Pedagogical Content Knowledge and Scaffolds:
Measuring Teacher Knowledge of Equivalent Fractions
in a Didactic Setting

by

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Abstract

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Alexis Chestnut-Andrews

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The purpose of this study was to document the mathematical knowledge and pedagogical content knowledge teachers used while tutoring elementary school students in equivalent fractions. Specifically, the study defined the scaffolding procedures used by more knowledgeable classroom teachers (those with excellent knowledge of equivalent fractions) in working with individual students. An additional purpose of this study was to define the scaffolding procedures used by more knowledgeable and less knowledgeable classroom teachers in working with individual students. Using a three-phase assessment process developed in this study, the cognitive and teaching patterns of teachers with more pedagogical content knowledge were compared with the cognitive and teaching patterns of teachers with less pedagogical content knowledge. The study combined techniques (cognitive analyses) used in expert/novice research in cognitive psychology, work on assessments (standardized assessments and interview protocols) of teacher content knowledge, and transcripts of individual tutoring scores.

Sixty elementary teachers from a large suburban school district participated in a summer mathematics staff development program. In Stage I, as part of the staff development program, fifty-three teachers completed the twenty-item Equivalence Assessment A and a Demographic Questionnaire. Thirty-two teachers volunteered for Stage II of the study in which the teachers completed Equivalence Assessment B, a Written Concept Map and a Verbal Think Aloud Concept Map I. The verbal concept map was modeled after a “think aloud” interview, where

teachers were asked to state their ideas about the concepts they believed are important. Teachers' verbal concept maps were audio taped, transcribed and analyzed.

Ten teachers were selected for Stage III, five more knowledgeable and five less knowledgeable. They were audio taped during tutoring sessions with individual students on equivalent fractions. After the tutoring sessions were complete, teachers constructed a verbal concept map using two of their transcribed tutoring scripts. These tutoring sessions were examined for pedagogical content scaffolds and these results were compared with the teachers' knowledge of equivalent fractions. Pedagogical content scaffolds were defined by 1) types of models used (area, set or linear); 2) model translations (partitioning, fair shares, fraction strips); 3) use of relationships (part/whole, part/part and relationships among different fractions) and 4) instructional strategies (conceptual or numeric).

The results show that teachers who possess more knowledge about equivalent fractions are able to demonstrate that knowledge across both traditional (Equivalence Assessment A and B) and conceptual (Written Concept Map and Verbal Think Aloud Map I and II) measures as well as instructional (Tutoring Transcripts) measures. Additionally, case studies of four teachers were developed and their data was examined for differences across the measures.

The case studies allowed the differences between the teachers to be examined closely. Teachers with more knowledge were able to demonstrate their knowledge both in assessments of their own knowledge and in their instructional practice. The teachers with more knowledge demonstrated a deeper understanding of equivalent fractions evidenced by their highly connected concept maps, lack of errors in their knowledge base and their ability to use a variety of representations of models when tutoring a student. On the other hand, teachers with less knowledge performed poorly across all measures. The teachers with less knowledge achieved lower scores on Equivalence Assessment A and B (traditional assessments) and the Written Concept Map. All their concept maps contained errors and misconceptions, as well as having fewer categories and fewer pedagogical content scaffolds during tutoring instruction.

The three phase assessment process developed in this study supported the comparison of the cognitive and teaching patterns of these with more pedagogical content knowledge and with less pedagogical content knowledge. In particular, the development of pedagogical content scaffolds used in tutoring and the scoring procedures of the tutoring transcripts themselves are a contribution to the research in these areas.

Acknowledgements

I have many people to extend my thanks to during this endeavor of completing this dissertation. First, I must sincerely thank Dr. Carol Tittle, my adviser through almost nine years of constant work for her humor, insight and incredible intelligence. I could not have completed my program without her continuous encouragement. Additionally, I must thank Dr. Helen Johnson for both encouraging me to enter this program and supplying some of the best instruction I received at the Graduate Center. Dr. Johnson also served on my committee along with Dr. Hope Hartman and I must thank them both for their time and constructive suggestions in this project. Special thanks to Dr. Gerard Nuesell for his encouragement and his humor during our time in graduate school together.

To my family, I must express my gratitude for accepting gracefully the coursework, the papers and exams that kept me from attending many functions over the years. Without your support, my success would have been bittersweet.

Finally, to all those teachers struggling to give their students the best educational experience possible, I applaud you. I know how difficult the task is and what little thanks you receive.

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CHAPTER 1

Introduction

Effective teaching in mathematics has been the subject of research, teacher development and national debate for many years. Since the 1980's, research on teaching has focused on defining the knowledge teachers need to be effective within the instructional setting, including knowledge of pedagogy, subject matter and pedagogical content knowledge. Teachers' instructional settings frequently include one-on-one teaching. These tutoring sessions can also be studied for examples of particular teacher knowledge – pedagogical, subject matter and pedagogical content knowledge. Classroom tutorial sessions offer unique research opportunities.

The purpose of the present study is to describe and contrast the in-class teacher tutoring sessions of more knowledgeable and less knowledgeable teachers. The study analyzed the specific forms of knowledge used by elementary teachers tutoring students in fractional equivalence. As Hiebert, Gallimore and Stigler (2002) suggested, specific examples of teacher practice will contribute to the needed knowledge base for teacher development of effective teaching of elementary school mathematics.

Pedagogical Content Knowledge

Pedagogical content knowledge was first proposed by Shulman (1986) and developed with colleagues in the *Knowledge Growth in Teaching* project as a broader perspective model for understanding teaching and learning (Shulman & Grossman, 1988). This project studied how novice teachers acquired new understandings of their content, and how these new understandings influenced their teaching. These researchers described pedagogical content knowledge as the knowledge formed by the synthesis of

three knowledge bases: subject matter knowledge, pedagogical knowledge, and knowledge of context. Pedagogical content knowledge was unique to teachers and separated, for example, a science teacher from a scientist. Shulman (1987) defined pedagogical content knowledge as a blending of general pedagogical knowledge with content specific knowledge. General pedagogical knowledge consists of the knowledge of classroom management and common teaching strategies while content knowledge is viewed as knowledge of subject matter. In the teaching of mathematics general pedagogical knowledge involves the conceptual and procedural elements that lead to teacher actions which assist students in the development of their understanding of those elements (Kane & Nisbet, 1996).

In an effort to solidify the concept of pedagogical content knowledge in the domain of mathematics, researchers have approached the task from two main perspectives. The first view focuses its attention on the pedagogical content knowledge in the domain of mathematics as being framed by the instruction that is specific to the subject matter (Ball, 1988). The other view structures pedagogical content knowledge around teachers' beliefs of mathematics teaching and how those beliefs interact with the mathematics those teachers are required to teach (Pajares, 1992; Raymond, 1997). However, few of these studies provide detailed information concerning these components in practicing teachers of mathematics. Recently, Ball, Lubienski and Mewborn (2001) reviewed the literature for articles exclusively on teachers' mathematical knowledge and found that only five percent of the articles probed how current teachers' mathematical understanding affected their practice. Similarly, even fewer studies (2%) examined how teachers' mathematical understanding affected their students' learning. Much of the

research is focused on preservice teacher's depth of pedagogical content knowledge in specific mathematical domains.

Teacher Knowledge and the Expert-Novice Paradigm

Research based in varying theoretical frameworks has begun to formalize the operationalization of a teacher's mathematical knowledge based on both concepts: subject matter knowledge and pedagogical content knowledge. The expert-novice framework was used by Leinhardt and Greeno (1986) to compare the difference in instructional models between expert and novice teachers. The study showed that there were distinct differences between the lesson structures of expert and novice teachers. These differences revolved mainly around the presentation of the teachers' agendas and they stated that their study focused mainly on the lesson structure portion of instructional skill. While this study is considered a classic within the framework of the expert-novice paradigm, there are two important points to consider when evaluating this study. One, the label of "expert" was made solely based on the growth of students' test scores over a five year period and "novice" teachers were selected from an undergraduate student teaching pool. Secondly, since each teacher's pedagogical content knowledge of mathematics is unique, it might have been better to analyze lessons in one common topic. This might have allowed for greater generalizations across the three teachers. Regardless, this study was one of the first that recognized the importance of connecting teacher actions to teacher subject matter knowledge.

The Expert-Novice Connection to Tutoring

The expert-novice paradigm has been used most often to compare the pedagogical content knowledge of teachers within the whole classroom setting. However, in the

setting of the whole classroom pedagogical content knowledge cannot be isolated from the pedagogical knowledge such as knowledge of students and curriculum. Therefore, researchers began analyzing instructional skills of one-to-one interactions during tutoring sessions. By linking the expert-novice paradigm to the instructional practice of tutoring, researchers were able to focus more specifically on the actions of the tutor. Researchers expanded on two main factors in the tutoring research. First, in most of the research, tutors do not have formal training in pedagogical skills. Researchers (Chi, Siler, Jeong, Yamauchi & Hausmann, 2001) analyzed the tutoring process as consisting of the pedagogical skills of knowing when to give feedback, scaffolding and explanations to undergraduate college students. Secondly, while tutors (undergraduate students) did not have expertise in pedagogical skills, most had expertise in the subject matter for which they were tutoring.

The Tutor

Grasser, Person and Magliano (1995) further identified five steps in a typical tutoring dialogue of undergraduate students that they described as a tutoring frame. The five steps include the tutor asking an initiation question, the student answering, tutor feedback, tutor scaffolding and the tutor gauging the student's understanding. While the first three steps are generally known both in classroom and tutoring research as the IRE (i.e. Initiation, Response, Evaluation), it is the tutor scaffolding that has the potential of extending student's learning.

The concept of scaffolding within an instructional environment has been discussed primarily in three types of research: human tutoring research, intelligent tutoring research and developmental research (Chi et al., 2001). It is from the

developmental research area that scaffolding has been studied as an instructional tool in educational settings.

Building on Vygotsky's definition of the zone of proximal development (ZPD), researchers began to focus on the mechanism or tool that allowed a higher level of development to exist. Wood, Bruner, and Ross (1976) originally coined the term "scaffolding" to describe the tutorial process where an adult or "expert" helps someone who is less knowledgeable. They stated the role of the expert was essential because it allows the novice to solve a problem or carry out a task that would be beyond the novice's unassisted efforts. From a Vygotskian (1978) perspective, scaffolding is a type of exchange where the adult guides the child to develop to their full potential within the zone of proximal development (ZPD). The zone of proximal development is the range between what a child can accomplish independently and what a child can accomplish with assistance. Thus, a tutoring scenario is similar to the more general adult-child interaction Vygotsky discusses. As such, it allows an in-depth look at the guidance a student receives. Where "scaffolding" is the tutorial process, a scaffold is considered to be an action by the "expert" that facilitates the "novice's" understanding. These actions can include but are not limited to hints, questioning and clarifications. The concepts of "scaffolding" and the ZPD have become important guideposts in the area of educational research. However, research focusing on the nature of scaffolds and their functions in specific elementary school contexts is limited (Bliss & Askew, 1996). Further, the introduction of concepts of subject matter knowledge (domain specificity) and pedagogical content knowledge (combined subject matter and pedagogical knowledge) are likely to be helpful, providing links to broaden the area of research on teachers.

Scaffolding and Teacher Pedagogical Content Knowledge

Providing scaffolding to children in a school environment depends heavily on the expertise of the teacher. Scaffolding requires that an "expert" have domain-specific expertise. In mathematics, the effectiveness of teaching depends a great deal on a teacher's pedagogical content knowledge (Schulman, 1986). Studies have revealed that a teacher's pedagogical content knowledge is a significant determinant of their teaching practices in mathematics (Lee, Lee, & Meadows, 2003). Research has also shown that both pedagogical knowledge and content knowledge are critical elements of effective teaching (Leinhardt, 1986). However, studies have shown there is great variability in the amount of pedagogical content knowledge a teacher brings to instruction. In Bliss and Askew's (1996) study of teachers' scaffolding in mathematics classrooms, it was found that the lack of subject matter knowledge in the domains of geometry and fractions resulted in the teachers' misunderstanding of student errors. As a result, the teachers presented few scaffolds specific to subject matter domains. The reason hypothesized was that the teachers' lack of content knowledge meant that they were only steps ahead of their students.

While research on teachers' mathematical pedagogical content knowledge has provided some foundations for teacher education research, more research with typical teachers of mathematics is necessary. There has been a difference in the settings for classroom and tutoring research. Previously, research has focused on either the whole class **or** a one-on-one tutoring scenario. The elementary classroom is one setting in which combining the expert-novice paradigm with the structural components of the recent tutoring research could prove useful. In the elementary classroom it is not uncommon for

a teacher to move from whole class to one-on- one instruction during the course of a lesson. That is, the elementary school teacher identifies a student who requires "reteaching" for a concept. The structure of the classroom permits this tutorial. However, both bodies of previous research lack a rigorous method by which teachers' knowledge of mathematics can be measured. The use of practicing teachers of mathematics as tutors in one-to-one student interactions will allow for a detailed analysis of pedagogical content knowledge. The development of a detailed assessment process will contribute to defining the teacher-tutor's subject matter knowledge of equivalent fractions.

Statement of Purpose

The purpose of this dissertation is to further develop an analysis to explain teacher pedagogical content knowledge in the context of teacher tutoring. In my pilot study, two third-grade teachers each tutored one third-grade student for three sessions in the concept of equivalent fractions. The transcripts of these tutoring sessions were analyzed. As a result, several types of scaffolds were identified and defined as indicators of teacher pedagogical content knowledge and teacher expertise. By building on the theoretical foundations of expert-novice models and tutoring research, this study can provide a model for future researchers to examine how that knowledge is presented, both in the contexts, classroom and tutoring.

Since a teacher's knowledge base is a personal and unique entity, an attempt to operationalize it is recognized as difficult. In an effort to make a meaningful comparison of teacher pedagogical content knowledge and expertise, a specific topic was selected. This study focuses on the equivalence of fractions as taught in the elementary grades within a tutoring context. Fractions offer a wide variety of concepts, relationships and

procedures from which to gather information about teacher pedagogical content knowledge. A tutoring setting was chosen based upon research that showed teachers working with individual students as an ideal setting in which to research how teachers communicate the knowledge of subject matter to their students (Shavelson & Stern, 1981).

The following questions guided the research:

1. What does pedagogical content knowledge consist of in the tutoring of equivalence of fractions? What are some examples?
2. How does pedagogical content knowledge of equivalence of fractions differ between more knowledgeable or less knowledgeable teachers?
3. What types of scaffolds are used by teachers during a tutoring session of equivalent fractions?
4. Can a three-phase assessment process of fraction equivalence contribute to understanding of the teacher knowledge base used in teaching equivalence of fractions?

Chapter 2

Review of the Literature

The following literature review examines the research in teacher knowledge and how this knowledge is used instruction. First, an overview of the research on teachers' knowledge bases is presented. Next, research on the components of teacher knowledge is examined from different perspectives, including the expert-novice paradigm. The issue of teacher knowledge in the area of mathematics is reviewed extensively including specific case studies by researchers Ma, Leinhardt, Lampert and Ball. Next, because this research study involves the concept of fractional equivalence, an overview of foundations of the concept is examined. The construct of scaffolds and their role in instruction is considered. Finally, the link between teacher knowledge and specific scaffolding strategies in mathematics is scrutinized.

Defining Teacher Knowledge

As critical as the analyses of teachers' knowledge bases are to understanding expertise, both educators and researchers struggle to define the term "knowledge base." In the empirical struggle to define "knowledge bases" it becomes clear that the structures of knowledge bases must also be defined. Addressing the issues and dilemmas related to defining "knowledge bases," Griffin (1983) noted that some scholars question the legitimacy of defining "essential" knowledge in teaching, maintaining that adequate definition depends on the ends considered desirable--a matter of judgment, not science. In contrast, others have argued that, properly understood, the knowledge base is a framework that consists of several different types of knowledge, including statements

about valued ends and the methods used in evaluating or justifying them (Shulman, 1986, 1987).

Therefore, in examining the mathematics knowledge of a teacher, researchers must analyze both the "ideas and understandings" (subject matter knowledge) and the "effective actions" (strategies, practices, methods, or approaches). Shulman (1987) suggested that knowledge of pedagogical structures and procedures should be separated from subject matter knowledge or knowledge of curriculum. He observed that effective teaching consisted of development of sophisticated conceptual and procedural knowledge structures within the subject matter.

Research on elementary teachers' subject matter knowledge has intensified in the last thirty years, and a number of important issues have surfaced as a result. The earlier research focused on the connections between broadly defined teachers' knowledge and student achievement, defining a teacher's knowledge as the number of courses taken in a particular domain. For the most part, these studies were not able to document a significant correlation between teacher knowledge and their students' achievement. Fennema and Franke (1992) noted that the propensity to use correlational techniques to assess the relationship between teacher knowledge and student performance resulted in little being known about the directionality of any existing relationships. Further, these measures could have concealed any true relationship between teacher knowledge and student learning.

Other studies have focused on the preparation of elementary teachers in mathematics. Studies found that many elementary teachers had taken relatively few mathematics courses in their undergraduate and graduate years. As a result, these studies

were able to highlight weaknesses in teachers' knowledge (defined still by number of courses taken) in specific mathematical content areas such as geometry, numeracy and fractions. Many of these studies found teachers possessed primarily a basic procedural knowledge of school mathematics, while their conceptual knowledge of mathematics was considerably weaker (Kilpatrick, 1992). In an attempt to further examine whether this lack of conceptual knowledge was endemic only to elementary teachers in the United States, comparative research was also conducted. What followed was a series of studies that compared elementary and secondary teachers in the United States and China, as well as teachers around the world. (Mullens, Murnane & Willett, 1996; Blume & Simon, 1994; Ma, 1999).

Another approach to understanding teacher knowledge was also developed during this time period. This approach focused on comparisons of teachers who were designated as "expert" and "novice." Leinhardt and Smith (1985) developed their theory of teacher knowledge based on their study of expert-versus-novice differences in mathematics teaching. The expert-novice research paradigm is used most frequently to study the nature of expertise through the analysis of cognitive tasks. By analyzing the performance of an expert during tasks, it was hoped that these analyses might provide guidance for teacher novices. Leinhardt and Smith identified two components of knowledge of teaching: lesson structure knowledge and subject matter knowledge. Lesson structure knowledge included planning and delivering an effective lesson while subject matter knowledge included concepts, algorithmic operations, and connections between different algorithmic procedures, as well as understanding of errors and curriculum presentation.

Because this research focused on teaching of mathematics, it will be discussed further in the section on mathematics teaching presented below.

Subject matter knowledge has been found to be an important component in effective teaching; however, subject matter knowledge is not unique to teachers. Many people possess the knowledge of procedures and concepts in a particular area, but do not possess the knowledge necessary to teach these procedures and concepts. Teachers also possess other types of knowledge related specifically to teaching. Shulman (1987) developed a useful taxonomy of this type of teacher knowledge.

Shulman described three categories of teacher knowledge (Figure 1).

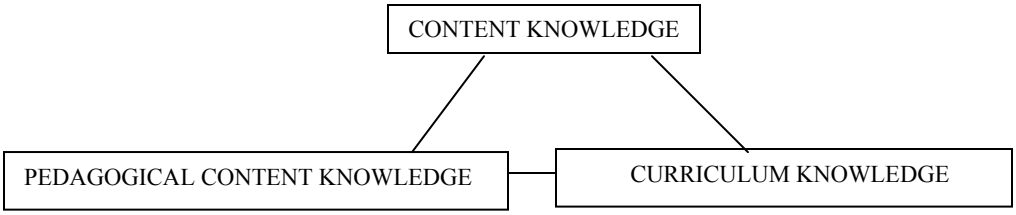


Figure 1. Categories of Teacher Knowledge (Shulman, 1987)

The first category was ***content knowledge,*** which included procedural and conceptual knowledge about the domain being taught. The second category, ***pedagogical content knowledge***, included information on representations of specific content ideas as well as what makes learning a specific topic a challenge for students (Wilson, Shulman, & Richert, 1987). Pedagogical content knowledge also refers to the integration of pedagogical knowledge and content knowledge. Teachers may possess general pedagogical knowledge about how students learn, which is not specific to a particular subject area, as well as subject-matter-specific knowledge about a particular domain (e.g., equivalent fractions). A teacher might have a toolbox full of activities, examples and previous teaching experience, all of which deals specifically with a single topic, such as

equivalent fractions. Thus, pedagogical content knowledge refers to the way teachers transform their subject matter knowledge into an understanding of particular topics, problems, or issues in order to present information to diverse learners of varying abilities and interests (Fuller, 1996). ***Curriculum knowledge*** was the third category and includes ways of using curriculum and how those topics are arranged within the school year.

The second of Shulman's three types of knowledge, pedagogical content knowledge, has been particularly useful in the study of the relationship between teacher knowledge and teacher practice. When subject matter knowledge and general pedagogical knowledge are combined into the construct researchers label as pedagogical content knowledge, it becomes a lens by which to analyze instructional practice.

Relating Teacher Knowledge and Teacher Practice

The connections between teacher knowledge, classroom practice and student performance have been particularly challenging for researchers. Shulman (1986) suggested that what teachers know is one of the most important predictors of what is done in the classroom. Quinn (1997) stated that teachers' knowledge of the subject is necessary and serves as a fundamental element for quality teaching. Researchers began to examine the characteristics of subject matter across different academic fields. Wilson and Wineburg (1988) considered how teachers' subject matter background in social studies affected the way they presented that information to their students. Numerous studies in school mathematics identified what teachers should know in order to be successful at teaching school mathematical subjects (e.g., fractions, division and geometry) and noted that this was very different from what the average person need to know about the subjects

(e.g., Ball, 1988, 1990, 1991; Brown & Borko, 1992; Eisenhart, Behm, & Romagnano,1991; Leinhardt & Smith, 1985).

Despite consensus in the research that teachers' knowledge of the subject matter they are teaching is important, there is no standard as to ***what and how much*** knowledge is required in the subject matter to teach effectively. There seem to be separate research approaches to the question: What is it and how much is enough? The first approach gained attention in the late 1970's. This approach focused on how many mathematics courses teachers had taken, the type of degrees they obtained, and certification within the area of their specialty. This approach had many supporters in the research and education fields.

However, in 1979, Begle conducted an extensive study on the effects of teacher knowledge variables on student achievement. The results of his study surprised both researchers and educators. Begle (1979) found that teachers' mathematics course experience produced on average a positive gain of 10% in student achievement. Similarly, he found that having a teacher who majored or minored in mathematics in college had a positive effect of 9% on student achievement. Begle also completed an analysis of the relationship between the number of mathematics courses taken by the teacher and student performance and showed a positive effect of 23%. In his study, Begle showed that the more mathematics courses a teacher had taken, the more likely their students would be successful at their end of year standardized achievement tests. Even with these strong results, Begle stated that the issue of teachers' knowledge could not be "quantitified" further without the construct of teachers' knowledge being better defined.

This approach was not replaced but rather enhanced by a second research approach.

This second approach functioned on the understanding that taking a large number of mathematics courses did not guarantee that a mathematics teacher would be effective (Marks, 1990). While acknowledging that an understanding of subject matter was important, equally important to effective teaching is an understanding of representations and knowledge of how students learn. The second approach embraced Shulman's (1987) theory of subject matter knowledge as part of the important whole of pedagogical content knowledge. The historical research on the general concept of teacher knowledge has yielded very complex studies that focus on teacher knowledge and its components. Shulman's (1987) categories of teacher knowledge—content knowledge, pedagogical content knowledge, knowledge of students and curriculum knowledge, formed a springboard from which more current research has expanded and refined the concept.

Teacher Knowledge and Practice in Mathematics Teaching

Problems and Issues Specific to Knowledge of Fraction Concepts

Nearly every national study of student achievement and every major research project leads to the conclusion that students do not understand fractions even though sound knowledge of fraction concepts are the basis for higher mathematical concepts. The National Council of Teachers of Mathematics (NCTM, 1989) published literature on what areas needed to be in place in order to understand fractions deeply. The NCTM identified seven areas, including (a) identifying and representing fractions in a variety of forms (i.e. operators), (b) translating among common representation of fractions, (c) ordering fractions, (d) understanding how basic operations on fractions are related, (e) computing with fractions, (f) estimating the results of fraction computations and (g) connecting an understanding of fractions to whole number and rational number concepts.

In 1983, the Rational Number Project-RNP (Behr, Lesh, Post & Silver, 1983)

investigated children's development of rational number concepts. The researchers designed their study around six interpretations of the concept of rational numbers they felt children must have in order to development a deep, working knowledge of rational numbers. These interpretations of rational number included: part-whole, decimal, ratio, division, operator, and measure. The RNP expanded Kieren's (1976, 1980, 1992) epistemological analysis of rational numbers. Each of Kieren's subconstructs is a different interpretation of the quotient $a \div b$. *Measures* refer to whole quantities divided into equal parts (e.g., ½ of a cup of flour); *ratios* express relations between two different sets (e.g., 2 boys for every 3 girls); *operators* unite multiplication and division in a single operation (e.g., ⅔ of 12 pizzas); decimals indicate the numeral result of division; and *quotients* express an uncompleted division (e.g., 2 pizzas to be divided among 3 people). Although he recognized that these subconstructs were not strictly distinct, Kieren (1976) argued that each was a site for building rational number knowledge and students must learn each subconstruct in order to develop a full understanding of rational numbers and that research on learning should be conducted within each subconstruct.

