Complex Earthquakes are Holomorphic

by

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Abstract

COMPLEX EARTHQUAKES ARE HOLOMORPHIC

by

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Earthquakes on compact Riemann surfaces have been studied extensively. They are mappings of the Teichmuller space of a compact Riemann surface to itself. It is a result of Kerckhoff that an earthquake path is a real analytic path in the Teichmuller space of a compact surface. We give a generalization of Kerckhoff's result to the Teichmuller space of any Riemann surface, in fact, to the Universal Teichmuller Space.

We start from a bounded measure on the hyperbolic plane and the corresponding earthquake path parameterized by the positive real numbers. We extend the parameterization to a neighborhood of the real line in the complex plane. The extension is a holomorphic map in the parameter and, for a fixed parameter, it is a one to one map of the unit circle. Hence, the complex earthquake path, with the parameter in the given neighborhood of the real line, is a holomorphic motion of the unit circle. By Slodkowski's theorem, it is extendible to a holomorphic motion of the complex plane. Then, for a fixed positive parameter, the earthquake map is the restriction to the unit

circle of a quasiconformal map of the complex plane preserving the unit disk. Thus an earthquake with a bounded measure is quasisymmetric. We also prove that a quasisymmetric earthquake has bounded measure.

The above results taken together show that for an earthquake the following are equivalent:

1. The measure of an earthquake is bounded,
2. An earthquake is quasisymmetric,
3. An earthquake path is a part of a holomorphic motion of the unit circle.

Moreover, an earthquake path with bounded measure is a real analytic path in the Universal Teichmuller Space.

To my parents,

Jasna and Mirko

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1 Introduction

An earthquake is a bijective, but not continuous, piecewise hyperbolic isometry of the hyperbolic plane $\mathbb{H}^2$ with certain compatibility properties. The set of discontinuity lies in a geodesic lamination $\mathcal{L}$. $\mathcal{L}$ is a collection of non-intersecting geodesic lines whose union forms a closed set in $\mathbb{H}^2$. The earthquake induces a non-negative transverse measure supported on the lines of $\mathcal{L}$.

An earthquake extends to a homeomorphism of the boundary S^1 of $\mathbb{H}^2$. Thurston (see [13]) showed, conversely, that any homeomorphism of S^1 can be obtained as the restriction of an earthquake and this earthquake is essentially unique. In particular, the lamination $\mathcal{L}$ is unique and the assignments of hyperbolic isometries to the complementary components of $\mathcal{L}$ and to the geodesics of $\mathcal{L}$ determine a unique nonnegative transverse measure. Gardiner, Hu and Lakic gave different proof (see [6] and [7]).

We say that the transverse measure is bounded if the measures of all closed segments of hyperbolic length one transverse to the lamination are bounded by a constant. Given a geodesic lamination and a nonnegative, locally finite measure on it, we can always get a map which satisfies all the properties of an earthquake except surjectivity. If the measure is bounded, however, then the map is surjective and indeed is an earthquake. This fact, originally stated by Thurston, is a consequence of results proved in this paper.

Thurston first introduced earthquakes in the context of Teichmüller spaces of compact Riemann surfaces. An earthquake is defined on the base Riemann surface as a piecewise hyperbolic isometry discontinuous along a geodesic lamination and satisfying a compatibility condition; its lift to the hyperbolic plane is an earthquake which is invariant under the action of the Fuchsian group representing the fundamental group. Kerckhoff [10] proved that if the transverse measure is multiplied by a positive parameter, the resulting path in any finite dimensional Teichmüller space depends real analytically on the parameter.

If we start with an arbitrary nonnegative, bounded measure μ supported on a lamination $\mathcal{L}$ in $\mathbb{H}^2$ and multiply it by a positive parameter t we get a path of bounded measures. Normalized earthquakes associated to this path form an earthquake path in the universal Teichmüller space. In this paper, we multiply μ by a complex parameter τ. If the imaginary part of τ is small enough, we can define a piecewise hyperbolic isometry on each component of $\mathbb{H}^2 - \mathcal{L}$ and on each leaf of $\mathcal{L}$ into hyperbolic three space $\mathbb{H}^3$. The images of the components of $\mathbb{H}^2 - \mathcal{L}$ are glued together along the images of the leaves of $\mathcal{L}$ at an angle determined by the imaginary part of τ. This map is often called a quakebend [4]. It extends to a homeomorphism of S^1 onto its image in S^2. The quakebend, restricted to S^1, can thus be considered as a function of two variables.

Our main theorem is:

Theorem A *Fix a bounded transverse measure and a point on S^1. Then the complex earthquake path depends holomorphically on the parameter τ.*

This theorem is part of the following more general result:

Theorem B *The following three statements are equivalent:*

1. *The restriction of an earthquake to the boundary of the hyperbolic plane is quasisymmetric,*
2. *The measure of an earthquake is bounded,*
3. *An earthquake path is the restriction of the holomorphic motion of the boundary of the hyperbolic plane. Furthermore, the domain of definition D_μ of the parameter τ of the holomorphic motion depends only on the bound on transverse measure and contains the real axis.*

The main arguments of the paper depend on the cone lemma of Keen and Series [8], Lemma 6.3, on the crescent lemma, Lemma 2.7, and on the theorem of Slodkowski on holomorphic motions [12].

The plan of the paper is the following:

In section 2 we give the definitions of an earthquake E_μ with measure μ and an earthquake path $E_{t\mu}$, and the basic lemmas needed for the rest of the paper. In section 3 we find a sequence of finite earthquakes E_{μ_n} which approximate E_μ. We show that the sequence of paths $E_{t\mu_n}$ is also a good approximation for the path $E_{t\mu}$. In section 4 we use the lemmas of section

2 and the finite earthquake approximations of section 3 together with the Keen-Series cone lemma to prove our theorems.

2 Definition and Properties of Earthquakes

Following Thurston [13], we give the definition of an earthquake.

Definition 2.1 A geodesic lamination $\mathcal{L}$ on the hyperbolic plane $\mathbb{H}^2$ is a closed subset $\mathcal{L}$ of $\mathbb{H}^2$ which is a union of disjoint geodesics called *leaves*. The components of the complement of $\mathcal{L}$ are called *gaps*. The gaps are geodesic polygons with vertices only on the boundary of $\mathbb{H}^2$. The gaps and leaves of $\mathcal{L}$ are called the *strata* of $\mathcal{L}$. $\mathbb{H}^2$ is partitioned by the strata of $\mathcal{L}$.

Definition 2.2 Let $\mathcal{L}$ be a geodesic lamination of $\mathbb{H}^2$. A *left earthquake* E along $\mathcal{L}$, the support of E, is a (possibly discontinuous) one to one map from $\mathbb{H}^2$ onto $\mathbb{H}^2$ which is a hyperbolic isometry $E|A$ on each stratum A of $\mathcal{L}$. A geodesic l separates two sets A and B if any path connecting a point a in A to a point b in B intersects l. The map E must satisfy the condition that for any two strata A and B of $\mathcal{L}$, the comparison isometry

$$cmp(A, B) = (E|A)^{-1} \circ (E|B) : \mathbb{H}^2 \to \mathbb{H}^2$$

is a hyperbolic translation whose axis separates A and B and which translates B to the left as viewed from A. A right earthquake E along $\mathcal{L}$ is defined as above if we change the word "left" to the word "right".

In this paper by an earthquake we mean a left earthquake unless stated otherwise.

Remark 2.1 If A is a gap for $\mathcal{L}$ and l is a geodesic on the boundary of A, which implies that $l \subset \mathcal{L}$, then $E|A$ and $E|l$ are not always the same; in this case the axis of the comparison isometry is l.

In general, a geodesic lamination $\mathcal{L}$ can have infinitely, even uncountably many leaves. Since it is difficult to conceptualize general earthquakes it is useful to define finite earthquakes.

Definition 2.3 An earthquake E whose support $\mathcal{L} = \{l_1, l_2, \ldots, l_k\}$ contains only finitely many leaves is called a *finite earthquake*.

In this paper we utilize finite earthquakes to approximate earthquakes with non-finite support.

The strata of a finite lamination are its gaps $\{A_1, A_2, \ldots, A_n\}$ and its leaves $\{l_1, l_2, \ldots, l_k\}$ which separate these gaps. For any two adjacent gaps A_i and A_j, with common boundary l, the comparison isometry $cmp(A_i, A_j)$ of an earthquake E is a hyperbolic translation with axis l. Given the translation length of $cmp(A_i, A_j)$ and $E|A_i$ we can recover $E|A_j$ by the formula $E|A_j = (E|A_i) \circ cmp(A_i, A_j)$ (see Thurston [13]), but we cannot recover $E|l$. The ambiguity can be seen as follows. The condition that $cmp(A_i, l)$ and $cmp(l, A_j)$ are hyperbolic translations with axis l which translate to the left as viewed from A_i, and the condition $cmp(A_i, l) \circ cmp(l, A_j) = cmp(A_i, A_j)$ forces $E|l$ to be equal to $E|A_i$ post composed with a hyperbolic translation whose axis is l and whose translation length is between 0 and the translation length of $cmp(A_i, A_j)$.

A finite earthquake E extends continuously to the boundary S^1 of $\mathbf{H}^2$, the map $E|S^1$ is piecewise Möbius. If we are only interested in $E|S^1$ then the ambiguity described above is not important. A finite earthquake can be given by finitely many disjoint geodesics in $\mathbf{H}^2$ and weights, positive numbers, assigned to them. The way to recover an earthquake E given a finite lamination $\mathcal{L}$ and weights on the leaves of $\mathcal{L}$ is to take one gap A and define $E|A = id$. Let B be a gap of $\mathcal{L}$. Let $\{l_1, l_2, \dots, l_k\}$ be the leaves of $\mathcal{L}$ which separate A and B in the given order as viewed from A and let $\{a_1, a_2, \dots, a_k\}$ be the assigned weights. Denote the hyperbolic translation with axis l, oriented toward the attracting fixed point, and translation length a, by T_l^a. Define

$$E|B = T_{l_1}^{a_1} \circ T_{l_2}^{a_2} \circ \cdots \circ T_{l_k}^{a_k},$$

where l_i are oriented to the left as seen from A. $E|B$ is a hyperbolic translation whose attracting fixed point is in between the attracting fixed points of $T_{l_1}^{a_1}$ and $T_{l_k}^{a_k}$, and whose repelling fixed point is in between the repelling fixed points of $T_{l_1}^{a_1}$ and $T_{l_k}^{a_k}$.

Let l be a geodesic leaf of $\mathcal{L}$. Then l is on the boundary of two gaps B_1 and B_2. We can define $E|l$ to be $E|B_1$ or $E|B_2$. If we are interested in the restriction of an earthquake E to S^1 then this ambiguity is not important.

From now on we will consider two earthquakes to be the same if their restrictions to S^1 are the same maps. Any two finite earthquakes E_1 and E_2 with the same underlying finite lamination $\mathcal{L}$ and equal weights on the leaves differ by postcomposition with a hyperbolic isometry. To see this we fix a gap A of $\mathcal{L}$ and normalize E_1 and E_2 by postcomposing them with hyperbolic isometries such that $E_1|A = E_2|A = id$. If A_1 is an adjacent gap to A with common boundary l_1 and assigned weight a_1 then $E_1|A_1 = E_2|A_1 = T_{l_1}^{a_1}$, because $cmp_{E_1}(A, A_1) = cmp_{E_2}(A, A_1) = T_{l_1}^{a_1}$ and because of the above normalization. If A_2 is adjacent to A_1 we get that $cmp_{E_1}(A_1, A_2) = cmp_{E_2}(A_1, A_2) = T_{l_2}^{a_2}$. Also from the above $E_1|A_1 = E_2|A_1$. Combining these two equalities, we get $E_1|A_2 = E_2|A_2$. Continuing as above we get that the normalized E_1 and E_2 agree on each gap of $\mathcal{L}$. Hence the original E_1 and E_2 differ by a postcomposition with a hyperbolic isometry.

Finite earthquakes are dense in the set of all earthquakes in the topology of uniform convergence on S^1(see Thurston [13] ,Gardiner-Lakic [6]). A proof of this fact is essentially given in section 3 of this paper.

A geodesic lamination $\mathcal{L}$, together with the atomic measure on $\mathcal{L}$, defines a transverse measure for any geodesic arc transverse to the leaves of $\mathcal{L}$. This is the special case of the following:

Definition 2.4 Let $\mathcal{L}$ be a geodesic lamination. A *transverse measure* μ with support $\mathcal{L}$ is an assignment of a positive Borel measure to any closed geodesic segment I which transversally intersects $\mathcal{L}$. The support of the measure on I is $I \cap \mathcal{L}$. If I' is a subset of I then the measure on I' is equal to the restriction of the measure on I. Also if I is homotopic to J by any homotopy which preserves the leaves of $\mathcal{L}$ then the measure on I is equal to the pushforward of the measure on J by the homotopy map.

Assume that an earthquake E has a general lamination $\mathcal{L}$ as its support. Thurston(see [13]) showed the existence of a transverse measure μ for $\mathcal{L}$ associated to E.

We need the following definition:

Definition 2.5 Let $\mathcal{L}$ be a geodesic lamination. Let $I = [x, y]$ be any closed geodesic arc which intersects $\mathcal{L}$ transversally. A set $\mathcal{P} = \{A_1, A_2, \dots, A_k\}$ consisting of finitely many strata of $\mathcal{L}$ is called an *allowable set* for I if each A_i intersects I, $A_1 = A_x$ contains x, $A_k = A_y$ contains y, and if the order in which the A_i intersect I is given by their enumeration. Further, if A_x is a geodesic, and if B is a gap of $\mathcal{L}$ whose boundary contains A_x, and if $B \cap I = \emptyset$ then we add B to $\mathcal{P}$ as the initial stratum and similarly for A_y. By a *refinement* of an allowable set for I we mean adding finitely many strata of $\mathcal{L}$ which intersect I to $\mathcal{P}$ such that the new set is again an allowable set for I. A sequence of refinements $\mathcal{P}_n$ of an allowable set $\mathcal{P}$ for I is called a *good sequence of refinements* if the union of the elements in $\mathcal{P}_n$, over all n, is dense in I.

Remark 2.2 If we start from an allowable set $\mathcal{P}$ we can get different good sequences of refinements $\mathcal{P}'_n$ and $\mathcal{P}^*_n$. Also, starting from different allowable sets $\mathcal{P}'$ and $\mathcal{P}^*$, corresponding sequences of good refinements $\mathcal{P}'_n$ and $\mathcal{P}^*_n$ can agree after some n_0.

Definition 2.6 Let E be an earthquake whose support is a geodesic lamination $\mathcal{L}$. Let I be a closed geodesic arc which intersects $\mathcal{L}$ transversally and let $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ be an allowable set for I. Let $tr.l.\{T\}$ denote the translation length of the hyperbolic translation T. The *approximate measure* of I with respect to $\mathcal{P}$ is denoted by $\mu_{\mathcal{P}}(I)$, and is defined as

$$\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\}.$$

The main tool in the proof of the existence of a transverse measure is the following lemma, whose statement and proof is given by Thurston (see [13]). The details of the proof are given by Gardiner, Hu and Lakic (see [7]). It gives us a way to estimate the effect of an earthquake on a region between two strata in terms of the comparison map for those strata.

Lemma 2.1: Let E be an earthquake and let $d > 0$. Let $I = [x, y]$ be any closed geodesic arc of length less than or equal to d which intersects strata of E transversally. Let $\mathcal{P} = \{A_1 = A_x, A_2, \ldots, A_k = A_y\}$ be an allowable set for I. Set $T_I = tr.l.\{cmp(A_x, A_y)\}$. Then there exists $M > 0$ such that

$$\left|\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\} - T_I\right| \leq M \cdot \left(\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\}\right) \cdot d^2.$$

Moreover

$$\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\} \text{ decreases as we refine } \mathcal{P}.$$

In particular, $\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\} \leq T_I$.

Pf. Suppose that g_1 and g_2 are two hyperbolic translation whose axes are distance $d > 0$ apart. Further, suppose g_1 and g_2 translate in the same direction and assume $\log \lambda_1 = tr.l.\{g_1\} \geq \log \lambda_2 = tr.l.\{g_2\}$.

