\textcircled{A}$. This gives the exact triangle

$$\begin{array}{ccc} \overline{H}\underset{\sim}{C}(i_{\#}i^{\#}A) & \xrightarrow{i_*} & \overline{H}\underset{\sim}{C}A \\ \partial \nwarrow & & \swarrow j_* \\ & \overline{H}\underset{\sim}{C}(p\ A) & \end{array}$$

which is just (7.7) since $\overline{H}\underset{\sim}{C}(i_{\#}i^{\#}A) = \overline{H}\underset{\sim}{C}'(i^{\#}A)$ by (7.3).

<u>Lemma 7.2</u>. Let $f:(X,X') \to (Y,Y')$ be a morphism in $\textcircled{C}'$ and let $\tilde{f} = f|X'$. Then for any $B \in \textcircled{A}^{Y}$,

(7.8) $$H_*(X';i^{\#}f^{\#}B) = H_*(X';\tilde{f}^{\#}i^{\#}B),$$

where i denotes the inclusion maps $X' \to X$ and $Y' \to Y$.

For, the diagram

$$\begin{array}{ccc} \textcircled{A}^{Y} & \xrightarrow{i^{\#}} & \textcircled{A}^{Y'} \\ f^{\#}\downarrow & & \downarrow \tilde{f}^{\#} \\ \textcircled{A}^{X} & \xrightarrow{i^{\#}} & \textcircled{A}^{X'} \end{array}$$

commutes.

<u>Proposition 7.3</u>. Every morphism $f:(X,X') \to (Y,Y')$ in $\textcircled{C}'$ induces a morphism $\overline{f}_* = \{\overline{f}_q\}$ in homology which "maps" the homology sequence of (X,X') into that of (Y,Y'). i.e. f gives rise to a morphism of exact triangles.

For, in the diagram

$$\begin{array}{ccccccccc} \ldots \to H_q(X';\tilde{f}^{\#}i^{\#}B) & \xrightarrow{i_*} & H_q(X;f^{\#}B) & \xrightarrow{j_*} & H_q(X,X';f^{\#}B) & \xrightarrow{\partial} & H_{q-1}(X';\tilde{f}^{\#}i^{\#}B) & \to \ldots \\ \downarrow \tilde{f}_q & & \downarrow f_q & & \downarrow \overline{f}_q & & \downarrow \tilde{f}_{q-1} & \\ \ldots \to H_q(Y';i^{\#}B) & \xrightarrow{i_*} & H_q(Y;B) & \xrightarrow{j_*} & H_q(Y,Y';B) & \xrightarrow{\partial} & H_{q-1}(Y';i^{\#}B) & \to \ldots \end{array}$$

the top row is the homology sequence of (X,X') with $H_*(X';i^{\#}f^{\#}B)$ replaced by $H_*(X';\tilde{f}^{\#}i^{\#}B)$. The proposition is just a restatement of the one for chain homology.

Proposition 7.4. H_* defines a functor on $\textcircled{C'}$ in the sense that for simplicial maps $(X,X') \xrightarrow{f} (Y,Y') \xrightarrow{g} (W,W')$ and any $E \in \textcircled{A}^W$, we have

$$(gf)_* = g_* f_* : H_*(X,X';f^\# g^\# E) \to H_*(W,W';E) .$$

This follows from the fact $(gf)^\# = f^\# g^\#$ and the usual argument for chain homology.

8. THE FUNCTORS H_* AND Tor

In this section, we shall generalize the tensor functor on modules and show that the homology functor H_* is isomorphic to the generalized torsion functor Tor.

For each $A \in \textcircled{A}^X$, let TA denote the co-kernel of the morphism d_1 defined by (7.1). Then T defines an additive functor $T: \textcircled{A}^X \to \textcircled{A}$. Clearly, T is co-continuous (i.e. T is right exact and preserves co-products). This means that T has the formal properties of the tensor product functor on modules. We call T the generalized tensor functor. The left derived functor of T, written Tor, is called the generalized torsion functor.

Lemma 8.1. $\overline{H}_q \underset{\sim}{C} U = 0$ for $q > 0$.

Proof. By (7.3), $\overline{H}\underset{\sim}{C}(x_\# P^n) = \overline{H}\underset{\sim}{C}P^n$. It is well-known that $\underset{\sim}{C}P^n$, $n = 0,1,2, \ldots$, are acyclic, so are $\underset{\sim}{C}(x_\# P^n)$. Hence

$$\overline{H}_q \underset{\sim}{C}(\amalg_{x \in X} x_\# P^n) \approx \amalg_{x \in X} \overline{H}_q \underset{\sim}{C}(x_\# P^n)$$

shows that $\overline{H}_q \underset{\sim}{C} U = 0$ for $q > 0$.

Proposition 8.2. $\overline{H}\underset{\sim}{C} \approx \mathrm{Tor}: \textcircled{A}^X \to \textcircled{A}^\infty$.

Proof. $\overline{H}_o \underset{\sim}{C} = T = \mathrm{Tor}_o$ by definitions. It suffices to show that $\overline{H}_q \underset{\sim}{C} A = 0$ for $q > 0$ and A projective in $\textcircled{A}^X$. Since

U is a projective generator, a projective A is a summand of a co-product of copies of U. The proof follows from lemma 8.1.

This shows that $H_*(X,X';A)$ can be computed by $\mathrm{Tor}(p_o A) = \overline{H}T(\underset{\sim}{P} p_o A)$ where $\underset{\sim}{P} p_o A$ is a projective resolution of $p_o A$.

Now, let $f\colon X \to Y$ be a simplicial map and let B be in $Ⓐ^Y$. Then f induced a natural morphism $f_B\colon \mathrm{Tor}_X f^{\#} B \to \mathrm{Tor}_Y B$ in the following way:

Let $\underset{\sim}{P}'$ be a projective resolution of $f^{\#}B$, then $f_{\#}\underset{\sim}{P}'$ is a projective resolution of $f_{\#}f^{\#}B$. Since $f_{\#}$ is the co-adjoint of $f^{\#}$, there is a natural transformation $t\colon f_{\#}f^{\#} \to 1\colon Ⓐ^Y$ such that for each $B \in Ⓐ^Y$, t_B is the morphism corresponding to $1\colon f^{\#}B$ under the isomorphism $Ⓐ^Y(f_{\#}f^{\#}B,B) \approx Ⓐ^X(f^{\#}B,f^{\#}B)$. If $\underset{\sim}{P}$ denotes a projective resolution of B, then, by the comparison theorem, there is a chain map $f\colon f_{\#}\underset{\sim}{P}' \to \underset{\sim}{P}$ lifting t_B. This chain map induces a morphism $f_B\colon \overline{H}T(f_{\#}\underset{\sim}{P}') \to \overline{H}T\underset{\sim}{P}$. But since $Tf_{\#}\underset{\sim}{P}' = T\underset{\sim}{P}'$, we have

$$\overline{H}T(f_{\#}\underset{\sim}{P}') = \overline{H}T\underset{\sim}{P}' = \mathrm{Tor}_X f^{\#}B$$

and thus $f_B\colon \mathrm{Tor}_X f^{\#}B \to \mathrm{Tor}_Y B$.