This view conceptualized learning as a process of synthesizing and abstracting from knowledge of different subdomains of rational number. However, the RNP (Behr et al., 1983) found that the part-whole interpretation of the rational number was the foundation for understanding all the other interpretations. Specifically, equivalence and partitioning must be mastered before students' can progress successfully to the other subconstructs of rational number. Later, research in the area of fractions supported the RNP's statement. Lester (1984) identified equivalency as one of the main concepts where teachers should focus their instruction. In their study of classroom teaching methods and

fraction knowledge, Gearhart, Saxe, and Seltzer (1999) found that students with a limited understanding of the basic concepts of equivalency and partitioning did not perform as well on achievement measures.

The issue of equivalence is one of the main tenets of rational numbers. Each rational number is expressed in an infinite number of equivalent numerical forms. For example, the fractions 1/2 and 4/8 look different, but they represent the same rational number: *one half*. Each rational number is an infinite set of equivalent fractions; the equivalence set for *one half* is {1/2, 3/6, 4/8 . . .}. This many-to-one relation between fraction numerals and the rational numbers they represent in turn complicates judgments of order (Smith, 1995).

Judged from a natural number perspective, the order and equivalence of rational numbers look very different and more complex. Rational numbers cannot be uniquely symbolized, even within the fractions representation. Rational numbers satisfy the density property: for any two distinct rationals, an infinite number of different rationals are between them. Because there is no "successor," no "next" rational number, listing rational numbers in order is substantially more complicated. And then comes the most basic question of all: what is a fraction? In this proposal, we will use Smith's (1995) definition of both fractions and rational numbers. The term *fraction* will be used in contexts involving specific numerals (e.g., 8/11). The term *rational number* will be used in discussions of the numerical relations that are represented by but are more general than specific fractions (e.g., 1.22).

Fractions are a very difficult concept for elementary students to master. There appear to be very distinct cognitive obstacles in learning fractions for elementary

students. The first obstacle is the concept of equal-partitioning. Young children come to school with an informal knowledge of equal partitioning of the whole. That is, without formal instruction they are able to partition wholes into equal parts (Lamon, 1996). However, even after formal instruction on equal partitioning, students have difficulty with formal representations of equal-partitioning (Lamon, 1996). There are many research studies that document extensively the difficulty a typical elementary school student encounters when learning fractions. Behr, Harel, Post and Lesh (1992) conducted an analysis based on data gleaned from Bezuk's study of elementary students' understanding of equivalent fractions. In Bezuk's study students were asked to show 2/3 of the two shapes as shown in Figure 2.

Figure 2. Students' partitioning rectangles (2/3) (Behr et al., 1992)

While all seventy-seven of the students shaded the first rectangle correctly, only 25% of the same number was able to show the correct representation of 2/3 on the second rectangle. Additionally, in extensive interviews, Bezuk documented the students thinking on partitioning of a circle with the following diagram:

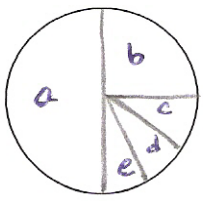


Figure 3. Students' partitioning circles for equivalency (Behr, et al., 1992)

The resulting transcripts showed that students had difficulty generalizing equivalency. That is, if the figure did not show explicitly how the parts were related to the whole, students could not infer equivalence from the information given. Kamii (1994) explains the difficulty Bezuk's students had with equivalent fractions as the result of the students' inability to conserve the whole. The students in Bezuk's study were asked to demonstrate the equivalence of three sections of a circle and they were not successful. Kamii hypothesized that the students could not conserve the whole of 1 *and* the part of ¼ and thus could not see three sections of the circle as 3/12.

Another cognitive obstacle to learning fractions is the invariance of the whole. Piaget (Lavatelli, 1973) stated that fractions are both parts of the whole and wholes themselves. This is an important concept which is the building block for quantitative subdivision. A fraction is used to express a relation between two quantities of magnitude, the numerical value of which can be less than one. The understanding of the relationship between the quantities is knowledge of the equal-whole (Hart, 1988).

Equivalent fractions represent two related aspects of operative thinking identified by Piaget: 1) multiplicative thinking and 2) conservation of the whole and parts (Piaget, 1987). Kamii (1994) described multiplicative thinking as a hierarchical structure and simultaneous structure. She gave the example of 4 X 3 which involves thinking on two levels simultaneously as shown in the figure below.

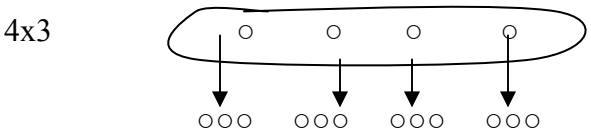


Figure 4. Multiplicative thinking: hierarchical and simultaneous structure (Kamii, 1994)

She extends this hierarchical structure to the understanding of equivalent fraction based on the research conducted by Olive (1993). Olive (1993) demonstrated the hierarchy of equivalent fractions as shown below.

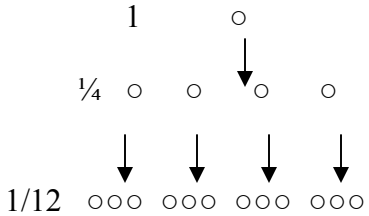


Figure 5. Hierarchical level of equivalent fractions (Kamii, 1994)

Kamii conducted her own study of 5th and 6th graders' understanding of the equivalence of the unit fractions ½, ¼, and 1/8. The students were given two tasks. The first task presented two equal rectangles which had been cut first in half with a vertical line so that each part of the rectangle was equal and then on the diagonal of the rectangle. The students were then asked questions about the relationships of the four parts of the two rectangles labeled a, b, c and d respectively as shown in Figure 6.

Figure 6. Students' identifying equivalence – Partitioning rectangle task I (Kamii, 1994)

Of the sixty-one fifth graders in the sample, only 44% stated the parts of the rectangle would be equal, while 38% of the students felt that the part c (triangle portion)

was more than a (rectangle portion). The sixth graders' performance was comparable to the fifth graders. Fifty-six percent of the sixth graders correctly stated that the triangle portion and the rectangle portion were equal.

In the second task, the fifth and sixth graders were asked to perform a problem solving task using two equal rectangles. One of the rectangles was separated equally into fourths and the other rectangle into eighths as shown in Figure 7. The problem required the students to use the both rectangles to solve ¾ = ?/8. Only 32% of the sixth graders in Kamii's study correctly reasoned independently that ¾ = 6/8.

Figure 7. Students' identifying equivalence – Partitioning rectangle task II (Kamii, 1994)

Understanding Fractional Representations

To fully understand fractional arithmetic, children must understand what fractions represent and link this understanding to symbolic representations (Rittle-Johnson & Siegler, 1998). In working with equivalency, children need to know that the same amount can be represented with fractions that include different numbers (e.g. ½=3/6). There have been a number of studies which have supported the importance of procedural and conceptual knowledge in understanding fractional arithmetic.

Byrnes and Wasik (1991) found that fourth- and sixth-graders who had a better conceptual understanding of fractions were more successful working with fractions in other operations. They assessed understanding of basic fraction concepts by asking the children to identify the written fraction that represented the shaded portion of the object

and to choose an alternative pictorial representation of a stated fraction. The researchers found that students with a strong conceptual knowledge were more successful at the procedural aspects of the task. Additionally, Kazemi and Stipek (2001) conducted a study which examined the instructional practices which resulted in improved conceptual thinking of students involved in learning addition of fractions. The researchers describe interactions between the teacher and students that fostered explanations that were not simply a procedural explanation. The procedural explanation had a place in the interaction as it served as the springboard to deeper conceptual thinking.

Relating Pedagogical Content Knowledge to Teaching Practice in Mathematics

Leinhardt and Smith (1985) examined teacher knowledge and its impact on instruction using the expert-novice paradigm. Four experts and four novice teachers' knowledge of equivalent fraction concepts was established through interviews and a card sort activity. Four of the experts were in service teachers, while the four novices were undergraduate student teachers who were identified as excellent by their supervisors. The researchers designed a conceptual map for fractions based on each teacher's data. Finally, they compared the conceptual maps of the teachers specifically for instructional differences that could be related to differences in the teachers' knowledge base.

The study revealed wide differences between the teachers' knowledge bases and instructional practices. The differences in pedagogical content knowledge were not just between the expert and novice teachers but also appeared among the experts. Most of the teachers thought about fractions only as they related to a part-to-whole relationship. As a result, all of the novice teachers and one expert teacher were unable to extend the concept of equivalence into a problem with larger numbers than the one they had been given.

Leinhardt and Smith (1985) noted the dissonance between the teachers' procedural knowledge and their attempts to conceptually explain fractional representations. Puzzled by the imbalance, the researchers observed the three expert teachers teach a lesson on reducing fractions. They found that their expert teachers exposed the students to different conceptual representations. Thus, the students' learning experience varied greatly and depended on which expert teacher presented the lesson.

Leinhardt and Smith (1985) discuss at length the representations used by an expert fourth grade teacher during her lesson on equivalent fractions. As the analysis is of particular relevance to this study, the analysis of representations is presented in detail. Ms. Konrad's (Ms. K) goal for her lesson was to show that two different fractions can be used to show the same amount (1/2 = 2/4) and to demonstrate how to operate on a given fraction in order to produce an equivalent fraction (multiply or divide by a fractional equivalent of one). During her lesson, Ms. K used three different representations of fractions: a number line, regions of figures, and sets. Ms. K used two number lines; each number line was divided into different fractional parts. She arranged the number lines so that the equivalent fractions of one number line were aligned with the other number line. Using this representation of equivalent fractions emphasized the bidirectionality of equivalence.

Ms. K also used a regional model of equivalence in her lesson. She drew two equal size rectangles on the board. One rectangle was shaded to show ½ and the other was shaded to show 3/6. Ms. K asked her students to compare the area covered in each rectangle. Finally, Ms. K used a set representation that expanded on the shapes used in her regional representation.

The extensive use of representations during instruction fueled additional research by Leinhardt. In 1991, Leinhardt, Putnam, Stein, and Baxter reexamined expert and novice teachers' use of representations within their curriculum scripts. Representations are defined as the entity used to explain something. These entities can include analogies, pictures, or manipulatives. Leinhardt et al. supported the analysis of representations as a window into a teacher's subject matter knowledge. Representations are embedded in instructional explanations and as a result, the goals and actions of the explanation influence the particular representation a teacher chooses as well as how the teacher uses the representation in instruction. Therefore, the effective use of the appropriate representations requires that teachers' have deep subject matter knowledge of the topic they are teaching.

Leinhardt et al., (1991) also hypothesized that the two fundamental knowledge systems required by teachers of mathematics were knowledge of lesson structure and knowledge of subject matter. Knowledge of lesson structure consists of knowledge necessary for conducting lessons while knowledge of subject matter knowledge supports the lesson structure by providing subject matter content to be taught.

Using detailed classroom observations of expert and novice teachers, the researchers suggested specific points in instruction where subject matter knowledge was crucial. First, subject matter knowledge impacts how a teacher's agendas or mental plans for a lesson are structured. The more subject matter knowledge a teacher has, the more detailed their mental agendas are. In order to ascertain the structure of the mental agendas, the researchers asked the teachers in the study what they were going to do and what they expected would happen.

Another area where the researchers found that subject matter knowledge was crucial was within teacher curriculum scripts. The researchers defined a curriculum script as the goals and actions for teaching a particular topic. Putnam (1985) had introduced the concept of curriculum scripts in his study of teacher tutoring. In the study, teachers presented content to students and responded to their errors while tutoring in a didactic setting. The teachers in the study made adjustments during their instruction in response to individual student needs. Expert teachers' curriculum scripts showed characteristics not present in the novice teachers' scripts. Expert teachers' curriculum scripts showed overall teaching goals that were clear, but sub goals and actions that were more flexible. The expert teachers' goals were organized in a network rather than a linear sequence. This type of organization allows the teacher more flexibility in responding to student input. However, the flexibility in the curriculum scripts requires that a teacher have extensive subject matter knowledge.

Leinhardt et al., (1991) illustrated how the lack of knowledge of a representation of multiplication in a curriculum script disrupted a teacher's lesson on multiplication: Ms. Benny attempted to represent the equivalence of *3 x 9* and *9 x 3* for her students. However, she presented the combination of *9 x 3* on a 9 by 9 array, effectively confounding not only herself but her students as well. It was hypothesized that if Ms. Benny had a richer knowledge of representation in her curriculum script, the confusion would have been avoided.

Leinhardt et al., (1991) also expanded the concept of explanations embedded in curriculum scripts of expert and novice teachers. The concept of teacher explanation was introduced in Leinhardt's (1988, 1989) research and reviewed in Leinhardt (2001).

Leinhardt (1988) defined explanations as complex actions that occur in response to several goals. Teachers construct explanations in order to instruct and help students learn new material. There are several instructional situations in which explanations might be warranted. Teachers can offer explanations to clear up a student misconception, connect previous knowledge to new knowledge or to demonstrate a procedure. Regardless of the use, teacher explanations are one place where subject matter knowledge can be examined closely.

Leinhardt (1989) constructed semantic webs of expert and novice teacher explanations. The expert teacher's semantic web showed a connectedness between major concepts of a subtraction lesson while the novice's web showed very little connectedness between her explanations of multiplication, the expert teacher connected three representations of regrouping with different procedures. The expert teacher used sticks, felt strips and numerals to represent regrouping, in contrast, the novice teacher used arrays incorrectly by attempting to represent *3 x 9* on a 9 by 9 array. Additionally, the semantic web of the expert teacher had much more details than the novice's web. Leinhardt emphasized the connectedness and clarity of the expert's knowledge. She stated that when a teacher has a deep knowledge of a particular content topic it allows the teacher to keep the lessons going by staying away from confusing interactions. An example of a well-supported knowledge of equivalent fractions would include knowledge of part-whole versus ordered-pair interpretation or the concept of bounded infinite subdivisions.

Leinhardt and her colleague's research in the area of teacher knowledge and its connection with subject matter knowledge revealed important findings. Subject matter

knowledge can be documented by examining a teacher's curriculum script. Embedded in the curriculum script are the foundations of a teacher's understanding of subject matter. Leinhardt and her colleagues also distinguished the difference in subject matter knowledge between expert and novice teachers based on the analysis of their curriculum scripts for lessons and their semantic webs.

Expert and Novice Teachers' Subject Matter Knowledge

Differences in teacher knowledge were documented by analyzing lesson scripts and semantic webs and linking those differences to instructional practice. Although Leinhardt et al., (1991) could document differences in the teacher knowledge of experts and novices; they could not definitively state how much knowledge in a particular area was enough to teach the subject matter effectively. Other researchers have sought to further define teacher knowledge in mathematics.

Related research was conducted by Deborah Ball. Ball (1991) focused her studies on the knowledge of particular domains of mathematics as well as the procedural and conceptual understanding necessary for teachers of mathematics. These research studies are especially relevant to this study since they focused mainly on fraction concepts and are presented in detail.

Ball (1988) began her research of teacher knowledge in mathematics by interviewing two teachers around the topic of division by zero. Both teachers interviewed thought about the division by zero simply in procedural terms. This was evidenced by their statements such as "can't do it," "it's just a rule you have to remember," and "something you can't do in mathematics even in calculus." Both teachers relied on their

procedural knowledge of mathematics without understanding the mathematical reasoning behind the rule.

Ball also tested teacher's knowledge of place value and numerations by setting up three-digit multiplication algorithms with errors. She asked the teachers to explain what they would do if their own students were making errors. Of the three prospective teachers interviewed, only one teacher talked about the problem *123 x 44* as really being *123 x 40 and 123 x 4,* thereby recognizing the conceptual understanding of expanded notation in the multiplication algorithm. The remaining two teachers focused solely on the procedural learning of "lining the numbers up correctly."

Also, in Lampert and Ball (1998), Ball argued that while subject matter knowledge does not stand alone in the foundation of a teacher's understanding of mathematics, its influence affects the other building blocks of pedagogical content knowledge. She points out that what a teacher needs to understand about shapes or proofs depends on what the teacher thinks the point of teaching geometry is and in turn is connected to understanding mathematics in general and geometry in particular. Ball supported her views on the importance of subject matter knowledge using case studies of three fourth grade teachers. Each of the teachers taught the same topic (long multiplication) and the same age students. Each of the classes received very different instruction on long multiplication. Ball hypothesized that the variance in instruction was the result of the teachers' different levels of knowing of and about mathematics. Two of the teachers demonstrated multiplication lessons in their classrooms which were based solely on procedural learning. Emphasis on rote computation and extensive practice with algorithms were evident in both of the classrooms. The third teacher in Ball's study,

whose teaching displayed an in-depth knowledge of mathematical knowledge, was Magdalene Lampert.

Lampert is also a prominent mathematics researcher whose publications include *Talking Mathematics in School: Studies of Teaching and Learning* (2001) and *Teaching, Multimedia and Mathematics* (Lampert & Ball, 1998). Since the 1970's, Lampert's work has been influential in changing the paradigm in teaching and research of teaching in mathematics. Her research writings have served as a discussion point for the complicated relationship of theory and practice. Because Lampert is both a classroom teacher and a researcher, her work has been praised as an example of a unique view, between the worlds of research and the primary classroom. Lampert (1988) extensively used classroom transcripts, audiotape and videotape to closely examine the mathematics learning in her own fifth grade elementary classroom.

Ball (1990) continued her line of research on teacher subject matter knowledge by specifically examining teachers' knowledge in the domain of fraction division. By observing teachers while they delivered lessons of fraction division, she proposed a distinction between *knowledge of mathematics* and *knowledge about mathematics***.** Ball defined knowledge of mathematics includes knowledge of concepts, ideas, and procedures, while knowledge about mathematics includes as how one decides a claim is true or a solution is complete. In Ball's (1988) earlier study, preservice teachers were asked to complete a questionnaire covering a variety of mathematical concepts. Two hundred fifty-two teachers were asked to solve the fraction division problem *1 ¾ divided by ½* and to give an appropriately correct representation and story problem. Most of the preservice teachers could solve the problem procedurally, using the "invert and multiply"

method. However, almost all struggled to give an appropriate representation. Ball stated that these actions showed a lack of conceptual knowledge of fraction division. She pointed out that the teachers wanted to give their students meaningful experiences with fractions but could not because their subject matter knowledge was insufficient to accomplish this task.

Several researchers had similar results when they examined teachers' knowledge of fraction division. Borko (1992) examined an elementary teacher's attempt to explain a similar problem to her students. Her explanations lacked fundamental understanding of rational numbers and she struggled to make connections between important fraction concepts. Borko proposed that the teacher's inadequate explanations were due to a lack of conceptual understanding around fraction division. A similar study by Simon (1993) found that teachers were deficient in the conceptual understanding of division of fractions. This deficit was evidenced by the teacher's use of a partitive representation when a measurement representation would have been more efficient.

More recently, Ma (1999) used comparisons of United States and Chinese teachers to propose the idea of a *profound understanding of mathematics*, which she defined as connectedness, multiple perspectives, basic mathematical ideas, and longitudinal coherence during teaching. She revisited teachers' knowledge of fraction division in her in-depth study of Chinese and United States educators. In her book, *Knowing and Teaching Mathematics*, Ma documents the differences between Chinese and U.S. teachers' knowledge of mathematics for teaching and suggests that Chinese teachers' understanding of mathematics and of its teaching contributes to their students' success. The book was an extension of her dissertation, which found that although

United States teachers had been exposed to more advanced mathematics in their high school and college education Chinese teachers displayed a more comprehensive knowledge of the mathematics taught in elementary school (Ma, 1999).

In her dissertation, Ma used the *Teacher Education and Learning to Teach* (TELT) project interview questions and data on United States teachers. The TELT project was a five-year longitudinal study conducted by The National Center for Research on Teacher Learning (NCRTL) during its first years of operations. The TELT project focused on a sample of more than 700 teachers and teacher candidates at 11 program sites throughout the United States who completed questionnaires before, during, and after their participation in formal teacher education programs. A subset of 160 teachers was also interviewed and their teaching was observed.

Ball (1988) developed the TELT mathematics instruments for her own dissertation research. She designed the interview tasks to investigate teachers' knowledge of mathematics in the context of common actions teachers complete during the course of teaching. The division problem developed by Ball (1988), *1 ¾ ÷ ½*, was designed to represent the mathematics of division by fractions in the context of the familiar task of teaching by generating a representation, a real-world context, or a diagram for the specific topic.

Ma (1999) identified 23 United States teachers from the TELT project for comparative purposes. Eleven of the teachers were experienced teachers who were participating in the Summer Math for Teachers Program at Mount Holyoke College while the other twelve teachers were participating in the Graduate Intern Program run by a school district at the University of New Mexico. Seventy-two Chinese teachers, chosen

from five elementary schools with which Ma was familiar before she came to the United States, also participated. Three schools were located in Shanghai and two schools were located in a province near Shanghai.

Ma (1999) presented Ball's division problem, *1 ¾ ÷ ½* to the Chinese teachers in the study. All of the Chinese teachers could correctly answer the question, explain the answer fully and generate appropriate representations of the problem. Of the 23 United States teachers in the sample, only 43% could perform the computation correctly. The standard procedure most of the teachers used for solving this problem was "invert and multiply." Additionally, although 43% of the U.S. teachers successfully calculated the answer, almost all failed to come up with a representation of division by fractions. Among the 23 teachers, 6 could not create a story and 16 made up stories with misconceptions. Only one teacher provided a conceptually correct but pedagogically problematic representation. The teachers displayed various misconceptions about the meaning of division by fractions. In fact, when trying to construct a mathematics problem that represents *1 ¾ ÷ ½*, the teacher that provided a conceptually correct problem did so by changing the problem to *2 ¼ ÷ ½*. She explained that she would take 2 ¼ Twinkies and give each child ½ of the Twinkies. Mathematically, the representation is correct, but the answer of 4 ½ children is pedagogically unsatisfactory. Since we teach children with realistic word problems, the answer of 4 ½ children would be considered incorrect because you cannot have ½ of a child.

Ma's (1999) study clearly showed that the United States teachers' lack of subject matter knowledge around the division by fractions prevented them from using appropriate representations. The teachers correctly used area models (pizza and pies), but

those representations did not assist them in their understanding of the problem. Ma points out that in order to utilize representations in the domain of division by fractions, a teacher must first know *what* to represent as well as *how* to represent it.

Many of the teachers in Ma's study created representations for the problem *1 ¾ ÷ ½*. The teachers used money, recipes, apples and whatever else they could think of that would fit with the problem. These representations are appropriate models used in teaching fractions. However, most of the teachers made a mistake by multiplying by, rather than dividing by, ½. This lack of subject matter knowledge could not be made up by knowing which representations to use in this problem. Because of their misconceptions about the meaning of division by fractions, these teachers failed to create correct representations for their students. It is because of the strong connection between subject matter knowledge and the instructional choices teachers make in their interactions with students that the issue of instructional practice warrants closer scrutiny.

Cognitively Guided Instruction and Related Research

Cognitively Guided Instruction (CGI) is a professional development program for teachers that examined the development of children's mathematical thinking within specific content domains. CGI (Carpenter, Fennema & Franke, 1996) is guided by two main principles. The first principle is that children bring an intuitive knowledge of mathematics to school with them and that this knowledge should serve as the basis for developing the formal mathematics instruction in primary school. The second principle is that the professional development of teachers should be based on their existing knowledge. Using these two principles, CGI provides a framework for understanding children's mathematical knowledge.

The researchers (Fennema, Franke, Carpenter, & Carey, 1993) conducted a case study of a second grade CGI teacher. In one assessment given by the CGI researchers, the teacher showed strong pedagogical knowledge in the teaching of addition and subtraction, as well as fractions. However, the subject matter measurements showed a range of differences between the subject matter knowledge of addition and subtraction and the subject matter knowledge of fractions. The second grade teacher was observed teaching lessons on addition, subtraction and fractions. There were noticeable instructional differences between the teachings in these two subject matters. When teaching addition and subtraction, the teacher used a problem solving approach, adjusted her teaching to the students' questions and encouraged detailed discussion around the subject matter. However, when the teacher was observed during her lesson on fractions she limited her examples to part-whole representations, demonstrated more teacher-directed activities and limited the students' discussion. The differences in instructional practice were interpreted as due to the differences in the knowledge of the subject matter (Fennema, et al., 1993).

In 1992, Lehrer and Franke also examined teachers' knowledge of fractions and its effect on instructional practice. Two teachers participated in the study—one fifth grade teacher and one second grade teacher. The researchers used both observations and a set of fraction problems to determine how the teachers' knowledge would affect their instruction. First, the set of fractions problems was presented to the teachers and they were asked to respond to pedagogical questions designed to determine whether their instruction was conceptually or procedurally inclined. A conceptually inclined teacher would be more prone to using multiple representations, discussion and real world

analogies to present information, while a procedurally inclined teacher would focus on the rules and algorithms to assist student understanding. The second grade teacher was shown to be more conceptually inclined in her instruction, while the fifth grade teacher showed she would rely more on procedural explanations. After the teachers' instructional inclinations were identified, the researchers observed the teachers in their classrooms. During the classroom observations, the second grade teacher displayed a greater tendency to offer representations that were appropriate and familiar to her students. On the other hand, the fifth grade teacher gave the students problems from the textbook and was unable to offer different representations when her students had difficulty with the textbook examples.

