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THE EARLY DEVELOPMENT OF ARITHMETIC REASONING:

NUMERATIVE ACTIVITIES AND LOGICAL OPERATIONS

by

ALICE KLEIN

A dissertation submitted to the Graduate Faculty in Psychology
in partial fulfillment of the requirements for the degree
of Doctor of Philosophy, The City University of New York.

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Abstract

THE EARLY DEVELOPMENT OF ARITHMETIC REASONING: NUMERATIVE ACTIVITIES AND LOGICAL OPERATIONS

by

Alice Klein

Adviser: Professor Harry Beilin

This study addressed the role of numerative activities in the development of children's reasoning about arithmetic operations. The theoretical approach of this study was based on two proposals. First, children's ability to solve an arithmetic problem is influenced by both the numerative activities that they employ on a problem and their logical operations of addition and subtraction. Second, children's knowledge of addition and subtraction constitutes a system of interrelated operations.

Preschool children between 4 and 6 years of age participated in two experiments, a counting experiment and a figural correspondence experiment. Each experiment comprised tasks that assessed two fundamental arithmetic abilities: (1) the ability to make an inference about the outcome of an arithmetic operation (judgment task), and (2) the ability to "undo" the outcome of an arithmetic operation (inverse task). The main difference between these two experiments concerned the numerative activities (counting objects or establishing one-to-one correspondence between objects) by which children constructed numerical representations of the arithmetic problems.

A set of arithmetic problems was presented to children in both experiments, and these problems varied on two principal dimensions: (1) the logical form or the arithmetic operations, and (2) the quantitative relationship between the initial collections. Children received an initial interview and a final interview (judgment and inverse tasks) on each problem. In addition, a pretest was administered in order to determine children's level (I-III) of number development with respect to their logical operations and their understanding of the numerative activities in the study.

The results indicate that children in the two experiments constructed different forms of numerical representation of the arithmetic problems through their activities of counting objects or establishing one-to-one correspondence between objects. Furthermore, the particular properties (cardinal values or correspondence relations) of each form of numerical representation of the problems influenced how children solved the judgment task and the inverse task. The results also indicate that the dimensions of the arithmetic problems influenced children's ability to reason about these problems. Children solved one-way function problems earlier than two-way function problems, and equality problems earlier than inequality problems. Finally, the findings in both experiments reveal that children's arithmetic reasoning changed qualitatively over levels, both in their understanding of different numerative activities and in their ability to solve different types of arithmetic problems. These results are discussed in relation to the theoretical proposals of the study.

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Table of Contents

List of Tables

List of Figures

INTRODUCTION

An important lesson from recent research on early cognitive development is that infants and preschool children have a larger set of capacities than was initially posited. The development of mathematical thinking is a case in point. An extensive repertoire of numerical and arithmetic abilities has been revealed in young children, and precursors of these abilities may be present even in infancy. Is this repertoire of early mathematical abilities necessarily problematic for theoretical accounts of qualitative change in cognitive development? Young children could possess extensive mathematical knowledge, and yet, this knowledge could still differ qualitatively from the mathematical knowledge of older children. Accordingly, the following summary of the literature examines the early development of arithmetic reasoning with a view toward assessing children's abilities as well as their limitations.

The literature on early arithmetic reasoning is replete with evidence that preschool children have some understanding of addition and subtraction. It is difficult to draw general conclusions from this literature, however, because the types of arithmetic problems and the conditions under which they have been presented to children vary greatly among the studies. Therefore, this review is organized around three issues that are relevant to the present study.

The first issue concerns children's understanding of arithmetic operations on a single collection of objects as compared with arithmetic operations on the quantitative relationship between two collections of objects. Research has indicated that the ability to reason about addition and subtraction operations on a single collection is evident early in development. Langer (1980) revealed precursors of this ability in infants' use of pragmatic addition and subtraction of objects to construct 2- and 3-point iterative series. Starkey (1983) employed a nonverbal measure, retrieval of objects from a screened container, to assess the arithmetic knowledge of 2-year-old children. He found that they solved arithmetic problems involving small numbers of objects. Similar capacities were also demonstrated with 3- to 5-year-old children using a verbal measure (Starkey & Gelman, 1982). These children were able to state the precise number of objects in a screened array on which an addition or subtraction operation had been performed. Note that in both studies the final arrays of the problems were screened from view. Thus, it can be concluded that children's accurate solutions were not merely due to their numerical evaluation of the final collections, but rather reflected their understanding of arithmetic operations on a single collection.

In contrast to children's early understanding of arithmetic operations on a single collection, their understanding of these operations on the quantitative relationship between two collections continues to develop beyond the preschool years. In order to reveal the protracted development of this ability, it is useful to distinguish among three types of arithmetic problems in the literature. One type requires only directional reasoning about addition and subtraction operations (i.e., addition to a collection yields more, and subtraction from a collection yields less). A second type requires the integration of directional reasoning with the quantitative relationship between two collections. A third type requires reciprocal reasoning about a joint addition-subtraction transfer operation (i.e., subtraction from one collection and addition to the other collection simultaneously yields less and more).

There is considerable evidence that children as young as 3 years of age exhibit directional reasoning about addition and subtraction operations under a variety of presentation conditions. Note that this evidence is consistent with the previous conclusion that young children understand arithmetic operations on a single collection. Children's directional understanding of arithmetic operations has been revealed both when the final collections were visible (Beilin, 1968; Cooper et al., 1978) and when they were screened from view (Brush, 1978; Smedslund, 1966). It has also been demonstrated when children were required to infer the occurrence of a screened addition or subtraction operation by comparing the initial and final collections of a problem (Gelman, 1972; Cooper et al., 1978).

Preschool children's abilities on directional reasoning problems can be contrasted with their limitations on integration problems and reciprocal reasoning problems. Brush (1978) found with 4- to 7-year-old children that problems which required the integration of an addition or subtraction operation with an inequality relation were significantly more difficult than directional reasoning problems. Similar findings were reported by Blevins et al. (1981) who distinguished between "primitive" errors and "qualitative" errors on integration problems. Primitive errors were committed primarily by younger preschoolers, and reflected their directional reasoning about addition and subtraction operations. By comparison, qualitative errors were committed primarily by older preschoolers, and indicated an incipient form of non-quantitative integration on these problems. Nevertheless, both types of errors attest to one limitation of young children's arithmetic reasoning -- an inability to integrate reasoning about an arithmetic operation with the quantitative relationship between two collections.

A second limitation is evident from young children's performance on reciprocal reasoning problems. Piaget and his collaborators (Piaget et al., 1977) found that children did not solve addition-subtraction transfer problems involving a two-way (i.e., reciprocal) function until about 7 years of age. However, at this age they solved these problems by using a computational procedure. It was not until several years later that children solved these problems by making a deductive inference. Thus, it can be concluded that although preschool children possess a considerable understanding of arithmetic operations on a single collection, their understanding of these operations on the quantitative relationship between two collections is initially limited to one-way directional reasoning.

The second issue in this review is whether young children's understanding of addition and subtraction constitutes a system of arithmetic knowledge, or simply a number of unrelated abilities. One of the most consistent findings that supports the view of an early arithmetic system is children's comparable understanding of addition and subtraction operations. In fact, this finding is so common in the literature (e.g., Brush, 1978; Siegler, 1981) that its significance is frequently overlooked. Nonetheless, it suggests an arithmetic system of related operations. There is also evidence that preschool children have some understanding of the inverse operation, albeit not an understanding of a precise inverse operation (Gelman, 1977; Ginsburg & Russell, 1981). Gelman (1977) demonstrated that children as young as 3 years of age were capable of using a "solvability principle" to return a collection to its initial numerosity after a screened arithmetic operation had been performed on that collection. Children often succeeded through a series of trial-and-error additions to and subtractions from a collection.

On the one hand, these findings support the view that young children's understanding of addition and subtraction constitutes a developing arithmetic system. On the other hand, evidence from other studies indicates that such an arithmetic system is still incomplete. For example, Gréco (1962b) found that an understanding of commutativity was not part of a "pre-operatory" arithmetic. Similarly, Morf (1962) demonstrated that 5- to 7-year-old children did not understand the "connexity" of the series of whole numbers generated by iterative additions of one unit. However, it has been found that children in this age range were able to make local inferences of "one more" for a particular number in the whole number series (Morf, 1962; Walters, 1983). Thus, it can be argued that young children possess an arithmetic system, but the degree of completeness of this system remains to be determined.

The final issue to be considered concerns the role of numerative activities, such as counting objects or establishing one-to-one correspondence between objects, in the development of numerical cognition. The traditional position taken by Piaget (1965) is that neither counting nor one-to-one correspondence is necessary for the construction of number, specifically number conservation. Piaget argued to the contrary that the acquisition of number conservation, and its underlying logical groupings, provide the basis for the operational use of these numerative activities. However, Piaget's position has been challenged by several different accounts of the role of numerative activities in numerical development.

Gelman and Gallistel (1978) have proposed a model of numerical development that is based on the distinction between numerical abstraction processes (e.g., counting) and numerical reasoning processes (e.g., numerical identity). A fundamental claim of their model is that children obtain representations of specific numerosities through counting, and these numerical representations, in turn, access particular numerical reasoning principles for solving a problem. In contrast to Gelman and Gallistel's approach, Gréco (1962a) and Saxe (1979a) have offered constructivist accounts of the role of one-to-one correspondence and counting, respectively, in numerical development. These accounts share the view that numerative activities construct numerical knowledge by virtue of their relationship to general cognitive operations. In support of this view, research has indicated that both counting and one-to-one correspondence matching facilitate the acquisition of number conservation (Fuson, Secada, & Hall, 1983; Saxe, Cohen, & Rindskopf, 1980). Thus, contrary to the traditional Piagetian position, it is evident that numerative activities influence the development of numerical cognition. The role of these activities in the development of arithmetic cognition is examined in the present study.

Theoretical Approach

The intent of this section is to explicate the assumptions underlying this inquiry, and to develop a theoretical approach to the two principal problems of the study. First, each problem is discussed, and the position taken on each is compared with alternative theoretical positions. Then the definitions of some key terms in the study are provided to conclude this section.

Principal Problems of the Study

The first problem concerns the relationship between numerative activities and logical operations in the development of arithmetic reasoning. It is proposed here that knowledge obtained through each of these activities has a particular effect on children's ability to solve an arithmetic problem. Furthermore, the relationship between numerative activities and logical operations is reciprocal rather than unidirectional.

The proposal that numerative activities and logical operations differentially influence children's arithmetic reasoning is related to the contrast between figurative and operative aspects of cognitive activity in Piagetian theory (Piaget, 1970). Piaget distinguished between the operative activities of the child that transform reality and the figurative activities that attempt to reproduce or represent reality. Operative activities construct logical knowledge by transforming environmental events, whereas figurative activities represent selected features of those events. Thus, Piaget's distinction between figurative and operative activities is useful for conceptualizing the difference between numerative activities and logical operations, respectively. However, it does not capture the reciprocal relationship between these activities.

The position taken in the present study is more consistent with that of Langer (1974). Langer proposed that there is a continuous interaction between figurative and operative activities, termed accommodatory figurations and assimilatory operations, during the course of a person's development. Moreover, both figurational and operational activities construct knowledge, albeit different forms of knowledge. Accommodatory figurations construct empirical knowledge by extracting and representing information about reality, whereas assimilatory operations construct theoretical knowledge by transforming the information represented through accommodatory figurations. According to Langer's position, a person's stage of development is determined by the interaction between his or her figurational and operational levels of cognitive functioning.

This position has several implications for the theoretical approach of the present study. First, it emphasizes the constructive role of both numerative activities and logical operations in children's arithmetic reasoning. On the one hand, numerative activities such as counting objects construct a representation of the numerical information extracted from a given arithmetic problem. On the other hand, logical operations construct numerical knowledge by transforming this numerical representation during the process of reasoning about an arithmetic problem.

Second, it recognizes the reciprocal nature of the relationship between numerative activities and logical operations. That is, not only do logical operations influence children's use of numerative activities to represent an arithmetic problem, but also numerative activities differentially influence children's ability to reason about an arithmetic problem. This point can be stated more generally in terms of the representational function of numerative activities such as a counting system. To the extent that new mathematical knowledge can be deduced from the internal relations of a counting system, it can be argued that the representation constructed through such a numerative activity facilitates the formation of mathematical thought (see Beilin, 1975, for a more extensive discussion of the representational functions of mathematical languages).

Finally, this position indicates that a person's stage of development is comprised of a figurational level and an operational level of functioning. Thus, in order to determine children's level of development with respect to a particular concept, it is necessary to assess both their figurational and operational functioning on that concept.

The second principal problem of the study concerns the structure of children's arithmetic knowledge prior to formal instruction in school. This problem entails two distinct issues. One issue is whether young children's arithmetic knowledge constitutes a system of interrelated abilities, or whether it is comprised of a number of independent abilities. If, in fact, young children's arithmetic knowledge is organized as a system, then the issue of the generality of such a system is raised. This issue concerns whether an emerging arithmetic system is related to a domain-general system, or whether it constitutes a domain-specific system of knowledge.