This shows that every simplicial map $f\colon(X,X') \to (Y,Y')$ determines a morphism $f_B\colon H_*(X;f^{\#}B) \to H_*(Y;B)$ and thus a morphism $\overline{f}_B\colon H_*(X,X';\ f^{\#}B) \to H_*(Y,Y';B)$. We observe that f_B and $\overline{f}_B$ are the induced morphisms f_* and $\overline{f}_*$, respectively.

9. EXCISION, ADDITIVITY, DIMENSION

From now on, all coefficients for homology are normalized. It is clear that p_o and all pull-back functors preserve normalization.

Let X', X'' be simplicial subsets of X and let $(X; X', X'')$ be the triad with inclusion maps

$$(X', X' \cap X'') \xrightarrow{i} (X' \cup X'', X'') \xrightarrow{h} (X, X' \cup X'')$$
$$(X'', X' \cap X'') \xrightarrow{j} (X' \cup X'', X') \xrightarrow{k} (X, X' \cup X'') ,$$

where i and j are called excision maps. A triad is proper if, for any $A \in (NA)^X$, the induced morphisms

(9.1)
$$i_*: H_*(X', X' \cap X''; i^\# h^\# A) \to H_*(X' \cup X'', X'; h^\# A) ,$$
$$j_*: H_*(X'', X' \cap X''; j^\# k^\# A) \to H_*(X' \cup X'', X''; k^\# A)$$

are isomorphisms.

Proposition 9.1. Excision maps i and j induce isomorphisms (9.1).

For, the chain complexes $\underset{\sim}{C}p_o i^\# h^\# A$ and $\underset{\sim}{C}p_o h^\# A$ are the same. Similar argument for j_* .

Thus all triads of simplicial sets are proper.

Proposition 9.2. Let $\{X_\alpha\}$ be a set of simplicial subsets of X such that $X = (\cup X_\alpha) \cup X'$ and $X_\alpha \cap X_\beta \subset X'$ if $\alpha \neq \beta$, where X' is a fixed simplicial subset of X . Let $X'_\alpha = X_\alpha \cap X'$ and let $h_\alpha : (X_\alpha, X'_\alpha) \to (X, X')$ be the inclusion map, then, for any $A \in (NA)^X$,

(9.2)
$$H_*(X, X'; A) \approx \amalg_\alpha H_*(X_\alpha, X'_\alpha ; h_\alpha^\# A)$$

canonically.

This is clear since $\underset{\sim}{C}(p_o A) = \amalg_\alpha \underset{\sim}{C}(p_o h_\alpha^\# A)$.

For the special case that $X' = \emptyset$ we have

Corollary 9.3. H_* is (infinitely) additive. i.e. homology of disjoint union of a set of simplicial sets is isomorphic to the co-product of homology of the individual simplicial sets.

Now, let Δ^n , $\dot{\Delta}^n$, Δ^{n-1} , V , and C^{n-1} denote the simplicial analogue of the standard affine n-simplex, its boundary, an (n-1)-face, the vertex of Δ^n opposite Δ^{n-1} , and the closed star of V in $\dot{\Delta}^n$, respectively. Then the only non-degenerate simplexes of Δ^n are faces of δ^n as described before. V is point-like in the sense that $H_q(V;A) = 0$ for $q \neq 0$ and $H_o(V;A) \approx Av$, where $A \in \text{(NA)}^V$ and $v \in V$. C^{n-1} is the simplicial subset of Δ^n consisting of all faces of Δ^n excepting Δ^n and Δ^{q-1} . Also

(9.3) $$\dot{\Delta}^n = \Delta^{n-1} \cup C^{n-1} , \quad \dot{\Delta}^{n-1} = \Delta^{n-1} \cap C^{n-1} ,$$

and then $(\Delta^n ; \Delta^{n-1} , C^{n-1})$ is a proper triad. (cf. [2], p. 78).

Proposition 9.4. Let Δ^x be the simplicial analogue of a (non-degenerate) simplex $x \in NX$ and let $\dot{\Delta}^x$ be the "boundary simplicial subset" of Δ^x . If $i_x: \Delta^x \to X$ denotes the inclusion map, then, for $A \in \text{(NA)}^X$,

(9.4) $$H_q(\Delta^x, \dot{\Delta}^x; i_x^{\#}A) = 0 \quad \text{if} \quad q \neq \dim x .$$

For, the normalized chain complex of $p_o(i_x^{\#}A)$ has zero in all dimensions q except possibly for $q = \dim x$.

10. HOMOTOPY

Let I_n be the simplicial analogue of the closed interval $[n,n+1]$ and let $W = \bigcup_{n=0}^{\infty} I_n$ be the "simplicial half line". Denote the n-skeleton of X by X^n , $n = 0,1,2,\ldots$, and write $L = \bigcup_{n=0}^{\infty} X^n \times I_n$. We shall show that for any $A \in \text{(NA)}^X$, the obvious projection $p: L \to X$ induces isomorphic homology in all dimensions. Some notations of previous sections will be used without reference.

<u>Lemma 10.1</u>. For any constant co-stack $E \in \text{(NA)}^X$, the projection $p:(X\times W,\ X'\times W) \to (X,X')$ defined by $p(x,\sigma) = x$ induces a chain equivalence

$$(10.1) \qquad \underset{\sim}{C}p:\ \underset{\sim}{C}(X\times W,\ X'\times W;\ p^{\#}E) \to \underset{\sim}{C}(X,X';E)\ .$$

<u>Proof</u>. Let $\otimes: \text{(A)} \times \text{(Ab)} \to \text{(A)}$ be the tensor functor defined by Freyd [3, p. 86] and let $\underset{\sim}{C}(X,X';Z)$ be the usual free chain complex of (X,X') . Then, if $\widetilde{E}$ denotes the constant value of the co-stack E , we have

$$\underset{\sim}{C}(X,X';E) \approx \widetilde{E} \otimes \underset{\sim}{C}(X,X';Z)\ ,$$

$$\underset{\sim}{C}(X\times W,X'\times W;p^{\#}E) \approx \widetilde{E} \otimes \underset{\sim}{C}(X\times W,X'\times W;Z)\ .$$

It is well-known that p induces a chain equivalence of the free chain complexes. This gives rise to the chain equivalence (10.1). Q.E.D.