Although this study supported the researchers' hypothesis about teacher knowledge and instructional practice, the use of teachers in different grade levels posed a methodological issue. It may have been easier to generalize the results had the teachers taught the same grade level and content.

Examination of an Instructional Practice: Scaffolding

One instructional strategy that has received considerable research attention is instructional scaffolding. However, research focusing on the nature of scaffolds and their functions in specific school contexts is limited (Bliss & Askew, 1996).

Scaffolds are forms of support provided by the teacher (or another student) to help students bridge the gap between their current abilities and the intended goal (Palincsar & Brown, 1984; Paris, Wixson, & Palincsar, 1986; Tobias, 1992; Wood, Bruner, & Ross, 1976). The purpose of scaffolding is to provide the learner with support during the developmental phases of learning. Once structures are in place and the learner can apply

strategies and big ideas independently, these external supports are gradually removed. Scaffolds help bridge the gap between a learner's current performance and the intended goal. A scaffold also allows the learners "to participate at an ever-increasing level of competence" (Palincsar & Brown, 1984, p.122). Scaffolding procedures can reduce the complexities of problems, breaking them down into manageable chunks that a student has a chance of solving (Bickhard, 1992). Examples of teachers' scaffolds include (a) providing simplified problems, (b) modeling of the procedures, (c) and thinking aloud as they solve the problem. Scaffolds may also be tools, such as cue cards or checklists.

Early Studies of Instructional Scaffolds

Some of the first research on scaffolding was conducted by Wood, Bruner, and Ross (1976). They used the term to refer to the instructional process, whereby an adult controls, "those elements of the task that are initially beyond the learner's capacity, thus permitting the learner to concentrate upon and complete only those elements that are within his range of competence" (Wood et al., 1976). Several scaffolding procedures identified by Wood et al. include (a) reducing the complexity of the task to manageable limits, (b) maintaining student interest, marking critical features, and (c) demonstrating solutions where the learner can recognize the process of the explanation.

Although scaffolds can be applied to the teaching of all skills, they are particularly useful, and often indispensable, for teaching higher-level cognitive strategies, where many of the steps or procedures necessary to carry out these strategies cannot be specified. Instead of providing explicit steps, one supports, or scaffolds, the students as they learn the skill.

The teaching of cognitive strategies is an example of working in a child's zone of proximal development. Vygotsky (1978) observed that a child has two developmental levels. One is the "actual developmental level," the level at which children can independently deal with tasks, and the other is the "level of potential development," the level at which a child can solve a problem with the assistance of a teacher or in collaboration with other children. The zone of proximal development is the distance between the level of potential development and the actual developmental level of the child (Vygotsky, 1978). Vygotsky emphasized the role of instruction in fostering development. Vygotsky was describing the action of instructional scaffolding. The research on instructional scaffolding suggests that the focal point of teaching is when a teacher assists a student in such a way that their learning is first supported and then extended. The identification of scaffolds as a vehicle by which learning can take place is a pivotal concept in education.

Although Vygotsky did not provide much information on how one might instruct children in their zone of proximal development (Palincsar, David, Winn, Snyder, & Stevens, 1989), other investigators have developed instructional procedures which involve scaffolding children’s learning. Langer and Applebee (1986) and Applebee (1986) further clarified the concept of instructional scaffolding as a process through which the learner can gradually internalize routines and procedures to complete tasks from the social and cultural contexts in which learning takes place. They suggested that tasks and activities in the classroom should provide an orderly, coherent set of experiences for the student at appropriate levels of challenge and support. For example, an instructional scaffold in teaching language might involve having a more experienced

language learner assist the language learner with a task by modeling the language task to be learned verbally and/or in writing. Scaffolding is also provided by asking probing questions to extend or elaborate on the knowledge the student already possesses.

Research has identified specific types of scaffolds that facilitate children's learning. A new concept in scaffolding is that of providing students with procedural prompts. These prompts are a special type of scaffold, one that is specific to the cognitive strategy being taught, yet flexible enough to allow application to a variety of contexts. Examples include prompts that help students generate questions, prompts that suggest procedures for summarizing, and questions to consider when planning an essay. In summary, because higher-level cognitive strategies cannot be taught as a series of explicit steps, there is a need for a great deal of student practice to help overcome this problem. This practice can take a variety of forms, yet all of them apply the procedures of scaffolding and fading. As guided practice begins, the majority of the task is completed by the teacher through modeling and thinking aloud. As practice progresses, students take on more responsibility in completing the task, and teacher support is gradually withdrawn. As guided practice nears completion, the student completes the entire task independently, with little or no support provided by the teacher. Previously withdrawn scaffolds can be temporarily reintroduced if students encounter difficulties (Rosenshine & Meister, 1997).

Research on Scaffolds in Reading and Language Learning Studies

Stahl and Hayes (1997) conducted a study that identified the instructional procedures and scaffolds that have been developed to help students to learn and apply cognitive strategies in reading. They analyzed 50 intervention studies in which cognitive

strategies were taught to students in an effort to improve their reading comprehension. At the end of an intervention study, the comprehension scores of students in the experimental group were compared with the scores of students in the control group. They studied the methods section of these studies, and abstracted from them the instructional procedures and scaffolds that were used to teach the cognitive strategy. Although this study focused solely on instructional practice and scaffolding in the domain of reading, the general explanation of how scaffolds are used to teach cognitive strategies is important to the current study.

Cazden (1983) adopted Bruner's concept of scaffolding in her research on learning language through discourse. However, she initially distinguished between two types of instructional scaffolding: a) vertical scaffolding and b) sequential scaffolding. Cazden defined vertical scaffolding as an adult extending a child's language development through questioning. For example, if a child says "dog," the adult might say "Yes, that is a dog, what does the dog say?" Sequential scaffolding is interaction with that occurs through children's games and interaction with their parents. For example, during bath time a parent might state "First, we put the water in, next the bubbles…" effectively giving the child clues to the next step in the process.

Cazden continued her research on instructional scaffolding for many years. In writing about classroom discourse as a scaffold for learning, Cazden (2001) gave examples of both dyadic and group scaffolding. As part of her research on dyadic scaffolding in school settings, Cazden analyzed videotapes made in New Zealand of participants in a Reading Recovery Program. The Reading Recovery Program is an intense one to one reading intervention for children who have not learned to read at

appropriate levels. The Reading Recovery Program is geared only to first graders. Cazden videotaped a six-year old's progress during three points in the half-hour, 12 to 20 week individual tutoring sessions. The child's progress was marked by her success at writing down stories at the beginning, the middle and the end of the program. Cazden and Clay (1992) documented this progress by distinguishing between the child's attempts with and without the teacher's assistance. The teacher's assistance or scaffolding is achieved through specific scaffolding techniques which included questioning, modeling actions and directing the child's practice. The researchers described both the child's and the teacher's actions during use of those scaffolding techniques. One particular scaffolding technique the authors noted was the teacher calling attention to the sounds in the spoken words as shown below:

T in Week 1: "Do you know how to start writing *little*"

T in Week 3: [After child has written *s* for *swim*] "Let's listen to it [saying the word slowly]. What can you hear?"

Cazden and Clay stated that examining a one to one tutorial relationship such as Reading Recovery makes the structure and the function of the teacher's scaffolding more apparent than if the teacher was working with a large group. Further, the researchers state that the child's writing shows clear evidence of her functioning within the zone of proximal development because she completed tasks alone at the end of the sessions in which she needed the teacher's assistance at the beginning of the sessions.

Cazden pointed out the importance of the idea of a scaffold with the zone of proximal development. Research is needed to expand the foundations of classroom discourse and give educators specific strategies that support student learning. In fact,

research (Wood, et al, 1976) on effective scaffolding showed that the absence of scaffolding in many lessons was due to two important factors: teaching strategies used by teachers and their subject knowledge. In their study, teachers overestimated the success of their lessons compared to the understanding demonstrated by their students. Additionally, three out of four teachers in the study lacked confidence in teaching with a specific content area. This lack of specific content knowledge confidence contributed to the lack of specific strategies in instruction and the uncertainty of lesson goals.

Research on Scaffolds in Mathematics Teaching

Bliss and Askew (1996) conducted research on identifying instructional scaffolding within the school setting across design and technology, math, and science. Their case studies in of mathematics instruction the school setting are of particular relevance to this study and are described here.

Bliss and Askew's study was undertaken in order to develop a taxonomy of scaffolding strategies used in the context of school mathematics. The researchers conducted a qualitative study which employed three basic methods. First, observations and videotapes of class instruction were made and corresponding field notes of the observations were written. Next, interviews were conducted with both the teacher and the students in order to compare each group's perceptions of the lessons. Finally, the teacher in the study participated in discussions with the researchers in which the audiotapes from the classroom lessons were used. The study showed that only 22% percent of the teacher-student interactions demonstrated successful scaffolds. The majority of the scaffolds were categorized as approvals, encouragements or structuring of work. Prop scaffolds and localized scaffolds were also identified in the interactions. Bliss and Askew defined prop

scaffolds as the action that takes place when a teacher provides a suggestion that will help students throughout the tasks while localized scaffolds are when the teacher scaffolds one part of a more complex task or concept.

Of particular interest to this study is the description of scaffolds during a lesson on equivalent fractions. One of the lessons the researchers transcribed and analyzed was a session with one teacher and five students on the concept of equivalent fractions. As part of the instruction, the teacher used an apple and two cardboard circles to teach the concept of fractions in the denomination of halves, quarters, eighths, thirds and sixths. The instruction begins with the teacher introducing the cardboard circles as "cakes" that have to be divided into thirds. She then introduces a "clock face" as a scaffold for one of the students to understand the concept of breaking an object into thirds. By using a "clock face" analogy to introduce equivalent concepts the students become confused and cannot demonstrate proficiency in equivalent fractions. Furthermore, the teacher then engages the students in a conversation about the appropriate tool to measure the clock face and completely loses the students. Stronger pedagogical content knowledge might have helped this teacher. While the did have a number of tools to teach fractions, i.e., circles and a clock face, she did not foresee that the clock face representation might have not been the best one with which to introduce the concepts of equivalent fractions. Students should already have a basic understanding of the relationship between ½ and ¼ before using the clock face as a tool. The researchers found that the teacher's understanding of the lesson subject matter content was sometimes limited and confused and as a result, the students had difficulty understanding lessons.

The researchers' preliminary results identified a lack of scaffolding by the teachers in most of the lessons observed. Based on the audiotapes and field notes, the researchers hypothesized that the lack of scaffolds was due to two significant factors: the teachers' lack of instructional strategies and lack of sufficient subject matter knowledge.

Summary and Methodological Issues

Providing scaffolding to children in a school environment depends heavily on the expertise of the teacher. Scaffolding requires that an "expert" have domain-specific expertise. In mathematics, the effectiveness of teaching depends a great deal on a teacher's pedagogical content knowledge (Schulman, 1986). Studies have revealed that a teacher's pedagogical content knowledge is a significant determinant of their teaching practices in mathematics (Lee, Lee, & Meadows, 2003). Similarly, research has shown that both pedagogical knowledge and content knowledge are critical elements of effective teaching (Leinhardt, 1986). However, studies have also shown there is great variability in the amount of pedagogical content knowledge and content knowledge a teacher brings to instruction.

The question of what knowledge a teacher may possess and how they use this knowledge to facilitate student learning has guided the research on teacher knowledge and instructional practice. The specific sub domain of fraction division has received a great deal of attention in recent years. The understanding of fraction division is a particularly difficult concept to master and it is for that reason it has been the focal point for research in the early grades (elementary and/or primary schools). Most of the research has focused on two distinct samples: preservice and in-service teachers. Even with this distinction the research results are similar. Overall, both preservice and in-

service teachers' knowledge of division by fractions was weak. The teachers relied on procedural computations to solve problems, lacked depth in their mathematical knowledge of fractions, and had difficulty explaining the conceptual side of the domain. Their instructional practice mirrored the weakness in their knowledge base.

Even with these strong results, there are aspects of the research which can be improved. In Ball's 1990 research, preservice teachers were used as a sample. As stated earlier, it takes a minimum of three years to develop general expertise in teaching; therefore, Ball's sample was already at a disadvantage in that the findings would be difficult to generalize for the teaching population. A cross-national comparison yielded similar results (Ma, 1999). Ma's sample included both United States and Chinese in service teachers and her study also gave teachers specific problems to solve. For example, teachers were asked to solve *1 ¾ ÷1/2* and she found substantial differences between the Chinese teachers and United States teachers.

The connections made between teacher knowledge and observed instructional practice were better examined in the earlier research by Leinhardt and Smith (1985) and extended in the series of studies by Leinhardt, et al. (1991). Their analysis of teacher knowledge through the lens of the expert-novice paradigm set the standard for the types of methods that could be used in teacher knowledge research. The use of interviews, card sort activities and conceptual maps for in service teachers provided valuable data by which teacher knowledge could be analyzed. In particular, their extensive examination of representations used during instructional practice offered a specific location in which to explore teacher subject matter knowledge. However, in both Leinhardt and Smith's (1985) study and the Leinhardt et al., (1991) study, the teacher sample used could have

been adjusted in order to obtain more robust findings. In Leinhardt and Smith's (1985) study, the researchers observed the top three experts in their sample but did not include novices in the analysis of the representations. The Leinhardt et al., (1991) study did include both experts and novices, but analyzed the use of representations across different mathematical domains. Similarly, Lehrer and Franke's (1992) study of teacher knowledge of fractions and its effect on instructional practice involved the same domain, fractions, but examined teaching at different grade levels. Finally, case studies by Fennema, Carpenter, Franke & Carey (1992) involved different teachers across two different domains; addition and subtraction and fractions.

In the research analyzing teacher knowledge and instructional practice, each study seems to have a different format. Research on teacher knowledge and instructional practice may be most meaningful when the grade level and the domain are constant. This would allow more similarity for comparisons and as a result, a stronger study.

Developing Assessments of Teacher Content and Pedagogical Content Knowledge of Mathematics

Darling-Hammond (2000) asserted that although content knowledge is important to effective teaching, it is pedagogical content knowledge that exerts a stronger influence on overall teaching performance. Past studies investigating expert and novice teachers have used different definitions to identify experts such as improved student achievement (Leinhardt and Greeno, 1986) and/or supervisor recommendations (Borko & Livingston, 1989; Clarridge & Berliner, 1991; Swanson, O'Connor & Cooney, 1990). Additionally, extensive teaching experience is not sufficient when used as the only criterion to differentiate between an expert and novice.

In this study, content knowledge and pedagogical content knowledge measures are used to identify "experts" and "novices." An assessment of teacher knowledge of content and representations was constructed (Equivalence Assessment A) and was used to place teachers in instructional groups for professional development. After the identification of the instructional groups, four additional assessments were developed to further study and identify "experts" and "novices". The four assessments were: Equivalence Assessment B, Verbal Think Aloud Concept Map I, Verbal Think Aloud Concept Map II and the Written Concept Map.

In the following section each of the measures used in this study are discussed in detail. Additionally, the research background of each measure is summarized.

Equivalence Assessment A

The Equivalence Assessment A (Appendix D) was a twenty-item assessment of fraction knowledge used in Stage I of this study and adapted from assessments used by researchers in the Rational Number Project (Post et al., 1991). The instrument was field-tested and revised in preparation for use in this study. The Rational Number Project (RNP) was funded by the National Science Foundation (NSF) in 1979. The project's earlier intent was to describe children's rational number development from its beginnings to its formal operational level in well-defined instructional settings. The RNP also included work with in-service teachers in the development of a model middle school mathematics teacher education program. As part of the middle school mathematics teacher education program a three-part assessment of fraction knowledge was developed and administered. Part 1 consisted of short answer items (adapted for the present study), Part 2 requested pedagogical explanations of solutions generated, and Part 3 consists of a

two-hour interview, all relating to rational number concepts: part-whole, decimals, ratios and percents, proportionality, and multiplication and division.

Equivalence Assessment B

Equivalence Assessment B (Appendix F) was developed for this study and adapted from Niemi's (1996) study of fraction knowledge of fifth grade students. Niemi developed and administered several measures of fraction knowledge to 540 fifth-grade students in 22 classrooms. Students were asked to represent their conceptual knowledge in several different task contexts and formats, and performance was compared across tasks. He found that the level of representational knowledge predicted performance on problem solving, justification, and explanation tasks. It was also found that explanations and justifications could be reliably and efficiently judged and that the assessments provided diagnostic information on the level and quality of individual students' understanding. The representational fluency and the problem solving/justification measures developed are discussed in detail below.

This present study adaptation changed Niemi's (1996) procedures of *selecting* representations to *direct questions* asking teachers to construct fraction representations. Additionally, procedures for problem-solving and justification were adapted. Teachers were asked to solve a set of word problems and give equivalence for expressions.

Written Concept Maps

A written concept map has been used in varying forms in the research on teacher knowledge. A written concept map allows teachers the opportunity to solve specific problems and/or draw a schematic representation of their thinking processes. In the present study, the teachers were asked to draw a concept map of key fraction concepts. A

series of five questions were used to prompt their thinking on key concepts. The Written Concept Map (Appendix G) used in the present study was adapted from an Explanation Task developed by Niemi (1996) in his assessment of conceptual understanding in fractions. The Explanation Task was adapted from an extensively validated content knowledge assessment model developed by Baker and colleagues (Baker, Aschbacher, Niemi & Sato, 1992; Baker, Freeman, & Clayton, 1991). In their study, the explanation prompt asked participants to imagine that they had been recruited to explain fractions on a television show, to write out what they would say and draw pictures to support their explanations. In the present study, the explanation prompt (Appendix G) asked teachers to imagine a new teacher asked them to do a lesson in their classroom on equivalent fractions. The new teacher has never taught equivalent fractions before and does not feel comfortable teaching mathematics.

To infer cognitive activities, we asked teachers to think aloud while mapping. The think-aloud technique has been used to reveal cognitive activities in performing a variety of tasks (Ericsson & Simon, 1993), for example, problem solving (Baxter & Glaser, 1998), multiple-choice test taking (Levine, 1998), concept-map construction (Ruiz-Primo et al., 2001b), and performance.

The knowledge structure of experts and successful learners is characterized by elaborate, highly integrated frameworks of related concepts (e.g., Chi, Glaser, & Far, 1988; Mintzes, Wandersee, & Novak, 1997), which facilitate problem solving and other cognitive activities (e.g., Baxter, Elder, & Glaser, 1996).

A concept map includes nodes (terms or concepts), linking lines (usually with a unidirectional arrow from one concept to another), and linking phrases which describe

the relationship between nodes. Linking lines with linking phrases are called labeled lines. Two nodes connected with a labeled line are called a proposition. Moreover, concept arrangement and linking line orientation determine the structure of the map (e.g., hierarchical or nonhierarchical).

Verbal Think Aloud Concept Maps I and II

Several researchers have used a **"**verbal think aloud**"** protocol to demonstrate differences in expert and novice knowledge. The verbal think aloud protocol has been used in conjunction with analyzing videotapes, specific problems and solutions within a content area and in specific content scenarios. In these studies, the verbal think aloud was audio taped and videotaped and then transcribed. For example, expert and novice teachers were shown a videotaped classroom lesson (Sabers, Cushings & Berliner, 1991). The experimental set-up involved three screens that showed simultaneous events occurring throughout the classroom (the left, center, and right). During part of the session, the expert and novice teachers were asked to talk aloud about what they were seeing. Later, they were asked questions about classroom events. Overall, the expert teachers had very different understandings of the events they watched than did the novice teachers. The expert teachers noticed the effectiveness of the classroom environment and the students' reaction to instruction, while the novice teachers' seemed confused and unable to reflect on the videotape in the same manner. The idea that experts recognize features and patterns that are not noticed by novices is potentially important for improving instruction. When viewing instructional texts, slides, and videotapes, for example, the information noticed by novices can be quite different from what is noticed by experts (e.g., Sabers et al., 1991; Bransford, Sherwood, Vye & Rieser, 1986).

Chi et al. (1981) studied experts' and novices' knowledge in the domain of physics using two verbal sorting tasks. The researchers defined the experts as individuals currently employed as physicists and the novices as college students. The participants were asked to categorize physics word problems into groups based on solution similarity. The sorting tasks revealed that the experts mention physics principles underlying the word problem such as Newton's Law of Motion, while novices generally looked to surface features for grouping similarity. Additionally, while both experts and novices shared a common general knowledge of physics, the experts' verbalized a complexity in solution methods which derived from both implicit and explicit knowledge of the subject.

Scripts

In the existing research literature, analysis of teacher classroom instruction has yielded concepts such as instructional scripts. These may provide a model useful in developing teacher-tutoring transcripts.

Leinhardt and Greeno (1986) used instructional scripts to determine the differences between expert and novice teachers' knowledge. Instructional scripts are the lessons that cover the topical material to be taught. Instructional scripts were made from audio tapes of the participants' lessons. The tapes were transcribed in order to provide a script of the teachers' lessons. Researchers compared the expert and novice teachers' scripts for clarity and connectedness. The results showed that expert teachers' scripts were clearer and contained more connection between concepts than the novice teachers' scripts.

Additionally, the use of scripts in the expert-novice research has allowed an examination of teacher actions during the instructional process. The research suggests

that expert and novice teachers use different types of cues to inform their instructional decision-making. Housner and Griffey (1985) found that expert teachers (defined by researchers as more experienced) change instructional strategies based on student achievement while novices (less experienced) focus more on class interest and ability. Novice teachers abandoned their lesson plan when their students' interest or enjoyment decreased in order to maintain classroom management.

Both the Leinhardt and Greeno (1986) and the Housner and Griffey (1985) studies suggest findings that may apply to the present study of less knowledgeable and more knowledgeable teachers.

Summary

The research on teacher knowledge and its effect on instructional practice is complex. In an attempt to clearly define the construct of teacher knowledge, several different perspectives have emerged. In the earliest research, teacher knowledge was defined by how many courses a teacher had completed in a particular subject area. Later, that research perspective was shown to be incomplete, that subject matter knowledge is an important component to teacher knowledge, but also that subject matter knowledge is not sufficient. Shulman (1987) proposed three distinct categories of teacher knowledge: content knowledge, pedagogical content knowledge and curriculum knowledge. These three categories were widely accepted in the field of teacher research until recently.

It is from the area of research of teaching and learning mathematics that more specific understandings of teacher knowledge and its effects on instructional practice has been developed. Building on the work done by Shulman and other researchers, researchers in the area of teaching and learning mathematics began to examine what it

meant to "know" and "teach" mathematics. The research of Ball, Leinhardt, Lampert and Ma, through their different research perspectives has enhanced an understanding of what it means to teach mathematics. Ball developed case studies of prospective teachers of mathematics which highlighted the tendency to rely on procedural explanations rather than conceptual understanding. The work of Leinhardt and her colleagues examined instructional differences between "expert" and "novice" mathematics teachers. Through her work, the use of semantic webs and lesson structures indicated that "novice" teachers' understanding of specific mathematics topics was not as connected as "experts" and as a result, their lesson were less effective.

Lampert combined the knowledge of a researcher and an elementary mathematics teacher as a methodology for examining the impact of teacher knowledge on student understanding. Since the 1970's, Lampert's work has been influential in changing the paradigm in teaching and research of teaching in mathematics. Her research writings have served as a discussion point for the complicated relationship of theory and practice. Because Lampert is both a classroom teacher and a researcher, her work has been praised as an example of a unique view, between the worlds of research and the primary classroom. Lampert (1988) extensively used classroom transcripts of audiotapes and videotapes to closely examine the mathematics learning in fifth grade elementary school classroom. Her extensive use of audiotapes during mathematics lessons allowed discrete analysis of the teacher-student exchange.

Finally, Ma coined the term *profound understanding of mathematics* to describe the subject matter knowledge of mathematics and the pedagogical content knowledge teachers possess in bringing a deeper understanding of mathematical concepts to their

students. While recent research has brought us closer to the question, "How much knowledge must a teacher have in order to effectively teach mathematics?" the question has not fully been answered. As the research has shown, the question cannot be answered in the general sense; it must be framed more specifically. Each sub domain of mathematics has different knowledge requirements. Fennema, et al. (1993) study involving a second grade teacher trained in CGI demonstrates this point well. The second grade teacher used a variety of instructional techniques to bridge her students' understanding of addition and subtraction but when the same teacher switched to a fraction lesson, she relied solely on a part-whole explanation. Thus, her students did not receive the benefit of different representations in their instruction of fractions.

Purpose of the Study and Hypotheses

This dissertation extends the research on teacher knowledge by documenting the pedagogical content knowledge used by teachers during one-on-one tutoring sessions. In a pilot study (Chestnut-Andrews, 2005) it was shown that pedagogical content knowledge may have an influence on the type and quality of scaffolds used during instruction. Teacher-student interactions during one-on-one tutoring sessions were audio taped, transcribed and coded. After the sessions had been transcribed, two independent readers read and coded the transcripts using the following codes: episode, exchange and scaffold. After all transcripts were coded, the total number of episodes, exchanges and scaffolds were analyzed. Teacher-student interactions coded as scaffolds were further differentiated as either general scaffolds or pedagogical content scaffolds. Pedagogical content scaffolds were identified by 1) the types of models used (area, set or linear); 2) model translations (partitioning, fair shares, fraction strips); 3) use of relationships (part/whole,

part/part, and relationships among different types of fractions) and instructional strategies (conceptual or numeric). By combining the methodologies used in the expert/novice research and as well as recent teacher knowledge assessments, this study of teacher classroom tutoring provides an example of how content and pedagogical content knowledge of teachers can be quantified and analyzed. For the purposes of this study, pedagogical content scaffolds were examined using types of models (area, set, linear, numeric expressions and numeric operations).

