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A SYSTEM OF EXTENDED SET THEORY

by

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PREFACE

A system K of set theory is defined which contains classes, called "multiplicities," and sets, called "extended classes." Certain extended classes, called "extended sets," are singled out; these are characterized by the property that their cardinalities are less than a fixed cardinal number α_0, which is shown to be strongly inaccessible. The terminology and the axioms are those of Friedman, J.I.: Proper Classes as Members of Extended Sets, Math. Ann. 183, 232-240 (1969), with the maximality axiom replaced by the cardinality restriction on extended sets. Analogously to the von Neumann-Bernays-Gödel system (NBG), the system K is obtainable from an extended version of the von Neumann function. In fact, for each ordinal number β, the extended classes M_β and K_β given by the axiom of universes, and consisting respectively of extended sets and proper extended classes, coincide respectively with extended classes S_β and C_β arising from the inductive definition of the extended von Neumann function. Some results on cardinal numbers are obtained which involve the α_β, M_β and K_β, where the α_β are the ordinal numbers of K which are not extended sets. Again analogously with NBG, the extended von Neumann function together with the inacessible ordinals α_β with $\beta > 0$ (if any) yield the natural models of K. The system K is relatively consistent with NBG together with the hypothesis of the existence of two inaccessible ordinals. Gödel's construction which shows the relative consistency of the axiom of constructibility with NBG is extended to K.

Bracketed numbers refer to the bibliography. Within each section a number cited in a proof refers to an earlier theorem in that section. All notation used may be found in [1], [2] and [3].

I wish to thank Professor Elliott Mendelson of Queens College of the City University of New York for suggesting that I axiomatize and study a system based upon the extended von Neumann function, and for suggesting much of the reading I have done in set theory and mathematical logic over several years.

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1. Introduction. The first objective is to modify the extended set theory of [1] so that any extended set of the resulting theory K will have the same cardinality as some extended set of M_0. By analogy with the von Neumann function of NBG, the extended von Neumann function φ given below (without formal justification) is required to be definable as a function of K, and to yield all of the extended classes and extended sets of K. The ordinal numbers α in the definition of $\varphi_\beta{}'\alpha$ may be regarded as those of M_0; note that φ_0 is the ordinary von Neumann function.

$$\varphi_0{}'0 = 0 \qquad\qquad \varphi_\beta{}'0 = \bigcup_{\gamma<\beta} S_\gamma \cup C_\gamma$$

$$\varphi_0{}'\alpha+1 = P(\varphi_0{}'\alpha) \qquad\qquad \varphi_\beta{}'\alpha+1 = \{u : u \subseteq \varphi_\beta{}'\alpha \wedge u \prec S_0\}$$

$$\mathit{Lim}(\lambda) \rightarrow \varphi_0{}'\lambda = \bigcup \varphi_0{}''\lambda \qquad\qquad \mathit{Lim}(\lambda) \rightarrow \varphi_\beta{}'\lambda = \bigcup \varphi_\beta{}''\lambda$$

$$S_0 = \bigcup \varphi_0{}''On_M \qquad\qquad S_\beta = (\bigcup \varphi_\beta{}''On_M) - \bigcup_{\gamma<\beta} C_\gamma$$

$$C_0 = P(\bigcup \varphi_0{}''On_M) - S_0 \qquad\qquad C_\beta = P(\bigcup \varphi_\beta{}''On_M) - S_\beta$$

Write U_α for $\bigcup \varphi_\alpha{}''On_M$, P_α for $P(\bigcup \varphi_\alpha{}''On_M)$. Then by induction,

$$S_\alpha = U_\alpha - \bigcup_{\lambda<\alpha} (P_\lambda - U_\lambda),$$

$$C_\alpha = P_\alpha - S_\alpha = P_\alpha - (U_\alpha - \bigcup_{\lambda<\alpha} (P_\lambda - U_\lambda)).$$

$$\therefore\ S_\alpha \cup C_\alpha = (U_\alpha - \bigcup_{\lambda<\alpha} (P_\lambda - U_\lambda)) \cup P_\alpha.$$

$$\therefore\ \bigcup_{\alpha<\beta} S_\alpha \cup C_\alpha = \bigcup_{\alpha<\beta} (U_\alpha - \bigcup_{\lambda<\alpha} (P_\lambda - U_\lambda)) \cup \bigcup_{\alpha<\beta} P_\alpha = \bigcup_{\alpha<\beta} U_\alpha \cup \bigcup_{\alpha<\beta} P_\alpha, \text{ i.e.,}$$

$$\bigcup_{\alpha<\beta} S_\alpha \cup C_\alpha = \bigcup_{\alpha<\beta} (\bigcup \varphi_\alpha{}''On_M) \cup \bigcup_{\alpha<\beta} P(\bigcup \varphi_\alpha{}''On_M).$$

2. Definition of the theory K. K is a first-order theory with a 2-place predicate ϵ and a 1-place predicate M. Upper-case letters are used for the individual variables and will be said to denote "multiplicities." Lower-case letters are used for individual variables which denote multiplicities X such that (EY)(XϵY). Such multiplicities will be called "extended classes;" their properties are given by the "general axioms," which in form are the axioms for NBG. (Thus the multiplicities of K correspond to the classes of NBG, and the extended classes of K to the sets of NBG.) The comprehension schema of NBG gives

the existence of the multiplicity $\{x:Mx\}$, also denoted by M, and the multiplicity $\{x:x=x\}$, denoted by V. If $x \in M$, x is called an "extended set." The extended sets satisfy the "special axioms" (in addition to the "general axioms"); the first six of these are as in [1].

General axioms:

Extensionality $X=Y \to (X \in Z \leftrightarrow Y \in Z)$.

Pairing $(x)(y)(Ez)(u)(u \in z \leftrightarrow u=x \vee u=y)$.

Class existence axioms B1-B7 as in [2].

Sum $(x)(Ey)(u)(u \in y \leftrightarrow (Ev)(u \in v \wedge v \in x))$.

Replacement $(x)(Un(X) \to (Ey)(u \in y \leftrightarrow (Ev)(\langle v,u\rangle \in X \wedge v \in x)))$.

Power $(x)(Ey)(u)(u \in y \leftrightarrow u \subseteq x)$.

Strong axiom of choice $(EX)(Fnc(X) \wedge (u)(u \neq 0 \to X'u \in u))$.

Restriction $(X)(X \neq 0 \to (Ey)(y \in X \wedge y \cdot X = 0))$.

Infinity $(Ex)(0 \in x \wedge (u)(u \in x \to u \cup \{u\} \in x))$.

Special axioms:

Subsets$_M$ $x \in M \to x \cdot Z \in M$.

Union$_M$ $x \in M \to \cup_M x \in M$ $(\cup_M x = \cup(x \cdot M))$.

Power$_M$ $x \in M \to Px \in M$.

Pairing$_M$ $x,y \in M \to \{x,y\} \in M$.

Infinity$_M$ $(Ex)(x \in M \wedge Tx \subseteq M \wedge 0 \in x \wedge (u)(u \in x \to u \cup \{u\} \in x))$ $(Tx = x \cup \cup x \cup \cup\cup x \cup \cdots)$.

Universes $\{x: Tx \subseteq M \cup z\} \in V$.

Cardinality $x \in M \leftrightarrow Ey \in M, Ty \subseteq M \wedge x \approx y$. (This is the requirement that extended sets have "small" cardinality.)

Notation: The axiom of universes permits the definition of the extended classes $K_\beta = \{x \in V-M: Tx \subseteq M \cup \bigcup_{\gamma<\beta} K_\gamma\}$ and $M_\beta = \{x \in M: Tx \subseteq M \cup \bigcup_{\gamma<\beta} K_\gamma\}$.

3. Some immediate results. As in [1], the following are proved from the general axioms and the first six of the special axioms.

1. $K_\gamma M_\beta = 0$; $\gamma \leq \beta \to (K_\gamma \subseteq K_\beta;\ M_\gamma \subseteq M_\beta)$.

2. $V = \bigcup_{\beta \in O_n} M_\beta \cup K_\beta$; $M = \bigcup_{\beta \in O_n} M_\beta$; $V-M = \bigcup_{\beta \in O_n} K_\beta$.

3. $M_\beta \in K_\beta$; $M_\beta \prec K_\beta$ (proof uses axiom of choice).

4. $x, y \in M_\beta \to x \cdot \mathcal{Z}$, $\cup_\mu x$, Px, $\{x, y\} \in M_\beta$.

4. Formal definition of the extended von Neumann function.

1. Let $On_\mu = On \cdot M$, $On_{\mu_0} = On \cdot M_0$. Then $On_\mu = On_{\mu_0}$. (For otherwise, E least $x \in On_\mu - On_{\mu_0}$. Then $x \in M \to Ew \in M_0, x \approx w$. $\therefore y \in x \to Ev \leq w, y \approx v$. Since $Tv \leq Tw$, $v \in M_0$. $\therefore y \in M$. $\therefore y \in M_0$. $\therefore x \leq On_{\mu_0} \leq M_0 \leq M$. If $z \in \cup x$, $Ey \in x$, $z \in y$. $\therefore z \in x$, since $z \in y \in x$. $\therefore \cup x \leq x \leq M$. In general, $Tx \leq M$. Since $x \in M$, $x \in M_0$, $\to\leftarrow$.) $\therefore On_\mu = On_{\mu_0} = On \cdot M_0 \leq M_0 \in V$, so $On_\mu \in V$.

2. There is a function ψ with domain On_μ, $\psi'0 = 0$, $\psi'\alpha + 1$ $P(\psi'\alpha)$, $Lim(\lambda) \to \psi'\lambda = \cup\psi''\lambda$. (Proof is by the general axioms, i.e., by NBG. By 1., On_μ is an extended class, and by definition of On and M_0, $u \in On_{\mu_0} \wedge w \in u \to w \in On_{\mu_0}$. $\therefore On_{\mu_0}$ is an ordinal number.) Let $S_0 = \cup\psi''On_\mu$.

3. $(X_0)(X_1)(X_2)(E_1 Y)(Fnc(Y) \wedge D(Y) = On \times On_\mu \wedge (\beta)(Y'\langle\beta, 0\rangle = X_0'(\beta \times On_\mu \upharpoonright Y) \wedge (\alpha)(Y'\langle\beta, \alpha'\rangle = X_1'Y'\langle\beta, \alpha\rangle \wedge (Lim(\alpha) \to Y'\langle\beta, \alpha\rangle = X_2'(\{\beta\} \times \alpha \upharpoonright Y)))))$. (Proof is by the general axioms; the fact $On \not\in V$ is used.)

Proof: Let $Y_1 = \{u; Fnc(u) \wedge (Ev \in On)(D(u) = v \times On_\mu \wedge (\beta)(\beta < v \to u'\langle\beta, 0\rangle = X_0'(\beta \times On_\mu \upharpoonright u) \wedge (\alpha \in On_\mu)(u'\langle\beta, \alpha'\rangle = X_1'u'\langle\beta, \alpha\rangle \wedge (Lim(\alpha) \to u'\langle\beta, \alpha\rangle = X_2'(\{\beta\} \times \alpha \upharpoonright u)))))\}$. If $u_1, u_2 \in Y_1$, then either $u_1 \leq u_2$ or $u_2 \leq u_1$. For let $v_1 \times On_\mu = D(u_1)$, $v_2 \times On_\mu = D(u_2)$. Assume $v_1 \leq v_2$. Let $w = \{\alpha : \alpha < v_1 \wedge (E\xi \in On_\mu)(u_1'\langle\alpha, \xi\rangle \neq u_2'\langle\alpha, \xi\rangle)\}$. If $w \neq 0$ let η be the least ordinal in w. Then $u_1'\langle\eta, 0\rangle = X_0'(\eta \times On_\mu \upharpoonright u_1) = X_0'(\eta \times On_\mu \upharpoonright u_2)$ $u_2'\langle\eta, 0\rangle$. If $E\alpha, u_1'\langle\eta, \alpha\rangle \neq u_2'\langle\eta, \alpha\rangle$, let ϵ be the least such. Then $\epsilon > 0$. If $\epsilon = \nu'$, $u_1'\langle\eta, \epsilon\rangle = X_1'u_1'\langle\eta, \nu\rangle = X_1'u_2'\langle\epsilon, \nu\rangle = u_2'\langle\eta, \epsilon\rangle$. If $Lim(\epsilon)$, then $u_1'\langle\eta, \epsilon\rangle = X_2'(\{\eta\} \times \epsilon \upharpoonright u_1) = X_2'(\{\eta\} \times \epsilon \upharpoonright u_2) = u_2'\langle\eta, \epsilon\rangle$. Hence $(\xi \in On_\mu)(u_1'\langle\eta, \xi\rangle = u_2'\langle\eta, \xi\rangle)$. $\therefore u_1 = (v_1 \times On_\mu) \upharpoonright u_1 = (v_1 \times On_\mu) \upharpoonright u_2 \leq u_2$. Thus any two functions in Y_1 agree on their common domain. Let $Y = \cup Y_1$. (Y is a multiplicity as it is defined by a predicative formula.) Then Y is a function. For $Y \leq V \times V$; and $Un(Y)$ since $\langle\alpha, \beta_1\rangle \in Y \wedge \langle\alpha, \beta_2\rangle \in Y \to Eu_1, u_2 \in Y_1$, $\langle\alpha, \beta_1\rangle \in u_1 \wedge \langle\alpha, \beta_2\rangle \in u_2$; but $u_1 \leq u_2$ or $u_2 \leq u_1$. For some $\beta \in On$, $D(Y) = \beta \times On_\mu$, or $D(Y) = On \times On_\mu$. For $D(Y) = Z \times On_\mu$ for some $Z \leq On$. Let $x \in Z$. Show $x = Seg(Z, x)$.

