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DIFFRACTION OF ELECTROMAGNETIC WAVES IN A UNIAXIALLY ANISOTROPIC MEDIA

by

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Chapter 1

Introduction and Background

In this chapter some physical phenomena, mathematical model of which leads to anisotropic electromagnetic problems, are considered. After introducing the terminology and the notations for anisotropic medium, a review of solved anisotropic electromagnetic problems will be presented. At the end of this chapter, an outline of the work of this dissertation will be given.

1.1 Physical phenomena leading to Anisotropic Electromagnetic Problems.

It is known that Maxwell's electromagnetic equations are not complete in material media; namely the two curl equations connect four instead of two unknown vector quantities. The additional information, called the constitutive equations of the media, must be specified independently of Maxwell's equations. In vacuum the constitutive equations reduce to a simple form

$$\underline{B} = \mu_0 \underline{H} \tag{1.1}$$

$$\underline{D} = \epsilon_0 \underline{E} \tag{1.2}$$

where $\underline{B}$, $\underline{D}$, $\underline{H}$, and $\underline{E}$, are the monochromatic magnetic and electric vectors respectively, and μ_0 and ϵ_0 are the permeability and the permittivity of vacuum. However, in general, physical materials do not obey such simple dependencies.

It has been suggested that in general constitutive equations are nonlinear, for example, of the form [1]

$$B_i = \mu H_i + \mu_{ij} H_j + q_{ijk} H_j H_k + \ldots \tag{1.3}$$

$$D_i = \epsilon E_i + \epsilon_{ij} E_j + f_{ijk} E_j E_k + \ldots \tag{1.4}$$

where ϵ, μ, q and f are all spatial as well as frequency dependent parameters[1]. However, in this discussion, we will consider linear terms only, neglecting all second and higher order terms, and constant parameters for the media. This is not a severe restriction as long as we consider low monochromatic signal levels and homogeneous media. There are many problems of practical importance that can be modeled using these assumptions. A few such problems will now be described in some detail.

a. Communications from a reentering space vehicle. During reentry into the earth atmosphere, space vehicles form a plasma sheet around the vehicle that is known to be detrimental to communications to and from the vehicle. [2, 3] A reasonable model for the study of this phenomena is to view the vehicle as a grounded surface above which there is an antenna radiating into the plasma medium. It is known that plasmas can be modeled as a homogeneous anisotropic medium. Therefore, the calculation of the radiation from an antenna mounted on a space vehicle leads us to consider an anisotropic radiation problem.

b. Ferrite-filled resonators and waveguides.

There are many devices that use ferrite materials as component part. Since ferrites are usually contained in wave guides or resonator, the calculation of

[1](N. B. the Einstein summation convention is used for the indexies.)

such wave guide or resonating modes are of continued practical interest [4]. Again, since ferrites can be modeled by a linear, homogeneous anisotropic constitutive relation, these mode calculations are another example of an anisotropic electromagnetic problem.

c. Crystal-filled laser cavity.

A problem of recent interest is the calculation of the stationary modes of oscillation of a laser cavity filled with a solid, crystalline host material media. The cavity may be formed, for example, by an open structure consisting of two parallel plates (the so-called Fabry-Perrot resonator) with ruby crystal material media between the mirrors. In this context a stationary mode is defined as an electro-magnetic wave that starting from mirror 1, travels to mirror 2, and back to mirror 1, reproducing itself. The calculation of these modes is again an example of a linear, anisotropic electromagnetic problem [5].

1.2 Terminology and Notations

In this dissertation the anisotropy will be assumed to be only in the electric rather than the electric and magnetic quantities. Therefore,

$$\underline{B} = \mu \underline{H} \tag{1.5}$$

$$\underline{D} = \underset{\approx}{\epsilon} \underline{E} \tag{1.6}$$

where μ is a scalar magnetic permeability and in Eq. (1.6), the permittivity matrix $\underset{\approx}{\epsilon}$ relates the cartesian components of $\underline{D}$ and $\underline{E}$ vectors. This is a convenient assumption since if the anisotropy is in the magnetic quantities alone then the problem can be solved by the principle of duality. The question of both electric and magnetic anisotropy has no practical interest as yet.

From an energy consideration [6], it can be shown that the $\underset{\approx}{\epsilon}$ matrix of a linear, homogeneous, passive, lossless anisotropic medium must be Hermetian (with real and positive eigenvalues). There are a number of special cases of the $\underset{\approx}{\epsilon}$ matrix which are of practical interest.

A medium is called a uniaxially gyro-electric medium if its permittivity matrix can be represented in the following form, where $j=\sqrt{-1}$,

$$\underset{\approx}{\epsilon} = \begin{pmatrix} \epsilon_1 & jg & 0 \\ -jg & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}.$$

The dual form can be used to represent a mathematical model of a ferrite element. In this discussion we will consider, however, $\underset{\approx}{\epsilon}$ matrix with real elements only. For such a matrix it is possible, by orthogonal rotation of the coordinate axis, to transform the matrix into a diagonal form as follows

$$\underset{\approx}{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{1.7}$$

The new coordinate axes are called the principal axes of the medium, and such a medium is said to be a biaxial medium. If along the diagonal $\epsilon_1 = \epsilon_2$ then the medium is said to be a uniaxial medium. The axis correpsonding to ϵ_3 is called the distinguished or the optic axis of the medium. The uniaxially gyro-electric medium can be brought into the form of a biaxial medium by a complex rotation of the coordinate axis.

1.3 A Discussion of Methods Used in Solving Anisotropic Electromagnetic Problems and a Review of Some Solved Cases.

We now turn our attention to the problem of diffraction in an anisotropic medium. The two Fourier transformed Maxwell's equations

$$\nabla \times \underline{E} = -j\omega \underline{B} - \underline{M} \tag{1.8}$$

$$\nabla \times \underline{H} = j\omega \underline{D} + \underline{J} \tag{1.9}$$

and the two constitutive relations (1.5) and (1.6) contain sufficient information such that if the sources ($\underline{J}$ the electric, and $\underline{M}$, the magnetic current densities) are given then, in principle, the four vector fields $\underline{B}$, $\underline{D}$, $\underline{E}$, $\underline{H}$, can be computed. The practical problem lies in the fact that the above four equations are vector equations, each of which represent three scalar equations; that is, we have a total of twelve coupled scalar partial differential equations with arbitrary boundary conditions. Obviously, to solve such a problem analytically, in general, is impossible. To find the class of problems that may be solved, first we discuss those that are amenable to analytical study in a simple medium, like vacuum, and then try to extend the techniques of solution to the anisotropic problems.

In vacuum, the simplest vector problem is the calculation of the radiation from an arbitrarily oriented dipole radiating into an infinite space. This question is important not only because the dipole type problem is simple for visualization, but also the dipole response is the Green's function of an infinite medium. The method used for solution of the vacuum problem is the introduction of the so-called auxiliary potentials. These potentials satisfy a well-known wave equation whose sources are the scalar components of the given vector sources. We consider the case of electric dipole as the prototype for the discussion since the magnetic dipole case is then just the dual of the electric problem.

The method of solution is to decompose the electric dipole into its cartesian components, solve for the three scalar potentials and then reconstruct the field from the components. It is also known that [7] in principle, it is possible to decompose, uniquely, an arbitrary electromagnetic field into two partial fields one of which is transverse magnetic (TM) having no longitudinal magnetic field, and the other of which is transverse electric (TE) having no longitudinal electric field. Furthermore, each component of the partial field can be derived from a single scalar potential, called the TM and TE potential, that satisfies a wave equation. The difficulty with this decomposition is that given an arbitrary source distribution, there is no general procedure available to decompose a given source into partial sources one of which will generate a TM field and the other a TE field. However, in the case of an arbitrarily oriented dipole, such decomposition is available [8]. Thus this technique provides a method of solution to the dipole problem by the use of two rather than the three scalars required by the previous technique. There are however, certain special cases where an arbitrarily prescribed source is already in the desired form (for example, surface currents flowing on planar, cylindrical or spherical boundaries) and then the TE and TM modes can be found [9]. It is known that these are the only cases that are amenable to this type of solution [10]. In the case of a two dimensional problem, say no z dependency, a TE and TM decomposition with respect to z, can always be performed [11].

Having discussed the vacuum case, we now turn our attention to an anisotropic problem. Again, as the prototype, we inquire about the radiation characteristic of an arbitrarily oriented dipole radiating into an infinite anisotropic space. To keep the discussion simple, as a start, let us take the simplest type of anisotropy, the uniaxially anisotropic medium. Following the line of reasoning used in the vacuum problem, first we look for a decomposition of the anisotropic fields into TE and TM potentials each of which satisfies a wave equation. This decomposition turns out to be possible [12], where the coefficients of the wave equation, in cartesian coordinates, are different for the TE and TM potentials. Since by scaling the cartesian variables, the anisotropic wave equation can be brought into a standard form, that is the vacuum form, the field of a dipole radiating into an infinite uniaxially anisotropic media can be found from an equivalent scaled vacuum problem. Generalizing this idea, we can state that if for a given vacuum problem the sources can be resolved into two partial sources and the corresponding field problem can be solved, then there corresponds to this vacuum problem a scaled uniaxially anisotropic problem [13]. This statement may be generalized to cylindrical surface currents if the optic axis is along the axis of the cylinder [14]. In the case of two dimensional problems, a scaling can always be found that will convert a uniaxially anisotropic boundary value problem to an equivalent vacuum boundary value problem [15]. The use of three scalars, (vector potential) in an anisotropic problem leads to a complicated wave equation for one of the scalars. Although, by the judicious choice of gauge condition [16] this wave equation can be simplified, as yet, in general, this technique has not shown promise of success.

We now inquire about the radiation characteristic of a dipole immersed in a biaxial, or its equivalent, uniaxially gyroelectric medium. Again, following the line of reasoning used in uniaxially anisotropic media, first we look for a decomposition of the anisotropic fields into two partial fields one which is TE and the other is TM with respect to some longitudinal direction. It has been stated [17] , however, that for biaxial media no such decomposition is possible. If we take the twelve coupled scalar partial differential equations and by the process of elimination solve for one of the cartesian components of one of the vectors, say E_x, then we obtain a fourth order partial differential equation with constant coefficients. Hence the implication is clear; if the media is uniaxial then the fourth-order partial differential operator decomposes into a product of two second-order wave operators with constant coefficients, but in general, such decomposition is not available. Since a linear differential operator with constant coefficients has exponentials as its characteristic functions, by an expansion of the fields in terms of a spectrum of inhomogeneous plane waves, the so called angular spectrum, the problem can be converted into an eigen-value problem [18]. The corresponding inverse spatial Fourier transforms, in general, do not lend themselves to closed form solution. There are, however, a number of approximations available that provide physical insight into the mechanism of radiation in a biaxially anisotropic medium. A general approximation technique has been developed [19], that is based on a stationary phase approximation for two dimensional integrals, which is able to predict, using quasi-optical arguments, the nature of power flow in a biaxially anisotropic medium. In the case of a dipole, it is shown [20] that the inverse Fourier transform integral can be separated into two parts; a part that is singular but integrable, and thus displaying the singularity of the fields, and a part that is finite but does not lend itself to closed form evaluation.

1.4 An Outline of the Work of this Dissertation.

Approximation techniques notwithstanding, the general radiation problem in an anisotropic medium is quite complex. Thus in order to gain better insight into the radiation mechanism, we need a model that is both physical and at the same time analytically tractable. Obviously, infinite medium without boundaries or two dimensional problems with boundaries are hardly physical, but are studied because they are amenable to mathematical analysis. The aim of this dissertation is to extend the class of problems of radiation in an anisotropic medium which are capable of analytical solution and have some physical significance. Because uniaxial anisotropy is the simplest type of anisotropy, and furthermore, 3-dimensional vacuum electromagnetic boundary value problems that are solved have plane, spherical, conical or rotationally symmetric boundaries, in this dissertation attention will be focused on rotationally symmetric boundaries with rotationally symmetric boundary conditions or sources immersed in a uniaxially anisotropic host medium. A contribution of this dissertation is the demonstration that if the optic axis of the medium coincides with the axis of symmetry of the boundaries then by proper scaling, the uniaxial problem can be reduced to an equivalent vacuum boundary value problem. Moreover, as a practical application of the theory, the total field due to an axially aligned electric dipole immersed in a uniaxially anisotropic medium above a conductive prolate spheroid will be obtained. This field will also be calculated for a paraboloid of revolution. In these calculations it is assumed that optical axis is aligned with the axis of symmetry of the conducting boundaries. For the case of prolate spheroid, 'exact' high frequency and long wavelength asymptotic scattering will be calculated.

Chapter 2

Rotationally Symmetric Potentials

In this chapter we examine a class of isotropic electromagnetic boundary value problems with the aim of obtaining solutions to problems in anisotropic media.

2.1 A Derivation of Rotationally Symmetric Potentials

It is known [21] that, under special symmetry to be specified, it is possible to decompose, in vacuum, an arbitrary electromagnetic field into two partial fields, a decomposition different from Bromwich's [22], each of which is derivable from a single scalar that satisfies a second order partial differential equation. These scalars, designated as Rotationally Symmetric Potentials (RSP), reduce to a special case of Generalized Axially Symmetric Potentials (GASP) when the frequency is set to zero. Such potentials appear in many facets of engineering [23, 24].

The method of decomposition is based on the contention that if there is a preferred axis of symmetry, say the z axis, for the sources as well as the boundaries, then the fields should not depend on the angle of symmetry. This source, for example, could be a ring or axially aligned current element, with boundaries which are figures of revolution: spheroid or paraboloid, for example. Furthermore, it will be shown that in such a case, Maxwell's equations decouple into two independent sets of equations, one of which gives rise to a transverse electric potential (TE), and the other to a transverse magnetic potential (TM) with respect to z. Both of these potentials satisfy a second-order partial differential equation that is related to the scalar Helmholtz equation.

To derive these potentials, by the use of (1.1), (1.2), (1.8) and (1.9), we have in the source free case

$$\frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2} h_3 E_3 - \frac{\partial}{\partial u_3} h_2 E_2\right] = -j\omega\mu H_1 \qquad (2.1)$$

$$\frac{1}{h_1 h_3}\left(\frac{\partial}{\partial u_3} h_1 E_1 - \frac{\partial}{\partial u_1} h_3 E_3\right) = -j\omega\mu H_2 \qquad (2.2)$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1} h_2 E_2 - \frac{\partial}{\partial u_2} h_1 E_1\right) = -j\omega\mu H_3 \qquad (2.3)$$

$$\frac{1}{h_2 h_3}\left(\frac{\partial}{\partial u_2} h_3 H_3 - \frac{\partial}{\partial u_3} h_2 H_2\right) = j\omega\epsilon E_1 \qquad (2.4)$$

$$\frac{1}{h_1 h_3}\left(\frac{\partial}{\partial u_3} h_1 H_1 - \frac{\partial}{\partial u_1} h_3 H_3\right) = j\omega\epsilon E_2 \qquad (2.5)$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1} h_2 H_2 - \frac{\partial}{\partial u_2} h_1 H_1\right) = j\omega\epsilon E_3 \qquad (2.6)$$

where u_1, u_2, and u_3 are the generalized orthogonal coordinates with metrics h_1, h_2, h_3. The subscripts on the vectors refer to the components of the vector in that particular direction. Let u_3 denote the angle of symmetry - φ, with the corresponding metric $h_3 = \rho$ where ρ is the cylindrical distance variable in the x-y plane. If neither the sources nor the boundaries are functions of φ, (the metrics do not depend on φ), then the field components cannot have any φ dependency, and the derivative with respect to φ must be identically equal to zero. Setting the derivative with respect to φ equal to zero in Maxwell's equation, we have:

$$\frac{1}{h_2 h_3}\frac{\partial}{\partial u_2} h_3 E_3 = -j\omega\mu H_1 \qquad (2.7)$$

$$\frac{-1}{h_1 h_3}\frac{\partial}{\partial u_1} h_3 E_3 = -j\omega\mu H_2 \qquad (2.8)$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1} h_2 E_2 - \frac{\partial}{\partial u_2} h_1 E_1\right) = -j\omega\mu H_3 \qquad (2.9)$$

$$\frac{1}{h_2 h_3}\frac{\partial}{\partial u_2} h_3 H_3 = j\omega\epsilon E_1 \qquad (2.10)$$

$$\frac{-1}{h_1 h_3} \frac{\partial}{\partial u_1} h_3 H_3 = j\omega\epsilon E_2 \tag{2.11}$$

$$\frac{1}{h_1 h_2} \left(\frac{\partial}{\partial u_1} h_2 H_2 - \frac{\partial}{\partial u_2} h_1 H_1 \right) = j\omega\epsilon E_3 \tag{2.12}$$

Notice that (2.7) through (2.12) can be resolved into two independent set of equations. Equations (2.7), (2.8) and (2.12) have H_1, H_2, E_3 as their dependent variables, whereas (2.10), (2.11) and (2.9) have E_1, E_2, H_3 as their dependent variables. The first set has as its only electric field component $E_3 = E_\varphi$. Thus the electric field is transverse to the z axis and hence this set produces the TE mode. Similarly, the second set has as its only magnetic field component $H_3 = H_\varphi$. Thus the magnetic field is transverse to the z axis and hence this set produces the TM mode. Let us write out these equations as a TE set

$$\frac{1}{h_2 h_3} \frac{\partial}{\partial u_2} h_3 E_3 = -j\omega\mu H_1 \tag{2.7}$$

$$\frac{-1}{h_1 h_3} \frac{\partial}{\partial u_1} h_3 E_3 = -j\omega\mu H_2 \tag{2.8}$$

$$\frac{1}{h_1 h_2} \left(\frac{\partial}{\partial u_1} h_2 H_2 - \frac{\partial}{\partial u_2} h_1 H_1 \right) = j\omega\epsilon E_3 \tag{2.12}$$

and a TM set

$$\frac{1}{h_2 h_3} \frac{\partial}{\partial u_2} h_3 H_3 = j\omega\epsilon E_1 \tag{2.10}$$

$$\frac{-1}{h_1 h_3} \frac{\partial}{\partial u_1} h_3 H_3 = j\omega\epsilon E_2 \tag{2.11}$$

$$\frac{1}{h_1 h_2} \left(\frac{\partial}{\partial u_1} h_2 E_2 - \frac{\partial}{\partial u_2} h_1 E_1 \right) = -j\omega\mu H_3 \tag{2.9}$$

Now by the duality transformations, that is $E_i \longrightarrow H_i$ and $H_i \longrightarrow -E_i$ and $\epsilon \leftrightarrows \mu$, it is easy to see that the two sets are equivalent. Therefore, for the purpose of this discussion, the TM set will be considered. In the TM set we have three simultaneous equations with three unknowns and thus, by the process of elimination, we can generate a single partial differential equation for one of the unknowns.

The RSP potential is

$$\psi = h_3 H_3 \tag{2.13}$$

and is that scalar from which the vector components can be generated. To eliminate E_1 and E_2 from (2.9), multiply (2.10) by h_1 and take the derivative with respect to u_2 and similarly in (2.11) multiply by h_2 and take the derivative with respect to u_1. Substituting these equations into (2.9) we have

$$\frac{\partial}{\partial u_1}\left(\frac{h_2}{h_1 h_3}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1}{h_2 h_3}\frac{\partial \psi}{\partial u_2}\right) + \omega^2 \epsilon\mu \frac{h_1 h_2}{h_3}\psi = 0 \tag{2.14}$$

Hence ψ satisfies a second order partial differential equation of elliptic type, which in general, has variable coefficients. For future reference, in Table 1, Equation (2.14) is depicted in some commonly occuring figure of revolution coordinate systems. In order to determine the nature of the solutions of (2.14), in the next section, the relation between it and most widely studied second order scalar wave equation, the Helmholtz equation, will be demonstrated. A comparison between the so-called Bromwich potentials and RSP will be also presented.

	INDEPENDENT VARIABLES	METRICS	ROTATIONALLY SYMMETRIC WAVE EQUATION
CIRCULAR-CYLINDER	$u_1 = z$ $u_2 = \rho$ $u_3 = \varphi$	$h_1 = 1$ $h_2 = 1$ $h_3 = \rho$	$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}\right) + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$
SPHERE	$u_1 = r$ $u_2 = \Theta$ $u_3 = \varphi$	$h_1 = 1$ $h_2 = r$ $h_3 = r \sin\Theta$	$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\Theta}{r^2}\frac{\partial}{\partial \Theta}\left(\frac{1}{\sin\Theta}\frac{\partial \psi}{\partial \Theta}\right) + k^2 \psi = 0$
PROLATE SPHEROID	$u_1 = \xi$ $u_2 = \eta$ $u_3 = \varphi$	$h_1 = c\sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}}$ $h_2 = c\sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}}$ $h_3 = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}$	$(\xi^2 - 1)\frac{\partial^2 \psi}{\partial \xi^2} + (1 - \eta^2)\frac{\partial^2 \psi}{\partial \eta^2} + k^2 c^2 (\xi^2 - \eta^2)\psi = 0$
PARABOLOID	$u_1 = u$ $u_2 = v$ $u_3 = \varphi$	$h_1 = \sqrt{u^2 + v^2}$ $h_2 = \sqrt{u^2 + v^2}$ $h_3 = \sqrt{uv}$	$\sqrt{u}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{u}}\frac{\partial \psi}{\partial u}\right) + \sqrt{v}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{v}}\frac{\partial \psi}{\partial v}\right) + k^2(u^2 + v^2)\psi = 0$

Table 1

Metrics, transformation, and RSP Wave Equation in Some Rotational Coordinate Systems

2.2 Discussion on Bronwich and Rotationally Symmetric Potentials.

The Rotationally Symmetric Potentials in cylindrical coordinates satisfy Eq. (2.14). In these coordinates the equation is

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial \rho}\right) + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0 \qquad (2.14a)$$

We now show that solution to Eq. (2.14) may be obtained from solutions of the scalar Helmholtz equation.

In cylindrical coordinates, the Helmholtz equation is

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + k^2 \phi = 0 \qquad (2.15)$$

This equation can be brought to the form of (2.14a) by the change of a variable. Let $\phi(\rho, \varphi, z)$ be a solution of (2.15) and of the form

$$\phi(\rho, \varphi, z) = f(\rho, z) e^{j\varphi} . \qquad (2.16)$$

Then, as is shown below,

$$\psi(\rho, \varphi, z) = \rho f(\rho, z) \qquad (2.17)$$

is a solution of (2.14a).

Substituting (2.16) into (2.15), we have for $f(\rho, z)$ the equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{\partial^2 f}{\partial z^2} + \left(k^2 - \frac{1}{\rho^2}\right) f = 0 \qquad (2.18)$$

Let $g(\rho, z) = \rho f(\rho, z)$ be a new dependent variable in (2.18).

Then with this, dependent variable (2.18) is

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial g}{\partial \rho}\right) + \frac{\partial^2 g}{\partial z^2} + k^2 g = 0 \qquad (2.19)$$

Comparing (2.19) with (2.14a), we see that $g(\rho, z) = \psi(\rho, z)$. Thus we have demonstrated that the RSP's may be obtained from the solutions of the scalar Helmholtz equation.

The general technique of decomposing the vector electromagnetic problem into a set of scalar problems is based on the ability of separating Maxwell's equations into two sets of independent equations. The fields of these independent sets of equations may then be derived from a scalar that satisfies a second-order partial differential equation. It is generally known that such resolution is feasible in cartesian coordinate system [22] where the scalar, referred to as Bromwich potential, satisfies a scalar Helmholtz wave equation.

It is also known that the scalar Helmholtz equation separates into sets of ordinary differential equations in eleven coordinate systems. However, in spite of the fact that the scalar Helmholtz equation separates, for example, in ellipsoidal coordinate system, in such coordinate systems the vector wave equation of the electromagnetic problem cannot be solved in a simple form in terms of solutions of scalar wave equation. The reason, for this apparent anomaly, is that the scalar, from which the fields are generated, is not related to the tangential field components in a simple fashion in the coordinate system mentioned above. In fact, out of the eleven coordinates in which the Helmholtz equation separates, only five of them (plane, circular - , elliptic -, parabolic cylinder, sphere and cone) give rise to scalars that are simply related to the tangential field components. Out of these five systems only the spherical coordinates deal with fully 3-dimensional problems. If, however, there is an angle of symmetry, both in the sources as well as the boundaries, then, as it was shown, Maxwell's equations may be separated in additional coordinate systems. For example, spheroid and paraboloid of revolution coordinates, systems in which Bromwich potentials do not exist.

Chapter 3

Anisotropic Rotationally Symmetric Boundary Value Problems and Their Scaling to an Equivalent Vacuum Problem

In this chapter we extend the concept of rotationally symmetric potential for application to electromagnetic problems with uniaxially anisotropic host medium. It is assumed that the optic axis is aligned with the symmetry axis of sources and boundaries. After discussing the physical interpretation of the assumed boundary conditions in problems with rotational symmetry in isotropic media, a scaling will be presented that reduces the anisotropic boundary value problem to an equivalent isotropic problem.

3.1 Extension of the Concept of RSP for use in Anisotropic Media.

In this section it will be shown that for anisotropic medium whose optic axis is aligned with the symmetry axis of the rotationally symmetric boundary, the $\underline{\underline{\epsilon}}$ matrix of the new constitutive relation, that is the matrix that connects the curvilinear components of the $\bar{D}$ vector to the components of the $\bar{E}$ vector, does not depend on the angle of symmetry φ. Furthermore, the only coupling of D_φ to E_φ is through a constant, independent of the coordinate system. This fact will allow us to use the same reasoning applied in the case of isotropic medium to resolve Maxwell's equation into two sets of equations, one of which is TE and the other TM with respect to the z axis, the axis of symmetry. The TE mode will be a vacuum mode, that is it will satisfy the same equations as the vacuum case. However, the TM mode is an anisotropic mode and hence the duality applicable to the vacuum case will not hold.