Both the domain-general and the domain-specific positions share the assumption of system constraints (in addition to the assumption of information-processing limitations) on the development of arithmetic knowledge. The fundamental difference between these positions concerns the degree of generality of the cognitive system that is assumed to constrain the development of arithmetic knowledge. Are the constraints specific to the domain of number, or do they apply more generally to the domain of logico-mathematical thought or, at the most general level, do they reflect the abstract properties of a cognitive system encompassing the logico-mathematical, physical, and social domains?

It is proposed here that preschool children's arithmetic knowledge constitutes a system, albeit a system that may be incomplete in respect to several properties (see Gréco, 1962b, on the commutativity of addition, and Gelman & Gallistel, 1978, on the concept of infinity). Furthermore, the development of this arithmetic system is assumed to be related, in part, to an emerging system of logico-mathematical thought.

The implications of these proposals for the present study were twofold. First, the development of arithmetic knowledge was examined in respect to a fundamental property of an arithmetic system, the relation between direct and inverse operations. Second, in order to reveal some constraints imposed on the development of arithmetic knowledge by a system of logico-mathematical thought, different logical forms of arithmetic problems were included in the study.

The proposal that preschool children's arithmetic knowledge is constrained by the general form of their logico-mathmatical thought is characterized as the domain-general position. It is based on the function research conducted by Piaget and his collaborators (Piaget et al., 1977) with children from 3 to 7 years of age. They demonstrated that young children's thinking constitutes a system which can be described by a "logic of functions." Such a description accounts for the considerable evidence of children's ordered or one-way reasoning on causal and logico-mathematical problems, in particular, their one-way reasoning about addition and subtraction on arithmetic problems. Nevertheless, this system of pre-operatory or "constitutive" functions is incomplete, according to Piaget's formulation, because these functions comprise one-way structures that lack reversibility.[1]

In contrast to the domain-general position, Gelman (1980) has argued that number appears to be a "natural" and universal human ability. This view is described here as the domain-specific position. It is based on the following evidence: some numerical (e.g., counting) abilities and arithmetic abilities are present in all cultures studied to date, these abilities emerge early in development without formal schooling, and they do not change qualitatively over age. Keil (1981) has taken this argument one step further. He contends that there are abstract constraints on human number knowledge. Although these constraints are as yet unspecified, Keil proposes that such constraints restrict the class of "naturally learnable" number concepts and thereby facilitate the acquisition process. However, until these hypothetical constraints have been explicitly formalized, it is not possible to determine whether they are truly specific to the domain of number.

The proposal that children possess a system of arithmetic knowledge can be contrasted with the opposing view that children possess a number of independent arithmetic abilities. The latter view is designated as the "skills" position. Cooper, Campbell, and Blevins (1983) and Siegler (1981) have advocated different versions of this position in respect to the development of arithmetic knowledge. The following discussion first summarizes each version, and then indicates some general inadequacies of both versions of the skills position.

Cooper, Campbell, and Blevins have proposed a model of the development of numerical skills and how this development leads to changes in children's numerical representations. Their model begins in infancy with independent skills that yield representations of small, absolute numerosities, and culminates around 7 years of age in a quantitative system of skills. Fundamental to this model is the distinction between estimator and operator skills (see Gelman, 1972). Estimator skills, such as subitizing and counting, provide representations of absolute or relative numerosities. Operator skills, such as addition and subtraction, provide representations of the outcome of a transformation on absolute or relative numerosities. Accordingly, the interaction of estimator and operator skills results in the development of numerical representations.

Their model specifies two parallel developmental sequences, a sequence of skills in mastering different estimators and a sequence of rules (skills) in understanding addition and subtraction operators. Moreover, the developmental sequence of addition and subtraction rules appears to have some generality beyond the number domain. Goth (1980) found that children employed the same sequence of rules to reason about addition and subtraction on length and substance (solid quantity) tasks, but their performance was developmentally more advanced on number tasks than on length and substance tasks. In summary, Cooper, Campbell, and Blevins view preschool children's arithmetic knowledge in terms of individual rules that are used to represent the outcome of addition and subtraction. It is proposed that these rules develop into a quantitative system of operator skills through a process of interaction with the empirical information provided by estimator skills.

Siegler (1981) has proposed a model of the acquisition of conservation that includes knowledge of addition and subtraction as an integral component of the model. A fundamental assumption of his model is that there is a developmental shift from a perceptual approach to a transformational approach in the solution of conservation problems. A perceptual approach involves judgments based on one or more configurational dimensions of the problems (e.g., length or density on number conservation problems). In contrast, a transformational approach involves judgments based on the type of transformation performed in the problems, that is, between quantity-relevant transformations such as addition and subtraction and quantity-irrelevant transformations such as spatial relocation. Thus, this model assumes that the acquisition of conservation requires an understanding of addition and subtraction as well as spatial transformations. Children who possess such an understanding know that addition and spatial transformations of a quantity yield more (than before), subtraction and spatial transformations of a quantity yield less, and spatial transformations alone yield the same quantity.

It is necessary to summarize Siegler's model of the development of arithmetic knowledge before comparing his model with that of Cooper, Campbell, and Blevins. Siegler's model is based on several experiments that sought to determine the sequence of understandings in children's acquisition of three conservation concepts -- number, liquid quantity, and solid quantity. Siegler found number conservation to be the earliest of the concepts investigated, and he specified a sequence of four knowledge states in the acquisition of number conservation. These knowledge states refer to children's understanding of the effects of quantity-relevant and quantity-irrelevant transformations in number conservation problems with small and large sets. Each knowledge state is described in terms of a unique rule that accounts for children's performance on these problems at a particular point in the acquisition process. Thus, in respect to the development of arithmetic knowledge, Siegler proposed a gradual shift from the use of perceptual rules to the use of transformational rules in judging the effects of addition and subtraction on small and large number problems.

However, Siegler suggested that attainment of the fourth knowledge state (i.e., use of transformational rules on small and large sets) in number conservation was initially based on empirical solutions to these problems. Children employed counting and pairing (one-to-one correspondence between objects) strategies to solve number conservation problems before they relied on knowledge of the type of transformation without empirical verification. This ability to rely solely on the type of transformation in number conservation has a dual significance in Siegler's model. It not only marks a complete understanding of the number conservation concept, but it also constitutes the developmental prerequisite for transferring knowledge of transformations to other quantity conservation concepts.

The purpose in summarizing the models of Cooper, Campbell, and Blevins and of Siegler was to reveal the assumptions common to both of them, and to indicate some inadequacies of the skills position. First, the models assume that children's arithmetic knowledge is comprised of individual rules used to interpret the effects of addition and subtraction on numerical sets of different sizes. Each rule is unique, and it accounts for children's performance on a particular type of addition or subtraction problem. Second, the models characterize the development of arithmetic knowledge as a sequence of rules for solving different types of addition or subtraction problems. Through some unspecified process, less advanced rules are replaced by more advanced rules in the sequence. It is assumed that these advanced rules are initially employed in understanding the number concept, and then knowledge of these rules is <u>transferred</u> in understanding other quantity concepts. Finally, the models emphasize the role of experiential data in the development of arithmetic knowledge. Recall that Cooper, Campbell, and Blevins attributed advancements in operator skills to the empirical information provided by estimator skills. Similarly, Siegler proposed that the use of empirical strategies facilitates the acquisition of transformational rules. Thus, both models assume that knowledge of addition and subtraction is obtained inductively through empirical processes.

From the theoretical perspective of the present study, there are two fundamental inadequacies of the skills position as indicated by the preceding assumptions: (1) its inductive approach to the development of arithmetic knowledge, and (2) its inability to account for a fully developed system of arithmetic knowledge given only an initial understanding of individual rules. Both of these inadequacies reflect the empiricist epistemology of the skills position.

In respect to the first point, it has been cogently argued by theorists opposed to the empiricist tradition that induction from experience can not adequately explain either the development of logico-mathematical knowledge (Beth & Piaget, 1966) or the acquisition of linguistic knowledge (Chomsky, 1967, 1968). The "system" position advocated in the present study is consistent with such a view. It is proposed that children's knowledge of addition and subtraction comprises a deductive system of interrelated arithmetic operations. Furthermore, this deductive system is constructed during the course of development, rather than given as an a priori structure of the mind.

The second point concerns the issue of whether a body of knowledge as complex as arithmetic can be described in terms of a finite number of simple, individual rules. The position taken in the present study is that such a description of arithmetic knowledge is inadequate. In contrast, it is proposed here that children's arithmetic knowledge is structured as a system early in development. Although this system may be incomplete in several respects, it can account for preschool children's performance across a variety of arithmetic tasks better than a finite number of task-specific rules. Strauss and Levin (1981) cast this argument more generally in their recent critique of Siegler's rule-assessment approach to concept development: "... even if simple rules were to emerge, what strikes us as missing here is a framework or structure into which they can fit. It is this structure that gives the rules meaning, and without that meaning the rules seem somewhat vacuous" (p. 79).

Definitions of Terms

Numerative activities are those activities by which children construct numerical representations of the arithmetic problems in the study. Two different numerative activities are examined in this study -- counting objects and establishing one-to-one correspondence between objects. Consistent with the theoretical approach developed in the preceding section, numerical representations of arithmetic problems are constructed through an interaction of the operational and figurational aspects of numerative activities. The operational aspect of a numerative activity, such as counting a target collection of objects, entails the coordination of three activities: (1) establishing a unique one-to-one mapping between a symbolic vehicle and its referent, (2) performing a successive iteration of this mapping activity until the target collection has been exhausted, and (3) producing a summation of all of the iterated mapping activities (see Saxe, 1979a, for evidence on the development of each of these activities). According to this definition, numerative activities embody the fundamental relations of number -- cardinality and ordinality.

The above definition of numerative activities asserts that children construct numerical representations of the arithmetic problems. More specifically, they form representations of the numerical information obtained about collections of objects in the arithmetic problems. In view of the multiple usages of the term "representation" in the literature, it is necessary to clarify its usage in the present study.

Representation is primarily used here in the more restricted sense of a symbolic activity, that is, an activity that establishes a relationship between a symbolic vehicle and its referent. This sense of representation is distinguished from the broader sense of the term that is co-extensive with conceptual intelligence in Piagetian theory (Piaget, 1962), or refers to knowledge of the external world in other accounts of cognitive development (Mandler, 1983; Nelson, 1983). Some discussions of representation, however, do not make such a sharp distinction between these two senses. Symbolic activity is viewed as a particular form of representational cognition about nonpresent events (e.g., Forman, 1982; Sigel, 1981). Thus, in the present study the interrelatedness of these two senses of representation is recognized as well. On the one hand, numerative activities function in the symbolic sense to construct a numerical representation of the collections of objects in an arithmetic problem. On the other hand, the referent of this representation -- the child's knowledge of number -- is not in the collections of objects but in the mind.

Logical operations are mental operations that construct logico-mathematical knowledge. The term "operation" is used in a psychological sense that is consistent with Piagetian theory (Piaget, 1970). It is assumed that the logical operations of addition and subtraction transform numerical representations of the arithmetic problems in the study. Specifically, three logical forms of addition and subtraction are examined in this study -- one-way functions, two-way functions, and size axioms.[2] The distinction between numerative activities that construct numerical representations of the arithmetic problems, on the one hand, and logical operations that transform or manipulate these representations, on the other hand, appears in diverse accounts of the development of numerical thought (see Gelman & Gallistel, 1978; Klahr & Wallace, 1976; Saxe & Posner, 1983). This distinction is used in the present study to emphasize the constructive role of both numerative activities and logical operations in children's arithmetic reasoning.

Research Questions

Four research questions are addressed in this study. Each question and its principal hypotheses are stated below.

1. What is the role of numerative activities in the development of arithmetic reasoning? More specifically, how do different numerative activities by which children construct numerical representations of the arithmetic problems influence their reasoning about these problems?

It is hypothesized that different numerative activities, counting objects as contrasted with establishing one-to-one correspondence between objects, result in different forms of numerical representation of the arithmetic problems. Each form of numerical representation is characterized by a particular set of properties that influences how children solve these problems. Furthermore, the influence of each form of numerical representation is expected to change over levels as children develop an understanding of the particular numerative activity and how to use it in reasoning about arithmetic problems.

2. How do different types of arithmetic problems that are presented in the reasoning tasks influence children's ability to solve these tasks? In particular, the focus is on the influence of two dimensions of the problem types: (1) the logical form of the arithmetic problems, and (2) the quantitative relationship between the pairs of initial collections in the problems.[3]

It is hypothesized that children's arithmetic reasoning is influenced by both dimensions of the problem types -- the logical form of the arithmetic problems, and the quantitative relationship between the initial collections in the problems. Moreover, the influence of each dimension on children's arithmetic reasoning is expected to vary over levels. This expectation leads to two developmental predictions about different types of problems. The first prediction concerns the logical form of the arithmetic problems. It is hypothesized that two-way function problems and size axiom problems, whose logical forms entail a transformation, are more difficult than one-way function problems. Therefore, children should solve one-way function problems earlier than two-way function problems or size axiom problems. The second prediction addresses the quantitative relationship between the initial collections in the problems. It is hypothesized that inequality problems, which require the integration of two relations, are more difficult than equality problems. Therefore, for each logical form, children should solve equality problems earlier than inequality problems.