Since $x\in On$, $\{w: w\in Z \wedge w<x\}\subseteq x$. Let $w\in x$. $\mathrm{E}u\in Y_1 \wedge \alpha\in On_\mu, \langle x,\alpha\rangle\in D(u)$. $\therefore \langle w,\alpha\rangle\in D(u)$. $\therefore w\in z$. $\therefore x\ Seg(Z,x)$. $\therefore Z\in On \vee Z=On$. Suppose $Z\in On$. Let $W=Y\cup \{\langle\langle Z,0\rangle, X_0`(Z\times On_\mu \upharpoonright Y)\rangle\}\cup \{\langle\langle Z,\alpha'\rangle, X_1`Y`\langle Z,\alpha\rangle\rangle : \alpha\in On_\mu\}\cup \{\langle\langle Z,\alpha\rangle, X_2`(\{Z\}\times\alpha\upharpoonright Y)\rangle : \alpha\in On_\mu \wedge Lim(\alpha)\}$. Then $W\in Y_1$, so $D(W)\subseteq D(Y)$. But $D(Y)= Z\times On_\mu \subsetneq Z'\times On_\mu = D(W)$, →←.

Uniqueness of Y: Suppose Z also has the defining property of Y. If $Y\neq Z$, let α be the first ordinal, $\mathrm{E}\beta\in On, Y`\langle\beta,\alpha\rangle\neq Z`\langle\beta,\alpha\rangle$. Then $Y`\langle\beta,0\rangle = X_0`(\beta\times On_\mu\upharpoonright Y)=X_0`(\beta\times On_\mu\upharpoonright Z)=Z`\langle\beta,0\rangle$. $\therefore \alpha\neq 0$. If $\alpha=\gamma'$, $Y`\langle\beta,\alpha\rangle=X_1`Y`\langle\beta,\gamma\rangle= X_1`Z`\langle\beta,\gamma\rangle=Z`\langle\beta,\alpha\rangle$. $\therefore \alpha\neq\gamma'$. If $Lim(\alpha)$, $Y`\langle\beta,\alpha\rangle=X_2`(\{\beta\}\times\alpha\upharpoonright Y)=X_2`(\{\beta\}\times\alpha\upharpoonright Z) = Z`\langle\beta,\alpha\rangle$. $\therefore \sim Lim(\alpha)$. $\therefore Y=Z$.

4. In 3., take $X_0=\{\langle u,v\rangle : v=\bigcup R(u)\cup \bigcup_{x\in D(D(u))} P(\bigcup R(u\cdot(\{x\}\times V\times V)))\}$, $X_1= \{\langle u,v\rangle : v=\{x: x\subseteq u \wedge x\prec S_0\}\}$, $X_2=\{\langle u,v\rangle : v=\bigcup R(u)\}$. Then Y is the required function φ. The notation $\varphi_\beta`\alpha$ is used for $\varphi`\langle\beta,\alpha\rangle$. By NBG, $u\in S_0\to u\prec S_0$, so that $\varphi_0=\psi$. ($u\in S_0\to$ E minimal α, $u\subseteq\psi`\alpha$. If $\psi`\alpha\approx S_0$ then $F(\psi`\alpha)\succ S_0$. But $F(\psi`\alpha)=\psi`\alpha+1\subseteq S_0$, →←.)

5. From the general axioms it follows that the extended classes $S_\alpha\cup C_\alpha$ constitute a multiplicity, and hence there is a multiplicity $H=\bigcup_{\alpha\in On_\mu} S_\alpha\cup C_\alpha$.

5. <u>Some properties of the S_α and C_α; $S_\alpha=M_\alpha$ and $C_\alpha=K_\alpha$.</u>

1. $\alpha<\beta \to \bigcup \varphi_\alpha''On_\mu\subseteq\bigcup \varphi_\beta''On_\mu$.

Proof: Let $x\in\bigcup \varphi_\alpha''On_\mu$. If $x\in S_\alpha$ then $x\in \varphi_\beta`0\subseteq\bigcup \varphi_\beta''On_\mu$. If $x\notin S_\alpha$, then by definition of $S_\alpha$, $x\in C_\gamma$ for some $\gamma<\alpha$. $\therefore x\in\varphi_\beta`0\subseteq\bigcup \varphi_\beta''On_\mu$.

2. $(\alpha)(\beta)(C_\alpha\cdot S_\beta=0)$.

Proof: If $\alpha=\beta$, $C_\alpha\cdot S_\beta=0$ by definition of C_α. If $\alpha<\beta$, $C_\alpha\cdot S_\beta=0$ by definition of S_β. Assume $\beta<\alpha$. True if $\alpha=0$. Assume for ordinals $<\alpha$. Suppose $x\in S_\beta\cdot C_\alpha$. Then $x\in C_\alpha\to x\notin S_\alpha$. By 1., $x\in S_\beta\to x\in\bigcup \varphi_\alpha''On_\mu$. By definition of S_α, $x\notin S_\alpha\to x\in C_\gamma$ for some $\gamma<\alpha$. If $\gamma\leq\beta$ then $x\in S_\beta\cdot C_\gamma=0$ by the previous cases, →←. If $\beta<\gamma$, $x\in S_\beta\cdot C_\gamma=0$ by induction hypothesis, →←.

3. $\alpha<\beta\to S_\alpha\subseteq S_\beta \wedge C_\alpha\subseteq C_\beta$.

Proof: If $x\in S_\alpha$ then $x\in\varphi_\beta`0\subseteq\bigcup \varphi_\beta''On_\mu$, and $x\notin\bigcup_{\gamma<\beta} C_\gamma$ by 2. $\therefore x\in S_\beta$. If $x\in C_\alpha$

then $x \in \bigcup \psi_\alpha``On_\mu \subseteq \bigcup \psi_\beta``On_\mu$, and $x \notin S_\beta$ by 2. ∴ $x \in C_\beta$.

4. $u \subseteq H \to u \in H$.

Proof: Let $u \in V \wedge u \subseteq \bigcup_{\beta \in \sigma_\mu} S_\beta \cup C_\beta$. If $x \in u$, E least β, $x \in S_\beta \cup C_\beta$. By the axiom of replacement, these β form an extended class. Let λ be greater than all such β. Then $x \in u \to x \in S_\beta \cup C_\beta \subseteq S_\lambda \cup C_\lambda$, by 3. ∴ $u \subseteq S_\lambda \cup C_\lambda = \psi_{\lambda+1}`0$. If $u \notin S_{\lambda+1}$, then $u \in P(\bigcup \psi_{\lambda+1}``On_\mu) - S_{\lambda+1} = C_{\lambda+1}$. ∴ $u \in \bigcup_{\alpha \in \sigma_\mu} S_\alpha \cup C_\alpha = H$.

5. $V=H$.

Proof: If $V-H \neq 0$, $\exists y \in V-H, y \cdot (V-H)=0$. Hence by 4., $y \in H$.

6. $S_0 \subseteq M$.

Proof: (i) Assertion: $\beta \in On_\mu \to \psi_0`\beta \in M$. (Proof by induction.)

(a) $\beta=0$. In $Subsets_\mu$ take $x \in M$, x is infinite and $Z=0$.

(b) $\beta=\gamma+1$. By $Power_\mu$, $\psi_0`\beta = P(\psi_0`\gamma) \in M$.

(c) Lim(β). Since $\beta \in V$, $\{\psi_0`\xi : \xi<\beta\} \in M$ by Cardinality. Also, $\{\psi_0`\xi : \xi<\beta\} \subseteq M$ by induction hypothesis. ∴ $\psi_0`\beta \in M$ by $Union_\mu$.

(ii) Assertion: $\beta \in On_\mu \to \psi_0`\beta \subseteq M$. (Proof by induction.)

(a) $\beta=0$. True ∵ $\psi_0`0=0$.

(b) $\beta=\gamma+1$. If $x \in \psi_0`\beta$ then $x \subseteq \psi_0`\gamma$. By (i), $\psi_0`\gamma \in M$, so $x \in M$ by $Subsets_\mu$.

(c) Lim(β). Let $x \in \psi_0`\beta$. E $\gamma<\beta$, $x \in \psi_0`\gamma$. By induction hypothesis, $\psi_0`\gamma \subseteq M$, so $x \in M$.

7. $S_0 \approx On_\mu$.

Proof: By induction on α, $\alpha \in On_\mu \to \alpha \in \psi_0`\alpha+1$. ∴ $On_\mu \subseteq S_0$. But $S_0 \preceq On_\mu \times On_\mu$. For let $x \in S_0$. E minimal $\alpha$, $x \in \psi_0`\alpha$. By 6.(i), $\psi_0`\alpha \in M$, so E $\gamma$ which well-orders $\psi_0`\alpha$. By Cardinality, $\gamma \approx \psi_0`\alpha \to \gamma \in M$. Let x correspond to $\langle \alpha, \beta \rangle$, where $\beta$ is associated with x by the well-ordering of $\psi_0`\alpha$. Thus $S_0 \preceq On_\mu \times On_\mu \approx On_\mu$. ∴ $S_0 \approx On_\mu$.

8. $u \in \bigcup_{\gamma \in \sigma_\mu} C_\gamma$ iff $u \succeq S_0$. (Hence $u \in \bigcup_{\gamma \in \sigma_\mu} S_\gamma$ iff $u \prec S_0$, by 5.)

Proof: Let $u \in \bigcup_{\gamma \in \sigma_\mu} C_\gamma$. E minimal γ, $u \in C_\gamma$. For each $x \in u$ E minimal α, $x \in \psi_\gamma`\alpha$. Let A be the extended class of such α. Then $A \preceq u$. If $A \approx On_\mu$, then $S_0 \preceq u$ by 7. Suppose $A \prec On_\mu$. Since On_μ is an ordinal number, $A \approx \xi$

for some $\lambda\in On_M\leq M_0$. Since $A\leq M_0$, $TA\leq M_0$, so $A\in M$. $\therefore\ \bigcup A\in M$ by Union$_M$, so that A has a strict upper bound $\lambda\in On_M$. May assume $Lim(\lambda)$. Then $u\leq \varphi_\gamma{}'\lambda$. If $u\prec S_0$ then $u\in\varphi_\gamma{}'\lambda+1$. Since $u\notin\bigcup_{\tau<\gamma} C_\tau$, $u\in S_\gamma$, $\rightarrow\leftarrow$. $\therefore\ u\not\prec S_0$.

Conversely, assume $u\in\bigcup_{\gamma\in On_M} S_\gamma$. E minimal γ, $u\in S_\gamma$. E minimal α, $u\in\varphi_\gamma{}'\alpha+1$. $\therefore\ u\leq\varphi_\gamma{}'\alpha\wedge u\prec S_0$.

9. $M=\bigcup_{\alpha\in On_M} S_\alpha$. (Thus $V-M=\bigcup_{\alpha\in On_M} C_\alpha$, by 6.)

Proof: (i) Assertion: $\alpha\in On\rightarrow S_\alpha\leq M$. True if $\alpha=0$, by 6. Assume for all $\beta<\alpha$. Let $x\in S_\alpha$. By induction hypothesis, may assume α minimal, $x\in S_\alpha$. E minimal β, $x\in\varphi_\alpha{}'\beta$. By minimality of α, $\beta\neq 0$, and by minimality of β, $\sim Lim(\beta)$. $\therefore\ \beta=\gamma+1$. $\therefore\ x\leq\varphi_\alpha{}'\gamma\wedge x\prec S_0$. $\therefore$ E$y\leq S_0$, $y\prec S_0\wedge x\approx y$. By 3., $y\in C_0\rightarrow y\not\prec S_0$. $\therefore\ y\in S_0\leq M$. $\therefore\ x\in M$ by Cardinality.

(ii) Assertion: $M\leq\bigcup_{\alpha\in On_M} S_\alpha$. Let $x\in M_0$. E α which well-orders x, so $x\approx\alpha$. By Cardinality, $\alpha\in M$. $\therefore\ \alpha\in On_M\leq S_0$. Since $\alpha\in S_0$, $\alpha\prec S_0$. $\therefore\ x\prec S_0$. $\therefore\ x\in\bigcup_{\alpha\in On_M} S_\alpha$, by 8., i.e., $M_0\leq\bigcup_{\alpha\in On_M} S_\alpha$. Let $y\in M$. By Cardinality, E$z\in M$, $Tz\leq M\wedge y\approx z$. Since $z\in M_0$, $z\prec S_0$ by 3. $\therefore\ y\in\bigcup_{\alpha\in On_M} S_\alpha$ by 5.