Now, for convenience in calculating, we use the upper half plane model of $\mathbb{H}^2$. We normalize by conjugation such that $g_1(z) = \lambda_1 z$ and $g_2(z) = \frac{\lambda_2(b-a)(z-a)}{(\lambda_2-1)(z-a)+(b-a)} + a$, where $0 < a < 1$ is the repelling fixed point of g_2 and $b = \frac{1}{a}$ is the attracting fixed point of g_2. Then $g_1 \circ g_2(z) = \frac{\lambda_1\lambda_2(b-a)(z-a)}{(\lambda_2-1)(z-a)+(b-a)} + \lambda_1 a$. The repelling fixed point of $g_1 \circ g_2$ is a point x between 0 and a.

Note that the multiplier of a hyperbolic isometry g is equal to its derivative at the repelling fixed point. To see it is, let $z \to z_r$ in the formula $\frac{g(z)-z_r}{z-z_r} = \lambda\frac{g(z)-z_a}{z-z_a}$, where z_r is the repelling fixed point of g and z_a is the attracting fixed point of g. Hence if λ is the multiplier of $g_1 \circ g_2$ we have

$$\lambda = (g_1 \circ g_2)'(x) = \frac{\lambda_1\lambda_2(b-a)^2}{[(\lambda_2 - 1)(x-a) + (b-a)]^2}$$

so that

$$\log\lambda - \log\lambda_1 - \log\lambda_2 = -2\log\frac{(b-a)+(\lambda_2-1)(x-a)}{b-a}. \qquad (1)$$

Since $0 < x < a < b$ and $\lambda_2 > 1$ we have that

$$\frac{(b-a)+(\lambda_2-1)(x-a)}{b-a} < 1.$$

Because x is the repelling fixed point of $g_1 \circ g_2$ we have

$$\frac{x-\lambda_1 a}{\lambda_1\lambda_2(x-a)} = \frac{b-a}{(\lambda_2-1)(x-a)+(b-a)}.$$

The expression on the left side of the above equality is positive because $0 < x < a$ and $\lambda_1 > 1$. Hence

$$0 < \frac{(b-a)+(\lambda_2-1)(x-a)}{b-a} < 1$$

and

$$\log\frac{(b-a)+(\lambda_2-1)(x-a)}{b-a} < 0.$$

The last inequality together with (1) implies that

$$\log\lambda_1 + \log\lambda_2 < \log\lambda.$$

Now we estimate $|\log\frac{(b-a)+(\lambda_2-1)(x-a)}{b-a}| = |\log(1+(\lambda_2-1)\frac{x-a}{b-a})|$. Since $|x-a| < a$ and $b = \frac{1}{a}$ we have

$$|\frac{x-a}{b-a}| \leq \frac{a^2}{1-a^2}.$$

If d is the distance between the axes of g_1 and g_2, and θ is the angle between the axis of g_1 and the Euclidean line passing through 0 which is tangent to the axis of g_2 we have that

$$\sinh d = \tan\theta,$$

by Beardon [2], section 7.20. Using elementary formulas we compute that for $a < \frac{1}{2}$

$$\tan\theta = a \cdot \frac{\sqrt{a^2+3}}{1-a^2} < 4a.$$

Then for d small, $d \sim \sinh d$ and a is small; in fact $d \sim \tan\theta \sim \sqrt{3}a$. Then for λ_2 in a bounded interval and d small we have that

$$|\log\left[1+(\lambda_2-1)(\frac{x-a}{b-a})\right]| = O(d^2\log\lambda_2).$$

Consequently,

$$\begin{aligned}&\log\lambda_1+\log\lambda_2 \le \log\lambda\\ &\le \log\lambda_1+\log\lambda_2+O(d^2\cdot\log\lambda_2).\end{aligned} \tag{2}$$

In words this says that the translation length of the composition of two hyperbolic translations differs from the sum of the translation lengths of these two translations by at most a constant times the smaller translation length times the square of the distance between the axes. The last two statements of the lemma follow directly from the left part of the above inequality.

Now we prove the first statement of the lemma. First by (2) we have

$$|tr.l.\{cmp(A_1, A_{k-1})\} + tr.l.\{cmp(A_{k-1}, A_k)\} - T_l|$$

$$\le M \cdot tr.l.\{cmp(A_{k-1}, A_k)\} \cdot d^2.$$

Then, also by (2) we have

$$|tr.l.\{cmp(A_1, A_{k-2})\} + tr.l.\{cmp(A_{k-2}, A_{k-1})\} - tr.l.\{cmp(A_1, A_{k-1})\}|$$
$$\leq M \cdot tr.l.\{cmp(A_{k-2}, A_{k-1})\} \cdot d^2.$$

Combining the above two inequalities and using the triangle inequality we get

$$|tr.l.\{cmp(A_1, A_{k-2})\} + tr.l.\{cmp(A_{k-2}, A_{k-1})\} +$$
$$+tr.l.\{cmp(A_{k-1}, A_k)\} - T_I|$$
$$\leq M \cdot (tr.l.\{cmp(A_{k-2}, A_{k-1})\} + tr.l.\{cmp(A_{k-1}, A_k)\}) \cdot d^2.$$

Continuing in the same fashion we get

$$|\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\} - T_I| \leq M \cdot (\sum_{i=2}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\}) \cdot d^2.$$

□

Now we can state and prove a proposition of Thurston on existence of transverse measures.

Proposition 2.1: (Thurston) Let E be an earthquake with support $\mathcal{L}$. Let I be a closed geodesic arc intersecting $\mathcal{L}$ transversally. Then the infimum of the approximate measure of I with respect to all allowable sets is equal to the limit of the approximate measure of I with respect to a sequence of good refinements.

Corollary 2.1: The limit of the approximate measures of I with respect to a sequence of good refinements does not depend on the particular choice of the sequence.

Pf. of Prop. 2.1 If only finitely many strata of E intersect I then the infimum is attained by taking all of them in an allowable set and the proof is obvious.

We assume that infinitely many strata of E intersect $I = [x, y]$. Let m be the infimum of the approximate measures over all allowable sets for I. Fix a sequence of good refinements $\mathcal{P}_n$. Let $\epsilon > 0$ be given. Then there exists an allowable set $\mathcal{P}' = \{A_1, A_2, \ldots, A_k\}$ such that $|\mu_{\mathcal{P}'}(I) - m| < \epsilon$. We can choose $n_0 > 0$ big enough such that for any $A_i \in \mathcal{P}'$ there exist B_{j_i} and B_{j_i+1} in $\mathcal{P}_{n_0}$, where A_i is in between them and $dist(I \cap B_{j_i}, I \cap B_{j_i+1}) < \frac{\epsilon}{k}$. Note that $x \in A_1 = B_1$ and $y \in A_k = B_n$. By lemma 2.1

$$\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\}$$
$$\geq \sum_{i=1}^{k-1} \Big(tr.l.\{cmp(A_i, B_{j_i+1})\}$$
$$+tr.l.\{cmp(B_{j_i+1}, B_{j_{i+1}})\} + tr.l.\{cmp(B_{j_{i+1}}, A_{i+1})\}\Big).$$

The right-hand side of the above inequality can be rewritten as

$$\sum_{i=1}^{k-1} tr.l.\{cmp(B_{j_i+1}, B_{j_{i+1}})\}$$
$$+\sum_{i=2}^{k-2} \Big(tr.l.\{cmp(B_{j_i}, A_i)\} + tr.l.\{cmp(A_i, B_{j_i+1})\}\Big).$$

Again by lemma 2.1

$$
\begin{aligned}
& tr.l.\{cmp(B_{j_i}, A_i)\} + tr.l.\{cmp(A_i, B_{j_i+1})\} \\
& \geq tr.l.\{cmp(B_{j_i}, B_{j_i+1})\} - O(T_I \frac{\epsilon^2}{k^2}),
\end{aligned}
$$

because

$$
T_I = tr.l.\{cmp(B_1, B_n)\} \geq \max\left\{tr.l.\{cmp(B_{j_i}, A_i)\}, tr.l.\{cmp(A_i, B_{j_i+1})\}\right\}.
$$

Hence

$$
\begin{aligned}
& \sum_{i=2}^{k-2} \left(tr.l.\{cmp(B_{j_i}, A_i)\} + tr.l.\{cmp(A_i, B_{j_i+1})\}\right) \\
& \geq \sum_{i=2}^{k-2} tr.l.\{cmp(B_{j_i}, B_{j_i+1})\} - (k-2)O(T_I \frac{\epsilon^2}{k^2}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\} \\
& \geq \sum_{i=1}^{k-1} tr.l.\{cmp(B_{j_i+1}, B_{j_{i+1}})\} + \sum_{i=2}^{k-2} tr.l.\{cmp(B_{j_i}, B_{j_i+1})\} \\
& -(k-2)O(T_I \frac{\epsilon^2}{k^2}) \\
& \geq \sum_{j=1}^{n-1} tr.l.\{cmp(B_j, B_{j+1})\} - kO(T_I \frac{\epsilon^2}{k^2}).
\end{aligned}
$$

Hence the approximate measure of I with respect to $\mathcal{P}_{n_0}$ is close to the infimum m. Because the approximate measure decreases as n increases, the limit is as close as we want to m. Thus the limit is equal to m. □

We use above corollary to define the measure of a closed geodesic arc.

Definition 2.7 The measure of a closed geodesic arc I is the limit of the approximate measures of I with respect to a sequence of good refinements.

It is obvious that if a closed geodesic arc I is homotopic to a closed geodesic arc J by a homotopy respecting the leaves of $\mathcal{L}$ then the measure of I is equal to the measure of J.

In order to have a Borel positive measure on a closed geodesic arc we need to specify the measure on each half-open interval. To do that we will define the measure of a single leaf of $\mathcal{L}$. Let l be a leaf of $\mathcal{L}$ and let $I = [x, y]$ be a closed geodesic arc whose interior intersects l in one point. Let $I_n = [x_n, y_n]$ be a sequence of closed geodesic arcs such that the intersection of the interior of I_n with l is one point, $I_n \subset I_{n-1} \subset I$ for all $n > 1$, and the length of I_n tends to zero as n converges to infinity.

Lemma 2.2: The limit as $n \to \infty$ of the measures of the closed geodesic arcs I_n does not depend on the choice of I and I_n as long as they satisfy the above properties.

Pf. Take I and I' as closed geodesic arcs whose interiors intersect l. Denote by I_n and I'_n sequences of closed geodesic arcs corresponding to I and I' with the above properties. For a fixed n, there exists m such that I'_m can be homotoped to a subset of I_n. The same is true if we reverse the roles of I_n and I'_n. Since the measures are decreasing as n increases we see that the limits of the measures of I_n and I'_n are the same. □

Now we can define the measure of an open interval of finite length as the measure of its closure minus the measure of its endpoints. The measure of a half-open interval is defined similarly. Hence we obtain a positive Borel measure on each closed geodesic arc which also respects homotopy. It is also obvious that the restriction of the measure to a subinterval is equal to its measure considered as an interval in its own right.

These lemmas allow us to make the following definition:

Definition 2.8 To an earthquake E there is a naturally associated transverse measure. The measure of a closed geodesic arc I is equal to the limit of the approximate measures of I with respect to a sequence of good refinements. The measure of a single leaf in the support of E is equal to the limit of the measures of a sequence of closed geodesic arcs whose intersection with the leaf is equal to a point.

We want to show that the map which associates a transverse measure to a normalized earthquake is one to one.

Define the norm of a hyperbolic isometry $T = \frac{az+b}{cz+d}$ with $ad - bc = 1$ by $\|T\| = \max\{|a| + |b|, |c| + |d|\}$.

We need the following lemma.

Lemma 2.3: Let k be a closed geodesic arc and $N > 0$ fixed number. Let l_1 and l_2 be two fixed disjoint geodesics passing through the endpoints of k. Let m_1 and m_2 be any two disjoint geodesics between l_1 and l_2. Let g_1 and g_2 be two hyperbolic translations which translate in the same direction with the same translation length such that the axes of g_1 and g_2 (possibly intersecting) lie between m_1 and m_2. Set $d = dist(m_1 \cap k, m_2 \cap k)$. There exists $M > 0$, depending only on k and N, and choice of l_1 and l_2, such that

$$\|g_1 \circ g_2^{-1} - id\| \leq M \cdot tr.l.\{g_1\} \cdot d,$$

for all g_1, g_2 with $tr.l.\{g_1\} \leq N$.

Pf. It is enough to prove the lemma for d sufficiently small, by the compactness. We work with the upper half plane model and represent isometries by elements of $PSL_2(R)$. For readability we use the same notation for a hyperbolic isometry and for its representative in $PSL_2(\mathbf{R})$.

By conjugation we assume that $k = [i, ai]$, $a > 1$, is an arc on the imaginary axis. Then l_1 passes through i and l_2 passes through ai. Assume that $m_1 = [x_1, y_1]$ and $m_2 = [x_2, y_2]$ are passing through endpoints of a closed subarc $k' = [bi, ci]$, $a \geq c > b \geq 1$, of arc k whose length is d. We want to estimate $|x_2 - x_1|$ and $|y_2 - y_1|$ in terms of d.

Since k is a fixed closed geodesic arc in $\mathbf{H}^2$ the hyperbolic distance on k is comparable with the Euclidean distance. So for small $d > 0$, we can consider k' as an interval of fixed Euclidean length d. It is obvious that $|y_2 - y_1|$ will be maximal if m_1 and m_2 have an endpoint in common, $x_1 = x_2 = x$, and this endpoint coincides with the endpoint of l_1 and if k' is at the highest position in k (which means $c = a$).

Observe the triangle Δ_1 with vertices x, bi and 0, and the triangle Δ_2 with vertices x, $ci = ai$ and 0. Let C_1 and C_2 be the Euclidean centers of semicircles m_1 and m_2; let P_1 and P_2 be the midpoints of the segments $[x, bi]$ and $[x, ai]$. Then the triangle Δ_1' with vertices x, C_1 and P_1 is similar to Δ_1; and triangle Δ_2' with vertices x, C_2 and P_2 is similar to Δ_2. Let r_1 be the radius of m_1 and r_2 be the radius of m_2. From the similarity of the above triangles we get $r_1 = \frac{|x-bi|^2}{2|x|}$ and $r_2 = \frac{|x-ai|^2}{2|x|}$. Since $|y_2 - y_1| = r_2 - r_1$ we get

$$|y_2 - y_1| = \frac{|x - bi|^2 - |x - ai|^2}{2|x|}.$$

From the triangle inequality $|x - ai| - |x - bi| \leq a - b = d$ and from the above we obtain

$$|y_2 - y_1| \leq K(l_1) \cdot d,$$

where $K(l_1)$ is a constant which depends on l_1. In the similar way we get

$$|x_2 - x_1| \leq K(l_2) \cdot d,$$

where $K(l_2)$ is a constant which depends on l_2.

Let us denote by x_{g_1} and y_{g_1} the fixed points of g_1 and by x_{g_2} and y_{g_2} the fixed points of g_2. Then, for $K = \sup\{K(l_1), K(l_2)\}$,

$$|x_{g_2} - x_{g_1}| \leq |x_2 - x_1| \leq K \cdot d$$

and

$$|y_{g_2} - y_{g_1}| \leq |y_2 - y_1| \leq K \cdot d.$$

Let $\lambda = \frac{1}{2} tr.l.\{g_1\} = \frac{1}{2} tr.l.\{g_2\}$ and

$$E = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}.$$

Set

$$A = \begin{pmatrix} \frac{y_{g_1}}{\sqrt{y_{g_1} - x_{g_1}}} & \frac{x_{g_1}}{\sqrt{y_{g_1} - x_{g_1}}} \\ \frac{1}{\sqrt{y_{g_1} - x_{g_1}}} & \frac{1}{\sqrt{y_{g_1} - x_{g_1}}} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{y_{g_2}}{\sqrt{y_{g_2} - x_{g_2}}} & \frac{x_{g_2}}{\sqrt{y_{g_2} - x_{g_2}}} \\ \frac{1}{\sqrt{y_{g_2} - x_{g_2}}} & \frac{1}{\sqrt{y_{g_2} - x_{g_2}}} \end{pmatrix}.$$

We estimate

$$\|AEA^{-1}BE^{-1}B^{-1} - id\|.$$

It is obvious that $\|AEA^{-1}BE^{-1}B^{-1} - id\|$ is a real analytic function of λ, A and B. It is zero for $\lambda = 0$ and any A and B; and it is zero for $A = B$ and any λ. Hence we get

$$\|AEA^{-1}BE^{-1}B^{-1} - id\| \leq const \cdot \lambda \cdot \|A - B\|.$$

We prove that $\|A - B\| \leq const \cdot d$, which gives the result. In order to prove this we subtract corresponding entries of A and B.