The hypotheses for the study are stated below.

The first hypothesis examines the influence of knowledge of equivalent fractions on the use of scaffolds during instruction.

H1: Teachers with more knowledge of equivalent fractions will use more scaffolds during instruction than teachers with less knowledge of equivalent fractions.

The second hypothesis examines the influence of knowledge of equivalent fractions on the variety of models (area, linear, set, numeric expressions and numeric operations) used when teachers present scaffolds during instruction.

H2: Teachers with more knowledge of equivalent fractions will use a greater variety of models when presenting scaffolds than teachers with less knowledge of equivalent fractions.

The third hypothesis examines the relationship of the number of years teaching and pedagogical content scaffolds used during instruction.

H3: There will be no relationship between number of years of teaching experience and the number of pedagogical content scaffolds used during tutoring instruction.

The fourth hypothesis examines the relationship of the number of years teaching and the variety of models when presenting scaffolds during tutoring instruction.

H4: There will be no relationship between number of years teaching and the variety of models used during tutoring instruction.

CHAPTER 3

Methodology

This section includes details on study participants, procedures, measures and data analysis.

Participants

Sixty elementary teachers from a middle and upper socioeconomic status suburban school district in a Northeastern state participated in a summer mathematics staff development program. The district population is approximately five thousand elementary students, 95% of whom are Caucasian. As part of the staff development program, fifty-three teachers completed an assessment of fraction equivalence (Appendix D) and a demographic questionnaire (Appendix E). On the basis of their scores, thirty-two teachers were asked to volunteer for Stage II of the study. Following the completion of the Stage II assessment, ten teachers were selected for Stage III of the study. These teachers audio taped tutoring sessions with individual students in their classrooms on equivalent fractions.

Procedures

Stage I: Screening of Participants

During Stage I, fifty-three elementary (kindergarten through fifth) teachers completed the Equivalence Assessment A and the Demographic Questionnaire. Based on the analysis of the results from the Equivalence Assessment A, sixteen teachers were identified as either possessing a high level of knowledge (17-20 points) or a low level of knowledge (8 – 14 total points) in the domain of equivalent fractions. Teachers from these two groups were asked to volunteer for the second stage of the study.

Stage II: Informed Consent and Assessment

Teachers (n=32) who agreed to participate in Stage II of the study were sent consent letters (Appendix A). In Stage II, the thirty-two volunteers were asked to complete the Equivalence Assessment B (Appendix F), the Written Concept Map (Appendix G) and the Verbal Think Aloud Concept Map I (Appendix H). Equivalence Assessment B consisted of two parts. Part I contained two questions regarding fraction representations and Part II contained five word problems which required the teachers to provide a detailed written explanation of how they solved the problem. Further explanations of Equivalence Assessment B, the Written Concept Map and the Verbal Think Aloud Concept Map I appear in the Measures section below.

Stage III: Teacher-student data collection

In Stage III, ten teachers were identified as more or less knowledgeable about equivalent fractions based on their scores on Equivalence Assessment B (Appendix F), the Written Concept Map (Appendix G) and the Verbal Think Aloud Concept Map I (Appendix H) were asked to participate in Stage III of the study. In Stage III, each teacher identified a student in their mathematics classroom for whom she provided two individual tutoring sessions on equivalent fractions and audio taped the sessions. The student was selected by the teacher based on the student's performance in a lesson on equivalent fractions. That is, the teacher identified the student based on the student's misconception of the concepts taught during the lesson.

It is the district policy that the parents of each registered student complete a media release form. This media release form gives parental/guardian consent for a student to be audio taped and/or videotaped during the course of the school year. Additionally, a

parent/guardian consent form (Appendix B) and an assent form for the students being tutored (Appendix C) were forwarded to the parents of identified students by the teacher. The investigator enclosed a self-addressed stamped envelope for the return of the forms.

Identification of the teacher and student pairs was made using the codes T1 – T10 (Teacher 1 – Teacher 10). Students were coded S1, S2 and etc. (Student 1, Student 2 and etc.). During Stage III, ten teachers audio taped two lessons on equivalent fractions. The audiotapes were transcribed and coded using a Transcript Analysis and Coding measure developed for this study and described below. After the tutoring sessions were completed, the teachers were asked to construct the Verbal Think Aloud Concept Map II (Appendix J). The Verbal Think Aloud Concept Map II was modeled after a "think aloud" interview. In their "think alouds", teachers were asked to state the ideas about the concepts they believed are important for students to know about fractions and equivalent fractions. The teachers' think aloud concept maps were audio taped and transcribed for analysis.

At the completion of the study, participating teachers were given a summary of the study results.

Measures

Stage I: Equivalence Assessment A

The Equivalence Assessment A (Appendix D) consisted of twenty items designed to measure knowledge of equivalence including the subtopics of equivalence applications, concept of unit and missing value. The Equivalence Assessment A used in Stage I was adapted from assessments used by researchers in the Rational Number Project (Post, et al. 1991). The instrument was field-tested and revised in preparation for

use in this study.

Stage I: Demographic Questionnaire

The Demographic Questionnaire (see Appendix E) consisted of questions regarding teachers' educational background, years teaching, specific mathematics training/courses and educational philosophy regarding the teaching and learning of mathematics for elementary students. The items for the questionnaire were selected and adapted from a teacher questionnaire used by the University of Michigan in their Study of Instructional Improvement (Rowan, Schilling, Ball, & Miller, 2001). Several of these selected items have been used in many studies and are of interest for a possible relationship with Equivalence Assessment A. For example, years of teaching experience has been shown to have little relationship to pedagogical content knowledge (Hanushek, 1986; Strauss & Vogt, 2001, March).

Stage II: Equivalence Assessment B

Equivalence Assessment B was a two-part assessment designed to measure teachers' knowledge of fractional representations and problem solving (Appendix F). Equivalence Assessment B was adapted from measures used in prior research on conceptual understanding of mathematics (Niemi, 2001). The first part of the assessment requires teachers to draw graphic representations for a given symbol. The second part of the assessment requires that teachers solve five symbolically presented fraction problems.

Scoring for Section I of Equivalence Assessment B examines teacher knowledge of part/part and part/whole relations as evidenced by types of models (area, set, linear) as well as numeric representations (expressions and/or operations). The total number of points possible in Section I was five points for each question. One point was given for

each type of model or numeric expression written. Credit was not given if more than one type of the same model or numeric expression was written. Specifies examples of teacher knowledge are shown in Table 1.

Table 1

Scoring Criterion for Section I Equivalence Assessment B

Types of Models	Examples	Numeric Representations	Examples
Area Model	Partitioning Squares	Expressions	½ + ⅓
Set Model	Fair Shares of Sets	Operations	½ + ⅛= ⅝
Linear Model	Fraction Strips	n/a	n/a

Below are scored exemplars from Section I of Equivalence Assessment B. The exemplars show an example of a five point response and a three point response for question number 1 *(How many different ways can you represent the fraction 5/3?)*. For example, the five point response contains area, set, linear, numeric expression and numeric operation models.

Five point response- Question 1 – Teacher 6

Three point response- Question 1- Teacher 21

Figure 8. Teacher Response – Equivalent Assessment B Question 1

Scoring for Part II (problem-solving) of the Equivalence Assessment B was based on a rubric which scores each item based on two points. One point was given for the correct answer and another point was given for correct representation of the problem. The total number of points that could be obtained on Part II was ten points. Below are scoring exemplars from Part II of Equivalence Assessment B Question 3 showing a two point response and a one point response *(Justin has a measuring cup with a capacity of 2/3 cup. How many times will he fill his measuring cup with flour to measure 3 1/3 cups?)*

Two point response – Question 3- Teacher 15

One point response – Question 3 – Teacher 23

Figure 9. Teacher Response Equivalence Assessment B Question 3.

Stage II: Written Concept Map

The teachers were given the Written Concept Map (Appendix G) and one blank sheet of 8 ½ x 11 paper. The instructions and subsequent directions were read aloud to the teachers. The teachers were asked to draw a concept map of key fraction concepts. A series of five questions were used to prompt their thinking on key concepts.

Scoring for the Written Concept Map (Appendix L) was adapted from Niemi's (1996) Explanation Task Rubric used to analyze student responses about fractions. The rubric was based on a previously validated content-knowledge scoring rubric (Baker, Aschbacher, Niemi, & Sato, 1992; Baker, Freeman, & Clayton, 1991; Niemi, 1993, 1994). The rubric had five separately scored dimensions: (a) general quality of content knowledge, (b) concepts and principles, (c) knowledge of facts and procedures, (d) misconceptions, and (e) integration of knowledge. Two independent raters scored the papers. After reviewing the assessment tasks and scoring rubric, raters selected and used model responses illustrating score points on each of the dimensions to be scored. The Explanation Task Rubric included in Appendix L.

Stage II: Verbal Think Aloud Concept Map I

The researcher read the instructions for the Verbal Think Aloud Concept Map I which contained prompts (Appendix H). The teachers were given two blank sheets of 8 ½ x 11 paper and instructions (Appendix I). The think aloud administration was audio taped and transcribed for analysis

The process used to construct the concept maps was adapted from Leinhardt and Smith (1985). They observed mathematics lessons and created transcripts for those lessons. Leinhardt and Smith (1985) then created a map showing the main topics of the lessons and the relationships of the topics as links on the map. Two independent readers reviewed the sixteen transcripts from and developed concept maps using this technique.

The researcher and two independent raters scored a sample of the verbal concept maps and then the scores were compared. Differences in scores on the differentiation of concepts attribute were discussed further. These discussions led to fine tuning of the

descriptors for the attributes in the rubric. The two raters then scored a sample of 10 maps independently using the revised rubric. The scores were entered on score sheets. One of the raters was an elementary teacher and one was a middle school math teacher.

Stage III: Tutoring Transcripts

The Transcript Analysis and Coding Measure (Appendix M) was developed in connection with identifying teacher pedagogical content scaffolds in tutoring. For those teachers who were selected to progress to Stage III of the study, a Transcript and Coding Measure was used to analyze the transcripts. Five scores were derived from the transcript analysis: total number of exchanges, total number of scaffolding episodes, total number of pedagogical content scaffolds, total number of models used and variety of representations used. The Transcript and Coding Measure is further explained in detail in Appendix M.

Stage III: Verbal Think Aloud Concept Map II

The researcher read the instructions for the Verbal Think Aloud Concept Map II (Appendix K). The teachers were given two blank sheets of 8 ½ x 11 paper and instructions that included verbal prompts from Appendix J. Additionally, the teachers were given two sections of the tutoring transcripts from their tutoring sessions. These sections were selected by the researchers based on the number of scaffolding episodes counted in each exchange. One exchange section of the transcript contained seven or more scaffolding episodes while the other selected exchange section contained three or less scaffolding episodes. The think aloud administration was audio taped and transcribed for analysis. The scoring procedure for the Verbal Think Aloud Concept Map II followed the same protocol for the Verbal Think Aloud Concept Map I.

Data Analysis

Data Analyses for each stage of the study are described below:

Stage I:

In Stage I, the fifty-three teachers completed both the Equivalence Assessment A and the Demographic Questionnaire. Frequency distributions and descriptive statistics including the range, mean and standard deviation were obtained. Regression analyses were used to examine the relationship between the Equivalence Assessment A scores and the following variables in the Demographic Questionnaire: years of teaching, number of mathematics courses completed and grade level taught (K-5). T tests were used to further examine the characteristics of the teachers with more knowledge of equivalent fractions and the teachers with less knowledge.

Stage II:

In Stage II, the total number of teachers completing the Equivalence Assessment B, the Written Concept Map and the Verbal Think Aloud Concept Map I was thirty-two (sixteen teachers from the more knowledgeable group and sixteen from the less knowledgeable group).

Scores were obtained on Equivalence Assessment B for: Section I, Section II, the total for both sections, model representations and numeric expressions. Frequency distributions and descriptive statistics including range, mean and standard deviation were obtained. Additionally, the scores of teachers with more knowledge (N=16) and less knowledge (N=16) for Equivalence Assessment A scores were calculated and compared.

Stage III:

Transcript Analyses were made for ten teachers, five teachers from each group. The analyses for each teacher generated four scores. These scores are: total number of exchanges, total number of scaffolding episodes, total number of pedagogical content scaffolds and total number of representations.

The Verbal Think Aloud Concept Map II was analyzed and descriptive statistics were completed. Additionally, case studies for four of the teachers in this stage were constructed using information from all data sources in Stages I, II and III (reported in Results Section). The purpose of the case studies was to verify the initial categorizations of more knowledgeable and less knowledgeable teachers in Stages I and II. Additionally, the case studies were used to identify the knowledge bases and their connections in the domain of equivalent fractions.

Hypotheses

The proposal's hypotheses and pertinent data analysis are described in detail below.

Hypothesis 1: Teachers with more knowledge of equivalent fractions will use more scaffolds during instruction than teachers with less knowledge of equivalent fractions. A t test of the differences in the mean number of pedagogical content scaffolds of the two groups was used.

Hypothesis 2: Teachers with more knowledge of equivalent fractions will use a greater variety of models when presenting scaffolds than teachers with less knowledge of equivalent fractions. This hypothesis was tested, using a t test, comparing the mean number of model types used in pedagogical content scaffolds for the teachers with greater

and lesser knowledge of equivalent fractions.

Hypothesis 3: There will be no relationship between number of years of teaching experience and the number of pedagogical content scaffolds used during tutoring instruction. This hypothesis was examined by correlating the number of years teaching and the number of pedagogical content scaffolds used during tutoring instruction.

Hypothesis 4: There will be no relationship between number of years teaching and the variety of models used during tutoring instruction. This hypothesis was tested using a correlation coefficient.

Chapter 4

Results

This chapter is organized with results from the analyses at each stage of the study. Additionally, the hypotheses of the study are examined.

Stage I Demographic Questionnaire and Equivalence Assessment A

Demographic Data and Equivalence Assessment A Scores

Fifty-three teachers participated in Stage I of the study (n=53). All participants were certified elementary education teachers. Of the fifty-three participants, forty-four participants (83%) were female and nine were male (12%). The mean score on Equivalence Assessment A for the participants (n=53) was 15.42 (SD=2.7). The scores on Equivalence Assessment A ranged from 8 to 20 points.

There was no significant difference in the Equivalence A scores between the males (Mean=15.89, SD=2.7) and the females (Mean= 15.32, SD=2.7).

Teaching Experience

The participants (n=53) varied in the total number of teaching years and grade levels taught. The participants' years of teaching ranged from 3 to 15 years (Mean= 6, SD=2.7). Two were kindergarten teachers (3.8%), twelve first grade (22.6%), four second grade (7.5%), twelve third grade (22.6%), eleven fourth grade (20.8%) and twelve grade five (22.6%).

Educational Background

The fifty-three participants possessed a range of educational backgrounds. Almost one-half (47%) of the participants were undergraduate education majors, while the remainder of participants (53%) held undergraduate majors in English, social/behavioral

science, math and foreign language. The education undergraduate majors had the lowest mean score on Equivalence Assessment A (Mean=13.80, SD=2.3). The participants with English undergraduate majors had a mean score of 16.88 (SD= 2.1). The tests for differences in means among undergraduate majors on Equivalence Assessment A were not significant.

Math Courses Completed (Undergraduate and Graduate)

Twenty-seven of the participants (50.9%) had completed four to six undergraduate mathematics courses, while the remainder of participants (49.1%) had completed one to three undergraduate mathematics courses. Six participants (11.3%) had never completed an undergraduate mathematics methods course while forty-seven (88.7%) had completed between one and three courses.

Of the fifty-three participants (n=53), thirty-seven (69.8%) had not taken a graduate mathematics course, while sixteen (30.2%) had taken between one and three graduate courses in mathematics. Additionally, seven participants (13.2%) had completed no graduate mathematics methods courses, while forty-six (86.8%) had completed between one and three graduate mathematics methods courses.

Equivalence Assessment A, Years of Teaching, Math Courses and Grade Level Taught

Table 2

Correlation of Equivalence Assessment A Scores with Years of Teaching, Total Number of Math Courses and Grade Level Taught

	Score on Equivalence Assessment A	Years Teaching	Math Courses	Grade Level Taught
Score on Equivalence Assessment A	1	.191	.472(**)	.410(**)
Years Teaching		1	.560(**)	.099
Math Courses			1	.352(**)
Grade Level Taught				1

**Correlation significant at the 0.01 level (2-tailed).

Table 2 shows the correlations between each variable. There were a number of significant relationships. There are significant correlations between Equivalence Assessment A scores and the grade level taught (r=.41), as well as total number of math courses taken (undergraduate and graduate) (r=.56). The number of years of teaching did not correlate with the Equivalence Assessment A but correlated with the total number of math courses taken (r=.35).

Regression analysis. A simultaneous regression was performed, with the dependent variable of Equivalence A scores and the independent variables of number of years teaching experience, total number of undergraduate and graduate math courses taken and grade level of the students taught. The resulting model accounted for 29% of the variance in Equivalence Assessment A (R=.542, p=.01).

As shown in Table 3, the total number of math courses taken is the best predictor of Equivalence A scores and this predictor is statistically significant (p=.010). The next strongest predictor is grade level of students taught which is statistically significant (p=.042). The total number of years teaching experience was not a significant predictor.

Table 3

Coefficients Table for Math Courses, Grade Level Taught and Years of Teaching

	B	Std. Error	Beta	P value
Math Courses	1.507	.564	.415	.010
Grade Level Taught	-.068	.145	-.069	.042
Years of Teaching	.459	.219	.270	.640

n=53

Teacher Self-Reported Instructional Practice

In the Demographic Questionnaire participants (n=53) reported on their mathematics instructional practice by responding to two questions (Questions 9 and 10) regarding their students mathematics activities and one question (Question 11) regarding their preparation for their mathematics lessons. Appendix N shows the analysis of the participants' responses by percentage and number.

An analysis of Questions 9 and 10 shows the majority of teachers presented definitions or steps for a procedure and had their students perform tasks requiring methods already introduced at least one to two times per week. Seventy-three percent of the participants (N=40) did not have their students work on a mathematics investigation/problem for several days. Forty-five percent (N=24) and fifty-four percent (N=29) respectively had their students work on mathematics problems that have multiple answers or solution methods, and had their students discuss math problems in pairs or groups.

An analysis of Question 11 shows that all of the participants agree or strongly agree that they refer to their teacher guides of curriculum materials when preparing their lessons. However, fifty-three percent (N=30) do not refer to or use information found in curriculum frameworks or standards documents. Finally, sixty-eight percent of the participants (N=36) do not refer to the content of assessments when preparing their mathematics lessons.

Stage II Equivalence Assessment B, Written Concept Map and Verbal Think Aloud Concept Map I

In Stage II of the study, teachers were selected and grouped by their scores on the Equivalence Assessment A administered in Stage I. Sixteen teachers (n=16) who scored between 8 and 14 on Equivalence Assessment A were designated as Group A (less knowledgeable-LK). Group A consisted of six kindergarten grade teachers, four first grade teachers, three second grade teachers, two third grade teachers and one fourth grade teacher. The sixteen teachers (n=16) who scored between 17 and 20 points on Equivalence Assessment A were designated as Group B (more knowledgeable-MK). Group B consisted of two second grade teachers, four third grade teachers, four fourth grade teachers, and six fifth grade teachers. A t test was calculated on the Equivalence Assessment A scores for Group A and Group B. The results were significant, $t(30)=13.07$, $p<.0001$.

Both Group A (LK) and B (MK) were administered the Equivalence Assessment B, the Written Concept Map and the Verbal Think Aloud Concept Map I. For each measure, the data was analyzed and are described below.

Equivalence Assessment B

The first part of the assessment asked teachers to draw graphic representations for a given symbol. The second part of the assessment required that teachers solve five fraction problems. Scoring for Section I of Equivalence Assessment B examines teacher knowledge of part/part and part/whole relations as evidenced by types of models (area, set, linear) as well as numeric representations (expressions and/or operations). The total number of points possible in Section I was five points for each question. One point was

given for each type of model or numeric expression written. Credit was not given if more than one type of the same model or numeric expression was written.

Group A (LK) total scores on Equivalence Assessment B ranged from 12 to 19, with the mean score of 14.3 (SD=1.78). Group B (MK) scores on Equivalence Assessment B ranged from 15 to 20, with a mean score of 17.4 (SD=1.73). The t tests for means for Group A and Group B were significant, $t(30) = 4.93$, $p < .0001$. Teacher scores for each section are listed in Appendix O. Appendix P shows the number of teachers using the models and numeric representations used to solve the problems for both sections of Equivalence Assessment B. In comparing the results of Group A (LK) and Group B (MK), the table shows that Group B (MK) used a greater number and more variety of models in response to the questions. Also, Appendix P shows that in Section II of Equivalence Assessment B Group B (MK) consistently used model representations in conjunction with numeric representations in solving the word problems. In contrast, Group A (LK) used numeric operations to solve the word problems. Appendix P shows that Group B (MK) had a more consistent performance on Equivalence Assessment B than did Group A (LK). Through the consistent performance on Equivalence Assessment B, Group B (MK) demonstrated a more extensive knowledge base than Group A (LK).

Written Concept Map. Group A (n=16) and Group B (n=16) answers to the Written Concept Map (Appendix G) were scored on five dimensions: (a) general quality of content knowledge (GQCK), (b) concepts and principles (C/P), (c) knowledge of facts and procedures (FACT), (d) misconceptions (MIS), and (e) integration of knowledge (INT).

Table 4

Written Concept Map Scores Mean and SD on Five Dimensions for Group A and Group B (N=32)

Dimension	Group A (LK)		Group B (MK)	
	Mean	*SD*	Mean	*SD*
GQCK	1.26	.99	3.31	.90
C/P	.31	.57	2.31	1.57
FACT	.81	1.00	2.28	1.22
MIS	2.16	.94	1.23	.85
INT	1.45	.69	2.75	1.01

A t test was calculated on the Written Concept Maps scores for Group A and Group B. The results were significant for all variables: GQCK: $t(30)=6.13$, $p<.0001$, C/P: $t(30)=4.80$, $p<.0001$, FACT: $t(30)= 3.70$, $p<.0009$, MIS: $t(30)=2.93$, $p<.0065$ and INT: $t(30)= 4.27$, $p<.0002$. The differences were in the expected direction with the less knowledgeable teachers having the most misconceptions. Examples of scored Written Concept Maps are included in Appendix Q.

Verbal Think Aloud Concept Map I

Two independent raters reviewed Group A (LK) and B (MK) Verbal Think Aloud Concept Map I. Individual concepts maps were constructed based on the teachers' transcripts. The teachers' transcripts were read several times in order to group the concepts discussed into main categories and subcategories. A total of thirty-two (N=32) were constructed and four (N=4) were selected for more in depth analysis. The four concept maps were selected because of the wider range of responses; two concepts maps from two individual teachers (MK) and two individual teachers (LK). These concept maps with written examples are included in their entirety in Appendix R.

Group A (LK) Verbal Think Aloud Concept Map I

Group A (LK) concept maps contained four main categories and five subcategories. The breakdown of main categories and subcategories as well as examples of both is shown in Table 5.

Table 5

Group A (LK) Verbal Think Aloud Concept Map I Categories

Main Categories	*Subcategories*	*Examples*
Mathematics of Fractions	*Operations*	*Adding Like/Unlike Denominators* *Inverse Operations*
	Part/Whole	*What part of whole is the piece*
Teachers' Knowledge	*Curriculum*	*Sequence of Lessons*
Teachers' Beliefs Fractions	*Students*	*Ability to Learn Fractions* *Facility with Multiplication and Division*
Students' Knowledge	*Prior Experiences*	*Success in earlier grade* *Parent support*

Note N=16

An example of a concept map from Teacher 4 in Group A (LK) is included in Appendix R. The numbers of concepts discussed as well as the connections between those concepts were minimal.

Group B (MK) Verbal Think Aloud Concept Map I

Group B (MK) concept maps contained five main categories and eleven subcategories. The main categories and subcategories, as well as examples of both, are shown in Table 6.

Table 6

Group B (MK) Verbal Think Aloud Concept Map I Categories

Main Categories	*Subcategories*	*Examples*
Mathematics of Fractions	*Operations*	*Adding, Subtraction, Multiplication Division, Inverse Operations*
	Equivalence	*Adding Like/Unlike Denominators*
		Part/Part
		Part of Fraction
	Part/Whole	*Part of 1, Part of the Whole*
	Proportions	*In Relationship to fractions*
	Decimals	
Knowledge of Content	*Curriculum*	*Differentiation of Lessons*
		Building on Student Knowledge
		Deep Understanding of Mathematics
		Connections between Numbers
Instructional Knowledge	*Teaching Strategies*	*Sequence of Lessons*
		Representing Fractions in different ways
		Building on Students' Knowledge
Teachers' Beliefs of Mathematics	*Fractions*	*Ability to Learn Fractions*
		Facility with Multiplication and Division
Students' Growth	*Prior Experiences*	*Connecting Prior Knowledge and New Knowledge*

Note N=16

An example of a concept map from Teacher 7 in Group B (MK) shown in Appendix R The numbers concepts and connections between those concepts were more extensive.