10. (a) $\alpha\in On\rightarrow C_\alpha\succ S_0$ (hence $C_\alpha\nleq M$); (b) $\beta<\alpha\rightarrow C_\beta\in C_\alpha$.

Proof: (a) $C_\alpha\supseteq C_0=P(S_0)-S_0$, and $P(S_0)\succ S_0$.

(b) $C_\beta\leq\varphi_\alpha{}'0\leq\bigcup\varphi_\alpha{}''On_M$, and $C_\beta\nleq C_\alpha$ by (a).

11. $\bigcup\varphi_\alpha{}''On_M$ is transitive.

Proof: True for $\alpha=0$. Assume for ordinals $<\alpha$. Let $x\in\bigcup\varphi_\alpha{}''On_M$. Then $x\in\varphi_\alpha{}'\tau$ for some minimal τ. By definition of φ, τ is not a limit ordinal. If $\tau=0$ then E $\beta<\alpha$, $x\in S_\beta\cup C_\beta$. If $x\in S_\beta$, $x\in\bigcup\varphi_\beta{}''On_M$. By induction hypothesis and 1., $x\leq\bigcup\varphi_\beta{}''On_M\leq\bigcup\varphi_\alpha{}''On_M$. If $x\in C_\beta$, then $x\leq\bigcup\varphi_\beta{}''On_M\leq\bigcup\varphi_\alpha{}''On_M$. If $\tau=\eta+1$ and $x\in\varphi_\alpha{}'\tau$, then $x\leq\varphi_\alpha{}'\eta\leq\bigcup\varphi_\alpha{}''On_M$.

12. $x\in S_\beta\cup C_\beta\rightarrow Tx\leq S_\beta\cup\bigcup_{\gamma<\beta} C_\gamma$.

Proof: True if $\beta=0$. Assume for ordinals $<\beta$.

(i) $x\in S_\beta$. E minimal α, $x\in\varphi_\beta{}'\alpha$. If $\alpha=0$ then $x\in S_\tau$ for some $\tau<\beta$. $\therefore\ Tx\leq S_\tau\cup\bigcup_{\gamma<\tau} C_\gamma\leq S_\beta\cup\bigcup_{\gamma<\beta} C_\gamma$, by 3. If $\alpha>0$ then $\alpha=\tau+1$ for some τ. Since $x\leq\varphi_\beta{}'\tau\leq\bigcup\varphi_\beta{}''On_M$, $Tx\leq\bigcup\varphi_\beta{}''On_M$ by 11. Let $y\in Tx$. If $y\notin C_\gamma$ for any $\gamma<\beta$ then $y\in S_\beta$, by definition of S_β. ∴ $Tx \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$.

(ii) $x \in C_\beta$. Since $x \subseteq \bigcup \varphi_\beta''On_\mu$, $Tx \subseteq \bigcup \varphi_\beta''On_\mu$, by 11. Let $y \in Tx$. If $y \notin C_\gamma$ for any $\gamma<\beta$ then $y \in S_\beta$ by definition of S_β. ∴ $Tx \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$.

13. $x \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma \to Tx \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$.

Proof: Sufficient to show $\bigcup x \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$, and then iterate this result. Let $y \in \bigcup x$. $\exists z \in x, y \in z$. By hypothesis, $z \in S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$. If $z \in S_\beta$ then $z \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$, by 12. ∴ $y \in S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$. Assume $z \notin S_\beta$. Then $\exists$ minimal $\gamma<\beta$, $z \in C_\gamma$. ∴ $z \subseteq \bigcup \varphi_\gamma''On_\mu$. ∴ $y \in \varphi_\gamma'\alpha$ for some α. By definition of S_γ, $y \in S_\gamma$ or $y \in C_\eta$ for some $\eta<\gamma$. ∴ $y \in S_\gamma \cup \bigcup_{\xi<\gamma} C_\xi \subseteq S_\beta \cup \bigcup_{\xi<\beta} C_\xi$.

14. $x \in \bigcup_{\beta\in\theta_\mu} S_\beta \wedge Tx \subseteq S_\alpha \cup \bigcup_{\gamma<\alpha} C_\gamma \to x \in S_\alpha$.

Proof: Let $y \in x$. If $x \in S_\alpha$ then $y \in \bigcup \varphi_\alpha''On_\mu$. If $y \in C_\gamma$ for some $\gamma<\alpha$ then $y \in \varphi_\alpha'0$. For each $y \in x$, $\exists$ minimal τ, $y \in \varphi_\alpha'\tau$. Since $x \in \bigcup_{\beta\in\theta_\mu} S_\beta$, $x \in S_\delta$, so that there is a limit ordinal $\eta \in On_\mu$ which is greater than all such τ. Then $y \in x \to y \in \varphi_\alpha'\tau \subseteq \varphi_\alpha'\eta$. ∴ $x \subseteq \varphi_\alpha'\eta$. ∴ $x \in \varphi_\alpha'\eta+1$. ∴ $x \in S_\alpha$.

15. $x \in \bigcup_{\gamma\in\theta_\mu} S_\gamma \wedge Tx \subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma \cup \bigcup_{\gamma<\beta} C_\gamma \to x \in S_\beta$.

Proof: Sufficient to show α minimal, $x \in S_\alpha \to \alpha$ minimal, $Tx \subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma \cup \bigcup_{\gamma<\alpha} C_\gamma$. (For suppose $x \in \bigcup_{\gamma\in\theta_\mu} S_\gamma \wedge Tx \subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma \cup \bigcup_{\gamma<\beta} C_\gamma$. $\exists$ minimal α, $x \in S_\alpha$. By 12., $Tx \subseteq S_\alpha \cup \bigcup_{\gamma<\alpha} C_\gamma \subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma \cup \bigcup_{\gamma<\alpha} C_\gamma$. ∴ $Tx \subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma \cup \bigcup_{\gamma<\beta} C_\gamma \to \alpha \leq \beta$. ∴ $x \in S_\beta$, by 3.) Use double induction on α and η, where η is defined as follows. $\exists$ minimal ρ, $x \in \varphi_\alpha'\rho$. By minimality of α, $\rho \neq 0$. By definition of $\varphi_\alpha'\rho$, $\sim Lim(\rho)$. ∴ $\exists \eta$, $\rho=\eta+1$. ∴ $x \subseteq \varphi_\alpha'\eta$, and η is minimal with this property.

(i) $\alpha=0$. Then $Tx \subseteq S_0$, so $Tx \subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma$.

(ii) $\alpha=\beta+1$.

(a) $\eta=0$. Then $x \in S_\beta \cup C_\beta$. If $x \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$, then $Tx \subseteq S_\beta \cup \bigcup_{\gamma<\beta} C_\gamma$ by 13. ∴ $x \in S_\beta$ by 14., ↯ minimality of α. ∴ $\exists y \in x$, $y \in C_\beta \wedge y \notin C_\gamma$ for any $\gamma<\beta$. ∴ $Tx \subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma \cup \bigcup_{\gamma<\delta} C_\gamma \to \delta>\beta$, i.e., $\delta \geq \alpha$.

(b) $\eta>0$. Then $x \notin \varphi_\alpha'0 = S_\beta \cup C_\beta$. But by 12., $x \subseteq Tx \subseteq S_\alpha \cup \bigcup_{\gamma<\alpha} C_\gamma \subseteq S_\alpha \cup C_\beta$. ∴ $\exists y \in x$, $y \in S_\alpha \wedge y \notin S_\gamma$ for any $\gamma<\beta$. Now $y \in \varphi_\alpha'\eta$. Since α is minimal, $y \in S_\alpha$, $\exists$ minimal $\xi<\eta$, $y \subseteq \varphi_\alpha'\xi$. By induction hypothesis, $\delta<\alpha \to Ty \not\subseteq \bigcup_{\gamma\in\theta_\mu} S_\gamma \cup \bigcup_{\gamma<\delta} C_\gamma$.

Since $Ty \subseteq Tx$, $Tx \notin \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\xi} C_\nu$.

(iii) Lim(α).

(a) $n=0$. Then $x \subseteq \bigcup_{\nu<\alpha} S_\nu \vee C_\nu$. By minimality of α, $\beta<\alpha \rightarrow x \notin S_\beta \vee C_\beta = \varphi_{\beta+1}{}'0$ (for otherwise $x\in M \rightarrow x \in \varphi_{\beta+1}{}'1$, so $x \in S_{\beta+1}$). $\therefore$ E $y\in x$ and $\xi>\beta$, $y \in S_\xi \vee C_\xi$ and ξ is minimal with $y \in S_\xi \vee C_\xi$. By induction hypothesis, $y \in S_\xi \rightarrow \xi$ is minimal, $Ty \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\xi} C_\nu$. Since $Ty \subseteq Tx$, $Tx \notin \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\beta} C_\nu$. If $y \in C_\xi$, then $x \notin \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\beta} C_\nu$. $\therefore Tx \notin \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\beta} C_\nu$.

(b) $n>0$. Then $x \notin \bigcup_{\nu<\alpha} S_\nu \cup C_\nu$. $\therefore \beta<\alpha \rightarrow x \notin S_\beta \vee C_\beta$. Proceed as in case (a).

16. $x \in \bigcup_{\nu\in\theta_\mu} C_\nu \wedge Tx \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\beta} C_\nu \rightarrow x \in C_\beta$.

Proof: Sufficient to show α minimal, $x \in C_\alpha \rightarrow \alpha$ minimal, $Tx \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\alpha} C_\nu$. (For suppose $x \in \bigcup_{\nu\in\theta_\mu} C_\nu \wedge Tx \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\beta} C_\nu$. E minimal α, $x\in C_\alpha$. By 12., $Tx \subseteq S_\alpha \vee \bigcup_{\nu<\alpha} C_\nu \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\alpha} C_\nu$. $\therefore Tx \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\beta} C_\nu \rightarrow \alpha \leq \beta \therefore x \in C_\beta$, by 3.)

(i) $\alpha=0$. Then $Tx \subseteq S_0$, so $Tx \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu$.

(ii) $\alpha=\beta+1$. Required to show E $w\in Tx$, $w\in C_\beta$ and $w \notin C_\nu$ for any $\nu<\beta$. Since $x \notin \bigcup \varphi_\beta{}''On_\mu$, E $y\in x$, $y \notin \bigcup \varphi_\beta{}''On_\mu$. $\therefore y \notin S_\beta$. By 12., $x \in C_{\beta+1} \rightarrow y \in Tx \subseteq S_{\beta+1} \vee \bigcup_{\nu\leq\beta} C_\nu$. Suppose $y \in S_{\beta+1}$. Then $\beta+1$ is minimal with this property, so as in the first sentence of the proof of 15., E $z\in Ty$, $z\in C_\beta$ and $z\notin C_\nu$ for any $\nu<\beta$. Since $z\in Tx$, z is as required. If $y \notin S_{\beta+1}$ then $y \in C_\beta$. If E $\nu<\beta$, $y \in C_\nu$, then $y \in \varphi_\beta{}'0$. $\therefore y \in \bigcup \varphi_\beta{}''On_\mu$, $\rightarrow\leftarrow$. Hence y is as required.

(iii) Lim(α). For each $\beta<\alpha$, required to show E $w\in Tx$, $w \in C_\xi$ for some minimal $\xi\geq\beta$. Since $x \notin \bigcup \varphi_\beta{}''On_\mu$, E $y\in x$, $y \notin \bigcup \varphi_\beta{}''On_\mu$. $\therefore y \notin S_\beta$. If $y \in \bigcup_{\nu\in\theta_\mu} S_\nu$ then $y \in S_\tau$ for a minimal $\tau>\beta$. As in the first sentence of the proof of 15., $Ty \notin \bigcup_{\nu\in\theta_\mu} S_\nu \vee \bigcup_{\nu<\beta} C_\nu$. $\therefore$ E $z\in Ty$, $z\in C_\xi$ for some $\xi\geq\beta$ and $z \notin C_\nu$ for any $\nu<\xi$. Since $Ty \subseteq Tx$, z is as required. Suppose $y \in \bigcup_{\nu\in\theta_\mu} C_\nu$. E minimal ν, $y\in C_\nu$. If $\nu<\beta$, then $y \in \varphi_\beta{}'0$. $\therefore y \in \bigcup \varphi_\beta{}''On_\mu$, $\rightarrow\leftarrow$. $\therefore \nu\geq\beta$, so y is as required.

17. $\alpha \in On \rightarrow M_\alpha = S_\alpha \wedge K_\alpha = C_\alpha$.

Proof: $M_0 \overset{\text{DEFINITION}}{=} \{x\in M : Tx \subseteq M\} \overset{9.}{=} \{x \in \bigcup_{\nu\in\theta_\mu} S_\nu : Tx \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu\} \overset{12.,15.}{=} S_0$.