Set $C = A - B$. We estimate $C(1,1)$ by

$$\frac{y_{g_2}}{\sqrt{y_{g_2}-x_{g_2}}} - \frac{y_{g_1}}{\sqrt{y_{g_1}-x_{g_1}}} = \frac{y_{g_2}\sqrt{y_{g_1}-x_{g_1}} - y_{g_1}\sqrt{y_{g_2}-x_{g_2}}}{\sqrt{y_{g_2}-x_{g_2}}\sqrt{y_{g_1}-x_{g_1}}}$$
$$\leq \frac{y_{g_2}\sqrt{y_{g_2}-x_{g_2}+2Kd} - (y_{g_2}-Kd)\sqrt{y_{g_2}-x_{g_2}}}{\sqrt{y_{g_2}-x_{g_2}}\sqrt{y_{g_1}-x_{g_1}}}$$
$$\leq \frac{y_{g_2}\sqrt{y_{g_2}-x_{g_2}}(1+Kd) - y_{g_2}\sqrt{y_{g_2}-x_{g_2}} + Kd\sqrt{y_{g_2}-x_{g_2}}}{\sqrt{y_{g_2}-x_{g_2}}\sqrt{y_{g_1}-x_{g_1}}}$$
$$= \frac{K(y_{g_2}+1)}{\sqrt{y_{g_1}-x_{g_1}}} \cdot d.$$

In a similar way we get an estimate for $C(1,2)$.

For $C(2,1)$ we get

$$\frac{1}{\sqrt{y_{g_1}-x_{g_1}}} - \frac{1}{\sqrt{y_{g_2}-x_{g_2}}} = \frac{\sqrt{y_{g_2}-x_{g_2}} - \sqrt{y_{g_1}-x_{g_1}}}{\sqrt{y_{g_2}-x_{g_2}}\sqrt{y_{g_1}-x_{g_1}}}$$
$$\leq \frac{\sqrt{y_{g_1}-x_{g_1}}(1+Kd) - \sqrt{y_{g_1}-x_{g_1}}}{\sqrt{y_{g_1}-x_{g_1}}\sqrt{y_{g_2}-x_{g_2}}} = \frac{K}{\sqrt{y_{g_2}-x_{g_2}}} \cdot d.$$

Finally $C(2,1) = C(2,2)$. □

Proposition 2.2: (Thurston) Let E_1 and E_2 be two earthquakes with the same underlying geodesic lamination $\mathcal{L}$ and the same transverse measure μ. Then there exists a hyperbolic isometry g such that the earthquakes $g \circ E_1$ and E_2 are essentially the same.

Pf. We call an earthquake E trivial if it has only one stratum, the hyperbolic plane, and is therefore a hyperbolic isometry of $\mathbb{H}^2$ or the identity.

We first prove that an earthquake E with zero associated transverse measure is trivial. Fix a stratum A of E. Let B be any other stratum of E and let I be a closed geodesic arc connecting A to B. It is enough to prove that $cmp(A, B) = id$. Let $T_I = tr.l.\{cmp(A, B)\}$ and let $\mathcal{P}_n$ be a sequence of good refinements. By lemma 2.1

$$|T_I - \mu_{\mathcal{P}_n}(I)| \leq M \cdot [length(I)]^2 \cdot \mu_{\mathcal{P}_n}(I).$$

Since by assumption $\mu_{\mathcal{P}_n}(I) \to \mu(I)$ as $n \to \infty$, we get that $T_I = 0$. Since $cmp(A, B)$ is a hyperbolic translation with translation length zero, it is the identity. Hence, $E|B = E|A$ for all B. Thus E is trivial.

Next, we assume that E_1 and E_2 are two earthquakes with the same support $\mathcal{L}$ and the same transverse measure μ. We prove that $E_1 \circ E_2^{-1}$ is a hyperbolic isometry. It is not clear that the map $E_1 \circ E_2^{-1}$ is even an earthquake, but it certainly is a hyperbolic isometry on each strata of the geodesic lamination $\mathcal{L}^* = E_2(\mathcal{L})$. We take two strata A^* and B^* of $\mathcal{L}^*$ and connect them by a closed geodesic arc I^*. Let $A = E_2^{-1}(A^*)$ and $B = E_2^{-1}(B^*)$, and let I be a closed geodesic arc connecting A to B. Then $(E_1 \circ E_2^{-1})|A^* = (E_1|A) \circ (E_2^{-1}|A^*) = (E_1|A) \circ (E_2|A)^{-1}$ and $(E_1 \circ E_2^{-1})|B^* = (E_1|B) \circ (E_2^{-1}|B^*) = (E_1|B) \circ (E_2|B)^{-1}$. The comparison maps for $E_1 \circ E_2^{-1}$ is defined by

$$cmp(A^*, B^*) = [(E_1 \circ E_2^{-1})|A^*]^{-1} \circ [(E_1 \circ E_2^{-1})|B^*] =$$

$$(E_2|A) \circ (E_1|A)^{-1} \circ (E_1|B) \circ (E_2|B)^{-1} =$$

$$(E_2|A) \circ (E_1|A)^{-1} \circ (E_1|B) \circ (E_2|B)^{-1} \circ (E_2|A) \circ (E_2|A)^{-1}.$$

We can take the representation of the above isometries in $PSL_2(\mathbf{R})$. Then we have

$$cmp(A^*, B^*) =$$

$$= (E_2|A)\{[(E_1|A)^{-1}(E_1|B)(E_2|B)^{-1}(E_2|A)] - id\}(E_2|A)^{-1} + id.$$

Let $\mathcal{P}_n = \{A_1^* = A^*, A_2^*, \ldots, A_k^* = B^*\}$ be an allowable set for I^* with respect to $\mathcal{L}^*$ such that $dist(I \cap A_i, I \cap A_{i+1}) \leq \frac{1}{n}$.

Then

$$cmp(A^*, B^*) = cmp(A_1^*, A_2^*) \circ cmp(A_2^*, A_3^*) \circ \ldots \circ cmp(A_{k-1}^*, A_k^*) \quad (3)$$

and

$$\begin{aligned} &cmp(A_i^*, A_{i+1}^*) = \\ &(E_2|A_i)\{[(E_1|A_i)^{-1}(E_1|A_{i+1})(E_2|A_{i+1})^{-1}(E_2|A_i)] - id\}(E_2|A_i)^{-1} \\ &+id. \end{aligned} \quad (4)$$

We want to apply lemma 2.3 to $(E_1|A_i)^{-1}(E_1|A_{i+1})$ and $(E_2|A_i)^{-1}(E_2|A_{i+1})$, but they do not have the same translation lengths. In aplying lemma 2.3 constant will depend on the boundary geodesics of strata A and B. Let T be a hyperbolic translation with axis equal to the axis of $(E_1|A_i)^{-1}(E_1|A_{i+1})$ and with the translation length equal to $tr.l.\{(E_2|A_i)^{-1}(E_2|A_{i+1})\}$.

If $T_1 = T_t^{a_1}$ and $T_2 = T_t^{a_2}$ then

$$\|T_1 T_2^{-1} - id\| \leq const \cdot |a_1 - a_2|. \quad (5)$$

We apply this to $(E_1|A_i)^{-1}(E_1|A_{i+1})$ and T. Let I_i be a closed subarc of I which connects A_i and A_{i+1}. For $j = 1, 2$, by lemma 2.1, we have

$$|tr.l.\{(E_j|A_i)^{-1}(E_j|A_{i+1}\} - \mu(I_i)| \leq const \cdot \mu(I_i) \cdot \frac{1}{n^2}.$$

Thus by the triangle inequality

$$\begin{aligned}&|tr.l.\{(E_1|A_i)^{-1}\circ(E_1|A_{i+1})\} - tr.l.\{(E_2|A_i)^{-1}\circ(E_2|A_{i+1})\}|\\ &\le |tr.l.\{(E_1|A_i)^{-1}\circ(E_1|A_{i+1})\} - \mu(I_i)|\\ &+|\mu(I_i) - tr.l.\{(E_2|A_i)^{-1}\circ(E_2|A_{i+1})\}|\\ &\le const\cdot\mu(I_i)\cdot\tfrac{1}{n^2}.\end{aligned} \tag{6}$$

By (5) and (6) we have

$$\|(E_1|A_i)^{-1}(E_1|A_{i+1})T^{-1} - id\| \le const\cdot\mu(I_i)\cdot\frac{1}{n^2}. \tag{7}$$

Then by lemma 2.3, where k is subarc of I connecting A_i and A_{i+1}, l_1 is boundary leaf of A_i closest to A_{i+1}, l_2 is boundary leaf of A_{i+1} closest to A_i and $N = tr.l.\{cmp(A_1, A_k)\}$; by (7) and by the triangle inequality

$$\begin{aligned}&\|(E_1|A_i)^{-1}(E_1|A_{i+1})(E_2|A_{i+1})^{-1}(E_1|A_{i+1}) - id\|\\ &= \|(E_1|A_i)^{-1}(E_1|A_{i+1})T^{-1}T(E_2|A_{i+1})^{-1}(E_2|A_i) - id\|\\ &\le \|[(E_1|A_i)^{-1}(E_1|A_{i+1})T^{-1} - id]T(E_2|A_{i+1})^{-1}(E_2|A_i)\|\\ &+\|T(E_2|A_{i+1})^{-1}(E_2|A_i) - id\|\\ &\le const_1\cdot\mu(I_i)\cdot\tfrac{1}{n^2} + const_2\cdot tr.l.\{(E_2|A_{i+1})^{-1}(E_2|A_i)\}\cdot\tfrac{1}{n}\\ &\le const_3\cdot tr.l.\{(E_2|A_{i+1})^{-1}(E_2|A_i)\}\cdot\tfrac{1}{n}.\end{aligned} \tag{8}$$

Using (8) in (4) we get

$$\|cmp(A_i^*, A_{i+1}^*) - id\| \le const\cdot tr.l.\{(E_2|A_{i+1})^{-1}(E_2|A_i)\}\cdot\frac{1}{n}.$$

Replacing each $cmp(A_i^*, A_{i+1}^*)$ with $[cmp(A_i^*, A_{i+1}^*) - id] + id$ in (3) we get

$$\begin{aligned}&cmp(A^*, B^*) =\\ &[cmp(A_1^*, A_2^*) - id]cmp(A_2^*, A_3^*)\cdots cmp(A_{k-1}^*, A_k^*)\\ &+[cmp(A_2^*, A_3^*) - id]cmp(A_3^*, A_4^*)\cdots cmp(A_{k-1}^*, A_k^*)\\ &+\cdots+[cmp(A_{k-1}^*, A_k^*) - id] + id.\end{aligned}$$

By the above, by the triangle inequality and by (8) we obtain

$$\begin{aligned}&\|cmp(A^*, B^*) - id\|\\ &\le \|[cmp(A_1^*, A_2^*) - id]cmp(A_2^*, A_3^*)\cdots cmp(A_{k-1}^*, A_k^*)\|\\ &+\|[cmp(A_2^*, A_3^*) - id]cmp(A_3^*, A_4^*)\cdots cmp(A_{k-1}^*, A_k^*)\|\\ &+\cdots+\|[cmp(A_{k-1}^*, A_k^*) - id]\|\\ &\le const\cdot\textstyle\sum_{i=1}^{k-1} tr.l.\{(E_2|A_i)^{-1}(E_2|A_{i+1})\}\cdot\tfrac{1}{n}.\end{aligned}$$

For a fixed A^* and all strata B^* intersecting I^*, as n converges to infinity, we see that $cmp(A^*, B^*)$ converges to the identity. Hence $E_1 \circ E_2^{-1}$ is equal to a fixed hyperbolic isometry $E_1 \circ E_2^{-1}|A^*$. □

Remark 2.3 The surjectivity property of earthquakes is not used in the proof of the above proposition. Hence for mappings which satisfy all the properties of earthquakes except surjectivity it is still true that two of them with the same support and the same transverse measure differ by postcomposition with a hyperbolic isometry.

Notation: If x and y are two points in the hyperbolic plane, then $[x, y]$ is the hyperbolic geodesic arc which joins x and y. Also $length([x, y])$ is the length of the arc $[x, y]$. Let A and B be two sets in the hyperbolic plane. Then $dist(A, B) = \inf_{a \in A, b \in B} length([a, b])$.

Definition 2.9 An earthquake E is bounded if there exists $M > 0$ such that for any two strata A and B of E with $dist(A, B) \leq 1$, their comparison map, $cmp(A, B)$, has translation length less than or equal to M.

Definition 2.10 A transverse measure μ is bounded if there exists a number $a > 0$ such that the measure of any geodesic arc of length 1 is less than or equal to a. If $\mathcal{F}$ is the family of all geodesic arcs of length 1 and μ is a bounded measure we define the norm of μ to be $\|\mu\| = \sup_{I \in \mathcal{F}} \mu(I)$.

We prove that for an earthquake these two notions are equivalent:

Proposition 2.3: Let E be an earthquake. Then E is bounded if and only if E has bounded transverse measure.

Pf. Let I be a closed geodesic arc of length 1 and let $\mathcal{P} = \{A_1, A_2, \dots, A_k\}$ be an allowable set for I. Then by lemma 2.1

$$|\sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\} - T_I| \leq M \cdot \sum_{i=1}^{k-1} tr.l.\{cmp(A_i, A_{i+1})\},$$

for a constant $M > 0$ depending on I and $T_I = tr.l.\{cmp(A_1, A_k)\}$. Recalling definition 2.6, this inequality is just inequality about approximate measures for a given refinement. Taking a sequence of good refinements of $\mathcal{P}$ we have the same inequality for each refinement. The limit of approximate measure of I under good refinements of $\mathcal{P}$ is the measure of I.

Hence

$$|\mu(I) - T_I| \leq const \cdot \mu(I).$$

Thus if the measure of I is bounded then E is a bounded earthquake. Conversely, if E is a bounded earthquake then directly from lemma 2.1 E has bounded transverse measure. □

Definition 2.11 Let E_n, $n = 1, 2, \dots$ be a sequence of earthquakes and μ_n their corresponding transverse measures. Let E be a single earthquake and μ its corresponding measure. The sequence μ_n converges in the weak* sense to μ if for any closed geodesic arc I and any continuous function f,

$$\lim_{n\to\infty}\int_I f d\mu_n = \int_I f d\mu.$$

In order to prove that a certain sequence of earthquakes converges to a given earthquake map we need the following basic estimate given by Epstein and Marden (see [4]).

Notation: Let l be a geodesic in $\mathbb{H}^3$ with endpoints x and y in $\overline{\mathbb{C}} = \partial\mathbb{H}^3$, and a a complex number. Assume that l is oriented from x to y. T_l^a is the loxodromic element of $PSL_2(\mathbb{C})$ defined by $\frac{T_l^a(z)-x}{T_l^a(z)-y} = e^a\frac{z-x}{z-y}$, for $z \in \overline{\mathbb{C}}$. Note that $T_l^a = T_l^{\Re a} \circ T_l^{i\Im a} = T_l^{i\Im a} \circ T_l^{\Re a}$.

Lemma 2.4: Let us fix a closed geodesic arc I of length $d > 0$ and a number $N > 0$. Then there exists a constant $M > 0$ such that for any finite set of geodesics $\{l_1, l_2, ..., l_k\}$ intersecting I in the given order and oriented to the left as viewed from one of the endpoints of I, and any finite set of complex numbers $\{a_1, a_2, ..., a_k\}$, with $\sum_{i=1}^k |\Re(a_i)| \le N$, and any geodesic l intersecting I that is disjoint from or inside the set $\{l_1, l_2, \ldots, l_k\}$,

$$\|T_{l_1}^{a_1} \circ T_{l_2}^{a_2} \circ \cdots \circ T_{l_k}^{a_k} - T_l^{\sum_{i=1}^k a_i}\| \le M \cdot d \cdot \sum_{i=1}^k |a_i|. \quad \square$$

Definition 2.12 Let μ be a finite transverse measure with support $\{l_1, l_2, \ldots, l_k\}$, $\mu(l_i) = a_i > 0$ and h any homeomorphism of the boundary of the hyperbolic plane. Let l_i' be a hyperbolic geodesic whose endpoints are images under h of the endpoints of the l_i, for $i = 1, 2, \ldots k$. The pushforward of μ by h is the finite transverse measure μ^* with support $\{l_1', l_2', \ldots, l_k'\}$ and $\mu^*(l_i') = a_i$.

Let μ be a finite transverse measure. Then there exists an earthquake E_μ whose measure is μ. For all $t > 0$, $t\mu$ is a family of finite transverse measures. There exists a family of earthquakes $E_{t\mu}$ whose measures are $t\mu$. Earthquakes $E_{t\mu}$ have the same strata as E_μ and we normalize them to be identity on a fixed stratum A. The family $E_{t\mu}$ is called an earthquake path. Because μ is a finite measure, if $t < 0$ we define $E_{t\mu}$ as the earthquake obtained by translating to the right on each of the finite strata.