The following section is divided into two parts; in the first part, the constitutive equations for rotationally symmetric vectors will be given, and in the second part, RSP will be extended to uniaxial medium with optic axis parallel to the axis of symmetry.

3.11 Constitutive Equations

The constitutive equations for an uniaxially anisotropic medium can be specified, in matrix form, as follows

$$\underline{D_x} \triangleq \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \epsilon \begin{pmatrix} 1 & & \\ & 1 & \\ & & K \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \triangleq \underline{\underline{\epsilon}} \underline{E_x} \tag{3.1}$$

$$\underline{B_x} \triangleq \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \mu \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} \triangleq \underline{\underline{\mu}} \underline{H_x} \tag{3.2}$$

where the optical axis is parallel to the z axis, and $\underline{D_x}$, $\underline{E_x}$, $\underline{B_x}$ and $\underline{H_x}$ are the magnetic and the electric vectors and the subscripts refer to the cartesian components of these vectors, $\underline{\underline{\epsilon}}$ is the electric permittivity and $\underline{\underline{\mu}}$ is the magnetic permeability and K is the relative permittivity of the medium. Let the transformation from cartesian to an arbitrary rotationally symmetric coordinate system be given by:

$$x = q(u_1, u_2) \cos\varphi \tag{3.3}$$

$$y = q(u_1, u_2) \sin\varphi \tag{3.4}$$

$$z = f(u_1, u_2) \tag{3.5}$$

where φ refers to the angle measured from the x axis in the x-y plane, and u_1 and u_2 refer to the rotationally symmetric coordinates. Introducing the following notation,

$$g_1 = \frac{\partial g}{\partial u_1} \quad , \quad g_2 = \frac{\partial g}{\partial u_2} \tag{3.6}$$

$$f_1 = \frac{\partial f}{\partial u_1} \quad , \quad f_2 = \frac{\partial f}{\partial u_2} \tag{3.7}$$

let h_1 be the metric corresponding to u_1, and h_2 be the metric corresponding to u_2, and h_3 be the metric of the angle φ, then

$$\underline{a}_{\underline{x}} \triangleq \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \frac{g_1}{h_1}\cos\varphi & \frac{g_2}{h_2}\cos\varphi & -\sin\varphi \\ \frac{g_1}{h_1}\sin\varphi & \frac{g_2}{h_2}\sin\varphi & \cos\varphi \\ \frac{f_1}{h_1} & \frac{f_2}{h_2} & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_\varphi \end{pmatrix} \triangleq \underline{\underline{A}}\, a_{\underline{u}} \tag{3.8}$$

where the a's refer to unit vectors corresponding to the subscripts as indicated. Since the transformation is unitary, the corresponding inverse of $\underline{\underline{A}}$ is

$$\underset{\approx}{A}^{t} = \underline{\underline{A}}^{-1} \tag{3.9}$$

Now using (3.1), (3.8) and (3.9), and defining $\underset{\approx}{\epsilon}'$ such that

$$D_{\underline{u}} \triangleq \begin{pmatrix} D_1 \\ D_2 \\ D_\varphi \end{pmatrix} = \begin{pmatrix} \epsilon'_{11} & \epsilon'_{12} & \epsilon'_{13} \\ \epsilon'_{21} & \epsilon'_{22} & \epsilon'_{23} \\ \epsilon'_{31} & \epsilon'_{32} & \epsilon'_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_\varphi \end{pmatrix} \triangleq \underline{\underline{\epsilon}}' E_{\underline{u}} \tag{3.10}$$

we have

$$\underline{\underline{\epsilon}}' = \underline{\underline{A}}^{-1}\, \underline{\underline{\epsilon}}\, \underline{\underline{A}} \tag{3.11}$$

which is the required permittivity matrix. Performing the indicated matrix multiplication yields

$$\epsilon'_{11} = \frac{\epsilon}{h_1^2}\left(g_1^2 + K f_1^2\right) \tag{3.12}$$

$$\epsilon'_{21} = \epsilon'_{12} = \frac{\epsilon}{h_1 h_2}\left(g_1 g_2 + K f_1 f_2\right) \tag{3.13}$$

$$\epsilon'_{22} = \frac{\epsilon}{h_2^2}\left(g_2^2 + K f_2^2\right) \tag{3.14}$$

$$\epsilon'_{32} = \epsilon'_{23} = 0 \tag{3.15}$$

$$\epsilon'_{31} = \epsilon'_{13} = 0 \tag{3.16}$$

$$\epsilon'_{33} = \epsilon \tag{3.17}$$

Note that for arbitrary rotationally symmetric coordinates, the transformation from cartesian to rotationally symmetric coordinates is analytic, that is it satisfies Cauchy-Riemann equations [25]

$$\frac{1}{h_1}\frac{\partial g}{\partial u_1} = \frac{1}{h_2}\frac{\partial f}{\partial u_2} \tag{3.18}$$

$$\frac{1}{h_2}\frac{\partial g}{\partial u_2} = -\frac{1}{h_1}\frac{\partial f}{\partial u_1} \tag{3.19}$$

Because of this property if the medium is vacuum, i.e. K = 1, as expected, (3.12) through (3.17) reduces to a constant multiple of a unit matrix. As indicated above $\epsilon'_{33} = \epsilon_{\varphi\varphi} = \epsilon$ is a constant, independent of the coordinate system.

3.12 RSP for Uniaxial Medium with Optical Axis Parallel to the Axis of Symmetry.

To derive the RSP mode equations, we use Maxwell's equations in conjunction with the derived constitutive equations. The Fourier transform ($\frac{\partial}{\partial t} \rightarrow j\omega$) of Maxwell's equation in conjunction with the constitutive equations (3.2) and (3.10) are in the following form

$$\frac{1}{h_2 h_3}\left(\frac{\partial}{\partial u_2} h_3 E_\varphi - \frac{\partial}{\partial \varphi} h_2 E_2\right) = -j\omega\mu H_1 \tag{3.20}$$

$$\frac{1}{h_1 h_3}\left(\frac{\partial}{\partial \varphi} h_1 E_1 - \frac{\partial}{\partial u_1} h_3 E_\varphi\right) = -j\omega\mu H_2 \tag{3.21}$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1} h_2 E_2 - \frac{\partial}{\partial u_2} h_1 E_1\right) = -j\omega\mu H_\varphi \tag{3.22}$$

$$\frac{1}{h_2 h_3}\left(\frac{\partial}{\partial u_2} h_3 H_\varphi - \frac{\partial}{\partial \varphi} h_2 H_2\right) = j\omega\left(\epsilon'_{11} E_1 + \epsilon'_{12} E_2\right) \tag{3.23}$$

$$\frac{1}{h_1 h_3}\left(\frac{\partial}{\partial \varphi} h_1 H_1 - \frac{\partial}{\partial u_1} h_3 H_\varphi\right) = j\omega\left(\epsilon'_{21} E_1 + \epsilon'_{22} E_2\right) \tag{3.24}$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1} h_2 H_2 - \frac{\partial}{\partial u_2} h_1 H_1\right) = j\omega\epsilon E_\varphi \tag{3.25}$$

Using the same reasoning as used in the case of vacuum, and assuming that the boundaries (and hence the metrics) as well as the sources are independent of the angle of symmetry $-\varphi$, and from the fact that $\underset{\approx}{\epsilon}'$ is independent of φ, we conclude that the fields cannot have φ dependency. Thus the derivative with respect to φ must vanish. Therefore (3.20) through (3.25) separate into two independent rotationally symmetric modes, one of which is TE with respect to z and the other is TM with respect to z. From (3.20), (3.21) and (3.25) we have the TE equations

$$\frac{1}{h_2 h_3} \frac{\partial}{\partial u_2} h_3 E_\varphi = -j\omega\mu H_1 \tag{3.26}$$

$$\frac{-1}{h_1 h_3} \frac{\partial}{\partial u_1} h_3 E_\varphi = -j\omega\mu H_2 \tag{3.27}$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1} h_2 H_2 - \frac{\partial}{\partial u_2} h_1 H_1\right) = j\omega\epsilon E_\varphi \tag{3.28}$$

and from (3.23), (3.24) and (3.22) the TM equations

$$\frac{1}{h_2 h_3} \frac{\partial}{\partial u_2} h_3 H_\varphi = j\omega\left(\epsilon_{11}' E_1 + \epsilon_{12}' E_2\right) \tag{3.29}$$

$$\frac{-1}{h_1 h_3} \frac{\partial}{\partial u_1} h_3 H_\varphi = j\omega\left(\epsilon_{21}' E_1 + \epsilon_{22}' E_2\right) \tag{3.30}$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1} h_2 E_2 - \frac{\partial}{\partial u_2} h_1 E_1\right) = -j\omega\mu H_\varphi \tag{3.31}$$

Comparing the TE set with (2.7), (2.8) and (2.12), the TE vacuum equations, we recognize (3.26), (3.27) and (3.28) to be in the same form as the vacuum equations. However, this is not true for the TM equations. As indicated, therefore, the duality applicable to the vacuum case does not hold. We have now shown that the concept of RSP may be extended to uniaxial medium.

If we again define

$$\psi = h_3 H_3 \tag{3.32}$$

then by eliminating E_1 and E_2 from (3.31) by the use of (3.29) and (3.30) and using the relationship

$$\epsilon'_{11}\epsilon'_{22} - \epsilon'^{2}_{12} = K\epsilon^2$$

(which is obtained from Eqs. (3.12) through (3.19)) we have a single partial differential equation for ψ which is

$$\frac{\partial}{\partial u_1}\left(\frac{h_2(g_1^2 + Kf_1^2)}{h_1^3 h_3}\frac{\partial \psi}{\partial u_1} + \frac{g_1 g_2 + Kf_1 f_2}{h_1 h_2 h_3}\frac{\partial \psi}{\partial u_2}\right)$$

$$+\frac{\partial}{\partial u_2}\left(\frac{g_1 g_2 + Kf_1 f_2}{h_1 h_2 h_3}\frac{\partial \psi}{\partial u_1} + \frac{h_1(g_2^2 + Kf_2^2)}{h_2^3 h_3}\frac{\partial \psi}{\partial u_2}\right) \tag{3.34}$$

$$+\ \omega^2 \epsilon \underline{\mu} K \frac{h_1 h_2}{h_3}\psi = 0$$

The vacuum mode equation (2.14) in conjunction with the equation (3.34) reduces the solution of the vector anisotropic boundary value problem to a solution of two scalar problems. Note that if in (3.34) K = 1, as expected, it reduces to (2.14), the vacuum case. As opposed to the isotropic equation, (3.34) does not separate into ordinary differential equations except in cylindrical coordinates. Before we discuss the boundary conditions of the scalar equations, we also note that (3.34) may be generalized to stratified as well as inhomogeneous medium where the inhomogeneity does not depend on φ , the angle of symmetry.

3.2 Vacuum Boundary Conditions

In this section three types of boundary conditions, Dirichlet, Neumann and Robin conditions, for the above potentials, will be discussed. It will be shown that in the case of vacuum, the specification of a homogeneous Dirichlet boundary condition is physically interpreted as a lack of impressed currents whereas the vanishing of the normal derivative, the Neumann condition, at a boundary represents an ideal conductor for the boundary surface. Furthermore, the notion of a surface impedance will be introduced. This impedance unifies the Dirichlet and the Neumann problems.

3.21 Dirichlet Problem

Again, for the purpose of discussing the boundary conditions, only the TM case will be considered. The interpretation of the TE boundary condition follows from the principle of duality. Since the only magnetic field component in the TM case is H_φ, from Maxwell's equation we have

$$\nabla \times H_\varphi \underline{a}_\varphi = j\omega\epsilon \left(E_1 \underline{a}_1 + E_2 \underline{a}_2\right) \qquad (3.35)$$

Using Stokes' theorem,

$$\int_A \nabla \times H_\varphi \underline{a}_\varphi \cdot \underline{n}\, da = \oint_C H_\varphi \underline{a}_\varphi \cdot d\underline{\ell} \qquad (3.36)$$

But

$$j\omega \int_A D_h \, da = \int_0^{2\pi} h_3 H_\varphi \, d\varphi = 2\pi \psi \qquad (3.37)$$

or

$$\psi = \frac{j\omega Q}{2\pi} = \frac{I}{2\pi} \qquad (3.38)$$

Therefore, the specification of impressed electric current on the u_1 = constant surface leads to a Dirichlet problem for ψ .

3.22 Neumann Problem

Let u_1 be the normal and u_2 be the tangential coordinate. From Eq. 2.11 reproduced below

$$-\frac{1}{h_1 h_3}\frac{\partial}{\partial u_1}\psi = j\omega\epsilon E_2$$

we have

$$-\frac{\partial\psi}{\partial n} = j\omega\epsilon h_3 E_t \tag{3.39}$$

But on an ideal conductor the tangential electric field is zero, hence specification of an ideal conducting boundary leads to a homogeneous Neumann problem.

3.23 Robin Problem

Let a surface impedance Z be defined as

$$Z \triangleq \left.\frac{E_2(u_2)}{H_\varphi(u_2)}\right|_{u_1 = u_0} \tag{3.40}$$

Substituting (3.39) into (3.40) we have

$$Z = -\left.\frac{\frac{1}{j\omega\epsilon}\frac{\partial\psi}{\partial n}}{h_3 H_4}\right|_{u_1 = u_0} \tag{3.40a}$$

Rearranging (3.40a) yields

$$\frac{\partial\psi}{\partial n} + \sigma(u_2)\psi = 0 \tag{3.41}$$

where

$$\sigma(u_2) = j\omega\epsilon Z(u_2)$$

This type of boundary value problem where a linear combination of the function and its normal derivative are specified on a coordinate surface is referred to as a Robin problem. Note that if in (3.41) we set $Z = 0$ then the homogeneous Robin problem reduces to a homogeneous Neumann problem, as expected. If, however, we let $Z \to \infty$, while keeping $\frac{\partial \psi}{\partial n}$ finite, the homogeneous Robin problem reduces to a homogeneous Dirichlet problem. Thus we have shown, as indicated above, that the concept of surface impedance unifies the Dirichlet and the Neumann conditions.

3.3. Scaling of the Anisotropic Boundary Value Problem.

To note the effect of the anisotropy of the medium on the mode equation (3.34), consider (3.34) in cylindrical coordinate system which is

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial \rho}\right) + K \frac{\partial^2 \psi}{\partial z^2} + K k^2 \psi = 0 \qquad (3.42)$$

where k is the isotropic wave-number. Let

$$z' = \frac{z}{\sqrt{K}} \qquad (3.42a)$$

be the transformed coordinate. Substituting (3.42a) into (3.42), equation (3.42) reduces to an equivalent isotropic mode equation (2.14a) with $k'^2 = K k^2$ as its new wavenumber. Thus by a scaling of the coordinate axis, the mode equation (3.42) as well as the boundary is transformed in a form resembling an isotropic problem. The only question remaining is the effect of such scaling upon the boundary condition. Here, the idea of surface impedance is utilized. It is shown that by the introduction of a direction dependent surface impedance, the anisotropic boundary value problem is reduced to an equivalent isotropic boundary value problem.

Let the anisotropic surface impedance Z be defined as

$$Z \triangleq \left. \frac{E_{\ell}}{H_{\varphi}} \right|_{f(\rho, z) = 0} \tag{3.43}$$

where $f(\rho, z)=0$ is the equation of the constant coordinate surface in cylindrical coordinate system. See Figure 1 for the geometry of the scaling. From Figure 1, however,

$$E_{\ell} = E_z \sin \alpha + E_{\rho} \cos \alpha \tag{3.44}$$

Using (3.29) and (3.30), rewritten for reference in cylindrical coordinates as

$$\frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = j\omega\epsilon K E_z \tag{3.29}$$

$$-\frac{1}{\rho} \frac{\partial \psi}{\partial z} = j\omega\epsilon E_{\rho} \tag{3.30}$$

and substituting these equations in conjunction with (3.44) into (3.43) yields for

$$Z = \frac{\dfrac{\sin \alpha}{j\omega\epsilon K} \dfrac{\partial \psi}{\partial \rho} - \dfrac{\cos \alpha}{j\omega\epsilon} \dfrac{\partial \psi}{\partial z}}{\psi} \tag{3.45}$$

Rearranging, we have

$$K \cos \alpha \frac{\partial \psi}{\partial z} - \sin \alpha \frac{\partial \psi}{\partial \rho} + j\omega\epsilon K Z \psi = 0 \quad @ \quad f(\rho, z) = 0 \tag{3.46}$$

If we let K = 1 in (3.46) then, as expected, it reduces to (3.41), the isotropic Robin condition. Note that Z is the isotropic surface impedance. Introducing the following scaling transformations in (3.46), see Figure 1, let

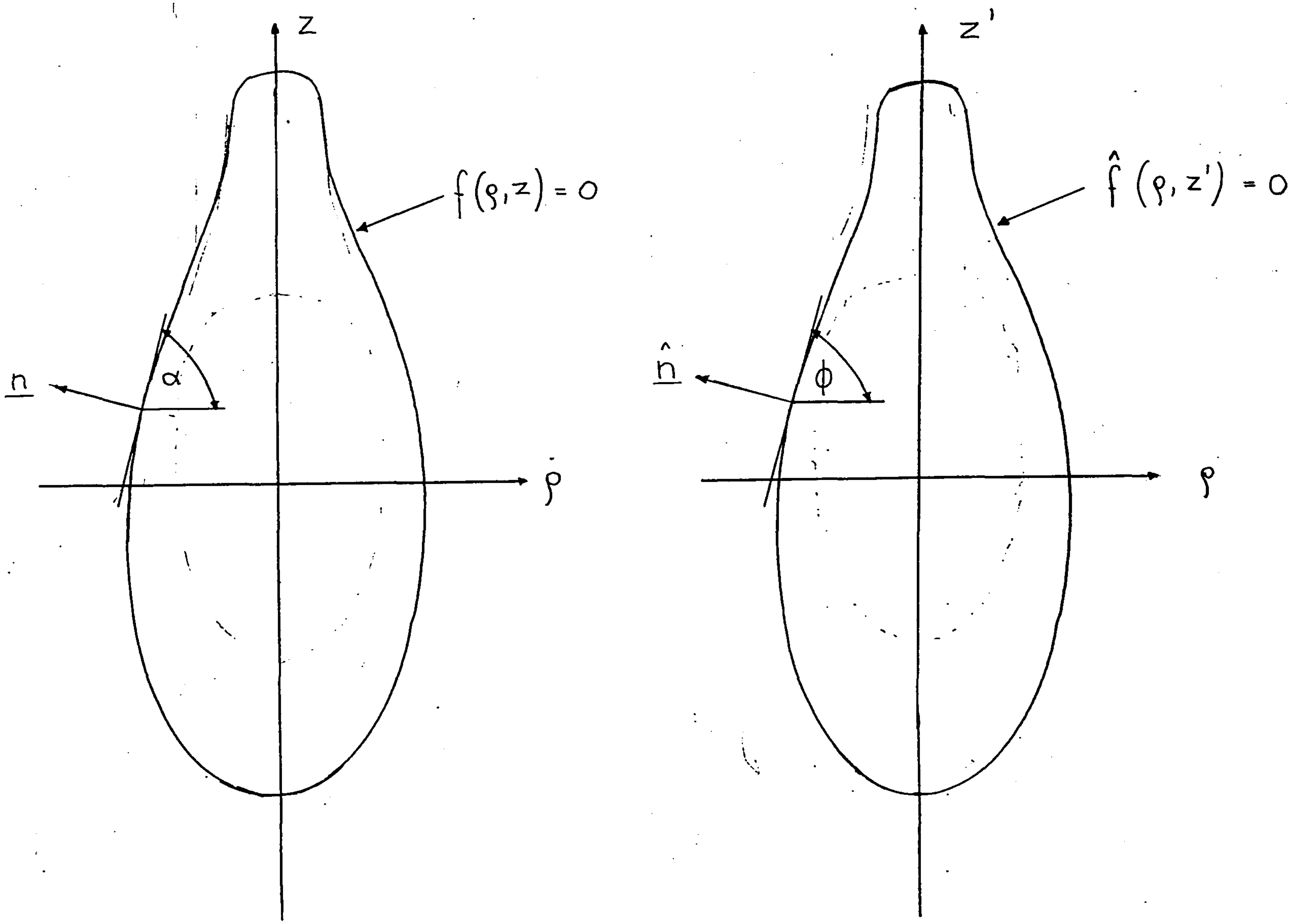

Figure 1. The Geometry of the Scaling

$$\sqrt{K} \cot \alpha = \cot \phi$$

we have, by using equation (3.42a)

$$\frac{\partial \hat{\psi}}{\partial \hat{n}} + j\omega\epsilon \hat{Z} \hat{\psi} = 0 \qquad @ \qquad \hat{f}'(\rho, z') = 0 \tag{3.47}$$

where

$$\hat{Z} = \frac{K}{(\sin^2\alpha + K\cos^2\alpha)^{1/2}} Z$$

and the caret denotes the scaled variables. Thus, as indicated, the anisotropic boundary problem is reduced, by the introduction of a direction dependent surface impedance $\hat{Z}$, to an equivalent isotropic boundary value problem. From (3.47) we note, that if the original boundary surface is an ideal conductor, that is $Z = 0$, the scaled boundary surface is an ideal conductor as well. Hence, the homogeneous Neumann condition in an anisotropic medium transforms to an equivalent isotropic homogeneous Neumann problem. It is also noted that the form of the Dirichlet boundary condition does not change by the process of scaling. To show that this is the case, consider $\psi[\rho, g(\rho)]$ to be the anisotropic Dirichlet condition on the boundary surface $z = g(\rho)$. Since the only scaling is along the z axis, the form of ψ cannot change and thus the form of the anisotropic Dirichlet condition on the old surface is the same as the form of the equivalent isotropic Dirichlet condition on the scaled surface. As an example, let the boundary be a unit sphere whose equation is

$$z^2 + \rho^2 = 1 \tag{3.48}$$

and the Dirichlet boundary condition, say,

$$\psi(\theta) = \cos\theta \tag{3.49}$$

Since (3.49) in cylindrical coordinates is $\psi(\rho, z) = \dfrac{z}{\sqrt{z^2+\rho^2}}$

with the use of (3.48)

$$\psi(\rho) = \sqrt{1-\rho^2} \tag{3.50}$$

Since the scaling along the z axis does not affect the variable ρ, the Dirichlet condition on the scaled coordinate surface is

$$\hat{\psi}(\rho) = \sqrt{1-\rho^2} \tag{3.51}$$

To summarize, it has been shown that in a uniaxial anisotropic medium the concept of RSP can be extended for the use of anisotropic electromagnetic boundary value problems and the resulting scalar anisotropic boundary value problem is transformable, by proper scaling, to an equivalent scalar isotropic boundary value problem. In the subsequent chapters, some practical anisotropic boundary value problems and their solutions are considered.

Chapter 4

Scattering by a Conductive Spheroid and Paraboloid of Revolution

The scaling technique developed in the previous chapter will now be applied to some problems of practical interest. In particular, the total (incident plus scattered) field due to an electric dipole aligned with the axis of a rotationally symmetric boundary and the optic axis of an anisotropic medium will be calculated. To be considered in the discussion are spheroidal, rotationally symmetric ellipsoidal and paraboloidal boundaries. The spheroidal shaped surface is of interest because it is the most general finite surface for which Maxwell's equations can be solved analytically. The paraboloidal surface is included because its usefulness in antenna applications. Since by proper scaling of the isotropic scattering problem anisotropic solutions can be generated, attention will be focused on the vacuum problem. The anisotropic problems and their solutions will be obtained by scaling the vacuum solutions.

4.1 Scattering by a Conducting Prolate Spheroid.

The intent of Abraham [26], the orginiator of RSP, was to use RSP modes to study the natural electromagnetic oscillations of a spheroidal cavity with perfectly conducting walls. Since his treatment of the problem, a number of researchers have contributed to this as well as other electromagnetic problems involving spheroidal geometry [27, 28, 29, 30, 31]. Only recently [32] spheroidal conducting surfaces have been proposed as an approximating model for the calculations of the resonant modes of a confocal mirror laser resonator. The functions, (the solutions of the separated RSP mode equation in spheroidal coordinate system) that arise in these problems are spheroidal wave functions[1] that are of interest in many fields of engineering. For example: Fourier analysis, mean-square estimation, acoustic problems [33], to name a few. Thus the problem chosen for illustrative purpose is not only of interest for the reason stated but as a representative of a much wider class of problems, namely, problems involving spheroidal wave functions. In the next section, the problem of calculating the total field due to an electric dipole above a conducting spheroid is formulated and solved. The solution is Eqs. (4.7), (4.8), (4.10) and (4.15).

4.11 Statement of the Problem.

Let ξ and η denote the radial and the angle prolate spheroidal coordinates, respectively. Figure 2 shows a conductive prolate spheroid centered at the origin with coordinate $\xi = \xi_0$ with parameter c as its semi-interfocal distance. Let an electric dipole whose axis is coincident with the symmetry axis of a spheroid be located at a point $\xi = \alpha$ and $\eta = 1$. Let the electric dipole and conductive prolate spheroid be immersed in a uniaxially anisotropic host medium whose optic axis is parallel with the symmetry axis of the conducting spheroid. The problem is to find the total electromagnetic field, consisting of an incident dipole field and a scattered field.