Stage III. Tutoring Transcripts and Verbal Think Aloud Concept Map II

Tutoring Transcripts

In Stage III of the study, cognitive analyses were made for ten teachers (N=10), five teachers (N=5) from the group designated more knowledgeable (MK-two third grade teachers, one fourth grade teacher and two fifth grade teachers) and five teachers (N=5)

from the group designated less knowledgeable (LK-two kindergarten teachers, two first grade teachers, one second grade teacher). The identification of the teachers was made using the codes T1 – T5 (MK group) and T6-T10 (LK) group. Each of the ten teachers audio taped two lessons on equivalent fractions. The resulting twenty transcripts were coded using the Transcript Analysis and Coding measure described in Appendix M. Five scores for each teacher were derived from the transcript analysis: total number of exchanges (EX), total number of scaffolding episodes (SE), total number of pedagogical content scaffolds (PCS), the total number of variety of types of models used (M) and the total number of representations (R) (See Appendix S). The variety of types of models (M) used during the tutoring sessions was designated using category coding: a) area model, b) set model, c) linear model, d) area and set models, e) area and linear models, f) set and linear models and g) area, set and linear models. For example, in Appendix S, Teacher 1 (MK) used area and set models (category D) while Teacher 6 (LK) used only area models (category A).

The tests for differences in means for Group A (LK) and Group B (MK) were significant for all categories as follows: Exchanges (EX): $t(8)= 4.63$, $p<.002$, Scaffolding Episodes (SE): $t(8)= 4.39$, $p<.002$, Pedagogical Content Scaffolds (PCS): $t(8)= 6.93$, $p<.005$ and Number of Representations (R): $t(8)= 4.81$, $p<.001$.

Verbal Think Aloud Concept Map II

The teachers (N=10) were asked to complete the Verbal Think Aloud Concept Map II (Appendix J). The teachers were given two sections of their tutoring transcripts selected by the researcher based on the number of scaffolding episodes recorded. One section of the tutoring transcripts contained seven or more scaffolding episodes while the

other section contained three or less scaffolding episodes.

There were differences in the manner that Group A (LK) and Group B (MK) responded to the Verbal Think Aloud Concept Map II. The responses were analyzed into two categories: teacher directed statements and student directed statements. The teacher directed statements were defined as statements that emphasized teacher actions and/or knowledge. For example, a teacher directed statement was: "Right here I should have used a simpler example to help him understand." The student directed statements were defined as statements that emphasized student actions or knowledge. For example, a student directed statement was "I don't know why he wasn't understanding that part…"

Group B (MK) made a total of seventy-two teacher directed statements and twenty-four student directed statements. Group A (LK) made a total of forty-eight teacher directed statements and fifty-six student directed statements. Appendix T contains the Verbal Think Aloud Concept Map II and a sample from the transcript used to build the concept map. The four examples given are taken from the case studies conducted.

Introduction to Teacher Case Studies

Case studies were based on four teachers using the data collected in Stage I, II and III of the study. The purpose of these case studies was to verify the initial categorizations of less knowledgeable and more knowledgeable teachers as well as to describe in more detail the differences in the knowledge base for more knowledgeable teachers and less knowledgeable teachers. The differences are more evident in each of the assessment procedures used in this study.

The four teachers were selected for more in depth analysis based on the range of their responses on the assessments and how those responses reflected strongly on their

knowledge base. Table 7 shows the assessment data in each stage of the study for four teachers.

Table 7

Summary of Assessment Scores for Teachers 10 and 7(MK) and 1 and 9 (LK)

	Teacher 10 3rd Grade	Teacher 7 5th Grade	Teacher 1 2nd Grade	Teacher 9 1st Grade
Assessment A	19	20	10	10
Assessment B	19	20	14	13
Written Concept Map	top 1/3	top 1/3	lower 1/3	lower 1/3
Verbal Think Aloud Concept Map I	7 main	6 main	3 main	3 main
Transcripts				
Exchanges	8	8	5	5
Scaffolds	16	14	8	7
Pedagogical content scaffolds	12	12	6	6
Models	all	all	area	area
Representations	5	5	2	2
Verbal Think Aloud Concept Map II				
Teacher Directed Comments	7	6	4	2
Student Directed Comments	3	3	8	5

Presentation of Case Studies –Demographic Information and Teaching Experience

Case Study 1 Teacher 10

Teacher 10 was a third grade teacher who had been teaching for a total of 5 years. She taught fifth grade for two years before moving to the third grade classroom to which she had been assigned for the past two years. Teacher 10 had a Bachelors degree in education and a Masters degree in special education. Questions 9 and 10 of the Demographic Questionnaire suggested that Teacher 10 had a more constructivist approach in her views of teaching mathematics, e.g. students building their knowledge

through mathematical experiences.

Case Study 2 Teacher 7

Teacher 7 was a fifth grade teacher who had been teaching for a total of 11 years. She taught third grade for six years before moving to the fifth grade. Teacher 7 had a Bachelors degree and a Masters degree in education. Questions 9 and 10 of the Demographic Questionnaire suggest that Teacher 7 had a balance of traditional as well as constructivist views on teaching mathematics. For example, while her students spend a great deal of time practicing skills in worksheet form, there were also opportunities for group work and extended discussion.

Case Study 3 Teacher 1

Teacher 1 was a second grade teacher who had been teaching for a total of five years. She had a Bachelors degree in early childhood education and a Masters degree in education. Questions 9 and 10 of the Demographic Questionnaire suggested that Teacher 1 was more traditional in her views of teaching mathematics. For example, Teacher 1 stated that "It's not good for students to work together all the time, sooner or later they have to do it on their own and how do you know who really knows it or not if they are always working together?"

Case Study 4 Teacher 9

Teacher 9 was a first grade teacher who had been teaching for a total of 7 years. She taught kindergarten for three years before moving to the first grade classroom to which she had been assigned for the past four years. Teacher 9 had a Bachelors degree in psychology and a Masters degree in education. Questions 9 and 10 of the Demographic

Questionnaire suggested that Teacher 9 was more "traditional" in her views of teaching mathematics. Teacher 9 stated "The basic of mathematics – adding and subtracting numbers is important in the first grade; they have to understand the basics before they can get the harder stuff."

Summary of Case Studies – Assessment Data Stages I-III

The four teacher case studies provide a more in depth analyses of the assessment procedures used to elicit teacher pedagogical content knowledge and math knowledge of fractions. A comparison of the data from the teachers designated as MK (more knowledgeable) will be discussed. Next, the data from the teachers designated as LK (less knowledgeable) will be examined.

Teacher 10 and Teacher 7 Equivalence Assessment A (MK)

As described earlier, the scores from Equivalence Assessment A in Stage I of the study were used to divide a group of teachers (N=53) based on their knowledge into two differentiated staff development groups. For the purpose of this study and staff development, the two groups were labeled as LK (less knowledgeable) and MK (more knowledgeable). Teacher 10 scored a total of nineteen points and Teacher 7 scored a total of twenty points (maximum 20 points) on the Equivalence Assessment A in Stage I of the study. As a result of the scores, both teachers were designated as belonging to the MK (more knowledgeable) group. The teachers' responses to Equivalence Assessment A demonstrated an ability to answer questions using both algorithm procedures and conceptual methods to solve problems.

Table 8 shows examples of Teacher 10 and Teacher 7 responses to questions on Equivalence Assessment A.

Table 8

Teacher 10 and Teacher 7 Responses to Equivalence Assessment A

Question	Teacher 10	Teacher 7
#1 – Equivalent or Not Equivalent ½ and 1/3 (Make a drawing)		
#4-Equivalent or Not Equivalent 2/3 and ¼ (Draw fraction strips)		

Teacher10 and Teacher 7 responses show both a numeric and model answer. Their answers demonstrate that they were able to represent fractions using different representations. Particularly, in drawing appropriate fraction strips to represent 2/3 and ¼ the teachers demonstrate the knowledge both of the manipulative tool of fractions strips and how to use them to demonstrate equivalency.

Teacher 10 and Teacher 7 Equivalence Assessment B

Equivalence Assessment B (Appendix F) was administered in Stage II of the study to thirty-two volunteers (N=32) of the original fifty-three participants (N=53). Equivalence Assessment B was a two-part assessment designed by the researcher to measure teachers' knowledge of fractional representations and problem-solving. The total number of points possible in Section I was five points for each question. The total number of points that could be obtained on Section II of Equivalence Assessment B was ten points.

Teacher 10 scored a total of 19 out of 20 points on Equivalence Assessment B. On Section I, she scored 9 points out of a possible 10 points. Teacher 7 scored a total of 20

out of 20 points on Equivalence Assessment B. She scored the maximum number of points on each section. Similar to their responses on Equivalence Assessment A, the responses were both procedurally and conceptually complete.

Teacher 10 and Teacher 7 Written Concept Map

The Written Concept Map was scored on five different dimensions: (a) general quality of content knowledge (GQCK), (b) concepts and principles (C/P), (c) knowledge of facts and procedures (FACT), (d) misconceptions (MIS), and (e) integration of knowledge (INT). Teacher 10's scores placed her in the upper third of the group of sixteen (Group B MK).

Teacher 10's Written Concept Map revealed her strong base of knowledge of fractions. Specifically, the concepts and principles she referred to in her concept map were coherent, correct and not repetitive. Additionally, each concept or principle was demonstrated by an appropriately correct procedure or fact. She displayed no serious errors or misconceptions in her explanations and examples. Finally, Teacher 10 successfully integrated her facts, procedures, principles and concepts into a complete conceptual model, therefore scoring the maximum number of five points on the Integration/Argumentation dimension.

Teacher 7's Written Concept Map total score across the five dimensions was in the upper half of the group of sixteen more knowledgeable teachers. Similar to Teacher 10, Teacher 7's concepts were successfully linked with appropriate subcategories and these subcategories had written examples.

Teacher 10 and Teacher 7 Verbal Think Aloud Concept Map I

In response to the Verbal Think Aloud Concept Map I, Teacher 10 drew a concept map (Appendix R) that contained seven main categories: fractions, improper fractions, proper fractions, operations, part-whole, part-part and equivalence. Under each of these main concepts she listed two to three minor categories and included numerical examples of all of the concepts. As shown below, the concept map is well defined and all categories linked to each other.

Teacher 7 drew a concept map (Appendix R) that contained six main categories: fractions, improper fractions, proper fractions, operations, part-whole, and equivalence. Under each of these main concepts she listed two to three minor categories and included numerical examples of all of the concepts. The concept map is well defined and all categories linked to each other.

Teacher 10 and Teacher 7 Verbal Think Aloud Concept Map II

Teacher 10 provided an extensive concept map demonstrating her knowledge of fraction concepts. She used the transcripts provided almost exclusively to build the Verbal Think Aloud Concept Map II. Each concept or point on the concept map was supported by a section of her tutoring transcripts. Also, only Teacher 10 added a category of student learning to her concept map after reviewing the tutoring transcripts. The Verbal Think Aloud Map II she constructed is shown in Appendix T. Teacher 7 also developed an extensive concept map demonstrating her knowledge of fraction concepts. Mathematical concepts as well as instructional strategies were included in the concept map. Additionally, Teacher 7 used the tutoring transcripts as an assessment of how well

she had taught the concepts that the student found difficult. The Verbal Think Concept Aloud Map II she constructed is shown in Appendix T.

Teacher 10 and Teacher 7 Transcript Analyses

The analysis of Teacher 10 tutoring transcripts showed that there were eleven exchanges completed, sixteen scaffolding episodes and fourteen pedagogical content scaffolds used. The fourteen pedagogical content scaffolds used during the tutoring session represented area, linear and set models. Additionally, five different were representations used.

The analyses of Teacher 7 tutoring transcripts showed that there were eight exchanges completed, nine scaffolding episodes and twelve pedagogical content scaffolds used. The twelve pedagogical content scaffolds used during the tutoring session consisted of all types of models (i.e., area, linear, and set) and five different representations (e.g., partitioning rectangles, fraction strips).

Summary of Teacher 10 and Teacher 7 Data

In Stage I of the study, Teacher 10 and Teacher 7 were placed in the group of teachers designated as more knowledgeable. This designation was based solely on the teachers' scores on Equivalence Assessment A. Teacher 10 scored 19 out of 20 points on the assessment while Teacher 7 scored 20 out of 20 points. Therefore, it was expected that Teacher 10 and Teacher 7 would perform equally as well on Equivalence Assessment B. Equivalence Assessment B was a two-part assessment designed by the researcher to measure teachers' knowledge of fractional representations and problem-solving. Both Teacher 10 and Teacher 7 performed well on Equivalence Assessment B. Specifically, the teachers completed each problem showing both numerical and

conceptual processes. For example, question one in Section I asked the teachers to show the fraction 5/3 in as many ways as possible. Both Teacher 10 and Teacher 7 drew five different representations – area model, set model, linear model, a numeric representation and an expression. The thoroughness of their answers demonstrated a depth of knowledge of fraction representations.

The Written Concept Maps drawn by both of the teachers provided further confirmation that the teachers had a firm knowledge base from which to draw. Neither teacher had serious misconceptions or errors in their understanding of fractions. For example, a common error, not made by these teachers, was the belief that there were no fractions between 8/9 and 1. Each teacher had cohesive concept maps with accurate subcategories that were well linked with examples. The Written Concept Maps drawn by both teachers are included in Appendix Q.

The connectedness of the teachers' knowledge continued to be evidenced in their responses to the Verbal Think Aloud Concept Map I (Appendix R). Each of the categories listed in the Verbal Think Aloud Concept Maps was successfully supported with minor categories. For example, Teacher 7's concept map moved from the broad category of equivalence to the minor category of the reciprocal of 1 as a method to compute fractions.

Teacher 10 and Teacher 7 responses on the Verbal Think Aloud Concept Map II (see Appendix T) continued to demonstrate the depth of the teachers' knowledge and how they used that knowledge to instruct their students. Teacher 10 referred to her transcripts when building her Verbal Think Aloud Concept Map II. For example, the transcript read: Teacher 10, "When we were talking about the fraction ¾ earlier you said

there was an easy way that you could remember an equivalent fraction. Can you tell me about it?" Student "Yeah, I remember cause three quarters see you get it three quarters is seventy-five cents out of a dollar. So the fraction is seventy-five over a hundred." Teacher 10 recognized that the student used the knowledge about money and connected that information to equivalence. Teacher 10, therefore, included student knowledge about other math concepts as a category. Similarly, Teacher 7 talked through her transcripts and remembered reviewing a particular concept of fair shares with her student. The transcript notes: Teacher 7 said, "Let's look at this problem, four brownies and five people…let's talk through it together." When Teacher 7 reviewed the transcripts she stated, "This concept was tough for him…he needs more experiences with partitioning…I could develop similar problems…it might be best to look at this again with a similar but easier problem."

Teacher 1 and Teacher 9 Equivalence Assessment A (LK)

The scores from Equivalence Assessment A in Stage I of the study were used to divide a group of teachers (N=53) based on their knowledge into two differentiated staff development groups. For the purpose of this study and staff development, the two groups were labeled as LK (less knowledgeable) and MK (more knowledgeable). Both Teacher 1 and Teacher 9 scored a total of ten points (maximum 20 points) on the Equivalence Assessment A in Stage I of the study. As a result of the scores, both teachers were designated as belonging to the LK (less knowledgeable) group. The teachers' responses to Equivalence Assessment A were primarily procedural. That is, even when the directions specified to make a drawing or to show fractions strips both teachers displayed the answers numerically, e.g. ½ x ¾ and ¼ x ¾. When questioned, each teacher gave

similar responses about their use of algorithms. The researcher asked, "Taking a look at Questions 1 and 4 on Equivalence Assessment A, I noticed that you used algorithms to solve both problems even though the directions stated to make a drawing and draw fraction strips, can you tell me more about that choice?" Teacher 1 answered "I made a drawing, I just did it with numbers and since I don't use fraction strips, I just did the computation." Teacher 9 answered, "Drawing and stuff like that is not the most efficient way to solve the problem, I teach my students to be efficient and that's how to approach problems."

Teacher 1 and Teacher 9 Equivalence Assessment B

Teacher 1 scored 14 out of 20 points while Teacher 9 scored 13 out of 20 points. In an analysis of the teachers' responses for Section I of Equivalence Assessment B, neither teacher scored higher than five points on this section. Section I of Equivalence Assessment B asked the teachers to show different representations of certain fractions. In Section II of Equivalence Assessment B, teachers were asked to solve five word problems. Teacher 1 and Teacher 9 each answered most of the questions accurately. In each case, however, the teachers' responses were solely procedural. There was no attempt on either teachers' part to solve the word problems any other way.

Teacher 1 and Teacher 9 Written Concept Map

The participants' Written Concept Map was scored on five dimensions: (a) general quality of content knowledge (GQCK), (b) concepts and principles (C/P), (c) knowledge of facts and procedures (FACT), (d) misconceptions (MIS), and (e) integration of knowledge (INT). Teacher 1's Written Concept Map scores were in the lower third of the group of sixteen.

Teacher 9's Written Concept Map total score also placed her in the lower third of the group. The Written Concept Map revealed serious misconceptions in her knowledge of fractions. Specifically, her definition of a fraction was limited to the concept of a fraction as part of a whole. The number of facts and procedures written on her concept map contained many repetitions of that same concept.

Teacher 1 and Teacher 9 Verbal Think Aloud Concept Map I

Teacher 1's response to the Verbal Think Aloud Concept Map I showed eight major concepts with fifteen connections between the main concepts of her map. In response to the Verbal Think Aloud Concept Map I, Teacher 9 drew a concept map that contained three main categories: Related Concepts in Fractions, Important Concepts and Difficult to Learn. Under each of these main concepts she listed two to three minor categories and included numerical examples of some of the concepts. As shown in Appendix R, the concept map is not well defined nor are all categories linked to each other.

Teacher 1 and Teacher 9 Transcript Analyses

The analyses of Teacher 1 tutoring transcripts showed that there were five exchanges completed, seven scaffolding episodes and six pedagogical content scaffolds used. The six pedagogical content scaffolds used during the tutoring session were all area models and only two different representations were used.

The analyses of Teacher 9 tutoring transcripts showed that five exchanges were completed, and seven scaffolding episodes and six pedagogical content scaffolds were used. The six pedagogical content scaffolds used during the tutoring session were all area models and only two different representations were used. (Samples from Teacher 1

and Teacher 9's transcripts are included in Appendix T – Verbal Think Aloud Concept Map II).

Teacher 1 and Teacher 9 Verbal Think Aloud Concept Map II

Teacher 1 designed a concept map demonstrating her knowledge of fraction concepts. Mathematical concepts as well as instructional strategies were included in the concept map. Teacher 9 made a concept map that resembled both the Written Concept Map I and the Verbal Think Aloud Concept Map I in the main categories it contained. The concepts of fractions did not differ in the Verbal Think Aloud Concept Map II. Even though Teacher 9 was instructed to use the transcript analyses from the tutoring sessions in the concept map, she did not. Teacher 9 did review the transcripts but they did not provided a stimulus for thinking about her instructional practice. The Verbal Think Aloud Map II she constructed is shown in Appendix T.

Hypotheses

In this section, each of the hypotheses will be examined and the corresponding data discussed.

Data for Hypothesis 1 and Hypothesis 2 was obtained from the tutoring transcripts scores of Group A and Group B. The mean and standard deviations for the tutoring transcripts scores are included in Table 9.

Table 9

Means, Standard Deviations and t Values for Tutoring Transcript Scores

	Group A		Group B				
Measure	M	SD	M	SD	t	df	p
Exchanges (EX)	4.40	1.14	7.40	0.89	4.63	8	.00*
Scaffolding Episodes (SE)	8.80	1.48	13.40	1.82	4.39	8	.00*
Pedagogical Content (PCS) Scaffolds	6.60	1.14	11.60	1.14	6.93	8	.00*
Variety of Models	3.40	.55	4.40	.55	2.88	8	.02*
No. of Representations (R)	3.40	1.14	7.00	1.22	4.81	8	.00*

N=5 teachers in each group, *Significance at p<.05, alpha level, .05.

Hypothesis 1: Knowledge of equivalent fractions and scaffolds

Hypothesis 1: Teachers with more knowledge of equivalent fractions will use more scaffolds during instruction than teachers with less knowledge of equivalent fractions. A t test of the differences in the mean number of pedagogical content scaffolds of the two groups was used. The t tests of the differences was significant, t=6.93, p<.00. Therefore, Group B (More knowledgeable) used more pedagogical content scaffolds during instruction.

Hypothesis 2: Knowledge of equivalent fractions and models

Hypothesis 2: Teachers with more knowledge of equivalent fractions will use a greater variety of models when presenting scaffolds than teachers with less knowledge of equivalent fractions. The variety of models was coded 1 to 5 such that 1 = one type of model (i.e. area, set, linear, numeric expression or numeric operation), 2 = any two model, 3= any three models, 4=any four models and 5= all five models. This hypothesis was tested, using a t test, comparing the mean number of model types used in

pedagogical content scaffolds for the teachers with greater and lesser knowledge of equivalent fractions. The t tests of differences were significant, $t=2.88$, $p<.02$.

Hypothesis 3: Years of teaching and pedagogical content scaffolds

Hypothesis 3: There will be no relationship between number of years of teaching experience and the number of pedagogical content scaffolds used during tutoring instruction. This hypothesis was examined by correlating the number of years teaching mean=6, SD = 2.91) and the number of pedagogical content scaffolds (mean = 9.1, SD = 2.90) used during tutoring instruction. The resulting correlation was r =.25. The correlation shows there is little relationship between the number of years of teaching experience and the number of pedagogical content scaffolds.

Hypothesis 4: Years of teaching and variety of models

Hypothesis 4: There will be no relationship between number of years teaching and the variety of models used during tutoring instruction. This hypothesis was examined by correlating the number of years teaching (mean=6, SD = 2.91) and the variety of models (mean = 3.9, SD = 2.90) used during tutoring instruction. The resulting correlation was r=.2. This correlation shows there is little relation between the number of years teaching and the number of different models of representation used during tutoring instruction.

Although, this study is based on a small sample of teachers, all four hypotheses were supported by the data. That is, for this sample, teachers' knowledge of equivalent fractions and how they applied that knowledge during tutoring was based on their integrated knowledge rather than how many years they had been teaching.

Chapter 5

Discussion and Conclusions

Overview

This chapter contains: 1) a discussion and summary of major findings, 2) study strengths, 3) limitations, 4)and suggestions for future directions.

The purpose of this study was to define the scaffolding procedures, the mathematical knowledge of equivalent fractions and the pedagogical content scaffolds used by more knowledgeable and less knowledgeable classroom teachers in working with individual students. A three-phase assessment process was developed in this study. The pedagogical content knowledge and cognitive and tutoring patterns of teachers with more knowledge were compared with the cognitive and teaching patterns of teachers with less knowledge. The study was intended to extend current research in the area of teacher knowledge, both in assessment of teacher knowledge and in the context area of classroom tutoring. Quantitative and qualitative analyses within each stage of the study were conducted and the results are discussed below. In addition, the theoretical framework of the current study as well as strengths and limitations of the study's methodology are reviewed. Suggestions for future research are also addressed in that discussion. Finally, the theoretical and educational implications of this study are presented.

The Three- Phase Assessment Process

In the first two phases of this study, the more traditional assessments (Equivalence A and B), provided initial data which assisted in identifying more and less knowledgeable teachers. These assessments focused on the teachers' mathematical knowledge of equivalent fractions. A conceptual framework developed in a pilot study

defined types of models of equivalent fractions as area, set, linear, numeric expression and numeric operations (see Table 2). The conceptual framework was used to develop scoring criteria for Equivalence Assessment A and B. These types of models also formed a basis for identification of teacher pedagogical content knowledge used in scaffolds during tutoring sessions.

Also, the second phase of the assessments, the Written Concept Map was designed. The Written Concept Maps were scored on five dimensions: (a) general quality of content knowledge (GQCK), (b) concepts and principles (C/P), (c) knowledge of facts and procedures (FACT), (d) misconceptions (MIS), and (e) integration of knowledge (INT) (see Table 4). Significant differences were noted on all five dimensions between less knowledgeable and more knowledgeable teachers.

Additionally, a verbal think aloud procedure was used to capture the teachers' conceptual framework for equivalent fractions and provide a comparison of these conceptual frameworks for more and less knowledgeable teachers (see Tables 5 and 6). As shown in Tables 5 and 6, the more knowledgeable teachers had more main categories, subcategories and examples. In addition, the more knowledgeable teachers included categories of instructional knowledge and student growth in their verbal think aloud while the less knowledgeable teachers did not include these categories. Also, the more knowledgeable teachers' subcategories of mathematics of fractions were linked from the part/whole concept to decimals and proportions. The more knowledgeable teachers' subcategories ranged from all aspects of operations of fractions, to teaching strategies and to recognition of students' prior knowledge. On the other hand, the less knowledgeable teachers' subcategories included operations as a broad statement, teachers' beliefs about

students, and curriculum as a sequence of lessons (See Table 5 and 6). These findings support the research of Leinhardt and Smith (1985) in their study of differences in expert and novice teachers' knowledge of equivalent fractions.

In the third and final phase, transcripts were made of the teachers tutoring an individual student on equivalent fractions. Building on previous research, a conceptual framework for the identification of teacher pedagogical content knowledge of equivalent fractions was structured (Bliss & Askew, 1996; Cazden, 1983). It is in this context that the concept of pedagogical content scaffolds was defined and as part of the framework of teachers' pedagogical content knowledge, exchanges and scaffolding episodes (Chi, et al., 2001) of tutoring transcripts were also identified. In this study, the framework is also used to situate teachers' mathematical knowledge and pedagogy as linked in pedagogical content knowledge (Shulman, 1986).