Denote by μ_t^* the pushforward of the measure μ by $E_{t\mu}$. We need the following result from Gardiner-Hu-Lakic [7]. We include the proof for completness and note that we obtain an explicit expression for a constant with $\|\mu\|$.

Lemma 2.5: Let μ be a finite transverse measure and for $t \in \mathbf{R}$, let μ_t^* be the pushforward of the measure μ by $E_{t\mu}$. Then

$$\|\mu_t^*\| \leq 8e^{\frac{\|\mu\|}{2}|t|}.$$

Pf. Let E_μ be an earthquake with transverse measure μ. Let l_1 and l_2 be two geodesics in the support of a finite transverse measure μ. Let J be the closed common orthogonal geodesic arc connecting l_1 and l_2; assume the length of J to be at least 1. Now we prove that the distance between the geodesics $l'_1 = E_\mu(l_1)$ and $l'_2 = E_\mu(l_2)$ is at least $\frac{1}{3}e^{-\frac{\mu(J)}{2}}$.

We work with the upper half-plane model. By a conjugation, we can assume that l_1 is a geodesic with endpoints 0 and ∞ fixed by E, and l_2 is a geodesic with endpoints 1 and $c > 1$. Then the common orthogonal geodesic is a half-circle with center at 0 and orthogonal to l_2. Because of the formula for the hyperbolic metric, we get

$$d = \int_{\theta_0}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta} = -\log\tan\frac{\theta_0}{2},$$

where d is the hyperbolic distance of l_1 and l_2, and θ_0 is the angle between the x-axis and the Euclidean line tangent to l_2 and passing through 0. By elementary Euclidean geometry we get

$$\sin\theta_0 = \frac{c-1}{c+1}.$$

From the above two formulas we obtain

$$c = (\frac{e^d+1}{e^d-1})^2,$$

from which

$$e^d = \frac{\sqrt{c}+1}{\sqrt{c}-1}.$$

Suppose that a finite earthquake E_μ has n support lines in between l_1 and l_2, inclusive. Then the action of E_μ on l_2 is given by $B_1 \circ B_2 \circ \cdots \circ B_n$, where

B_i are hyperbolic translations whose axes are in the support of μ and in between l_1 and l_2. Set $b_i = tr.l.\{B_i\}$. Let $a = E_\mu(1) > 1$ and $b = E_\mu(c) > c$. We can write down the distance from $l_1' = l_1$ to l_2' in terms of a and b. To estimate it in terms of c we use Lemma 2.1. Let $A(z) = \frac{1}{a}z$ be the hyperbolic translation with axis l_1 which maps a to 1. Then $A \circ E_\mu$ is an earthquake fixing $0, 1$ and ∞. Denote, again, by $l_1' = l_1$ and l_2' the images of l_1 and l_2 under $A \circ E_\mu$. The maximal value of $A \circ E_\mu(c)$ will give us the smallest possible distance from l_1' to l_2'. We can write

$$A \circ E_\mu | l_2 = A \circ B_1 \circ B_2 \circ \cdots \circ B_n$$

$$= (A \circ B_1 \circ A^{-1}) \circ (A \circ B_2 \circ A^{-1}) \circ \cdots \circ (A \circ B_n \circ A^{-1}) \circ A.$$

Then the image of c under $A \circ E_\mu$ is equal to the image of $\frac{c}{a}$ under $(A \circ B_1 \circ A^{-1}) \circ (A \circ B_2 \circ A^{-1}) \circ \cdots \circ (A \circ B_n \circ A^{-1})$. However, by lemma 4.6, $(A \circ B_1 \circ A^{-1}) \circ (A \circ B_2 \circ A^{-1}) \circ \cdots \circ (A \circ B_n \circ A^{-1})$ translates $\frac{c}{a}$ less than the hyperbolic translation $B = T_{l_1}^{\sum_{i=1}^{n} b_i}$. Recall J is the common orthogonal to l_1 and l_2. Then $B(\frac{c}{a}) \leq B(c) = e^{\mu(J)}c$. Denote by d_1 the distance between l_1' and l_2'. Then by the above

$$e^{d_1} \geq \frac{\sqrt{c}e^{\frac{\mu(J)}{2}} + 1}{\sqrt{c}e^{\frac{\mu(J)}{2}} - 1} \geq \frac{\sqrt{c}e^{\frac{\mu(J)}{2}}}{\sqrt{c}e^{\frac{\mu(J)}{2}} - 1}.$$

By taking logarithms in the above inequality we get

$$d_1 \geq \log \frac{\sqrt{c}e^{\frac{\mu(J)}{2}}}{\sqrt{c}e^{\frac{\mu(J)}{2}} - 1}$$

and consequently

$$d_1 \geq \frac{1}{\sqrt{c}e^{\frac{\mu(J)}{2}}}.$$

Since we know that the distance from l_1 to l_2 is at least 1 we get $\sqrt{c} \leq \frac{e+1}{e-1}$ and $\frac{1}{\sqrt{c}} \geq \frac{e-1}{e+1} \geq \frac{1}{3}$. Hence $d_1 \geq \frac{1}{3}e^{-\frac{\mu(J)}{2}}$.

Now we take a closed geodesic arc J' of length 1 and estimate $\mu_t^*(J')$. Let l_1' and l_2' be the leftmost and the rightmost geodesics in the support of the μ_t^* which intersect J'. Let l_1 and l_2 be the corresponding geodesics for the measure μ. Let I' be the closed common orthogonal geodesic arc joining l_1 and l_2. Then $\mu_t^*(J') = \mu(I')$. In between the geodesics l_1 and l_2 we choose k geodesics $\{\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_k\}$ of the support lamination of μ such that $\bar{l}_1 = l_1$ and $\bar{l}_k = l_2$; and if I_i is the closed common orthogonal geodesic arc connecting $\bar{l}_i$ and $\bar{l}_{i+1}$, then for $i = 1, 2, \ldots k-1$,

$$\mu(I_i) \leq 2\|\mu\|$$

and the length of I_i is at least 1, except possibly I_{k-1} whose length could be less than 1.

Suppose $k > 2$. Let I_i^* denote the closed common orthogonal geodesic segment connecting $\bar{l}_i^* = E_{t\mu}(\bar{l}_i)$ and $\bar{l}_{i+1}^* = E_{t\mu}(\bar{l}_{i+1})$. Then, by the above, the length of I_i^* is greater than or equal to $\frac{1}{3}e^{-\frac{|t|\|\mu\|}{2}}$. Thus

$$dist(l_1', l_2') \geq \sum_{i=1}^{k-1} length(I_i^*) \geq \frac{1}{3}(k-2)e^{-\frac{\|\mu\|}{2}|t|}.$$

Hence $\frac{1}{3}(k-2)e^{-\frac{\|\mu\|}{2}|t|} \leq 1$, thus $k-2 \leq 3e^{\frac{\|\mu\|}{2}|t|}$, from which $k-1 \leq 4e^{\frac{\|\mu\||t|}{2}}$.

Also,

$$\mu(I') \leq \sum_{i=1}^{k-1} \mu(I_i) \leq 2(k-1)\|\mu\|$$

and it follows that

$$\mu_t^*(J') = \mu(I') \leq 8e^{\frac{\|\mu\|}{2}\cdot|t|}\|\mu\|.$$

Suppose $k = 2$. Then the length of I' is less than or equal to 1. Immediately $\mu_t^*(J') \leq \|\mu\|$. The statement follows from the above two inequalities. □

Given an earthquake we have its transverse measure. The converse is not true; given a lamination and a transverse measure on it we do not necessarily get an earthquake. The induced map on $\mathbb{H}^2$ may fail to be surjective. See Thurston [14] for a counterexample and also see Gardiner-Hu-Lakic [7]. Thurston(see [13]), however, states the following result.

Proposition 2.4: For any bounded transverse measure μ, there is an earthquake E_μ having μ as its transverse measure.

Pf. Let A be a stratum of μ. We define an earthquake E.

Let $E|A = id$. Let B be any other stratum of μ and let I be a closed geodesic arc connecting A to B. Let $\mathcal{P}_n = \{l_1, l_2, \ldots l_{k(n)}\}$ be a sequence of finite sets of leaves of μ such that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, and such that $I \cap (\cup_n \mathcal{P}_n)$ is dense in $I \cap support(\mu)$. Fix n. Then, for $i = 1, 2, \ldots k-1$, we take x_i to be the mid point of $(I \cap l_i)$ and $(I \cap l_{i+1})$; x_0 to be the left endpoint of

I, and x_k to be the right endpoint of I. The weights on l_i are defined as $a_i = \mu([x_i, x_{i+1}]) - \frac{1}{2}(\mu(x_i) + \mu(x_{i+1}))$. Then we define $E|B = \lim_{n\to\infty} T_{l_1}^{a_1} \circ T_{l_2}^{a_2} \circ \cdots \circ T_{l_k}^{a_k}$. The limit exists (see section 3). It has the property that for any two strata B and C of μ, $(E|B)^{-1} \circ (E|C)$ is a hyperbolic translation whose axis separates B and C, and which translates C to the left as viewed from B. Hence, E is defined on the intersection points of the strata of μ with S^1. These are dense in S^1. Also, E is monotone on this set. In order to prove that E maps $\mathbb{H}^2$ onto $\mathbb{H}^2$ it is enough to prove that $E|S^1$ is onto. To prove that $E|S^1$ is onto it is enough to prove that E extends to a map of S^1. In other words, we prove that E can be extended by continuity to the points of S^1 which do not lie on the boundary of some strata of μ.

Let $x \in S^1$ be such a point. For each leaf g of μ we define a half plane h_g whose boundary contains g and the point x. There exists a sequence $\{g_n\}$ of leaves of μ such that $h_{g_{n+1}} \subset h_{g_n}$, for all n, and the intersection of the closures of h_{g_n} is equal to the point x. In particular, the Euclidean size of h_{g_n} goes to zero. We prove that the Euclidean size of the image of h_{g_n} goes to zero. Let k be a geodesic ray starting from the stratum A and ending at a point x. Then, after some n_0, k intersects all of the g_n, for $n > n_0$. Out of $\{g_n\}_{n\geq n_0}$ we choose a subsequence, call it $\{g_n\}_{n\geq 1}$, such that $dist(g_n, g_{n+1}) \geq 1$. Define $g_n^* = E(g_n)$. By the first part of the proof of lemma 2.4

$$dist(g_n^*, g_{n+1}^*) \geq \frac{1}{3}e^{-\frac{\|\mu\|}{2}} = C.$$

Since

$$dist(A, g_n^*) \geq \sum_{i=1}^{n-1} dist(g_i^*, g_{i+1}^*) \geq (n-1)C \to \infty, \text{ as } n \to \infty,$$

and $h_{g_n^*} \supset h_{g_{n+1}^*}$, for all n, we conclude that the Euclidean size of $h_{g_n^*}$ goes to zero. Hence, we can extend the map E continuously to x.

As in section 3, the transverse measure of E is equal to μ. □

We recall a lemma given in Epstein-Marden [4] in the case of a hyperbolic translation and extended in Keen-Series [8] to the case of a loxodromic translation. Let $\mathbf{H}^3$ be the upper half space $\{(x, y, t) \in \mathbf{R} : t > 0\}$ with the standard hyperbolic metric. Identify the boundary of $\mathbf{H}^3$ with $\overline{\mathbf{C}} = \{z = x + iy\} \cup \{\infty\}$. Let $\mathbf{H}^2$ be the geodesic plane whose boundary meets the boundary of $\mathbf{H}^3$ on the x-axis and ∞, i.e. $\mathbf{H}^2 = \{(x, 0, s) : s > 0\}$. In this embedding the point $i = (0, 1) \in \mathbf{H}^2$ corresponds to $j = (0, 0, 1) \in \mathbf{H}^3$. Denote by $T^1\mathbf{H}^3$, the unit tangent bundle of $\mathbf{H}^3$ and write (ζ, v) for its elements, where ζ is in $\mathbf{H}^3$ and v is a unit vector. Then every element of $PSL_2(\mathbf{C})$ extends to a map from $T^1\mathbf{H}^3$ onto itself. The distance on $T^1\mathbf{H}^3$ is given by:

$$d_{T^1\mathbf{H}^3}((\zeta, v), (\xi, w)) = d_{\mathbf{H}^3}(\zeta, \xi) + \|T_\xi(v) - w\|,$$

where T_ξ is the parallel translation from ζ to ξ along the geodesic joining them. Elements of $PSL_2(\mathbf{C})$ are isometries for this metric.

Lemma 2.6:[Epstein-Marden, Keen-Series] Let $K \subset \mathbf{H}^3$ be a compact set and let $L \subset PSL_2(\mathbf{C})$ also be a compact set. There exists a constant $M > 0$ such that if $p = (\zeta, v)$ is a point in K and, T_1 and T_2 are any two elements of L then

$$d_{T^1\mathbf{H}^3}(T_1((\zeta,v)), T_2((\zeta,v))) \leq M \cdot \|T_1 - T_2\|. \ \square$$

Instead of using this lemma we derive a similar lemma that better fits our needs. Recall that a loxodromic translation T_l^{t+is} in $\mathbf{H}^3$ consists of a hyperbolic translation along its axis T_l^t in $\mathbf{H}^3$ followed by a hyperbolic rotation T_l^{is} along the same axis. The transformation is determined by the given data.

In lemma 2.7 we take a loxodromic transformation with small rotation angle, small real translation length and a point p of $T^1\mathbf{H}^3$ a bounded distance from the axis of the transformation. Then we obtain a similar but more uniform estimate where the constant $M > 0$ does not depend on the compact subset L of $PSL_2(\mathbf{C})$.

Definition 2.13 Let $l \subset \mathbf{H}^3$ be a geodesic. For $d > 0$ define

$$D(l;d) = \{(\zeta, v) \in T^1\mathbf{H}^3 | \text{distance from } \zeta \text{ to } l \text{ is less than or equal to d in the hyperbolic metric of } \mathbf{H}^3\}.$$

Lemma 2.7:(Crescent Lemma) Given $d > 0$ and $\epsilon > 0$ there exists $M > 0$ depending only on ϵ and d such that for any loxodromic translation

$T = T_l^a$ with $|a| \leq \epsilon$, if $p \in D(l; d)$, then

$$d_{T^1\mathbb{H}^3}(p, T(p)) \leq M|a|.$$

Pf. Any $T \in PSL_2(\mathbf{C})$ is conjugate by an isometry $R \in PSL_2(\mathbf{C})$ to

$$S(z) = e^a z$$

(i.e. $R \circ T \circ R^{-1} = S$ and the axis l_1 of S is the $t - axis$ in $\mathbf{H}^3$). Since

$$\begin{aligned} d_{T^1\mathbb{H}^3}(p, T(p)) &= d_{T^1\mathbb{H}^3}(R(p), R \circ T(p)) \\ &= d_{T^1\mathbb{H}^3}(R(p), R \circ T \circ R^{-1}(R(p))) = d_{T^1\mathbb{H}^3}(R(p), S(R(p))) \end{aligned}$$

and

$$R(p) \in R(D(l; d)) = D(R(l); d) = D(l_1; d),$$

it is enough to prove the lemma for S.

Denote an arbitrary point of $D(l_1; d)$ by $p = ((z, t), v)$, where $z = x + iy \in \mathbf{C}$ and $t > 0$. Then $dist((z, t), l_1) \leq d$ and $S((z, t), v) = ((e^a z, e^{\Re a} t), e^a v)$. By Beardon (see [2] section 7.35):

$$d_{\mathbb{H}^3}((z, t), (e^{\Re a} z, e^{\Re a} t)) = 2\sinh^{-1}[\cosh(dist((z, t), l_1)) \sinh(\frac{1}{2}|\Re a|)].$$

Since $2\sinh^{-1}[\cosh(dist((z, t), l_1)) \sinh(\frac{1}{2}|\Re a|)]$ is a differentiable function of $|\Re a|$, and it is equal to zero for $\Re a = 0$ and $|\Re a| \leq \epsilon$, there exists $M_1 > 0$ such that

$$d_{\mathbb{H}^3}((z, t), (e^{\Re a} z, e^{\Re a} t)) \leq M_1 |\Re a|.$$

Also to estimate the rotation from above we integrate along a Euclidean segment joining the points to get

$$\begin{aligned} d_{\mathbb{H}^3}((e^{\Re a}z, e^{\Re a}t), (e^{a}z, e^{\Re a}t)) &\leq \frac{|z|}{t}|\Im a| \\ &= \cosh[dist((z,t), l_1)]|\Im a|. \end{aligned}$$

The equality follows by the formula for the distance of a point to a geodesic (see [2], section 7.20]). Using the triangle inequality and the two inequalities above we obtain

$$d_{\mathbb{H}^3}((z,t), S(z,t)) \leq M_2|a|.$$

For the second part of $d_{T^1\mathbb{H}^3}$ we get

$$\|T_p(e^a v) - v\| = \|e^{i\Im a}v - v\| = |e^{i\Im a} - 1|\|v\| \leq M_3|\Im a|.$$

Putting last two inequalities together we obtain the desired inequality. □

Lemma 2.8: Let E be an earthquake and $K \subset \mathbb{H}^2$ a compact set. For a given $\epsilon > 0$ there exist only finitely many strata of E which intersect K and whose measure is greater than or equal to ϵ.