<u>Remark</u>. Whenever the induced morphisms in homology are concerned, the coefficients of the homology are suitably related by pull-back functors. To save writing, we shall omit some coefficients which are unambiguous from the content. On the other hand, we shall sometimes write the same coefficients for the homology of a simplicial pair and those of its "subpairs" to save writing the pull-back functors determined by the inclusion maps. In any case, the genuine coefficients can be inferred from the context of the statements.

<u>Lemma 10.2</u>. Let NX_n denote the set of all non-degenerate n-simplexes of X . Then for any coefficients

$$(10.2) \qquad H_*\ (X^n,X^{n-1}) \approx \amalg_{x \in NX_n} H_*(\Delta^x,\dot{\Delta}^x)$$

canonically.

This follows immediately from proposition 9.2.

Proposition 10.3. $p: X \times W \to X$ induces isomorphism

(10.3) $$p_*: H_*(X \times W; p^{\#}A) \to H_*(X;A)$$

for any $A \in (NA)^X$.

Proof. First, we shall show by induction that

(10.4) $$H_*(X^n \times W; p^{\#}A) \approx H_*(X^n;A)$$

for any non-negative integer n . The crucial point is the fact that $(p^{\#}A)(x,\sigma) = Ap(x,\sigma) = Ax$ for all $\sigma \in W$ and then $H_*(\Delta^x, \dot{\Delta}^x)$ and $H_*(\Delta^x \times W, \dot{\Delta}^x \times W)$ have constant coefficients for any fixed $x \in NX$.

For the case $n = 0$, $H_*(X^o \times W) = \amalg_{x \in X_o} H_*(\Delta^x \times W)$ is isomorphic to $\amalg_{x \in X_o} H_*(\Delta^x)$ since, by lemma 10.1, each summand $H_*(\Delta^x \times W)$ is isomorphic to $H_*(\Delta^x)$. Hence we have $H_*(X^o \times W) \approx H_*(X^o)$.

Assume inductively that $H_*(X^r \times W) \approx H_*(X^r)$ for $r = 1,2,\dots,n-1$, and consider the commutative diagram

$$\begin{array}{ccccccccc}
\dots \to & H_q(X^{n-1} \times W) & \to & H_q(X^n \times W) & \to & H_q(X^n \times W, X^{n-1} \times W) & \to & H_{q-1}(X^{n-1} \times W) & \to \dots \\
& \downarrow 2 & & \downarrow 3 & & \downarrow 4 & & \downarrow 5 & \\
\dots \to & H_q(X^{n-1}) & \to & H_q(X^n) & \to & H_q(X^n, X^{n-1}) & \to & H_{q-1}(X^{n-1}) & \to \dots ,
\end{array}$$

where the maps 2 and 5 are isomorphisms. Since

$$H_*(X^n \times W, X^{n-1} \times W) \approx \amalg_{x \in NX_n} H_*(\Delta^x \times W, \dot{\Delta}^x \times W)$$

and $H_*(X^n, X^{n-1}) \approx \amalg_{x \in X_n} H_*(\Delta^x, \dot{\Delta}^x)$ by lemma 10.2, it follows from lemma 10.1 that the map 4 is an isomorphism. Hence, by the five lemma, the map 3 is an isomorphism. This proves (10.4) and, of course, the case when X is finite dimensional.

Now, suppose that X is infinite dimensional, $X^o \subset X^1 \subset X^2 \subset \dots \subset X$. Clearly, $H_q(X^n) = H_q(X^{n-1}) = \dots = H_q(X)$ for $n > q+1$. This and (10.4) prove (10.3).

Corollary 10.4. Let $p: X \times I \to X$ be the simplicial map defined by $p(x,\sigma) = x$ for $x \in X$ and any $\sigma \in I$, then for any $A \in Ⓝⓐ^X$, $p_*: H_*(X \times I; p^\# A) \to H_*(X;A)$ is an isomorphism.

For, we have retractions $X \times W \xrightarrow{r'} X \times I \xrightarrow{r} X \times [0]$ such that $r_* r'_* = (rr')_*$ is an isomorphism.

Thus homotopic simplicial maps induce the same morphisms.

Proposition 10.5. For any $A \in Ⓝⓐ^X$, the projection $p: L \to X$ defined by $p(x,\sigma) = x$, where $(x,\sigma) \in X^n \times I_n$, $n = 0,1,2,\ldots$, induces isomorphism

$$p_*: H_*(L; p^\# A) \to H_*(X;A) . \tag{10.5}$$

Proof. Let $L^n = \bigcup_{r=0}^{n} X^r \times I_r$ and let $LX^n = L^n \cup (X^n \times [n+1,\infty])$, then $L^n \subset LX^n \subset L$. Since LX^n is a deformation retract of $X^n \times W$, $H_q(LX^n) \approx H_q(X^n \times W)$. Hence, by proposition 10.3,

$$H_q(LX^n) \approx H_q(X^n) \approx H_q(X) \quad \text{for} \quad n > q+1 .$$

Thus for any $q \ge 0$, there is $n > q+1$ such that

$$H_q(X) \approx H_q(LX^n) \approx H_q(LX^{n+1}) \approx \ldots \approx H_q(L) .$$

The proof is complete.

CHAPTER IV. HOMOLOGY THEORY ON $Ⓒ'_K$

11. K-PAIRS AND AXIOMS FOR HOMOLOGY

Let K be a fixed simplicial set. A K-pair is a simplicial pair $(X,X') \in Ⓒ'$ together with a simplicial map $\varphi: X \to K$. Such a K-pair is denoted by $(X,X')_\varphi$. $(K,K')_1$ is written (K,K') and $(X,\emptyset)_\varphi$ is written X_φ. When $\varphi = \sigma^\delta: \Delta^q \to K$, the subscript σ^δ is abbreviated by σ.

Given two K-pairs $(X,X')_\varphi$ and $(Y,Y')_\psi$, a K-map $f:(X,X')_\varphi \to (Y,Y')_\psi$ is, by definition, a simplicial map $f:(X,X') \to (Y,Y')$ such that $\varphi = \psi f$. In particular, an inclusion map $i:(Y,Y') \to (X,X')$ is a K-map $i:(Y,Y')_{\varphi i} \to (X,X')_\varphi$ for any simplicial map $\varphi:X \to K$. $(Y,Y')_{\varphi i}$ is called a K-subpair of $(X,X')_\varphi$. We shall omit the inclusion map in the notation of a K-subpair. e.g. write $(Y,Y')_\varphi$ for $(Y,Y')_{\varphi i}$, X'_φ for $X'_{\varphi i}$, X_φ for $X_{\varphi i}$, $(\Delta^x, \dot{\Delta}^x)_\varphi$ for $(\Delta^x, \dot{\Delta}^x)_{\varphi i}$, etc.

K-pairs form a category, denoted by $Ⓒ'_K$, with morphisms K-maps. Any K-pair of the form (K,K') is a terminal object (i.e. right zero object).