3. How do configurational properties of the final collections influence children's ability to reason about arithmetic problems that have been represented through different numerative activities -- counting objects as contrasted with establishing one-to-one correspondence between objects?

It is hypothesized that the influence of configurational properties on children's arithmetic reasoning is determined by two factors: (1) the numerative activity by which children construct a numerical representation of the arithmetic problems, and (2) the screening procedure under which the arithmetic problems are presented to the children. The predictions associated with each factor are discussed below.

First, configurational properties of the final collections should influence children's arithmetic reasoning in the figural correspondence experiment (Experiment 2) but <u>not</u> in the counting experiment (Experiment 1). Second, configurational properties should influence children's arithmetic reasoning in the figural correspondence experiment <u>only</u> under post-transformation screening in which the final collections are perceived by the children. These predictions are based on the following argument. The numerative activity of establishing one-to-one correspondence between objects leads children to construct a form of numerical representation of the arithmetic problems which entails a spatial medium. Moreover, reasoning about arithmetic problems that are represented in a spatial medium is more likely to be influenced by configurational properties than reasoning about arithmetic problems that are represented in a nonspatial medium. Therefore, perceiving the configurations of the final collections, which contain misleading spatial alignment cues, should influence children's ability to reason about the arithmetic problems in the figural correspondence experiment but <u>not</u> in the counting experiment.

4. How does the developmental factor, level of number development, interrelate with the three other factors in this study -- numerative activities, types of arithmetic problems, and configurational properties of the final collections?

It is hypothesized that level of number development influences children's arithmetic reasoning in relation to all three factors. The principal developmental hypotheses for each factor are as follows. First, a given numerative activity, such as counting objects, yields a form of numerical representation of the arithmetic problems that is expected to change over levels as children's understanding of the particular numerative activity develops. Developmental changes in the form of numerical representation should influence children's ability to reason about these arithmetic problems and the processes by which they do so.

Second, different types of arithmetic problems are expected to be solved at different levels as predicted from the dimensions of the problem types. The developmental predictions of two dimensions are particularly important (see Research Question 2 for a detailed statement of these predictions). One dimension is the logical form of the arithmetic problems. Children should solve one-way function problems at an earlier level than two-way function problems or size axiom problems. The other important dimension is the quantitative relationship between the initial collections in the problems. Children should solve equality problems at an earlier level than inequality problems for each logical form.

Third, configurational properties of the final collections should influence children's arithmetic reasoning differentially over levels in the figural correspondence experiment. However, this effect is expected only under post-transformational screening in which the final collections are perceived by the children. It is predicted that the influence of configurational properties on children's arithmetic reasoning should be greatest at level I, and then it should decrease over levels II and III. Furthermore, because the configurations of the final collections contain misleading spatial alignment cues, perceiving these configurations should <u>reduce</u> the accuracy of children's arithmetic reasoning at level I as compared to levels II and III.

Footnotes

[1] In this pre-operatory logic a separate coordinator, permutator C, is proposed to account for the inverse of a one-way functional scheme.

[2] Each logical form of addition and subtraction is specified in Chapter 2.

[3] A complete description of the dimensions of the problem types and the arithmetic problems under each problem type is given in Chapter 2.

2

PLAN OF THE STUDY

Two experiments investigated the early development of arithmetic reasoning, a counting experiment and a figural correspondence experiment. The focus of the experiments was on children's knowledge of addition and subtraction operations as well as their knowledge of the inverse of these operations. This chapter constitutes a bridge between the theoretical approach and the empirical findings of the study. The first section outlines some theoretical considerations that determined the general design of the two experiments. The second section provides a psychological and mathematical analysis of the arithmetic problems presented to children in both experiments.

Rationale for the Experiments

Several theoretical considerations informed the plan of the study. Each consideration and how it was translated into the design of the two experiments are discussed in this section.

First, in order to address the relationship between numerative activities and logical operations in the development of arithmetic reasoning, arithmetic problems which differed in their logical properties were presented to children in the two experiments. The principal difference between the experiments concerned the numerative activity by which children constructed a numerical representation of the arithmetic problems. Children counted the objects in the initial collections of the problems in the counting experiment, whereas they established one-to-one correspondence between objects in the initial collections of the problems in the figural correspondence experiment.

Second, consistent with the theoretical approach of this study, children were categorized into one of three levels of number development. Level was defined by children's figurational and operational functioning within the conceptual domain of number. A pretest was employed in both experiments to determine children's level of number development. This pretest assessed children's understanding of the numerative activities of counting and one-to-one correspondence between objects (i.e., measure of figurational level) and their logical operations with respect to the number concept (i.e., measure of operational level).

Third, arithmetic reasoning was examined in respect to two fundamental abilities: (1) the ability to make an inference about the outcome of an arithmetic operation, and (2) the ability to perform the inverse of an arithmetic operation. These abilities were assessed by tasks designated as the judgment task and the inverse task, respectively.

Consider first the judgment task which assessed the ability to make an arithmetic inference. An arithmetic inference is defined as a numerical judgment about the final collections which is based on the integration of a numerical judgment about the initial collections with an addition or subtraction operation on these collections. Given this definition, a critical requirement of the judgment task was to screen the collections from view during the final interview about the outcome of an arithmetic operation. The screening procedure was necessary to ensure that children's judgments were based on an <u>inference</u> about the outcome of an arithmetic operation, rather than on an empirical determination of the outcome through a numerative activity such as counting.

Now consider the inverse task which assessed the ability to "undo" the outcome of an arithmetic operation. An inverse operation is defined as an arithmetic action which precisely negates a direct arithmetic action. It was assumed in this task that children's actions of undoing the outcome of an arithmetic operation reflected their knowledge of the relation between direct and inverse operations within an arithmetic system.

In summary, the two experiments examined children's arithmetic reasoning on the judgment task and the inverse task for a specific set of arithmetic problems. The next section provides an analysis of this set of problems.

Arithmetic Problems and Problem Types

This section describes the arithmetic problems and problem types included in the counting experiment and the figural correspondence experiment. The section is organized as follows. First, the dimensions specifying the problem types are introduced. Next, the numerical properties of the arithmetic problems are discussed. Finally, the problem types and the arithmetic problems under each type are detailed.

A set of 12 arithmetic problems was presented to children in both experiments. This set constituted six problem types with two arithmetic problems exemplifying each problem type. Table 1 summarizes the three dimensions of the problem types: (1) logical form of the arithmetic problems, (2) quantitative relationship between the pairs of initial collections in the arithmetic problems, and (3) number of arithmetic operations in the problems. Each problem type was generated by a unique combination of these three dimensions, but all arithmetic problems under a given type exemplified the same combination of dimensions.

The first dimension specifies the logical properties of the arithmetic problems for all problem types. As indicated in Table 1, there were three logical forms of arithmetic problems -- one-way functions, two-way functions, and size axioms. Each logical form was examined in two problem types. The second dimension concerns the quantitative relationship between the pairs of initial collections in the arithmetic problems. Specifically, three problem types (I, III, V) were comprised of equality problems, and three problem types (II, IV, VI) were comprised of inequality problems. Note that this dimension was orthogonal to the first dimension, and thus each logical form occurred with both equality and inequality problems. The third dimension specifies the number of arithmetic operations in the problems. An arithmetic operation was performed on one or both initial collections in a problem. The third dimension was partly related to the first dimension in that the definition of each logical form entailed a certain number of arithmetic operations. However, it was possible to study these two dimensions independently for the subset of problem types (III, IV, V, VI) in which two arithmetic operations were performed.

Table 1

Dimensions of the Problem Types

Problem Type	Logical Form of Arithmetic Problems	Quantitative Relationship Between Initial Collections	Number of Arithmetic Operations
I	One-Way Function	Equality	1
II	One-Way Function	Inequality	1
III	Two-Way Function	Equality	2
IV	Two-Way Function	Inequality	2
V	Size Axiom	Equality	2
VI	Size Axiom	Inequality	2

Table 2 describes the 12 addition and subtraction problems with respect to the number of elements in the initial collections, the arithmetic operations, and the final collections. The size of the problems was determined by two factors: (1) the number of elements comprising the initial collections, and (2) the number of elements participating in the arithmetic operations on the initial collections.

The first factor was the number of elements comprising the initial collections. There were two constraints imposed on this factor. First, the initial interview on each problem in the counting experiment required the child to count and to compare accurately the initial collections. Although corrective feedback was provided during the initial interview, it was important to minimize the errors committed in counting these collections. Thus, an _upper_ limit was placed on the size of the problems such that the number of elements comprising the initial collections was within the range that could be counted accurately by children as young as four years of age.

Second, a _lower_ limit was placed on the size of the problems in order to restrict children's numerative activities. Specifically, it was necessary to exclude the potential role of subitizing in order to ensure the use of counting in the counting experiment, and the use of one-to-one correspondence between objects in the figural correspondence experiment. Research has demonstrated that preschool children are able to subitize numerical information for collections containing up to 4 elements (Cooper & Starkey, 1977; Klahr & Wallace, 1976; Silverman & Rose, 1980). Thus, in the present study the number

Table 2

Description of the Addition and Subtraction Problems

	Initial Collections		Arithmetic Operations		Final Collections	
Problem Number[a]	Child	Interviewer	Child	Interviewer	Child	Interviewer
1 (I)	8	8	+2		10	8
2 (I)	8	8		-2	8	6
4 (II)	10	7		+2	10	9
5 (II)	10	7	-2		8	7
3 (III)	8	8	-2	+2	6	10
7 (III)	8	8	-2	+2	6	10
6 (IV)	10	5	-2	+2	8	7
8 (IV)	10	5	-2	+2	8	7
9 (V)	8	8	+2	+2	10	10
10 (V)	8	8	-2	-2	6	6
11 (VI)	10	7	+2	+2	12	9
12 (VI)	10	7	-2	-2	8	5

[a] Numerals in parentheses indicate the problem types for the addition and subtraction problems.

of elements comprising the initial collections exceeded the subitizing range of preschoolers.

It followed from these two constraints that the size of the problems had to be small enough to permit accurate counting, yet large enough to preclude subitizing by preschool children. Therefore, an intermediate problem size of 7-10 elements was chosen for the initial collections. Equality problems consisted of pairs of 8-element collections, and inequality problems consisted of pairs of 7- and 10-element collections.

The second factor that determined the size of the problems was the number of elements participating in the arithmetic operations on the initial collections. This factor also involved two constraints. First, the size of the arithmetic operations was required to be large enough to encourage the use of computational procedures in solving the problems. Second, the size of the arithmetic operations was required to be uniform across all problems. Considerable evidence has indicated that preschool children are able to employ computational procedures, such as counting algorithms, when 2 elements are added or subtracted (Gelman, 1977; Groen & Resnick, 1977; Siegler & Robinson, 1981). Therefore, a size of 2 elements was chosen for the arithmetic operations on the initial collections.

The logical properties of the six problem types and the arithmetic problems under each type are detailed in the next section.

Problem Type I

The logical form of the problems under the first problem type is a dependence relation between the number of elements transferred by an arithmetic operation on one of two collections and the resulting difference between the two collections. Specifically, in the case of equality problems, the final difference between two collections which results from this arithmetic transfer is equivalent to the number of elements added to or subtracted from one collection.

According to Piaget's formulation of the logic of functions (Piaget, Grize, Szeminska, & Vinh Bang, 1977), this dependence relation can be expressed as a one-way function. If x is the number of elements transferred by an arithmetic operation on one collection and y is the resulting difference between two collections, then the general function $y = f(x)$ takes the form of a one-way function $y = x$ for the arithmetic problems under the first problem type. In summary, the dependence relation expressed by a one-way function in the first problem type entails a replacement of the number of transferred elements.

Problem 1 illustrates a one-way function for Problem Type I. The two initial collections comprising problem 1 were equal, and both collections contained 8 elements. With respect to the arithmetic operation, 2 elements were taken from the reserve collection and were added to one of the initial collections (i.e., $x = 2$). The final collections which resulted from this arithmetic transfer contained 10 and 8 elements, respectively, a difference of 2 elements (i.e., $y = 2$). Therefore, the resulting difference between the two collections in problem 1 was equivalent to the number of elements added to one collection, or $y = x$. The arithmetic problems under the first problem type are described below.

Problem 1 began with two initial collections which were equal. The child's collection contained 8 elements and the interviewer's collection contained 8 elements. Then an external arithmetic operation was performed on the child's collection. Specifically, 2 elements were taken from the reserve collection and were added to the child's collection. This arithmetic operation was designated as "external" because the source of the elements in the addition operation (i.e., the reserve collection) was outside of the two collections comprising the problem. The outcome of the arithmetic operation was consistent with the particular operation performed by the interviewer. Addition to the child's collection yielded a greater quantity (10 elements) than the interviewer's collection (8 elements).

Problem 2 began with two initial collections which were equal. The child's collection contained 8 elements and the interviewer's collection contained 8 elements. Then an external arithmetic operation was performed on the interviewer's collection. Specifically, 2 elements were subtracted from the interviewer's collection and were placed in the reserve collection. This arithmetic operation was also designated as "external" because the destination of the elements in the subtraction operation (i.e., the reserve collection) was outside of the two collections comprising the problem. The outcome of the arithmetic operation was consistent with the particular operation performed by the interviewer. Subtraction from the interviewer's collection yielded a lesser quantity (6 elements) than the child's collection (8 elements).