Pf. Note that if B is a gap then $\mu(B) = 0$.

Then only geodesics can have nonzero measure. Assume to the contrary that we have a sequence of geodesics $\{g_n\}$ with $\mu(g_n) \geq \epsilon$ and $K \cap g_n \neq \emptyset$, for all n. Then a subsequence of g_n, call it g_n again, converges to the geodesic g. Hence any closed geodesic arc which intersects g transversally also intersects

g_n transversally for all $n \geq n_0$ and some big n_0. By the countable additivity of μ we get $\mu(I) = \infty$. Contradiction. $\square$

3 Approximation by Finite Earthquakes

For the duration of this section let $D = \{z : |z| < 1\}$ with the standard hyperbolic metric be our model of the hyperbolic plane. Then the boundary is $S^1 = \{z : |z| = 1\}$. Let μ be a bounded transverse measure. Denote by E_μ the corresponding earthquake. We can normalize E_μ to be the identity on any stratum of μ. In this section we construct a sequence of finite transverse measures μ_n whose corresponding earthquakes E_{μ_n} converge to E_μ under proper normalization.

Lemma 3.1 Let E be a bounded earthquake, μ its measure and A a fixed stratum of E. Assume that $E|A = id$. There exists a sequence of finite transverse measures μ_n, which have A as a subset of one of their strata A_n, and whose corresponding earthquakes $E_{t\mu_n}$ are normalized such that $E_{t\mu_n}|A_n = id$, with the following properties:

1. for all $t > 0$, $E_{t\mu_n} \to E_{t\mu}$ uniformly on S^1, as $n \to \infty$,
2. $\mu_n \to \mu$ in the weak* sense as $n \to \infty$,
3. $\|\mu_n\| \to \|\mu\|$ as $n \to \infty$.

Pf. We divide the proof into several steps.

Step I. (Inductive definition of μ_n) In order to define μ_n, we need to give finitely many nonintersecting geodesics and weights on them. Let G_n be the hyperbolic disk of radius n around 0. Let $\mathcal{L}$ be an underlying geodesic lamination for μ. Let $\mathcal{L}_n$ be the family of all leaves of $\mathcal{L}$ which intersect G_n. From amoung the $\mathcal{L}_n$ we choose finitely many leaves whose union is $\frac{1}{n}$-dense in $\mathcal{L}_n \cap G_n$. To this finite set of geodesics we add all geodesics in $\mathcal{L}_n$ whose measure is greater than or equal to $\frac{1}{n}$. By lemma 2.8 we have only finitely many of these. If A is a geodesic we add it to this set. If A is a geodesic polygon then we add the finitely many sides of A which intersect G_n to this set. Call this finite set of geodesics S_n. In the case that A is a geodesic or a geodesic polygon with finitely many sides then A is a stratum of S_n, for all n big enough. If A is a geodesic polygon with infinitely many sides then A is a proper subset of some stratum A_n of S_n.

We specify weights on the elements of S_n. The complement of S_n consists of a set of geodesic polygons with finitely many sides whose vertices are on S^1. Some of them are just hyperbolic halfplanes. Let $\{g_1, g_2, \cdots, g_k\}$ be the geodesics in S_n which bound these halfplanes. Fix a point $\xi \in A$. Connect ξ to g_i with a closed geodesic segment s_i, for $i = 1, 2, \ldots, k$. Any geodesic g in S_n intersects at least one s_i from the set $\mathcal{S}_n = \{s_1, s_2, \ldots, s_k\}$. Let $\{h_{i_1}, h_{i_2}, \ldots, h_{i_r}\}$ be the geodesics in S_n which intersect s_i in the given order. Then $h_{i_r} \in \{g_1, g_2, \cdots, g_k\}$ so that h_{i_r} bounds a halfplane in the complement of S_n; h_{i_1} could be A, if A is a geodesic. Set $x_{i_0} = \xi$ and set $x_{i_j} = h_{i_j} \cap s_i$ for $j = 1, 2, \ldots, r$. Let $y_{i_0} = x_{i_0}$, let y_{i_j} be the midpoint of the segment $[x_{i_j}, x_{i_{j+1}}]$ for $j = 1, 2, \ldots, r-1$, and let $y_{i_r} = x_{i_r}$.

Some of the elements of S_n can intersect more than one s_i. For each such $g \in S_n$ let s_i be the closed geodesic segment with smallest index i which intersects g. For any other $k > i$, for which g intersects s_k, we change the definition of some of the y_{k_j}. Let $[y_{i_l}, y_{i_{l+1}}]$ be the segment on s_i which contains $g \cap s_i$ and let $[y_{k_l}, y_{k_{l+1}}]$ be the segment on s_k which contains $g \cap s_k$. Then we change the definition of y_{k_l} and $y_{k_{l+1}}$ to the points on s_k which are the images of y_{i_l} and $y_{i_{l+1}}$ under a homotopy respecting $\mathcal{L}$.

At this point we give the definition of the measure on each $g \in S_n$. We fix i and define the measure for each h_{i_j}. For readibility we drop the first subscript and write h_j instead of h_{i_j} etc.

If $h_1 = A$ then $\xi = x_0 = x_1 = y_0$. Define

$$\mu_n(h_1) = \mu(A),$$

$$\mu_n(h_2) = \mu((y_0, y_2]) - \frac{1}{2}\mu(y_2),$$

$$\mu_n(h_j) = \mu([y_j, y_{j+1}]) - \tfrac{1}{2}[\mu(y_j) + \mu(y_{j+1})], \text{ for } j = 3, 4, \ldots, r-1,$$

and

$$\mu_n(h_r) = \mu([y_r, y_{r+1}]) - \frac{1}{2}\mu(y_r).$$

If $h_1 \neq A$ then $y_0 = x_0$ and $x_1 \neq y_0$. Define

$$\mu_n(h_1) = \mu([y_0, y_1]) - \frac{1}{2}\mu(y_1),$$

$$\mu_n(h_j) = \mu([y_j, y_{j+1}]) - \tfrac{1}{2}[\mu(y_j) + \mu(y_{j+1})], \text{ for } j = 2, 3, \ldots, r-1,$$

and

$$\mu_n(h_r) = \mu([y_r, y_{r+1}]) - \frac{1}{2}\mu(y_r).$$

For the convergence step we will need $\mu((x_j, x_{j+1})) < \frac{1}{n}$. To achieve this we add more leaves of $\mathcal{L}$ to S_n. For s_i, if $\mu((x_j, x_{j+1})) > \frac{1}{n}$ then we add to S_n finitely many leaves of $\mathcal{L}$ which intersect (x_j, x_{j+1}) such that the measure μ of each of the complementary open intervals for the new division is less than $\frac{1}{n}$. To choose these leaves, we divide $\mu|(x_j, x_{j+1})$ into an atomic part and an absolutely continuous part. Choose finitely many leaves from the support of the atomic part of $\mu|(x_j, x_{j+1})$ such that the measure μ of the remaining atomic leaves is less than $\frac{1}{2n}$. Then we choose finitely many leaves of the support of the absolutely continuous part of $\mu|(x_j, x_{j+1})$ such that the absolutely continuous part of the measure μ of each open interval in the complement of the union of the chosen leaves in (x_j, x_{j+1}) is less than $\frac{2}{n}$. We add these leaves to S_n and call it S_n again. Use the above procedure to define x_j and y_j for the new S_n and to define the weights on the leaves of the new S_n. Now $\mu((x_j, x_{j+1})) < \frac{1}{n}$.

We form S_{n+1} by adding leaves of μ to S_n.

Step II. (convergence of μ_n) For any point $z \in D$, we prove $\mu_n(z) \to \mu(z)$ as $n \to \infty$. If $\mu(z) > 0$ then the leaf on which z lies is included in S_n, for n big enough. If $\mu(z) = 0$, then z is either on a leaf of μ with 0 measure or z is in a gap of μ. If z is in a gap of μ then z is in a gap of μ_n, for all n.

In that case $\mu_n(z) = 0 \to \mu(z) = 0$ as $n \to \infty$. If z is on a leaf of μ but not on a leaf of μ_n, for all n, then $\mu_n(z) = 0 \to \mu(z) = 0$ as $n \to \infty$. If z is on a leaf of μ and on a leaf of μ_n then $\mu_n(z)$ is equal to $\mu(I_n) - \frac{1}{2}[\mu(x_n) + \mu(y_n)]$, where $I_n = [x_n, y_n]$ is a closed geodesic arc containing z of length at most $\frac{2}{n}$. Then $\mu_n(z) \leq \mu(I_n) \to \mu(z) = 0$. Hence $\mu_n(z) \to \mu(z)$ as $n \to \infty$.

Let I be a closed geodesic segment which intersects the support $\mathcal{L}$ of μ transversally, and hence, for sufficiently large n, also intersects the support S_n of μ_n transversally. Now we prove that $\mu_n(I) \to \mu(I)$ and $\|\mu_n\| \to \|\mu\|$, as $n \to \infty$. Let $\{h_1, h_2, \ldots, h_m\}$ be the geodesics of S_n which intersect I in order. In general, however, there are other leaves of $\mathcal{L}$, not in the set $\{h_1, h_2, \ldots, h_m\}$, which intersect I.

Suppose that all geodesics in the set $\{h_1, h_2, \ldots, h_m\}$ intersect a single geodesic segment $s_1 \in \mathcal{S}_n$. Let

$$\{h_{-r}, h_{-r+1}, \ldots, h_{-1}, h_0, h_1, \ldots, h_m, h_{m+1}, \ldots, h_p\}$$

be the complete set of geodesics in S_n which intersect s_1. Let x_j be the points of the intersection of h_j with s_1. Let y_j be the midpoints of the $[x_j, x_{j+1}]$ for $j = -r, -r+1, \ldots, p-1$ and let $y_p = x_p$. Each of the segments $[y_j, y_{j+1}]$, for $j = 2, 3, \ldots, m-2$, can be homotoped to the subsegment of I by a homotopy respecting L, but $[y_0, y_1]$ and $[y_{m-1}, y_m]$ might not. In other words, the image of $\cup_{j=0}^{m-1}[y_j, y_{j+1}]$ under this homotopy map may contain the whole interval I. The other possibility is that not all points of $\mathcal{L} \cap I$ can be covered by this homotopy of $\cup_{j=0}^{m-1}[y_j, y_{j+1}]$.

Let I' be the subsegment of I which joins h_1 and h_m. Then by the definition of μ_n and by the homotopy

$$\begin{aligned}
\mu_n(I') &= \sum_{j=0}^{m-1}\left\{\mu([y_j, y_{j+1}]) - \frac{1}{2}[\mu(y_j) + \mu(y_{j+1})]\right\} \\
&= \sum_{j=0}^{m-1}\mu([y_j, y_{j+1}]) - \frac{1}{2}[\mu(y_0) + \mu(y_m)] \\
&= \left\{\sum_{j=1}^{m-2}\mu([y_j, y_{j+1}]) + \mu([x_1, y_1)) + \mu((y_{m-1}, x_m])\right\} \\
&+ \mu([y_0, x_1)) + \mu((x_m, y_m]) - \frac{1}{2}[\mu(y_0) + \mu(y_m)] \\
&\leq \mu(I') + \frac{2}{n} \leq \mu(I) + \frac{2}{n}.
\end{aligned}$$

Also $\mu(I') \leq \mu_n(I')$, because $\mu_n(I')$ is greater than or equal to the measure μ of some open geodesic arc containing I'. For fixed I, we can find n big enough such that $I \subset G_n$. If the endpoints of I have nonzero measure μ then they will be included in the set S_n for n big enough. If not, then by our choice of G_n, the set $I - I'$ can be homotoped to two half open intervals on some $s_i \in S_n$ not containing the points x_j. Thus $\mu(I) \leq \mu(I') + \frac{2}{n}$. By the above and $\mu_n(I') = \mu_n(I)$ we have $\mu(I) - \frac{2}{n} \leq \mu_n(I) \leq \mu(I) + \frac{2}{n}$. Hence, $\mu_n(I) \to \mu(I)$ as $n \to \infty$.

Suppose that not all geodesics in the set $\{h_1, h_2, \dots, h_m\}$ intersect one geodesic segment s_i. Then we take two geodesic segments s_1 and s_m which join ξ to h_1 and h_m, respectively. We divide I into two parts, one homotopic to a subsegment of s_1 and other homotopic to a subsegment of s_m, by the homotopies preserving L, and we apply the above reasoning to each of them separately. Then again we get the same inequality as above.

For any closed geodesic segment I of length 1, $\mu_n(I) \le \mu(I) + \frac{2}{n}$. Then $\|\mu_n\| \le \|\mu\| + \frac{2}{n}$. Hence $\|\mu_n\| \to \|\mu\|$ as $n \to \infty$ and μ_n converges in the weak* sense to μ.

Step III. (convergence of $E_{t\mu_n}$) We define finite earthquakes $E_{t\mu_n}$. The measure $t\mu_n$ is just the multiple by $t > 0$ of the weights of the measure μ_n on the geodesics in S_n. Let $A_n \supset A$ be a stratum of S_n. We normalize $E_{t\mu_n}$ such that $E_{t\mu_n}|A_n = id$. If A is a geodesic, we take D_n, a gap of S_n, on whose boundary A is, and define $E_{t\mu_n}|D_n = id$. For all $m > n$, there exists D_m, a gap of S_m, such that $D_m \subset D_n$ and A is on the boundary of D_m. We define $E_{t\mu_m}|D_m = id$.