Let Ⓐ be a co-complete abelian category with a fixed projective generator P and having the property AB4. A homology theory on $Ⓒ'_K$ with values in Ⓐ is a sequence of functors $H_*: Ⓒ'_K \to A$, $* = 0,1,2,\ldots$, together with a family of natural transformations $\partial_q: H_q(X,X')_\varphi \to H_{q-1}X'_\varphi$, $q > 0$, satisfying the following axioms:

<u>Axiom 1. (Exactness axiom).</u> For each $(X,X')_\varphi$ with inclusion maps $X'_\varphi \xrightarrow{i} X_\varphi \xrightarrow{j} (X,X')_\varphi$ there is an <u>exact triangle of</u> $(X,X')_\varphi$,

$$\begin{array}{ccc} H_*X'_\varphi & \xrightarrow{\ i_*\ } & H_*X_\varphi \\ & {}_{\partial}\nwarrow & \downarrow j_* \\ & & H_*(X,X')_\varphi \end{array} \tag{11.1}$$

where $i_* = H_*i$, $j_* = H_*j$.

<u>Axiom 2. (Homotopy axiom).</u> If W and P are defined as in lemma 10.1, then

$$H_*p = p_* : H_*(X\times W \ , \ X'\times W)_{\varphi p} \to H_*(X,X')_\varphi$$

is an isomorphism.

Let $j_o, j_1 : (X,X') \to (X\times I, X'\times I)$ and $p:(X\times I, X'\times I) \to (X,X')$ be simplicial maps defined by $j_o x = (x,0)$, $j_1 x = (x,1)$, and $p(x,\sigma) = x$, respectively, where $x \in X$, $\sigma \in I$. Then for any simplicial map $\varphi: X \to K$, j_o, j_1 , and p are K-maps as shown in the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\ j_\alpha\ } & X\times I & \xrightarrow{\ p\ } & X \\ & {}_{\varphi}\searrow & \downarrow \varphi p & \swarrow_{\varphi} & \\ & & K & , \ \alpha = 0,1 \ . & \end{array} \tag{11.2}$$

Two K-maps $f,g:(X,X')_\varphi \to (Y,Y')_\psi$ are K-<u>homotopic</u> if there is a K-map $h:(X\times I, X'\times I)_{\varphi p} \to (Y,Y')_\psi$, called the K-<u>homotopy</u> of f and g , such that $f = hj_o$, $g = hj_1$.

The homotopy axiom implies (by corollary 10.4)

<u>Axiom 2'. (Weak homotopy axiom).</u> p_* induced by the K-projection $p:(X\times I, X'\times I)_{\varphi p} \to (X,X')_\varphi$ is an isomorphism, or equivalently, if f and g are K-homotopic, then $f_* = g_*$.

<u>Axiom 3. (Excision axiom).</u> The excision maps i and j (defined in section 9), regarded as K-maps, induce isomorphisms

$H_* i = i_*$, $H_* j = j_*$.

It follows from this axiom and (9.3) that $(\Delta^q; \Delta^{q-1}, C^{q-1})_\sigma$ is a proper triad with respect to H_* . This and the exactness axiom give rise to the diagram

(11.3)
$$\begin{array}{ccccc} H_q(\Delta^q,\dot\Delta^q)_\sigma & \xrightarrow{\partial} & H_{q-1}(\dot\Delta^q)_\sigma & \xrightarrow{h_*} & H_{q-1}(\dot\Delta^q, C^{q-1})_\sigma \\ & \searrow^{F^1} & & & \downarrow j_*^{-1} \\ & & & & H_{q-1}(\Delta^{q-1},\dot\Delta^{q-1})_{d\sigma} \end{array}$$

where h is an inclusion map, j is an excision map, and $F^1 = j_*^{-1} h_* \partial$.

<u>Axiom 4. (Dimension axiom)</u>. For any $x \in NX_q$ with $\varphi x = \sigma$, $x_*^\delta : H_q(\Delta^q,\dot\Delta^q)_\sigma \to H_q(\Delta^x,\dot\Delta^x)_\varphi$ is an isomorphism and $H_n(\Delta^q,\dot\Delta^q)_\sigma = 0$ if $n \neq q$. Moreover, if $\sigma = s^1\tau$, then F^1 defined by (11.3) is an isomorphism.

<u>Axiom 5. (Additivity axiom)</u>. Let $(X_\alpha, X'_\alpha)_\varphi$ be K-subpairs of $(X,X')_\varphi$ defined as in proposition 9.2, then

$$H_*(X,X')_\varphi \approx \coprod_\alpha H_*(X_\alpha, X'_\alpha)_\varphi$$

canonically.

<u>Axiom 6. (Deformation axiom)</u>. $p_* = H_* p$ of the K-map $p: L_{\varphi p} \to X_\varphi$ (defined as in proposition 10.5) is an isomorphism.

As usual, $H_q(X,X')_\varphi$ is called the <u>q-dimensional relative homology of</u> X_φ <u>modulo</u> X'_φ , ∂q , or simply written ∂ , the <u>boundary operator</u>, and $f_* = H_* f$ the morphism <u>induced</u> by f . The exact triangle (11.1) is called the <u>homology sequence</u> of $(X,X')_\varphi$.

12. EXISTENCE THEOREM, COEFFICIENT CO-STACKS

Let **A** be a co-stack on K with values in Ⓐ , **A** ∈ $(\text{NA})^K$.

For each $(X,X')_\varphi \in \textcircled{C'}_K$, let

(12.1) $$\widetilde{H}_*((X,X')_\varphi;A) = H_*(X,X';\varphi^\# A),$$

the right hand side is the homology of the simplicial pair (X,X') with coefficients in $\varphi^\# A \in \textcircled{NA}^X$ as defined in the previous chapter.

If $f:(X,X')_\varphi \to (Y,Y')_\psi$ is a K-map, then $\psi f = \varphi$ and so $f^\# \psi^\# = \varphi^\#$. We then have

$$\widetilde{H}_*((X,X')_\varphi;\ A) = H_*(X,X';\ f^\# \psi^\# A).$$

Let $\widetilde{H}_* f$ be the morphism

$$H_* f = f_*: H_*(X,X';\ f^\# \psi^\# A) \to H_*(Y,Y';\ \psi^\# A)$$

and write $\widetilde{H}_* f = f_*: \widetilde{H}_*((X,X)_\varphi;\ A) \to H_*((Y,Y');\ A)$. Then the functorial properties of H_* show that $\widetilde{H}_*: \textcircled{C'}_K \to A$ is a functor.

Indeed, $\widetilde{H}_*$ and H_* are essentially the same thing; the co-stack A supplies coefficients for the homologies of simplicial pairs. In particular, when K is a point (i.e. $K = V$), a K-pair is just a simplicial pair and its homology has constant coefficients.