The data from the tutoring transcripts suggests that more knowledgeable teachers were able to instruct their students in a more complete and in depth manner. These teachers had more exchanges and scaffolding episodes during tutoring than did their counterparts (see Table 8). Additionally, more knowledgeable teachers initiated more pedagogical content scaffolds during instruction than did less knowledgeable teachers.

Two concepts maps were developed to further define the differences between more and less knowledgeable teachers, Verbal Think Aloud Concept Map I and II. Of particular interest is the second map because teachers looked at two sections of tutoring transcripts that contained examples of pedagogical content scaffold, i.e., their interactions with students. As shown earlier shown in Table 6, comparing two of the most knowledgeable teachers with two of the less knowledgeable teachers, the comments of

more knowledgeable teachers were focused on their instruction practice. In contrast, the less knowledgeable teachers focused more on the characteristics of the students than in their own interaction. For example, a less knowledge teacher stated "He should have known that…see I explained it in the first session." By contrast, the most knowledgeable teacher's statements focused on actions that she could take to improve the student's understanding of the content. An example was "The idea of partitioning the rectangle seemed easier for him…maybe I could use the same idea to help him better with this problem."

Summary

This study's hypotheses and data both support and extend the current research on the identification of teachers' knowledge primarily in the area of equivalent fractions. The purposeful design of the assessments in this study may serve to inform researchers on practical measures of quantifying teacher knowledge.

Comparing the asssessments

Equivalence Assessment A and B were designed as more traditional forms of assessment of teacher knowledge. Both assessments offered the participants an opportunity demonstrate both their procedural and conceptual knowledge of equivalent fractions. Research (Lampert & Ball, 1998) has demonstrated that a teacher's instruction varies depending on the knowledge of mathematics they possessed. The researchers documented fourth grade teachers instructing their students on the concept of multiplication using extensive algorithm practice and rote learning. When teachers possess strong procedural knowledge but lack conceptual understanding of a concept, the teacher's instruction lacks depth (Ball, 1990; Borko, 1992).

Equivalence Assessment B focused on measuring a teacher's fractional knowledge as evidenced by responses that required fractional representations and problem-solving.

The Concept Maps

Concept maps (Written Concept Map, Verbal Think Aloud Map I and II) were designed for this study as a method to elicit and analyze the connections among concepts in teachers' knowledge in the area of equivalent fractions. The concept maps permit examination of the relationships among concepts. The scores of assessments of pedagogical content knowledge (e.g., tutoring transcripts and Verbal Think Aloud Concept Maps I and II), show disparity between the two groups. These findings are consistent with prior research (Leinhardt & Smith, 1985) which found differences in pedagogical content knowledge between the expert and novice teachers as demonstrated by comparing conceptual maps of the two groups on the differences related to their knowledge base. In this study, the three concept maps were used to document teachers' knowledge of equivalent fractions as demonstrated in three different contexts.

Use of Models and Representations

In this study, the results show that there significant differences in the quantity (number of models) and the variety of models and representations used during tutoring sessions (See Table 7 and 9) between less knowledgeable teachers and more knowledgeable. The differences were also evident in the case studies analyzed. Prior research has noted the importance of models and representations. Leinhardt and Smith (1985) and others (Ma, 1999 and Ball, 1991) noted the more knowledgeable (expert) teachers provided their students with a variety of conceptual representations. In the case

studies, in contrast, both less knowledgeable teachers used only the area model and only two representations in their tutoring sessions. These results are consistent with the research reported in Leinhardt and Smith's (1985) study in which most teachers conceptualized fractions in a part-to-whole relationship.

The findings here also support by earlier research of Leinhardt, Putnam, Stein and Baxter (1991) who stated that analysis of representations used during instruction could be used to determine a teacher's subject matter knowledge. The research (Leinhardt, et al., 1991; Leinhardt and Smith, 1985) identified and analyzed representations in a qualitative within an instructional explanation. In contrast, the present study defined specific representations used for the instruction of equivalent fractions and compared more knowledge and less knowledgeable teachers use of representations during tutorial instruction.

Major Strengths

A major strength of this study involved the development and use of a series teacher knowledge about equivalent fractions assessments that were designed and administered in this study, the identification of pedagogical content scaffolds as a particular type of teacher knowledge in tutoring and the use of tutoring transcripts as a means to elicit a teacher's knowledge in the specific context of classroom tutoring of an individual student.

Another strength of this study was the development and recognition of pedagogical content scaffolds as a form of teacher knowledge. In a pilot study, the concept of pedagogical content scaffolds (PCS) was identified and defined. PCS are an extension of the concept of scaffolding student learning in their zone of proximal

development. When students are learning within a specific domain (e.g. fractions), there are specific mathematical concepts or ideas that, when linked to instruction, become pedagogical content scaffolds.

Finally, the use of tutoring transcripts to build concept maps appears to be unique to this study. Using tutoring transcripts as part of a protocol for a teacher talk aloud (Verbal Think Aloud Concept Map II) enables the participant to review the actual lesson. This process facilitates teacher reflection of the interaction with the student in the tutoring context.

Study Limitations

There were several limitations to this study. First, the final sample size of ten participants is too small to consider the possible meaning of the findings beyond the particular teachers who participated in the study. Additionally, there can be no assumption that the sample is representative of all elementary school teachers. The purpose of the study was exploratory; to develop and test a method by which teachers' pedagogical content knowledge could be meaningfully examined in the context of classroom tutoring.

Another limitation of the study is that the three-phase assessment process is a time consuming process when considering the results of each phase of assessment, and further reflection on the data, concept maps and actual teaching transcripts provide more information on the depth of a teacher's pedagogical content knowledge.

However, there is also strength in the three-phase process since because it allows teachers' pedagogical content knowledge to be viewed both traditional assessment tasks which have been used in past research to measure teacher knowledge, as well as

performance assessments which examine teachers' cognitive processes. By using a three-phase process, it may be possible to obtain a more complete picture of a teacher's pedagogical content knowledge and how this specific is used during the instructional tutoring process.

Future Directions

While there are many different avenues of research that can be supported based on the results of this study (e.g. relationships of teacher content knowledge, and students' development of mathematical knowledge), the research in this study may prove useful both in the area of research of teacher professional development and practice.

Hiebert, et al., 2002 recommended that teacher pedagogical content knowledge must be examined further. The researchers suggested that one way of working toward a unified goal of building teacher knowledge is through producing and verifying professional knowledge. They forecasted a new type of knowledge about improving classroom practice and that such knowledge would culminate into a professional knowledge base for teaching and support long-term continuing improvement in teaching. More school districts offer their teachers extensive staff development opportunities in order to improve their instruction. This study began with fifty-three teachers with different educational experiences and backgrounds. In the first stage of the study, these teachers received differentiated staff development in the area of equivalent fraction based upon the analysis of their answers.

While phases two and three of the study were very time consuming, it seems that the assessments in these phases were able to describe teachers' strengths and weaknesses in their knowledge base as well as their instructional processes in more detail than in the

phase one assessments. The identification of specific misconceptions (e.g. the number of fractions between 8/9 and 1) and areas of weakness in instruction (e.g. using only area models when discussing equivalence) could be a powerful tool to pinpoint the areas of knowledge that need to be further developed. Also, the use of the concept maps, both written and verbal could assist in organizing and extending the prior knowledge a teacher possesses, to identify and support the professional development of teachers.

List of Appendixes

Ph.D. Program in Educational Psychology

Appendix A

The Graduate School and University Center
The City University of New York
365 Fifth Avenue
New York, NY 10016-4309
TEL 212.817.8285 FAX 212.817.1516

(letterhead)

Teacher Consent Form

Dear Colleague:

My name is Alexis Chestnut-Andrews and I am a Ph.D. student in Educational Psychology at the City University of New York, Graduate School and Principal Investigator of this project, entitled "Pedagogical Knowledge and Scaffolds: Measuring Teacher Knowledge of Equivalent Fractions." The purpose of this study is to examine how teachers teach their students equivalent fractions. The information from the Demographic Questionnaire and the Equivalence Assessment you completed this summer assisted in identifying you as a participant for this part of the study. I would like permission to interview you about your experiences teaching equivalent fractions. The interview should take from thirty to forty-five minutes. I would also like to audiotape two tutoring sessions in which you instruct one of your students in equivalent fractions. The audiotape of each tutoring session should take between ten and twenty minutes.

Your participation in this study is voluntary and you are free to withdraw from participation at any time. Participation is **NOT** related to your professional standing in any way. All information will be kept strictly confidential by assigning a code which will substitute for your name on all materials. No teachers' names will be used in any report of the research. The audio tapes will be used for research purposes only and stored in a locked file in my office.

There is no known risk associated with participation in this study. The results will be used to further the understanding of teacher knowledge used in teaching mathematics.

If you would like a summary of the major findings, I will be pleased to send you one at the conclusion of the story.

If you have any questions related to this study, please feel free to contact me at (914) 261-2859, gecko394@juno.com or my faculty advisor, Professor Carol Tittle at (212) 817-8288, ctittle@gc.cuny.edu. If you have any questions regarding your rights as a participant in this study, you may contact Kay Powell, IRB Administrator, The Graduate Center/City University of New York at (212) 817-7525, kpowell@gc.cuny.edu.

Thank you for participating in this study. I will give you a copy of this form to take with you.

I agree to have my interviews and tutoring sessions audio-taped please [circle one]:
Yes No

Participant's signature Date Investigator's signature Date

Graduate School & University Center / CUNY
IRB
Approved
05-07-0835
Approved: 9/26/05 To 9/25/06 Expired:

Ph.D. Program in Educational Psychology

Appendix B

The Graduate School and University Center
The City University of New York
365 Fifth Avenue
New York, NY 10016-4309
TEL 212.817.8285 FAX 212.817.1516

Parent/Guardian Consent Form

Dear Parent/Guardian:

My name is Alexis Chestnut-Andrews and I am a Ph.D. student in Educational Psychology at the Graduate School of the City University of New York. I would like your assistance in carrying out the study for my dissertation entitled, "Pedagogical Knowledge and Scaffolds: Measuring Teacher Knowledge of Equivalent Fractions." The purpose of this study is to examine the language used between teacher and student during tutoring in mathematics. This analysis requires audio taping the individual tutorial sessions.

Children sometimes receive individual help from their teachers. The purpose of my study is to examine the assistance that teachers give students during an individual math tutorial. With your permission, two sessions will be audio taped.

Your child's participation in the study is voluntary. You and your child are free to withdraw from the study at any time. Whether or not you give permission for your son or daughter to participate, his or her academic standing will not be affected in any way. All information will be kept strictly confidential and your child's name will not appear on any documents or reports. The audio tapes will be used for research purposes only. They will be secured in a locked file cabinet in the researcher's office.

There is no risk in participating in this study and the results will be used to further the understanding of teacher knowledge of mathematics. The benefits of your child's participation is that it may further knowledge about the assistance teachers give to students in math tutorials.

THE GRADUATE CENTER
UNIVERSITY OF NEW YORK

Ph.D. Program in Educational Psychology

Appendix C

The Graduate School and University Center
The City University of New York
365 Fifth Avenue
New York, NY 10016-4309
TEL 212.817.8285 FAX 212.817.1516
A. Chestnut

Student Assent Form

STUDENT ASSENT

(To be read to student by teacher)

Your parent has given permission for you to help me learn more about how teachers help children learn math.

I will ask you to work on some math problems with me. If you have any questions with problems I will help you.

Would you like to be in the study?

If you agree to be in this study now, it's okay if you feel differently later on and can ask to stop being in the study.

Appendix D
Equivalence Assessment A

The purpose of this assessment is to explore how teachers think about the concept of fraction equivalence.

Directions:

1. Spend as much time as you need on the assessment and do not use a calculator.
2. Work independently on all of the items.

PART I

Are the following fractions equivalent or not equivalent? Make a drawing to support your answer.

1. ½ and 2/3

Solve the following problem:

2. May had ½ yard of string. She cut it into 4 equal pieces. How long was each piece?

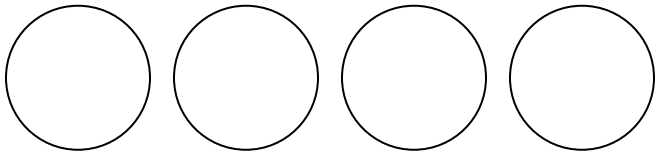
Write equivalent or not equivalent to describe the fractions. Draw fraction strips to support your answer for Questions 3 and 4.

3. 1/3 and 2/8

4. 2/5 and ¼

Appendix D
Equivalence Assessment A
(continued)

5. Shade 3/8 of these circles:



6. 8/14 = ____/12

7. 4/6 = 6/____

8. What will happen to the value of the fraction a/b if "a" is increased four times and "b" is halved?

Appendix D
Equivalence Assessment A
(continued)

9. ½=___/3

10. Jake had three containers of punch. Four people drank equal amounts and finished all of the punch. What fraction of one whole container did each person drink?

11. John has 5/4 pounds of peanuts. He wants to put them into small bags containing ¼ pound each. How many bags can he make?

12. How many fractions are between 8/9 and 1?

13. Given 9 circles are ⅔ of some unit, what is the whole unit?

Appendix D
Equivalence Assessment A
(continued)

14. xxxx xxxx = 4/3 of a unit of some unit, draw the whole unit.

15. Jen spent ½ of her money and then spent ⅓ of what was left. If she spent $100.00, how much money did she have to start with?

16. Construct as many different representations as possible for the following:

    a. 5/3

    b. ½ +⅓

17. What happens to 9/8 if the numerator is tripled and the denominator divided by 4?

Appendix D
Equivalence Assessment A
(continued)

18. Dave cut a circle into 6 equal pieces. He colored ½ of the pieces purple and ½ yellow. How many sixths are yellow?

19. Write a word problem that represents 1 ¾ ÷ 2. Solve the problem showing all of your work.

Thank you for answering these questions.

Appendix E
Demographic Questionnaire

Please answer the following questions:

1) **Are you: Circle one answer.**

   a) Female b) Male

2) **How many years have you worked as a teacher? Record whole years, not fractions or months. Round up to the nearest whole number and include the current school year.**

   Number of years____

3) **What was your undergraduate major field of study? Circle one.**

   a) Do not have an undergraduate degree
   b) Education
   c) English
   d) Social or Behavioral Sciences
   e) Foreign Language
   f) Mathematics
   g) Other_____ (Fill in blank)

4) **What type of teaching certification do you hold from the state where you teach? Circle all that apply.**

   a) Permanent or standard certification
   b) Probationary certification
   c) Temporary, provisional or emergency certification
   d) Alternative certification
   e) Not certified

5) **About how many undergraduate classes have you taken at college or university in the following areas?**

	None	1-3 classes	4-6 classes	more than 7 classes
Mathematics	□	□	□	□
Methods of teaching mathematics	□	□	□	□

Appendix E
Demographic Questionnaire
(continued)

6) About how many graduate classes have you taken at college or university in the following areas?

	None	1-3 classes	4-6 classes	more than 7 classes
Mathematics	□	□	□	□
Methods of teaching mathematics	□	□	□	□

7) What grade level(s) are the majority of the students in your mathematics class?

□ Kindergarten

□ 1st

□ 2nd

□ 3rd

□ 4th

□ 5th

8) On a typical day, how many minutes do you teach mathematics?
Print the number of minutes on the line below.

_____ total minutes

9) This year, how often did the students in your math class do the following?

	Less than once a month	1-3 times per month	1-2 times per week	3-4 times per week	Never
Work on mathematics textbook, worksheet or board work exercises for practice or review	□	□	□	□	□
Work on mathematics problems that have multiple answers or solution methods	□	□	□	□	□
Discuss mathematical ideas, problems solutions or methods in pairs or small groups	□	□	□	□	□
Write extended explanations of mathematical ideas, solutions or method	□	□	□	□	□
Work on a mathematics investigation, problem or project for several days	□	□	□	□	□

Appendix E
Demographic Questionnaire
(continued)

10) This year, how often did the students in your math class do the following? Mark (X) EACH item.

	Less than once a month	1-3 per month	1-2 times per week	3-4 times per week	Never
Listen to me present the definition of a term or the steps of a procedure	□	□	□	□	□
Perform tasks requiring methods or ideas already introduced to students to students	□	□	□	□	□
Assess a problem and choose a method to use from those already introduced to students	□	□	□	□	□
Perform tasks requiring methods or ideas not already introduced to students	□	□	□	□	□
Explain an answer or a solution method for a particular problem	□	□	□	□	□
Analyze similarities and differences among representations, solutions or methods	□	□	□	□	□
Prove that a solution is valid or that a method works for all similar cases	□	□	□	□	□

11) To what extent do you agree or disagree with the following statements about how you prepare mathematics lessons class? Mark (X) each item.

	Strongly Disagree	Disagree	Agree	Strongly Agree	NA
I frequently refer to and use information found in curriculum frameworks or standards documents	□	□	□		□
I frequently refer to and use information from the teachers' guides associated with the curriculum materials adopted by my school	□	□	□		□
I frequently refer to the content of assessments	□	□	□		□

This completes the questionnaire. Thanks again for completing this questionnaire. Your help is greatly appreciated.

Equivalence Assessment B

Please complete the following assessment. The assessment is divided into two sections. Read the directions carefully.

Section I

How many different ways can you represent the fraction 5/3? Please use the space below to show your representations. Please show as many representations as you can.

How many different ways can you represent the fraction 2/3? Please use the space below to show your representations.

Appendix F
Equivalence Assessment B
(continued)

Section II

Please solve the following problems. **Show all of your work for each question.**

1. Elissa and Matt bought ⅞ of a yard of fabric to make a class project. They used only ⅓ of a yard. How much fabric was left? Show your work using numbers, pictures and words.

2. The expression 1/15 ÷ (4/15 + ⅓) is equivalent to:

3. Justin has a measuring cup with a capacity of ⅔ cup. How many times will he fill his measuring cup with flour to measure 3 ⅓ cups? Show all your work using numbers, pictures and words.

Appendix F
Equivalence Assessment B
(continued)

4. Tikka made bamboo fishing poles for her sister and herself. She used 3/5 of a 10-foot stick of bamboo for her pole and the rest for her sister's. How long is her sister's fishing pole? Show your work using numbers, pictures and words.

5. In the first race of the season, Carol swam her event in 5 minutes and 15 seconds. Her goal was to be able to swim the in ⅔ the time by the end of the season. If she reaches her goal, how long will it take for her to swim the race at the end of the season? Show your work using numbers, pictures and words.

Appendix G
Written Concept Map

Imagine a new teacher asked you to do a lesson in their classroom on equivalent fractions. This new teacher has never taught equivalent fractions before and does not feel comfortable teaching mathematics. Draw the new teacher map or web of how the concepts of fractions are related. Use the blank sheets of paper you have been given to draw the map. Use as many pictures and words as you like.

Below are the questions the new teacher should be able to answer using the written concept map you make:

What is a fraction?

Why are there two numbers in a fraction?

How can you tell if two fractions are equivalent?

How can you add two fractions?

What other important ideas should this teacher know about equivalent fractions?

Appendix H
Verbal Think Aloud Concept Map I
(Participants Version)

I am also interested in how you would "map out" **your** knowledge of fractions and equivalent fractions. I would like for you to talk aloud about your knowledge. Talk aloud about these concepts and how they related or linked to each other. As you talk about these concepts please draw out your thoughts on the paper provided.

Appendix I
Verbal Think Aloud Concept Map I
Researcher Copy

I am also interested in how you would "map out" **your** knowledge of fractions and equivalent fractions. I would like for you to talk aloud about your knowledge about these concepts and how they related or linked to each other.

Take a moment to gather your thoughts and when you are ready to proceed let me know. Please draw out a map of the concepts as you talk about them.

The investigator may use the following prompts:

"What is it about fractions that makes it so difficult to teach or to learn?"

"What concepts in fractions are related?"

"What concepts are important for students to know about fractions?"

"Is there any else you would like to include?"

Appendix J
Verbal Think Aloud Concept Map II

Participants' Version

I am also interested in how you would "map out" **your** knowledge of fractions and equivalent fractions. I would like for you to talk aloud about your knowledge about these concepts and how they related or linked to each other. Here are two of your tutoring transcripts that you may refer to when talking about the concepts of fractions and equivalent fractions.

Take a moment to gather your thoughts and when you are ready to proceed let me know. Please draw out a map of the concepts as you talk about them.

If you haven't already, please take a moment to read over the sections from your tutoring transcripts.

In these sections of the transcripts, what could you have done differently during the instructional piece to help the student understand?

Specifically, what other ways could you taught this concept differently?

"Is there any else you would like to include?"

Appendix K
Verbal Think Aloud Concept Map II

Researcher Version

I am also interested in how you would "map out" **your** knowledge of fractions and equivalent fractions. I would like for you to talk aloud about your knowledge about these concepts and how they related or linked to each other. Here are two of your tutoring transcripts that you may refer to when talking about the concepts of fractions and equivalent fractions.

Take a moment to gather your thoughts and when you are ready to proceed let me know. Please draw out a map of the concepts as you talk about them.

The investigator will use the following prompts:

"What is it about fractions that makes it so difficult to teach or to learn?"

"What concepts in fractions are related?"

"What concepts are important for students to know about fractions?"

If you haven't already, please take a moment to read over the sections from your tutoring transcripts.

The investigator will use the following prompts:

"In these sections of the transcripts, what could you have done differently during the instructional piece to help the student understand?

Specifically, what other ways could you taught this concept differently?

"Is there any else you would like to include?"

Appendix L
Explanation Scoring Rubric for Written Concept Map

1. General Impression of Content Quality (GICQ)

How much does the teacher know about this mathematical topic?
(1-5 Global Rating: 1=No Knowledge, 5=Highest level of understanding)

2. Number of Principles or Concepts (PN)

Record the number of principles or concepts that the student uses with comprehension. Principles and concepts are general, abstract ideas. To be counted, the idea must be clearly and explicitly stated; do not guess whether the teacher knows what s/he is talking about.

Examples related to fractions include the following ideas:

1. Between any two numbers you can find an infinite number of fractions
2. For any fractions you can find an infinite number of equivalent fractions
3. When you partition or measure some object in order to find a fraction of it, the partitions or measurement units must be equal in size
4. You can partition any quantity into as many equal-sized parts as you want; to get more parts, you just have to make them smaller
5. A fraction is a number that shows a relation between two other numbers

Note: A definition of a fraction as a part of a whole is conceptually limited and will not be counted as a concept in this rubric. This definition only covers common fractions.

3. Number of Facts and Procedures

Record the number of facts and/or procedures to the problem or topic.
Count one point for each piece of information or each procedure that the teacher demonstrates or explains. Facts include definitions and other statements. Typical procedures include: adding two fractions, finding equivalent fractions, drawing a circular representation of a fraction.

Do not additional points for multiple instances of the same procedures e.g. if the teacher draws twenty different circle and rectangle representations of the same fraction, this would count as knowledge about two procedures how to draw an area and a circle representation. A number line representation would count as an extra point.

4. Misconception/Errors

Award score points according to this scheme:

1- One or more serious misconception
2- One or more factual or procedural errors
3- No errors or misconceptions

A serious misconception is a conceptual errors such as believing that there are only a few fractions between 0 and 1. Factual errors are mistakes in definitions or in descriptions of procedures. Procedural errors are mistakes in carrying out procedures.

5. Integration/Argumentation (INT)

How well does the teacher integrate facts, procedures, principles and concept to develop a conceptual model? (1-5 point global rating:1=no integration, 5= highest level of integration) For

Appendix L
Explanation Scoring Rubric for Written Concept Map
(continued)

example teachers who successfully integrate graphics and text to make a point should receive at least a 2. Review anchor papers to determine other score points.

Appendix M

Transcript Analysis and Coding Measure

Pilot Study

In my pilot study, eight transcripts of teacher-student interaction during several tutoring sessions were analyzed. There were two overarching goals in the analysis of the transcripts: first, to categorize specific scaffolding behaviors on the part of the tutors and secondly, to attempt to define the depth of knowledge that the tutors possessed.

The author and an independent coder repeatedly read the transcripts of teachers and students engaged in one-on-one tutoring sessions. During the reading process, the readers referred to the study by Chi, Hausmann, Jeong, Siler, & Yamauchi, (2001) of expert human tutors to supplement the analysis and to assist in the coding scheme for the observed behaviors. Chi et al. (2001) conducted a study on the effectiveness on one-to-one tutoring. Since their study was similar to our pilot study in both method and procedure, we felt it beneficial to use their coding protocol as a reference for our own. Chi et al. (2001) transcribed twenty hours of one-on-one tutoring sessions. After they read the transcripts in study, the researchers came up with three comprehensive codings for those interactions. After the initial comprehensive coding, each of the three major codings was reread and additional codings of sections of these transcripts were made.

Similarly, after examining this study's transcripts and reviewing the study's hypothesis, the readers came to a consensus on three major codes to describe the actions taking place during tutoring. These major codes are ***the exchange code***, ***the scaffolding episode*** and ***the scaffold code.***

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

Exchange Coding.

The exchange coding defines an interaction between the tutor and the student that surrounds a main topic, concept or problem. During the exchange coding, the interaction was placed in brackets. For example:

[Teacher2: Let's look at this first problem..Ok, it says..Terry had 18 cookies, she wanted to share them equally with two friends. How many cookies would each friend get?