Take a stratum B of $\mathcal{L}$. Then there exists a stratum B_n of μ_n which contains B. Connect $\xi \in A$ to B_n by a closed geodesic segment s. Let $\{h_1, h_2, \ldots, h_{r(n)}\}$ be all geodesics from S_n which intersect s in order, where $r(n)$ is an integer function of n. Then $E_{t\mu_n}|B_n = T_{h_1}^{t\mu_n(h_1)} \circ T_{h_2}^{t\mu_n(h_2)} \circ \cdots \circ T_{h_{r(n)}}^{t\mu_n(h_{r(n)})}$. Because $\|\mu_n\| \to \|\mu\|$ as $n \to \infty$, given a closed geodesic segment s, there exists a constant $K > 0$, such that $\mu_n(s) < K$ for all n. Since $\mu_n(s) < K$, for some constant $K > 0$ and all $n > 0$, and all geodesics $\{h_1, h_2, \ldots, h_{r(n)}\}$ intersecting the segment s, the sequence of hyperbolic isometries $E_{t\mu_n}|B_n$, for a fixed $t > 0$, is compact. Hence, there exists a subsequence that converges. Also, for all n and for all $1 \le i \le j \le n$, the set of hyperbolic isometries $T_{h_i}^{t\mu_n(h_i)} \circ T_{h_{i+1}}^{t\mu_n(h_{i+1})} \circ \cdots \circ T_{h_j}^{t\mu_n(h_j)}$ is compact. Hence there exists a constant $N > 0$ such that, for all n and for all $1 \leq i \leq j \leq n$, $\|T_{h_i}^{t\mu_n(h_i)} \circ T_{h_{i+1}}^{t\mu_n(h_{i+1})} \circ \cdots \circ T_{h_j}^{t\mu_n(h_j)}\| < N$. Let $n < m$. Then $S_n \subset S_m$. For each $T_{h_i}^{t\mu_n(h_i)}$ in the formula for $E_{t\mu_n}|B_n$, we have the corresponding composition $T_{h_{i_1}}^{t\mu_m(h_{i_1})} \circ T_{h_{i_2}}^{t\mu_m(h_{i_2})} \circ \cdots \circ T_{h_{i_q}}^{t\mu_m(h_{i_q})}$ in the formula for $E_{t\mu_m}|B_m$, where $h_{i_1}, h_{i_2}, \ldots, h_{i_q}$ are the geodesics in S_m which intersect the segment $[y_i, y_{i+1}] \subset s$. By the triangle inequality and by lemma 2.4 we have

$$\begin{aligned}
&\|T_{h_i}^{t\mu_n(h_i)} - T_{h_{i_1}}^{t\mu_m(h_{i_1})} \circ T_{h_{i_2}}^{t\mu_m(h_{i_2})} \circ \cdots \circ T_{h_{i_q}}^{t\mu_m(h_{i_q})}\| \\
&\leq \|T_{h_i}^{t\mu_n(h_i)} - T_{h_i}^{t\mu_m(h_{i_1})+t\mu_m(h_{i_2})+\cdots t\mu_m(h_{i_q})}\| \\
&+\|T_{h_i}^{t\mu_m(h_{i_1})+t\mu_m(h_{i_2})+\cdots t\mu_m(h_{i_q})} - T_{h_{i_1}}^{t\mu_m(h_{i_1})} \circ T_{h_{i_2}}^{t\mu_m(h_{i_2})} \circ \cdots \circ T_{h_{i_q}}^{t\mu_m(h_{i_q})}\| \\
&\leq C \cdot t \cdot |\mu_n(h_i) - \mu_m(h_{i_1}) - \mu_m(h_{i_2}) - \cdots - \mu_m(h_{i_q})| + \\
&+M \cdot t \cdot [\mu_m(h_{i_1}) + \mu_m(h_{i_2}) + \cdots + \mu_m(h_{i_q})] \cdot \frac{1}{n} \leq \\
&C \cdot t \cdot \frac{2}{n} + M \cdot t \cdot K \cdot \frac{1}{n} \leq C_1 \cdot \frac{1}{n}.
\end{aligned}$$

Applying the triangle inequality and the above inequality $r(n)$ times, we get that $\|E_{t\mu_n}|B_n - E_{t\mu_m}|B_m\| \leq C_1 \cdot \frac{N^2}{n}$. Hence, for fixed $t > 0$, the whole sequence $E_{t\mu_n}|B_n$ converges to a hyperbolic isometry $F|B$. Piecing the isometries $F|B$ we get a map F of $\mathbf{H}^2$ into $\mathbf{H}^2$. Let B and C be any pair of strata for F, and take $B_n \supset B$ and $C_n \supset C$ in S_n. The axis of the approximations $(E_{t\mu_n}|B_n)^{-1} \circ (E_{t\mu_n}|C_n)$ separates B_n and C_n and moves C_n to the left as viewed from B_n. Moreover the translation length of these approximations is bounded below by the measure of a closed geodesic arc connecting the strata B and C. Therefore F satisfies the separation property. Hence it is also a one to one map of $\mathbb{H}^2$ into $\mathbb{H}^2$. We can define the measure σ for F in the same way as we did for E.

Next we prove that measure σ which corresponds to F is equal to $t\mu$. Then by the uniqueness of earthquakes with given transverse measure, we conclude that $F = E_{t\mu}$ (see remark 2.3). Take a closed geodesic interval $I = [x, y]$ which intersects strata of E_μ transversally. Let $\epsilon > 0$ be a real number. Choose finitely many strata $\{A_1 = A_x, A_2, ..., A_k = A_y\}$ of E_μ which intersect I such that A_x contains x, and A_y contains y, and such that they are $\sqrt{\frac{\epsilon}{3TM}}$ dense in I. Set $T = tr.l.\{(F|A_x)^{-1} \circ (F|A_y)\}$ and let M be the constant from lemma 2.1. By lemma 2.1

$$\left|\sum_{i=1}^{k-1} tr.l.\{(F|A_i)^{-1} \circ (F|A_{i+1})\} - \sigma(I)\right| \leq \frac{\epsilon}{3}.$$

Since $F|A_i$ is the limit of $E_{t\mu_n}|A_i$, for $n = n(k)$ big enough, the translation length of $(F|A_i)^{-1} \circ (F|A_{i+1})$ is estimated by the translation length of $(E_{t\mu_n}|A_i)^{-1} \circ (E_{t\mu_n}|A_{i+1})$ up to an error of $\frac{\epsilon}{3k}$. Thus

$$\left|\sum_{i=1}^{k-1} tr.l.\{(F|A_i)^{-1} \circ (F|A_{i+1})\} - \sum_{i=1}^{k-1} tr.l.\{(E_{t\mu_n}|A_i)^{-1} \circ (E_{t\mu_n}|A_{i+1})\}\right| \leq \frac{\epsilon}{3}.$$

Note that $(E_{t\mu_n}|A_i)^{-1} \circ (E_{t\mu_n}|A_{i+1})$ is just the composition of finitely many hyperbolic translations whose axes are elements of S_n which intersect I. Denote them by $\{l_{i_1}, l_{i_2}, ..., l_{i_l}\}$. Then $(E_{t\mu_n}|A_i)^{-1} \circ (E_{t\mu_n}|A_{i+1}) = T_{l_{i_1}}^{t\mu_n(l_{i_1})} \circ T_{l_{i_2}}^{t\mu_n(l_{i_2})} \circ \cdots \circ T_{l_{i_l}}^{t\mu_n(l_{i_l})}$. In our choice of strata $\{A_1 = A_x, A_2, ..., A_k = A_y\}$

of μ we ensure that the distance between $A_i \cap I$ and $A_{i+1} \cap I$ is less than or equal $\frac{length(I)}{k-1}$, for all $k \geq 3$. Then by lemma 2.1

$$|tr.l.\{(E_{t\mu_n}|A_i)^{-1} \circ (E_{t\mu_n}|A_{i+1})\} - \sum_{j=i_1}^{i_t} t\mu_n(l_j)|$$
$$\leq \frac{length(I)^2 \cdot M \cdot T}{(k-1)^2}.$$

Using the triangle inequality $k-1$ times we obtain

$$\Big|\sum_{i=1}^{k-1} tr.l.\{(E_{t\mu_n}|A_i)^{-1} \circ (E_{t\mu_n}|A_{i+1})\}$$
$$-\sum_{i=1}^{k-1}\sum_{j=i_1}^{i_t} t\mu_n(l_j)\Big| \leq \frac{length(I)^2 \cdot M \cdot T}{k-1}.$$

Choose k so large that $\frac{length(I)^2 \cdot M \cdot T}{k-1} < \frac{\epsilon}{3}$. Then from the above three inequalities, $|\sigma(I) - t\mu(I)| < \epsilon$, for any ϵ. Hence $\sigma(I) = t\mu(I)$. Thus we get $F = E_{t\mu}$.

Since $E_{t\mu_n}|B \to F|B$ for any stratum B of μ, and $F = E_{t\mu}$ we get that $E_{t\mu_n}|B \to E_{t\mu}|B$. The same is true for the boundary points of B on S^1. This means that $E_{t\mu_n}$ converges to $E_{t\mu}$ on a dense set of S^1. Since the maps $E_{t\mu}|S^1$ are monotone, the convergence is uniform. □

4 Function $E_{\tau\mu}(x)$

Let E be a bounded earthquake with associated transverse measure μ. For $t > 0$, denote by $E_{t\mu}$ an earthquake path with measure $t\mu$. In this section we prove that for fixed x in $\mathbf{R}$ the function $E_{t\mu}(x)$ is the restriction of a holomorphic function $E_{\tau\mu}(x)$, for $\tau = t + is$ in a neighborhood of $\mathbf{R}$, and that, for τ fixed, $E_{\tau\mu}(x)$ is a one to one map of $\overline{\mathbf{R}}$ into $\overline{\mathbf{C}}$.

Notation. Let $b > 0$ be a real number and l a geodesic in $\mathbf{H}^3$. Then we define $R_l^b = T_l^{ib}$ as the rotation of an angle b about l. If l is a loxodromic element with axis l that moves a point on l a distance a we write $T_l^{a+ib} = T_l^a \circ R_l^b = R_l^b \circ T_l^a$. We call $a + ib$ the complex translation length $a + ib$.

This notation will be convenient to differentiate between the translation and rotation components of a loxodromic transformation. In the following lemma we show how to interchange the order of the composition of a hyperbolic translation and hyperbolic rotation, with disjoint axes.

Lemma 4.1: Let l_1 and l_2 be two non-intersecting geodesics in

$$\mathbf{H}^3 = \{(x, y, t) | x, y \in \mathbf{R}, t > 0\}$$

contained in a geodesic plane, and oriented in the same direction. Let $a > 0$ and $b > 0$ be two real numbers. Then $T_{l_1}^a \circ R_{l_2}^b = R_{l_2'}^b \circ T_{l_1}^a$, where $l_2' = T_{l_1}^a(l_2)$.

Pf. Let g be an isometry which maps the plane containing l_1 and l_2 into the plane $\{(x,0,t)|t>0, x\in\mathbf{R}\}\subset\mathbf{H}^3$, $g(l_1)=\{(0,0,t)|t>0\}$ and $g(l_2)$ is the geodesic with endpoints $(c,0,0)$ and $(d,0,0)$ where $0<c<d$. We conjugate $T^a_{l_1}$ and $R^b_{l_2}$ by the isometry g and call them $T^a_{l_1}$ and $R^b_{l_2}$ again. Then $T^a_{l_1}(z)=\lambda z$, $\lambda=e^a\neq 1$ and $\frac{R^b_{l_2}(z)-d}{R^b_{l_2}(z)-c}=e^{ib}\frac{z-d}{z-c}$ are actions on $\overline{\mathbf{C}}$, the boundary of $\mathbf{H}^3$. It follows that $R^b_{l_2}(z)=\frac{(d-e^{ib}c)z-(1-e^{ib})cd}{(1-e^{ib})z+de^{ib}-c}$ and

$$\begin{aligned}T^a_{l_1}\circ R^b_{l_2}(z) &= \frac{\lambda(d-e^{ib}c)z-\lambda(1-e^{ib})cd}{(1-e^{ib})z+de^{ib}-c}\\ &= \frac{(\lambda d-e^{ib}\lambda c)(\lambda z)-(1-e^{ib})\lambda c\lambda d}{(1-e^{ib})(\lambda z)+\lambda de^{ib}-\lambda c}=R^b_{l'_2}\circ T^a_{l_1}(z),\end{aligned}$$

where l'_2 has endpoints $\lambda c=T^a_{l_1}(c)$ and $\lambda d=T^a_{l_1}(d)$. □

Definition 4.1 Let μ be a finite transverse measure and A a fixed stratum of μ which contains ∞. Let $E_{t\mu}$ be a finite earthquake with the transverse measure $t\mu$ normalized such that $E_{t\mu}|A=id$. Let B be a stratum of μ and let $\{l_1,l_2,\ldots,l_k\}$ be the geodesics in the support of μ between A and B. Then $E_{t\mu}|B=T^{t\mu(l_1)}_{l_1}\circ T^{t\mu(l_2)}_{l_2}\circ\cdots\circ T^{t\mu(l_k)}_{l_k}$. Let $\tau=t+is\in\mathbf{C}$, and let $T^{\tau\mu(l_i)}_{l_i}$ be the loxodromic element $T^{t\mu(l_i)}_{l_i}\circ R^{s\mu(l_i)}_{l_i}$. We define a finite complex earthquake $E_{\tau\mu}$, the extension of $E_{t\mu}$, as $E_{\tau\mu}|B=T^{\tau\mu(l_1)}_{l_1}\circ T^{\tau\mu(l_2)}_{l_2}\circ\cdots\circ T^{\tau\mu(l_k)}_{l_k}$. Also $E_{\tau\mu}|A=id$.

If $t<0$ and $s=0$ then $E_{\tau\mu}=E_{t\mu}$ is a right earthquake i.e. the comparison maps translate to the right.

An earthquake $E_{t\mu}$ acts on $\mathbf{H}^2$. We embed $\mathbf{H}^2 = \{(x,t)|t > 0, x \in \mathbf{R}\}$ into $\mathbf{H}^3 = \{(x,y,t)|t > 0, x \in \mathbf{R}, y \in \mathbf{R}\}$ such that $(x,t) \to (x,0,t)$, and call it $\mathbf{H}^2$ again. Denote by $\mathbf{R}$ the boundary $\{(x,0,0)|x \in \mathbf{R}\}$ of $\mathbf{H}^2 \subset \mathbf{H}^3$.

We consider a complex earthquake $E_{\tau\mu}$ as a mapping of the embedded $\mathbf{H}^2$ into $\mathbf{H}^3$. $E_{\tau\mu}$ extends to the boundary $\overline{\mathbf{R}}$ of $\mathbf{H}^2$ and maps it into the boundary $\overline{\mathbf{C}}$ of $\mathbf{H}^3$; for τ real it preserves $\overline{\mathbf{R}}$.

Lemma 4.2: Let $E_{\tau\mu}$ be a finite complex earthquake. Then, for x in $\mathbf{R}$, the map $E_{\tau\mu}(x)$ is a holomorphic function in τ.

Pf. Define $T = T_l^{\tau a}$. If r_T is the repelling fixed point and a_T the attracting fixed point then $\frac{T(z)-r_T}{T(z)-a_T} = e^{\tau a}\frac{z-r_T}{z-a_T}$. From the above equation we see that the entries of a matrix representing T can be written as holomorphic functions of τ. $E_{\tau\mu}$ is a composition of finitely many elements $T_{l_i}^{\tau a_i}$. The conclusion follows. □

Definition 4.2 Let p be a point in $\mathbf{H}^3$ and let α be a real number between 0 and $\frac{\pi}{2}$. Let l be an oriented geodesic ray starting at the point p. Then the *hyperbolic cone* $C(p,l,\alpha)$ is the pencil of geodesic rays starting at p which subtend an angle not greater than α with l. The *shadow* of $C(p,l,\alpha)$ is the set of endpoints of rays in $C(p,l,\alpha)$. The *boundary* of $C(p,l,\alpha)$ is the set of geodesic rays which subtend an angle equal to α with l.

Let v be a unit vector at p. A vector v determines the geodesic ray l_v

starting at p. For simplicity we set $C(p, v, \alpha) = C(p, l_v, \alpha)$.

Definition 4.3 Let $S_{\epsilon,\mu} = \{\tau = t + is \in \mathbf{C} : |\Im\tau| < \min\{\frac{\epsilon}{C(t)M_\mu}, \epsilon\}\}$ be an open neighborhood of $\mathbf{R}$ in $\mathbf{C}$, where $M = M(\epsilon, 1)$ is the constant of Crescent Lemma, for $\epsilon > 0$ and $d = 1$, $a > 0$ is an arbitrary real number and $C(t) = 8e^{\frac{|t|\cdot\|\mu\|}{2}}$ is the function from lemma 2.5. Let γ_x be the geodesic ray that joins $j = (0,0,1)$ in $\mathbf{H}^3$ to the point x on the real axis. Let V be equal to the union, over all $x \in \mathbf{R}$, of the shadows of the hyperbolic cones $C(j, \gamma_x, \alpha)$. Thus V is a neighborhood of $\mathbf{R}$ in $\mathbf{C}$.

Definition 4.4 Let μ be a finite transverse measure and let $\tau = is$, $s \in \mathbf{R}$. Then the complex earthquake $E_{\tau\mu}$ with measure $s\mu$ is called a *pure bend* or simply a *bend*.

Lemma 4.3 (Keen-Series Cone Lemma): Let μ be a finite transverse measure and A a stratum of μ which contains $j \in \mathbf{H}^2 \subset \mathbf{H}^3$ and ∞. Let α be a real number between 0 and $\frac{\pi}{2}$, and let γ_x be the geodesic ray from j to $x \in \mathbf{R}$. Let $E_{is\mu}$ be the bend with measure $s\mu$ normalized such that $E_{is\mu}|A = id$. Then there exists a real number $\epsilon > 0$, depending only on α, such that the bent geodesic $E_{is\mu}(\gamma_x)$ is contained in the cone $C(j, \gamma_x, \alpha)$, for all real s with $|s| < \frac{\epsilon}{M\|\mu\|}$, where $M = M(\epsilon, 1)$ is the constant from Crescent Lemma.

Pf. Orient γ_x from j to x. Divide the geodesic ray γ_x into intervals of length 1. Let $x_0 = j, x_1, ..., x_i, ...$ be the division points and $v_0, v_1, ..., v_i, ...$ the unit tangent vectors in the direction of γ_x at the points $x_1, x_2, \ldots, x_i, \ldots$. Since the hyperbolic distance from x_i to x_{i+1} is 1, $\mu([x_i, x_{i+1}]) \leq \|\mu\|$.