Thus, all arguments of chapter III can be carried over with H_* and $\textcircled{C}$ replaced by $\widetilde{H}_*$ and $\textcircled{C'}_K$ and then we have

<u>Theorem 12.1. (Existence theorem)</u>. For every $A \in \textcircled{NA}^K$, there is a homology theory $\{\widetilde{H}_*, \partial\}$, simply written $\widetilde{H}_*$, on $\textcircled{C'}_K$ defined by the chain homology functor as $\widetilde{H}_*((X,X')_\varphi;A) = \overline{H}_* \underset{\sim}{C}(p_\bullet \varphi^\# A)$, or equivalently, by the generalized torsion functor as $\widetilde{H}_*((X,X')_\varphi;A) = \mathrm{Tor}_*(p_\bullet \varphi^\# A)$.

Now, let $\{H_*, \partial\}$, simply written H_*, be a homology theory on $\textcircled{C'}_K$. The <u>coefficient co-stack</u> A of H_* is defined by letting $A\sigma = H_q(\Delta^q, \dot{\Delta}^q)_\sigma$, $A(d^i) = F^i$, and $A(s^i) = (F^i)^{-1}$. We observe that the coefficient co-stack of $\widetilde{H}_*$ in theorem 12.1 is just that A.

13. UNIQUENESS THEOREM

The object of this section is to show that there is essentially one homology theory on $\mathcal{C}_K$, namely the theory $\widetilde{H}_*$ of theorem 12.1.

Let H_* be any homology theory on $\mathcal{C}_K$ and let $\emptyset = X^{-1} \subset X^1 \subset X^2 \subset \ldots \subset X_\varphi$ (the subscripts φ in X^n_φ are omitted) be the increasing filtration of X_φ by skeletons. Consider the diagram

(13.1)

$$\begin{array}{ccccccc}
 & & H_{q+1}(X^{q+1},X^q) & & & & \vdots \\
 & & \downarrow \partial_{q+1} & \searrow & & & \downarrow \\
\cdots \longrightarrow & & H_q(X^q) & \xrightarrow{i_*} & H_q(X^q,X^{q-1}) & \xrightarrow{\partial_q} & H_{q-1}(X^{q-1}) \longrightarrow \cdots \\
 & & \downarrow & & & \searrow & \downarrow j_* \\
 & & \vdots & & & & H_{q-1}(X^{q-1},X^{q-2})
\end{array}$$

where the row and columns are homology sequences of (X^q, X^{q-1}) , (X^{q+1}, X^q) , and (X^{q-1}, X^{q-2}) , respectively. We have $\partial_q i_* = 0$. Let $d_{q+1} = i_*\partial_{q+1}$ and $d_q = j_*\partial_q$, then $d_q d_{q+1} = j_*\partial_q i_*\partial_{q+1} = 0$. Thus there is a chain complex $\underset{\sim}{C}^H(X_\varphi) = \{\underset{\sim}{C}^H_q , d\}$ with $\underset{\sim}{C}^H_q = H_q(X^q , X^{q-1})$.

<u>Lemma 13.1. (Eilenberg).</u> If X is finite dimensional, $X_\varphi = X^r$, and if $H_n(X^q , X^{q-1}) = 0$ for $n \neq q$, then $\overline{H}\underset{\sim}{C}^H(X_\varphi) \approx H_*(X_\varphi)$ naturally.

<u>Proof</u>. By the hypotheses, the homology sequences of the triples $(X^{q+2} , X^{q+1} , X^{q-2})$ and $(X_\varphi , X^{q-2} , X^{q-3})$ show that

$$H_q(X^{q+1} , X^{q-2}) \approx H_q(X^{q+2} , X^{q-2}) ,$$
$$H_q(X_\varphi , X^{q-2}) \approx H_q(X_\varphi , X^{q-3}) .$$

Proceeding inductively, we have $H_q(X_\varphi) \approx H_q(X^{q+1} , X^{q-2})$. Thus it suffices to show that $H_q(X^{q+1} , X^{q-2}) \approx \overline{H}_q\underset{\sim}{C}^H(X_\varphi)$ for $q = 0,1,2,\ldots$.

Consider the diagram

$$
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
H_{q+1}(X^{q+1},X^{q}) & \xrightarrow{h} & H_{q}(X^{q},X^{q-2}) & \longrightarrow & H_{q}(X^{q+1},X^{q-2}) & \longrightarrow & 0 \\
\downarrow \partial & \searrow^{d_{q+1}} & \downarrow & & & & \\
H_{q}(X^{q}) & \xrightarrow{i_{x}} & H_{q}(X^{q},X^{q-1}) & \xrightarrow{\partial} & H_{q-1}(X^{q-1}) & & \\
 & & \downarrow & \searrow^{d_{q}} & \downarrow j_{*} & & \\
 & & H_{q-1}(X^{q-1},X^{q-2}) & = & H_{q-1}(X^{q-1},X^{q-2}) & &
\end{array}
\tag{13.2}
$$

obtained by adjoining to (13.1) the middle column and the top row which are homology sequences of the triples $(X^{q}, X^{q-1}, X^{q-2})$ and $(X^{q+1}, X^{q}, X^{q-2})$, respectively. We have $H_{q}(X^{q}, X^{q-2}) = \operatorname{Ker} d_{q}$, the image of h isomorphic to the image of d_{q+1}, and then $H_{q}(X^{q+1}, X^{q-2}) \approx \bar{H}_{q}\underset{\sim}{C}^{H}$. This completes the proof.

<u>Theorem 13.2. (Uniqueness theorem)</u>. For any homology theory H_{*} on $\mathcal{C}'_{K}$, there is a natural isomorphism

$$H_{*}(X,X')_{\varphi} \approx \widetilde{H}_{*}((X,X')_{\varphi};\ A)\ , \tag{13.3}$$

where A is the coefficient co-stack of the theory H_{*}.

<u>Proof</u>. First, we prove this for finite dimensional case, $X_{\varphi} = X^{r}$. By the additivity axiom, $H_{*}(X^{q},X^{q-1}) \approx \amalg_{x} H_{*}(\Delta^{x},\dot{\Delta}^{x})$, all $x \in NX_{q}$. We have then $H_{n}(X^{q},X^{q-1}) = 0$ for $n \neq q$ from the dimension axiom. Hence $H_{*}(X_{\varphi}) \approx \bar{H}\underset{\sim}{C}^{H}$ by lemma 13.1.