Problem Type II

The logical form of the problems under the second problem type is the same as the first problem type. There is a dependence relation between the number of elements transferred by an arithmetic operation on one of two collections and the resulting difference between the two collections. However, in the case of inequality problems, the final difference between two collections which results from this arithmetic transfer is equivalent to the initial numerical difference between unequal collections less the number of elements added to or subtracted from one collection.

Analogous to the first problem type, this dependence relation can be expressed as a one-way function. If d is the initial numerical difference between unequal collections, x is the number of elements transferred by an arithmetic operation on one collection and y is the resulting difference between two collections, then the general function $y = f(x)$ takes the form of a one-way function $y = d - x$ for the arithmetic problems under the second problem type. Thus, the logical form of the problems under the first and second problem types is the same. The dependence relation in both problem types entails a replacement of the number of transferred elements. The principal distinction between these problem types is the quantitative relationship between the pairs of initial collections in the arithmetic problems. For the problems under the second problem type, the dependence relation is integrated with the initial numerical difference between unequal collections.[1]

Problem 4 is an example of a one-way function for Problem Type II. The two initial collections comprising problem 4 were unequal such that one collection contained 10 elements and the other contained 7 elements, a difference of 3 elements (i.e., $d = 3$). With respect to the arithmetic operation, 2 elements were taken from the reserve collection and were added to the collection with 7 elements (i.e., $x = 2$). The final collections which resulted from this arithmetic transfer contained 10 and 9 elements, respectively, a difference of 1 element (i.e., $y = 1$). Therefore, the resulting difference between the two collections in problem 4 was equivalent to the initial numerical difference between unequal collections less the number of elements added to one collection, or $y = d - x$. The arithmetic problems under the second problem type are described below.

Problem 4 began with two initial collections which were unequal. The child's collection contained 10 elements and the interviewer's collection contained 7 elements. Then an external arithmetic operation was performed on the interviewer's collection. Specifically, 2 elements were taken from the reserve collection and were added to the interviewer's collection. The outcome of the arithmetic operation was not consistent with the particular operation performed by the interviewer. Addition to the interviewer's collection yielded a lesser quantity (9 elements) than the child's collection (10 elements).

Problem 5 began with two initial collections which were unequal. The child's collection contained 10 elements and the interviewer's collection contained 7 elements. Then an external arithmetic operation was performed on the child's collection. Specifically, 2 elements were subtracted from the child's collection and were placed in the reserve collection. The outcome of the arithmetic operation was *not* consistent with the particular operation performed by the interviewer. Subtraction from the child's collection yielded a greater quantity (8 elements) than the interviewer's collection (7 elements).

Problem Type III

The logical form of the problems under the third problem type is a dependence relation, but the form of this dependence relation is different from the first and second problem types. For the third problem type, the dependence relation holds between the number of elements transferred by arithmetic operations on *two* collections and the resulting difference between the two collections. Specifically, in the case of equality problems, the final difference between two collections which results from these arithmetic transfers is equivalent to the number of elements subtracted from one collection plus the number of elements added to the other collection, or twice the number of elements transferred by these arithmetic operations.

Piaget et al. (1977) hold that this dependence relation can be expressed as a two-way function. If x is the number of elements transferred by arithmetic operations on two collections and y is the resulting difference between two collections, then the general function $y = f(x)$ takes the form of a two-way function $y = 2x$ for the arithmetic problems under the third problem type. In summary, the dependence relation expressed by a two-way function in the third problem type entails a transformation of the number of transferred elements by a twofold proportion.

Problem 7 illustrates a two-way function for Problem Type III. The two initial collections comprising problem 7 were equal, and both collections contained 8 elements. With respect to the arithmetic operations, 2 elements were subtracted from one collection and the same two elements were added to the other collection (i.e., $x = 2$). The final collections which resulted from these arithmetic transfers contained 6 and 10 elements, respectively, a difference of 4 elements (i.e., $y = 4$). Therefore, the resulting difference between the two collections in problem 7 was equivalent to twice the number of elements transferred by these arithmetic operations on two collections, or $y = 2x$.

The arithmetic problems under the third problem type were identical in logical form, number of elements in the initial and final collections, and sequence of arithmetic operations (i.e., subtraction followed by addition). However, these problems differed in respect to the source/destination of the elements transferred by the arithmetic operations. The arithmetic operations in problem 7 were designated as "internal" because the source of the elements in the addition operation (i.e., the child's collection) and the destination of the elements in the subtraction operation (i.e., the interviewer's collection) were within the two collections comprising the problem. Thus, problem 7 constituted the standard form of a two-way function problem with equal initial collections (see Piaget, 1980; Piaget et al., 1977).

In contrast, the arithmetic operations in problem 3 were designated as "external" because the source of the elements in the addition operation (i.e., the reserve collection) and the destination of the elements in the subtraction operation (i.e., the reserve collection) were outside of the two collections comprising the problem. Problem 3 served as the control form of a two-way function problem under the third problem type. Since both problems had the same logical form but differed as to the internal or external context of the arithmetic operations, it was possible to examine the context of these operations apart from their logical form. The arithmetic problems under the third problem type are described below.

Problem 3 began with two initial collections which were equal. The child's collection contained 8 elements and the interviewer's collection contained 8 elements. Then external arithmetic operations were performed on both collections. Specifically, 2 elements were subtracted from the child's collection and were placed in the reserve collection, and 2 other elements were taken from the reserve collection and were added to the interviewer's collection. The outcome of each arithmetic operation was consistent with the particular operation performed by the interviewer. Subtraction from the child's collection yielded a lesser quantity (6 elements) than the interviewer's collection, and addition to the interviewer's collection yielded a greater quantity (10 elements) than the child's collection.

Problem 7 began with two initial collections which were equal. The child's collection contained 8 elements and the interviewer's collection contained 8 elements. Then internal arithmetic operations were performed on both collections. Specifically, 2 elements were subtracted from the child's collection and the same 2 elements were added to the inteviewer's collection. The outcome of each arithmetic operation was consistent with the particular operation performed by the interviewer. Subtraction from the child's collection yielded a lesser quantity (6 elements) than the interviewer's collection, and addition to the interviewer's collection yielded a greater quantity (10 elements) than the child's collection.

Problem Type IV

The logical form of the problems under the fourth type is the same as the third problem type. There is a dependence relation between the number of elements transferred by arithmetic operations on two collections and the resulting difference between the two collections. However, in the case of inequality problems, the final difference between two collections which results from these arithmetic transfers is equivalent to the initial numerical difference between unequal collections less twice the number of elements transferred by these arithmetic operations.

Analogous to the third problem type, this dependence relation can be expressed as a two-way function. If d is the initial numerical difference between unequal collections, x is the number of elements transferred by arithmetic operations on two collections, and y is the resulting difference between two collections, then the general function $y = f(x)$ takes the form of a two-way function $y = d - 2x$ for the arithmetic problems under the fourth problem type. Thus, the logical form of the problems under the third and fourth problem types is the same. The dependence relation in both problem types entails a transformation of the number of transferred elements by a twofold proportion. The principal distinction between these problem types is the quantitative relationship between the pairs of initial collections in the arithmetic problems. For the problems under the fourth problem type, the dependence relation is integrated with the initial numerical difference between unequal collections.[2]

Problem 8 illustrates a two-way function for Problem Type IV. The two initial collections comprising problem 8 were unequal such that one collection contained 10 elements and the other collection contained 5 elements, a difference of 5 elements (i.e., $d = 5$). With respect to the arithmetic operations, 2 elements were subtracted from one collection and the same 2 elements were added to the other collection (i.e., $x = 2$). The final collections which resulted from these arithmetic transfers contained 8 and 7 elements, respectively, a difference of 1 element (i.e., $y = 1$). Therefore, the resulting difference between the two collections in problem 8 was equivalent to the initial numerical difference between unequal collections less twice the number of elements transferred by these arithmetic operations on two collections, or $y = d - 2x$.

The arithmetic problems under the fourth problem type were identical in all respects except the source/destination of the elements transferred by the arithmetic operations. The arithmetic operations in problem 8 were internal, whereas the arithmetic operations in problem 6 were external (see Problem Type III for a complete description of internal and external arithmetic operations). Thus, problem 8 constituted the standard form of a two-way function problem with unequal initial collections, and problem 6 served as the control form of this problem. The purpose of the control problem for the fourth problem type was the same as for the third problem type. Since both problems had the same logical form, the internal or external context of the arithmetic operations could be examined apart from their logical form. The arithmetic problems under the fourth problem type are described below.

Problem 6 began with two initial collections which were unequal. The child's collection contained 10 elements and the interviewer's collection contained 5 elements. Then external arithmetic operations were performed on both collections. Specifically, 2 elements were subtracted from the child's collection and were placed in the reserve collection, and 2 *other* elements were taken from the reserve collection and were added to the interviewer's collection. The outcome of each arithmetic operation was *not* consistent with the particular operation performed by the interviewer. Subtraction from the child's collection yielded a greater quantity (8 elements) than the interviewer's collection, and addition to the interviewer's collection yielded a lesser quantity (7 elements) than the child's collection.

Problem 8 began with two initial collections which were unequal. The child's collection contained 10 elements and the interviewer's collection contained 5 elements. Then internal arithmetic operations were performed on both collections. Specifically, 2 elements were subtracted from the child's collection and the *same* 2 elements were added to the interviewer's collection. The outcome of each arithmetic operation was *not* consistent with the particular operation performed by the interviewer. Subtraction from the child's collection yielded a greater quantity (8 elements) than the interviewer's collection, and addition to the interviewer's collection yielded a lesser quantity (7 elements) than the child's collection.

In respect to the preceding description of Problem Types I-IV, it is useful to characterize the distinctions between two-way function problems (Problem Types III and IV) and one-way function problems

(Problem Types I and II). There are three principal distinctions: (1) the transformational or nontransformational property of the dependence relation expressed by the different functions in these problems, (2) the number of arithmetic operations in the problems, and (3) the source/destination of the elements transferred by the arithmetic operations.

The first distinction concerns the transformational or nontransformational property of the dependence relation expressed by the different functions in these problems. Recall that in two-way function problems the dependence relation entails a transformation of the number of transferred elements by a twofold proportion. However, in one-way function problems there is simply a replacement, without a transformation, of the number of transferred elements. Thus, the transformational property of the dependence relation is present only in two-way function problems.

The second distinction follows directly from a procedural description of these problems. Two-way function problems involve double arithmetic operations on the initial collections (i.e., subtraction from one collection and addition to the other collection). In contrast, one-way function problems involve a single arithmetic operation on the initial collections.

The final distinction bears on the source/destination of the elements transferred by the arithmetic operations. In two-way function problems, the arithmetic operations were performed in both internal and external contexts. The arithmetic operations in problems 7 and 8 were internal, and the arithmetic operations in problems 3 and 6 were external. By comparison, one-way function problems (problems

1, 2, 4, 5) were performed only in an external context. In summary, although the logical form of all of these problems is a dependence relation, two-way function problems can be distinguished from one-way function problems on the basis of the transformational property of the dependence relation, the double arithmetic operations, and the internal arithmetic operations performed on specific two-way function problems.

Problem Type V

The logical form of the problems under the fifth problem type is a quantitative relation between two collections that remains invariant through the arithmetic transfer of an equal number of elements to or from the two collections. Specifically, in the case of equality problems, the equivalence relation between two initial collections is preserved through the addition of an equal number of elements to the two collections or the subtraction of an equal number of elements from the two collections.

According to Inhelder and Piaget's (1963) analysis of numerical inferences, this invariance relation can be expressed as the size axiom for equal quantities.[3] If a and b refer to two collections such that $a = b$, and a' and b' refer to the number of elements transferred by arithmetic operations on two collections such that $a' = b'$, then the equal size axiom takes the following forms for the problems under the fifth problem type: (1) $a + a' = b + b'$, and (2) $a - a' = b - b'$. In summary, the invariance relation expressed by the equal size axiom in the fifth problem type entails a transformation of the number of elements in the two collections. Nevertheless, the equivalence relation between two initial collections is preserved because the transformation combines conservation with arithmetic operations.

Problem 9 illustrates the equal size axiom for Problem Type V. The two initial collections comprising problem 9 were equal, and both collections contained 8 elements (i.e., $\underline{a} = \underline{b}$). With respect to the arithmetic operations, 2 elements were taken from the reserve collection and were added to one collection; then 2 more elements were taken from the reserve collection and were added to the other collection (i.e., $\underline{a}' = \underline{b}'$). The final collections which resulted from these arithmetic transfers were equal, and both collections contained 10 elements. Therefore, the equivalence relation between two initial collections in problem 9 was preserved through the addition of an equal number of elements to two initial collections, or $\underline{a} + \underline{a}' = \underline{b} + \underline{b}'$.

The arithmetic problems under the fifth problem type are described below. Note that the arithmetic operations in both problems 9 and 10 were designated as "external" because the source of the elements in the addition operations (i.e., the reserve collection) and the destination of the elements in the subtraction operations (i.e., the reserve collection) were outside of the two collections comprising the problem.