[1] There are many books, tables, and articles on the subject of spheroidal wave functions. Some of their properties are reviewed in Appendix A on Prolate Spheroidal Wave Functions. The following books were found useful on the subject:

Stratton, Hunter, Little, Chu, and Corbato, Spheroidal Wave Functions, John Wiley, New York, 1956.

C. Flammer, Spheroidal Wave Functions, Stanford U. Press, Stanford, Calif. 1957.

Meixner, and Schäfke, Mathieusche Funktionen and Sphäroidfunktionen, Springer-Verlag, Berlin 1954.

L. Robin, Fonctions Spheriques de Legendre et Functions Spheroidales, Gauthier-Villars, Paris, vol. 3., 1957.

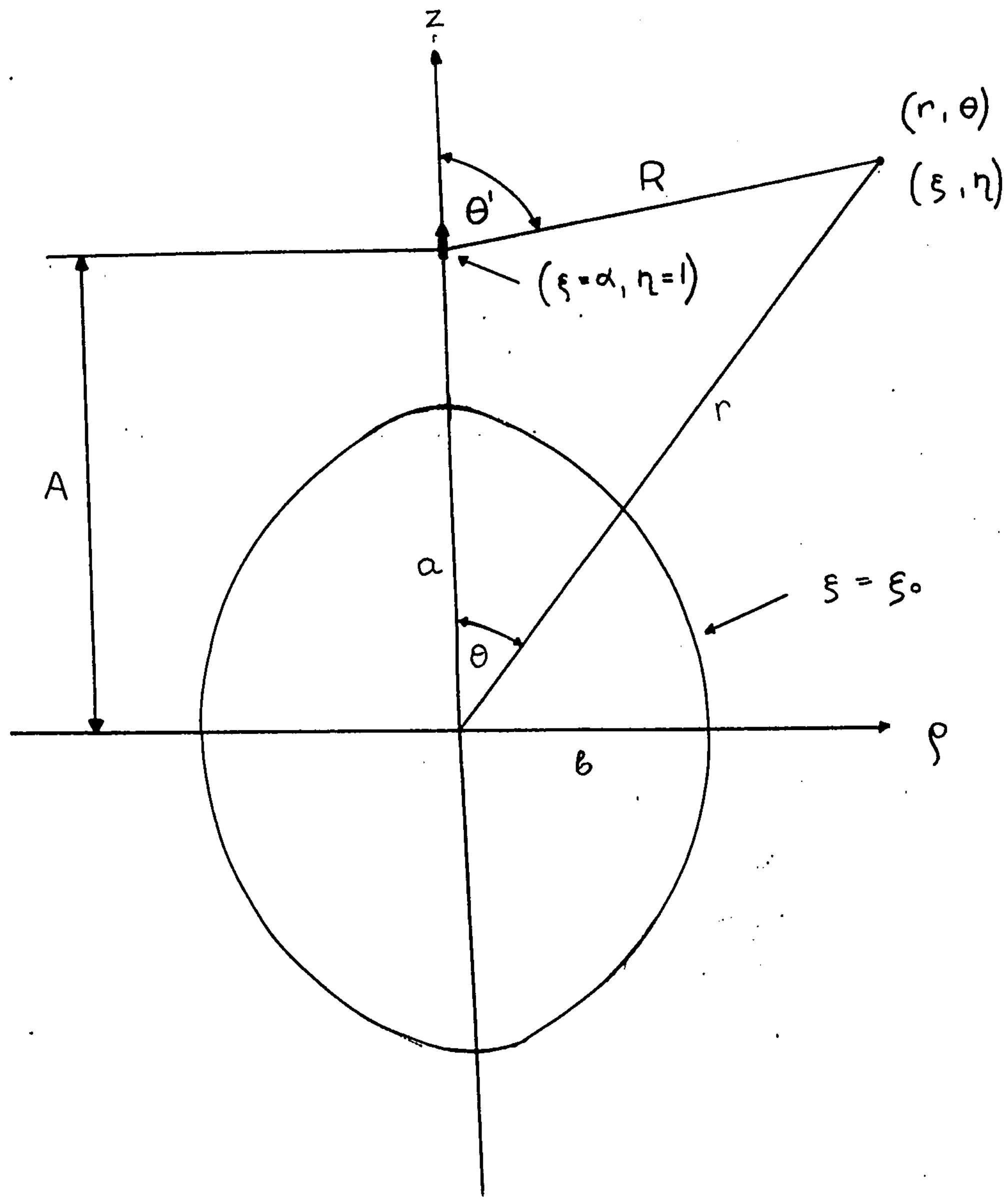

Figure 2. Geometry of the Spheroidal Scattering Problem

4.12 Solution in Isotropic Media

The method of solution of this scattering problem follows the usual procedure. First the incident dipole field is expanded as a sum of prolate spheroidal waves. Then by assuming the scattered field to be a sum of outgoing spheroidal waves of the same form as the incident field but with unknown coefficients, referred to as the scattering coefficients, the form of the total field is found. By satisfying the requirement that the tangential electric field on the conductive surface should vanish, the scattering coefficients are determined and, in accordance with the uniqueness theorem, the solution is complete.

To determine the spheroidal wave expansion of the incident dipole field, following Meixner and Schäfke [34], we note that the field of an electric dipole, on and aligned with the z axis, is rotationally symmetric, as well as transverse magnetic. Thus the only magnetic field component is

$$H_\varphi^{(i)} = \frac{A'}{4\pi} \frac{\omega e^{-jkR}}{R} \left(\frac{j}{R} - k \right) \sin \theta' \tag{4.1}$$

where (i) stands for the incident field, k is the wave number, R is the distance measured from the observer to the source (see fig. 2), θ' is the angle measured from the z axis with the dipole as the origin and A' is a complex constant signifying the magnitude of the electric dipole moment. For future reference, we note that (4.1) can be written as (see Ref. 34)

$$H_\varphi^{(i)} = -\tilde{A} \omega k^2 \psi_1^{(4)} (kR)\, P_1' (\cos \theta') \tag{4.1a}$$

where $\tilde{A} = \frac{A'}{4\pi}$ and $\psi_1^{(4)}$ is the spherical Bessel function of fourth kind, first degree and P_1' is the associated Legendre polynomial of first order and first degree, and $\cos\theta'$ and R are given in terms of the spheroidal coordinates by

$$\cos\theta' = \frac{\xi\eta - \alpha}{\sqrt{\xi^2 + \eta^2 + \alpha^2 - 2\xi\eta\alpha - 1}} \tag{4.2}$$

$$R = c\sqrt{\xi^2 + \eta^2 + \alpha^2 - 2\xi\eta\alpha - 1}$$

where c is the semi-interfocal distance of the spheroid, and (ξ, η) are the spheroidal coordinates of the field variables. As shown in Chapter 2, however, the solutions of the scalar Helmholtz equation with m=1 (spheroidal wave functions of the first order in spheroidal coordinates) are valid modes for the rotationally symmetric magnetic field. With this in mind, we let

$$H_\varphi^{(i)} = \sum_{n=1}^{\infty} C_n(\xi)\, S_{1n}(\eta, \gamma) \tag{4.3}$$

where

$$\gamma \triangleq kc \tag{4.3a}$$

and $S_{1n}(\eta, \gamma)$ is the spheroidal angle wave function of first order and nth degree and C_n is independent of η. To evaluate C_n, multiply (4.3) with $S_{1n}(\eta, \gamma)d\eta$ and integrate between -1 and +1. By the use of the orthogonality property of spheroidal angle functions

$$\int_{-1}^{1} S_{mn}(\eta, \gamma)\, S_{mn'}(\eta, \gamma)\, d\eta = N_{mn}(\gamma)\, \delta_{nn'} \tag{4.4}$$

where $N_{mn}(\gamma)$ is* a mornalization constant and $\delta_{nn'}$ is the Kronecker symbol,

*$N_{mn}(\gamma)$ is tabulated for small m, n and γ [35].

we thus have

$$C_n = \frac{1}{N_{1n}(\gamma)} \int_{-1}^{1} H_\varphi^{(i)} S_{1n}(\eta, \gamma)\, d\eta \qquad (4.5)$$

In order to evaluate the integral, we note that

$$P_n^{m}(x) = (-1)^m \frac{(n+m)!}{(n-m)!} P_n^{-m}(x) \qquad (A.38)$$

and by rewriting (4. 5) and substituting (4. 1a) and(A.38), yields

$$C_n = \frac{2\omega \tilde{A} k^2}{N_{1n}(\gamma)} \int_{-1}^{1} \psi_1^{(4)}\left(\gamma\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1}\right) P_1^{-1}\left(\frac{\xi\eta-\alpha}{\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1}}\right) S_{1n}(\eta,\gamma)\, d\eta$$

Using the following identity (see Appendix A), [1]

$$\int_{-1}^{1} \psi_1^{(4)}\left(\gamma\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1}\right) P_1^{-1}\left(\frac{\xi\eta-\alpha}{\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1}}\right) S_{1n}(\eta,\gamma)\, d\eta \qquad (A.40)$$

$$= -\frac{1}{\gamma}(\alpha^2-1)^{-\frac{1}{2}} R_{1n}^{(4)}(\xi,\gamma) R_{1n}^{(1)}(\alpha,\gamma) \qquad \text{for } \alpha < \xi$$

where $R_{1n}^{(1)}$ and $R_{1n}^{(4)}$ are the radial prolate spheroidal wave functions of the first order, nth degree, first and fourth kind, we have

$$C_n = -\frac{2\omega k \tilde{A}}{c(\alpha^2-1)^{1/2} N_{1n}(\gamma)} \begin{cases} R_{1n}^{(4)}(\xi,\gamma) R_{1n}^{(1)}(\alpha,\gamma) & \xi > \alpha \\ R_{1n}^{(4)}(\alpha,\gamma) R_{1n}^{(1)}(\xi,\gamma) & \xi < \alpha \end{cases} \qquad (4.6)$$

[1]Here the normalization condition of Stratton [41] is used;

i. e. $\sum_{\ell=0,1}^{\infty}{}' d_\ell^{mn} \frac{(\ell+2m)!}{\ell!} = \frac{(n+m)!}{(n-m)!}$

This step then completes the determination of the incident dipole field as a sum of prolate spheroidal waves. The result is

$$H_\varphi^{(i)} = -\frac{2\omega k \tilde{A}}{c(\alpha^2-1)^{1/2}} \sum_{n=1}^{\infty} \frac{1}{N_{1n}(\gamma)} S_{1n}(\eta,\gamma) \begin{cases} R_{1n}^{(1)}(\xi,\gamma) R_{1n}^{(4)}(\alpha,\gamma) & \xi < \alpha \\ R_{1n}^{(1)}(\alpha,\gamma) R_{1n}^{(4)}(\xi,\gamma) & \xi > \alpha \end{cases} \tag{4.7}$$

To evaluate the scattered field, we assume it to be in the same form as the incident field but with outgoing spheroidal waves, and unknown constants - a_n, the scattering coefficients. Let $H_\varphi^{(s)}$ be the scattered magnetic field given as

$$H_\varphi^{(s)} = \frac{2\omega k A}{c(\alpha^2-1)^{1/2}} \sum_{n=1}^{\infty} \frac{a_n}{N_{1n}(\gamma)} S_{1n}(\eta,\gamma) R_{1n}^{(4)}(\xi,\gamma) R_{1n}^{(4)}(\alpha,\gamma) \tag{4.8}$$

Since the tangential electric field must vanish on an ideal conductor, from Eq. (2.11),

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)^{1/2} H_\varphi^{(t)}\right]\Bigg|_{\xi=\xi_0} = 0 \tag{4.9}$$

where (t) refers to the total magnetic field. By substituting (4.7) and (4.8) into (4.9), we have for the scattering coefficients

$$a_n = \frac{\frac{d}{d\xi}\left[(\xi^2-1)^{1/2} R_{1n}^{(1)}(\xi,\gamma)\right]}{\frac{d}{d\xi}\left[(\xi^2-1)^{1/2} R_{1n}^{(4)}(\xi,\gamma)\right]}\Bigg|_{\xi=\xi_0} \tag{4.10}$$

The total isotropic magnetic field is therefore now determined. Next we obtain the solution to our anisotropic problem.

4.13 Solution in Anisotropic Media

As shown in Chapter 3, the anisotropic solution is obtained by scaling the vacuum solution. Since the scaling consists of contracting the distances along the z axis by the factor $\sqrt{K}$, the spheroidal boundary surface defined by ξ_0 and interfocal distance 2c, transforms to another spheroidal surface defined by λ_0 with an interfocal distance 2ℓ.

To find the transformation from the original confocal spheroids to the scaled confocal spheroids, note that the original boundary satisfies the equation

$$\frac{z^2}{c^2\xi_0^2} + \frac{\rho^2}{c^2(\xi_0^2-1)} = 1 \tag{4.11}$$

with major axis

$$a = c\xi_0 \tag{4.11a}$$

and minor axis

$$b = c\left(\xi_0^2-1\right)^{\frac{1}{2}} \tag{4.11b}$$

by the introduction of the scaled axis $z' = \frac{z}{\sqrt{K}}$, the new equation is

$$\frac{z'^2}{\frac{c^2\xi_0^2}{K}} + \frac{\rho^2}{c^2(\xi_0^2-1)} = 1 \tag{4.12}$$

where now the major axis is

$$a' = \frac{c\xi_0}{\sqrt{K}} \tag{4.12a}$$

the minor axis is

$$b = b' = c\left(\xi_0^2-1\right)^{\frac{1}{2}}, \tag{4.12b}$$

and the semi-interfocal distance

$$\ell \triangleq \left(a'^2 - b'^2\right)^{\frac{1}{2}} = \frac{c}{\sqrt{K}}\left[\xi_0^2(1-K) + K\right]^{\frac{1}{2}} . \qquad (4.13)$$

The transformation between the original spheroidal coordinates (ξ, η) and the scaled coordinates (λ, ν) is obtained by first noting that

$$\begin{aligned} \rho &\triangleq \ell\left[(1-\nu^2)(\lambda^2-1)\right]^{\frac{1}{2}} \triangleq c\left[(1-\eta^2)(\xi^2-1)\right]^{\frac{1}{2}} \\ z' &\triangleq \ell\nu\lambda \triangleq \frac{c}{\sqrt{K}}\xi\eta . \end{aligned} \qquad (4.14)$$

By considering either (ξ, η) or (λ, ν) as fixed, equations for the remaining pair (λ, ν) or (ξ, η), respectively, are found. The roots of these equations are the transformation. For example, eliminating λ or ν from (4.14) yields

$$\left\{\begin{matrix}\lambda^4\\ \nu^4\end{matrix}\right\} - \left\{1 + \frac{z'^2 \mp \rho^2}{\ell^2}\right\}\left\{\begin{matrix}\lambda^2\\ \nu^2\end{matrix}\right\} + \frac{z'^2}{\ell^2} = 0 \qquad (4.15)$$

the roots of which are the transformation relation from (ξ, η) to (λ, ν).

The solution of the anisotropic problem is obtained by substituting the new variables (λ, ν) for (ξ, η) in (4.7), (4.8) and (4.10).

4.2 Scattering by a Conducting Paraboloid of Revolution

As a second example, to the scaling technique presented in Chapter 3, the total field (incident plus reflected) established by an axially aligned electric dipole and scattered by a conducting paraboloid of revolution is presented. The medium is taken to be uniaxially anisotropic whose optic axis is aligned with symmetry axis. The technique of solution is based on scaling of Pinney's [35] isotropic solution. For this purpose first, the paraboloidal coordinates and wave functions are introduced and then the isotropic solution to the scattering problem is given. This solution is then scaled in accordance to arguments presented in Chapter 3.

4.21 Introduction to Laguerre Wave Functions

Let the transformation from cartesian to paraboloid of revolution coordinate system be defined as

$$\begin{aligned} x &= 2\sqrt{uv}\,\cos\varphi \\ y &= 2\sqrt{uv}\,\sin\varphi \\ z &= -u+v \end{aligned} \tag{4.16}$$

(here the definitions of Pinney [36] are used).

Eliminating u or v from (4.16) we have

$$\rho^2 = 4uv = 4u^2 + 4uz = 4v^2 - 4vz \tag{4.17}$$

and therefore for $u \geq 0$ and $v \geq 0$

the u = constant and the v = constant coordinates correspond to paraboloids of revolution about the z axis, open in the positive and negative directions. These family of paraboloids are confocal, with the origin as focus, and have focal distances u and v. In this system of coordinates, the scalar Helmholtz equation is

$$\frac{\partial}{\partial u}\left(u\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(v\frac{\partial\phi}{\partial v}\right) + \frac{u+v}{4uv}\frac{\partial^2\phi}{\partial\varphi^2} + k^2(u+v)\phi = 0 \tag{4.18}$$

Using the usual separation of variables technique, the solution to (4.18) is of the form

$$\phi(u,v,\varphi) = \phi_\nu^m(\pm u)\,\phi_\nu^m(\mp v)\,e^{-jm\varphi} \tag{4.19}$$

The separated equation for $\phi_\nu^m(x)$ is

$$\frac{d}{dx}\left(x\frac{d}{dx}\phi_\nu^m\right) + \left(k^2x + h - \frac{m^2}{4x}\right)\phi_\nu^m(x) = 0 \tag{4.20}$$

If we let,

$$\phi_\nu^m(x) = x^{\frac{m}{2}} e^{-jkx} y_\nu^m(2jkx)$$

with

$$h \triangleq jk(m + 2\nu + 1) \tag{4.21}$$

Then y_ν^m satisfy a confluent hypergeometric equation

$$\left[x\frac{d^2}{dx^2} + (m+1-x)\frac{d}{dx} + \nu\right] y_\nu^m(x) = 0 \tag{4.22}$$

whose solutions are the Laguerre functions. These functions are defined for arbitrary ν as

$$L_\nu^m(x) \triangleq -\frac{\sin\pi\nu}{\nu}\Gamma(1+m+\nu)\sum_{p=0}^{\infty}\frac{\Gamma(p-\nu)}{\Gamma(p+m+1)}\frac{x^p}{p!} \tag{4.23}$$

where Γ denotes the gamma function.

If $\nu = n =$ integer then the Laguerre functions reduce to their corresponding polynomials. The properties of these solutions are known and are available [36, 37] in the literature. The functions of interest, in this sequal, are the so-called Laguerre wave functions, defined as

$$S_\nu^m(x) \triangleq x^{\frac{1}{2}m} e^{-\frac{1}{2}x} L_\nu^m(x) \tag{4.24}$$

$$V_\nu^m(x) \triangleq x^{\frac{1}{2}m} e^{\frac{1}{2}m} U_\nu^m(x) \tag{4.25}$$

where U_ν^m (x) are the second solutions of (4.20). [N. B. to conform with Pinney's notation, the symbol for the wave functions of the first kind are denoted by S_ν^m. This symbol should not be confused with S_{mn}, the angle prolate spheroidal wave-functions used in the previous problem.] The following property of the Laguerre wave functions are noted [36];

$S_\nu^m(-2jku)\,V_\nu^m(2jkv)$ represents a wave traveling in outward direction

$V_\nu^m(2jku)\,S_\nu^m(-2jkv)$ represents a wave traveling in outward direction

$S_\nu^m(2jku)\,V_\nu^m(-2jkv)$ represents a wave traveling in inward direction

$V_\nu^m(-2jku)\,S_\nu^m(2jkv)$ represents a wave traveling in inward direction

$S_\nu^m(2jku)\,S_\nu^m(-2jkv)$ represents a wave traveling in +z direction

$V_\nu^m(-2jku)\,V_\nu^m(2jkv)$ represents a wave traveling in +z direction

$S_\nu^m(-2jku)\,S_\nu^m(2jkv)$ represents a wave traveling in -z direction

$V_\nu^m(2jku)\,V_\nu^m(-2jkv)$ represents a wave traveling in -z direction

With this short introduction to Laguerre wave functions, we are ready to formulate and present the solution to the isotropic scattering problem.

4.22 Statement and Solution of the Problem

In the paraboloidal coordinate system, considered in the previous paragraph, let an electric dipole be at the origin with its moment parallel to the z axis. Let the conductive paraboloid of revolution to be at the coordinate $u = u_o$. The problem is to find the total field. Since the electromagnetic field is TM as well as rotationally symmetric, the only magnetic field component is along the φ direction and from this component all other field components may be generated. The result for the total magnetic field is [38]

$$H_\varphi = -4\pi j k^2 \omega \epsilon \sum_{n=1}^{\infty} \frac{[V_n'(2jku) - B_{n+1} S_n'(2jku)] S_n'(-2jkv)}{n+1} \tag{4.26}$$

where the scattering coefficient B_n is obtained by requiring that at $u = u_o$ the total tangential electric field shall vanish. The result for B_n is

$$B_n = n \frac{V_n(2jku_o) + V_{n-1}(2jku_o)}{S_n(2jku_o) + S_{n-1}(2jku_o)} \tag{4.27}$$

Having presented the isotropic scattering solution, we turn to the anisotropic problem. By scaling the distances along z direction by the factor $1/\sqrt{K}$, the anisotropic Neumann problem is reduced to an equivalent vacuum problem. In a manner similar to procedure used in the previous problem, we find that the transformations

$$uv = u'v' , \qquad \frac{-u+v}{\sqrt{K}} = -u' + v' \tag{4.28}$$

are the desired connecting relations between the scaled and the unscaled coordinate system.

Chapter 5.

Rayleigh Scattering by a Conductive Spheroid

The form of the solution, Eqs. (4.7), (4.8) and (4.10), of the spheroidal scattering problem, although valid for all frequencies, is only useful for small γ, since in this case both the spheroidal wave functions as well as the summations in terms of these functions rapidly converge. [N. B. For small $\gamma = kc = 2\pi \frac{c}{\lambda} \ll 1$, where c is the semi-interfocal distance of the spheroid scatterer is referred to as a Rayleigh Scatterer.] Furthermore, in the case of an anisotropy along the symmetry axis, the scaled confocal spheroidal wave functions may be expanded into an infinite series of the "unscaled" spheroidal wave functions, by the use of an addition theorem for spheroidal wave functions [34], and thereby adding additional difficulty in visualizing the anisotropic solution. However, for low frequency or large anisotropy, the scatterer becomes a Rayleigh-type of scatterer and thus it is feasible to expand the scattering solution into a Taylor series in powers of γ. This expansion, in addition to being a generalization of a corresponding calculation for a conducting sphere in the vacuum case [39], is also a check on frequency perturbational calculations [40]. For example, the calculations of the scattered field due to an arbitrarily incident electromagnetic wave on a conducting spheroid [41], where an 'exact' solution is not available.

5.1 Formulation of the Problem

In this chapter we evaluate the far-field limit of scattered magnetic field, referred to as the scattering amplitude. It is defined as

$$g(\theta, A, B, a, b, k) \triangleq r e^{jkr} \lim_{c\xi \to \infty} H_\varphi^{(s)}(\xi, \gamma) \tag{5.1}$$

where A is the major and B is the minor semi-axis of the spheroid describing the location of the electric dipole and θ is the angle measured from the z axis (see Figure 2). The final result contains no reference to special functions and are given in terms of physical parameters; A, B, a, b and κ. The final results are tabulated in Tables 2 and 3.

5.2 Evaluation of the Scattering Amplitude

To evaluate the scattering amplitude, the following procedure is followed. First, the scattered magnetic field, Eq. (4.8), is reduced to its far-field component, and by the use of (5.1), the scattering amplitude is found. Second, based on the long wavelength formulae of the spheroidal wave functions and their derivatives, developed by Burke [42] for the scalar (acoustic) scattering problem, an expansion of the first two terms of an infinite series expansion of the scattering amplitude, in a series of integral power of k, is presented.

Following the indicated procedure, from (4.8), the scattered field is

$$H_\varphi^{(s)} = \frac{2\omega k A}{B} \sum_{n=1}^{\infty} \frac{a_n(\gamma)}{N_{1n}(\gamma)} S_{1n}(\eta,\gamma) R_{1n}^{(4)}(\alpha,\gamma) R_{1n}^{(4)}(\xi,\gamma) \tag{4.8}$$

Using the asymptotic property of the prolate spheroidal radial wave functions (see Appendix A)

$$R_{1n}^{(4)}(\xi,\gamma) \xrightarrow{\xi\to\infty} \frac{e^{-j\gamma\xi}}{\gamma\xi}\, e^{j(n+1)\frac{\pi}{2}} \tag{A.15}$$

and noting that as $c\xi \to \infty$, $c\xi$ approaches r, the radial distance, and η approaches cos θ, (see A.2) we have, by using (5.1), for the scattering amplitude

$$g(\theta,A,B,a,b,k) = \sum_{n=1}^{\infty} \frac{a_n(\gamma)}{N_{1n}(\gamma)} \frac{2\omega A}{B} e^{j(n+1)\frac{\pi}{2}} S_{1n}(\cos\theta,\gamma) R_{1n}^{(4)}(\alpha,\gamma)$$

, (5.2)

For future use, we define

$$g(\theta, A, B, a, b, k) \triangleq \sum_{n=1}^{\infty} g_n(\theta, A, B, a, b, k) \tag{5.3}$$

where the expression for g_n is found by comparing (5.3) and (5.2). In the rest of this chapter we evaluate g_1 and g_2 for isotropic and anisotropic medium in terms of the physical parameters of the problem.