Now, $\underset{\sim}{C}^{H}_{q} = H_{q}(X^{q},X^{q-1}) \approx \amalg_{x} H_{q}(\Delta^{x},\dot{\Delta}^{x})$, we have, by the dimension axiom and the definition of A,

$$\underset{\sim}{C}^{H}_{q} \approx \amalg_{x} H_{q}(\Delta^{q},\dot{\Delta}^{q})_{\varphi x} = \amalg_{x} A\varphi(x)\ ,\ x \in NX_{q}\ .$$

Thus $\underset{\sim}{C}^{H}_{q} \approx \amalg_{x}(\varphi^{\#}A)x$. From the constructions of A and $\underset{\sim}{C}^{H}_{q}$, we observe that the two chain complexes $\underset{\sim}{C}^{H}$ and $\underset{\sim}{C}$ are isomorphic in dⒶ . We have

$$H_*(X_\varphi) \approx \overline{H}\underset{\sim}{C}(\varphi^{\#}A) = \widetilde{H}_*(X_\varphi;\ A) \ .$$

(13.3) follows from the exactness axiom and the five lemma. The naturality of the isomorphism is obvious.

Next, suppose that X is infinite dimensional. We have seen that it suffices to prove the isomorphism for the absolute case. From the first part of this proof, we see that for any fixed integer $q \geq 0$ and any integer $n > q$ there is a canonical isomorphism $H_q(X^n) \approx \widetilde{H}_q(X^n;A)$. But

(13.4) $$\widetilde{H}_q(X^n;A) \approx \widetilde{H}_q(X^{n+1};A) \approx \dots \approx \widetilde{H}_q(X_\varphi;A) \ ,$$

we have a direct system

(13.5) $$H_q(X^o) \xrightarrow{i_*^o} H_q(X^1) \xrightarrow{i_*^1} H_q(X^2) \xrightarrow{i_*^2} \dots$$

with isomorphisms i_*^n for $n > q+1$.

Now, use axioms 1,2',3,5, and 6 and proceed as in [8], we get a Mayer-Vietoris sequence

$$\begin{array}{ccc} \coprod_{n=0}^{\infty} H_*(X^n) & \xrightarrow{f} & \coprod_{n=0}^{\infty} H_*(X^n) \\ \partial' \nwarrow & & \swarrow g \\ & H_*(X_\varphi) & \end{array}$$

With Coker $f_q = \varinjlim H_q(X^n)$. Dual to the lemma 2 of [8], denote the kernel of f_q by $\underset{\sim}{L}'\{H_{q-1}(X^n)\}$ and call $\underset{\sim}{L}'$ the derived functor of $\varinjlim$, then there is an exact sequence

$$0 \to \varinjlim H_q(X^n) \to H_q(X_\varphi) \to \underset{\sim}{L}'\{H_{q-1}(X^n)\} \to 0$$

and a similar one for $\widetilde{H}_*$. Apply (13.4) and (13.5), we have $\underset{\sim}{L}'\{H_{q-1}(X^n)\} = 0$ and $H_q(X_\varphi) \approx \widetilde{H}_q(X_\varphi;A)$.

CHAPTER V. CO-STACKS OF ABELIAN GROUPS

14. THE CATEGORY $\mathbf{Ab}^K$ AND THE USUAL HOMOLOGY THEORY

Recall that the category of abelian groups, $\mathbf{Ab}$, is a complete Grothendieck category with a projective generator Z which plays the role of P in A. $\mathbf{Ab}$ has enough projectives and injectives. It has also the property $AB4^*$ of [4], i.e. product of a family of short exact sequences is a short exact sequence.

Let K be a fixed simplicial set. By the pointwise arguments for a functor category, we observe that the category of co-prestacks of abelian groups, $\mathbf{Ab}^K$, shares all the properties of $\mathbf{Ab}$ described above. $U = \amalg_{\sigma \in K} \sigma_{\#} Z^n$ is a projective generator of $\mathbf{Ab}^K$. Also, $\mathbf{Ab}^K$ has all the properties of $\mathbf{A}^X$ shown in the previous chapters.

The category of co-stacks of abelian groups, $\mathbf{NAb}^K$, has a projective generator $U^* = N^* \amalg_{\tau \in NK} (\tau_{\#} Z^n)$. We shall exhibit here the structure of U^* when K is a simplicial complex. Observe that for every $\tau \in NK_n$, the simplicial map $\tau^{\delta}: \Delta^n \to K$ is an isomorphism of Δ^n onto Δ^{τ}. Therefore $\tau_{\#} Z^n$ is the co-stack on K with integral supports on Δ^{τ}. It follows that $\amalg_{\tau \in NK} \tau_{\#} Z^n$ is normalized and then $U^* = \amalg_{\tau \in NK} \tau_{\#} Z^n$. We observe also that $(\tau_{\#} Z^n)\sigma = Z$ if $\sigma \leq \tau$, i.e. if σ is a face of τ, and it is zero otherwise. Hence, for a fixed $\sigma \in NK$,

$$U^*\sigma = \amalg_{\tau \in NK} (\tau_{\#} Z^n)\sigma = \amalg_{\sigma \leq \tau} Z(\tau) , \tag{14.1}$$

where $Z(\tau)$ is the infinite cyclic group generated by the symbol τ.

A homology theory on $\textcircled{C}'_K$ with coefficients in $A \in \textcircled{NAb}^K$ is said to *have local coefficients in groups*. We have

Theorem 14.1. There is a unique homology theory H_* on $\textcircled{C}'_K$ with local coefficients in groups.

The theory H_* is defined by chain complexes of groups or by the generalized torsion functor Tor as discussed before.

In particular, when K is a point, we have

Corollary 14.2. (Eilenberg-Steenrod). The usual homology of simplicial pairs is the only homology theory with constant group coefficients.

15. THE CATEGORY $\textcircled{Ab}_K$ AND K-MODULES

Let $\textcircled{C}_K$ be the full subcategory of $\textcircled{C}'_K$ consisting of all K-pairs of the form X_φ . We shall use (E,π) , or simply E if there is no danger of ambiguity, to denote an object in $\textcircled{C}_K$ and call (E,π) or E a K-*complex*. Morphisms of $\textcircled{C}_K$ are still called K-maps.

In this section, we shall define the notion of a $\textcircled{Ab}_K$-complex as well as that of a K-module and show that there is an isomorphism from the category of $\textcircled{Ab}_K$-complexes to $\textcircled{Ab}^K$ under which the category of K-module is identified as $\textcircled{NAb}^K$.