$K_0 \overset{\text{DEFINITION}}{=} \{x\in V-M : Tx \subseteq M\} \overset{9.}{=} \{x \in \bigcup_{\nu\in\theta_\mu} C_\nu : Tx \subseteq \bigcup_{\nu\in\theta_\mu} S_\nu\} \overset{12.,16.}{=} C_0$. Assume for all $\nu<\alpha$.

$M_\alpha \overset{\text{DEFINITION}}{=} \{x\in M : Tx \subseteq M \vee \bigcup_{\nu<\alpha} K_\nu\} \overset{9.,\text{ INDUCTION HYPOTHESIS}}{=}$

$\{x \in \bigcup_{\gamma \in \theta_\mu} S_\gamma : Tx \subseteq \bigcup_{\gamma \in \theta_\mu} S_\gamma \cup \bigcup_{\gamma < \alpha} C_\gamma\} \overset{12.,15.}{=} S_\alpha .$

$K_\alpha \overset{\text{DEFINITION}}{=} \{x \in V - M : Tx \subseteq M \cup \bigcup_{\gamma < \alpha} K_\gamma\} \overset{9.,\ \text{INDUCTION HYPOTHESIS}}{=}$

$\{x \in \bigcup_{\gamma \in \theta_\mu} C_\gamma : Tx \subseteq \bigcup_{\gamma \in \theta_\mu} S_\gamma \cup \bigcup_{\gamma < \alpha} C_\gamma\} \overset{12.,14.}{=} C_\alpha .$

6. Further properties of K; some cardinality relations.

1.(The following are true also for the system in [1].)

(a) $\{x\} \in M$ ($\therefore \{x_1, \cdots, x_n\} \in M$, by induction).

(b) $x \in K_\beta \rightarrow \{x\} \in M_{\beta+1}$.

(c) $M \notin V$.

Proof: Suppose $M \in V$. Since $M \notin M$, $M \in C_\alpha$ for some minimal α. $\therefore M \subseteq \bigcup \varphi_\alpha{}'' On_\mu$. Since $\{C_\alpha\} \in M$, $\{C_\alpha\} \in \varphi_\alpha{}'\beta$ for some minimal β. Suppose $\beta = 0$. Then $\{C_\alpha\} \in \bigcup_{\gamma < \alpha} S_\gamma \cup C_\gamma$. $\therefore \{C_\alpha\} \in S_\gamma$ for some minimal $\gamma < \alpha$. $\therefore \{C_\alpha\} \in \varphi_\gamma{}'\eta + 1$ for some minimal η. $\therefore \{C_\alpha\} \subseteq \varphi_\gamma{}'\eta$, so $C_\alpha \in \varphi_\gamma{}'\eta$. If $\eta = 0$, $C_\alpha \in \bigcup_{\tau < \gamma} S_\tau \cup C_\tau$. Since $C_\alpha \succ S_0$, $C_\alpha \notin S_\tau$. $\therefore C_\alpha \in C_\tau$ for some $\tau < \alpha$. But $\tau < \alpha \rightarrow C_\tau \in C_\alpha$, $\rightarrow\!\leftarrow$. $\therefore \eta \neq 0$. $\therefore \eta = \tau + 1$ for some τ. Since $C_\alpha \in \varphi_\gamma{}'\eta$, $C_\alpha \subseteq \varphi_\gamma{}'\tau \wedge C_\alpha \prec S_0$. But $C_\alpha \succ S_0$, $\rightarrow\!\leftarrow$. $\therefore \beta \neq 0$. $\therefore \beta = \gamma + 1$ for some γ. $\therefore \{C_\alpha\} \subseteq \varphi_\alpha{}'\gamma$. $\therefore C_\alpha \in \varphi_\alpha{}'\gamma$. If $\gamma = 0$, $C_\alpha \in \bigcup_{\tau < \alpha} S_\tau \cup C_\tau$. If $C_\alpha \in S_\tau$, $C_\alpha \prec S_0$, $\rightarrow\!\leftarrow$. $\therefore C_\alpha \in C_\tau$. But $\tau < \alpha \rightarrow C_\tau \in C_\alpha$, $\rightarrow\!\leftarrow$. $\therefore \gamma \neq 0$. $\therefore \gamma = \xi + 1$ for some ξ. Since $C_\alpha \in \varphi_\alpha{}'\gamma$, $C_\alpha \subseteq \varphi_\alpha{}'\xi \wedge C_\alpha \prec S_0$, $\rightarrow\!\leftarrow$. $\therefore M \notin V$.

(d) $M \approx V$.

Proof: $x \rightarrow \{x\}$ is 1-1 from V into M.

(e) $x \prec On_\mu \rightarrow x \in M$.

Proof: $On_\mu \subseteq M_0 = S_0$, so $x \prec S_0$.

2. In the axiom of replacement, if $x \in M$ then $y \in M$.

Proof: Let $x_1 = x \cdot D(X)$. Since $u \in y \leftrightarrow Ev \in x_1, \langle v, u \rangle \in X$, the correspondence $v \rightarrow u$ maps x_1 onto y. $\therefore y \preceq x_1 \preceq x \prec S_0$.

Notation: $\alpha_0 < \alpha_1 < \cdots$ denotes the sequence of ordinal numbers not in M.

3. $\beta \in On \rightarrow \alpha_\beta \in C_\beta \wedge (\gamma < \beta \rightarrow \alpha_\beta \notin C_\gamma)$.

Proof: (i) $\beta = 0$. If $\gamma < \alpha_0$ then $\gamma \in S_0$, so $\alpha_0 \subseteq S_0$. Since $\alpha_0 \notin S_0$, $\alpha_0 \in C_0$.

Assume for all ordinals $< \beta$.

(ii) $\beta=\gamma+1$. Then $\alpha_\beta=\alpha_\gamma+1$. By induction hypothesis, $\alpha_\gamma\in C_\gamma$, so $\alpha_\gamma\subseteq \bigcup\varphi_\gamma``On_\mu\subseteq\bigcup\varphi_{\gamma+1}``On_\mu$. Also, $\alpha_\gamma\in C_\gamma\subseteq\varphi_{\gamma+1}`0\subseteq\bigcup\varphi_{\gamma+1}``On_\mu$, so $\alpha_\beta=\alpha_\gamma\cup\{\alpha_\gamma\}$ $\subseteq\bigcup\varphi_{\gamma+1}``On_\mu$. ∴ $\alpha_\beta\in C_\beta$. Let $\xi<\beta$. To show $\alpha_\beta\notin C_\xi$ it is sufficient to show $\alpha_\beta\notin C_\gamma$. If $\alpha_\beta\in C_\gamma$, then $\alpha_\gamma\cup\{\alpha_\gamma\}\subseteq\bigcup\varphi_\gamma``On_\mu$. ∴ $\alpha_\gamma\in\bigcup\varphi_\gamma``On_\mu$. Since by induction hypothesis, $\alpha_\gamma\notin C_\tau$ for $\tau<\gamma$, $\alpha_\gamma\in S_\gamma$, →←.

(iii) Lim(β). Then $\alpha_\beta=\bigcup_{\gamma<\beta}\alpha_\gamma$. By induction hypothesis, $\gamma<\beta\to\alpha_\gamma\subseteq\bigcup\varphi_\gamma``On_\mu$ $\subseteq\bigcup\varphi_\beta``On_\mu$. ∴ $\alpha_\beta\subseteq\bigcup\varphi_\beta``On_\mu$. Since $\gamma<\beta\to\alpha_\gamma\succeq S_0$, $\alpha_\beta\notin\bigcup_{\gamma\in\theta_\mu}S_\gamma$. ∴ $\alpha_\beta\in C_\beta$. Suppose $\delta<\beta\wedge\alpha_\beta\in C_\delta$. Then $\alpha_{\delta+1}\subseteq\alpha_\beta\subseteq\bigcup\varphi_\delta``On_\mu$. ∴ $\alpha_{\delta+1}\in C_\delta$, →←.

4. $K_\beta\preceq M_{\beta+1}$.

Proof: $x\to\{x\}$ is 1-1 from C_β into $S_{\beta+1}$.

5. $x\in V-M\to\{u:u\subseteq x\wedge u\prec S_0\}\approx x$.

Proof: Since $x\succeq S_0$, $\alpha>0\wedge\alpha\in S_0\to x^\alpha\approx x$, by induction on α. ∴ $\{u:u\subseteq x\wedge u\prec S_0\}$ $\approx\bigcup_{\alpha\in S_0}\{u:u\subseteq x\wedge u\approx\alpha\}\preceq\bigcup_{\alpha\in S_0}x^\alpha\preceq x\times S_0\approx x$.

6. $S_\alpha\approx\bigcup\varphi_\alpha``On_\mu$.

Proof: $S_0=\bigcup\varphi_0``On_\mu$. Let $\alpha>0$. For all β, $C_\beta\succ S_0$, so if $\alpha>0$, $\varphi_\alpha`0=\bigcup_{\gamma<\alpha}S_\gamma\cup C_\gamma$ $\succ S_0$. ∴ $\varphi_\alpha`0\in V-M$. Assume $\varphi_\alpha`\xi\in V-M$ for all $\xi<\eta$. If $\eta=\beta+1$, $\varphi_\alpha`\eta\approx\varphi_\alpha`\beta$ by 5. ∴ $\varphi_\alpha`\eta\in V-M$. If Lim($\eta$), $\varphi_\alpha`\eta=\bigcup\varphi_\alpha``\eta$, so $\varphi_\alpha`\eta\in V-M$. Now $\varphi_\alpha`0=\bigcup_{\gamma<\alpha}S_\gamma\cup C_\gamma\approx$ $\bigcup_{\gamma<\alpha}C_\gamma$, since $C_\gamma\succ S_\gamma$. Assume $\varphi_\alpha`\xi\approx\bigcup_{\gamma<\alpha}C_\gamma$ for all $\xi<\eta$. If $\eta=\beta+1$, $\varphi_\alpha`\eta\approx\varphi_\alpha`\beta$ by 5., so $\varphi_\alpha`\eta\approx\bigcup_{\gamma<\alpha}C_\gamma$. If Lim($\eta$), $\varphi_\alpha`\eta=\bigcup\varphi_\alpha``\eta\approx(\bigcup_{\gamma<\alpha}C_\gamma)\times\eta\approx\bigcup_{\gamma<\alpha}C_\gamma$. ∴ $\bigcup\varphi_\alpha``On_\mu$ $\approx(\bigcup_{\gamma<\alpha}C_\gamma)\times On_\mu\approx(\bigcup_{\gamma<\alpha}C_\gamma)\times(On_\mu-\{0\})\approx\bigcup\varphi_\alpha``(On_\mu-\{0\})\approx\bigcup\varphi_\alpha``On_\mu-\bigcup_{\gamma<\alpha}C_\gamma=S_\alpha$.

7. $x\in C_\alpha\to x\preceq S_\alpha$.

Proof: $x\subseteq\bigcup\varphi_\alpha``On_\mu\approx S_\alpha$ by 6.

8. $C_\alpha\approx PS_\alpha$.

Proof: $C_\alpha=P(\bigcup\varphi_\alpha``On_\mu)-S_\alpha\approx PS_\alpha-S_\alpha$ by 6. But $PS_\alpha\succ S_\alpha$.

9. $C_\alpha\approx S_{\alpha+1}$.

Proof: $\varphi_{\alpha+1}`0\approx C_\alpha$. For all $\beta\in On_\mu$, $\varphi_{\alpha+1}`\beta\approx C_\alpha$ by 5. and induction on β. ∴ $\bigcup\varphi_{\alpha+1}``On_\mu\approx C_\alpha\times S_0\approx C_\alpha$. But $S_{\alpha+1}\approx\bigcup\varphi_{\alpha+1}``On_\mu$ by 6.

10. $\alpha<\beta\to S_\alpha\prec S_\beta$.

Proof: $\alpha+1\leq\beta$, so $S_\alpha\preceq S_{\alpha+1}\preceq S_\beta$. But $S_\alpha\prec S_{\alpha+1}$ by 9.

11. $\nu \le \alpha_\nu$; $\alpha_\nu \preceq S_\nu$.

Proof: $\nu \le \alpha_\nu$ by induction on ν. By 3., $\alpha_\nu \in C_\nu$, so by 7., $\alpha_\nu \preceq S_\nu$.

12. ν is not an initial ordinal $\to \nu \prec S_\nu$.

Proof: ∃ initial ordinal $\beta < \nu, \beta \approx \nu$. By 11., $\beta \preceq S_\beta$. ∴ $\nu \approx \beta \preceq S_\beta \prec S_\nu$, by 10.

13. $Lim(\alpha) \to S_\alpha = \bigcup_{\nu<\alpha} S_\nu$.

Proof: $\bigcup_{\nu<\alpha} S_\nu = \bigcup_{\nu<\alpha} C_\nu$ since $S_\beta \prec C_\beta \preceq S_{\beta+1}$ for all β. By the proof of 6., $\Psi_\alpha{}'\beta = \bigcup_{\nu<\alpha} C_\nu$ for all $\beta \in On_M$. ∴ $\bigcup \Psi_\alpha{}'On_M = (\bigcup_{\nu<\alpha} C_\nu) \times \omega_0 = \bigcup_{\nu<\alpha} C_\nu$. ∴ $S_\alpha = \bigcup_{\nu<\alpha} C_\nu$ by 6.