Problem 9 began with two initial collections which were equal. The child's collection contained 8 elements and the interviewer's collection contained 8 elements. Then external arithmetic operations were performed on both collections. Specifically, 2 elements were taken from the reserve collection and were added to the child's collection, and 2 more elements were taken from the reserve collection and were added to the interviewer's collection. The outcome of each arithmetic operation was not consistent with the particular operation performed by the interviewer. Nonetheless, these addition operations together preserved the equivalence relation between two initial collections. Addition to the child's collection yielded the same quantity (10 elements) as addition to the interviewer's collection (10 elements).

Problem 10 began with two initial collections which were equal. The child's collection contained 8 elements and the interviewer's collection contained 8 elements. Then external arithmetic operations were performed on both collections. Specifically, 2 elements were subtracted from the child's collection and were placed in the reserve collection, and 2 elements were subtracted from the interviewer's collection and were placed in the reserve collection. The outcome of each arithmetic operation was not consistent with the particular operation performed by the interviewer. Nonetheless, these subtraction operations together preserved the equivalence relation between two initial collections. Subtraction from the child's collection yielded the same quantity (6 elements) as subtraction from the interviewer's collection (6 elements).

Problem Type VI

The logical form of the problems under the sixth problem type is the same as the fifth problem type. There is a quantitative relation between two collections that remains invariant through the arithmetic transfer of an equal number of elements to or from the two collections. However, in the case of inequality problems, the nonequivalence relation between two initial collections is preserved through the addition of an equal number of elements to the two collections or the subtraction of an equal number of elements from the two collections.

Analogous to the fifth problem type, this invariance relation can be expressed as the size axiom for unequal quantities.[4] If a and b refer to two collections such that $a > b$, and a' and b' refer to the number of elements transferred by arithmetic operations on two collections such that $a' = b'$, then the unequal size axiom takes the following forms for the problems under the sixth problem type: (1) $a + a' > b + b'$, and (2) $a - a' > b - b'$. Thus, the logical form of the problems under the fifth and sixth problem types is the same. The invariance relation in both problem types entails a transformation of the number of elements in two collections with a preservation of the quantitative relation between them. The principal distinction between these problem types is the quantitative relationship between the pairs of initial collections in the arithmetic problems. For problems under the sixth problem type, the addition of an equal number of elements to two unequal collections or the subtraction of an equal number of elements from two unequal collections yields unequal collections with a constant numerical difference between them.

Problem 11 illustrates an unequal size axiom for Problem Type VI. The two initial collections comprising problem 11 were unequal such that one collection contained 10 elements and the other collection contained 7 elements (i.e., $a > b$). With respect to the arithmetic operations, 2 elements were taken from the reserve collection and were added to one collection; then 2 more elements were taken from the reserve collection and were added to the other collection (i.e., $\underline{a}' = \underline{b}'$). The final collections which resulted from these arithmetic transfers were unequal, and these collections contained 12 and 9 elements, respectively. Therefore, the nonequivalence relation (as well as the numerical difference) between two initial collections in problem 11 was preserved through the addition of an equal number of elements to two initial collections, or $\underline{a} + \underline{a}' > \underline{b} + \underline{b}'$. The arithmetic problems under the sixth problem type are described below.

Problem 11 began with two initial collections which were unequal. The child's collection contained 10 elements and the interviewer's collection contained 7 elements. Then external arithmetic operations were performed on both collections. Specifically, 2 elements were taken from the reserve collection and were added to the child's collection, and 2 more elements were taken from the reserve collection and were added to the interviewer's collection. The outcome of the arithmetic operation on the child's collection was consistent, but the outcome of the arithmetic operation on the interviewer's collection was not consistent with the particular operation performed by the interviewer. Nonetheless, these addition operations together preserved the nonequivalence relation between two initial collections. Addition to the child's collection yielded a greater quantity (12 elements) than the interviewer's collection, and addition to the interviewer's collection yielded a lesser quantity (9 elements) than the child's collection.

Problem 12 began with two initial collections which were unequal. The child's collection contained 10 elements and the interviewer's collection contained 7 elements. Then external arithmetic operations were performed on both collections. Specifically, 2 elements were subtracted from the child's collection and were placed in the reserve collection, and 2 elements were subtracted from the interviewer's collection and were placed in the reserve collection. The outcome of the arithmetic operation on the interviewer's collection was consistent, but the outcome of the arithmetic operation on the child's collection was <u>not</u> consistent with the particular operation performed by the interviewer. Nonetheless, these subtraction operations together preserved the nonequivalence relation between two initial collections. Subtraction from the interviewer's collection yielded a lesser quantity (5 elements) than the child's collection, and subtraction from the child's collection yielded a greater quantity (8 elements) than the interviewer's collection.

To conclude this description of the arithmetic problems and problem types, it is useful to compare the size axiom problems (Problem Types V and VI) with the function problems (Problem Types I-IV). Two points of comparison are most relevant: (1) the transformational or nontransformational property of the different logical forms of these problems, and (2) the source/destination of the elements transferred by the arithmetic operations.

The first point of comparison concerns the different logical forms of these problems. The invariance relation in size axiom problems entails a transformation of the number of elements in two collections. The dependence relation in two-way function problems, but not in one-way function problems, also entails a transformation. However, in the case of two-way function problems the transformation is performed on the number of transferred elements. Nevertheless, the logical forms of size axiom problems and two-way function problems are similar with respect to this transformational property.

The second point of comparison concerns the source/destination of the elements transferred by the arithmetic operations. On this point, size axiom problems are more similar to one-way function problems because the arithmetic operations in both types of problems are performed exclusively in an external context. Note, however, that a subset of two-way function problems (problems 3 and 6) also involve external arithmetic operations. Thus, these problems share a similarity with one-way function problems and size axiom problems in respect to this point.

The preceding description of the arithmetic problems and problem types indicates that these problem types differed on multiple dimensions. Three dimensions were of particular interest in this study: (1) the transformational or nontransformational property of the logical form of the arithmetic problems, (2) the quantitative relationship between the pairs of initial collections in the problems, and (3) the number of arithmetic operations in the problems. The psychological implications of these dimensions for young children's arithmetic reasoning were examined in two experiments. The principal developmental hypotheses about the problem dimensions were twofold.

First, it was predicted that problems in which the logical form entailed a transformation would be more difficult than those in which the logical form did not entail a transformation. According to this prediction, children would solve one-way function problems earlier than two-way function problems or size axiom problems.

Second, it was hypothesized for each logical form of the problems that children would solve equality problems earlier than inequality problems. This hypothesis followed from an analysis of the two logical forms of inequality function problems. Recall that in these problems the dependence relation (expressed as a one-way function or a two-way function) was integrated with the initial numerical difference between unequal collections. Moreover, the form of this integration (see Footnotes 1 and 2) was <u>not</u> consistent with the particular arithmetic operation performed by the interviewer (e.g., addition to one collection yielded a lesser quantity than the other collection). Therefore, the cognitive demands of encoding the initial numerical difference between unequal collections and integrating it with a particular form of the dependence relation led to the expectation that children would solve equality one-way function and equality two-way function problems earlier than inequality one-way function and inequality two-way function problems, respectively.

The development of arithmetic reasoning for the preceding set of problems was examined in the counting experiment and the figural correspondence experiment. Both experiments were designed to address the four research questions posed in the introduction to the study. Specifically, the experiments dealt with the influence of several factors on children's arithmetic reasoning -- numerative activities, types of arithmetic problems, configurational properties of the collections in the problems, and level of number development. The empirical findings of these experiments are discussed in the next two chapters.

Footnotes

[1] This integration of the dependence relation with the initial numerical difference between unequal collections can take two possible forms for Problem Type II. On the one hand, if an arithmetic operation is performed on one collection such that the outcome is consistent with the particular operation, the integration takes the form of $\underline{y} = \underline{d} + \underline{x}$ (i.e., the initial numerical difference plus the number of elements transferred by an arithmetic operation). On the other hand, if an arithmetic operation is performed on one collection such that the outcome is <u>not</u> consistent with the particular operation, the integration takes the form of $\underline{y} = \underline{d} - \underline{x}$ (i.e., the initial numerical difference less the number of elements transferred by an arithmetic operation). All problems under Problem Type II took the second form, that is, the outcome of the arithmetic operation was <u>not</u> consistent with the particular operation performed by the interviewer. The second form of integration was selected to provide a comparison with Problem Type I in which the outcome of the arithmetic operation was consistent with the particular operation performed by the interviewer.

[2] This integration of the dependence relation with the initial numerical difference between unequal collections can take two possible forms for Problem Type IV. It should be noted that these two forms are analogous to the two forms described for Problem Type II (see Footnote 1 for a detailed description of the two forms). On the one hand, if arithmetic operations are performed on two collections such that the outcomes are consistent with the particular operations, the integration takes the form of $\underline{y} = \underline{d} + 2\underline{x}$. On the other hand, if arithmetic operations are performed on two collections such that the outcomes are <u>not</u> consistent with the particular operations, the integration takes the form of $\underline{y} = \underline{d} - 2\underline{x}$. All problems under Problem Type IV took the second form, that is, the outcome of each arithmetic operation was <u>not</u> consistent with the particular operation performed by the interviewer. The second form of integration was selected to provide a comparison with Problem Type III in which the outcome of each arithmetic operation was consistent with the particular operation performed by the interviewer.

[3] In the case of two initial quantities that are equal, Inhelder and Piaget (1963) define a numerical inference based on the size axiom as follows: the addition of equal quantities to equal quantities or the subtraction of equal quantities from equal quantities conserve the equality of the resulting quantities. The equal size axiom corresponds to the "unique sums" law in a formal system of arithmetic (see Knopp, 1952, or Waismann, 1951, for a description of a formal arithmetic system). Specifically, under addition the <u>sum</u> of each pair of numbers $\underline{a}$ and $\underline{b}$ (i.e., $\underline{a} + \underline{b}$) is uniquely determined; inversely, under subtraction the <u>difference</u> between each pair of numbers $\underline{b}$ and $\underline{a}$ (i.e., $\underline{b} - \underline{a}$) is uniquely determined. Thus, the unique sums law of arithmetic can be expressed in two forms that correspond to the two forms of size axiom problems under Problem Type V: (1) If equals are added to equals, the sums are equal (problem 9), and (2) If equals are subtracted from equals, the remainders are equal (problem 10).

[4] In the case of the two initial quantities that are unequal, Inhelder and Piaget (1963) define a numerical inference based on the size axiom as follows: the addition of equal quantities to unequal quantities or the subtraction of equal quantities from unequal quantities conserve the inequality of the resulting quantities. The unequal size axiom corresponds to the "monotonic" law in a formal system of arithmetic (Knopp, 1952; Waismann, 1951). Specifically, under addition the order relation between each pair of numbers a and b (e.g., $a > b$) remains valid in the *sum* of an equal pair of numbers a' and b' (i.e., $a' = b'$); inversely, under subtraction the order relation between each pair of numbers a and b (e.g., $a > b$) remains valid in the *difference* of an equal pair of numbers a' and b' (i.e., $a' = b'$). Thus, the monotonic law of arithmetic can be expressed in two forms that correspond to the two forms of size axiom problems under Problem Type VI: (1) If equals are added to unequals, the sums are unequal (problem 11), and (2) If equals are subtracted from unequals, the remainders are unequal (problem 12).

3

COUNTING EXPERIMENT

The first experiment examines the role of the numerative activity of counting in the development of arithmetic reasoning. The general question of interest is how do young children use numerative activities to construct numerical representations of arithmetic problems and to reason about different logical forms of arithmetic operations on these problems? In order to address this question, the first experiment focuses on the developmental relations between understanding the numerative activity of counting and understanding the logical forms of arithmetic operations.

A pretest was administered to all children in the experiment. It assessed their understanding of counting and one-to-one correspondence between objects as numerative activities, and it categorized their logical functioning at one of three levels of number development. The experimental interview following the pretest examined children's arithmetic reasoning on the set of 12 addition and subtraction problems in respect to two tasks: (1) a judgment task which assessed the ability to make an inference about the outcome of an arithmetic operation, and (2) an inverse task which assessed the ability to "undo" the outcome of an arithmetic operation.

The analyses of children's arithmetic reasoning on these tasks addressed four principal issues. First, does the numerative activity of counting yield a particular form of numerical representation of the arithmetic problems, and does this form change over levels as children's understanding of counting develops? Second, what types of arithmetic inferences are children able to make at different levels, and how do the dimensions of the arithmetic problems influence children's ability to make these arithmetic inferences? Third, how do children at different levels use their counting activity to make arithmetic inferences on the judgment task? Finally, what types of inverse operations are children able to perform at different levels, and how does the numerative activity of counting influence the processes by which children solve the inverse task?

Method

Subjects

Thirty preschool children participated in the experiment. Children ranged in age from 4 years-0 months to 6 years-5 months with a mean age of 5 years-3 months. Three additional children were excluded from this experiment because their pretest performance was deficient. Specifically, two children (ages 4-1 and 4-11) were incorrect on the counting questions of the numerical comparison task, and one child (age 5-1) was incorrect on the comparison question of the same task. All children attended preschools in the Eugene, Oregon community, and they were from predominantly white, middle-class backgrounds.