We observe that $K = (K, 1_K)$ is a terminal object of $\textcircled{C}_K$ and that the pull-back $E \underset{K}{\times} E'$ with

$$(E \underset{K}{\times} E')_q = \{(e,e') \in E \times E' \mid \pi e = \pi' e'\}$$

is a product in $\textcircled{C}_K$. Hence group objects of $\textcircled{C}_K$ can be defined as usual as follows:

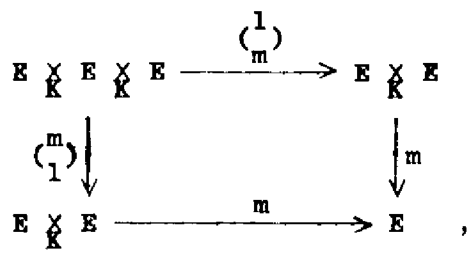
An (abelian) <u>group object</u> in $\mathbb{C}_K$ is a K-complex $E = (E,\pi)$ together with four K-maps

$$m\colon E \times_K E \to E\,, \qquad v\colon E \to E\,,$$

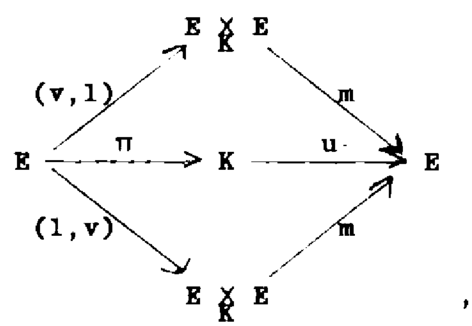
$$u\colon K \to E\,, \qquad t\colon E \times_K E \to E \times_K E\,,$$

where $t(e,e') = (e',e)$, such that the following diagrams are commutative:

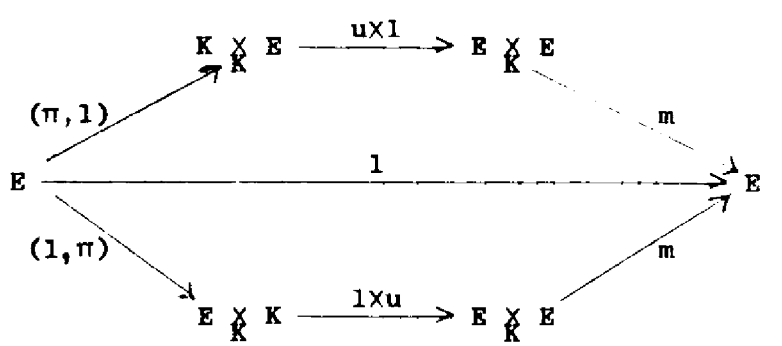
(E 1)



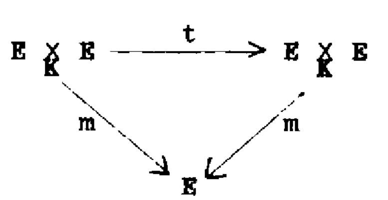
(E 2)



(E 3)



(E 4)



An (abelian) group object of $\mathbb{C}_K$ is called a Ab_K-complex. Clearly, in order that $E = (E,\pi)$ is a Ab_K-complex it is necessary that π is onto.

Let $\mathbf{E} = (E,\pi)$ be a K-complex such that π is onto. For each $\sigma \in K$, the subset $E_\sigma = \pi^{-1}\sigma$ of E is called a stalk (of $\mathbf{E}$) over σ.

If π is onto and $f:(E,\pi) \to (E',\pi')$ is a K-map, then π' is onto. f can be regarded as a family of maps $f_\sigma: E_\sigma \to E'_\sigma$ indexed by $\sigma \in K$, write $f = \{f_\sigma\}$.

We observe that the commutativity of the diagrams (E 1), (E 2), (E 3) and (E 4) implies that of the diagrams of stalks (E_σ 1), (E_σ 2), (E_σ 3) and (E_σ 4) for every $\sigma \in K$. Hence, if E_σ is a $\mathcal{Ab}_K$-complex, every E is an abelian group. On the other side, if every E_σ is an abelian group with group operations m_σ, v_σ, u_σ, t_σ and, if these operations form four K-maps $m = \{m_\sigma\}$, $v = \{v_\sigma\}$, $u = \{u_\sigma\}$, $t = \{t_\sigma\}$ (i.e. the operations commute with incidence maps). Then $\mathbf{E}$ is a $\mathcal{Ab}_K$-complex. We have the criterion

Lemma 15.1. $\mathbf{E} = (E,\pi)$ is a $\mathcal{Ab}_K$-complex if and only if

(M 1) every E_σ is an abelian group,

(M 2) every μ_σ is a homomorphism, where μ is an incidence map of $\mathbf{E}$ and $\mu_\sigma = \mu | E_\sigma$.

Proof. As we argued above, if $\mathbf{E}$ is a $\mathcal{Ab}_K$-complex, (M 1) is true. (M 2) follow from

$$\mu_\sigma(e_1 + e_2) = \mu_\sigma m_\sigma(e_1,e_2) = m_{\mu\sigma}\mu_\sigma(e_1,e_2)$$
$$= m_{\mu\sigma}(\mu_\sigma e_1,\mu_\sigma e_2) = \mu_\sigma e_1 + \mu_\sigma e_2 .$$

Conversely, condition (M 1) gives the commutativity of (E_σ1), (E_σ2), (E_σ3) and (E_σ4) for every $\sigma \in K$ while condition (M 2) implies that m_σ, v_σ, u_σ, and t_σ commute with μ_σ for every $\sigma \in K$. By the argument preceding the lemma, $\mathbf{E}$ is a $\mathcal{Ab}_K$-complex. Q.E.D.

Let E and E' be two $\mathbf{Ab}_K$-complexes A K-<u>homomorphism</u> $f\colon E \to E'$ is a K-map $f\colon E \to E'$ such that

$$\begin{array}{ccc} E \times_K E & \xrightarrow{\;m\;} & E \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ E' \times_K E' & \xrightarrow{\;m'\;} & E' \end{array}$$

commutes.

It is clear that

<u>Lemma 15.2.</u> A K-map $f\colon E \to E'$ of $\mathbf{Ab}_K$-complexes is a K-homomorphism if and only if for every $\sigma \in K$, $f_\sigma = f|E_\sigma$ is a homomorphism.

$\mathbf{Ab}_K$-complexes form a category, denoted by $\mathbf{Ab}_K$, with morphisms K-homomorphisms.

<u>Theorem 15.3.</u> $\mathbf{Ab}_K$ is isomorphic to $\mathbf{Ab}^K$.

This follows immediately from the lemmas above. For, if E is a $\mathbf{Ab}_K$-complex, then E determines a co-prestack, also denoted by E, by letting $E\sigma = E_\sigma$ and $E\mu_\sigma = \mu_\sigma$, the corresponding homomorphism μ_σ on E_σ. The correspondence $E \to E$ determines the required isomorphism of categories.

Let $\mathbf{M}_K$ be the subcategory of $\mathbf{Ab}_K$ corresponding to $\mathbf{NAb}^K$ under the isomorphism and call the objects of $\mathbf{M}_K$ K-<u>modules</u>. Then theorem 14.1 says that H_* is the only homology theory on $\mathbf{C'}_K$ with <u>coefficients in a K-module</u>. When K is a point, a K-module is represented by an abelian group and $\mathbf{NAb}_K \approx \mathbf{Ab}$.