The image of γ_x under $E_{is\mu}$ is a geodesic ray bent along the intersection points of γ_x with the support of μ; the angles at the bending points are less than or equal to s times the measure of the geodesic in the support of μ. Bent geodesics have two tangent vectors at the bending points. In the following we always take the forward tangent vector with respect to the orientation of γ_x.

There exists a real number β strictly greater than α such that the cone $C(x_1, v_1, \beta)$ is inside the cone $C(x_0, v_0, \alpha)$. To see this, take a hyperbolic plane P which contains γ_x. Form the hyperbolic triangle with vertices x_0, x_1 and third vertex $y \in \partial P$. The intersection of ∂P and the shadow of $C(x_0, v_0, \alpha)$ is an arc on C. We choose the vertex y to be one of the endpoints of this arc. The outer angle at x_1 of the triangle x_0x_1y is β. Then the inner angle is $\pi - \beta$. The sum of the angles in the triangle x_0x_1y is $\pi - \beta + \alpha$. Since the sum of the angles in a hyperbolic triangle is strictly less than π we obtain $\beta > \alpha$. A hyperbolic triangle with one finite side and two infinite sides is uniquely determined by the length of the finite side and one of the nonzero angles. Hence $\beta - \alpha > 0$ is a constant independent of the position of x_0 and x_1 (see the proof of lemma 6.3 in Keen-Series [8]).

By Crescent Lemma it follows that for a given $d = 1$ and $\epsilon > 0$ there exists $M = M(1,\epsilon) > 0$ such that

$$d_{T^1\mathbb{H}^3}((x_1, v_1), R_l^s(x_1, v_1)) < M|s|,$$

for all $|s| < \epsilon$ and any geodesic l whose distance from x_1 is less than 1.

Let $\{l_1, l_2, \dots, l_k\}$ be the support geodesics of measure μ which intersect the segment $[x_0, x_1]$ in order. Then

$$E_{is\mu}(x_1, v_1) = R_{l_1}^{s\mu(l_1)} \circ R_{l_2}^{s\mu(l_2)} \circ \cdots \circ R_{l_k}^{s\mu(l_k)}(x_1, v_1).$$

By the triangle inequality

$$\begin{aligned} d_{T^1\mathbb{H}^3}((x_1, v_1), E_{is\mu}(x_1, v_1)) &\leq d_{T^1\mathbb{H}^3}((x_1, v_1), R_{l_1}^{s\mu(l_1)}(x_1, v_1)) + \\ d_{T^1\mathbb{H}^3}(R_{l_1}^{s\mu(l_1)}(x_1, v_1), R_{l_1}^{s\mu(l_1)} \circ R_{l_2}^{s\mu(l_2)}(x_1, v_1)) &+ \cdots + \\ d_{T^1\mathbb{H}^3}(R_{l_1}^{s\mu(l_1)} \circ \cdots \circ R_{l_{k-1}}^{s\mu(l_{k-1})}(x_1, v_1), R_{l_1}^{s\mu(l_1)} &\circ \cdots \circ R_{l_k}^{s\mu(l_k)}(x_1, v_1)). \end{aligned} \tag{9}$$

Since x_1 has distance less than 1 from l_i, $d_{T^1\mathbb{H}^3}$ is invariant under the action of $PSL_2(\mathbf{C})$ and by Crescent Lemma we have

$$\begin{aligned} d_{T^1\mathbb{H}^3}(R_{l_1}^{s\mu(l_1)} \circ \cdots \circ R_{l_{i-1}}^{s\mu(l_{i-1})}(x_1, v_1), R_{l_1}^{s\mu(l_1)} \circ \cdots \circ R_{l_i}^{s\mu(l_i)}(x_1, v_1)) \\ = d_{T^1\mathbb{H}^3}((x_1, v_1), R_{l_i}^{s\mu(l_i)}(x_1, v_1)) \leq M|s|\mu(l_i), \end{aligned} \tag{10}$$

for all real s such that $|s| < \min\{\frac{\epsilon}{M\|\mu\|}, \epsilon\}$.

Using (10) in (9) we get

$$\begin{aligned} & d_{T^1\mathbb{H}^3}((x_1, v_1), E_{is\mu}(x_1, v_1)) \leq M|s| \textstyle\sum_{j=1}^{k} \mu(l_j) \\ & \leq M|s|\mu([x_0, x_1]) \leq M\|\mu\| \cdot |s| \\ & < M\|\mu\| \cdot \frac{\epsilon}{M\|\mu\|} = \epsilon. \end{aligned} \tag{11}$$

for all real s such that $|s| < \min\{\epsilon, \frac{\epsilon}{M\|\mu\|}\}$.

Define $(x_1', v_1') = E_{is\mu}((x_1, v_1))$. A hyperbolic cone $C(p, l, \alpha)$ and its shadow are continuous in its vertex p, direction l and angle α. By (11), for ϵ small, $C(x_1', v_1', \alpha)$ is close to $C(x_1, v_1, \alpha)$. Because $\beta > \alpha$, the cone $C(x_1, v_1, \alpha)$ is strictly inside the cone $C(x_1, v_1, \beta)$. Thus, for ϵ small enough, the shadow of the cone $C(x_1', v_1', \alpha)$ is inside the shadow of the cone $C(x_1, v_1, \beta)$ and x_1' is inside the cone $C(x_0, v_0, \alpha)$. Hence the cone $C(x_1', v_1', \alpha)$ is inside the cone $C(x_0, v_0, \alpha)$. Here the choice of ϵ depends only on α.

Define $(x_2', v_2') = E_{is\mu}((x_2, v_2))$. Let B be a stratum of μ which contains x_1. We prove that the cone $C(x_2', v_2', \alpha)$ is inside the cone $C(x_1', v_1', \alpha)$. The map $E_{is\mu}|B$ is just a Möbius transformation. Let $P = (E_{is\mu}|B)(\mathbb{H}^2)$ and $B_P = E_{is\mu}(B)$. Thus P is a hyperbolic plane in $\mathbb{H}^3$ and B_P is the image of B, under the bending $E_{is\mu}|B$ and is contained in P. In particular, B_P is a geodesic polygon with vertices at the boundary of P.

Let μ_P be the pushforward of the measure μ by $(E_{is\mu}|B)$. The support of the measure μ_P is on the hyperbolic plane P. Let $E_{is\mu_P}$ be the bending normalized such that $E_{is\mu_P}|B_P = id$. Then $E_{is\mu} = E_{is\mu_P} \circ (E_{is\mu}|B)$.

We repeat the above procedure for $(x_1', v_1') = (E_{is\mu}|B)(x_1, v_1)$, and $(\tilde{x}_2, \tilde{v}_2) = (E_{is\mu}|B)(x_2, v_2)$, and bending $E_{is\mu_P}$. Note that $(x_2', v_2') = E_{is\mu_P}(\tilde{x}_2, \tilde{v}_2)$. The geometry is the same and $C(x_2', v_2', \alpha)$ is inside $C(x_1', v_1', \alpha)$ for the same $\epsilon > 0$. Let $(x_j', v_j') = E_{is\mu}(x_j, v_j)$, for $j = 3, 4, \ldots$. Continuing as above we obtain that any subsequent cone with vertex x_j', direction v_j' and angle α is contained in the previous cone with vertex x_{j-1}', direction v_{j-1}' and angle α.

Thus the bent geodesic $E_{is\mu}(\gamma_x)$ is inside the first cone. □

Lemma 4.4: Let α be a real number between 0 and $\frac{\pi}{2}$. Assume μ is a finite transverse measure with stratum A containing $j \in \mathbf{H}^2$ and ∞. Then there exists a real number $\epsilon > 0$ depending only on α such that, for $\tau \in S_{\epsilon,\|\mu\|}$ and $x \in \mathbf{R}$, $E_{t\mu}(x)$ extends to the function $E_{\tau\mu}(x)$ holomorphic in τ whose image is contained in V.

Pf. Normalize $E_{\tau\mu}$ to be the identity on A. Take a point $x \in \mathbf{R}$ and connect it to j by the half geodesic γ_x and orient γ_x from j to x. Let B be a stratum of μ such that $x \in \partial B$. Let $\{l_1, l_2, \ldots, l_k\}$ be the leaves of μ between A and B in order. Take $\epsilon > 0$ from Lemma 4.3 (Cone Lemma). Let $\tau = t + is \in S_{\epsilon,\|\mu\|}$. By definition $E_{\tau\mu}|B = T_{l_1}^{\tau\mu(l_1)} \circ T_{l_2}^{\tau\mu(l_2)} \circ \cdots \circ T_{l_k}^{\tau\mu(l_k)}$. Write each term $T_{l_i}^{\tau\mu(l_i)}$ as $R_{l_i}^{s\mu(l_i)} \circ T_{l_i}^{t\mu(l_i)}$. Then $E_{\tau\mu}|B = R_{l_1}^{s\mu(l_1)} \circ T_{l_1}^{t\mu(l_1)} \circ R_{l_2}^{s\mu(l_2)} \circ T_{l_2}^{t\mu(l_2)} \circ \cdots \circ R_{l_k}^{s\mu(l_k)} \circ T_{l_k}^{t\mu(l_k)}$. By lemma 4.1, $T_{l_{i-1}}^{t\mu(l_{i-1})} \circ R_{l_i}^{s\mu(l_i)} = R_{l_i^*}^{s\mu(l_i)} \circ T_{l_{i-1}}^{t\mu(l_{i-1})}$, where $l_i^* = T_{l_{i-1}}^{t\mu(l_{i-1})}(l_i)$. Using the above and induction we obtain

$$E_{\tau\mu}|B = R_{l_1'}^{s\mu(l_1)} \circ R_{l_2'}^{s\mu(l_2)} \circ \cdots \circ R_{l_k'}^{s\mu(l_k)} \circ T_{l_1}^{t\mu(l_1)} \circ \cdots \circ T_{l_k}^{t\mu(l_k)},$$

where the l_i' are geodesics obtained as images of the l_i under the map $E_{(\Re\tau)\mu} = E_{t\mu}$.

The real earthquake $E_{t\mu}$ moves x to $x' = E_{t\mu}(x)$, support lines l_i to $l_i' = E_{t\mu}(l_i)$ and pushes forward the measure $s\mu$ to $s\mu_t^*$. Denote by $\gamma_{x'}$ the oriented geodesic ray from j to x'. Let $E_{is\mu_t^*}$ be the bending normalized such that $E_{is\mu_t^*}|A = id$. Then $E_{\tau\mu}|B = (E_{is\mu_t^*} \circ E_{t\mu})|B$ for every stratum B of μ. By Crescent Lemma, $\|\mu_t^*\| \le C(t)\|\mu\| = 8e^{\frac{|t|\|\mu\|}{2}}\|\mu\|$. By Cone Lemma the bent geodesic $E_{is\mu_t^*}(\gamma_{x'})$ is contained inside the cone $C(j, \gamma_{x'}, \alpha)$, for all real s such that $|s| < \min\{\epsilon, \frac{\epsilon}{M \cdot C(t) \cdot \|\mu\|}\}$. Hence the endpoint $E_{is\mu_t^*}(x')$ of the bent geodesic $E_{is\mu_t^*}(\gamma_{x'})$ is in the shadow of $C(j, \gamma_{x'}, \alpha)$. It follows that for $x \in \mathbf{R}$ and $\tau \in S_{\epsilon, \|\mu\|}$, $E_{\tau\mu}(x)$ is in the set V. □

Remark 4.1: Instead of normalizing E_μ to fix $j \in \mathbf{H}^2$ and ∞ it is possible to have some other point $p \in \mathbf{H}^2$ and $q \in \partial\mathbf{H}^2$ fixed. Lemma 4.4 is still true with the same ϵ, but a different neighborhood V of $\mathbf{R}$.

Remark 4.2: Let $a = \sup_{I \in \mathcal{F}_1} \mu(I)$, where $\mathcal{F}_1$ is the family of closed geodesic arcs of length 1 that lie on the geodesic rays from $j \in \mathbf{H}^2$ to the boundary points. Since $\mathcal{F}_1 \subset \mathcal{F}$, $a \le \|\mu\|$. It is clear from the proof of lemma 4.3, that $\frac{\epsilon}{M\|\mu\|}$ can be replaced by $\frac{\epsilon}{Ma}$.

Lemma 4.5: Let α be a real number between 0 and $\frac{\pi}{2}$. Let μ be a bounded transverse measure and A a stratum of μ which contains $j \in \mathbf{H}^2$ and ∞. Then there exists a real number $\epsilon > 0$ depending only on α such that, for $\tau \in S_{\epsilon, \|\mu\|}$ and $x \in \mathbf{R}$, $E_{t\mu}(x)$ extends to the function $E_{\tau\mu}(x)$ holomorphic in τ whose image is contained in V.

Pf. For a general earthquake E_μ we use the approximation by finite earthquakes E_{μ_n} from section 3. The sequence of measures μ_n converges weakly to μ, $\|\mu_n\| \to \|\mu\|$, and $E_{t\mu_n} \to E_{t\mu}$ uniformly on S^1, as $n \to \infty$. By lemma 4.4, for $\tau \in S_{\epsilon,\|\mu_n\|}$, the points $E_{\tau\mu_n}(x)$ belong to V. For simplicity, assume that $\|\mu_n\|$ decreases to $\|\mu\|$ as $n \to \infty$. Fix $x \in \mathbf{R}$ and $m > 0$. Since $\mathbf{C} - V$ contains more than two points it follows that $\{E_{\tau\mu_n}(x)\}_{n>m}$ is a normal sequence of holomorphic maps in the variable $\tau \in S_{\epsilon,\|\mu_m\|}$. By Montel's theorem there exists a convergent subsequence. For $\mu \neq 0$, however, the whole family converges for $\tau = t \in \mathbf{R}$ to a non constant function $E_{t\mu}(x)$. Hence the whole family converges to a holomorphic function $E_{\tau\mu}(x)$ and $E_{\tau\mu}(x)|(\tau \in \mathbf{R}) = E_{t\mu}(x)$. Since $E_{\tau\mu}(x)$ is a non constant holomorphic function in τ its image must be contained inside V. Also $\cup_{m=1}^{\infty} S_{\epsilon,\|\mu_m\|} = S_{\epsilon,\|\mu\|}$. Letting $m \to \infty$, the domain of definition of $E_{\tau\mu}(x)$ is $\mathbf{R} \times S_{\epsilon,\|\mu\|}$, and the image of $x \in \mathbf{R}$, for all $\tau \in S_{\epsilon,\|\mu\|}$, is contained in V. □

Lemma 4.6: Let μ be a bounded transverse measure and let $E_{\tau\mu}(x)$, for $\tau \in S_{\epsilon,\|\mu\|}$ and $x \in \mathbf{R}$, be the extension of $E_{t\mu}(x)$. Then, for fixed $\tau \in S_{\epsilon,\|\mu\|}$, $E_{\tau\mu}(x)$ is a one to one map from $\mathbf{R}$ into $\mathbf{C}$.

Pf. Take two different points x_1 and y_1 in $\mathbf{R}$.

If they belong to the boundary of a single stratum B of E_μ then x_1 and y_1 are mapped by the same loxodromic element $E_{\tau\mu}|B$. Since elements of $PSL_2(\mathbf{C})$ are one to one mappings on $\overline{\mathbf{C}}$, $E_{\tau\mu}(x_1) \neq E_{\tau\mu}(y_1)$.

Now assume that x_1 and y_1 do not lie on the boundary of one stratum. Also assume for the moment that μ is a finite earthquake. First apply the real earthquake $E_{t\mu}$ and then the pure bend $E_{is\mu_t^*}$. Define $x = E_{t\mu}(x_1)$ and $y = E_{t\mu}(y_1)$. Real earthquakes are one to one on the boundary of $\mathbf{H}^2$, hence $x \neq y$. Connect x and y with the geodesic γ. Then γ transversally intersects strata of μ_t^*. Take one leaf l_μ of μ_t^* which intersects γ. Fix a point q on $l_\mu \cap \gamma$. Denote by v_x the tangent vector to γ at the point q in the direction of x and denote by v_y the tangent vector to γ at the point q in the direction of y. The map $E_{is\mu_t^*}|l_\mu$ is just a Möbius transformation that maps $\mathbf{H}^2$ into some other embedded hyperbolic plane P, and maps γ, q, v_x and v_y onto γ', q', v_x' and v_y', respectively.