Student2: Um...18....how many people?

T2: Well, let's see..it's Terry and she wants to share with how many friends?

S2: Two...

T2: But, if she shares them with her two friends..how many people in total does she have to share them with? exchange continues....]

This part of the transcript is coded as an exchange (note brackets) because the conversation between the tutor and the students centered around one main problem. The unit of analysis for interaction was considered one exchange. All eight transcripts were analyzed and the number and nature of the exchanges were noted. The transcripts were separated by tutor. These results are listed below in Table 4.

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

Table 4. *Exchange Coding by Transcript and Teacher: Number of Exchanges*

	Teacher 1	Teacher 2
Transcript 1	4	3
Transcript 2	3	3
Transcript 3	4	3
Transcript 4	4	4
Total number of exchanges coded	15	13

As shown in Table 4, Teacher 1 had two more exchanges than Teacher 2.

Scaffold Episode Coding

The scaffold episode coding was first operationalized by Chi, et al. (2001) in their study of expert human tutors. Chi defined a scaffolding episode as an extended exchange dialogue that is initiated by the tutor's scaffolding moves and continue on the same concept, topic or problem for several turns until the concept is **left by the tutor**. The eight transcripts were reread and episodes annotated according to the construct. The episodes are marked by one asterisk at the beginning and end of the episode. The exchanges noted are included below:

Exchange start [Teacher2: Let's look at this first problem..Ok, it says..Terry had cookies, she wanted to share them equally with two friends. How many cookies would each friend get?

Student2: Um...18....how many people?

T2: Well, let's see..it's Terry and she wants to share with how many friends?

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

S2: Two...

Episode start *T2: But, if she shares them with her two friends..how many people in total does she have to share them with?

S2: Her and two people...

T2: So how many people in total are sharing the 18 cookies?

S2: Three?

T2: You don't sound so sure..are you sure?

S2: Yeah, cause its her (holds up one figure) and then two more (adds two more fingers) that's three.

Episode ends *T2: Ok...let's look at the rest of the problem..

Table 5. Scaffold Episode Coding by Teacher: Number of Scaffold Episodes

Teacher 1	Teacher 2
n=34	n=52

Scaffold Coding

Finally, the transcripts were reread in order to specify the types of scaffolds that the tutors used during instruction. Taking into account the various facets of scaffolding in past research, we choose a basic definition of a scaffold. In this study, a scaffold is a verbal prompt that pushes the

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

students' thinking or actions a bit further. Specific scaffolds are notated by double asterisks within the transcripts. An example of the scaffold code is noted below:

Exchange start [Teacher2: Let's look at this first problem..Ok, it says..Terry had cookies, she wanted to share them equally with two friends. How many cookies would each friend get?

Student2: Um...18....how many people?

T2: Well, let's see..it's Terry and she wants to share with how many friends?

S2: Two...that's nine cookies...

Episode start ***T2: But, if she shares them with her two friends..how many people in

Scaffold *** total does she have to share them with?

S2: Her and two people...

**Scaffold *****T2: So how many people in total are sharing the 18 cookies?

S2: Three?

Scaffold ***T2: You don't sound so sure..are you sure?

S2: Yeah, cause its her (holds up one figure) and then two more (adds two more fingers) that's three.

Episode ends *T2: Ok...let's look at the rest of the problem

Exchange continues until problem is solved. In the categorization of the scaffold coding scheme, two major subcategories emerged. The first scaffold subset was a general type of scaffold or attempt to further the students' thinking. One example of a general scaffold is the "leading question". A "leading question" can

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

be defined as a question directed to the student that causes a pause or reflection and that (depending on the answer) can offer the tutor further direction for instruction. An example is below:

Scaffold ***T2: You don't sound so sure..are you sure?

S2: Yeah, cause its her (holds up one figure) and then two more (adds two more fingers) that's three.

The tutor asks a "leading question" '...are you sure?'. This causes the student to pause and to defend the answer he has given. Because the student's answer was acceptable, the tutor can proceed with another problem or concept. If the student had not answered correctly, it would have signaled to the tutor that the concept or topic was not yet mastered. However, the teacher can either choose to ignore the incorrect answer or use the misconception to teach further. The key is the teacher's ability to assess the student's learning. Good and Grouws (1979) included a very limited examination of teachers' assessing student comprehension of subject matter. For a teacher to successively assess their students' knowledge of a particular subject matter they must make an instructional decision on when to ignore an error or to use that error as a springboard for further instruction.

The second type of scaffold subset categorized in the transcripts was the content specific scaffold. These content specific scaffolds dealt explicitly with the knowledge of part/whole and part/part relationships.These content specific scaffolds constitute the pedagogical content knowledge that each tutor brought to their content specific scaffolding episodes. Understanding that a teacher's knowledge base is connected and multi-faceted, we examined the transcripts for

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

patterns when that knowledge was presented. The transcripts were examined first for overall patterns of pedagogical knowledge concerning knowledge of part/whole and part/part relations. The patterns of knowledge observed where then categorized into four major headings: types of models, model translations, use of relationships and instructional strategies. Each of the major headings was then defined through further examination of the transcripts.

Table 6. *Observed Teacher Knowledge of Part/Whole & Part/Part Relations*

Types of Models	Model Translations	Use of Relationships	Instructional Strategies
Area Models	Partitioning Squares/circles Rectangles	Part/Part/Whole –What is the relationship between fraction, part and whole	Conceptual Strategies-gives students opportunity to consider whether/how methods for solving problems lead to fraction concepts-part/whole., part/part, equivalence
Set Models	Fair Shares of sets	Part/Part-Fractional Equivalents	
Linear Models	Fractions strips	Relationships among different representations of fractions	Numeric Strategies- gives students opportunities to use different mathematical representations for the concept of fractions

Models for Fractions.

There is substantial evidence that the use of models in fractions is important (Cramer & Henry, 2002). There are three main models used for the instruction of fractions: the region/area model, the length/measurement model and the set model. When using the discussion of sharing in fractions, all tasks involve sharing something that can be cut into smaller parts. The fractions are based on parts of an area or region. For example, the circle is the most commonly used area

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

model in elementary instruction of fractions. One advantage of using a circle area model is that it emphasizes the amount that is remaining to make up the whole. Other popular area models are rectangular regions, pattern blocks and geoboards.

In measurement models, lengths are compared instead of area. Either lines are drawn or subdivided or physical models are compared on the basis of length. Fractions strips and Cuisenaire rods are pieces that are in lengths of one to ten measured in terms of the smallest strip or rod. Each length is a different color to aid in identification. The number line is a more sophisticated measurement model (Bright, Behr, Post & Waachsmuth, 1988).

In set models, the whole is understood to be a set of objects, and subsets of the whole make up fractional parts. For example, three objects are one-fourth of a set of 12 objects. The set of 12, represents the whole of *1*. This idea of referring to a collection of counters as a single entity that makes set of models difficult for students.

Use of Relationships.

Three distinct uses of fractional relationships were categorized from our transcripts. The first use of relationship was the part-whole relationship. In a part-whole problem, the central question of the relationship is "What is the relationship between the fraction, the part and the whole?"

For example, Teacher 2 posed these questions during the course of transcripts:

1. "If this rectangle is one whole find ¼." [Whole, fraction, find the part]
2. "If this rectangle is 1/3, what could the whole look like? [Part, fraction, find the whole]

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

3. "What fraction of the big square does the small square represent?"[Whole, part, find the fraction]

The part-part relationship was used to compare equivalent fractions. The part-part interpretation is used when one quantity in a whole is compared to another quantity in a whole. For example, 2/3 and 5/6 can be compared as parts of the whole (1). Likewise, 2/3 of 6 and 5/6 of 12 can be compared as parts of different wholes (6) and (12).

Use of Instructional Strategies

There were two distinct instructional strategies categorized in the tutoring sessions: conceptual strategies and numerical strategies. As stated, conceptual strategies gives students the opportunity to consider whether and how methods for solving problems lead to fraction concepts-part/whole, part/part, and equivalence. In our transcripts, Teacher 1 asks: "If four children are sharing 10 brownies, so that each one will get the same amount how much would each child get?" This type of sharing problem allows the student to use an area model and distribute the whole (10) by giving each person one brownie and dividing the two remaining brownies in half. The final result would be each child receiving 2 ½ brownies. A numeric strategy for this problem might be "4 into 10 is 2 brownies with 2 left over". Each strategy-conceptual and numerical – affords the student certain interpretations while constraining other interpretations (Gearhart et al., 1999).

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Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

Scaffolds and Pedagogical Knowledge. Finally, the scaffold information for each teacher was linked to the type of pedagogical knowledge displayed during those scaffolding episodes. An example is described below:

Episode start *****T2: But, if she shares them with her two friends..how many people in

Scaffold *** total does she have to share them with?

S2: Her and two people...

Scaffold *** T2: So how many people in total are sharing the 18 cookies?

S2: Three?

Scaffold *** T2: You don't sound so sure..are you sure?

S2: Yeah, cause its her (holds up one figure) and then two more (adds two more fingers) that's three.

Table 7. Scaffold and Pedagogical Knowledge Teacher2

Scaffold	Type of Model/Translation	Use of Relationship
How many people in total are sharing the 18 cookies?	Fair Shares/Fair share of set of 18	Conceptual strategy

The combination of the scaffold as it relates to domain specific knowledge will be referred to as a pedagogical scaffold.

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

Pedagogical Scaffolds and Descriptions by Teacher

Table 5 **Teacher 1 Pedagogical Scaffolds**

n=48 Total Pedagogical Scaffolds – Types of Models

n=21 Area Model Scaffolds

n=27 Set Model Scaffolds

n=48 Total Pedagogical Scaffolds – Model Translations

n=18 Partitioning Squares/Rectangles

n=3 Partitioning of Circles

n=27 Fair Shares of Sets

n=48 Total Pedagogical Scaffolds- Use of Relationships

n=31 Part/Part/Whole

n=17 Part/Part Fractional Equivalents

n=48 Total Pedagogical Scaffolds – Instructional Strategies

n=19 Conceptual Strategies

n=31 Numeric Strategies

Teacher 1.

Teacher 1 made 48 scaffolds that could be considered pedagogical scaffolds in the domain of fractions. According to the transcripts, of the 48 pedagogical scaffolds, twenty-one of scaffolds used were area models scaffolds while twenty-seven of the scaffolds were set modes.

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

In using the area model scaffolds, Teacher 1 used the square or rectangle model translation almost exclusively. Nineteen of twenty-one model translations used were partitioning squares or rectangles.

Example 1 Teacher 1 Area Model:

T1 : Show me what 4/6 would look like….

S1 : Can I draw it here….

T1: Right….right there….draw like a rectangle right there…we'll start with this one first and we'll go from there…

Teacher 1 used a set model (n=27) as a pedagogical scaffold. Of the twenty-seven set models, all of the translations were fair shares of sets. Interestingly, Teacher 1 always used simple objects in the fair share problems she represented. She never presented a fair share problem that would have extended the students thinking such as: There are 40 students going on a trip. Each bus can hold 9 students. How many buses will they need? In this situation, the correct answer cannot be found simply by sharing the students among each bus.

Example 1 Teacher 1 Fair Shares:

T1: Ok..let's start with this problem….There are 5 brownies and 2 people…How many brownies does each person get?

Example 2 Teacher 1 Fair Shares:

T1: There are 12 pumpkins in the basket. I want to share them with my two friends..How many pumpkins will each of us get?

Appendix M

Transcript Analysis and Coding Measure
Pilot Study
(continued)

As discussed previously, use of relationships involves understanding part/part/whole, part/part and relationships among different representations of fractions. Of the three uses of relationships, Teacher 1 used only the part/part/whole and the part/part scaffolds to further the student's understanding.

Example 1 Teacher 1 Use of Relationships: Part/Part/Whole

T1: We have ten circles, if 6 of the those circles are red, what fraction of the chips are not red?

The use of conceptual and numeric models are both important in the understanding of fractions.

Table 6 Teacher 2 Pedagogical Scaffolds

n=62 Total Pedagogical Scaffolds – Types of Models

n=25 Area Model Scaffolds

n=27 Set Model Scaffolds

n=10 Linear Model Scaffolds

n=62 Total Pedagogical Scaffolds – Model Translations

n=14 Partitioning Squares/Rectangles

n=11 Partitioning of Circles

n=27 Fair Shares of Sets

n=10 Fraction Strips

n=62 Total Pedagogical Scaffolds- Use of Relationships

n=26 Part/Part/Whole

n=36 Part/Part Fractional Equivalents

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

n=62 Total Pedagogical Scaffolds – Instructional Strategies

n=45 Conceptual Strategies

n=17 Numeric Strategies

Teacher 2.

Teacher 2 made 62 scaffolds that could be considered pedagogical scaffolds in the domain of fractions. According to the transcripts, of the 62 pedagogical scaffolds, twenty-five were area model scaffolds, twenty-seven were set model scaffolds and ten were linear model scaffolds. In using the area model scaffolds, Teacher 2 used both square/rectangle model and circle model translations almost evenly (n=14;n=11). An interesting observation of Teacher 2's transcripts is that Teacher 2 used the square/rectangle model translation in representing fractions such as halves, fourths, sixths and eighths. Teacher 2 used the circle model translation for fractional representations of halves and thirds.

Example 1 Teacher 2: Area Model Translations

T2: Well…we are talking about a half of one whole…like we have here…
If you were going to show one half of a whole you could easily see that this circle is shaded so that half of it is blue and the other is red….

Example 2 Teacher 2: Area Model Translations

T2: So if we are trying to prove which one is bigger 2/3 or ½ how are we going to do that? Let's look at the ones we drew already….Put that one (2/3 shaded in a circle) next to this one (1/2 in a circle)...Which circle has more shaded?

Appendix M

Transcript Analysis and Coding Measure
Pilot Study
(continued)

Teacher 2 used a fair shares of sets model translation far more than any other model. This is especially evident because Teacher 2 presented far more word problems in the transcripts.

Example 1 Teacher 2: Fair Shares

T2: There are six cookies and four people...how many cookies would each person get?

Teacher 2 also used a linear model of fractions in the form of fraction strips. Fraction strips are a manipulative model produced by Cuiseinare. Teacher 2 used fractions strips to model addition and subtraction of fractions and comparison of equivalent amounts.

Example 1: Teacher2 Fractions Strips

T2: Let's try this...how about 1/3 +5/6..Can you work it out using the fraction strips...What would you do first?

S2: I can take the one third strip..and take five of the one-sixths and put them together...

Discussion of Pilot Study Results

T2: You are going to put the five-sixths together?

S2: Yeah, together with the other one..(1/3).

T2: Ok, then what...

S2: I don't know..

T2: Ok...well let's take a look at what we have ..can we put any of the sixths together to make the third?

S2: Yeah....one..two..two.

Exchange continued until finished.

Appendix M

Transcript Analysis and Coding Measure
Pilot Study
(continued)

The following questions about teachers' specific content knowledge and their instructional scaffolding strategies were examined:

1) Does a teacher's content knowledge about equivalent fractions affect the instructional strategies the teacher employs when extending the task in the child's zone of proximal development? and

2) Are there differences between the instructional concept maps of teachers with different content knowledge?

In reviewing the transcripts of the tutoring sessions for Teacher 1 and Teacher 2, there were some very distinct differences in how they approached the moment when their students required scaffolding. Both Teacher 1 and Teacher 2 are considered master teachers, however, an important difference is the amount of pedagogical content knowledge each possessed. Because research has shown that pedagogical content knowledge is a difficult construct to measure, this study chose to examine the link between the teachers' scaffolding and how they used their pedagogical content knowledge to make those scaffolding episodes successful. Therefore, by coding the interactions between teacher and student, a rough map of the teacher's knowledge of a particular domain, in the case fractions.

Teacher 1 and Teacher 2 both used area and set models in their scaffolding in basically the same amounts. However, Teacher 2 also used a linear model (fraction strips) to further their students' thinking. Teacher 1 chose not to use a linear model representation even though she had the same materials as Teacher 2. In the use of area model translations, Teacher 1 overwhelmingly used partitioning of squares/rectangles (n=18) and fair shares of sets (n=27).

Appendix M

Transcript Analysis and Coding Measure
Pilot Study
(continued)

With the use of only three models which involved partitioning of circles (3), Teacher 1 missed opportunities to expand her student's knowledge of fractions using a circle representation. Similarly, Teacher 2 employed a large number of fair shares of sets (n=28). This large number of fair shares of sets models might be due to the manner in which third grade mathematics curriculum is structured. Both teachers use Investigations in Number, Space and Data as their mathematics curriculum and the fraction unit is called "Fair Shares". It is possible the teachers knew the students would be exposed to a unit heavy on the idea of "fair shares" and it effected how they presented their instruction.

In the area of Use of Relationships Teacher 1 utilized the part/part/whole scaffold (n=31) and the part/part scaffold (n=17), while Teacher 2 used part/part/whole (n=26) and part/part (n=36). Teacher 2 showed more equity in the approaches while Teacher 1 relied heavily on the representation of part/part/whole.

Finally, both teachers are using the Investigations in Number, Space and Data curriculum in their everyday math instruction. This curriculum focuses on problem solving opportunities, hands on manipulatives and written explanations. However, when tutoring individual students only Teacher 2's instructional strategies reflect the conceptual nature of the interaction. Teacher 1 relied heavily on numeric strategies. Although both are important in solidifying students' understanding of fractional concepts, instruction that focus heavily on numeric strategies may not give student the opportunity to develop in depth understandings of subject matter.

These two teachers are different in their use of pedagogical content scaffolds during tutoring sessions. This difference may be attributed to their knowledge of conceptual and

Appendix M
Transcript Analysis and Coding Measure
Pilot Study
(continued)

procedural strategies of teaching. The transcription of teacher-student interaction as well as the identification of particular content scaffolds in the domain of fractions makes it possible to begin making some generalizations about and between the two teachers. Based on the data, Teacher 1 does not have the depth of knowledge in fractions that Teacher 2 possesses. Teacher 1's over reliance on certain models in favor of others could be clearly noted in the tables of pedagogical scaffolds. Teacher 2 used more types of pedagogical content scaffolds as well as more variety within the models to assist her student in understanding. Based on this particular study, in the domain of fractions Teacher 2 would be considered an "expert" while "Teacher 1" is still considered a novice in the domain of fractions.

Implications

The construct of pedagogical content knowledge in the educational field has had challenges. One of the challenges is a systematic measurement of the construct. Another challenge is the proven idea that one may have pedagogical content knowledge in one subject but not another. Pedagogical content knowledge being an important aspect educator's success with student learning, it causes researchers to looking at how pedagogical content knowledge is used more closely. One of the methods is by examining teacher-student interactions where the teacher is scaffolding the student's learning. But it is not enough to examine only the scaffolding or the pedagogical content knowledge in isolation, they must be examined concurrently. This study was an attempt to test the link between the two. As the data shows, some questions are raised the instruction that is taking place and how a teacher's knowledge impacts that instruction in zone of proximal development.

Appendix N
Responses to Questions 9, 10, 11 of
Demographic Questionnaire by Percent and Number
(continued)

Question 10: This year, how often did the students in your math class do the following?:

	Less than once a month	1-3 times per month	1-2 times per week	3-4 times per week	Never
Listen to me present the definition of a term or the steps of a procedure			100% (N=53)		
Perform tasks requiring methods or ideas already introduced to students			100% (N=53)		
Assess a problem and choose a method to use from those already introduced to students	58.5% (N=31)	13.2% (N=7)			28.3% (N=15)
Perform tasks requiring methods or ideas not already introduced to students			32.1% (N=17)	67.9% (N=36)	
Explain an answer or a solution method for a particular problem	50.9% (N=27)	49.1% (N=26)			
Analyze similarities and differences among representations, solutions or methods	100% (N=53)				
Prove that a solution is valid or that a method works for all similar cases	52.8% (N=28)		18.9% (N=10)	28.3% (N=15)	

Appendix N
Responses to Questions 9, 10, 11 of
Demographic Questionnaire by Percent and Number
(continued)

Question 10: This year, how often did the students in your math class do the following?:

	Less than once a month	1-3 times per month	1-2 times per week	3-4 times per week	Never
Listen to me present the definition of a term or the steps of a procedure			100% (N=53)		
Perform tasks requiring methods or ideas already introduced to students			100% (N=53)		
Assess a problem and choose a method to use from those already introduced to students	58.5% (N=31)	13.2% (N=7)			28.3% (N=15)
Perform tasks requiring methods or ideas not already introduced to students			32.1% (N=17)	67.9% (N=36)	
Explain an answer or a solution method for a particular problem	50.9% (N=27)	49.1% (N=26)			
Analyze similarities and differences among representations, solutions or methods	100% (N=53)				
Prove that a solution is valid or that a method works for all similar cases	52.8% (N=28)		18.9% (N=10)	28.3% (N=15)	

Appendix N
Responses to Questions 9, 10, 11 of
Demographic Questionnaire by Percent and Number
(continued)

Question 11: To what extent do you agree or disagree with the following statements about how you prepare mathematics lessons for class?

	Strongly Agree	Disagree	Agree	Strongly Agree	N/A
I frequently refer to and use information found in curriculum frameworks or standards documents	1.9% (N=1)	52.8% (N=28)	41.5% (N=22)	3.8% (N=2)	
I frequently refer to and use information from teachers' guides associated with the curriculum materials adopted by my school			58.5% (N=31)	41.5% (N=22)	
I frequently refer to the content of assessments	22.6% (N=12)	45.3% (N=24)	26.4% (N=14)	5.7% (N=3)	

Appendix O

Participants' Scores on Equivalence Assessment B by Group

Score on EQA	Teacher #	Section 1a	Section 1b	Section II					
		5	5	#1	#2	#3	#4	#5	TOTAL
Group LK									
8	**51**	3	3	0	0	2	2	2	12
10	**1**	2	2	2	2	2	2	2	14
10	**9**	2	3	1	1	2	2	2	13
11	**4**	2	3	2	2	2	2	2	15
11	**53**	3	2	2	2	1	2	2	14
12	**2**	2	3	2	2	2	2	2	15
12	**20**	2	3	2	2	2	2	2	15
13	**8**	2	3	2	2	2	1	2	14
13	**31**	2	2	2	2	2	2	2	14
13	**43**	3	3	2	2	2	2	2	16
13	**52**	3	3	2	2	1	2	2	15
13	**15**	3	3	2	2	1	2	2	15
13	**34**	4	4	2	2	2	1	2	17
13	**50**	3	4	2	2	2	2	2	17
14	**22**	5	4	2	2	2	2	2	19
14	**48**	4	4	2	2	2	2	2	18
									M=14.3

Group MK		Section 1a	Section 1b	Section II					Total
				#1	#2	#3	#4	#5	
17	**3**	3	2	2	2	2	2	2	15
17	**27**	3	2	2	2	2	2	2	15
17	**37**	4	3	2	2	2	2	2	17
17	**40**	4	4	2	2	2	2	2	18
17	**47**	3	3	2	2	2	2	2	16
18	**5**	4	4	2	2	2	2	2	18
18	**35**	5	4	2	2	2	2	2	19
19	**10**	5	4	2	2	2	2	2	19
19	**19**	5	5	2	2	2	2	2	20
19	**41**	5	5	2	2	2	2	2	20
19	**42**	5	5	2	2	2	2	2	20
19	**46**	5	5	2	2	2	2	2	20
19	**49**	5	5	2	2	2	2	2	20
20	**7**	5	5	2	2	2	2	2	20
20	**24**	5	4	2	2	2	2	2	19
20	**30**	5	4	2	2	2	2	2	19
									M=17.4

Appendix P

Number of Teachers Using Model and Numeric Representations
Teachers on Equivalence Assessment B

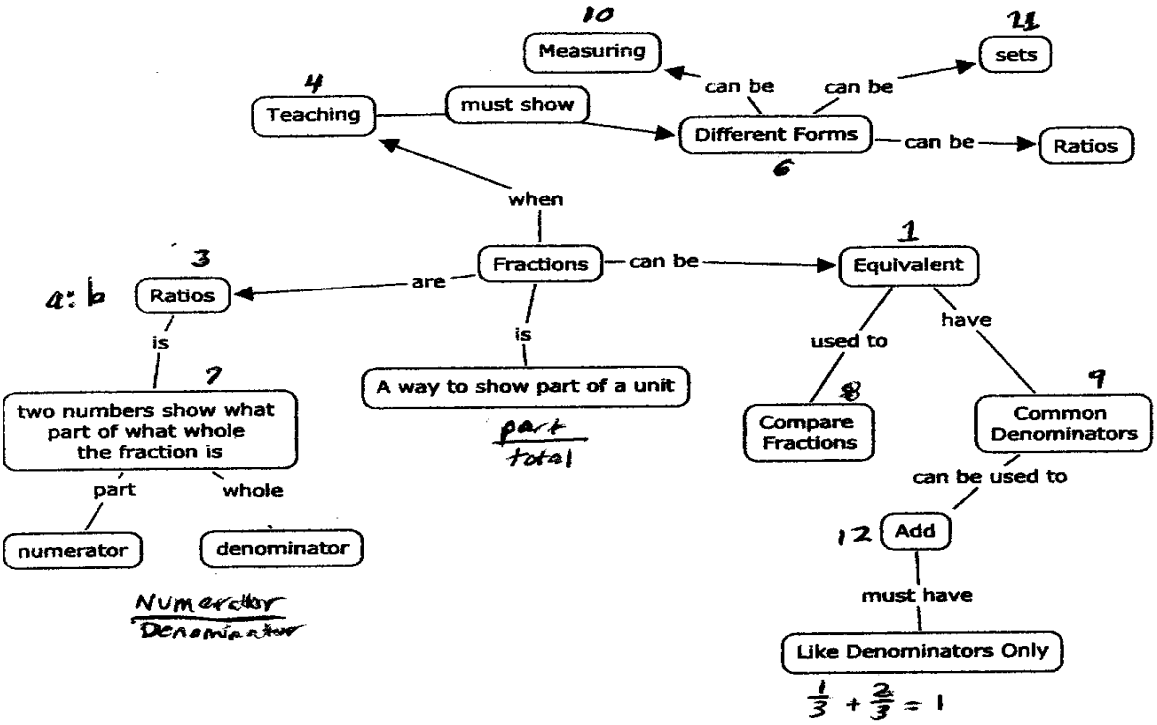
	Area	***Linear***	***Set***	***Expression***	***Operation***
Group A					
Section I					
Question 1	16	9	1	16	3
Question 2	16	7	3	16	2
Section II					
Question 1					16
Question 2					16
Question 3	9				16
Question 4					16
Question 5					16
Group B					
Section I					
Question 1	16	13	10	13	16
Question 2	15	13	6	15	11
Section II					
Question 1	10				16
Question 2		8			16
Question 3	15				16
Question 4			4		16
Question 5				5	