To evaluate g_1 and g_2, including terms of order γ^2 we note that, from Appendix A, taking into account up to the second power of γ

$$\begin{aligned} S_{11}(\cos\theta, \gamma) &= \sin\theta \left(1 + \frac{\gamma^2}{10}\sin^2\theta\right) \\ S_{12}(\cos\theta, \gamma) &= \frac{3}{2}\sin 2\theta \left(1 + \frac{\gamma^2}{14}\sin^2\theta\right). \end{aligned} \tag{5.4}$$

We also note [41] that the normalization constants (see 4.4) are

$$\begin{aligned} N_{11}(\gamma) &= \frac{4}{3}\left(1 + \frac{4}{25}\gamma^2\right) \\ N_{12}(\gamma) &= \frac{12}{5}\left(1 + \frac{4}{49}\gamma^2\right) \end{aligned} \tag{5.5}$$

In order to evaluate in $R_{1n}^{(4)}(\alpha, \gamma)$, we note that in terms of the spheroidal coordinates of the dipole $(\alpha, 1)$, the major and minor axis of its associated spheroid are

$$\begin{aligned} A &= c\,\alpha \\ B &= c\,(\alpha^2 - 1)^{\frac{1}{2}} \end{aligned} \tag{5.6}$$

Referring to the formulae of (A31) and (B.1a) from Appendix A and B, we define

$$\alpha_{11} = Re\left[R_{11}^{(4)}(\alpha, \gamma)\right] = R_{11}^{(1)}(\alpha, \gamma)$$

Using the relationships of (B. 1a) and substituting from (5. 6) into (A. 31), we have

$$\alpha_{11} = \frac{kB}{3}\left[1 + \frac{k^2}{50}\left(4a^2 - 4b^2 - 5A^2\right)\right] \tag{5.7a}$$

Similarly,

$$\begin{aligned}\alpha_{12} &= Re\left[R_{12}^{(4)}(\alpha, \gamma)\right] = R_{12}^{(1)}(\alpha, \gamma) \\ &= \frac{k^2 AB}{15}\left[1 + \frac{k^2}{98}\left(4a^2 - 4b^2 - 7A^2\right)\right]\end{aligned} \tag{5.7b}$$

$$\begin{aligned}\beta_{11} &= I_m\left[R_{11}^{(4)}(\alpha, \gamma)\right] = R_{11}^{(2)}(\alpha, \gamma) \\ &= \frac{3(AT - 2B^2)}{2(a^2-b^2)BTk^2}\left[1 - \right. \\ &\left. - \left(\frac{k}{20}\right)^2 \frac{8AT(5B^2 - a^2 + b^2) - 8B^2(33a^2 - 33b^2 + 5A^2 + 5B^2)}{AT - 2B^2}\right]\end{aligned} \tag{5.7c}$$

$$\begin{aligned}\beta_{12} &= I_m\left[R_{12}^{(4)}(\alpha, \gamma)\right] = R_{12}^{(2)}(\alpha, \gamma) \\ &= \frac{15}{2}\frac{T(3B^2 + a^2 - b^2) - 6AB^2}{(a^2-b^2)^2 BTk^3}\left[1 - \right. \\ &\left. - \left(\frac{k}{14}\right)^2 \frac{21A^2B^2T + T(a^2-b^2)(5B^2 - 3a^2 + 3b^2) - AB^2(17a^2 - 17b^2 + 21A^2 + 21B^2}{T(3B^2 - a^2 - b^2) - 6AB^2}\right]\end{aligned} \tag{5.7d}$$

where

$$T \triangleq \frac{4(a^2-b^2)^{\frac{1}{2}}}{\ln \dfrac{A - (A^2-B^2)^{\frac{1}{2}}}{A + (A^2-B^2)^{\frac{1}{2}}}} . \tag{5.8}$$

Let the real and the imaginary parts of the scattering coefficients be defined as

$$\begin{aligned} x_n(a, b, k) &\triangleq Re\, a_n \\ y_n(a, b, k) &\triangleq I_m\, a_n \end{aligned} \tag{5.9}$$

Substituting (5.4), (5.5), (5.6), and (5.7) into (5.2), the first two terms of the infinite series in the scattering amplitude is

$$g_1(\theta, A, B, a, b, k) = -\frac{3}{2}\omega \frac{\sin\theta}{B}(x_1 + jy_1)(\alpha_{11} + j\beta_{11}) \frac{1 + \frac{k^2c^2}{10}\sin^2\theta}{1 + \frac{4}{45}c^2k^2} \tag{5.10}$$

$$g_2(\theta, A, B, a, b, k) = -j\frac{5}{4}\omega \frac{\sin 2\theta}{B}(x_2 + jy_2)(\alpha_{12} + j\beta_{12}) \frac{1 + \frac{k^2c^2}{14}\sin^2\theta}{1 + \frac{4}{49}c^2k^2}$$

Performing the indicated operations in (5.10) we have, for the real and imaginary parts of g_1 and g_2, the equations

$$Re\, g_1 = -\frac{3}{2}\omega \frac{\sin\theta}{B}\left[1 + \frac{c^2k^2}{10}\left(\sin^2\theta - \frac{8}{9}\right)\right]\left[x_1\alpha_{11} - y_1\beta_{11}\right]$$

$$I_m\, g_1 = -\frac{3}{2}\omega \frac{\sin\theta}{B}\left[1 + \frac{c^2k^2}{10}\left(\sin^2\theta - \frac{8}{9}\right)\right]\left[y_1\alpha_{11} + x_1\beta_{11}\right] \tag{5.11}$$

$$Re\, g_2 = -\frac{5}{4}\omega \frac{\sin 2\theta}{B}\left[1 + \frac{c^2k^2}{14}\left(\sin^2\theta - \frac{8}{7}\right)\right]\left[y_2\alpha_{12} + x_2\beta_{12}\right]$$

$$I_m\, g_2 = -\frac{5}{4}\omega \frac{\sin 2\theta}{B}\left[1 + \frac{c^2k^2}{14}\left(\sin^2\theta - \frac{8}{7}\right)\right]\left[x_2\alpha_{12} - y_2\beta_{12}\right]$$

To evaluate (5.11), we need the expressions for the scattering coefficients (5.9). These coefficients are evaluated in Appendix B and the results are tabulated in Table 2. The results for these coefficients are valid for shapes ranging from conductive needles through disks. In particular, if the eccentricity of the spheroid tends to zero, they reduce to the known scattering coefficients of a conducting sphere [37]. Furthermore, for spheroids with small values of eccentricity, the coefficients may be expanded into a series of integral powers of the eccentricity. These expansions, for nearly spherical prolate as well as oblate spheroids, are also evaluated in Appendix B. If the oblate spheroid is almost a disk, then the coefficients may be expanded in a power series of a parameter ϵ, which is zero if the oblate spheroid is a disk.

	$x_1 = \mathrm{Re}\, a_1$	$y_1 = \mathrm{Im}\, a_1$
Spheroid	$-\frac{4}{81}\frac{a^2\tau^2c^4k^6}{(2a-\tau)^2}\left(1-\frac{k^2c^2}{25}\frac{36a-13\tau}{\tau-2a}\right)$	$-\frac{2}{9}\frac{a\tau c^2k^3}{2a-\tau}\left(1-\frac{k^2c^2}{50}\frac{36a-13\tau}{\tau-2a}\right)$
Nearly spherical prolate spheroid	$-\frac{4}{9}a^6k^6\left(1-\frac{4}{5}e^2\right)$	$-\frac{2}{3}a^3k^3\left(1-\frac{2}{5}e^2\right)\left\{1+\frac{3}{10}k^2a^2\left(1+\frac{4}{5}e^2\right)\right\}$
Nearly spherical oblate spheroid	$-\frac{4}{9}b^6k^6\left(1-\frac{9}{5}e^2\right)$	$-\frac{2}{3}b^3k^3\left(1-\frac{9}{10}e^2\right)\left\{1+\frac{3}{10}k^2b^2\left(1-\frac{8}{5}e^2\right)\right\}$
Disk type scatterer	$O\left(\epsilon^2\right)$	$-\frac{2}{9}\epsilon b^3k^3\left\{1-\frac{13}{50}k^2b^2\left(1-\frac{5}{26}\epsilon\pi\right)\right\}$

TABLE 2 - ELECTROMAGNETIC SCATTERING COEFFICIENTS FOR CONDUCTIVE SPHEROIDS IN CASE OF LOW FREQUENCY

	$y_2 = -\,\mathrm{Im}\, a_2$
Spheroid	$\dfrac{-2\tau b^2 c^4 (2a^2+b^2) k^5}{15^2(-9ab^2\tau + 12a^2b^2 + 6b^4)} \left\{ 1 - \dfrac{1}{3}\left(\dfrac{k}{14}\right)^2 \dfrac{a\tau(-6a^6+133a^4b^2-188a^2b^4-2b^6) - 84a^6b^2+46a^4b^4+124a^2b^6+40b^8}{(2a^2+b^2)(-3ab^2\tau + 4a^2b^2 + 2b^4)} \right\}$
Nearly spherical prolate	$-\dfrac{a^5k^5}{30}\left(1 - \dfrac{25}{21}e^2\right)\left\{1 - 24\left(\dfrac{ka}{14}\right)^2\left(1 - \dfrac{2515}{3024}e^2\right)\right\}$
Nearly spherical oblate	$-\dfrac{b^5k^5}{30}\left(1 - \dfrac{407}{224}e^2\right)\left\{1 - 24\left(\dfrac{kb}{14}\right)^2\left(1 - \dfrac{1063}{3024}e^2\right)\right.$
Disk type	$-\dfrac{4b^5k^5}{15^2 \cdot 3\pi}\left(1 + \dfrac{8}{\pi}\epsilon\right)\left\{1 - \dfrac{5}{3}\left(\dfrac{kb}{7}\right)^2\left(1 + \dfrac{29}{5\pi}\epsilon\right)\right\}$

TABLE 2 (continued)

Such an expansion is given in Appendix B. Based on the results of Appendix B, the real and imaginary parts of g_1 and g_2 are then calculated in Appendix C. The results of these calculations are collected in Table 3. From this table we note that the results for the isotropic scattering amplitudes do not contain any reference to special functions and are given directly in terms of the physical parameters of the problem. This step completes the long wavelength far-field solution of the isotropic problem.

Based on arguments presented in Chapter 3, to obtain the anisotropic low frequency scattering amplitude, a scaling along the z axis is performed. If we let, in Table 3,

$$A \longrightarrow \frac{A}{\sqrt{K}} \quad , \qquad a \longrightarrow \frac{a}{\sqrt{K}} \tag{5.12}$$

and therefore,

$$T \longrightarrow \frac{4(a^2 - K b^2)^{\frac{1}{2}}}{\sqrt{K}\, \ln \dfrac{A-(A^2-K B^2)^{\frac{1}{2}}}{A+(A^2-K B^2)^{\frac{1}{2}}}} \tag{5.13}$$

$$\Upsilon \longrightarrow \frac{4(a^2 - K b^2)^{\frac{1}{2}}}{\sqrt{K}\, \ln \dfrac{a-(a^2-K b^2)^{\frac{1}{2}}}{a+(a^2-K b^2)^{\frac{1}{2}}}}$$

the resulting amplitudes are the required anisotropic scattering amplitudes. Thus, as indicated above, the effect of the anisotropy may be observed directly by substituting Eqs. (5.12) and (5.13) for the scattering amplitude of Table 3. We here note, that if $K \longrightarrow 0$ the scatterer is of an equivalent disk type whereas if $K \longrightarrow \infty$ the scatterer is then a needle type of scatterer.

$$\operatorname{Re} g_1 = -\frac{1}{2}\,\frac{a\tau}{B^2 T}\,\frac{AT - 2B^2}{2a - \tau}\,k \sin\Theta + o(k^3)$$

$$\operatorname{Im} g_1 = \frac{1}{9}\,\frac{a\tau}{B^2 T}\,\frac{2aB^2(T-\tau) + \tau T(aA - B^2)}{(2a^2 - \tau)^2}\,c^2 k^4 \sin\Theta + o(k^6)$$

$$\operatorname{Re} g_2 = -\frac{3}{4}\,\frac{\tau b^2}{TB^2}\,\frac{(2a^2+b^2)\left[T(3B^2-c^2) - 6A^2B^2\right]}{-9ab^2\tau + 12a^2b^2 + 6b^4}\,c^2 k^2 \sin 2\Theta$$

$$\operatorname{Im} g_2 = \frac{1}{1350}\,\frac{Ab^2(2a^2+b^2)}{-9ab^2\tau + 12a^2b^2 + 6b^4}\,c^4 k^7 \sin 2\Theta$$

where

$$\tau = \frac{4c}{\ln\dfrac{a-c}{a+c}} \qquad T = \frac{4c}{\ln\dfrac{A-(A^2-B^2)^{\frac{1}{2}}}{A+(A^2-B^2)^{\frac{1}{2}}}}$$

TABLE 3. THE LONG WAVELENGTH SCATTERING AMPLITUDES FOR A CONDUCTIVE SPHEROID

Chapter 6

High Frequency Scattering by a Conductive Spheroid

In this chapter the high frequency electric current induced by an axially incident plane wave on the surface of a conducting prolate spheroid is calculated. The method of solution is based on the theory of complex resolvents and on the use of uniformly valid asymptotic approximations.

The solution to the scattering problem, Eqs. (4.7), (4.8), and (4.10), given in the form of an infinite sum of spheroidal waves, converges slowly for high frequencies. Although with the aid of computers numerical solutions may be obtained, in this chapter the aim is to find alternative ways of treating the problem which leads to a better understanding of the high frequency propagation and scattering phenomena.

A way of looking at the high frequency problem is to consider the scattering as a perturbation on a geometrical optics (short wavelength) solution. Such an approach has been applied to some extent, to the problem of the scattering of electromagnetic waves by a conductive prolate spheroid [43]. An alternative approach is to convert the infinite summation into a contour integration. (Such transformation was first given by Watson [44] and is called the Watson transformation.) By deforming the contour of the integral, in an appropriate manner, and summing its residues, the slowly converging series is converted into a rapidly converging series. Each residue term of the contour integral, called a mode, is interpreted in terms of two waves, the so-called 'creeping' waves. These waves propagate in opposite direction to each other, and dissipate energy along the surface of the scatterer ('leaky' surface waves). Such calculations, for the field of an electric dipole scattered by a conducting sphere, were carried out by Watson, and extended [45] by numerous researchers, and are useful, for example, in tropospheric radio wave propagation [46].

Recently a new technique has been introduced replacing the Watson transformation. This method, called the theory of complex resolvents, has been applied to the scalar scattering calculation from a prolate spheroid [47] . In the next section this technique is applied to the calculation of the high frequency electric current due to an axially incident electromagnetic wave impinging on a conducting prolate spheroid. The results and their physical interpretation may be found in section 5.14.

6.1 Formulation of the High Frequency Problem.

We now reformulate the problem, of finding the total field due to an electric dipole above and aligned with the axis of symmetry of a conductive prolate spheroid to be in a form applicable to the use of the theory of complex resolvents (Appendix D). Based on this theory, in the next section, a contour integral representation of the solution to the boundary value problem is presented. By deforming the contour of this integral and summing its new residues, the high frequency electric current on the surface of the conducting prolate spheroid is calculated.

We recall from Chapter 2, that the φ component of the magnetic field satisfies a scalar Helmholtz equation with m=1, which in spheroidal coordinate system is

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)\frac{\partial H_\varphi}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial H_\varphi}{\partial \eta}\right]$$
$$+\left[\gamma^2(\xi^2-1) + \gamma^2(1-\eta^2) - \frac{1}{\xi^2-1} - \frac{1}{1-\eta^2}\right]H_\varphi = 0 \tag{6.1}$$

with γ =kc where k is the wave number and c is the semi-interfocal distance of the spheroid. In order to be able to apply the complex resolvent theory, we rewrite (6.1) in the following form

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)\frac{\partial H_\varphi}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial H_\varphi}{\partial \eta}\right]+\left[\gamma_+^2(\xi^2-1)+\gamma_+^2(1-\eta^2)-\frac{1}{\xi^2-1}-\frac{1}{1-\eta^2}\right]H_\varphi(\xi,\alpha,\eta,1)$$

$$= \; -J(\xi,\eta)\,\delta(\eta-1)\,\delta(\xi-\alpha) \qquad (6.2)$$

where (using Kazarinoff and Goodrich's notation [48]), $\gamma_+ = \gamma - jsc$ with $s > 0$, and $J(\xi,\eta)\ \delta(\eta-1)\ \delta(\xi-\alpha)$ is the dipole source $\delta(x)\ \delta(y)\ \delta(z-z_0)$ transformed into prolate spheroidal coordinates with $J(\xi,\eta)$ as the Jacobian of the transformation. The solution of (6.2), subject to the boundary condition

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)^{\frac{1}{2}}\, H_\varphi(\xi,\eta)\right]\Bigg|_{\xi=\xi_0} = 0 \qquad (6.3)$$

in the limit as $s \to 0$, is the solution to the boundary value problem. The reason for the introduction of the complex γ_+ is that unless $s > 0$, the contour integral representation of the solution of (6.2) and (6.3) is not valid.

6.2 An Integral Representation for the Green's Function.

It is observed that (6.2) is in the form

$$(-L_\xi - L_\eta)H_\varphi(\xi,\alpha,\eta,1) = J(\xi,\eta)\,\delta(\eta-1)\,\delta(\xi-\alpha)$$

where

$$\begin{aligned} L_\xi &= -\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}\right]+\left[-\gamma_+^2(\xi^2-1)+\frac{1}{\xi^2-1}\right] \\ L_\eta &= -\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}\right]+\left[-\gamma_+^2(1-\eta^2)+\frac{1}{1-\eta^2}\right] \end{aligned} \qquad (6.4)$$

These type of operators are discussed in Appendix D, and satisfy the following conditions

$$\operatorname{Im}\left[-\gamma_+^2(\xi^2-1)+\frac{1}{\xi^2-1}\right] \geq 2ksc^2(\xi_0^2-1), \quad \xi_0 \leq \xi < \infty$$

$$\operatorname{Im}\left[-\gamma_+^2(1-\eta^2)+\frac{1}{1-\eta^2}\right] \geq 0 \qquad , \; -1 \leq \eta \leq 1$$

subject to the boundary condition

$$\frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}} H_\varphi(\xi,\eta)\right]\Bigg|_{\xi=\xi_0} = 0$$

The radial operator L_ξ is defined on an interval (ξ_0, ∞) with $p(\xi) = \xi^2 - 1$, and $p(\xi_0) \neq 1$. The homogeneous equation $(L_\xi - \lambda)y = 0$ where $\operatorname{Im} \lambda < 2ksc^2(\xi_0{}^2-1)$ has two linearly independent solutions $y_1(\xi, \lambda)$ and $y_2(\xi, \lambda)$ which are asymptotic as $\xi \to \infty$ to $(\xi^2-1)^{-\frac{1}{2}} e^{j\gamma\xi}$ and $(\xi^2-1)^{-\frac{1}{2}} e^{-j\gamma\xi}$, respectively. Since $\operatorname{Re}(j\gamma_+) > 0$, only the second of these solutions is square integrable, and L_ξ falls into Case I of the theorem of Appendix D. In order to determine the resolvent Green's functions, we must single out a solution that satisfies the boundary condition at ξ_0 and a solution which satisfies the radiation condition. We define these solutions as φ_1, and φ_2 respectively:

$$\begin{aligned}\varphi_1(\xi, \xi_0, \lambda) &= \frac{d}{d\xi_0}\left[(\xi_0^2-1)^{\frac{1}{2}} y_2(\xi_0)\right] y_1(\xi,\lambda) \\ &\quad - \frac{d}{d\xi_0}\left[(\xi_0^2-1)^{\frac{1}{2}} y_1(\xi_0)\right] y_2(\xi,\lambda)\end{aligned} \qquad (6.5)$$

$$\varphi_2(\xi,\lambda) = y_2(\xi,\lambda).$$

The resolvent Green's function, from the theorem of Appendix D, is therefore

$$G(\xi,\xi',\lambda) = \frac{1}{(\xi^2-1)\,W[\varphi_1(\xi',\lambda),\varphi_2(\xi',\lambda)]}\begin{cases}\varphi_1(\xi,\lambda)\,\varphi_2(\xi',\lambda) & \xi < \xi' \\ \varphi_1(\xi',\lambda)\,\varphi_2(\xi,\lambda) & \xi > \xi'\end{cases}$$

Evaluating the Wronskian with the aid of the asymptotic forms of the solutions y_1 and y_2, we obtain

$$G(\xi,\xi',\lambda) = \frac{1}{2j\gamma_+ \frac{d}{d\xi_0}\left[(\xi^2-1)^{\frac{1}{8}}\, y_2(\xi_0,\lambda)\right]}\begin{cases}\varphi_1(\xi,\lambda)\,\varphi_2(\xi',\lambda) & \xi < \xi' \\ \varphi_1(\xi',\lambda)\,\varphi_2(\xi,\lambda) & \xi > \xi'\end{cases} \tag{6.6}$$

For later use, we note that

$$\varphi_1(\xi_0,\lambda) = \sqrt{\xi_0^2-1}\; W\left[y_1(\xi_0,\lambda),\, y_2(\xi_0,\lambda)\right]$$

$$= -\frac{2j\gamma_+}{(\xi_0^2-1)^{\frac{1}{2}}} \tag{6.6a}$$

where the Wronskian has again been evaluated with the aid of asymptotic forms.

The operator L_η is to be considered on an interval (-1, 1) and it falls under the Case II of theorem of Appendix D. Let $\psi_2(\eta, -\lambda)$ be the solution of the homogeneous equation $(L_\eta - \lambda)y = 0$ which is regular at $\eta = 1$ and a second solution be defined as $\psi_1(\eta, -\lambda) = \psi_2(-\eta, -\lambda)$. With these definitions, the resolvent Green's function $\tilde{G}(\eta, \eta', -\lambda)$

is

$$\tilde{G}(\eta, \eta'; -\lambda) = \frac{1}{(1-\eta^2)\, W(\psi_1, \psi_2)} \begin{cases} \psi_1(\eta, -\lambda)\, \psi_2(\eta', -\lambda) & \eta < \eta' \\ \psi_1(\eta', -\lambda)\, \psi_2(\eta, -\lambda) & \eta > \eta' \end{cases}$$

(6.7)

We now are ready to derive the contour integral representation of $H_\varphi(\xi, \eta, \alpha, 1)$. Let R_λ and $\tilde{R}_{-\lambda}$ be the respective complex resolvents of L_ξ and L_η, and let Γ be a path in the complex λ-plane defined

$$\lambda = \delta + j\ell \quad \text{with real } \delta \text{ and } \ell$$

where

$$-\infty < \delta < \infty \quad \text{and} \quad 0 < \ell < 2\, kpc^2\,(\xi_0^2 - 1)$$

Rewrite (5.2) as

$$\left[(-L_\xi + \lambda) + (-L_\eta - \lambda)\right] H_\varphi(\xi, \eta, \alpha, 1)$$

$$= J(\xi, \eta)\, \delta(\xi - \alpha)\, \delta(\eta - 1) \qquad (6.8)$$

Applying R_λ and $\tilde{R}_{-\lambda}$ successively to (6.8), yields

$$H_\varphi(\xi, \eta, \alpha, 1) = \lim_{s \to 0^+} \frac{1}{2\pi j} \oint_\Gamma \tilde{R}_{-\lambda} R_\lambda \left[J(\xi', \eta')\, \delta(\xi' - \alpha)\, \delta(\eta' - 1)\right] d\lambda$$

(6.9)

Using (D.15) for the resolvent operators, we have

$$H_\varphi(\xi, \eta, \alpha, 1) = \lim_{s \to 0^+} \frac{1}{2\pi j} \oint_\Gamma d\lambda \int_{\xi_0}^{\infty} \int_{-1}^{1} G(\xi, \xi'; \lambda)\, \tilde{G}(\eta, \eta'; -\lambda)\, J(\xi', \eta')\, \delta(\xi' - \alpha)\, \delta(\eta' - 1)\, d\eta'\, d\xi'$$

Evaluating the delta function source results in

$$H_\varphi(\xi,\eta,\alpha,1) = \lim_{s \to 0^+} \frac{1}{2\pi j} \oint_\Gamma d\lambda\, G(\xi,\alpha,\lambda)\tilde{G}(\eta,1,-\lambda) \tag{6.10}$$

where $H_\varphi\,(\xi,\eta,\alpha,1)$ is the desired Green's function of the conductive prolate spheroid with an electric dipole aligned and on the axis of symmetry.

6.3 A Discussion of the Green's Function

The procedure outlined in this section follows the work of Kazarinoff [48, 49]. First, we consider a special case (6.10) by assuming that the dipole is at infinity. The solution is then the response of an axially incident plane wave. We then consider only the calculation of the electric current on the conducting prolate spheroid. The result will be compared with the scalar case and some similarities will be noted. Based on Langer's turning point analysis, (Appendix E) in the next section, the high frequency residues, the so-called modes, will be evaluated and the residue series presented.