Let μ_P be the pushforward of μ_t^* by $E_{is\mu_t^*}|l_\mu$. The leaves of the support of μ_P are in P. Let $E_{is\mu_P}$ be the pure bend of P normalized such that $E_{is\mu_P}|E_{is\mu_t^*}(l_\mu) = id$. Then $E_{is\mu_t^*} = E_{is\mu_P} \circ (E_{is\mu_t^*}|l_\mu)$. In order to find $E_{is\mu_t^*}(x)$ and $E_{is\mu_t^*}(y)$ we apply the pure bend $E_{is\mu_P}$ to the geodesic $(E_{is\mu_t^*}|l_\mu)(\gamma) \subset P$. This bending fixes the point $q' \in P$. The geometry of the bending is the same as if we had started from a fixed embedding of $\mathbf{H}^2$. For that reason the estimates and the conclusions of lemma 4.4 hold. In particular, for $0 < \alpha < \frac{\pi}{2}$ and for real numbers s such that $|s| < \min\{\frac{\epsilon}{C(t)M\|\mu\|}, \epsilon\}$, $E_{is\mu_t^*}(x)$ is contained in the shadow of the cone $C(q', v_x', \alpha)$ and $E_{is\mu_t^*}(y)$ is contained in the shadow of the cone $C(q', v_y', \alpha)$. Since $\alpha < \frac{\pi}{2}$, these shadows are disjoint. Hence $E_{is\mu_t^*}(x) \neq E_{is\mu_t^*}(y)$.

Fix α and suppose μ is bounded but not a finite transverse measure. As in section 3, take an approximating sequence μ_n of finite transverse measures. For each μ_n we have shown that there is a neighborhood $S_{\epsilon,\|\mu_n\|}$ such that $E_{\tau\mu_n}|\mathbf{R}$ is an injection. Since $\|\mu_n\| \to \|\mu\|$, given $\delta > 0$ there exists n_0 such that, for $n > n_0$, $E_{\tau\mu_n}$ is injective for $\tau \in S_{\epsilon,\|\mu\|+\delta}$. Also $E_{\tau\mu_n}|l_\mu \to E_{\tau\mu}|l_\mu$ which is an element of $PSL_2(\mathbf{C})$.

To see that $E_{\tau\mu}|\mathbf{R}$ is also an injection, again, let $x_1 \neq y_1 \in \mathbf{H}^2$ and let γ be the geodesic joining x_1 and y_1. Since real earthquakes are one to one, we can assume that $\tau = is$, so that $E_{\tau\mu} = E_{is\mu}$ is a pure bend. Let $x = E_{\tau\mu}(x_1)$ and $y = E_{\tau\mu}(y_1)$. For n_0 big enough, γ intersects some leaf l_μ in the support of μ_n, for $n > n_0$. By the definition of μ_n, l_μ is also in the support of μ. Let $q = l_\mu \cap \gamma$. Let γ^+ be the geodesic ray from q to x and let γ^- be the geodesic ray from q to y. Define

$$\gamma_n^+ = (E_{\tau\mu_n}|l_\mu)(\gamma^+),$$

$$\gamma_n^- = (E_{\tau\mu_n}|l_\mu)(\gamma^-),$$

$$q_n = (E_{\tau\mu_n}|l_\mu)(q),$$

$$x_n = (E_{\tau\mu_n}|l_\mu)(x)$$

and

$$y_n = (E_{\tau\mu_n}|l_\mu)(y).$$

Let

$$\gamma'_+ = (E_{\tau\mu}|l_\mu)(\gamma^+)$$

and

$$\gamma'_- = (E_{\tau\mu}|l_\mu)(\gamma^-).$$

Then $\gamma_n^+ \to \gamma'_+$, $\gamma_n^- \to \gamma'_-$, $x_n \to x$, $y_n \to y$ and $q_n \to q$ as $n \to \infty$. The cones are continuous in the vertices, central directions and angles. Thus $C(q_n, \gamma_n^+, \alpha) \to C(q, \gamma'_+, \alpha)$ and $C(q_n, \gamma_n^-, \alpha) \to C(q, \gamma'_-, \alpha)$ as $n \to \infty$. Hence the shadows of $C(q_n, \gamma_n^+, \alpha)$ converge to the shadow of $C(q, \gamma'_+, \alpha)$ and the shadows of $C(q_n, \gamma_n^-, \alpha)$ converge to the shadow of $C(q, \gamma'_-, \alpha)$. Since x_n is in the shadow of $C(q_n, \gamma_n^+, \alpha)$ and y_n is in the shadow of $C(q_n, \gamma_n^-, \alpha)$, by the continuity of the shadows, the point x is in the shadow of $C(q, \gamma'_+, \alpha)$ and the point y is in the shadow of $C(q, \gamma'_-, \alpha)$. Because $0 < \alpha < \frac{\pi}{2}$ these shadows are disjoint so that $x \neq y$. □

We state the main result of the paper:

Theorem 4.1: Let α be a real number between 0 and $\frac{\pi}{2}$. Let μ be a bounded transverse measure. Then there exists $\epsilon > 0$ depending only on α such that the real earthquake $E_{t\mu}(x)$, for $x \in \mathbf{R}$ and $t \in \mathbf{R}^+$, can be extended to the complex earthquake $E_{\tau\mu}(x)$, for τ in an open neighborhood $S_{\epsilon,\|\mu\|}$ of $\mathbf{R}$ and $x \in \mathbf{R}$. For $x \in \mathbf{R}$, the map $E_{\tau\mu}(x)$ is holomorphic in $\tau \in S_{\epsilon,\|\mu\|}$. For $\tau \in S_{\epsilon,\|\mu\|}$, the map $E_{\tau\mu}(x)$ is one to one on $\mathbf{R}$.

Pf. Given in lemmas 4.3, 4.4, and 4.5. □

Another way to state the main theorem is:

Theorem 4.1': If μ is a bounded transverse measure then there exists $\epsilon > 0$ such that the complex earthquake $E_{\tau\mu}(x)$, for $\tau \in S_{\epsilon,\|\mu\|}$ and $x \in \mathbf{R}$, is a holomorphic motion of the real line. □

As a corollary we have:

Corollary 4.1: Let μ be a bounded transverse measure. Then the earthquake $E_\mu|\mathbf{R}$ can be extended to a quasiconformal map of the upper half plane; that is, $E_\mu|\mathbf{R}$ is a quasisymmetric map.

Pf. By theorem 4.1' and by Slodkowski's theorem (see [12]) we can extend the holomorphic motion $E_{\tau\mu}(x)$ of the real line $\mathbf{R}$ to a holomorphic motion of the complex plane $\mathbf{C}$. Since $E_{t\mu}(x)$ maps $\mathbf{R}$ to $\mathbf{R}$ the extension for $\tau = t \in \mathbf{R}$ maps the upper half plane to itself. For τ fixed the extension of $E_{\tau\mu}(x)$, for $x \in \mathbf{R}$, to $E_{\tau\mu}(z)$, for $z \in \mathbf{C}$, is quasiconformal. It follows from Ahlfors' theorem (see [1]) that $E_\mu|\mathbf{R}$ is a quasisymmetric map. □

Lemma 4.7: Let $\lambda > 0$, $a \leq c < d \leq b$ belong to $\mathbf{R} \cup \{-\infty, \infty\}$, where a could be $-\infty$ and b could be ∞, but not at the same time. Let g be the hyperbolic translation with repelling fixed point a, attracting fixed point b and multiplier λ. Let h be the hyperbolic translation with repelling fixed point c, attracting fixed point d and multiplier λ. For any $x \in (c, d)$

$$g(x) \geq h(x).$$

Pf. Case 1. Assume $a = c$. Normalize by conjugation such that $a = c = -\infty$ and $0 = b \geq d$. Then $g(z) = \lambda z$, $h(z) = \lambda(z - d) + d$ and $0 < \lambda < 1$. For $x \in (-\infty, d)$, $g(x) = \lambda x$ and $h(x) = \lambda x + (1 - \lambda)d$. Hence $g(x) \geq h(x)$, with equality only if $b = d$.

Case 2. Assume $b = d$. Normalize by conjugation such that $0 = a \leq c$ and $b = d = \infty$. Then $g(z) = \lambda z$, $h(z) = \lambda(z-c)+c$ and $\lambda > 1$. For $x \in (c, +\infty)$, $g(x) = \lambda x$ and $h(x) = \lambda x + (1 - \lambda)c$. Hence $g(x) \geq h(x)$, with equality only if $a = c$.

Case 3. Assume $a \neq c$ and $d \neq b$. Let h_1 be the hyperbolic translation with repelling fixed point a, attracting fixed point d and multiplier λ. Then, for $x \in (c, d)$, $g(x) > h_1(x)$ by Case 2 and $h_1(x) > h(x)$ by Case 1, so that $g(x) > h(x)$. □

Theorem 4.2: Let μ be a transverse measure. If the corresponding earthquake $E_\mu : \mathbf{R} \to \mathbf{R}$ is quasisymmetric then μ is bounded.

Pf. Assume μ is an unbounded transverse measure. Then there exists a sequence of closed geodesic arcs $\{I_k\}$ such that length of I_k is 1, for all k, and $\mu(I_k) \to \infty$, as $k \to \infty$. Divide each I_k into two closed geodesic arcs of length $\frac{1}{2}$. For each k, one of them has measure not less than half the measure of I_k. Denote this arc by $I_{k,1}$. Then $\mu(I_{k,1}) \geq \frac{1}{2}\mu(I_k)$ and the length of $I_{k,1}$ is $\frac{1}{2}$. Divide each $I_{k,1}$ to two geodesic arcs of length $\frac{1}{4}$. Denote by $I_{k,2}$ one of the two arcs such that $\mu(I_{k,2}) \geq \frac{1}{2}\mu(I_{k,1})$. In the n-th step we get $I_{k,n}$whose length is $\frac{1}{2^n}$ and $\mu(I_{k,n}) \geq \frac{1}{2}\mu(I_{k,n-1})$. Then $\mu(I_{k,n}) \geq \frac{1}{2^n}\mu(I_k) \to \infty$, as $k \to \infty$, for n fixed. Using a diagonal process on the sequence $\{I_{k,n}\}$ choose $J_n = I_{k(n),n}$ such that $\mu(J_n)$ converges to ∞ as n converges to ∞.

Let A_n and B_n be two strata of μ which contain endpoints of J_n. Define $E_n = (E_\mu|A_n)^{-1} \circ E_\mu$ so that $E_n|A_n = id$. Let l_n be the geodesic on the boundary of A_n(it is possible that $A_n = l_n$) closest to B_n and let h_n be the geodesic on the boundary of B_n (possibly $B_n = h_n$) closest to A_n. Let γ_n be the element of $PSL_2(\mathbf{R})$ which maps l_n to the y-axis and the endpoints of h_n to the points a_n and b_n in $\mathbf{R}^+$, such that $b_n - a_n = 2$. Denote the new geodesics by l_n and h_n, again. Denote the map $\gamma_n \circ E_n \circ \gamma_n^{-1}$ by E_n again. Let θ_n be the angle of the crescent region around l_n which touches h_n. Using elementary trigonometry $a_n = \frac{1}{\cos\theta_n} - 1$. The map E_n is an earthquake which is the identity on $\gamma_n(A_n)$ and whose measure μ_n is the push-forward of μ by γ_n. For each closed geodesic arc I, $\mu(I) = \mu_n(\gamma_n(I))$ because $\gamma_n \in PSL_2(\mathbf{R})$.

Let g be the hyperbolic translation with axis h_n and with translation length $\log\lambda_n$ equal to the translation length of $E_n|\gamma_n(B_n)$: that is

$$g(z) = \frac{2\lambda_n(z - a_n)}{(\lambda_n - 1)(z - a_n) + 2} + a_n.$$

Since the translation length of $E_n|\gamma_n(B_n) = cmp(A_n, B_n)$ is not less than $\mu(J_n)$, the multiplier λ_n satisfies $\lambda_n \geq e^{\mu(J_n)}$. By lemma 4.7, for any point $x_0 \in (a_n, b_n)$,

$$(E_n|\gamma_n(B_n))(x_0) \geq g(x_0).$$

Since x_0 might not be on the boundary of $\gamma_n(B_n)$ and E_n is a left earthquake, $E_n(x_0) \geq (E_n|\gamma_n(B_n))(x_0)$. Also $E_n(-x_0) \geq -x_0$ because $E_n|\gamma_n(A_n) = id$ and E_n is a left earthquake. The expression for cheking quasisymmetry condition at $x = 0$ and above inequalities give

$$\frac{E_n(x+x_0) - E_n(x)}{E_n(x) - E_n(x-x_0)} = \frac{E_n(x_0)}{-E_n(-x_0)} \geq \frac{g(x_0)}{x_0}.$$

Let $x_n = a_n + \frac{1}{2^n}$. Obviously $x_n \in (a_n, b_n)$. In the above inequality we use x_n instead of x_0. We obtain

$$g(x_n) = \frac{\frac{\lambda_n}{2^{n-1}}}{\frac{\lambda_n - 1}{2^n} + 2} + a_n = \frac{2\lambda_n}{\lambda_n - 1 + 2^{n+1}} + a_n$$

and

$$\frac{g(x_n)}{x_n} = \frac{\frac{2\lambda_n}{\lambda_n - 1 + 2^{n+1}} + a_n}{a_n + \frac{1}{2^n}} \geq \frac{2\lambda_n}{(\lambda_n + 2^{n+1})(a_n + \frac{1}{2^n})} = \frac{2^{n+1}\lambda_n}{(\lambda_n + 2^{n+1})(2^n a_n + 1)}.$$

The expression $\frac{2^{n+1}\lambda_n}{\lambda_n + 2^{n+1}}$ converges to ∞ because $\lambda_n \to \infty$, as $n \to \infty$. Now $a_n = \frac{1}{\cos\theta_n} - 1$ and by the formula for the distance of a point to a line (see [2, section 7.20]), if d_n is the distance from l_n to h_n then $a_n = \cosh d_n - 1 \leq M d_n^2$. By our construction $d_n \leq \frac{1}{2^n}$. Hence $2^n a_n \leq \frac{M}{2^n} < 1$, for n big enough. It follows that $\frac{2^{n+1}\lambda_n}{(\lambda_n + 2^{n+1})(2^n a_n + 1)} \to \infty$ as $n \to \infty$. Hence

$$\frac{g(x_n)}{x_n} \to \infty$$

as $n \to \infty$.

The sequence $E_n|\mathbf{R}$ has a bounded constant of quasisymmetry. To see this, let f be a quasiconformal extension of $E_\mu|\mathbf{R}$. Then the sequence $f_n =$

$\gamma_n \circ (E_n|A_n)^{-1} \circ f \circ \gamma_n^{-1}$ has bounded constant of quasiconformality, because it is a pre and post composition by conformal maps of the quasiconformal map f, and it fixes ∞. Then by the Ahlfors' result(see [1]), $f_n|\mathbf{R}$ has bounded constant of quasisymmetry. But $f_n|\mathbf{R} = E_n|\mathbf{R}$ and $\frac{g(x_n)}{x_n} \to \infty$ gives a contradiction. Therefore μ must be bounded. □

Putting together Theorem 4.1 and Theorem 4.2 we easily obtain:

Theorem 4.3: Let E be an earthquake. The following three statements are equivalent:

1. $E|\mathbf{R}$ is quasisymmetric,

2. E is bounded,

3. there exists an $\epsilon > 0$ such that, for $x \in \overline{\mathbf{R}}$ and $t > 0$, the map $(x,t) \to E_{t\mu}(x)$ extends to a holomorphic motion $(x,\tau) \to E_{\tau\mu}(x)$ of $\overline{\mathbf{R}}$ for $\tau \in S_{\epsilon/\|\mu\|}$. □

Remark 4.3 There is a stronger version of the equivalence of 1. and 2. obtained by Gardiner-Hu-Lakic (see [7]). Let us denote by $\|h\|_{cr}$ the cross ratio norm of the quasisymmetric map $h : \mathbf{R} \to \mathbf{R}$. This norm is defined as

$$\sup \left| \log \frac{cr(h(Q))}{cr(Q)} \right|,$$

where $Q = (a,b,c,d)$ is a quadruple of points given in counterclockwise order on S^1, $cr(Q) = \frac{(d-c)(b-a)}{(c-b)(a-d)}$ and the supremum is taken over all Q with

$\sigma(Q) = -1$. Let $C_0 > 0$. For all μ with $\|\mu\| < C_0$, there exists $C > 0$ such that $\frac{1}{C}\|\mu\| \leq \|E_\mu|\mathbf{R}\|_\sigma \leq C\|\mu\|$.

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