N=16, each group

Appendix Q

Two Exemplars of Scored Responses for Written Concept Maps

Teacher 49 Grade 4 (MK)



Scored Response

General Impression of Content Quality (GICQ) – 4

Number of Principles/Concepts – 12 (As shown on Concept Map)

Number of Facts/Procedures - 4

Misconception/Errors – 0

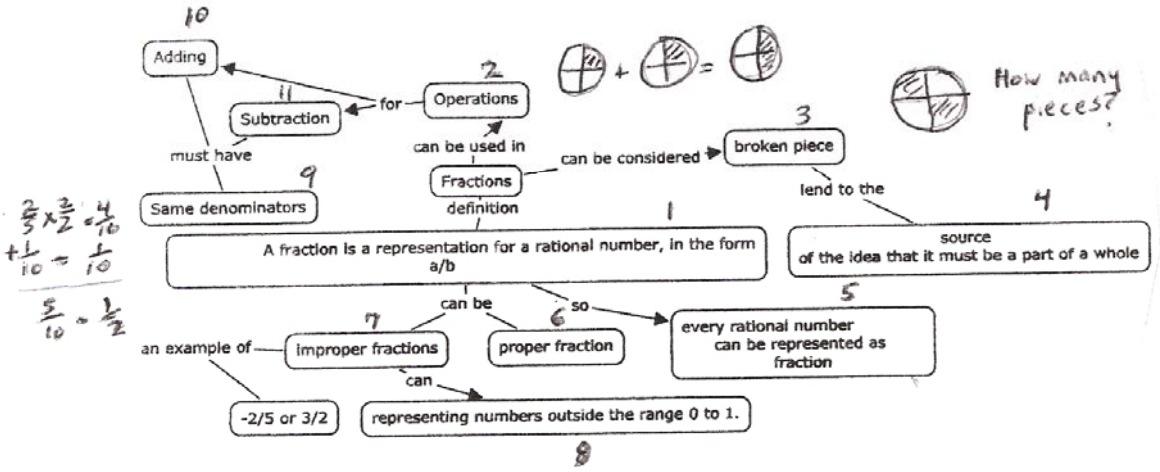
Integration/Argumentation - 4

Appendix Q

Two Exemplars of Scored Responses for Written Concept Maps

Teacher 35 Grade 3 (MK)

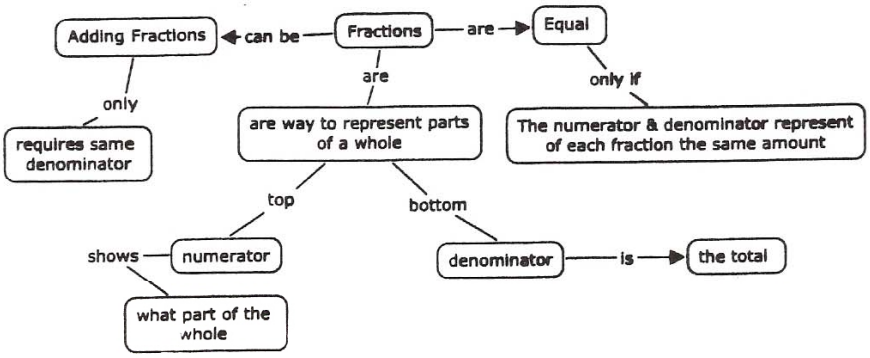
(continued)



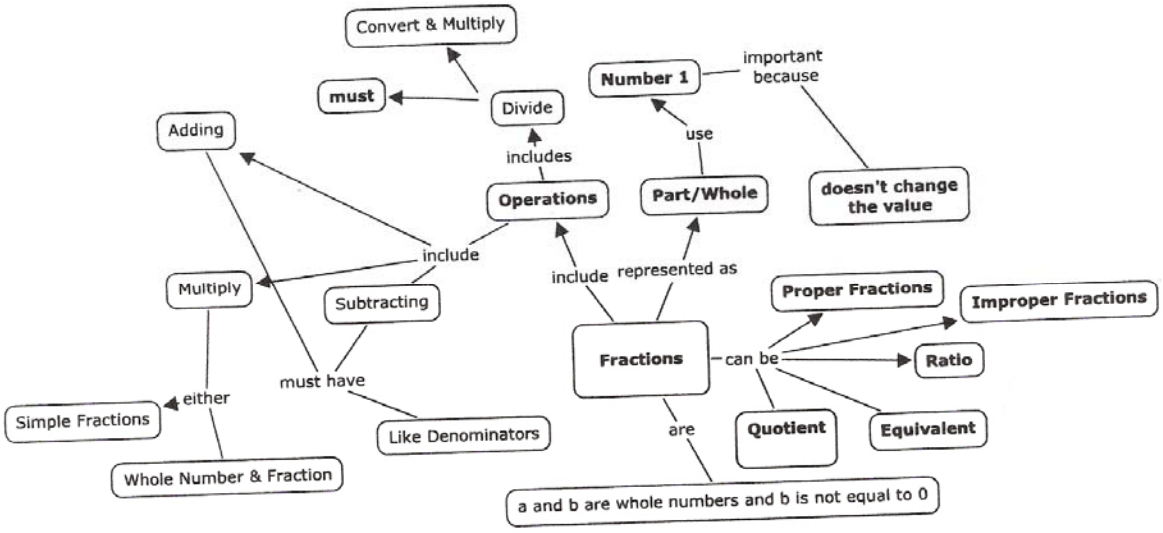
Scored Response

General Impression of Content (GICQ)- 3
Number of Principles and Concepts-10 (As shown on concept map)
Number of Facts/Procedures – 4
Misconceptions/Errors – 0
Intergration/Argumentation-4

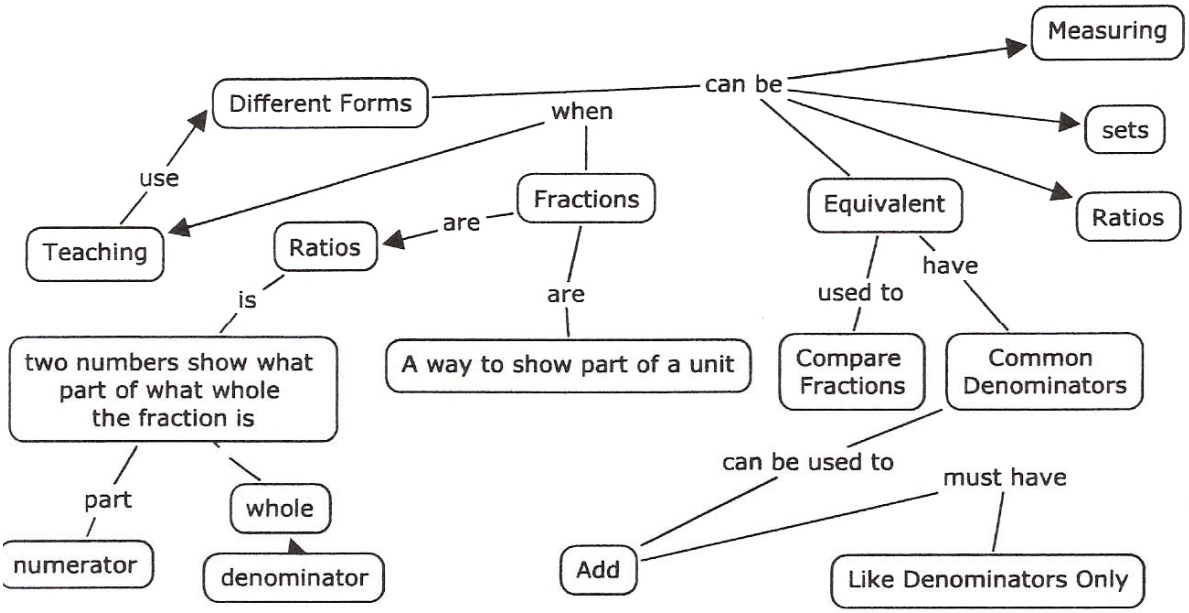
Appendix R
Verbal Think Aloud Concept Map I
Teacher 4 Kindergarten Less Knowledgeable
(continued)



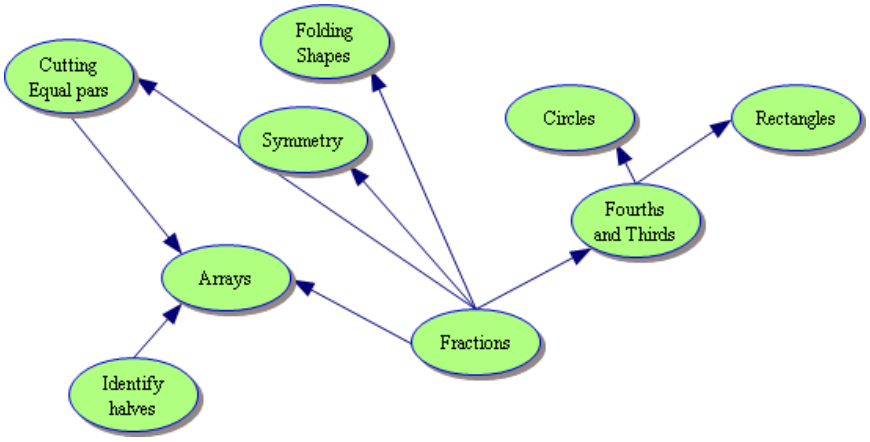
Appendix R
Verbal Think Aloud Concept Map I
Teacher 10 Grade 3 More Knowledge
(continued)



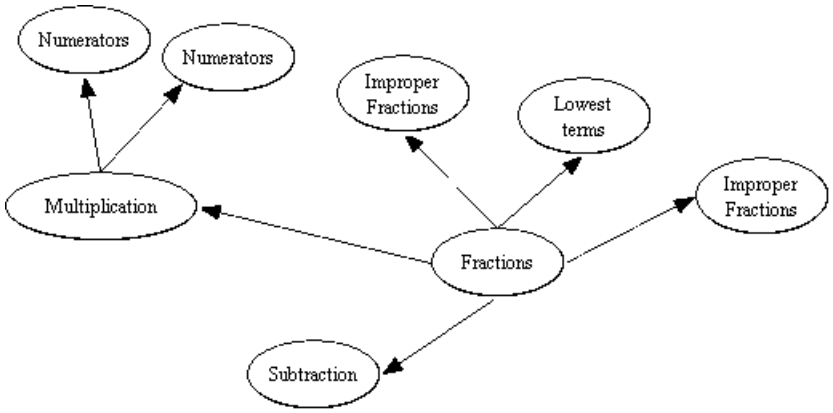
Appendix R
Verbal Think Aloud Concept Map I
Teacher 7 Grade 5
More Knowledgeable
(continued)



Appendix R
Verbal Think Aloud Concept Map I
Teacher 1 Grade 2
Less Knowledgeable
(continued)



Appendix R
Verbal Think Aloud Concept Map I
Teacher 9 Grade 1
Less Knowledgeable
(continued)



Appendix S

Scores for Tutoring Transcripts by Group by Teacher (T1-T10)

Teacher		EX (Exchanges)	SE (Scaffolding Episodes)	PCS (Pedagogical Content Scaffolds)	M (Models)	R (Number of representations)
Group B (MK)	T1	8	16	12	D	5
	T2	8	14	10	D	7
	T3	7	11	12	G	8
	T4	8	13	11	G	8
	T5	6	13	13	E	7
Group A (LK)	T6	4	9	8	A	3
	T7	6	11	7	A	3
	T8	4	8	7	C	4
	T9	5	7	6	A	2
	T10	3	9	5	C	5

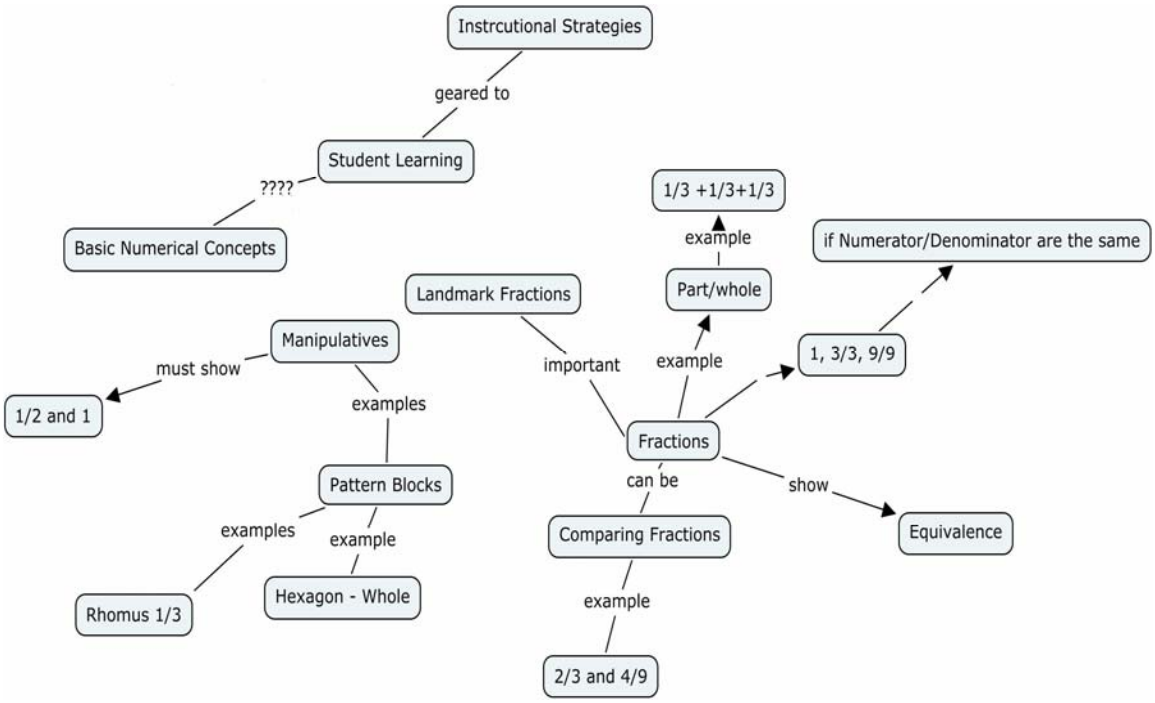
Note: Models coded as: a) area model, b) set model, c) linear model, d) area and set models, e) area and linear models, f) set and linear models and g) area, set and linear models

Appendix T

Participants' Verbal Think Aloud

Concept Map II

Teacher 10 Grade 3 (MK)



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Teacher 10 Tutoring Transcript – Lesson Estimation Equivalence

T10 – Let's take a look at what we did for this problem 2/3 and 4/9. It asks us to compare these two fractions and to see if they are equal or not. Let's take one fraction at a time Let's look at 2/3 first . Before we do anything can you tell me what you can about this fraction 2/3.

S: It's about thirds and it is almost the total.

T: What do you mean it's almost the total?

S: It's like thirds, than you have 1/3, / 1/3 and 1/3 and that is one.

T: Can you show me with pattern blocks what you mean and can you explain it to me please?

S: See the yellow one (hexagon) that is one and if you put 2 blue ones on it (rhombus) than the blue ones almost cover the whole yellow.

T: Do the two blue ones cover more or less that ½ of yellow hexagon?

Appendix T

Participants' Verbal Think Aloud

Concept Map II

Teacher 10 Grade 3 (MK)

(continued)

S; Umm it is like 3's so…

T: Ok well let's take off the blue pattern blocks and them think about it. What would be half of the hexagon. You can use the colored pencils to help you…

S: Can I write on the block?

T: Sure..

S: If I cut the hexagon in half it is this much

T: So that line ½ we are trying to find out what 2/3 looks like in comparison to the ½ of the hexagon. So what does it look like?

S: Here is the line for the half of the hexagon and the two blue blocks are a little more that half the line.

T: So how can we say that in mathematical language think first..

S: that 2/3is a little more than ½

T: Good alright so let's go back to the problem of 2/3 compared to 4/9. We have established that 2/3 is a little more than ½ what about 4/9? How can we find out if it is more or less that ½? What do you think?

S; Nines?

T:Ninths..don't get confused..we know that if the denominator and the numerator of the fraction are the same it is equal to what?

S: One

T: So if we have 4/9 what would make this fraction equal to one…

S: 9/9

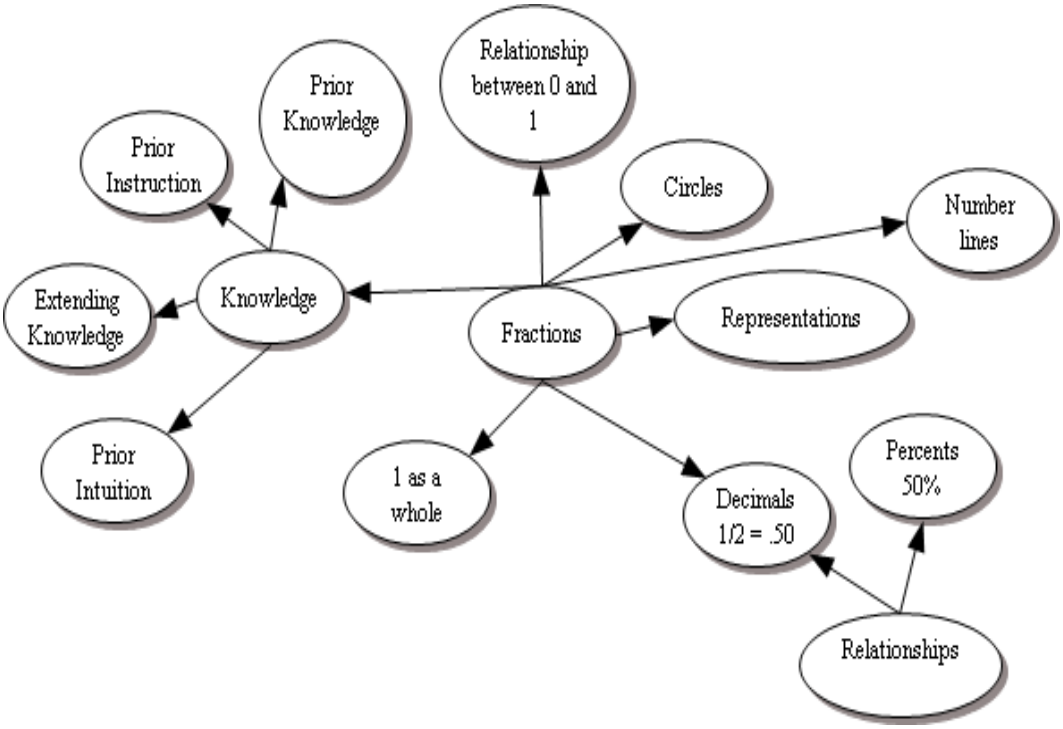
T: ok if 9/9 is equal to 1 would ½ of 9/9 be?

Appendix T

Participants' Verbal Think Aloud

Concept Map II

Teacher 7 Grade 5 (MK)



Teacher 5 Equivalence Decimals/Fractions

T5: We are exploring this game to help us understand how fractions and decimals are related and how they are equal.

S: um…

T5: So I noticed you were having problems with this one so I am going to play this hand with you so that we can learn together..so you pick the first card…what do you have?

S; 2/3..

T: Now you have to place the card somewhere between 0 and 1 and 2…so what percentage is 2/3 equal to?

S: It's like under 50%

T: Well yes..but lets us what we know…what percent is 1/3 equal to?

Appendix T

Participants' Verbal Think Aloud

Concept Map II

Teacher 7 Grade 5 (MK)

(continued)

S: 33 1/3 %

T: So using what you know…if 1/3 is 33 1/3% what would 2/3 be equal to?

S: 33 1/3 + 33 1/3…66 2/3….

T:okay good so place the 2/3 card on the number line…where do you think it should go between the 0 and 1 or the 1 and the 2?

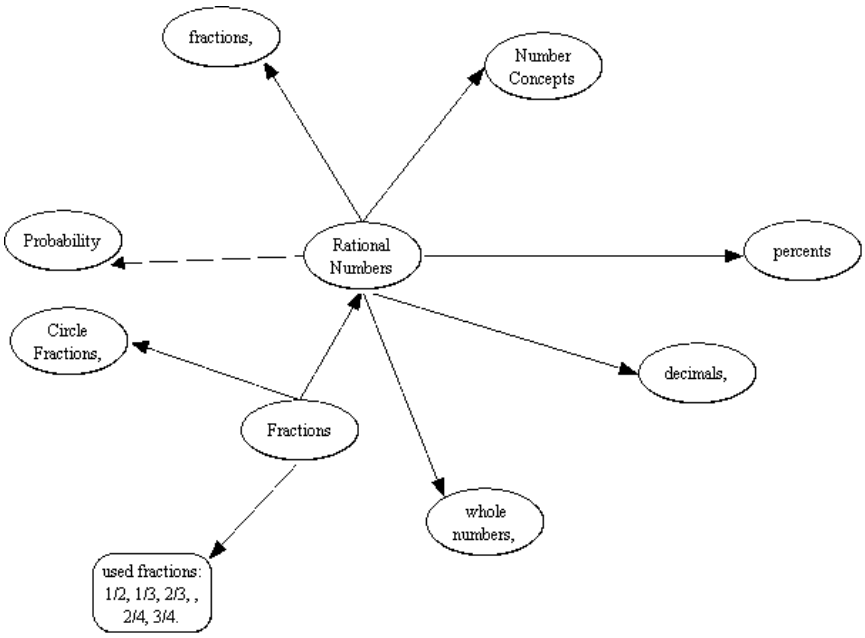
S: 2/3 is less than one but more than ½ so it will go closer to 1.

Appendix T

Participants' Verbal Think Aloud

Concept Map II

Teacher 1 Grade 2 (LK)



T: We are going to over the problem you had trouble with today…

S:yeah

T: what about this one it is ½ and you really have to know this.. if you don't learn anything else today I want you to learn to recognize 1/2

S; that is ½

T: yes but that includes dollars, pies anything,,,I want you to cut these in half….each one is a different shape…each one needs be cut in half..start with the circle…

S: this one too? (shows a rectangle?)

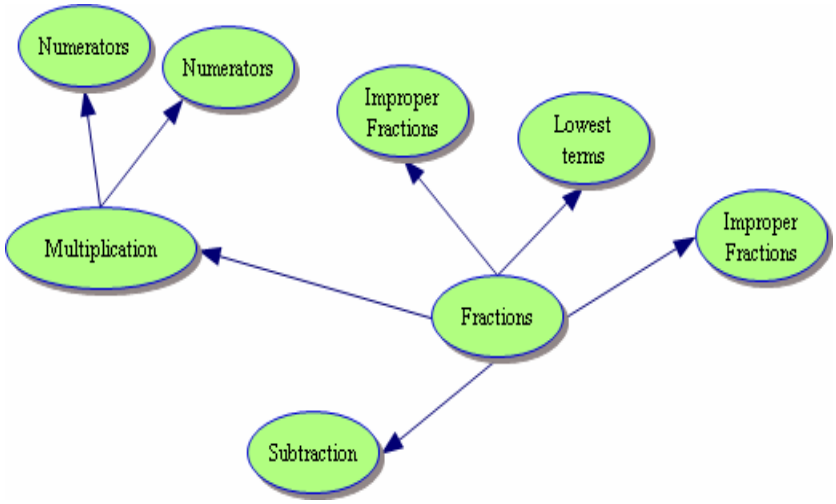
T: yes….show half on that one to…

Appendix T

Participants' Verbal Think Aloud

Concept Map II

Teacher 9 Grade 1 (LK)



T: We drew the problem on the chart paper..let's look at it…how many squares do we have all together here on the chart….

S: one..two..three…four…four

T: There are two blue and two green…we are talking about half…there are four and two are blue that means that half of the squares are blue….how about the other half..

S; They are red…

T: how many are there…

S: two

T: so two is half of four…two plus two equals four…how about here…there are six.. what is half..

S: Do I count them?

T: Is that going to help?

S:one…two…three…continues to six. Six red…

T: so how many is half..

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