14. ν not an initial ordinal and $Lim(\nu) \to \bar{\bar{S}}_\nu$ not regular.

Proof: ∃ a smaller (than $\bar{\bar{S}}_\nu$) increasing sequence of smaller (than $\bar{\bar{S}}_\nu$) cardinal numbers whose union is $\ge \bar{\bar{S}}_\nu$, viz., $\{\bar{\bar{S}}_\xi : \xi < \nu\}$, by 10., 12. and 13.

15. If $Lim(\nu)$, $\bar{\bar{S}}_\nu = \bigcup_{\alpha<\nu} \bar{\bar{S}}_\alpha$. If $\bar{\bar{S}}_\nu$ is also regular, $\bar{\bar{S}}_\nu = \nu$.

Proof: By 13., $\bar{\bar{S}}_\nu \le \bigcup_{\alpha<\nu}\bar{\bar{S}}_\alpha$, and by 10., $\bar{\bar{S}}_\alpha < \bar{\bar{S}}_\nu$. ∴ $\bigcup_{\alpha<\nu} \bar{\bar{S}}_\alpha \le \bar{\bar{S}}_\nu \cdot \bar{\bar{\nu}} \le \bar{\bar{S}}_\nu$ by 11. Hence, if $\bar{\bar{S}}_\nu$ is regular, $\bar{\bar{\nu}} \nless \bar{\bar{S}}_\nu$. ∴ $\bar{\bar{S}}_\nu = \nu$, by 11.

16. $x \prec S_\alpha \wedge Lim(\alpha) \to Px \prec S_\alpha$.

Proof: If $x \succ S_\nu$ for all $\nu<\alpha$ then $x \succeq \bigcup_{\nu<\alpha} S_\nu$. ∴ $x \succeq S_\alpha$ by 13., →←. ∴ $x \preceq S_\nu$ for some $\nu<\alpha$. ∴ $2^{\bar{\bar{x}}} \le 2^{\bar{\bar{S}}_\nu} \le \bar{\bar{C}}_\nu \le \bar{\bar{S}}_{\nu+1} < \bar{\bar{S}}_\alpha$ by 8., 9., and 10.

17. $\nu \in On_M \to \bar{\bar{S}}_\nu$ not regular or not $Lim(\nu)$.

Proof: Otherwise $\bar{\bar{S}}_\nu = \bar{\bar{\nu}}$ by 15., but $\bar{\bar{S}}_\nu \notin On_M$.

18. $\nu \in On_M \to \alpha_\nu \approx S_0$.

Proof: By 3. and 7., $\alpha_0 \approx S_0$. Assume for ordinals $< \nu$. If $\nu = \beta+1$, $\alpha_\nu = \alpha_\beta + 1 \approx \alpha_\beta \approx S_0$. If $Lim(\nu)$, $\alpha_\nu = \bigcup_{\beta<\nu}\alpha_\beta$, so $\bar{\bar{\alpha}}_\nu \le \bar{\bar{\nu}} \cdot \bar{\bar{S}}_0 = \bar{\bar{S}}_0$.

19. $\nu \in On_M \to \bar{\bar{S}}_{\alpha_\nu}$ not regular or not $Lim(\alpha_\nu)$.

Proof: Otherwise $\bar{\bar{S}}_{\alpha_\nu} = \bar{\bar{\alpha}}_\nu$ by 15. But $\bar{\bar{\alpha}}_\nu = \bar{\bar{S}}_0$ by 18. ∴ $\bar{\bar{S}}_{\alpha_\nu} = \bar{\bar{S}}_0$, →← 10.

20. $\bar{\bar{S}}_0$ is regular. (By 18., $\bar{\bar{S}}_0 = \alpha_0$. Also $\alpha_0 = \omega_{\alpha_0}$ and $Lim(\alpha_0)$. Hence α_0 is inaccessible; ∴ PS_0 is a model for NBG by [5].)

Proof: Let z be an increasing sequence of cardinal numbers ∋ $D(z) \prec S_0 \wedge (y \in R(z) \to y \prec S_0)$. Then $R(z) \in M \wedge R(z) \subseteq M$. ∴ $\bigcup R(z) \in M \cdot \bigcup R(z) \in M$. ∴ $\bigcup R(z) \prec S_0$.

Notation: $\iota_0 < \iota_1 < \cdots$ denotes the sequence of initial ordinals not in M.

21. If the generalized continuum hypothesis (GCH) holds for the ι_β, then $\overline{\overline{S}}_\beta = \iota_\beta$ for all β.

Proof: $\overline{\overline{S}}_0 = \iota_0$. Assume for all ordinals $<\beta$. Let $\beta=\gamma+1$. Then $\overline{\overline{S}}_\gamma=\iota_\gamma$, so $\iota_{\gamma+1} = 2^{\iota_\gamma} = 2^{\overline{\overline{S}}_\gamma} = \overline{\overline{S_{\gamma+1}}}$ by 8. and 9. Assume Lim(β). By 15., $\overline{\overline{S}}_\beta = \bigcup_{\gamma<\beta}\overline{\overline{S}}_\gamma = \bigcup_{\gamma<\beta}\iota_\gamma = \iota_\beta$.

22. If $\overline{\overline{S}}_\beta=\iota_\beta$ for all β, then GCH holds for the ι_β.

Proof: $PS_\beta \approx S_{\beta+1}$ by 8. and 9.

23. τ minimal, $\gamma \preceq S_\tau \to \tau$ minimal, $\alpha_\gamma \preceq S_\tau$. (Hence if $\gamma\approx S_\tau$ then $S_\gamma \preceq \alpha_\gamma$ by 11., so $\alpha_\gamma \approx S_\tau$.)

Proof: Show first $\alpha_\gamma \preceq S_\tau$. True if $\gamma=0$, by 18. Assume $\gamma=\beta+1$. Then $\gamma\approx\beta$, so $\gamma\preceq S_\tau \to \beta\preceq S_\tau$. Also, τ is minimal, $\beta\preceq S_\tau$; hence $\alpha_\beta \preceq S_\tau$. But $\alpha_{\beta+1}=\alpha_\beta+1\approx\alpha_\beta$. Assume Lim($\gamma$). Then $\alpha_\gamma=\bigcup_{\xi<\gamma}\alpha_\xi$. Since $\gamma\preceq S_\tau$, $\xi<\gamma\to\xi\preceq S_{\tau(\xi)}$ for some minimal $\tau(\xi)\le\xi$. $\therefore \alpha_\gamma \preceq \bigcup_{\xi<\gamma} S_{\tau(\xi)} \preceq S_\tau$. If $\beta<\tau$ and $\alpha_\gamma\preceq S_\beta$ then $\gamma\preceq S_\beta$ by 11., $\to\leftarrow$.

24. $\beta\approx\gamma\to\alpha_\beta\approx\alpha_\gamma$.

Proof: Use induction on γ. Assume $\beta<\gamma$. True if $\gamma=0$. Suppose $\gamma=\xi+1$. Then $\beta\le\xi \wedge \beta\approx\xi\to\alpha_\beta\approx\alpha_\xi$. But $\alpha_\gamma=\alpha_\xi+1\approx\alpha_\xi$. Suppose Lim($\gamma$). Then $\alpha_\gamma=\bigcup_{\tau<\gamma}\alpha_\tau=\bigcup_{\beta<\tau<\gamma}\alpha_\tau$. But $\beta<\tau<\gamma \wedge \beta\approx\gamma\to\beta\approx\tau$. $\therefore \alpha_\tau\approx\alpha_\beta$. $\therefore \overline{\overline{\alpha_\gamma}} \le \overline{\overline{\alpha_\beta}}\cdot\overline{\overline{\gamma}} = \overline{\overline{\alpha_\beta}}\cdot\overline{\overline{\beta}} = \overline{\overline{\alpha_\beta}}$.

25. $(\mathrm{E}n\in\omega \wedge \xi\in On_n)\left(\gamma = \alpha_{\alpha_{\cdot_{\cdot_{\alpha_\xi}}}}\right)\to \gamma\approx\alpha_0$.

Proof: Use induction on n. True for $n=0$, by 18. Let $\beta = \alpha_{\alpha_{\cdot_{\cdot_{\alpha_\xi}}}}$. By induction hypothesis, $\beta\approx\alpha_0$. $\therefore \alpha_\beta\approx\alpha_{\alpha_0}$, by 24. But $\alpha_{\alpha_0}=\bigcup_{\gamma<\alpha_0}\alpha_\gamma$, so $\overline{\overline{\alpha_{\alpha_0}}} \le \overline{\overline{\alpha_0}}\cdot\overline{\overline{\alpha_0}} = \overline{\overline{\alpha_0}}$.

26. $\alpha_{\iota_\gamma}=\iota_\gamma$ for all γ.

Proof: Suppose γ is the least ordinal, $\alpha_{\iota_\gamma}\neq\iota_\gamma$. Then $\alpha_{\iota_\gamma}>\iota_\gamma$ by 11. Now $\iota_\gamma=\alpha_\xi$ for some ξ, and $\xi\approx\iota_\beta$ for some β. $\therefore \iota_\gamma=\alpha_\xi\approx\alpha_{\iota_\beta}$ by 24. Since $\alpha_{\iota_\gamma}>\alpha_{\iota_\beta}$, $\iota_\gamma>\iota_\beta$. Hence $\gamma>\beta$. $\therefore \alpha_{\iota_\beta}=\iota_\beta$. $\therefore \iota_\gamma\approx\iota_\beta$. $\therefore \gamma=\beta$, $\to\leftarrow$.

27. $\iota_\gamma=\gamma \wedge (\mathrm{E}\beta)(\beta\le\xi<\gamma\to\xi<\iota_\xi)\to\gamma$ is not regular. (This holds in general for initial ordinals ω_γ.)

Proof: Let $\beta\le\xi<\gamma$. Then $\xi<\iota_\xi<\iota_\gamma=\gamma$. $\therefore \beta<\iota_\xi<\gamma\to\iota_\xi<\iota_{\iota_\xi}<\iota_\gamma=\gamma$. By iteration,

$\overbrace{\iota^{\iota^{\cdot^{\cdot^{\iota_1}}}}}^{n} < \gamma$. Suppose $\tau < \gamma$ and $\tau = \text{l.u.b.}\left\{\overbrace{\iota^{\cdot^{\cdot^{\iota_1}}}}^{n} : n \in \omega\right\}$. For some ρ, $\tau \neq \iota_\rho$, and $\rho \leq \iota_\rho$ by induction. Since $\overbrace{\iota^{\iota^{\cdot^{\cdot^{\iota_1}}}}}^{n} < \iota_\rho$, $\overbrace{\iota^{\cdot^{\cdot^{\iota_1}}}}^{n} < \rho$ for all $n \in \omega$. $\therefore \rho \geq \tau$. $\therefore \rho = \tau$, i.e.,

$\tau = \iota_\tau$. But $\beta < \iota_1 \leq \tau < \gamma$, so $\tau < \iota_\tau$, $\rightarrow\!\leftarrow$ $\therefore \gamma = \text{l.u.b.}\left\{\overbrace{\iota^{\cdot^{\cdot^{\iota_1}}}}^{n} : n \in \omega\right\}$, i.e., γ is the limit of a smaller(than γ) increasing sequence of smaller (than γ) ordinals.

7. Supercomplete inner models of K.

Let G denote the von Neumann function. Following [5], define multiplicity, extended class and extended set, respectively, of a supercomplete inner model of K by $X \leq W$, $x \in W$, $x \in W \cdot M$, where W is a multiplicity of K, X is a variable ranging over the multiplicities of K, and x is a variable ranging over the extended classes and extended sets of K. Since any model for K is a model for NBG, either $W = V$ or $W = G'\gamma$ where γ is an inaccessible ordinal of K. If $\gamma \in M$, then the model becomes an interpretation of K with the extended classes the same as the extended sets, since $G'\gamma \leq M$. Hence, assume $\gamma = \aleph_\rho$ for some ρ.

1. If $z \in G'\aleph_\beta$ and γ is minimal, $z \in M_\gamma \cup K_\gamma$, then $\gamma \leq \beta$. (Suppose Lim(τ) and $z \in G'\aleph_\tau$. Then $z \in G'\aleph_\beta$ for some $\beta < \tau$. Hence if γ is minimal, $z \in M_\gamma \cup K_\gamma$, $\gamma \leq \beta < \tau$.)