A pretest was administered to all children in the experiment. It assessed children's logical operations with respect to the number concept and their understanding of counting and one-to-one correspondence between objects as numerative activities. On the basis of their performance on the pretest, children were assigned to one of three levels of number development (see Pretest section below): Level I (mean age = 4-8; range = 4-1 to 5-8), Level II (mean age = 5-1; range = 4-0 to 6-4), and Level III (mean age = 5-11; range = 5-5 to 6-5). There were 10 children at each level equally divided between boys and girls. Analyses of the age differences between levels revealed that children at level III were significantly older ($\underline{M}$ = 5-11) than children at level II ($\underline{M}$ = 5-1), $\underline{t}(18) = 2.93$, $\underline{p} < .01$, and children at level I ($\underline{M}$ = 4-8), $\underline{t}(18) = 6.66$, $\underline{p} < .01$. There was no significant age difference between children at level II and children at level I.

Design

The design of the experiment comprised three main factors: Level of Number Development which refers to children's performance on the pretest, Screening Condition which refers to covering the collections of objects during the judgment task, and Arithmetic Problem Type. The two between-subject factors were Level of Number Development (I versus II versus III), and Screening Condition (pre-transformation screening versus post-transformation screening). The one within-subject factor was Arithmetic Problem Type (I - VI). This set of factors was examined in a 3(Level of Number Development) x 2(Screening Condition) x 6(Arithmetic Problem Type) mixed design with repeated measures on the third factor, yielding six groups with equal numbers of subjects in each group.

Two variables were counterbalanced in the design: (1) Sex (half were male and half were female at each level), and (2) Problem Order (half received Order 1 and half received Order 2 at each level). The two orders of 12 arithmetic problems were generated randomly with the constraint that two problems under the same problem type (e.g., problems 1 and 2) could not occur consecutively. The inverse task was administered on half of the problems in Order 1 and the other half in Order 2. Within each order, the inverse task was assigned randomly to problems such that it occurred on (1) one problem of each problem type, (2) equal numbers of addition, subtraction, and transfer problems, and (3) half of the equality problems and half of the inequality problems.

Materials

The addition and subtraction problems were presented to the children in the context of a "dinosaur game". Four sets of materials comprised the game: (1) initial collections of dinosaurs for each problem, (2) one reserve collection of dinosaurs used to perform external addition or subtraction operations, (3) boards to display the initial collections for each problem, and (4) screens to cover the collections during the final interview on each problem.

A problem began with two homogeneous collections of plastic dinosaurs which were approximately 2.5 in. high. The dinosaurs were mounted on 2.5 in. by 0.8 in. pieces of grey matboard in order to make them freestanding on the display boards. To maintain children's interest in the problems, six different types of dinosaurs were used such that no type occurred more than once during a testing session. A reserve collection of dinosaurs accompanied all problems. It served as a source of objects for external addition or subtraction operations on the problems. The reserve collection consisted of 24 dinosaurs, four of each of the six types. They were arranged in a 9 in.

circular container which was designated a "water hole" for the dinosaurs.

Each pair of initial collections comprising an arithmetic problem was presented on a 24 in. long by 9 in. wide wooden board. Since the size and configuration of the initial collections varied across problems, each problem required a different board. All boards were partitioned in half lengthwise by a 24 in. strip of wood.

Grey rectangular markers approximately 2.5 in. by 0.8 in. in size were affixed to the boards. The configuration of the markers on each board corresponded to the configuration of the initial collections of a particular problem with one constraint on subtraction problems. There could be no markers on the boards that corresponded to those dinosaurs to be subtracted from the initial collections. This constraint was imposed because such markers would have provided cues to the children during the inverse task. The markers, referred to as "rocks" in the dinosaur game, were used to construct the initial collections for each problem. The interviewer placed dinosaurs in all locations specified by markers and, in the case of subtraction problems, the interviewer also placed dinosaurs in certain locations specified by lines drawn underneath the boards.

The fourth set of materials was a pair of wooden screens to cover the collections during the final interview on each problem. The screens measured 24 in. long by 3 in. deep by 2.5 in. high, and they were composed of front, side, and top pieces without any bottom or back pieces. The construction of the screens allowed the dinosaurs to be seen and manipulated by the interviewer but not by the children.

The collections for all problems were presented in linear configurations oriented horizontally on the boards. Several considerations determined the configurations of the arithmetic problems in this experiment. First, the configurations were designed to facilitate the numerative activity of counting by which children represented numerical information about the initial collections of dinosaurs. Accordingly, configurational properties that suggested the numerative activity of one-to-one correspondence between objects (examined in Experiment 2) were eliminated as much as possible from the problems. Second, in order to assess arithmetic inferences about the final collections, children were required to make accurate numerical comparisons of the initial collections before the addition or subtraction operations were performed. Thus, the configurations of the initial collections contained spatial alignment cues which supported such numerical comparisons. Third, it was hypothesized that children might base their judgments on the configurations of the final collections rather than on arithmetic inferences about these collections. To determine the basis for children's judgments, the configurations of the final collections contained spatial alignment cues which conflicted with accurate arithmetic inferences.

The configurations of the initial and final collections for the arithmetic problems in this experiment are depicted in Figure 1. In view of the above considerations, the pairs of initial collections were constructed such that the objects in one collection were *not* in figural one-to-one correspondence with the objects in the other collection, but spatial alignment was a consistent cue for making accurate numerical comparisons of these collections. For the equality

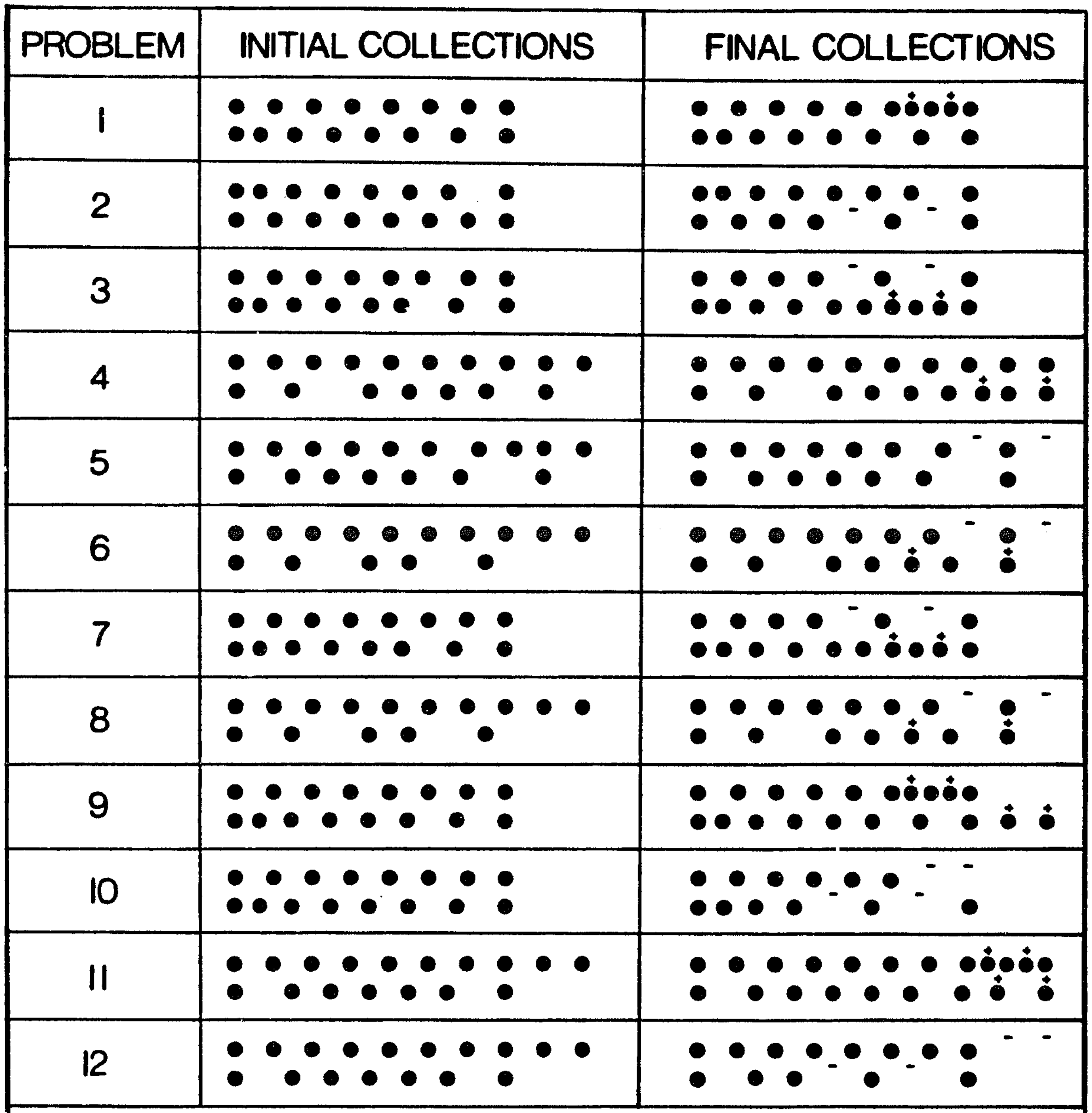

Figure 1. Configurations of the initial collections and final collections for the arithmetic problems in Experiment 1. (For each pair of collections, the top row denotes the child's collection and the bottom row denotes the interviewer's collection. + indicates the location of an addition operation on a collection, and - indicates the location of a subtraction operation on a collection.)

problems, the end-points of each pair of initial collections were aligned. For the inequality problems, the end-points were not aligned, and the collection with more objects was longer than the collection with fewer objects.

In contrast, spatial alignment was a misleading cue with respect to making accurate arithmetic inferences about the final collections. When addition or subtraction operations produced equal collections (problems 9 and 10), the end-points of each pair of final collections were not aligned, and one collection was randomly chosen to be longer than the other (the particular collection chosen was counterbalanced within each pair across problems). Conversely, when addition or subtraction operations produced unequal collections (problems 1-8, 11, and 12), the end-points were aligned.

Procedure

Each child was interviewed individually in a private room in the preschool over three to four sessions (depending on the attentiveness of the child). The pretest tasks were administered in the first testing session. The subsequent testing sessions were devoted to the experimental interview on the 12 addition and subtraction problems. The experimental interview was administered in two phases: an initial interview preceding the addition or subtraction operations, and a final interview immediately following these operations.

For all testing sessions, the child and the interviewer were seated at a table facing each other, and an observer was situated near the table so as to have a full view of the child and the materials. Three observers were trained to record the child's verbal and nonverbal behaviors during the testing sessions, and at least one observer was present at each session in addition to the interviewer. All testing sessions were recorded on audiotape.

The description of the procedure is organized as follows. The pretest is presented in the first section, and then the experimental interview on the addition and subtraction problems is detailed in the next two sections.

Pretest. The pretest assessed children's logical operations with respect to the number concept and their understanding of counting and one-to-one correspondence between objects as numerative activities. The pretest served several purposes. First, it provided a basis to categorize children into one of three levels of number development. Second, it ensured that all children in the study were able to use both counting and one-to-one correspondence between objects to make numerical comparisons for set sizes as large as those contained in the arithmetic problems of the experiments (i.e., set sizes of 7 - 10 objects). Finally, the pretest ascertained that children understood the language, particularly the comparative terms, employed in the interview on the arithmetic problems.

The pretest consisted of four tasks: (1) numerical comparison, (2) numerical equivalence, (3) quotité conservation, and (4) number conservation. Each task was administered twice, for a total of 8 trials. Tasks 1 and 2 screened children's abilities to use counting and one-to-one correspondence between objects to make numerical comparisons. These were the criterial tasks, and children were required to pass both tasks in order to participate in the study. Tasks 3 and 4 assessed children's operational notions of number with respect to the activities of counting and one-to-one correspondence.

Performance on all four tasks served to define the three levels of number development. The testing procedure and scoring criteria for each pretest task are detailed below, and then the three levels of number development are described.

Numerical comparison examined the abilities to count, to give the cardinal value, and to compare two homogeneous collections of objects. On one trial the pair of collections was equal (8 chips in both collections), and on the other trial the pair of collections was unequal (10 chips in one collection and 7 in the other). The pairs of collections were arranged so that the chips were not in figural one-to-one correspondence. The end-points of the equal collections were aligned; the end-points of the unequal collections were not aligned. Thus, the spatial configuration of the collections in this task was comparable to that of the initial collections of the arithmetic problems.

The equality trial illustrates this task. The interviewer constructed two rows of 8 chips, and asked the child to count each row: (1) "Can you count this row of chips?" (2) "How many chips are there in this row?" (3) "Can you count that row of chips?" (4) "How many chips are there in that row?" Then the child was asked to compare the two collections, "Do we have the same number of chips, or does one of us have more chips?" The child was required to answer all questions correctly on both the equality and inequality trials in order to to pass the numerical comparison task. This criterion assessed an understanding of counting that was comparable to the phase of quantitative counting described by Saxe (1979b).