To derive the electric current on the conductive prolate spheroid due to an axially incident electromagnetic wave, we consider the following limit of the radial resolvent Green's function

$$\lim_{\alpha \to \infty} G(\xi,\alpha,\lambda) = \frac{\varphi_1(\xi,\lambda)\,\lim_{\alpha\to\infty}\varphi_2(\alpha,\lambda)}{2j\gamma_+\frac{d}{d\xi_0}\left[(\xi_0^2-1)^{\frac{1}{2}}\,y_2(\xi_0)\right]} \tag{6.11}$$

As previously noted (6.6a)

$$\varphi_1(\xi_0,\lambda) = -\frac{2j\gamma_+}{(\xi_0^2-1)^{\frac{1}{2}}}$$

and

$$\lim_{\alpha \to \infty} \varphi_2(\alpha, \lambda) = \lim_{\alpha \to \infty} y_2(\alpha, \lambda) = -\frac{1}{\alpha} e^{-j\gamma_+\alpha}$$

we have by the usual plane wave normalization

$$G_p(\xi_0, \lambda) \triangleq \lim_{\alpha \to \infty} \alpha e^{j\gamma_+\alpha} G(\xi_0, \alpha, \lambda)$$

$$= \frac{1}{[(\xi_0^2 - 1)^{\frac{1}{2}}][\frac{d}{d\xi_0}(\xi_0^2 - 1)^{\frac{1}{2}} y_2(\xi_0)]} \qquad (6.12)$$

Substituting (6.12) into (6.10) yields

$$(\xi_0^2 - 1)^{\frac{1}{2}} H_{\varphi_p}(\xi_0, \eta) = \lim_{s \to 0^+} \frac{1}{2\pi j} \oint_\Gamma d\lambda \frac{\tilde{G}(\eta, 1, -\lambda)}{\frac{d}{d\xi_0}[(\xi_0^2 - 1)^{\frac{1}{2}} y_2(\xi_0, \lambda)]} \qquad (6.13)$$

Since the electric current on a prolate spheroid is

$$I(\xi_0, \eta) = 2\pi c [(\xi_0^2 - 1)(1 - \eta^2)]^{\frac{1}{2}} H_\varphi(\xi_0, \eta)$$

therefore

$$I_p(\xi_0, \eta) = \lim_{s \to 0^+} \frac{1}{2\pi j} \oint_\Gamma \frac{2\pi c [1 - \eta^2]^{\frac{1}{2}} \tilde{G}(\eta, 1, -\lambda)\, d\lambda}{\frac{d}{d\xi_0}[(\xi_0^2 - 1)^{\frac{1}{2}} y_2(\xi_0, \lambda)]} \qquad (6.14)$$

This is the desired contour integral representation of the electric current on the conductive prolate spheroid.

It is now instructive to compare the above solution with the solution of a scalar problem. If we let a scalar axial plane wave impinge on a hard prolate spheroid, the distribution on the surface of the spheroid is [49]

$$U_P(\xi_0,\eta) = \frac{1}{\xi^2-1} \lim_{s \to 0^+} \frac{1}{2\pi j} \oint_\Gamma \frac{\tilde{\tilde{G}}(\eta, 1, -\lambda)}{Z_2'(\xi_0,\lambda)}\, d\lambda \tag{6.15}$$

where $Z_2(\xi_0,\lambda) = R''_{on}(\xi_0)$ and $\tilde{\tilde{G}}(\eta, 1, -\lambda)$ is the angular resolvent Green's function constructed from $S_{on}(\eta)$ and $S_{on}(-\eta)$ functions. Thus the similarity is quite clear. The only difference between the two integrals (6.14) and (6.15), is that the mode functions, in the scalar case, are spheroidal wave functions of zero order and in the vector case, they are $(\xi^2-1)^{\frac{1}{2}}$ $R_{in}(\xi,\gamma)$ where now the spheroidal wave functions of the first order.

On this subject Kazarinoff and Ritt [50] comment

"The integrand which appears in the relation (6.15) has poles in the half-planes above and below Γ. These half-planes will hereafter be referred to as the upper and lower half-planes, respectively. The poles lying below arise from the singularities of $\tilde{\tilde{G}}(\eta, 1, -\lambda)$ for fixed η. The operator L_η is self-adjoint when s=0. Therefore, on the basis of Sturm-Liouville theory we may legitimately evaluate the integral in the relation (6.15) as a residue series involving the residues which arise from the poles of $\tilde{\tilde{G}}(\eta, 1, -\lambda)$. For large values of γ, the residue series will converge slowly -- it is the analog to the expansion in surface harmonics which occurs in the case of the sphere. It thus becomes necessary to consider the residue series contributed by singularities in the upper half-plane. These are precisely the zeros of $Z_2^1(\zeta_0,\lambda)$. Because the operator L_ζ, even with s = 0, is not self-adjoint when the radiation condition is imposed, the question whether or not the integral in the presentation (6.15) can be successfully evaluated as a residue series contributed by singularities in the upper half-plane, is one which can only be settled by considerations removed from Sturm-Liouville theory". The technique of solution follows the method of Kazarinoff and Ritt. Here is how they outline the procedure.

"From this point on the broad outline of our work follows that of Franz [51]. To determine the residue series, we must obtain a knowledge of the behavior of solutions of the equations

$$\left[(1-\eta^2)y_\eta\right]_\eta + \left[\gamma^2(1-\eta^2) - \lambda\right]y = 0 \tag{6.16}$$

and

$$\left[(\zeta^2-1)y_\zeta\right]_\zeta + \left[\gamma^2(\zeta^2-1)+\lambda\right]y = 0 \tag{6.17}$$

We first consider the contribution of the residue series for the right member of the relation (6.15) when $|\lambda| >> |\gamma|^2$. If $|\lambda| >> |\gamma|^2$, the solutions of Eq. (6.16) behave asymptotically in λ like the solutions of Legendre's equation of order ν, where $\nu(\nu+1) = -\lambda$. The resolvent Green's function associated with L_η is then essentially that obtained in the case of the sphere. A similar consideration applies to Eq. (6.17) but involves slightly more computation."

Similarly, in our case, for $|\lambda| >> |\gamma|^2$ the operators reduce to the corresponding vector spherical problem. Again, following their outline, "The preceding argument may be formalized so as to show that for $|\lambda| >> |\gamma|^2$, the integrand appearing in the relation (6.15) resembles that obtained in the case of the sphere closely enough that the argument of Franz [52] may be applied. The conclusion is that in the illuminated region, that is, for positive η, the residue series diverges; whereas in the shadow region, that is, for negative η, the residue series converges and represents the solution. This same argument shows that the contribution to the residue series for $|\lambda| >> |\gamma|^2$ may be neglected. To evaluate the integral in the representation (6.15) and give a physical meaning to the result, we must therefore do two things. We shall first compute the residues in the upper half-plane for values of λ which are "comparable to" γ^2. We shall then transform the residue series into a double series which, if the order of summation is interchanged, can be interpreted as a series of "creeping waves". We shall see that if the first creeping wave is deleted from this series, the remaining series converges even in the illuminated region. Finally we shall show heuristically that the terms which have been removed can be re-evaluated by the stationary phase technique, and thus find that they correspond to the optical contribution."

To summarize, we have reformulated the problem of finding a Green's function of a conducting prolate spheroid with an electric dipole that is on and aligned with the axis of symmetry of the boundary. This formulation is in the form suitable for the application of the theory of complex resolvents. By the use of this theory, the desired Green's function is evaluated and then specialized to planewave excitation. Based on this specialized Green's function the electric current, on the conducting prolate spheroid excited by an axial plane wave, is evaluated. This solution is then compared with a corresponding contour integral solution of the surface distribution of a hard (scalar) prolate spheroid. Here it is noted that the two solutions are identical in form and they differ only in their respective spheroidal functions appropriate to the problem. We shall refer to these functions as the mode functions. Based on Kazarinoff and Ritt's [56] reasoning, for large values of λ these mode functions approach their corresponding spherical mode functions, and the arguments of Franz [52] are applicable. In the next section, based on Langer's theory $^{+}$[53], uniformly valid asymptotic approximations of the radial mode functions are derived. Based on these approximations, the new poles of the contour integral are determined. By evaluating the residues of the complex contour integral (6.14) at these poles, the residue series is then obtained.

6.4 Turning Point Analysis and the Residue Series

To derive the uniformly valid asymptotic representations of the radial mode functions, we recall that from (2.14), the mode equation in prolate spheroidal coordinates is

$$(\xi^2-1)\frac{\partial^2\psi}{\partial\xi^2} + (1-\eta^2)\frac{\partial^2\psi}{\partial\eta^2} + \gamma^2(\xi^2-\eta^2)\psi = 0 \qquad (2.14)$$

Rewrite (2.14) in the form

$$(\xi^2-1)\frac{\partial^2\psi}{\partial\xi^2} + (1-\eta^2)\frac{\partial^2\psi}{\partial\eta^2} + \gamma_+^2(\xi^2-1)\psi + \gamma_+^2(1-\eta^2)\psi = 0 \qquad (6.18)$$

and let

$$\psi(\xi,\eta) = M(\xi)N(\eta) \qquad (6.19)$$

where $M(\xi)$ and $N(\eta)$ are the separated functions. Substituting (6.19) into

+ A review of Langer's method is presented in Appendix E.

(6.18) and separating variables, we have

$$(\xi^2-1)\frac{\partial^2 M}{\partial \xi^2} + \left[\lambda + \gamma_+^2(\xi^2-1)\right] M(\xi) = 0 \quad (6.20a)$$

$$(1-\eta^2)\frac{\partial^2 N}{\partial \eta^2} + \left[-\lambda + \gamma_+^2(1-\eta^2)\right] N(\eta) = 0 \quad (6.20b)$$

where λ is the separation constant. The solutions of (6.20a) are the required radial mode functions in Eq. (6.14). We now apply the Uniformly Valid Asymptotic Approximation discussed in Appendix E., to (6.20a).

Let ξ_1 be defined as

$$(\xi_1^2 - 1)\gamma_+^2 + \lambda = 0 \quad (6.21)$$

where $\xi_1 > 1$

Substituting (6.21) into (6.20a) results in

$$\frac{d^2 M}{d\xi^2} + \gamma_+^2\left(\frac{\xi^2-\xi_1^2}{\xi^2-1}\right) M(\xi) = 0 \quad (6.22)$$

We now note that (6.22) is in the form

$$\frac{d^2 M}{d\xi^2} + \gamma_+^2 N^2(\xi,\xi_1) M(\xi) = 0 \quad (6.22a)$$

where the 'index of refraction'

$$N^2(\xi,\xi_1) = \frac{\xi^2-\xi_1^2}{\xi^2-1}$$

has a simple zero at $\xi = \xi_1$ and therefore for large γ_+ Langer's turning point analysis is applicable. The solutions of (6.22a), described by Langer are

$$M^{(i)}(\xi) \sim C\left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{\pm j\frac{5\pi}{12}} \Psi(\xi) \rho^{\frac{1}{3}} H^{(i)}_{\frac{1}{3}}(\rho) \quad i = 1, 2 \tag{6.23}$$

where C is a constant, $H_{1/3}^{(i)}(\rho)$ are the Hankel function of the ith kind of fractional 1/3 order, the variable ρ is defined as

$$\rho = \gamma_+ \Phi(\xi) \tag{6.24}$$

where

$$\Phi(\xi) = \int_{\xi_1}^{\xi} N(\tau, \xi_1)\, d\tau \tag{6.25}$$

and

$$\Psi(\xi) = \Phi^{\frac{1}{6}} N^{-\frac{1}{2}} \tag{6.26}$$

The constant C is determined by the requirement that, as $\xi \rightarrow \infty$, the solutions of (6.23) are to reduce to the known asymptotic solutions of (6.20a). The calculation for finding C is performed in [54] and the result is

$$C = \gamma_+^{\frac{1}{6}} e^{j\gamma_+ f(\xi_1)} \tag{6.27}$$

where f (ξ_1) is defined as

$$f(\xi_1) = -\int_0^1 \left(\frac{\xi_1^2 - \tau^2}{1 - \tau^2}\right)^{\frac{1}{2}} d\tau \triangleq -\xi_1 E\left(\frac{\pi}{2}, \xi_1^{-1}\right) \tag{6.27a}$$

and E stands for the complete elliptic integral of the second kind.

Having determined the uniformly valid asymptotic representation of the radial mode functions, next an estimate of the zeros of the derivative of the spheroidal mode functions is obtained. These zeros represent the poles of the contour integral. The zeros of interest are the zeros of the equation.

$$\frac{d}{d\xi} Ai^{(2)}(\varphi, \lambda) \Big|_{\xi=\xi_0} + O(\gamma_+^{-1}) = 0 \tag{6.28}$$

where φ is given in (6.24) and (6.25) and $Ai^{(2)}(\varphi)$ is the tabulated Airy function [55]. Let the r-th zero of (6.28) be h_r, and the corresponding φ be φ_r. Following Kazarinoff and Ritt's [56] reasoning, at $\xi = \xi_0$ with λ considered as a variable, there exists a λ at which the value of the integral

$$\frac{h_r}{\gamma} = \int_{\xi_1(\lambda_r, \gamma_+)}^{\xi_0} \left[\frac{t^2 - \xi_1^2(\lambda_r, \gamma_+)}{t^2 - 1} \right]^{\frac{1}{2}} dt$$

is equal to h_r. We denote this λ by λ_r. Because the zeros h_r of $\frac{d}{dt}\left[t^{\frac{1}{3}} H_{\frac{1}{3}}^{(2)}(t) \right]$ are simple and this function is analytic in the neighborhood of each of its zeros, the values of φ_r which are determined by the condition (6.28) satisfy the relation

$$\varphi_r = h_r + O(\gamma_+^{-1}) \tag{6.29}$$

Using the definition of φ, (6.29) may be recast in the form

$$\frac{h_r}{\gamma} = \int_{\xi_1(\lambda_r, \gamma_+)}^{\xi_0} \left[\frac{t^2 - \xi_1^2(\lambda_r, \gamma_+)}{t^2 - 1} \right]^{\frac{1}{2}} dt \tag{6.29a}$$

This integral when expanded into powers of ($\xi_0 - \xi_1$) in conjunction with definition of λ_r (6.21) results [56] in

$$\lambda_r = -\gamma_+^2(\xi_0^2-1)\left[1+e^{-j\frac{\pi}{3}}\left(\frac{2\xi_0}{\xi_0^2-1}\right)^{\frac{2}{3}}\left(\frac{3h_r}{2\gamma_+}\right)^{\frac{2}{3}}+O\left(\gamma_+^{-\frac{4}{3}}\right)\right] \tag{6.30}$$

where the branch of cube root is chosen to satisfy the requirement that λ_r should be in the upper half of the complex λ plane. Eq. (6.30) is then the desired estimate of the locations of the poles of the complex contour integral in the upper half of the complex λ plane. Next, the residues at these poles are evaluated. This requires the computation of

$$\tilde{G}\,(\eta, 1, -\lambda)\Big|_{\lambda=\lambda_r} \tag{6.31}$$

To evaluate $\tilde{G}\,(\eta, 1, -\lambda_r)$ consider the mode equation (6.20b). Since $\lambda_r = (1-\xi_r^2)\,\gamma_+^2$, substituting into (6.20) we have

$$\frac{d^2N}{d\eta^2}+\ell^2\left[\frac{\xi_r^2-\eta^2}{(\xi_r^2-1)(1-\eta^2)}\right]N(\eta)=0 \tag{6.32}$$

where $\ell^2 = -\lambda_r$.

This equation fulfills the hypothesis necessary to the application of the theory in [57] on $-1 < \eta \leq 1$. Following Kazarinoff and Ritt [58] and based on Langer's theory, we define

$$P^2(\eta) = \frac{\xi_r^2 - \eta^2}{(1-\eta^2)(\xi_r^2 - 1)}$$

$$L(\eta) = \int_1^\eta P(\tau)\, d\tau \tag{6.33}$$

$$\sigma(\eta, \ell) = \ell L(\eta)$$

and

$$K(\eta) = \left[P(\eta) S(\eta)\right]^{-\frac{1}{2}}$$

with

$$\lim_{\eta \to 1} K(\eta) = K(1)$$

The solutions of (6.32) are

$$N_o = K \sigma J_o(\sigma)$$

and

$$N_i = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{\pm j \frac{\pi}{4}} K \sigma H_o^{(i)}(\sigma) \qquad i = 1, 2$$

$$N_i = K \sigma^{\frac{1}{2}} e^{\pm \sigma} \left[1 + O(\sigma^{-1})\right]$$

$$\text{as} \quad \sigma \longrightarrow \infty \quad |\arg \sigma| < \pi \tag{6.34}$$

Based on these solutions, the angle resolvent Green's function [58] is

$$\tilde{G}(\eta, 1, -\lambda_r) = \tag{6.35a}$$

$$-j\left(\frac{\pi}{2\ell}\right)^{\frac{1}{2}}\left[\frac{\xi_r^2 - 1}{(\xi_r^2 - \eta^2)(1-\eta^2)}\right]^{\frac{1}{4}}\left[\frac{\cos[\ell L(-\eta) - \frac{\pi}{4}]}{\cos[2\ell L(0)]}\right] \qquad \sigma(-\eta) \geq N$$

$$\tilde{G}(\eta, 1, -\lambda_r) =$$

$$-j\frac{\pi}{2}\ell^{-\frac{3}{2}}\left[\frac{\xi_r^2 - 1}{(\xi_r^2 - \eta^2)(1-\eta^2)}\right]\left[\frac{L^{\frac{1}{2}}(-\eta)\, J_0[\ell L(-\eta)]}{\cos[2\ell L(0)]}\right] \qquad \sigma(-\eta) < N$$

(6.35b)

Collecting results, we have for the residue series, for example

$$I_p(\xi_0,\eta) = \lim_{s\to 0^+} \sum_r \left[\frac{-j\,C(1-\eta^2)^{\frac{1}{2}}\,2\pi}{\frac{\partial^2}{\partial\lambda\partial\xi}M(\xi,\lambda)\Big|_{\xi_0,\lambda_0}} \left(\frac{\pi}{2\ell}\right)^{\frac{1}{2}} \frac{\cos(\ell L(-\eta)-\frac{\pi}{4})}{\cos[2\ell L(0)]} \right] \left[\frac{\xi_r^2-1}{(\xi_r^2-\eta^2)(1-\eta^2)} \right]^{\frac{1}{4}}$$

$$\sigma(-\eta) \geq N$$

(6.36)

Note that (6.36) can be rewritten as follows:

$$I_p(\xi_0,\eta) = \lim_{s\to 0^+} \sum_r A_r \left[e^{j\ell d(-\eta)} + e^{-j\ell d(-\eta)} \right]$$

where

$$A_r = -j\left(\frac{c^2\pi}{2\ell}\right)^{\frac{1}{2}} \left[\frac{2\pi}{\frac{\partial^2}{\partial\xi\partial\lambda}M(\xi,\lambda)\Big|_{\lambda_r,\xi_0}} \right] \left[\frac{(\xi_r^2-1)(1-\eta^2)}{\xi_r^2-\eta^2} \right]^{\frac{1}{4}} \qquad (6.37)$$

and

$$d(-\eta) = L(-\eta) - L(0)$$

Thus Eq. (6.36) is the required residue series. This equation rewritten in the form of Eq. (6.37) may be interpreted as a series of 'creeping' waves on the surface of the spheroid.

We note that the form of the modes in (6.37) are identical with modes found in the scalar case and only the nature of the residues are different, i.e. $\lambda_r{}'s$ are different. Therefore, the conclusion of Kazarinoff and Ritt applies to the vector case. Quoting "We find that the following description of the terms in the expansion can be given.

The leading term is pure imaginary and equals ks, where s is the arc length from the shadow boundary. The second term is complex and is proportional to $k^{\frac{1}{3}} s'$, where s' is an integral depending upon the local radius of curvature in exactly the manner predicted by Keller [59]."

"The next term in the expansion is proportional to $(k R_o)^{-\frac{1}{3}}$, where R_o is the radius of curvature at the tip of the spheroid. Thus, the approximations of Fock and Keller are applicable only when the wave length is small relative to R_o."

The calculation for the electric current at the tip of the spheroid as well as currents in the illuminated region follows the arguments presented by Kazarinoff and Ritt. These results apply to the vacuum high frequency scattering. In the case of anisotropic medium, based on the connecting relations between the scaled and unscaled boundary, derived in Chapter 4, Eq. (4.15), we may reinterpret the modes as 'creeping' waves traveling on the surface of the scaled spheroid.

To summarize, the problem of the scattering of the fields of an axial electric dipole by a conductive spheroid has been solved. It has been assumed that the medium is uniaxially anisotropic with optic axis aligned with the axis of symmetry. The solution is an infinite series of prolate spheroidal wave functions whose arguments are the scaled spheroidal variables. Although this form of solution is useful for numerical computations, to demonstrate the nature of the anisotropy special cases are considered. In particular, low and high frequency asymtotic limits are derived. In the low frequency case, it has been shown that the scattering amplitude may be expanded into a power series of k, where the coefficients directly show the parameters of the problem, in particular, the parameter of the anisotropy K . In the short wavelength case, the electric current on a conductive prolate spheroid induced by an axially incident plane wave has been evaluated. Here it is shown, that these currents may be interpreted in terms of a sum of 'creeping' waves propagating along the surface of the scaled spheroid.

Proposed Extensions

There are a number of problems that may be solved using the techniques used in this dissertation. A partial list of these is now given.

In solving for the total field due to an electric dipole above a conductive prolate spheroid, we have noted that in the case of low frequency, the solution may be expanded into a power series of the wave number k . As an example of this expansion technique, the low frequency far field of the magnetic field has been evaluated. However, by using the long wavelength formulae of Appendix A, far field as well as near field of all the field components may be evaluated. These results would then be valid for conductive oblate as well as prolate spheroids with their important special cases, the conductive sphere and disk. Such solution may be specialized to axial incident planewave excitation by moving the electric dipole at infinity. Based on these results, the low frequency scattering crossection of a prolate or oblate conductive spheroid may be evaluated.

In the case of high frequency, in this dissertation, the electric current induced by an axially incident plane wave has been evaluated by using the theory of complex resolvents. However, this theory may be used to evaluate the field off the prolate spheroid. Also, the Watson transformation, applicable in the case of scattering by a conductive sphere, is pertinent to conductive spheroids as well. Based on this type of high frequency calculation, the heuristic reasoning used in obtaining the field at the tip of the spheroid, the so-called Poisson-spot may be eliminated.

A further extension of our work is to consider the anisotropic, homogeneous medium to be lossy, that is κ, the relative permittivity, is now complex. Although, by extending the definition of the arguments of the mode functions into a complex plane, for some complex κ, the scaling technique can be utilized, these results are not so appealing because the scaled boundary cannot be interpreted as a real surface. Therefore further work is needed to understand the effect of loss on boundary value problem.

The scaling, for real κ, replaces the known special functions with functions appropriate to the anisotropic problem. These functions, of course, are not known functions, and because of it they obscure the effect of the anisotropy. It would be, therefore, of interest to approximate the anisotropic mode function in terms of special functions. For interior boundary value problem variational techniques, for example, Rayleigh-Ritz or the method of orthogonal projection, may be useful. In the case of small anisotropy, a perturbation in κ could be utilized to find the effect of anisotropy.

Appendix A. Prolate Spheroidal Wave Functions

In this appendix some properties of spheroidal wave functions will be noted. The knowledge of these functions are necessary for the understanding of the electric dipole problem discussed in Chapter 4.

I Prolate Spheroidal Coordinates and the Scalar Wave Equation.

Prolate spheroidal coordinates are related to cartesian coordinates by the transformation

$$\begin{aligned} x &= c\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi & \quad 1 < \xi < \infty \\ y &= c\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi & \quad -1 < \eta < 1 \\ z &= c\xi\eta & \quad 0 < \varphi < 2\pi \end{aligned} \tag{A.1}$$

Eliminating η from (A.1)

$$\frac{z^2}{c^2\xi^2} + \frac{\rho^2}{c^2(\xi^2-1)} = 1$$

the ξ =constant surface represents a prolate spheroid with major axis $c\xi$ and minor axis $c\sqrt{\xi^2-1}$. Similarly, eliminating ξ from (A.1), the η =constant surface are described by

$$\frac{z^2}{c^2\eta^2} + \frac{\rho^2}{c^2(\eta^2-1)} = 1$$

the equation of hyperboloid of revolution. Some typical coordinate surfaces are depicted in Figure 1A.