Proof: True for $\beta = 0$, since $G'\aleph_0 = M_0$. Assume for ordinals $< \beta$. Suppose $\beta = \xi + 1$. Then $z \leq G'\aleph_\xi$. Suppose $E\, \eta_1 < \gamma \wedge \eta_2 < \gamma$, $z \leq M_{\eta_1} \cup \bigcup_{\rho < \eta_2} K_\rho$. Then $Tz \leq M_{\eta_1} \cup \bigcup_{\rho < \eta_2} K_\rho \leq M_\eta \cup \bigcup_{\rho < \eta} K_\rho$, where $\eta = \max(\eta_1, \eta_2)$. $\therefore z \in M_\eta \cup K_\eta$, $\rightarrow\!\leftarrow$ minimality of γ . Hence either $Ey \in z$, γ is minimal with $y \in M_\gamma$, or $\eta < \gamma \rightarrow Ey \in z, y \in K_\rho$ for some minimal ρ with $\eta < \rho < \gamma$. Now $y \in G'\aleph_\xi$. Hence by induction hypothesis, in the first case $\gamma \leq \xi < \beta$, and in the second case $\eta < \rho \leq \xi < \beta$, so that $\gamma \leq \beta$. Suppose Lim(β). Then $z \in G'\aleph_\tau$ for some $\tau < \beta$. $\therefore \gamma \leq \tau < \beta$.

2. If $\eta < \beta$, then $M_\eta \leq G'\aleph_\beta$. (Suppose $\aleph_\tau$ is inaccessible and $\eta < \tau$. Let $\eta < \beta < \tau$. Then $M_\eta \leq G'\aleph_\beta$. Since $G'\aleph_\beta \in G'\aleph_{\beta+1} \leq G'\aleph_\tau$, $G'\aleph_\beta \in G'\aleph_\tau$, so $M_\eta \in G'\aleph_\tau$.)

Proof: True for $\beta = 0$. Assume for ordinals $< \beta$. Suppose Lim(β). Let $\eta < \tau < \beta$. Then $M_\eta \leq G'\aleph_\tau \leq G'\aleph_\beta$. Suppose $\beta = \tau + 1$. If $\eta < \tau$ then $M_\eta \leq G'\aleph_\tau \in G'\aleph_\tau + 1 = G'\aleph_{\tau+1}$. Hence it suffices to show $M_\tau \trianglelefteq G'\alpha_{\tau+1} = P(G'\alpha_\tau)$. True for $\tau = 0$. Assume for ordinals $< \tau$. If $\tau = \nu+1$, $M_\nu \trianglelefteq G'\alpha_\nu$, so $M_\tau = P(M_\nu) \trianglelefteq P(G'\alpha_\nu)$. If $\mathrm{Lim}(\tau)$, $\overline{\overline{M_\tau}} = \overline{\overline{\bigcup_{\nu<\tau} M_\nu}} \trianglelefteq \overline{\overline{\bigcup_{\nu<\tau} G'\alpha_{\nu+1}}} \trianglelefteq \overline{\overline{\tau}} \cdot \overline{\overline{G'\alpha_\tau}} \trianglelefteq \overline{\overline{G'\alpha_\tau}}$, since $\tau \leq \alpha_\tau$ for all τ and $\beta \trianglelefteq G'\beta$ by induction.

5. If α_ρ is an inaccessible ordinal of K and $\rho > 0$, then $G'\alpha_\rho$ determines a supercomplete inner model of K.

Proof: Since $G'\alpha_\rho$ is a model for NBG it is necessary only to check the special axioms. In the following proofs of the special axioms, let (*) denote the fact that the special axiom in question holds for the universe V of K, and (**) the fact that $G'\alpha_\rho$ is a model for NBG.

(a) Subsets$_M$: Let $x \in M \cdot G'\alpha_\rho$, $Z \in G'\alpha_\rho$. Then $x \cdot Z \in M$ by (*) and $x \cdot Z \in G'\alpha_\rho$ by (**).

(b) Union$_M$: Let $x \in M \cdot G'\alpha_\rho$. Then $x \cdot G'\alpha_\rho \in M$ by Subsets$_M$ for the system K, so $\bigcup x \cdot M \cdot G'\alpha_\rho \in M$ by (*). Also, $x \cdot M \cdot G'\alpha_\rho \in G'\alpha_\rho$ by (**), so $\bigcup x \cdot M \cdot G'\alpha_\rho \in G'\alpha_\rho$ by (**).

(c) Power$_M$: Let $x \in M \cdot G'\alpha_\rho$. Then $Px \in M$ by (*) and $Px \in G'\alpha_\rho$ by (**).

(d) Pairing$_M$: Let $x, y \in M \cdot G'\alpha_\rho$. Then $\{x,y\} \in M$ by (*) and $\{x,y\} \in G'\alpha_\rho$ by (**).

(e) Infinity$_M$: Let x be given by Infinity$_M$ for K. Then $x \in M_0 = G'\alpha_0 \subseteq G'\alpha_\rho$. $\therefore Tx \subseteq G'\alpha_\rho$.

(f) Cardinality: If $x \in M \cdot G'\alpha_\rho$, $\exists y \in M_0 \ni x \approx y$, by (*). $\therefore y \in G'\alpha_0 \subseteq G'\alpha_\rho$. The converse, viz., $(x \in G'\alpha_\rho \wedge \exists y \in M_0, x \approx y \rightarrow x \in M)$, also follows from (*).

(g) Universes: Let $z \in G'\alpha_\rho$. Required to show $y = \{x \in G'\alpha_\rho : Tx \subseteq M \cdot G'\alpha_\rho \vee z\} \in G'\alpha_\rho$. Sufficient to show $y \triangleleft G'\alpha_\rho$, for then $y \triangleleft \alpha_\rho$, so $y \approx \beta$ for some $\beta \in G'\alpha_\rho$, and the axiom of replacement applies. $\exists$ minimal $\alpha \ni z \in M_\alpha \vee K_\alpha$. Then $z \subseteq M_\alpha \vee K_\alpha$. $\therefore y \subseteq \{x : Tx \subseteq M \vee K_\alpha\} = M_{\alpha+1} \vee K_{\alpha+1} \trianglelefteq M_{\alpha+1}$. Hence to show $y \triangleleft G'\alpha_\rho$ it suffices to show $M_{\alpha+1} \triangleleft G'\alpha_\rho$. By 1., $\alpha < \rho$, so $\alpha + 1 < \rho$. By 2., $M_{\alpha+1} \triangleleft G'\alpha_\rho$.

8. <u>Relative consistency.</u>

An interpretation of K is constructed in NBG+(AI), where AI is the axiom of inaccessibility. This is done in two ways, with A. much simpler than B. (Apparently B. is superfluous.) Note that for A. the underlying set theory may be taken to be NBG+($\exists$ 2 inaccessible ordinals); the assumption AI is not necessary. Let G denote the von Neumann function.

A. For any ξ let $N_\xi = \{x : x \prec \xi\}$. If γ is minimal, $x \in G'\gamma$, call γ the index of x.

1. If ρ is a regular limit ordinal and $Tx \leq N_\rho$, then $x \leq G'\rho$.

Proof: Use induction on the index of x. True for index 1; assume for indices smaller than that of x and let x have index $\alpha+1$. Let $y \in x$ and let $\beta+1$ be the index of y. Since $Ty \leq Tx \leq N_\rho$ and $\beta+1 < \alpha+1$, $y \leq G'\rho$. Also, $y \prec \rho$. ∴ the indices of the elements of y can be arranged in an increasing sequence $(\beta_\gamma)_{\gamma<\pi}$ for some $\pi < \rho$, where $\beta_\gamma < \rho$ for all $\gamma < \pi$. By regularity, $\bigcup_{\gamma<\pi} \beta_\gamma < \rho$. But β is minimal, $y \leq G'\beta$. ∴ if $\gamma < \beta$, $Ez \in y, z \notin G'\gamma$. If the index of z is $\tau < \gamma$, then $z \in G'\tau \leq G'\gamma$. ∴ $\tau > \gamma$. ∴ $\bigcup_{\gamma<\pi} \beta_\gamma \geq \beta$. ∴ $\beta < \rho$. ∴ $y \in G'\beta+1 \leq G'\rho$.

2. Assume β inaccessible. (Also, assume γ inaccessible with $\gamma > \beta$.) Then V (G'γ) determines a model for K, where multiplicity, extended class and extended set of the model, respectively, are defined in NBG+(AI) by $X \leq V$($X \leq G'\gamma$), $x \in V$ ($x \in G'\gamma$), $x \in N_\beta$ ($x \in G'\gamma \cdot N_\beta$), where X is a class and x is a set in the sense of NBG+(AI).

Note: These are the only possible interpretations of K in NBG+(AI) with the notions of multiplicity, extended class and extended set defined by $X \leq W$, $x \in M$, $x \in U$ for some classes W, U of NBG+(AI); for in any such interpretation the universe of K must satisfy the axioms of NBG (hence must be V or G'γ), $N_\bullet$ must satisfy the axioms for NBG (hence must be G'β), and M must consist of the sets cardinally less than $N_\bullet$. Also, if K has such an interpretation in an extension NBG# of NBG, the $N_\bullet$ of the interpretation is a set in NBG# which determines a model for NBG, hence must be G'β, where β is an inaccessible ordinal in NBG#.

Proof: Since V (G'γ) satisfies the axioms for NBG, it is necessary only to check the special axioms. All of these are immediate but Universes and perhaps Union$_m$. For Union$_m$, let $x \in N_\beta$. Then $\overline{\overline{\bigcup x \cdot N_\beta}} \leq \bigcup_{z \in x \cdot N_\beta} \overline{\overline{z}}$. Since $\overline{\overline{x \cdot N_\beta}} \leq \overline{\overline{x}} \prec \beta$, $\bigcup_{z \in x \cdot N_\beta} \overline{\overline{z}}$ is the union of a smaller (than β) set of smaller (than β) cardinals, hence is $\prec \beta$. For Universes, let $z \in V$ ($z \in G'\gamma$) with index $\xi+1$. Then $\{x : Tx \leq N_\beta \cdot z\} \leq$

$\{x : Tx \subseteq N_\beta \cup G'\delta\} \subseteq \{x : Tx \subseteq N_\tau\}$ ($\{x \in G'\gamma : Tx \subseteq N_\beta \cup z\} \subseteq \{x \in G'\gamma : Tx \subseteq N_\beta \cup G'\delta\} \subseteq \{x \in G'\gamma : Tx \subseteq N_\tau\}$)
where $\bar{\bar{\tau}} = \max(\bar{\bar{\beta}}, \overline{\overline{G'\delta}})$. E regular limit ordinal $\rho, \tau < \rho$ ($\tau < \rho < \gamma$). Then by 1.,
$\{x : Tx \subseteq N_\tau\} \subseteq \{x : Tx \subseteq N_\rho\} \subseteq G'\rho+1$ ($\{x \in G'\gamma : Tx \subseteq N_\tau\} \subseteq \{x \in G'\gamma : Tx \subseteq N_\rho\} \subseteq G'\rho+1$).
$\therefore \{x : Tx \subseteq N_\beta \cup z\} \in V$ ($\{x \in G'\gamma : Tx \subseteq N_\beta \cup z\} \in G'\gamma$).

B. Let τ be an increasing sequence of inaccessible ordinals,

$R_{\tau_\beta} = G'\tau_\beta$, $M_0 = R_{\tau_0}$, $K_0 = PR_{\tau_0} - M_0$,

$M_\beta = \{x \in R_{\tau_\beta} : (Ey \in M_0, x \approx y) \wedge (y \in Tx \to (Ew \in M_0, y \approx w) \vee y \in \bigcup_{\gamma<\beta} K_\gamma)\}$,

$K_\beta = \{x \in PR_{\tau_\beta} - M_\beta : y \in Tx \to (Ew \in M_0, y \approx w) \vee y \in \bigcup_{\gamma<\beta} K_\gamma\}$.

Use the notation $W = \bigcup_{\beta \in On} M_\beta \cup K_\beta$ for the universe of the interpretation, (Ψ_β) for the statement that R_{τ_β} is a model for NBG, (Φ_β) for the statement $(y \in Tx \to (Ew \in M_0, y \approx w) \vee y \in \bigcup_{\gamma<\beta} K_\gamma)$, $x, y, \cdots$ for elements of W, and $X, Y, \cdots$ for subclasses of W.

Note: Since this interpretation of K is of the type described in A., the dependence of M_β and K_β upon the particular sequence (τ_β) is only apparent, and the interpretation is determined by τ_0. Hence by definition of M_β and K_β, $x \in M_\beta \to x \in R_{\tau^*_\beta} \wedge x \in K_\beta \to x \in R_{\tau^*_\beta}$ for any increasing sequence τ^* of inaccessible ordinals beginning with τ_0, in particular, for the sequence of all such.

1. $\delta \leq \beta \to M_\delta \subseteq M_\beta$.

Proof: $x \in M_\delta \to x \in R_{\tau_\delta} \subseteq R_{\tau_\beta}$.

2. $K_\beta \cdot M_\gamma = 0$.