Numerical equivalence examined the ability to compare two homogeneous collections on the basis of one-to-one correspondence relations between objects. The procedure was essentially the same as the first phase of the standard number conservation task (Piaget, 1965). The interviewer constructed one row of 10 chips, and asked the child to construct a row of the same number of chips without counting. After the child had constructed a collection in figural one-to-one correspondence with the objects in the interviewer's collection, the child was asked to compare the two collections, "Do we have the same number of chips, or does one of us have more chips?" The child was required to answer this question correctly on both trials in order to pass the numerical equivalence task.

Quotité conservation assessed the ability to infer correctly the number of objects in a collection that has been spatially transformed and screened after counting the objects in another (equivalent) collection. Gréco (1962a) found that an invariant concept of quotité (the counted quantity) precedes that of discrete quantity (the "countable" quantity). Furthermore, the development of an invariant concept of quotité parallels the development of an invariant concept of quantity with respect to both the judgments and the explanations given by children. Therefore, the conservation of quotité task was included in this pretest to identify a level at which the activity of counting leads to the invariance of the child's numerical schemes.

The quotité conservation task was adapted from procedures employed by Gréco (1962a) and by Inhelder, Sinclair and Bovet (1974). Both trials of the quotité conservation task were identical except that on one trial the child's collection was contracted and on the other trial the interviewer's collection was expanded. The first and second phases of this task followed the procedure of the number conservation task (described below), but the third phase was specific to the quotité task. After the interviewer had spatially transformed (i.e., expanded or contracted) one of two collections of 10 chips and the child had answered a set of number conservation questions, the interviewer asked the child to count and to give the cardinal value of the untransformed collection, "Can you count this row of chips?" and "How many chips are there in this row?" If an error was made at this point, the interviewer asked the child to recount the untransformed collection until a correct cardinal value was obtained. Then the interviewer screened the transformed collection and asked the child to make a judgment about the number of chips in the screened collection, "How many chips are there in this hidden row?" Finally, an explanation for the child's quotité judgment was sought, "How do you know?"

The child was required to make a correct judgment and to provide an adequate explanation on both trials in order to pass the quotité conservation task. An adequate explanation was defined as a reference to the irrelevance of the spatial transformation performed on one collection (e.g., "Even if you move them around we still got the same number"), or a reference to the absence of an arithmetic transformation on one collection (e.g., "I have 10, and you didn't take any away from your row").

Number conservation assessed the ability to infer that the numerical equivalence of two collections of objects placed in figural one-to-one correspondence remains invariant under a spatial transformation of one collection. In contrast to the quotité conservation task, this task required an invariant concept of discrete quantity which has been taken to be the criterion for an operational concept of number in Piagetian theory (Inhelder, Sinclair, and Bovet, 1974). Therefore, the number conservation task was included in this pretest to identify a level at which the activity of establishing one-to-one correspondence between objects leads to the invariance of the child's (discrete) quantity schemes.

Both trials of the number conservation task were identical, except that on one trial the child's collection was contracted and on the other trial the interviewer's collection was expanded. The interviewer constructed one row of 10 chips and asked the child to construct a row of the same number of chips. After the child had constructed a collection in figural one-to-one correspondence with the objects in the interviewer's collection, the child was asked to compare the two collections, "Do we have the same number of chips, or does one of us have more chips?" Then the interviewer performed a spatial transformation on one of the collections, and posed a set of three conservation questions. First, the child was asked to make a judgment, "Now, are there the same number of chips in your row as in my row, or does one of us have more chips?" Second, an explanation for the child's judgment was sought "How do you know?" Third, a counter-argument based on the child's judgment was presented, e.g., "But look how long my row of chips is. Aren't there more chips in my row?" (counter-argument to a conservation judgment) or "But when we started you put one chip in front of each of my chips, and you said that we had the same number of chips. What do you think?" (counter-argument to a nonconservation judgment). The child was required to make a correct judgment, to provide an adequate explanation (i.e., an explanation referring to identity, reversibility by inversion, or reversibility by compensation), and to resist a counter-argument on both trials in order to pass this task.

Children were assigned to one of three levels of number development based on their performance on the pretest tasks.

Level I. Children passed the numerical comparison and numerical equivalence tasks, but they failed the quotité conservation and number conservation tasks.

Level II. Children passed the numerical comparison, numerical equivalence, and quotité conservation tasks, but they failed the number conservation task.

Level III. Children passed all four tasks.

Initial interview. The pairs of initial collections comprising the arithmetic problems were constructed on individual boards prior to each testing session. All boards were divided in half lengthwise such that one collection in each pair was located on the part of the board nearest to the child, and the other collection was located on the part nearest to the interviewer. These different locations on the board were referred to as the child's "cave" and the interviewer's "cave," respectively.

The board displaying the two initial collections of the first problem was placed on the table in front of the child. The reserve collection of dinosaurs was also placed on the table in front of the child. The interviewer introduced the arithmetic problem in the form of a dinosaur game, and instructed the child as follows:

> Today we are going to play a dinosaur game. This is how we play the game. You and I each have a cave with dinosaurs in it. These are your dinosaurs standing on rocks in your cave. These are my dinosaurs standing on rocks in my cave. And here are some other dinosaurs drinking at their favorite water hole. Every time we play the game, I am going to make some dinosaurs go into the caves or go out of the caves. You must watch very carefully and try to remember how many dinosaurs go into the caves or go out of the caves. Then I am going to hide your dinosaurs and my dinosaurs with these cave covers and ask you some questions about the dinosaurs hiding in the caves.

In order to be certain that the instructions were understood, the interviewer asked the child to point to the dinosaurs in each of the caves and to the water hole before proceeding with the experimental interview. All children were successful on these preliminary questions.

The initial interview required the child to count and to compare the pair of collections. The purpose of this interview was to direct the child to use counting to construct a numerical representation of the initial collections. Recall that an arithmetic inference is defined as a numerical judgment about the final collections which is based on the integration of a numerical judgment about the initial collections with an addition or subtraction operation on these collections. By this definition, it was necessary to ensure that the child's judgment about the initial collections was accurate. Therefore, two procedures were followed in the initial interview. First, the configurations of the initial collections contained spatial alignment cues to facilitate accurate numerical comparisons. Second, if the child made an error on any initial interview question, the interviewer provided feedback and then repeated the question.

The initial interview began with a series of questions concerning the number of objects in each of the collections: (1)"Can you count the dinosaurs in your cave?" (2)"How many dinosaurs are there in your cave?" (3)"Can you count the dinosaurs in my cave?" (4)"How many dinosaurs are there in my cave?" Then the child was asked to make a numerical comparison of the initial collections, "Do we have the same number of dinosaurs, or do you have more dinosaurs, or do I have more dinosaurs?" Finally, an explanation was sought in order to determine the basis for the child's numerical comparison, "How do you know?"

Final interview. After completing the initial interview on a problem, the interviewer performed an addition or subtraction operation on one or both of the collections. The final interview examined the child's arithmetic reasoning on two tasks: a judgment task and an inverse task. The judgment task preceded the inverse task on all problems.

The judgment task assessed the ability to make an inference about the outcome of an arithmetic operation. In this task, the child was able to observe the arithmetic operation performed on the initial collections, but the final collections were screened from the child's view. The screening procedure was conducted under two conditions. For the _pre-transformation_ screening condition, the pair of initial collections was covered before an arithmetic operation was performed on them. The child in this condition did not view the final collections at any point during the judgment task. For the _post-transformation_ screening condition, the interviewer performed an arithmetic operation on the initial collections and then covered them prior to initiating the interview on the judgment task. This condition allowed the child to view the final collections for a few seconds while an arithmetic operation was being executed. Nevertheless, for both conditions the final collections were screened during the interview on the judgment task.

The purpose of the two screening conditions was to examine the effect of perceiving the configurations of the final collections on the ability to make an arithmetic inference. Since the final collections were briefly visible to the child in the post-transformation screening condition, it was possible for the child to evaluate their configurational properties. However, reliance on these properties _alone_ could not lead to a correct judgment because the configurations of the final collections contained misleading spatial alignment cues.

The interview on the judgment task was initiated differently for the two screening conditions. For the pre-transformation screening condition, the interviewer stated that she was going to hide the dinosaurs, and then placed the screens over the pair of collections. At this point, two questions from the initial interview were repeated to ensure that the child had not forgotten his or her numerical information about the initial collections. Specifically, the child was asked, "How many dinosaurs are there in your cave?" and "How many dinosaurs are there in my cave?" If the child made an error on either question, the screens were removed and the initial interview on the problem was administered again. This procedure was implemented to control for any difference in the memory demands of the two screening conditions. Very few children made errors on the memory probes, and none of them required more than one repetition of the initial interview.

The arithmetic operation was performed immediately after the memory probes. An appropriate explanation accompanied each arithmetic operation on the collections: (1) "These two dinosaurs are leaving the water hole to go to sleep in your [my] cave" (addition), (2) "These two dinosaurs are leaving your [my] cave to get a drink at the water hole" (subtraction), or (3) "These two dinosaurs are leaving your cave to visit their friends in my cave" (subtraction - addition transfer). The interview on the judgment task followed the arithmetic operation in the pre-transformation screening condition.

For the post-transformation screening condition, the arithmetic operation was performed on the collections as soon as the initial interview was completed. Each arithmetic operation was accompanied by the appropriate explanation described above. Then the interviewer indicated that she was going to hide the dinosaurs, and placed the screens over the pair of collections. The interview on the judgment task immediately followed the screening procedure in this condition.

The interview on the judgment task began with a request for the child to make an inference about the arithmetic operation performed on one or both collections, "Do we have the same number of dinosaurs, or do you have more dinosaurs, or do I have more dinosaurs?" This question is referred to as the <u>standard inference question</u>. Next, a justification for the child's judgment was sought, "How do you know?" If the child indicated on the standard inference question that a particular cave had more dinosaurs, then the difference question was asked, "How many more dinosaurs do you [I] have?"

The remaining questions on the judgment task concerned the anticipated outcome of imaginary arithmetic operations. The interviewer repeated the appropriate description of each arithmetic operation that had been performed on a problem: (1) "Let's pretend that two more dinosaurs leave the water hole to go to sleep in your [my] cave" (addition), (2) "Let's pretend that two more dinosaurs leave your [my] cave to get a drink at the water hole" (subtraction), or (3) "Let's pretend that two more dinosaurs leave your cave to visit their friends in my cave" (subtraction - addition transfer). While describing the imaginary arithmetic operation, the interviewer gestured to the appropriate cave(s), but did not transfer any concrete objects. Then an anticipatory inference question was posed based on the child's judgment on the standard inference question: (1) "Would you still have more dinosaurs?" (2) "Would I still have more dinosaurs?" or (3) "Would we still have the same number of dinosaurs?" Finally, the interviewer requested an explanation for the child's anticipatory judgment, "How do you know?"

Following the last question on the judgment task, the interviewer removed the screens covering the pair of final collections and administered the inverse task. The inverse task required the child to restore the collections to their initial numerical states, that is, prior to the arithmetic operation performed on them. This task yielded data in the form of actions for "undoing" the outcome of the arithmetic operation.

These data were informative about the child's arithmetic reasoning in three respects. First, the accuracy of the actions on different problems provided evidence of the child's knowledge of the inverse operation. Second, the actions also revealed diverse procedures for solving this task which were related to the child's knowledge of the inverse operation. Finally, the nonverbal data from the inverse task served to complement the verbal data from the judgment task in assessing the child's arithmetic competencies.

The interview on the inverse task consisted of two questions. The interviewer uncovered the collections and asked the child, "Can you fix the dinosaurs to make it the way it was at the beginning of this dinosaur game?" After the child's actions on a problem were completed, an explanation was requested to determine the basis for these actions, "How do you know that you fixed the dinosaurs to make it the way it was at the beginning of this game?"

No corrective feedback was provided to the child during the final interview. Each child's responses on the judgment task and the inverse task were recorded by a trained observer. The judgment task was also recorded on audiotape, and the observer's record was then verified against a transcription of the audiotape. Reliability for coding actions on the inverse task was determined by having a second trained observer record the arithmetic actions of 12 children in the study (total number of actions observed was 115). Interobserver agreement for Experiment 1 was 94%.

Results and Discussion

Initial Interview

In the initial interview on a problem, the child was required to count and give the cardinal value for each of two collections. Then the interviewer asked the child to make a numerical comparison of the collections and to provide an explanation for this judgment. Recall that if the child made an error on any question, the question was repeated. However, it was expected that very few errors would occur on these initial questions because all children in the experiment had passed the numerical comparison task of the pretest. This expectation was realized by the children's performance on the initial interview questions. There were no errors on the cardinal value questions or on the numerical comparison question. Occasionally, children miscounted a collection, but they did not require more than one repetition to achieve a correct count.

Children's explanations for their numerical comparisons provided evidence about how they initially represented the arithmetic problems. Thus, these explanations bear on a fundamental assumption of the present study: Different numerative activities lead children to construct different forms of numerical representation of arithmetic problems. Specifically, children's explanations were examined with respect to three issues. First, does the numerative activity of counting objects in the arithmetic problems result in a particular form of numerical representation of these problems? Second, if counting results in a particular form of numerical representation, does this form change over levels as children's understanding of counting develops? Third, how does the initial numerical representation constructed by children influence their subsequent reasoning on the judgment task and the inverse task?