For future reference, we list the transformation from prolate spheroidal to spherical coordinate system as

$$\begin{aligned} r &= c\sqrt{\xi^2+\eta^2-1} \\ \theta &= \cos^{-1}\frac{\xi\eta}{\sqrt{\xi^2+\eta^2-1}} \\ \varphi &= \varphi \end{aligned} \tag{A.2}$$

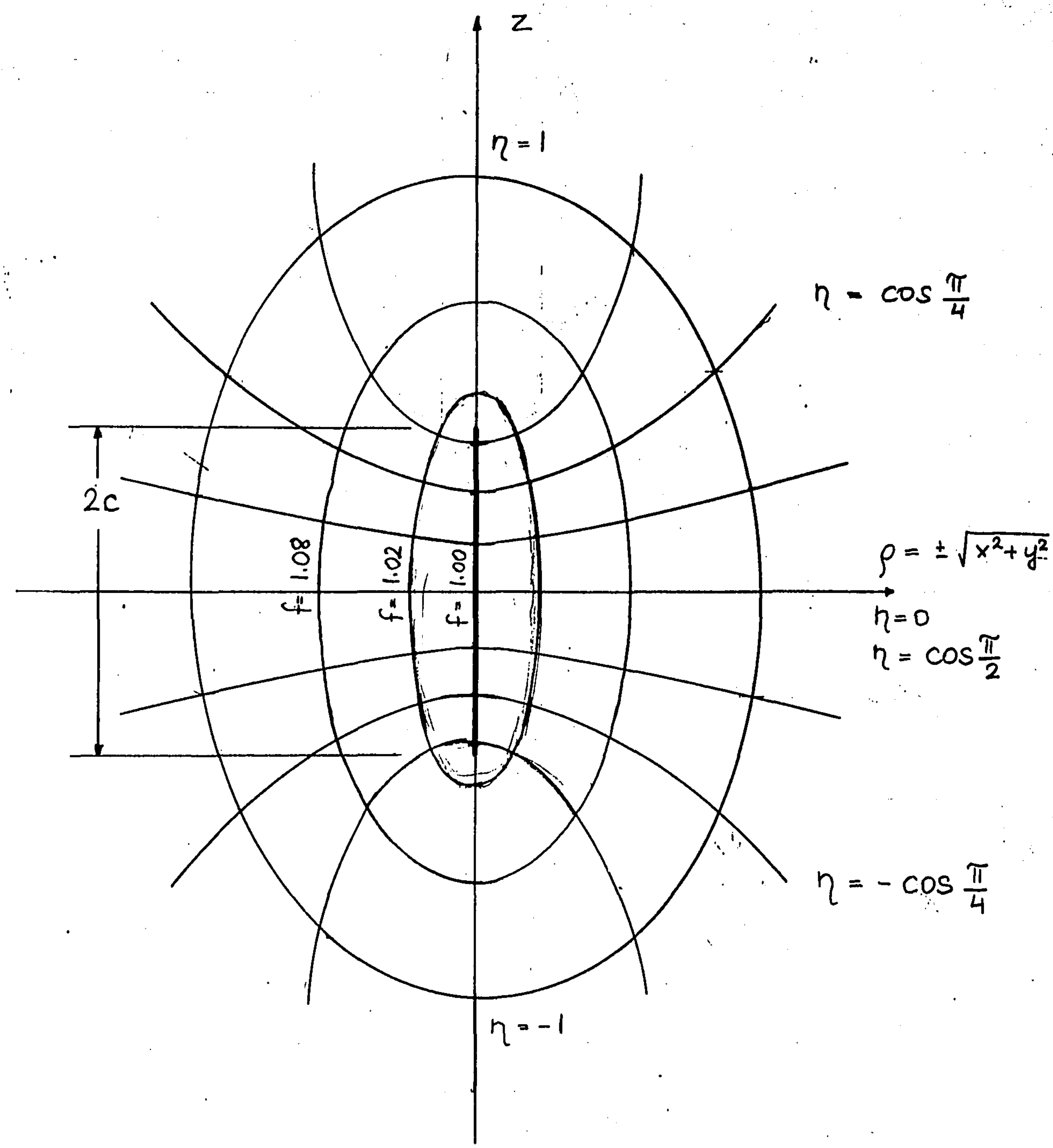

Figure 1A Prolate Spheroidal Coordinate System

The metrics of the spheroidal coordinate system can be found in the usual way; namely,

$$\begin{aligned}(ds)^2 &= \sum h_{\alpha\beta}\, dx^\alpha dx^\beta \\ &= (dx)^2 + (dy)^2 + (dz)^2 = h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 + h_\varphi^2 d\varphi^2\end{aligned} \tag{A. 3}$$

The result is

$$h_\xi = c\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}\,, \qquad h_\eta = c\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}\,, \qquad h_\varphi = c\sqrt{(\xi^2-1)(1-\eta^2)}$$

To obtain the prolate spheroidal wave functions, substitute (A3) into the Helmholtz equation in curvilinear coordinates and then separate variables. Hence

$$\frac{1}{c^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial\phi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\phi}{\partial\eta}\right] + \right.$$
$$\left. + \frac{\partial}{\partial\varphi}\left[\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\,\frac{\partial\phi}{\partial\varphi}\right]\right\} + k^2\phi = 0 \tag{A. 4}$$

is the Helmholtz equation in prolate spheroidal coordinate system. Assuming $\phi(\xi,\eta,\varphi) = R(\xi)S(\eta)e^{\pm jm\varphi}$ to be the separated form, where m is an integer and noting that

$$\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)} = \frac{1}{\xi^2-1} + \frac{1}{1-\eta^2}$$

we have for R (ξ) and S (η) the equations

$$\frac{d}{d\xi}\left[(\xi^2-1)\,\frac{d\,R_{mn}(\xi,\gamma)}{d\xi}\right] + \left[-\lambda_{mn}(\gamma) + \gamma^2\xi^2 - \frac{m^2}{\xi^2-1}\right]R_{mn}(\xi,\gamma) = 0 \tag{A. 5}$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dS_{mn}(\eta,\gamma)}{d\eta}\right] + \left[\lambda_{mn}(\gamma) - \gamma^2\eta^2 - \frac{m^2}{1-\eta^2}\right]S_{mn}(\eta,\gamma) = 0 \qquad \text{(A. 6)}$$

$\lambda_{mn}(\gamma)$ is the separation constant labelled according to its dependency on the parameters $\gamma \triangleq kc$, m and n.

IIA Angle Spheroidal Wave Functions

To delineate the role of the index n note that as $\gamma \longrightarrow 0$ subject to the condition that $S_{mn}(o, \pm 1)$ is finite, is an eigen-value problem whose solution for the eigen-values are

$$\lambda_{mn}(o) = n(n+1) \qquad \text{(A. 7)}$$

with the eigen functions given by the Associated Legendre Polynomials. It is customary to normalize the prolate spheroidal wave functions such that as $\gamma \longrightarrow 0$, $S_{mn}(o, \eta) = P_n^{|m|}(\eta)$. Using this knowledge, spheroidal angle functions can be expanded, at least for small γ, in terms of Associated Legendre Polynomials. Substituting

(1)

$$S_{mn}(\eta,\gamma) = \sum_{r=0,1}{}' d_r^{mn}(\gamma)\, P_{m+r}^{m}(\eta) \qquad \text{(A. 8)}$$

into (A. 6) we obtain a three term recursion relation of the form

$$A_1(\gamma,r,m)d_{r+2}^{mn} + A_2(\gamma,r,m)d_r^{mn} + A_3(\gamma,r,m)d_{r-2}^{mn} = 0 \qquad \text{(A. 9)}$$

(1) The prime notation represent summation on $\binom{\text{even}}{\text{odd}}$ integers if (n - m) is $\binom{\text{even}}{\text{odd}}$.

To assure convergence of (A. 8) the $\lim_{r\to\infty} \frac{d_{r+2}^{mn}}{d_r^{mn}}$ has to be zero resulting in a transcendental relation between λ, γ and m. There are denumerably infinite solutions for λ. These eigen-values are labelled with index n. Invoking Sturm-Liouville theory, the λ_{mn} are real and the corresponding eigen-functions are orthogonal.

For small γ, λ_{mn} can be expanded into a power series in γ, that is

$$\lambda_{mn}(\gamma) = n(n+1) + \sum_{k=1}^{\infty} l_{2k}^{mn} \gamma^{2k} . \tag{A. 10}$$

Various forms of l_{mn} as a function of m and n can be found in tables (see Flammer). These forms are fractional as well as in decimal forms. Using the values of λ_{mn} found from (A. 10) and substituting in (A. 9) the values of d_r^{mn} may be evaluated.

The d_r^{mn} that are important in the sequel are up to γ^6,

$$\begin{aligned}
d_0^{11} &= 1 + \frac{2}{25}\gamma^2 + \frac{89}{30625}\gamma^4 + \frac{3094}{3969}\left(\frac{\gamma}{5}\right)^6 \\
d_2^{11} &= -\frac{\gamma^2}{75} - \frac{4}{5625}\gamma^4 - \frac{178}{693}\left(\frac{\gamma}{5}\right)^6 \\
d_4^{11} &= \frac{\gamma^2}{11025} + \frac{50}{819}\left(\frac{\gamma}{5}\right)^6 \\
d_6^{11} &= -\frac{9}{35035}\left(\frac{\gamma}{3}\right)^6 \\
d_1^{12} &= 1 + \frac{2}{49}\gamma^2 + \frac{67}{64827}\gamma^4 + \frac{18188}{9801}\left(\frac{\gamma}{7}\right)^6 \\
d_3^{12} &= -\frac{3}{245}\gamma^2 - \frac{12}{26411}\gamma^4 - \frac{758}{715}\left(\frac{\gamma}{7}\right)^6 \\
d_5^{12} &= \frac{\gamma^4}{14553} + \frac{1274}{4455}\left(\frac{\gamma}{7}\right)^6 \\
d_7^{12} &= -\frac{9}{55055}\left(\frac{\gamma}{3}\right)^6
\end{aligned} \tag{A. 11}$$

The corresponding angle functions are up to γ^4.

$$S_{11}(\eta, \gamma) = P_1'(\eta) + \gamma^2\left[\frac{2}{25}P_1'(\eta) - \frac{1}{75}P_3'(\eta)\right] + \\ + \gamma^4\left[\frac{89}{30625}P_1'(\eta) - \frac{4}{5626}P_3'(\eta) + \frac{1}{11025}P_5'(\eta)\right] \tag{A.12}$$

$$S_{12}(\eta, \gamma) = P_2'(\eta) + \gamma^2\left[\frac{2}{49}P_2'(\eta) - \frac{3}{245}P_4'(\eta)\right] + \\ + \gamma^4\left[\frac{67}{64827}P_2'(\eta) - \frac{12}{26411}P_4'(\eta) + \frac{1}{14533}P_6'(\eta)\right]$$

The representation of angular spheroidal wave functions discussed above offers reasonable convergence for $\gamma < 10$. There are other representations that are useful for $\gamma \to \infty$, for high frequencies, however, these will not be used in Chapter 4 and thus will not be discussed. The problem of both γ and $n \to \infty$, will be encountered, however, and therefore will be discussed in Appendix E on Uniformly Valid Asymptotic Approximations.

IIIA Radial Spheroidal Wave Functions

To aid the understanding of the radial prolate spheroidal wave functions, we let $\gamma \to 0$ and $\xi \to \infty$ such that $x = \gamma\xi$ is finite. Substituting the new variable in (A.5) yields

$$\frac{d}{dx}\left[\left(x^2 - \gamma^2\right)\frac{dR_{mn}(x)}{dx}\right] + \left[-\lambda_{mn}(\gamma) + x^2 - \frac{m^2\gamma^2}{x^2 - \gamma^2}\right]R_{mn}(x) = 0 \tag{A.13}$$

Now let $\gamma \to 0$ and note from (A.7) that $\lambda_{mn}(0) = n(n+1)$ we have

$$\frac{d^2R_{mn}(x)}{dx^2} + \frac{1}{x}\frac{dR_{mn}}{dx} + \left[1 - \frac{n(n+1)}{x^2}\right]R_{mn}(x) = 0 \tag{A.14}$$

This is the spherical Bessel equation. From (A.2) it is seen that as $\xi \to \infty$, $x \to r$. A reasonable normalization of radial spheroidal wave functions is then given by the requirement that it should approach, in the far field, its corresponding Bessel, Neumann or Hankel function; they should satisfy the same radiation condition.

Thus

$$R_{mn}^{(1)}(\xi, \gamma) \to j_n(c\xi) \to \frac{\cos\left(c\xi - \frac{n+1}{2}\pi\right)}{c\xi}$$

$$R_{mn}^{(2)}(\xi, \gamma) \to h_n(c\xi) \to \frac{\sin\left(c\xi - \frac{n+1}{2}\pi\right)}{c\xi} \tag{A.15}$$

$$R_{mn}^{\substack{(3)\\(4)}}(\xi, \gamma) \to h_n^{\substack{(1)\\(2)}}(c\xi) \to \frac{e^{\pm j\left(c\xi - (n+1)\frac{\pi}{2}\right)}}{c\xi}$$

To obtain a large wavelength representation for the radial prolate spheroidal wave functions required in Chapter 4, first it is shown that such functions can be represented in terms of angle spheroidal wave functions. To this end we digress to give an integral representation of the spheroidal wave functions.

Digression: An Integral Representation.

<u>Theorem 1</u>

Let L_η be the linear, self-adjoint, differential operator

$$L_\eta = \frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}\right] - \frac{m^2}{1-\eta^2} - \gamma^2\eta^2 \tag{A.16}$$

and L_ξ be the operator obtained from L_η upon replacing η by ξ. Let $S_{mn}(\eta, \gamma)$ be an angle function of the first kind which satisfies the equation

$$\left[L_\eta + \lambda_{mn}(\gamma)\right] S_{mn}(\eta, \gamma) = 0 \tag{A.17}$$

let $K_m(\xi, \eta)$ along with its first and second derivatives be continuous functions of η and ξ; and let $K_m(\xi, \eta)$ satisfy the equation

$$\left(L_\xi - L_\eta\right) K_m(\xi, \eta) = 0 \tag{A.18}$$

in the domain D of the complex ξ-plane and in the domain of η-plane that includes the interval $a \leq \eta \leq b$, the end points of which are chosen so that

$$\int_a^b \left(S_{mn} L_\eta K_m - K_m L_\eta S_{mn}\right) d\eta = \left\{(1-\eta^2)\left[S_{mn}\frac{dK_m}{d\eta} - K_m \frac{dS_{mn}}{d\eta}\right]\right\}\Bigg|_a^b = 0 \tag{A.19}$$

Then

$$R_{mn}(\xi, \gamma) = \int_a^b K_m(\xi, \eta)\, S_{mn}(\eta, \gamma)\, d\eta \tag{A.20}$$

is a solution of the radial equation

$$\left[L_\xi + \lambda_{mn}(\gamma)\right] R_{mn}(\xi, \gamma) = 0 \tag{A.21}$$

for all ξ in D.

Proof

Apply operator L_ξ to (A.20)

$$L_\xi R_{mn}(\xi, \gamma) = \int_a^b S_{mn}(\eta, \gamma) L_\xi K_m d\eta$$

$$= \int_a^b K_m L_\eta S_{mn}(\eta) d\eta = -\lambda_{mn}(\gamma) R_{mn}(\xi, \gamma) \tag{A.22}$$

by the use of (A. 18) and (A. 19) and (A. 20). By the use of (A. 17) the proof is complete. The reason for the usefulness of this theorem is that the kernel $K_m(\xi, \eta)$ can easily be constructed.

Let ψ satisfy the scalar Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0 \tag{A.23}$$

where in prolate spheroidal coordinates

$$\psi = K_m(\xi, \eta) e^{\pm jm\varphi} \tag{A.24}$$

then $K_m(\xi, \eta)$ satisfies (A. 18). With this digression out of the way, we return to discuss the radial prolate spheroidal wave functions.

To derive the radial prolate spheroidal wave function in terms of the angle spheroidal wave function, let $K_m(\varphi, z) = \varphi^{+m} e^{jkz}$ be the kernel in (A. 20). Notice that the bilinear concommitant in (A. 19) vanishes for the following limits;

$$\begin{array}{lll} (i) & a = -1, & b = 1 \\ (ii) & a = j\infty, & b = 1 \\ (iii) & a = 1, & b = j\infty \end{array} \qquad \text{(A. 25)}$$

Thus, for example,

$$R_{mn}^{(1)}(\xi, \gamma) = \varphi_{mn} \int_{-1}^{1} (\xi^2 - 1)^{\frac{m}{2}} (1 - \eta^2)^{\frac{m}{2}} e^{j\gamma\xi\eta} S_{mn}(\eta, \gamma)\, d\eta \qquad \text{(A. 26)}$$

is a valid representation with φ_{mn} as a normalization constant. Using the following identity

$$P_n^m(\eta) = (-1)^m \frac{(n+m)!}{(n-m)!} P_n^{-m}(\eta) = (-1)^m \frac{(n+m)!}{(n-m)!} \frac{(1-\eta^2)^{-\frac{1}{2}m}}{2^n n!} \frac{d^{n-m}}{d\eta^{n-m}} (\eta^2 - 1)^n \qquad \text{(A. 27)}$$

in (A. 26) and integrating by part r times yields

$$R_{mn}^{(1)}(\xi, \gamma) = \rho_{mn} (\xi^2 - 1)^{\frac{m}{2}} \sum_{r=0,1}^{\infty}{}' d_r^{mn}(\gamma) \frac{(-1)^m (2m+r)! (\gamma\xi)^r}{2^{m+r} r! (m+r)!} \int_{-1}^{1} e^{j\gamma\xi\eta} (1-\eta^2)^{m+r} d\eta \qquad \text{(A. 28)}$$

Noting that

$$\int_{-1}^{1} e^{j\gamma\xi\eta} (1-\eta^2)^k d\eta = 2^{k+1} k! \frac{j_k(\gamma\xi)}{(\gamma\xi)^k}$$

$$\int_{j\infty}^{1} e^{j\gamma\xi\eta} (1-\eta^2)^k d\eta = 2^k k! \frac{h_k^{(1)}(\gamma\xi)}{(\gamma\xi)^k}$$

$$\int_{1}^{j\infty} e^{j\gamma\xi\eta} (1-\eta^2)^k d\eta = 2^k k! \frac{h_k^{(2)}(\gamma\xi)}{(\gamma\xi)^k} \qquad \text{(A. 29)}$$

where j_k, $h_k^{(1)}$ and $h_k^{(2)}$ are the Bessel, Hankel functions of the first and second kind. By using the normalization condition (A. 15), we have, for example,

$$R_{mn}^{(1)}(\xi, \gamma) = \frac{1}{\sum_{r=0,1}^{\infty}{}' d_r^{mn} \frac{(2m+r)!}{r!}} \left(\frac{\xi^2 - 1}{\xi^2}\right)^{\frac{1}{2}m} \sum_{r=0,1}^{\infty}{}' \, i^{m-n+r} \, d_r^{mn}(\gamma) \, \frac{(2m+r)!}{r!} \, j_{m+r}(\gamma \xi) \tag{A. 30}$$

For all other radial spheroidal wave functions, replace the Bessel functions with Neumann or Hankel functions. For small γ, the radial functions can be expanded in a power series of γ. The result for the radial spheroidal wave functic as well as their derivatives, for the case of interest, are

$$R_{11}^{(1)} = \frac{\gamma}{3} S_1 + \frac{\gamma^3}{150} S_1 \left[4 - 5C_1^2 + \frac{\gamma^2}{4900}\left(712 - 1400 C_1^2 + 875 C_1^4\right)\right]$$

$$R_{12}^{(1)} = \frac{\gamma^2}{15} S_1 C_1 + \frac{\gamma^4}{1470} S_1 \left[4C_1 - 7C_1^3 + \frac{\gamma^2}{5292}\left(536 C_1 - 1176 C_1^3 + 1029 C_1^5\right)\right]$$

$$R_{11}^{(2)} = -\frac{3 S_1}{2\gamma^2}\left\{C_1 S_1^{-2} - 2L - \left(\frac{\gamma}{20}\right)^2 \left[8C_1(5 - S_1^{-2}) - 8L(33 + 5C_2)\right.\right. \tag{A. 31}$$

$$\left.\left. - \frac{1}{196}\left(\frac{\gamma}{10}\right)^4 \left[85800 C_1 - 1750 C_3 + 712 C_1 S_1^{-2} - L(106324 - 76950 C_2 - 1750 C_4)\right]\right]\right\}.$$

$$R_{12}^{(2)} = -\frac{5 S_2}{4\gamma^3}\left\{3C_1^{-1}(3 + S_1^{-2}) - 18L - \left(\frac{\gamma}{14}\right)^2 \left[63 C_1 + 3C_1^{-1}(5 - 3S_1^{-2})\right.\right.$$

$$\left.\left. - L\,(51 + 63\, C_2)\right]\right\}$$

$$\frac{dR_{11}^{(1)}}{d\xi} = \frac{\gamma}{3} C_1 S_1^{-1} + \frac{\gamma^3}{150} C_1 S_1^{-1}\left[14 - 15 C_1^2\right.$$

$$\left. + \frac{\gamma^2}{4900}\left(3512 - 7700 C_1^2 + 4375 C_1^4\right)\right]$$

$$\frac{dR_{12}^{(1)}}{d\xi} = \frac{\gamma^2}{15}\left(2C_1^2 - 1\right)S_1^{-1} + \frac{\gamma^4}{1470} S_1^{-1}\left[-4 + 29C_1^2 - 28C_1^4\right.$$

$$\left. + \frac{\gamma^2}{5292}\left(-536 + 4600C_1^2 - 9849C_1^4 + 6174C_1^6\right)\right]$$

$$\frac{dR_{11}^{(2)}}{d\xi} = \frac{3}{2\gamma^2 S_1^3}\left\{1 - S_1^2\left(1 - 2C_1L\right) + \frac{\gamma^2}{100}\left[2C_1^2 + S_1^2\left(31 + 15C_2\right)\right.\right.$$

$$\left.\left. - S_1S_2L\left(23 + 15C_2\right)\right]\right\}$$

$$\frac{dR_{12}^{(2)}}{d\xi} = \frac{15}{4\gamma^3 S_1^3}\left\{2C_1\left(1 - 6S_1^2\right) + 12C_2S_1^2L\right. \tag{A. 32}$$

$$\left. + \frac{\gamma^2}{98}\left[6C_1 + 3S_1S_2\left(1 + 14C_2\right) - 2S_1^2L\left(17C_2 + 21C_4\right)\right]\right\}$$

with

$$\xi = C_1 = \cosh\mu \quad , \quad S_1 = \sinh\mu$$

$$C_n = \cosh n\mu \quad , \quad S_n = \sinh n\mu$$

$$L = \frac{1}{2}\ln\frac{1 + e^{-\mu}}{1 - e^{-\mu}}.$$

IV. An Integral Identity

Here an integral identity useful in the solution to the problem of scattering of a dipole above a conductive prolate spheroid is derived.

Let the kernel of the integral, in the integral identity proved in Theorem I, be the solution of the scalar wave equation in a spherical coordinate system. Thus, let

$$K_m(r, r', \theta, \theta') = \psi_n^{(j)}(kR)\, P_n^m(\cos\theta_{zz'}) \tag{A. 33}$$

be such kernel where $\psi_n^{(j)}$ represents the spherical Bessel function of the j^{th} kind n^{th} order, P_n^m the Associated Legendre Polynomial with indexes n and m, $R = |\underline{r} - \underline{r}'|$ and k is the wave number. Let the source coordinate be located on the $+z$ axis. Let the transformation from cartesian to prolate spheroidal coordinates of the source and the observer coordinates be represented by

$$\begin{aligned} x' &= 0 & x &= c\sqrt{(\xi^2-1)(1-\eta^2)}\ \cos\varphi \\ y' &= 0 & y &= c\sqrt{(\xi^2-1)(1-\eta^2)}\ \sin\varphi \\ z' &= c\alpha & z &= c\xi\eta \end{aligned} \tag{A. 34}$$

where the prime denotes the source coordinates.

Then

$$R = |\underline{r} - \underline{r}'| = c\sqrt{\xi^2 + \eta^2 + \alpha^2 - 2\xi\eta\alpha - 1}$$

$$\cos\theta_{zz'} = \frac{\xi\eta - \alpha}{\sqrt{\xi^2 + \eta^2 + \alpha^2 - 2\xi\eta\alpha - 1}} \tag{A. 35}$$

Consider now the following integral

$$I_1^{\binom{(3)}{(4)}}(\xi;\alpha,\gamma) =$$

$$\int_{-1}^{1}\psi_n^{(i)}\left(\gamma\sqrt{\xi^2+\alpha^2+\eta^2-2\xi\eta\alpha-1}\right)P_n^{-m}\left(\frac{\xi\eta-\alpha}{\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1}}\right)S_{mn}(\eta,\gamma)\,d\eta \qquad \text{(A. 36)}$$

For large ξ, and fixed α,

$$\cos\theta = \frac{\xi\eta-\alpha}{\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1}} \Longrightarrow \eta$$

and

$$\gamma\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1} \Longrightarrow \gamma\xi\left(1-\frac{\alpha\eta}{\xi}\right)$$

in the phase term of the Bessel function. Thus, using the asymptotic property of the radial spheroidal wave functions, we have

$$I^{\binom{(3)}{(4)}}(\xi;\alpha,\gamma) =$$

$$R_{mn}^{\binom{(3)}{(4)}}(\xi,\gamma)e^{\pm j(n-k)\frac{\pi}{2}}\int_{-1}^{1}e^{\mp j\gamma\alpha\eta}P_k^{-m}(\eta)S_{mn}(\eta)\,d\eta \qquad \text{(A. 37)}$$

By the use of identity (A. 27) with k=m yields

$$P_m^{-m}(\eta) = \frac{(1-\eta^2)^{\frac{m}{2}}}{2^m m!} \qquad \text{(A. 38)}$$

Substituting (A. 38) in (A. 37)

$$I^{\binom{(3)}{(4)}}(\xi;\alpha,\gamma) = \qquad \text{(A. 39)}$$

$$\frac{1}{2^m m!}S_{mn}^{\binom{(3)}{(4)}}(\xi,\gamma)\,e^{\pm j(n-m)\frac{\pi}{2}}\int_{-1}^{1}e^{\mp j\gamma\eta\alpha}(1-\eta^2)^{\frac{m}{2}}S_{mn}(\eta)\,d\eta$$

But from (A.26)

$$R^{(1)}_{mn}(\alpha,\gamma) =$$

$$\frac{1}{2}\,\frac{j^{m-n}\,\gamma^{m}}{\sum\limits_{r=0,1}^{\infty} d_r^{mn}\frac{(2m+r)!}{r!}}\int_{-1}^{1} e^{j\gamma\eta\alpha}(\alpha^2-1)^{\frac{m}{2}}(1-\eta^2)^{\frac{m}{2}}\,S_{mn}(\eta)\,d\eta$$

and hence the integral is evaluated. The result is

$$\int_{-1}^{1}\psi_m^{(j)}\left[\gamma\sqrt{\xi^2+\eta^2+\alpha^2-2\,\xi\eta\alpha-1}\right]P_m^m\left(\frac{\xi\eta-\alpha}{\sqrt{\xi^2+\eta^2+\alpha^2-2\xi\eta\alpha-1}}\right)S_{mn}(\eta,\gamma)\,d\eta \tag{A.40}$$

$$= \frac{(-1)^m(n-m)!}{2^{m-1}m!(n+m)!}\,\gamma^{-m}(\alpha^2-1)^{-\frac{m}{2}}\sum_{r=0,1}^{\infty} d_r^{mn}\frac{(2m+r)!}{r!}\,R^{(1)}_{mn}(\alpha,\gamma)R^{(4)}_{mn}(\xi,\gamma)$$

which is the identity sought.