16. FINITE K-MODULES

Let K be a fixed <u>finite</u> simplicial set, i.e. one such that NK is a finite set. Then U^* is a finite co-product. Compute

$$\mathbf{NAb}^K(U^*, A) \approx \mathbf{Ab}^K(U, A)\ ,\quad A \in \mathbf{NAb}^K\ ,$$

we have $\mathbf{NAb}^{K}(U^*,A) \approx \prod_{\sigma \in NK} A\sigma$ by (4,4). This is isomorphic to $\coprod_{\sigma \in NK} A\sigma$ since NK is finite. From this, it is easily verified that the functor $\mathbf{NAb}^{K}(U^*,-)$ preserves arbitrary co-products. This says that U^* is small.

Let R denote the ring of endomorphisms of U^*, then the additive group of R is

$$\mathbf{NAb}^{K}(U^*,U^*) \approx \mathbf{Ab}^{K}(U,U^*) \approx \coprod_{\sigma \in NK}(U^*\sigma) .$$

By (14.1), we see that if K is a simplicial complex this is $\coprod_{\sigma \leq \tau} Z(\sigma,\tau)$, where $\sigma,\tau \in NK$ and $Z(\sigma,\tau)$ is the infinite cyclic group generated by the symbol (σ,τ). We observe that the multiplication in R is defined by

$$(16.1) \qquad (\sigma,\tau)(\tau',\rho) = \begin{cases} 0 & \text{if } \tau \neq \tau' \\ (\sigma,\rho) & \text{if } \tau = \tau' . \end{cases}$$

Now, for each $A \in \mathbf{NAb}^{K}$, $\mathbf{NAb}^{K}(U^*,A)$ can be regarded as a right R-module by the composition $U^* \xrightarrow{r} U^* \xrightarrow{f} A$. In particular, when $A = Z^K$, the constant integer co-stack, we have

$$\mathbf{NAb}^{K}(U^*,Z^K) \approx \coprod_{\sigma \in NK} Z(\sigma)$$

regarding as a right R-module by the operation

$$(16.2) \qquad \sigma(\sigma',\tau) = \begin{cases} 0 & \text{if } \sigma \neq \sigma' \\ \tau & \text{if } \sigma = \sigma' . \end{cases}$$

This formula determines the way that R operates.

Identify $\mathbf{M}_K$ as $\mathbf{NAb}^{K}$. Then, by the well-known embedding theorem of Freyd-Gabriel-Mitchell, we have

Theorem 16.1. Let K be a finite simplicial set and let R be the ring of endomorphisms of U^*. If $\mathbf{M}_R$ denotes the category of right R-modules, then $h_U = \mathbf{M}_K(U^*,-)$ can be regarded as a functor from $\mathcal{M}_K$ to $\mathcal{M}_R$. As such, it is an equivalence of categories.

In the theorem if K is a simplicial complex, $R = \coprod_{\sigma \leq \tau} Z(\sigma,\tau)$, $\sigma, \tau \in NK$, with the multiplication table (16.1) and the operation rule determined by (16.2).

Now, let S be the inverse of h_U, i.e. S is a functor from $\mathcal{M}_R$ to $\mathcal{M}_K$ such that $h_U S$ and $S h_U$ are isomorphic to the identity functors of $\mathcal{M}_R$ and $\mathcal{M}_K$, respectively. Let T be the generalized tensor functor on $\mathcal{M}_K$ and let $T^* = TS: \mathcal{M}_R \to \mathcal{Ab}$. Then T^* is co-continuous. Thus, by a theorem of Eilenberg-Watts, T^* is isomorphic to a tensor functor. Precisely,

(16.3) $$T^* \approx - \otimes_R M^* ,$$

where $M^* = T^* R$ is regarded as a left R-module via the composite of the R-homomorphisms

(16.4) $$R \xrightarrow{\text{left multiplication}} \mathrm{Hom}_R(R,R) \xrightarrow{T^*} \mathrm{Hom}(M^*,M^*) .$$

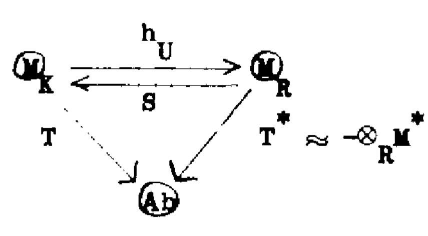
The details can be found in [1].

We observe that if K is a simplicial complex, $M^* = TU^* \approx \coprod_{\tau \in NK} T(\tau_\# Z^n) = \coprod_{\tau \in NK} Z(\tau)$ is indeed a left R-module with the operation

(16.5) $$(\sigma,\tau')\tau = \begin{cases} 0 & \text{if } \tau' \neq \tau \\ \tau & \text{if } \tau' = \tau . \end{cases}$$

<u>Theorem 16.2</u>. Let Tor^K denote the left derived functor of T, then $\mathrm{Tor}^K A$ is naturally isomorphic to $\mathrm{Tor}^R(h_U A, M^*)$ for every $A \in \mathcal{M}_K$.

<u>Proof</u>. In the diagram



,

Sh_U is isomorphic to the identity functor of $\mathfrak{M}_K$. We have $T \approx TSh = T^* h$. By (16.3) there is a natural isomorphism $TA \approx (hA) \otimes_R M^*$ which extends to a natural isomorphism

(16.6) $$\mathrm{Tor}^K A \approx \mathrm{Tor}^R(hA, M^*) \text{ , where } h = h_U \text{ .}$$

The proof is complete.

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AUTOBIOGRAPHICAL STATEMENT

Yuh-ching Chen was born on May 20, 1930 in Putien, Futien, China. He completed his primary and secondary education in Putien. He entered Taiwan Normal University, Taipei, Taiwan in 1948, graduating with B.Ed. in mathematics in 1953 after a one year teaching practice at Cheng Kwoun High School, Taipei. From 1953 to 1959 he was a full-time teaching assistant at Taiwan Normal University. He was married to Miss Jane Yun Huang in 1955. They had their first child Shwu-ming, a girl, on November 15, 1956. Their son Yie-ming was born on April 16, 1958.

He taught at Nanyang University, Singapore from 1959 to 1960, and spent the next year at Pei Yuan High School, Kampar, Malaya as a visiting teacher. From 1962 to 1964 he was a teaching assistant at University of Illinois, receiving his M.S. in Mathematics in 1963 and becoming a Ph.D. candidate in 1964. The next year he was an assistant professor at the University of Minnesota, Morris. He was admitted to The City University of New York in June, 1965 to complete his degree under his former academic advisor. He is a member of the American Mathematical Society.