Proof: By definition of K_β, $K_\beta \cdot M_\beta = 0$. $\therefore \gamma \leq \beta \to K_\beta \cdot M_\gamma \subseteq K_\beta \cdot M_\beta = 0$, by 1. Suppose $\beta < \gamma$. If $\gamma = 0$, $K_\beta \cdot M_\gamma = 0$. Assume for ordinals $< \gamma$. Let $\gamma = \delta+1$. If $\beta < \delta$ then $K_\beta \cdot M_\delta = 0$ by induction hypothesis. Let $\beta < \delta$ and assume $x \in K_\beta \cdot M_{\delta+1}$. Since $x \in K_\beta, x \notin M_\beta$. $\therefore x \notin R_{\tau_\beta} \vee \bar{E}y \in M_0, x \approx y$. But $x \in M_{\delta+1}$. $\therefore x \notin R_{\tau_\beta}$. Since $K_\beta \cdot M_\delta = 0$, $x \notin M_\delta$. $\therefore x \notin R_{\tau_\delta}$. $\therefore x \in R_{\tau_{\delta+1}} - R_{\tau_\delta}$. E minimal η, $x \in G'\eta+1$ and $\tau_\delta \leq \eta < \tau_{\delta+1}$. Since $x \in K_\beta$, $x \in G'\tau_\beta+1$. But $\tau_\beta < \tau_\delta \leq \eta$, $\rightarrow\leftarrow$ minimality of η. $\therefore \beta < \delta \to K_\beta \cdot M_{\delta+1} = 0$. Suppose $x \in K_\delta \cdot M_{\delta+1}$. Then $x \in R_{\tau_{\delta+1}} - R_{\tau_\delta}$, since $K_\delta \cdot M_\delta = 0$. Ey $\in M_0, x \approx y$, and E minimal $\eta < \tau_0$, $y \in G'\eta+1$, i.e., $y \subseteq G'\eta$. If $w \in y$, then E minimal $\rho \leq \eta$, $w \in G'\rho$. Since

$x \subseteq G'\tau_\xi$, the set of corresponding ρ for all $w \in x$ forms a sequence with domain $\preceq G'\eta$; but $G'\eta \in G'\tau_\xi$, so by (Ψ_ξ), $G'\eta \prec \tau_\xi$. Also, these ρ are all $< \tau_\xi$. By regularity of τ_ξ, the union ζ of these ρ is also $< \tau_\xi$. Assume $Lim(\mu) \wedge \zeta \le \mu < \tau_\xi$. If $w \in x$, E minimal $\alpha \ni w \in G'\alpha$. Since $G'\alpha \subseteq G'\mu$, $x \subseteq G'\mu$. ∴ $x \in G'\mu + 1 \subseteq R_{\tau_\xi}$, $\rightarrow\leftarrow$. ∴ $K_\xi \cdot M_{\xi+1} = 0$. Assume $Lim(\gamma)$, $\beta < \gamma$, $x \in K_\beta \cdot M_\gamma$. Since $x \in K_\beta$, $x \subseteq G'\tau_\beta$. ∴ $x \in G'\tau_\beta + 1 \subseteq G'\tau_{\beta+1}$. Also, $y \in Tx \rightarrow (Ew \in M_0 \ni y \approx w) \vee y \in \bigcup_{\nu<\beta} K_\nu$. Since $x \in M_\gamma$, $Ey \in M_0 \ni x \approx y$. ∴ $x \in M_{\beta+1}$, i.e., $x \in K_\beta \cdot M_{\beta+1}$, $\rightarrow\leftarrow$ induction hypothesis.

3. $\xi \le \beta \rightarrow K_\xi \subseteq K_\beta$.

Proof: $x \in K_\xi \rightarrow x \subseteq R_{\tau_\xi} \subseteq R_{\tau_\beta}$. By 2., $x \notin M_\beta$.

4. $v \subseteq W \rightarrow v \in W$ (where V is the universe of NBG+(AI) and $v \in V$).

Proof: If $x \in v$, E least $\beta \ni x \in M_\beta \vee K_\beta$. By the axiom of replacement these β form a set. Let λ be an upper bound of such β. Then $x \in v \rightarrow x \in M_\beta \vee K_\beta \subseteq M_\lambda \vee K_\lambda$ by 1. and 3., i.e., $v \subseteq M_\lambda \vee K_\lambda$. Let $y \in v$. If $y \in M_\lambda$, $Ew \in M_0 \ni y \approx w$; if $y \in K_\lambda$, $y \in \bigcup_{\nu<\lambda+1} K_\nu$. ∴ $y \in v \rightarrow (Ew \in M_0 \ni y \approx w) \vee y \in \bigcup_{\nu<\lambda+1} K_\nu$. Let $y \in \underbrace{\bigcup \cdots \bigcup}_{n} v$, $n > 0$. Then $Ez \in v \ni y \in \underbrace{\bigcup \cdots \bigcup}_{n-1} z$. Since $y \in Tz$ and $z \in M_\lambda \vee K_\lambda$, $(Ew \in M_0 \ni y \approx w) \vee y \in \bigcup_{\nu<\lambda} K_\nu$. ∴ $y \in Tv \rightarrow (Ew \in M_0 \ni y \approx w) \vee y \in \bigcup_{\nu<\lambda+1} K_\nu$. Since $v \subseteq M_\lambda \vee K_\lambda$, $v \subseteq R_{\tau_{\lambda+1}} = G'\tau_{\lambda+1}$. ∴ $v \in G'\tau_{\lambda+1} + 1 \subseteq \bigcup G''\tau_{\lambda+2} = R_{\tau_{\lambda+2}}$. ∴ $v \in R_{\tau_{\lambda+2}}$. If $Ew \in M_0 \ni v \approx w$ then $v \in M_{\lambda+2}$; otherwise $v \in K_{\lambda+2}$.

5. V=W. (Hence W satisfies the general axioms.)

Proof: If $V - W \neq 0$, $Ey \in V - W \ni y \cdot (V - W) = 0$. Hence by 4., $y \in W$, $\rightarrow\leftarrow$.

6. $x \subseteq y \in M_\beta \rightarrow x \in M_\beta$ (Subsets$_M$).

Proof: By (Ψ_β), $x \in R_{\tau_\beta}$. $Ez \in M_0 \ni y \approx z$. ∴ $Ew \subseteq z \ni x \approx w$, and $w \in M_0$ by (Ψ_0). If $z \in Tx$ then $z \in Ty$, so (Φ_β) holds for x.

7. $x, y \in M_\beta \rightarrow x \vee y \in M_\beta$.

Proof: By (Ψ_β), $x \vee y \in R_{\tau_\beta}$. $Eu, v \in M_0 \ni u \approx x$, $v \approx y$. By (Ψ_0), $u \vee v \in M_0$. If $z \in T(x \vee y)$ then $z \in Tx \vee Ty$, so (Φ_β) holds for $x \vee y$.

8. $x, y \in M_\beta \rightarrow \{x, y\} \in M_\beta$ (Pairing$_M$).

Proof: By (Ψ_β), $\{x\}, \{y\} \in R_{\tau_\beta}$. Obviously, (Φ_β) holds for $\{x\}$ and $\{y\}$. ∴ $\{x, y\} \in M_\beta$ by 7.

9. $x \in M_\beta \rightarrow \bigcup (x \cdot \bigcup_{\nu \in On} M_\nu) \in M_\beta$ (Union$_M$).

Proof: Since $x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma \leq x$, $x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma\in R_{\tau_\beta}$ by (Ψ_β). $\therefore \bigcup(x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma)\in R_{\tau_\beta}$ by (Ψ_β). Now $\bigcup(x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma)\leq\bigcup_{z\in x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma}\bar{\bar{z}}$. For each such z, $z\in\bigcup_{\gamma\in\theta_\mu} M_\gamma\rightarrow z\prec\tau_0$. Also, $x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma\in M_\beta$ by 6., so $x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma\prec\tau_0$. Hence by regularity, $\bigcup_{z\in x\cdot\bigcup_{\gamma\in\theta_\mu} M_\gamma}\bar{\bar{z}}\prec\tau_0$. Since $T(\bigcup x\cdot\bigcup_{\beta\in\theta_\mu} M_\beta))\subseteq T(x\cdot\bigcup_{\beta\in\theta_\mu} M_\beta)$, (Φ_β) for $\bigcup(x\cdot\bigcup_{\beta\in\theta_\mu} M_\beta)$ follows from (Φ_β) for $x\cdot\bigcup_{\beta\in\theta_\mu} M_\beta$. But the latter follows from 6.

10. $x\in M_\beta\rightarrow Px\in M_\beta$ (Power$_\mu$).

Proof: $Px\in R_{\tau_\beta}$ by (Ψ_β). $\exists y\in R_{\tau_0}$, $x\approx y$. By (Ψ_0), $Py\in R_{\tau_0}$. Also, $T(Px)-Px\subseteq Tx$. $\therefore(\Phi_\beta)$ holds for Px.

11. Infinity$_\mu$: $\omega\in M_0$.

12. Cardinality.

Proof: For any β, $x\in M_\beta\rightarrow\exists y\in M_0$, $x\approx y$, and $Ty\subseteq M_0$ by definition of M_0. Conversely, assume $x\in W$ has the property $\exists y\in\bigcup_{\beta\in\theta_\mu} M_\beta$, $Ty\subseteq\bigcup_{\beta\in\theta_\mu} M_\beta$, and $x\approx y$. To show $x\in\bigcup_{\beta\in\theta_\mu} M_\beta$ it is necessary to show $y\in M_0$. Suppose not. Since $y\in\bigcup_{\beta\in\theta_\mu} M_\beta$, $y\notin K_0$. $\therefore y\notin M_0$. $\therefore\exists y_1\in y$ with index $\beta_1>\tau_0$. Since $y_1\in\bigcup_{\beta\in\theta_\mu} M_\beta$, $y_1\notin K_0$. Since $\beta_1>\tau_0$, $y_1\notin M_0$. $\therefore y_1\notin M_0$. $\therefore\exists y_2\in y_1$ with index $\beta_2>\tau_0$. Since $y_2\in\bigcup_{\beta\in\theta_\mu} M_\beta$, $y_2\notin K_0$. Since $\beta_2>\tau_0$, $y_2\notin M_0$. $\therefore y_2\notin M_0$. $\therefore\exists y_3\in y_2$ with index $\beta_3>\tau_0$. Iteration yields descending $\in$-sequences $\cdots\in y_2\in y_1\in y$, $\cdots\beta_3<\beta_2<\beta_1$, $\rightarrow\leftarrow$. $\therefore y\in M_0$. $\exists$ minimal β, $x\in M_\beta\vee K_\beta$. If $x\in K_\beta$ then $x\notin R_{\tau_\beta}$. For each $w\in x$, $\exists$ minimal $\xi<\tau_\beta$, $w\in G'\xi$. Since $x\approx y$ for some $y\in M_0$, these ξ form a γ-sequence for some $\gamma<\tau_0$. Let η be the l.u.b. of these ξ. Then $x\subseteq G'\eta$, so $x\in G'\eta+1$. Since $x\notin R_{\tau_\beta}$, $\eta\geq\tau_\beta$, $\rightarrow\leftarrow$ regularity of τ_β. $\therefore x\in M_\beta$.

13. $z\in M_\beta\vee K_\beta\rightarrow z\subseteq M_\beta\vee K_\beta$.

Proof: $z\in M_\beta\vee K_\beta\rightarrow z\subseteq R_{\tau_\beta}$. $\therefore y\in z\rightarrow y\in R_{\tau_\beta}$. $\therefore y\subseteq R_{\tau_\beta}$. Also, $Ty\subseteq Tx$. If $\exists w\in M_0$, $w\approx y$, then $y\in M_\beta$; otherwise $y\in K_\beta$. $\therefore z\subseteq M_\beta\vee K_\beta$.

14. $z\in M_\beta\vee K_\beta\rightarrow\{x: Tx\subseteq\bigcup_{\gamma\in\theta_\mu} M_\gamma\vee z\}\subseteq\{x: Tx\subseteq\bigcup_{\gamma\in\theta_\mu} M_\gamma\vee\bigcup_{\gamma\leq\beta} K_\gamma\}$.

Proof: Assume $Tx\subseteq\bigcup_{\gamma\in\theta_\mu} M_\gamma\vee z$. Let $y\in Tx$. If $y\notin\bigcup_{\gamma\in\theta_\mu} M_\gamma$, then $y\in z$, so $y\in K_\beta$, by 13.

15. $\{x: Tx\subseteq\bigcup_{\gamma\in\theta_\mu} M_\gamma\vee\bigcup_{\gamma<\beta} K_\gamma\}=\{x: Tx\subseteq M_\beta\vee\bigcup_{\gamma<\beta} K_\gamma\}$.