Appendix B Calculation of the Low Frequency Asymptotic Limit of the Scattering Coefficients - a_n

Below the definition of the scattering coefficients, is given with a minus sign to conform with the spherical scattering coefficients. They are

$$a_n = -\frac{\frac{d}{d\xi_0}\left[(\xi_0^2-1)^{\frac{1}{2}}\,R^{(1)}_{14}(\xi_0,\gamma)\right]}{\frac{d}{d\xi_0}\left[(\xi_0^2-1)^{\frac{1}{2}}\,R^{(4)}_{14}(\xi_0,\gamma)\right]} \tag{B.1}$$

By using the high wavelength formulae of the spheroidal wave functions (Appendix A), the first two scattering coefficients will be evaluated. Recalling the following definitions:

$$R_{14}^{(4)}(\xi, \gamma) = R_{14}^{(1)}(\xi, \gamma) - j R_{14}^{(2)}(\xi, \gamma)$$

$$C_1 = \xi = \cosh\mu \quad , \quad C_n = \cosh n\mu$$

$$S_1 = \sinh\mu \quad , \quad S_n = \sinh n\mu \qquad \text{(B. 1a)}$$

$$L = \frac{1}{2} \ln \frac{1 - e^{-\mu}}{1 + e^{-\mu}}$$

and defining

$$\alpha_n = C_1 R_{1n}^{(1)}(\xi, \gamma) + S_1^2 \frac{dR^{(2)}(\xi, \gamma)}{d\xi}$$

$$\beta_n = C_1 R_{1n}^{(2)}(\xi, \gamma) + S_1^2 \frac{dR^{(2)}(\xi, \gamma)}{d\xi}$$

we have for

$$X_n = \operatorname{Re} a_n = - \frac{\alpha n^2}{\alpha n^2 + \beta n^2} \ , \quad y_n = \operatorname{Im} a_n = - \frac{\alpha_n \beta_n}{\alpha n^2 + \beta n^2} \qquad \text{(B. 2)}$$

To obtain a_n, first α_n and β_n will be evaluated. Let $a = cC_1$, be the major and $b = cS_1$, be the minor axis of the conducting spheroid. Substituting formulae(A. 31) and (A. 32) into definition (B. 1a) yields

$$\alpha_1 \triangleq C_1 R_{11}^{(1)} + S_1^2 \mathring{R}_{11}^{(1)}$$

$$= C_1 \left\{ \frac{\gamma S_1}{3} + \frac{\gamma^3 S_1}{150} \left[4 - 5C_1^2 + \frac{\gamma^2}{4900} \left(712 - 1400 C_1^2 + 875 C_1^4 \right) \right] \right\}$$

$$+ S_1^2 \left\{ \frac{\gamma}{3} S_1^{-1} C_1 + \frac{\gamma^3}{150} S_1^{-1} C_1 \left[14 - 15 C_1^2 + \frac{\gamma^2}{4900} \left(3512 - 7700 C_1^2 + 4375 C_1^4 \right) \right] \right\}$$

$$= \frac{2 \gamma C_1 S_1}{3} \left\{ 1 + \left(\frac{\gamma}{10} \right)^2 \left[18 - 20 C_1^2 + \frac{\gamma^2}{4900} \left(4224 - 9100 C_1^2 + 5250 C_1^4 \right) \right] \right\}$$

(B. 3)

$$= \frac{2}{3} k \frac{ab}{c} \left\{ 1 + \left(\frac{k}{10} \right)^2 \left[18a^2 - 18b^2 - 20a^2 + \right. \right.$$

$$\left. \left. + \frac{k^2}{4900} \left(4224 a^2 - 8448 a^2 b^2 + 4224 b^2 - 9100 a^4 + 9100 a^2 b^2 + 5250 a^4 \right) \right] \right\}$$

$$\alpha_1 = \frac{2}{3} k \frac{ab}{c} \left\{ 1 - \frac{k^2}{50} \left[a^2 + 9b^2 + \right. \right.$$

$$\left. \left. + \frac{k^2}{4900} \left(187 a^4 - 326 a^2 b^2 + 2112 b^4 \right) \right] \right\}$$

$$\alpha_2 = C_1 R_{12}^{(1)} + S_1^2 \mathring{R}_{12}^{(1)} =$$

$$= C_1\left\{\frac{\gamma^2 C_1 S_1}{15} + \frac{\gamma^4 C_1 S_1}{1470}\left[4 - 7C_1^2 + \frac{\gamma^2}{5292}\left(536 -\right.\right.\right.$$

$$\left.\left.\left.- 1176 C_1^2 + 1029 C_1^4\right)\right]\right\} + S_1^2\left\{\frac{\gamma^2}{15} S_1^{-1}\left(2C_1^2 - 1\right) +\right.$$

$$+ \frac{\gamma^2 S_1^{-1}}{1470}\left[-4 + 29C_1^2 - 28C_1^4 + \frac{\gamma^2}{5292}\left(-536 +\right.\right.$$

$$\left.\left.\left.+ 4600 C_1^2 - 9849 C_1^4 + 6174 C_1^6\right)\right]\right\}$$

$$= \frac{\gamma^2}{15} S_1\left(3C_1^2 - 1\right) + \frac{\gamma^4 C_1^2 S_1}{1470}\left[-4C_1^{-2} + 33 + 35 C_1^2 +\right.$$

$$\left.+ \frac{\gamma^2}{5292}\left(-536 C_1^{-2} + 5136 - 11025 C_1^2 + 7203 C_1^4\right)\right] \tag{B.4}$$

$$\alpha_2 = \frac{k^2 b}{15c}\left(2a^2 + b^2\right)\left\{1 - \frac{k^2}{98(2a^2 + b^2)}\left[6a^4 +\right.\right.$$

$$+ 25 a^2 b^2 + 4b^4 - \frac{k^2}{5292}\left(778 a^6 + 2361 a^4 b^2 +\right.$$

$$\left.\left.\left.+ 3528 a^4 b^2 + 536 b^6\right)\right]\right\}$$

$$\beta_1 = C_1 R_{11}^{(2)} + S_1^2 \mathring{R}_{11}^{(2)}$$

$$= -\frac{3 C_1 S_1}{2\gamma^2}\left\{C_1 S_1^{-2} - 2L - \left(\frac{\gamma}{20}\right)^2\left[8C_1\left(5 - S_1^{-2}\right) -\right.\right.$$

$$\left.\left.- 8L\left(33 + 5C_2\right)\right]\right\} + \frac{3S_1^2}{2\gamma^2 S_1^3}\left\{1 - S_1^2\left(1 - 2C_1 L\right) +\right.$$

$$+ \left(\frac{\gamma}{10}\right)^2 \left[2C_1^2 + S_1^2 (31 + 15 C_2) - S_1 S_2 L (23 + 15 C_2)\right]\Bigg\}$$

$$= -\frac{3S_1}{2\gamma^2}\Bigg\{2(1 - 2C_1 L) - \left(\frac{\gamma}{10}\right)^2 \Big[10C_1^2 - 2C_1 L(33 + 5C_2) +$$

$$+ 31 + 15 C_2 - S_1^{-1} S_2 L (23 + 15 C_2)\Big]\Bigg\}$$

$$= \frac{3 S_1 (2C_1 L - 1)}{\gamma^2} \Bigg\{ 1 - \frac{1}{2}\left(\frac{\gamma}{10}\right)^2 \frac{1}{1 - 2C_1 L}\Big[31 +$$

$$+ 25 C_1^2 + 15 S_1^2 - L C_1 (112 + 40 C_1^2 + 40 C_1^2)\Big]\Bigg\}. \qquad \text{(B. 5)}$$

$$\beta_1 = \frac{3b(2a - \tau)}{k^2 C^3 \tau}\Bigg\{1 - \frac{2}{\tau - 2a}\left(\frac{k}{10}\right)^2 \Big[\tau(14a^2 - 4b^2) -$$

$$- a(38a^2 - 18b^2)\Big]\Bigg\}$$

where $\tau = \left.\dfrac{C}{L}\right|_{\xi = \xi_1}$

$$\beta_2 = C_1 R_{12}^{(2)} + S_1^2 \mathring{R}_{12}^{(2)} = -\frac{5C_1 S_2}{4\gamma^3}\Bigg\{3C_1^{-1}(3 + S_1^{-2}) -$$

$$- 18L - \left(\frac{\gamma}{14}\right)^2 \Big[63C_1 + 3C_1^{-1}(5 - 3S_1^{-2}) -$$

$$- L(51 + 63C_2)\Big]\Bigg\} + \frac{15 S_1^2}{4\gamma^3 S_1^3}\Bigg\{2C_1(1 - 6S_1^2) +$$

$$+ 12 C_2 S_1^2 L + \frac{\gamma^2}{98}\Big[6C_1 + 3S_1 S_2 (14C_2 + 1) - 2S_1^2 L (17C_2 + 21C_4)\Big]\Bigg\}$$

$$= \frac{5}{4\gamma^3}\Big\{-3S_2(3+S_1^{-2}) + 6C_1S_1^{-1}(1-6S_1^2) + L(36C_2S_1 + 18C_1S_2) + \left(\frac{\gamma}{14}\right)^2\Big[63C_1^2S_2 + 3S_2(5-3S_1^{-2}) - C_1S_2L(51 + 63C_2) + 36C_1S_1^{-1} + 18S_2(14C_2+1) - 12S_1L(17C_2+21C_4)\Big]\Big\}$$

Recognizing that

$$S_2 = 2C_1S_1$$
$$C_2 = C_1^2 + S_1^2$$
$$C_4 = C_2^2 + S_2^2 = C_1^4 + 6C_1^2S_1^2 + S_1^4$$

we have for β_2

$$\beta_2 = \frac{5}{4\gamma^3}\Big\{-54C_1S_1 + 72C_1^2S_1L + 36S_1^3L + \left(\frac{\gamma}{14}\right)^2\Big[-18C_1S_1^{-1} + 66C_1S_1 + 630C_1^3S_1 + 504C_1S_1^3 - L(306C_1^2S_1 + 378C_1^4S_1 + 1638S_1^3C_1^2 + 204S_1^3 + 252S_1^5)\Big]\Big\} \tag{B. 6}$$

$$= \frac{15}{2c^2bk^3}\Big\{-9ab^2 + \frac{12a^2b^2}{\tau} + 6\frac{b^4}{\tau} + \left(\frac{k}{14}\right)^2\Big[-3a(a^2-b^2)^2 + 11ab^2(a^2-b^2) + 105a^3b^2 + 84ab^4 - \frac{51a^2b^2(a^2-b^2) + 63(a^4b^2) + 273a^2b^4 + 34b^4(a^2-b^2) + 42b^6}{\tau}\Big]\Big\}$$

Thus $$\beta_2 = \frac{15(-9ab^2\tau + 12a^2b^2 + 6b^4)}{2\tau bc^5k^3}\Big[1 + \left(\frac{k}{14}\right)^2\frac{\tau(-3a^5 + 122a^3b^2 + 70ab^4) - (114a^4b^2 + 256a^2b^4 + 8b^6)}{-9ab^2\tau + 12a^2b^2 + 6b^4}\Big]$$

To evaluate x_n's and y_n's , rewrite (B. 3) through (B. 6) in the following form:

$$\alpha_1 = A_1 k (1 - k^2 B_1) \quad , \quad \alpha_2 = A_3 k^2 (1 - k^2 B_3) \tag{B.7}$$

$$\beta_1 = \frac{A_2}{k^2}(1 - k^2 B_2) \quad , \quad \beta_2 = \frac{A_4}{k^3}(1 + k^2 B_4)$$

The following quantities are of interest

$$\frac{A_1}{A_2} = \frac{\frac{2}{3}\,\frac{ab}{c}}{\frac{3b(2a-\tau)}{c^3\tau}} = \frac{2}{9}\,\frac{a\tau c^2}{2a-\tau} \tag{B.8}$$

$$B_1 - B_2 = \frac{a^2+9b^2}{50} - \frac{1}{50}\,\frac{\tau(14a^2-4b^2) - a(38a^2-18b^2)}{\tau - 2a}$$

$$B_1 - B_2 = \frac{c^2}{50}\,\frac{-13\tau + 36a}{\tau - 2a} \tag{B.9}$$

$$\frac{A_3}{A_4} = \frac{\frac{b(2a^2+b^2)}{15c}}{\frac{15(-9ab^2\tau + 12a^2b^2 + 6b^4)}{2\tau bc^5}}$$

$$\frac{A_3}{A_4} = \frac{2\tau b^2c^4\,(2a^2+b^2)}{225(-9ab^2\tau + 12a^2b^2 + 6b^4)} \tag{B.10}$$

$$B_3 + B_4 = \frac{6a^4 + 25a^2b^2 + 4b^4}{98(2a^2+b^2)}$$

$$+ \frac{\tau(-3a^5 + 122a^3b^2 + 70ab^4) - (114a^4b^2 + 256a^2b^4 + 8b^6)}{14^2(-9ab^2\tau + 12a^2b^2 + 6b^4)}$$

(B. 11)

$$B_3 + B_4 = \frac{\tau(-6a^7 + 133a^5b^2 - 188a^3b^2 - 2ab^6)}{14^2(-9ab^2\tau + 12a^2b^2 + 6b^4)(2a^2+b^2)}$$

$$+ \frac{-84a^6b^2 + 46a^4b^4 + 124a^2b^2 + 40b^8}{14^2(-9ab^2\tau + 12a^2b^2 + 6b^4)(2a^2+b^2)}$$

By substituting (B. 7) into (B. 2), the real and the imaginary part of the scattering coefficient can be evaluated.

Hence

$$X_1 = -\frac{\alpha_1^2}{\alpha_1^2+\beta_1^2} = -\frac{A_1^2k^2(1-k^2B_1)^2}{A_1^2k^2(1-k^2B_1)^2 + \frac{A_2^2}{k^4}(1-k^2B_2)^2}$$

$$= -\left(\frac{A_1}{A_2}\right)^2 k^5 \frac{(1-k^2B_1)^2}{(1-k^2B_2)^2 + \frac{A_1^2}{A_2^2}k^6(1-k^2B_1)^2}$$

$$X_1 = -\frac{4}{81}\frac{a^2\tau^2c^4}{(2a-\tau)^2}k^6 + O(k^8)$$

(B. 12)

Similarly

$$y_1 = -\frac{\alpha_1\beta_1}{\alpha_1^2+\beta_1^2} = -\frac{A_1k(1-k^2B_1)(1-k^2B_2)\frac{A_2}{k^2}}{A_1^2k^2(1-k^2B_1)^2+\frac{A_2^2}{k^4}(1-k^2B_2)^2}$$

$$= -\frac{A_1}{A_2}k^3\frac{1-(B_1+B_2)k^2+O(k^4)}{1-2B_2k^2+O(k^4)}$$

$$y_1 = -\frac{A_1}{A_2}k^3\left[1-(B_1-B_2)k^2+O(k^4)\right]$$

$$y_1 = -\frac{2}{9}\frac{a\tau c^2k^3}{2a-\tau}\left[1-\frac{k^2c^2}{50}\frac{-13\tau+36a}{\tau-2a}\right]$$

(B. 13)

Using the same type of calculation for y_2 **yields**

$$y_2 = \frac{-2\tau b^2 c^4 (2a^2 + b^2)\, k^5}{15^2(-9ab^2\tau + 12a^2b^2 + 6b^4)} \Bigg[1 - \frac{1}{3}\left(\frac{k}{14}\right)^2 \frac{a\tau(-6a^6 + 133a^4b^2 - 188a^2b^4 - 2b^6) i + (-84a^6b^2 + 46a^4b^4 + 124a^2b^6 + 40b^8)}{(2a^2 + b^2)(-9ab^2\tau + 12a^2b^2 + 6b^4)} \Bigg] \qquad \text{(B.14)}$$

The result for x_2 is $O(k^8)$.

Thus the first two scattering coefficient has been evaluated in terms of physical parameters of a conductive spheroid. There are some special cases that are of interest and will be calculated subsequently.

1. Nearly Spherical Prolate Spheroid

If the prolate spheroid is nearly spherical then the parameters appearing in the scattering coefficients can be expanded into a power series of integral powers of the eccentricity. If then the eccentricity approaches zero, the spheroidal scattering coefficients will tend to the spherical scattering coefficient. This method thus provides a check on the spheroidal scattering calculations.

The eccentricity e, is defined as

$$e = \left(1 - \frac{m^2}{M^2}\right)^{\frac{1}{2}} \tag{B.15}$$

where m is the minor semiaxis and M is the major semiaxis of the spheroid. With this definition, for the prolate spheroid,

$$\begin{aligned} c &= ae \\ b^2 &= a^2(1-e^2) \\ \tau &= 2a\left(1 - \frac{1}{3}e^2 - \frac{1}{45}e^4 - \frac{44}{945}e^4\right) \end{aligned} \tag{B.16}$$

Substituting (B.16) into (B.8)

$$\frac{A_1}{A_2} = \frac{2}{9}\,\frac{a\tau c^2}{2a-\tau} = \frac{2}{9}\,\frac{a\,2a\left(1-\frac{1}{3}e^2\right)a^2e^2}{2a-2a\left(1-\frac{1}{3}e^2-\frac{1}{45}e^4\right)} =$$

$$= \frac{2}{9} a^3 \frac{e^2\left(1 - \frac{1}{3}e^2\right)}{e^2\left(\frac{1}{3} + \frac{1}{45}e^2\right)} = \frac{2}{3} a^3 \frac{1 - \frac{1}{3}e^2}{1 + \frac{1}{15}e^2}$$

$$\frac{A_1}{A_2} = \frac{2}{3} a^3 \left(1 - \frac{2}{5} e^2\right) \tag{B. 17a}$$

Similarly after algebraic manipulations we have

$$B_1 - B_2 = -\frac{3}{10} a^2 \left(1 + \frac{4}{5} e^2\right) \tag{B. 17b}$$

$$\frac{A_3}{A_4} = \frac{1}{30} a^5 \left(1 - \frac{25}{21} e^2\right) \tag{B. 17c}$$

$$B_3 + B_4 = \frac{24}{14^2} a^2 \left(1 - \frac{2515}{3024} e^2\right) \tag{B. 17d}$$

These results are then substituted into the x's and y's to obtain the nearly spheroidal prolate scattering coefficients. The coefficients are tabulated in Table 2. Note as $e \to 0$, the results agree with known values of spherical scattering coefficients. (Jones, 1966)

2. Nearly Spherical Oblate Spheroid.

For an oblate spheroid the eccentricity is defined, using the definition (B. 15), as

$$e = \left(1 - \frac{a^2}{b^2}\right)^{\frac{1}{2}}$$

Thus

$$c^2 = -b^2e^2$$

$$a^2 = b^2(1-e^2)$$

$$\tau = 2b\left(1 - \frac{1}{6}e^2 - \frac{17}{360}e^4 - \frac{367}{15120}e^6\right) \tag{B.18}$$

Using (B. 18) yields

$$\frac{A_1}{A_2} = \frac{2}{3}b^3\left(1 - \frac{9}{10}e^2\right) \tag{B.19a}$$

$$B_1 - B_2 = -\frac{3}{10}b^3\left(1 - \frac{9}{10}e^2\right) \tag{B.19b}$$

$$\frac{A_3}{A_4} = \frac{1}{30}b^5\left(1 - \frac{407}{224}b^2\right) \tag{B.19c}$$

$$B_3 + B_4 = \frac{24}{14^2}b^2\left(1 - \frac{1063}{3024}e^2\right) \tag{B.19d}$$

Results for the scattering coefficients for oblate, nearly spherical, spheroid are listed in Table 2. As the last special case, we discuss the case of nearly disk type scatterer.

3. Nearly Disk Type Scatterer.

For a disk, define the quantity

$$\epsilon = \frac{a}{b} \quad , \tag{B.20}$$

a parameter, that tends to zero for a perfect disk. Using the definition (B.20) and

$$\Upsilon = \frac{4b}{\pi}\left(1 + \frac{2\epsilon}{\pi}\right) \tag{B.21}$$

we have

$$\frac{A_1}{A_2} = \frac{2}{9}\,\epsilon\, b^2 \tag{B.22a}$$

$$B_1 - B_2 = \frac{13}{50}\, b^2\left(1 - \frac{5}{26}\pi\epsilon\right) \tag{B.22b}$$

$$\frac{A_3}{A_4} = \frac{4}{3}\,\frac{1}{15^2\pi}\, b^5\left(1 + \frac{8}{\pi}\epsilon\right) \tag{B.22c}$$

$$B_3 + B_4 = \frac{5b^2}{3(7^2)}\left(1 + \frac{29}{5\pi}\epsilon\right) \tag{B.22d}$$

The disk type scattering coefficients are listed in Table 2.

Appendix C. Calculations relating to Eq. (5.11)

Here the details for the calculation of the scattering amplitude is presented. Based on the formulae of Chapter 5 and Appendix B, the quantities

$$\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\left\{(x_i + jy_i)(\alpha_{1i} + j\beta_{1i})\right\}$$

and

$$\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\left\{g_i(\theta, k, A, B, a, b)\right\}$$

are calculated.

$$x_1\alpha_{11} - y_1\beta_{11} \doteq - y_1\beta_{11} = -\frac{1}{3}\,\frac{a\tau}{BT}\,\frac{AT-2B^2}{2a-\tau}\,k\left\{1-\frac{k^2}{50}\left[\frac{AT(5B^2-C^2)-B^2(33c^2+5A^2+5B^2)}{AT-2B^2} + \frac{36ac^2-13c^2\tau}{\tau-2a}\right]\right\}$$

$$= -\frac{1}{3}\,\frac{a\tau}{BT}\,\frac{AT-2B^2}{2a-\tau}\,k\left\{1-\frac{k^2}{50}\,\frac{AT(\tau-2a)(5B^2-c^2)-B^2(\tau-2a)(33c^2+5A^2+5B^2)+(36ac^2-13c^2\tau)(AT-2B^2)}{(AT-2B^2)(\tau-2a)}\right\}$$

But

$$I = AT(5B^2\tau - 10aB^2 - c^2\tau + 2ac^2) - B^2(33c^2\tau + 5A^2\tau + 5B^2\tau - 66ac^2 - 10aA^2 - 10aB^2) +$$

$$+ 36aAc^2T - 72aB^2c^2 - 13Ac^2\tau T + 26B^2c^2\tau$$

$$= -10aAB^2T + 2aAc^2T + 66aB^2c^2 + 10aA^2B^2 + 10aB^4 + 36aAc^2T - 72aB^2c^2 + 38aAc^2T - 6aB^2c^2 +$$

$$+ 5AB^2\tau T - Ac^2\tau T - 5A^2B^2\tau - 13Ac^2\tau T - 33B^2c^2\tau - 5B^4\tau + 26B^2c^2\tau - 14Ac^2\tau T - 7B^2c^2\tau$$

$$= a(-10AB^2T + 10A^2B^2 + 10B^4 + 38Ac^2T - 6B^2c^2) + A(5B^2\tau T - 5AB^2\tau - 14c^2\tau T) +$$

$$+ B^2(-5B^2\tau - 7c^2\tau)$$

$$= 2a(-5AB^2T + 5A^2B^2 + 5B^4 + 19Ac^2T - 3B^2c^2) + A\tau(5B^2T - 5AB^2 - 14c^2T) + B^2\tau(-5B^2 - 7c^2)$$

Therefore,

$$x_1\alpha_{11} - y_1\beta_{11} \doteq -\frac{1}{3}\frac{a\tau}{BT}\,\frac{AT-2B^2}{2a-\tau}\,k\left\{1 - \right. \tag{C.1}$$

$$\left. -\frac{k^2}{50}\left[\frac{2a(-5AB^2T+5A^2B^2+5B^4+19Ac^2T-3B^2c^2)+A\tau(5B^2T-5AB^2-14c^2T)+B^2\tau(-5B^2-7c^2)}{(AT-2B^2)(\tau-2a)}\right]\right\}$$

$$y_1\alpha_{11} + x_1\beta_{11} = \frac{2}{9}\,\frac{a\tau c^2 k^3}{2a-\tau}\,\frac{kB}{3} + \frac{4}{81}\,\frac{a^2\tau^2c^4k^6}{(2a-\tau)^2}\,\frac{3}{2}\,\frac{(AT-2B^2)c^2k^4}{c^2BTk^2} + O(k^6)$$

$$= \frac{2}{27}\,\frac{ABc^2\tau}{2a-\tau}k^4 + \frac{2}{27}\,\frac{a^2c^2\tau^2(AT-2B^2)}{BT(2a-\tau)^2}k^4 = \frac{2}{27}\,\frac{ac^2\tau k^4}{2a-\tau}\left[B + \frac{a\tau(AT-2B^2)}{BT(2a-\tau)}\right]$$

$$y_1\alpha_{11} + x_1\beta_{11} = \frac{2}{27}k^4\frac{a\tau c^2}{BT}\,\frac{2aB^2(T-\tau)+\tau T(aA-B^2)}{(2a-\tau)^2}$$

$$y_2\alpha_{12} + x_2\beta_{12} \doteq y_2\alpha_{12} \doteq \frac{2\tau b^2c^4(2a^2+b^2)k^5}{15^2(-9ab^2\tau+12a^2b^2+6b^4)}\,\frac{k^2AB}{15}\left[1 + \right. \tag{C.2}$$

$$\left. + \frac{k^2}{98}(4c^2-7A^2)\right]\left[1-\left(\frac{k}{14}\right)^2\frac{a\tau(-6a^6+133a^4b^2-188a^2b^4-2b^6)-84a^6b^2+46a^4b^4+124a^2b^6+40b^6}{(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right]$$

$$= \frac{2Ab^2Bc^4\tau\,(2a^2+b^2)\,k^7}{15^3(-9ab^2\tau+12a^2b^2+6b^4)}\left\{1+\left(\frac{k}{14}\right)^2\left[\frac{(4c^2-7A^2)^2}{98}\right.\right. -$$

$$\left.\left. - \frac{a\tau(-6a^6+133a^4b^2-188a^2b^4-2b^6)-84a^6b^2+46a^4b^4+124a^2b^6+40b^8}{14^2(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right]\right\}$$

But

$$(8c^2-14A^2)(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4) =$$

$$= (16a^2c^2-24a^2A^2-8b^2c^2-14A^2b^2)(-9b^2a\tau+12a^2b^2+6b^4) =$$

$$= -a\tau(16a^2b^2c^2-24a^2b^2A^2-8a^2b^4+8b^6-14A^2b^4) +$$

$$+ 6b^2(2a^2+b^2)(16a^4-16a^2b^2-8a^2b^2+8b^6-24a^2A^2-14A^2b^2)$$

$$y_2\alpha_{12} + x_2\beta_{12} \doteq \frac{2\tau b^2c^4(2a^2+b^2)AB\,k^7}{15^3(-9ab^2\tau+12a^2b^2+6b^4)}\left\{1+\left(\frac{k}{14}\right)^2\left[\frac{-a\tau(-6a^6+133a^4b^2-188a^2b^4-2b^6+16a^4b^2)}{(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right.\right.$$

$$\left.\left.\frac{-a\tau(-16a^2b^4-8a^2b^4+8b^6-24a^2A^2b^2-14A^2b^4)+2b^2(-42a^6+23a^4b^2+62a^2b^4+20b^6)+6b^2(32a^6-32a^4b^2-48a^4A^2-28a^2A^2b^2)}{(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right]\right\}$$

$$= \frac{2Ab^2Bc^2(2a^2+b^2)k^7}{15^3(-9ab^2\tau+12a^2b^2+6b^4)}\left\{1+\left(\frac{k}{14}\right)^2\left[\frac{-a\tau(-6a^6+149a^4b^2-212a^2b^4+6b^6-24a^2A^2b^2-14A^2b^4}{(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right.\right.$$

$$\left.\left.+\frac{2b^2(54a^6-73a^4b^2+36a^2b^4+44b^6-144a^4b^2-156a^2A^2b^2-42A^2b^4)}{(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right]\right\}$$

$$x_2\alpha_{12} - y_2\beta_{12} \doteq -y_2\beta_{12} \doteq \frac{2\tau b^2c^4(2a^2+b^2)k^5c^2k^2}{15^2(-9ab^2\tau+12a^2b^2+6b^4)}\,\frac{15}{2}\,\frac{(3B^2-c^2)T-6AB^2}{c^2TBk^3} \qquad \text{(C.3)}$$

$$\left\{1-\left(\frac{k}{14}\right)^2\left[\frac{a\tau(-6a^6+133a^4b^2-188a^2b^4-2b^6)-84a^6b^2+46a^4b^4+124a^2b^6+40b^8}{(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right]\right\}$$

$$\left\{1-\left(\frac{k}{14}\right)^2\left[\frac{T(21A^2B^2+5B^2c^2-3c^2)-AB^2(17c^2+21A^2+21B^2)}{(3B^2-c^2)T-6AB^2}\right]\right\}$$

$$x_2\alpha_{12} - y_2\beta_{12} \doteq - \frac{\tau b^2c^2(2a^2+b^2)\left[T(3B^2-c^2) - 6AB^2\right]}{15TB(-9ab^2\tau + 12a^2b^2 + 6b^4)}\, k^2 \Bigg\{ 1 - \left(\frac{k}{4}\right)^2 \left[\frac{T(21A^2B^2 + 5B^2c^2 - 3c^4) - AB^2(17c^2 + 21A^2 + 21B^2)}{(3B^2-c^2)T - 6AB^2} + \frac{a\tau(-6a^6 + 133a^4b^2 - 188a^2b^4 - 2b^6) - 84a^6b^2 + 46a^4b^4 + 124a^2b^6 + 40b^8}{(2a^2+b^2)(-9ab^2\tau + 12a^2b^2 + 6b^4)} \right] \Bigg\} \qquad (C.4)$$

$$\mathrm{Re}\; g_1 = \frac{3}{2}\,\frac{\sin\theta}{B}\left[1+\frac{k^2c^2}{10}\left(-\frac{8}{7}+\sin^2\theta\right)\right]\left[x_1\alpha_{11}-y_1\beta_{11}\right]$$

$$\mathrm{Re}\; g_1 = -\frac{1}{3}\,\frac{a\tau}{B^2T}\,\frac{AT-2B^2}{2a-\tau}\,\frac{3}{2}\,k\sin\theta\left\{1+\frac{k^2}{50}\left[-\frac{40}{7}+5\sin^2\theta-\right.\right.$$

$$\left.\left.-\frac{2a(-5AB^2T+5A^2B^2+5B^4+19Ac^2T-3B^2c^2)+A\tau(5B^2T-5AB^2-14c^2T)+B^2\tau(-5B^2-7c^2)}{(AT-2B^2)(\tau-2a)}\right]\right\} \tag{C. 5}$$

$$\mathrm{Im}\; g_1 = \frac{3}{2}\,\frac{\sin\theta}{B}\left[1+\frac{k^2c^2}{10}\left(-\frac{8}{7}+\sin^2\theta\right)\right]\left[y_1\alpha_{11}+x_1\beta_{11}\right]$$

$$= -\frac{2}{27}\,k^4\,\frac{a\tau c^2}{BT}\,\frac{2aB^2(T-\tau)+\tau T(aA-B^2)}{(2a-\tau)^2}\,\frac{3}{2}\,\frac{\sin\theta}{B}$$

$$\mathrm{Im}\; g_1 - \frac{1}{9}\,k^4\,\frac{a\tau c^2}{B^2T}\,\frac{2aB^2(T-\tau)+\tau T(aA-B^2)}{(2a-\tau)^2}\,\sin\theta \tag{C. 6}$$

$$\mathrm{Re}\, q_2 = \frac{5}{4}\,\frac{\sin\theta}{B}\left[1+\frac{c^2k^2}{14}\left(-\frac{8}{9}+\sin^2\theta\right)\right]\left[x_2\alpha_{12}-y_2\beta_{12}\right]$$

$$= \frac{5}{4}\,\frac{\sin\theta}{B^2}\,\frac{1}{15}\left(\frac{\tau}{T}\right)\frac{b^2c^2(2a^2+b^2)\left[T(3B^2-c^2)-6AB^2\right]}{B(-9ab^2\tau+12a^2b^2+6b^4)}\,k^2\left[1+\frac{c^2k^2}{14}\left(-\frac{8}{9}+\sin^2\theta\right)\right]$$

$$\left\{1-\left(\frac{k}{14}\right)^2\left[\frac{T(21A^2B^2+5B^2c^2-3c^4)-AB^2(17c^2+21A^2+21B^2)}{(3B^2-c^2)T-6A^2B^2}\right.\right.$$

$$\left.\left.+\frac{a\tau(-6a^6+133a^4b^2-188a^2b^4-2b^6)-84a^6b^2+46a^4b^4+124a^2b^6+40b^8}{(2a^2+b^2)(-9ab^2\tau+12a^2b^2+6b^4)}\right]\right\} \tag{C.7}$$

$$\mathrm{Im}\, q_2 \doteq \frac{5}{4}\,\frac{\sin^2\theta}{B}\left[1+\frac{c^2k^2}{14}\left(-\frac{8}{9}+\sin^2\theta\right)\right]\left[y_2\alpha_{12}+x_2\beta_{12}\right]$$

$$\mathrm{Im}\, q_2 \doteq \frac{1}{1350}\,\frac{AB^2c^4(2a^2+b^2)k^7\sin^2\theta}{(-9ab^2\tau+12a^2b^2+6b^4)} \tag{C.8}$$

Appendix D The Theory of Complex Resolvents

In order to gain physical insight into the theory of complex resolvent, first a heuristic discussion of a real resolvent operator is presented. Then a theorem on complex resolvents is noted.

Let

$$(L - \lambda)\, u(x) = -\, W(x) \tag{D.1}$$

be an equation where L is an ordinary differential operator with domain including the boundary condition and λ as its eigenparameter with source function W(x). The resolvent integral operator is then defined as

$$R_\lambda = (\lambda - L)^{-1} \quad \text{such that}$$

$$u(x) = R_\lambda W(x) \tag{D.2}$$

and

$$u(x) = -\frac{1}{2\pi j} \oint_\Gamma R_\lambda\, u(x)\, dx \tag{D.3}$$

To note the significance of these definitions, a connection between the resolvent operator and the resolvent Green's function will now be established.

Following Friedman [60], consider the homogeneous equation

$$L\, u(x) = \lambda\, u(x) \tag{D.4}$$

whose eigen function are $u_1, u_2, \ldots u_k$ with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$.

Consider the adjoint operator L^* with eigen functions $v_1, v_2, \ldots v_k$ and the same eigenvalues as L.

Let

$$u(x) = \sum_k \alpha_k u_k(x) \tag{D. 5}$$

where

$$\alpha_k = (v_k, u)$$

Applying the operator L to (D. 5) yields

$$L u(x) = \sum_k \alpha_k L u_k(x) = \sum_k \alpha_k \lambda_k u_k(x) \tag{D. 6}$$

Let f(t) be an analytic function of t.

Define

$$f(L)\, u(x) \triangleq \sum_k \alpha_k f(\lambda_k)\, u_k(x) \tag{D. 7}$$

Note that if f(t)=t (D. 7) reduces to (D. 4). Let f(t)= $\frac{1}{\lambda - t}$ such that (D. 7) is in the form

$$\frac{1}{\lambda - L} u(x) = \sum_k \alpha_k \frac{1}{\lambda - \lambda_k} u_k(x) \tag{D. 8}$$

Multiply by $\frac{d\lambda}{2\pi j}$ and integrate

$$\frac{1}{2\pi j} \oint_\Gamma \frac{d\lambda}{\lambda - L} u(x) = \sum_k \alpha_k u_k(x) \frac{1}{2\pi j} \oint_\Gamma \frac{d\lambda}{\lambda - \lambda_k} \tag{D. 9}$$

If Γ encloses all the eigenvalues of L, then (D. 9) reduces to (D. 3).

It is known, however, that

$$u(x) = R_\lambda w(x) = -\int_\xi G(x,\xi,\lambda)\, w(\xi)\, d\xi \tag{D. 10}$$

where $G(x,\xi,\lambda)$ is the resolvent Green's function of the boundary value problem. Thus

$$-\frac{1}{2\pi i}\oint_\Gamma R_\lambda w\, d\lambda = \frac{1}{2\pi i}\oint_\Gamma d\lambda \int_\xi G(x,\xi,\lambda)\, w(\xi)\, d\xi \tag{D. 11}$$

is the connection sought.

Having discussed some aspects of the resolvent operator, we now turn our attention to the resolvent operator to our boundary value problem. Note that (5.2) can be rewritten in the following form

$$\left(-L_\xi - L_\eta\right) H_\varphi = J(\xi',\eta')\,\delta(\eta'-1)\,\delta(\xi'-\xi_1) \tag{D. 12}$$

where

$$\begin{aligned} L_\xi &= -\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}\right] + \left[-\gamma_+^2(\xi^2-1) + \frac{1}{\xi^2-1}\right] \\ L_\eta &= -\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}\right] + \left[-\gamma_+^2(1-\eta^2) + \frac{1}{1-\eta^2}\right] \end{aligned} \tag{D. 13}$$

To construct complex resolvent operators to our boundary value problem the following theorem from [61] is noted.

<u>Theorem</u>

Let L be a formal differential operator defined by the identity

$$Ly = -\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y$$

$$(-\infty \le a \le x \le b \le \infty) \qquad \text{(D. 14)}$$

where p(x) is real, positive on the interval (a, b) and q(x) is a complex valued function such that $\operatorname{Im} q(x) \geq q_0$ on (a, b). Hereafter, the class of functions f that is absolutely square in $\int_a^b |f|^2 dx$ on the interval (a, b) will be denoted $\mathcal{L}^2(a,b)$. To construct a resolvent operator R_λ for the operator L there are two cases of interest;

Case I. Let

(a) $-\infty < a$ and $b = \infty$

(b) $p(a) \neq 0$

(c) for $\operatorname{Im} \lambda < q_0$

The homogeneous equation $(L - \lambda)\,y=0$ has exactly one linearly independent solution which is in $\mathcal{L}^2(a, \infty)$.

Under these conditions the resolvent R_λ has the representation

$$R_\lambda y = \int_a^\infty G(x, \xi, \lambda)\, y(\xi)\, d\xi \qquad \text{(D. 15)}$$

where the resolvent Green's function is defined by the formula

$$G(x, \tau, \lambda) = \frac{1}{p(x)\,W(y_1, y_2, \lambda)} \begin{cases} y_1(x)\, y_2(\tau) & x < \tau \\ y_1(\tau)\, y_2(x) & x > \tau \end{cases}$$

In this equation $(L-\lambda)y_i = 0$ with $i = 1, 2$, and y_1 satisfies the boundary condition at a, y_2 satisfies the radiation condition, and $W(y_1, y_2, \lambda)$ is the Wronskian of y_1, y_2 considered as a function of Υ.

Case II. Let

(a) $-\infty < a < b < \infty$

(b) $p(a) = p(b) = 0$

where a, b are singular points for $(L-\lambda)y = 0$

The representation of the resolvent and the resolvent Green's function follows the previous case, where now the boundary conditions are the finiteness of the solution at a and b. We here note that the radial operator falls under Case I and the angular operator under Case II.

Appendix E. Uniformly Valid Asymptotic Approximation

Here some background is presented for the topic of uniformly valid asymptotic approximation. First, the failure of the WKBJ (Wantzel-Kramer-Brilluoin-Jeffreys) approximation is noted for the case where there is a simple zero in the index of refraction, and then Langer's "turning point" analysis is briefly described.

E.1 WKBJ Method

Consider the following differential equation

$$\frac{d^2w}{dz^2} + k_0^2 N^2(z) w(z) = 0 \tag{E.1}$$

This equation can be interpreted as an inhomogeneous wave equation with the index of refraction N(z) and wavenumber k_0. In such case it is of interest to find asymptotic high frequency solution to (E.1), i.e. $k_0 \to \infty$.

Such solution is the WKBJ approximation

$$w(z) \sim \frac{A}{\sqrt{|N(z)|}} e^{\pm j k_0 \int^{z} |N(\tau)| d\tau} \tag{E. 2}$$

As an example to WKBJ theory, consider the Bessel equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)u = 0 \tag{E. 3}$$

If we let $u = w z^{\frac{1}{2}}$, then

$$\frac{d^2w}{dz^2} + \left[\nu^2\left(1 - \frac{1}{z^2}\right) + \frac{1}{4z^2}\right] w = 0 \tag{E. 4}$$

Thus for z not close to 1, the $\frac{1}{4z^2}$ term can be neglected and then (E. 4) is in the form of (E. 1). Using now the WKBJ theory, the large order Bessel function can be evaluated. For $z > 1$

$$w(z) \sim \frac{A}{(\sin\beta)^{\frac{1}{2}}} e^{\pm j\nu(\tan\beta - \beta)}$$

where $z = \sec\beta$, or

$$J_\nu(\nu\sec\beta) \sim \frac{1}{(\tan\beta)^{\frac{1}{2}}}\left[A_1 e^{j\nu(\tan\beta-\beta)} + B e^{-j\nu(\tan\beta-\beta)}\right] \tag{E. 5}$$

To evaluate the constants A_1 and B, note that if ν is finite and $z \to \infty$ the far field form of the Bessel function is

$$J_\nu(\nu z) \sim \left(\frac{2}{\pi\nu z}\right)^{\frac{1}{2}} \cos\left(\nu z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{E. 6}$$

Matching (E. 5) to (E. 6) as $z \to \infty$ requires that

$$A_1 e^{i\frac{\pi}{4}} = B e^{-i\frac{\pi}{4}} = \left(\frac{1}{2\pi\nu}\right)^{\frac{1}{2}}$$

Substituting in (E. 5) we have

$$J_\nu(\nu z) \sim \left(\frac{2}{\pi\nu\sqrt{z^2-1}}\right)^{\frac{1}{2}} \cos\left[\nu\left(\sec^{-1}z - \sqrt{z^2-1}\right) + \frac{\pi}{4}\right] \quad \text{(E. 7)}$$

which is known as Meissel's formula. Similarly for $z < 1$, we can match the WKBJ form of solution with the known Bessel function of small argument, i. e. $z \to 0$. The result is the Carlini's formula

$$J_\nu(\nu z) \sim \left(\frac{1}{2\pi\nu\sqrt{1-z^2}}\right)^{\frac{1}{2}} e^{\nu\left(\sqrt{1-z^2} - \mathrm{sech}^{-1}z\right)} \quad \text{(E. 8)}$$

Comparing the two forms of the large order Bessel function solution, we observe that (a) the two forms are different in different regions, and (b) both are unbounded as $z \to 1$. Thus a simple zero in the index of refraction represents a "turning point" in the behavior of the asymptotic solution. A solution that is finite and uniformly valid both above and below the turning point is called a uniformly valid asymptotic solution. The method that obtains such solution is called, after its inventor, the Langer's method.

E. 2 Langer's Method

The essence of the method is based on the following two observations;

(a) If the turning point is at $z = o$, say, then near that zero (E. 1) can be put in a form

$$\frac{d^2w}{dz^2} + k_o^2 N^2(o) z w = 0 \quad \text{(E. 9)}$$

which is the Airy equation. Therefore the solution of (E. 1) near the turning point will behave like Airy equation.

(b) The second observation is that the Airy differential equation

$$\frac{d^2y}{dx^2} + xy = 0 \tag{E. 10}$$

has a uniformly valid solution above and below $x = 0$. The method is thus to change the form of (E. 1), by change of both dependent and independent variables such that the new equation will conform to these requirements.

Let

$$w(z) = g(z)\,v(\xi) \tag{E. 11}$$

Substituting (E. 11) into (E. 1) yields

$$g\xi'^2 \frac{d^2v}{d\xi^2} + \left(g\xi'' + 2g'\xi'\right)\frac{dv}{d\xi} + \left(g'' + k_0^2 N^2 g\right)v = 0 \tag{E. 12}$$

Choose g so that $\frac{dv}{d\xi}$ term vanishes, that is

$$g\xi'' + 2g'\xi' = 0 \tag{E. 13}$$

and let the independent variable be determined by $\xi'^2\xi = N^2$. (E. 14)

Therefore

$$g = \frac{A_2}{\sqrt{\xi'}} \qquad \text{where} \qquad \xi'^2 = \frac{N^2}{\xi} \tag{E. 15}$$

Or

$$\xi^{\frac{1}{2}}\xi' = N \qquad \frac{2}{3}\,\xi^{\frac{3}{2}} = \int_0^z N(\tau)\,d\tau \tag{E. 16}$$

Note, for small z, $N(z) = N_o z^{\frac{1}{2}}$ and therefore $\xi \doteq N_o^{\frac{2}{3}} z$

Hence ξ corresponds to z near the turning point. By the use of condition (E. 13) and (E.14) we have

$$\frac{d^2 v}{d\xi^2} + \left(\frac{g''}{g\xi'^2} + k_o^2 \xi \right) v(\xi) = 0 \qquad \text{(E. 17)}$$

Where

$$W = A_3 \left(\frac{\xi}{N^2} \right)^{\frac{1}{4}} v(\xi) \qquad \text{(E. 18)}$$

Let $v = v_o(\xi)$ be solution of the equation

$$\frac{d^2 v_o}{d\xi^2} + k_o^2 \xi v_o(\xi) = 0 \qquad \text{(E. 19)}$$

Try a possible solution to (E. 18) as

$$v(\xi) = v_o(\xi) + \sum_{m=1}^{\infty} \frac{1}{k_o^{2m}} \left[a_m(\xi) v_o(\xi) + b_m(\xi) \dot{v}_o(\xi) \right] \qquad \text{(E. 20)}$$

It can be shown that under certain conditions, not only the series is asymptotic for complex values of ξ but also the error caused by terminating the series at the m-th term is $O\left(\frac{1}{k_o^{2m+1}}\right)$ independent of z. Moreover

$$\dot{v}(\xi) = v_o(\xi) \sum_{m=1}^{\infty} \frac{1}{k_o^{2m}} \left(\dot{a}_m - \xi b_{m+1} \right) + \dot{v}_o(\xi) \left[1 + \sum_{m=1}^{\infty} \frac{1}{k_o^{2m}} \left(a_m + \dot{b}_m \right) \right] \qquad \text{(E. 21)}$$

i.e. $\dot{v}(\xi)$ is given by taking the derivative of (E. 20) term by term.

Restricting our attention to the first term, w will be evaluated. Let

$$\xi = k_0^{-\frac{2}{3}} \eta \quad \text{, then}$$

$$\frac{d^2 v_0}{d\eta^2} + \eta v_0(\eta) = 0 \tag{E.22}$$

Let $2\eta^{\frac{3}{2}} = 3x$, $v_0 = x^{\frac{1}{3}} y$, then substituting in (E.22) yields

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{1}{9x^2}\right) y = 0 \tag{E.23}$$

This is a Bessel equation of order 1/3 with solution $J_{\frac{1}{3}}(x)$, $J_{-\frac{1}{3}}(x)$, $H_{\frac{1}{3}}(x)$ etc. It is convenient to define the solutions of (E.22), the Airy functions,as

$$Ai\left(\eta e^{j\frac{\pi}{3}}\right) \triangleq -j \frac{\eta^{\frac{1}{3}}}{2\sqrt{3}} H_{\frac{1}{3}}^{(2)}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right) \tag{E.24}$$

$$Ai\left(\eta e^{-j\frac{\pi}{3}}\right) \triangleq j \frac{\eta^{\frac{1}{3}}}{2\sqrt{3}} H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right).$$

Thus the uniformly valid asymptotic solutions of (E. 1) are

$$w \sim A\left(\frac{\xi}{N^2}\right)^{\frac{1}{4}} Ai^{(i)}\left(\eta e^{\pm j\frac{\pi}{3}}\right) \tag{E.25}$$

As an example to the theory, the uniformly valid asymptotic approximation of the Bessel equation (E. 4) is evaluated. The zero in this instance is at

$z = 1$. Thus by (E. 25) the solution of (E. 4) is

$$w \sim \left(\frac{\xi}{1-\frac{1}{z^2}}\right)^{\frac{1}{4}}\left[C\,Ai\left(\nu^{2/3}\xi\, e^{j\frac{\pi}{3}}\right) + D\,Ai\left(\nu^{2/3}\xi\, e^{-j\frac{\pi}{3}}\right)\right] \tag{E. 26}$$

where

$$\frac{2}{3}\xi^{\frac{3}{2}} = \int_1^z \left(1-\frac{1}{z^2}\right)^{\frac{1}{2}} dz = \left(z^2-1\right)^{\frac{1}{2}} - \sec^{-1} z \qquad z > 1\,, \quad \xi > 0$$

and

$$\frac{2}{3}\left(-\xi\right)^{\frac{3}{2}} = -\int_1^z \left(\frac{1}{z^2}-1\right)^{\frac{1}{2}} dz = \operatorname{sech}^{-1} z - \left(1-z^2\right)^{\frac{1}{2}} \qquad z < 1\,, \quad \xi < 0$$

By the asymptotic value $z \longrightarrow \infty$ of Airy function and the Bessel function, the constants C and D can be evaluated. The result is

$$w \sim 2^{\frac{3}{2}}\nu^{-\frac{1}{3}} e^{j\frac{\pi}{3}}\left(\frac{\xi}{z^2-1}\right)^{\frac{1}{4}} Ai\left(\nu^{\frac{2}{3}}\xi\, e^{j\frac{\pi}{3}}\right) \tag{E. 27}$$

This formula can be checked for $z \rightarrow 0$ and it does reduce to the known small argument Bessel function. More detail on Airy functions and uniformly valid asymptotic approximations of Bessel function can be found in the book by Jones "Theory of Electromagnetism". McMillan Co. New York 1964.

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