Proof: Assume $Tx\subseteq\bigcup_{\gamma\in\theta_\mu} M_\gamma\vee\bigcup_{\gamma<\beta} K_\gamma$. Suppose $Tx\nsubseteq M_\beta\vee\bigcup_{\gamma<\beta} K_\gamma$. Then $\exists$ minimal $\beta_1>\beta$, $Tx\cdot M_{\beta_1}\neq 0$. Let $x_1\in Tx\cdot M_{\beta_1}$ with $x_1\notin M_\gamma$ for any $\gamma<\beta_1$. If $x_1\subseteq R_{\tau_\beta}$ then $x_1\in K_\beta$,

→←. (Since $x_1 \in Tx$, (Φ_β) holds for x_1 by 12.) $\therefore Ex_2 \in x_1, x_2 \notin R_{\tau_\beta}$. If $x_2 \in K_\gamma$ for some $\gamma<\beta$ then $x_2 \subseteq R_{\tau_\gamma} = G'\tau_\gamma$. $\therefore x_2 \in G'\tau_\gamma+1 \subseteq R_{\tau_\beta}$, →←. $\therefore x_2 \in \bigcup_{\gamma\in\omega} M_\gamma$. $\therefore x_2 \in M_{\beta_2}$ for some minimal β_2. Since $x_2 \in x_1$, $\beta_2 \le \beta_1$. Since β_1 is minimal, $Tx\cdot M_{\beta_1} \neq 0$, $\beta_2 = \beta_1$. If $x_2 \subseteq R_{\tau_\beta}$ then $x_2 \in K_\beta$, →←. $\therefore Ex_3 \in x_2, x_3 \notin R_{\tau_\beta}$. By the same argument, with x_1 and x_2 replaced by x_2 and x_3, respectively, $x_3 \in M_{\beta_1}$. Iteration of this argument yields an infinite descending $\in$-sequence of elements of M_{β_1}, $\cdots \in x_3 \in x_2 \in x_1$, →←.

16. $M_\beta \cup \bigcup_{\gamma<\beta} K_\gamma$ is transitive.

Proof: Let $Y_\beta = M_\beta \cup \bigcup_{\gamma<\beta} K_\gamma$, $y \in x \in Y_\beta$. If $x \in M_\beta$, since $y \in Tx$, either $Ew \in M_\beta, y \in w$ or $y \in \bigcup_{\gamma<\beta} K_\gamma$. In the first case $y \in \bigcup_{\gamma\in\omega} M_\gamma$ by 12. Since $x \in R_{\tau_\beta}$, $y \in R_{\tau_\beta}$. Since $Ty \subseteq Tx$, $y \in M_\beta$. $\therefore y \in Y_\beta$. Suppose $x \in K_\delta$ for some $\delta<\beta$. Either $Ew \in M_\beta, y \in w$ or $y \in \bigcup_{\gamma<\delta} K_\gamma \subseteq \bigcup_{\gamma<\beta} K_\gamma$. In the first case $y \in \bigcup_{\gamma\in\omega} M_\gamma$ by 12. Since $x \subseteq R_{\tau_\delta}$, $y \in R_{\tau_\delta} \subseteq R_{\tau_\beta}$. Since $Ty \subseteq Tx$, $y \in M_\beta$. $\therefore y \in Y_\beta$.

17. M_β, $K_\beta \in W$.

Proof: $M_\beta \subseteq R_{\tau_\beta}$, and $M_\beta \notin M_\beta$. To show $M_\beta \in K_\beta$, it suffices to show $TM_\beta \subseteq Y_\beta = M_\beta \cup \bigcup_{\gamma<\beta} K_\gamma$. Let $x \in TM_\beta$. If $x \in M_\beta$ then then $x \in Y_\beta$. If $x \in \bigcup M_\beta$, $Ey \in M_\beta, x \in y$. By 16., $x \in y \subseteq Y_\beta$. If $x \in \bigcup\bigcup M_\beta$, $Ey_1 \in \bigcup M_\beta, x \in y_1$, and $Ey_2 \in M_\beta, y_1 \in y_2$. By 16., $y_1 \in y_2 \subseteq Y_\beta$ and $x \in y_1 \subseteq Y_\beta$. In general, if $x \in \underbrace{\bigcup\cdots\bigcup}_{n} M_\beta$ then $x \in Y_\beta$ by n applications of 16. $\therefore M_\beta \in W$. By definition, $K_\beta \subseteq PR_{\tau_\beta} = G'\tau_\beta+1 \subseteq G'\tau_{\beta+1}$, i.e., $K_\beta \in PR_{\tau_{\beta+1}}$. Also, $K_\beta \supseteq K_0$, so $K_\beta \supseteq K_0 \ni M_0$. $\therefore K_\beta \notin \bigcup_{\gamma\in\omega} M_\gamma$. Let $x \in TK_\beta$. If $x \in K_\beta$ then $x \in Y_{\beta+1}$. If $x \in \bigcup K_\beta$ then $Ey \in K_\beta, x \in y$. Since $y \in K_\beta \subseteq Y_{\beta+1}$, $x \in Y_{\beta+1}$ by 16. If $x \in \bigcup\bigcup K_\beta$ then $x \in y_1 \in y_2$ with $y_1 \in \bigcup K_\beta$ and $y_2 \in K_\beta \subseteq Y_{\beta+1}$. $\therefore y_1 \in Y_{\beta+1}$ by 16., so $x \in Y_{\beta+1}$ by 16. In general $TK_\beta \subseteq Y_{\beta+1}$. $\therefore K_\beta \in K_{\beta+1}$.

18. $\{x : Tx \subseteq M_\beta \cup \bigcup_{\gamma<\beta} K_\gamma\} \subseteq M_\beta \cup K_\beta$.

Proof: Assume $Tx \subseteq M_\beta \cup \bigcup_{\gamma<\beta} K_\gamma$. Let $y \in x$. If $y \in M_\beta$ then $y \in R_{\tau_\beta}$. Otherwise $y \in K_\gamma$ for some $\gamma<\beta$, so $y \subseteq R_{\tau_\gamma}$. $\therefore y \in G'\tau_\gamma+1 \subseteq R_{\tau_\beta}$. $\therefore x \subseteq R_{\tau_\beta}$. Also, (Φ_β) holds for x. Hence if $x \notin M_\beta$ then $x \in PR_{\tau_\beta} - M_\beta$, i.e., $x \in K_\beta$.

19. $\{x : Tx \subseteq \bigcup_{\beta\in\omega} M_\beta \cup z\} \in V$ (Universes).

Proof: By 14., 15., 18., 17.

20. $M_\beta=\{x\in\bigcup_{\gamma\in\theta_\alpha} M_\gamma : Tx\subseteq\bigcup_{\gamma\in\theta_\alpha} M_\gamma\cup\bigcup_{\gamma<\beta} K_\gamma\}$, $K_\beta=\{x\in W-\bigcup_{\gamma\in\theta_\alpha} M_\gamma : Tx\subseteq\bigcup_{\gamma\in\theta_\alpha} M_\gamma\cup\bigcup_{\gamma<\beta} K_\gamma\}$.

Proof: Let $x\in M_\beta\cup K_\beta$, $y\in Tx$. If $Ew\in M_0, y\approx w$, then $y\in\bigcup_{\gamma\in\theta_\alpha} M_\gamma$. $\therefore Tx\subseteq\bigcup_{\gamma\in\theta_\alpha} M_\gamma\cup\bigcup_{\gamma<\beta} K_\gamma$. Conversely, assume $Tx\subseteq\bigcup_{\gamma\in\theta_\alpha} M_\gamma\cup\bigcup_{\gamma<\beta} K_\gamma$. Then $Tx\subseteq M_\beta\cup\bigcup_{\gamma<\beta} K_\gamma$, by 15., so $x\in M_\beta\cup K_\beta$ by 18.

9. Some immediate relations between K and other systems.

1. Let A be a formula of K which does not involve M (and thus does not involve any M_α or K_α). Then A may also be regarded as a formula of NBG (although the variables of A as a formula of K and as a formula of NBG have different ranges). If $\vdash_{NBG} A$ then $\vdash_K A$, for a proof in NBG is a proof in K. But the converse need not hold.

Proof: In K it can be shown that Ex which determines a supercomplete model for NBG, viz., $x=M_0$. In NBG this is not provable, for it would imply the existence of an inaccessible ordinal .

2. Axiom 7 of [1] does not hold for the system K, since $K_\beta\in M_{\beta+1}$ follows from Axiom 7 of [1] (theorem 2.9 in [1]).

3. The theory K does not satisfy the axioms of the theory G_* of [4], for as observed in [1], $V\in V$ in G_*.

4. Suppose the model Δ for NBG+(GCH) described in [2] is constructed in K. Extend Δ to an interpretation of K by taking $\mathfrak{m}$ (the multiplicity of extended sets of the interpretation) to be $M\cdot L$. Then Δ is a model for K. (Hence the axiom of constructibility V=L and $\therefore$GCH are consistent with K.)

Proof: The notation is as in [2] and the numbers in parentheses refer to statements in [2]. All notions, operations, classes and variables occurring in the special axioms are shown in [2] to be absolute, except for the power class operation; hence the subscript L denoting relativization to Δ is omitted except for the power class operation.

(i) $M\cdot L$ is a constructible multiplicity.

Proof: It is required to show $F'\alpha\cdot M\cdot L\in L$ for all $\alpha\in On$(9.41). But $F'\alpha\subseteq L$ (9.51), so it is required to show $M\cdot F'\alpha\in L$. Assume for ordinals $<\alpha$ and let $\alpha = J_i\langle\beta,\gamma\rangle$, $0 \le i \le 8$. Then $\alpha > \mathrm{Max}\{\beta,\gamma\}$ if $i \neq 0$ (9.25).

(a) i=0. Then $F'\alpha = F''\alpha$ (9.35), so $F'\alpha \subseteq M$.

(b) i=1. Then $F'\alpha = \{F'\beta, F'\gamma\}$ (9.31), so $F'\alpha \subseteq M$.

(c) i=2. Then $F'\alpha = E \cdot F'\beta$ (9.32). But $E \subseteq M$, so $M \cdot F'\alpha = F'\alpha$.

(d) i=3. Then $F'\alpha = F'\beta - F'\gamma$ (9.33). $\therefore M \cdot F'\alpha = (M \cdot F'\beta) - F'\gamma \in L$ (9.6, i=3).

(e) $4 \le i \le 8$. Then $F'\alpha = F'\beta \cdot \mathcal{Q}_i''(F'\gamma)$ (9.34), so $M \cdot F'\alpha = (M \cdot F'\beta) \cdot \mathcal{Q}_i''(F'\gamma)$. By induction hypothesis, $E\delta, M \cdot F'\beta = F'\delta$, so $M \cdot F'\alpha = F'J_i\langle\gamma,\delta\rangle \in L$.

(ii) Subsets_M: $x \in M \cdot L \wedge \mathfrak{L}(Z) \rightarrow x \cdot Z \in M \cdot L$.

Proof: By Subsets_M, $x \cdot Z \in M$, and by definition of $\mathfrak{L}(Z)$ (9.41), $x \cdot Z \in L$.

(iii) Union_M; $x \in M \cdot L \rightarrow \bigcup x \cdot M \cdot L \in M \cdot L$.

Proof: By Union_M, $\bigcup x \cdot M \in M$, so $\bigcup x \cdot M \cdot L \subseteq \bigcup x \cdot M \rightarrow \bigcup x \cdot M \cdot L \in M$ by Subsets_M. Since $\mathfrak{L}(M \cdot L)$ by (i), $x \cdot M \cdot L \in L$ (9.41), so $\bigcup x \cdot M \cdot L \in L$ since Δ is a model for NBG.

(iv) Pairing : $x,y \in M \cdot L \rightarrow \{x,y\} \in M \cdot L$.

Proof: By Pairing_M, $\{x,y\} \in M$; since Δ is a model for NBG, $\{x,y\} \in L$.

(v) Infinity_M: $(Ex)(x \in M \cdot L \wedge Tx \le M \cdot L \wedge x$ is infinite).

Proof: The construction of K from the extended von Neumann function shows $\omega \in M \wedge T\omega \le M$. By (11.42), $\omega \in On \subseteq L$.

(vi) Universes: $z \in L \rightarrow \{y \in L : Ty \le M \cdot L \sim z\} \in L$.

Proof: Let $x = \{y \in L : Ty \le M \cdot L \sim z\}$. By K, $x \in V$, so $E\alpha \in On, \omega_\alpha > Od'y$ for all $y \in x$. Since $\omega_\alpha \in R(J_0)$ (*9.27), $x \subseteq F'\omega_\alpha$ (9.35). $\therefore x \in F'\omega_{\alpha+1}$ (12.2). $\therefore x \in L$ (9.51).

(vii) Cardinality: $x \in L \rightarrow (x \in M \cdot L \leftrightarrow Ey \in M \cdot L, Ty \le M \cdot L \wedge x \approx y)$.

Proof: If $Ey \in M \cdot L, Ty \le M \cdot L \wedge x \approx y$, then $x \in M$ by K. Conversely, let $x \in M \cdot L$. By K, $Ez \in M_0, x \approx z$. $E\alpha \in On_M, z = \alpha$. But $On \subseteq L$ (11.42), so that α is the required value of y.

(viii) Power_M: $x \in M \cdot L \rightarrow P_L x \in M \cdot L$.

Proof: Since $x \in M$, $Px \in M$ by Power_M. But $P_L x \subseteq Px$, so $P_L x \in M$ by Subsets_M. Since $x \in L$, $P_L x \in L$ $\because$ Δ is a model for NBG.

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