Seven types of explanations were identified on the initial interview. Each type is defined below with a sample explanation to illustrate the type.

1. Counting Activity. The child makes a verbal reference to the activity of counting the collections but does not state the cardinal number of objects in the collections. "I counted them."

2. Cardinal Number. The child makes a verbal reference to the cardinal number of objects in one or both collections. "'Cause I got 10 and you got 5."

3. Spatial Configuration. The child makes a verbal and/or gestural reference to the spatial alignment of the collections or to the spatial distribution of objects within the collections. "One row is shorter and the other row is longer."

4. Correspondence Relation. The child makes a verbal and/or gestural reference to a correspondence relation between objects in the two collections. This relation can take two forms: (1) object-to-object correspondence on equality problems, and (2) object-to-gap correspondence on inequality problems. Only object-to-gap explanations occurred in this experiment. An object-to-gap explanation entails a reference to the presence of objects in one collection that corresponds to the absence of objects in the other collection (i.e., a spatial "gap"). "Because there's none there and there."

5. Numerical Difference. The child makes a verbal reference to the numerical quantity of the inequality relation between two collections. The numerical difference can be expressed as a greater number of objects in one collection, or a lesser number of objects in the other collection. This type of explanation applies only to inequality problems. "Because you have 5 less than me and I have 5 more than you."

6. Combined Explanation. The child combines two or more types of explanations for the judgment on the numerical comparison question. "Mine go to here and yours only go to there; I have 10 and you have 5" (combination of Spatial Configuration and Cardinal Number explanations).

7. Irrelevant or None. The child makes one of three types of responses: (1) repetition of the judgment on the numerical comparison question, (2) an ambiguous explanation, or (3) no explanation. "They look like they just are."

Children's explanations revealed that counting the initial collections led them to construct a particular form of numerical representation of the arithmetic problems. Moreover, this form of numerical representation changed over levels such that by level II it entailed a fundamental property of the counting activity, namely cardinal number. These results are evident in Table 3 which presents the percentage of each type of explanation on the initial interview.

There were two predominant types of explanations over all levels and, taken together, these types accounted for 89% of the total number of explanations given on the initial interview. Children at level I either referred to the activity of counting the collections, or they

Table 3

Percentage of Each Type of Explanation on the Initial Interview in the Counting Experiment

	Type of Explanation						
Level	Counting Activity	Cardinal Number	Spatial Conf.[a]	Correspondence Relation	Numerical Difference	Combined	Irrelevant or none
I	45	40	8	3	0	3	2
II	23	72	1	0	1	1	3
III	16	73	0	0	3	3	0
All Levels	28	61	3	1	1	2	3

Note. Percentages were based on 120 explanations at each level.

[a] Spatial Configuration

referred to the cardinal number of objects in the collections. A clear developmental shift toward the latter explanation occurred at level II, and cardinal number remained the more frequent explanation thereafter.

It is also noteworthy that two other types of explanations were infrequent over all levels. First, children even at level I rarely mentioned the spatial configuration of the collections in their explanations. On the one hand, this finding is surprising in view of the extensive literature on the role of spatial indices in early numerical cognition (e.g., Beilin, 1968; Piaget, 1965, 1968; Zimiles, 1966). On the other hand, the lack of spatial explanations in this experiment is consistent with the argument that the figurational activity of counting involves the construction of a numerical representation of the arithmetic problems in a linguistic medium (i.e., verbal numerals or number names). Such a numerical representation does not entail a spatial medium. Thus, the child's numerical judgments should not be dependent on the spatial extension of the collections (Gréco, 1962a; Saxe, 1979a; Saxe, Cohen & Rindskopf, 1980).

Second, children gave few numerical difference explanations on the inequality problems. Although representing the numerical difference between two unequal collections would have been more efficient for making a subsequent inference on the judgment task, children even at level III referred primarily to the cardinal number of objects in the collections. It should be emphasized that a numerical representation of the cardinal values of the collections was <u>sufficient</u> for solving inequality problems, but it was less efficient because it required children to compute the numerical difference between the collections during the final interview on the problems.

In conclusion, counting the initial collections led children to construct a form of numerical representation that was particular to this numerative activity. Children's explanations indicated that they represented the summation of objects in the collections as cardinal values. Nevertheless, knowledge of the cardinal values of the collections, which was evident in three-fourths of the explanations at level II and at level III, was evident in less than half of the explanations at level I. Thus, the form of numerical representation that was revealed by these explanations changed from level I to level II as children developed an understanding of the implications of their counting activity.

Judgment Task

Standard inference question: Judgments.

The judgment task assessed children's ability to make an inference about the outcome of an arithmetic operation. To determine the effect of the principal factors in the experiment (i.e., level of number development, screening condition, and arithmetic problem type) on judgment accuracy, children's judgments on this question were scored as accurate/inaccurate and a set of analyses was conducted. Each of these analyses is discussed below. The discussion focuses on the types of arithmetic inferences that children make at different levels, and the strategies that underlie both their accurate and inaccurate inferences on this task.

Table 4 presents the mean accuracy on the standard inference question for each problem in the experiment. There are two patterns

Table 4

Mean Accuracy on the Standard Inference Question for Each Problem in the Counting Experiment

	Problem											
Level	1	2	3	4	5	6	7	8	9	10	11	12
I	1.00	1.00	1.00	.00	.30	.20	1.00	.00	.60	.70	.50	.60
II	1.00	1.00	1.00	.20	.20	.30	1.00	.30	1.00	1.00	.60	.50
III	1.00	1.00	1.00	.80	.50	.50	1.00	.70	1.00	1.00	1.00	.90

Note. Mean Accuracy was based on the proportion of correct judgments on the standard inference question.

to note in this table. First, children's accuracy was comparable for pairs of problems which differed by the particular arithmetic operation performed within problem types (e.g., problems 1 and 2 under problem type I). Thus, in the following analyses performance was summed over individual problems comprising the same problem type. Second, there was also no difference in accuracy between the pairs of standard and control problems within problem type III (problems 7 and 3) and problem type IV (problems 8 and 6). This finding indicates that the logical form of two-way function problems, rather than the internal or external context of the arithmetic operations on these problems (see section on Arithmetic Problem Types), determined the accuracy of children's judgments.

A 3 (level) x 2 (screening) analysis of variance (ANOVA) was performed on judgment accuracy over all problems. As expected, there was a significant effect for level, $\underline{F}(2,24)=9.89$, $\underline{p} < .001$, and no significant effect for screening condition. Children at level III made more correct judgments than children at levels I or II. Mean proportions of correct judgments for levels I, II, and III were .58, .68, and .87, respectively. An additional ANOVA revealed no significant effect for sex and, therefore, sex was not included as a factor in subsequent analyses.

One of the fundamental questions of the present study was how do the dimensions of the arithmetic problems, specifically the logical form of the problem and the quantitative relationship between the initial collections of the problem, influence children's ability to make arithmetic inferences. This question was examined in two analyses.

The first analysis addressed the effect of the logical form of the problem on judgment accuracy in a 3 (level) x 2 (screening) x 3 (problems: one-way function vs. two-way function vs. size axiom) multivariate analysis of variance (MANOVA). The mean accuracy on the standard inference question for the three logical forms of problems is reported in Table 5. An arcsin transformation was performed on the data because there was an unequal number of problems under problem types III and IV (control problems 3 and 6 under problem types III and IV, respectively, were not included in this analysis). A significant effect was obtained again for level such that children at level III made more correct judgments on the problems than children at levels I and II, $\underline{F}$ (2,24)=12.46, $\underline{p}<.001$. Mean proportions of correct judgments for levels I, II, and III were .56, .68, and .89, respectively. However, there were no significant effects for either the screening condition or the problems.

One possible interpretation for the lack of a significant effect for the logical form of the problem is that this dimension alone can not account for children's accuracy on the different types of arithmetic problems. It is evident from Table 5 that the quantitative relationship between the initial collections of the problem also contributed to problem difficulty. Thus, the next analysis examined the joint effect of the logical form of the problem and the quantitative relationship between the initial collections of the problem.

A 3(level) x 2(screening) x 4(problems: inequality one-way function vs. inequality two-way function vs. equality size axiom vs. inequality size axiom) MANOVA was performed on judgment accuracy.

Table 5

Mean Accuracy on the Standard Inference Question for One-Way Function, Two-Way Function, and Size Axiom Problems in the Counting Experiment

	Logical Form of Problem							
	One-Way Function[a]		Two-Way Function[b]				Size Axiom[c]	
Level	Equality	Inequality	Equality-S	Equality-C	Inequality-S	Inequality-C	Equality	Inequality
I	1.00	.15	1.00	1.00	.00	.20	.65	.55
II	1.00	.20	1.00	1.00	.30	.30	1.00	.55
III	1.00	.65	1.00	1.00	.70	.50	1.00	.95

Note. Mean Accuracy was based on the proportion of correct judgments on the standard inference question.

[a] One-Way Function Problems included Equality problems 1, 2 and Inequality problems 4, 5.

[b] S = standard form of problem; C = control form of problem. Two-Way Function Problems included Equality-S problem 7, Equality-C problem 3, Inequality-S problem 8, and Inequality-C problem 6.

[c] Size Axiom Problems included Equality problems 9, 10 and Inequality problems 11, 12.

Table 6 presents the data in this analysis. Note that equality one-way function problems and equality two-way function problems were not included in this MANOVA because there was no variation in judgment accuracy for these problems (i.e., children's performance was at ceiling for these problems). This analysis yielded two significant effects. First, there was a highly significant effect for problems, Hotelling T^2 (critical value 9.9799) = 67.69, $\underline{p} < .001$. Children's accuracy was greater for equality and inequality size axiom problems than for inequality one-way and two-way function problems. In order, mean proportions of correct judgments for equality size axiom, inequality size axiom, inequality one-way function, and inequality two-way function problems were .88, .68, .33, and .33. Second, there was an effect for level which replicated the finding of the preceding analyses. Judgment accuracy was greater at level III ($\underline{M} = .80$) than at levels I ($\underline{M} = .36$) and II ($\underline{M} = .51$), $\underline{F}(2,24)=9.89$, $\underline{p} < .001$.

To summarize, children's ability to make an arithmetic inference on this task emerged over levels with the greatest improvement in accuracy at level III. However, this ability was not uniform for all types of arithmetic problems. Recall that children's performance was at ceiling for the equality one-way function problems and the equality two-way function problems. Moreover, their accuracy was greater for size axiom problems than for inequality function problems. Even at level III, mean accuracy for inequality function problems was still only 63%. These results raise the following question: What is the basis of children's judgments when they do <u>not</u> make accurate arithmetic inferences, in particular, on inequality function

Table 6

Mean Accuracy on the Standard Inference Question under the Pre-Transformation and Post-Transformation Screening Conditions in the Counting Experiment

		Logical Form of Problem			
		One-Way Function[a]	Two-Way Function[b]	Size Axiom[c]	
Level	Screen Condition	Inequality	Inequality	Equality	Inequality
I	Pre	.20	.00	.80	.50
	Post	.10	.20	.50	.60
II	Pre	.10	.30	1.00	.50
	Post	.30	.30	1.00	.60
III	Pre	.70	.60	1.00	1.00
	Post	.60	.60	1.00	.90

Note. Mean Accuracy was based on the proportion of correct judgments on the standard inference question.

[a] One-Way Function Problems included Inequality problems 4, 5; Equality problems 1, 2 do not appear in the table because they were not included in the multivariate analysis.

[b] Two-Way Function Problems included Inequality problems 6, 8; Equality problems 3, 7 do not appear in the table because they were not included in the multivariate analysis.

[c] Size Axiom Problems included Equality problems 9, 10 and Inequality problems 11, 12.

problems? In order to answer this question, children's strategies for making judgments on this task were examined.

The most fundamental distinction in the strategy analysis was between *arithmetic* and *nonarithmetic* strategies. By definition, an *arithmetic* strategy was an accurate or inaccurate judgment based on the numerical information in a problem. In comparison, a *nonarithmetic* strategy was an inaccurate judgment based on spatial alignment or other irrelevant information in a problem.

The following example illustrates these two strategies. If the child judged one collection to be more because it underwent an arithmetic operation or because it contained a greater number of objects, then this judgment was categorized as an instance of an *arithmetic* strategy. In contrast, if the child judged one collection to be more because it was longer, then this judgment was categorized as an instance of a *nonarithmetic* strategy.

The second distinction in the strategy analysis was within the category of arithmetic strategies. *Operation* strategies were distinguished from *inference* strategies. An *operation* strategy was defined as an accurate or inaccurate judgment based on the arithmetic operation alone. It is evident that this type of judgment was not a "true" inference because it did not integrate numerical information about the initial collections with the arithmetic operation on these collections. An operation strategy led to an accurate judgment when the outcome of an arithmetic operation was consistent with the particular operation (e.g., problem type I), but it led to an inaccurate judgment when the outcome was *not* consistent with the particular operation (e.g., problem Type II). In comparison, an inference strategy was an accurate or inaccurate judgment that did integrate numerical information about the initial collections with the arithmetic operation on these collections. Depending on the accuracy of the computations by which children integrated the numerical information in a problem, an inference strategy led to either accurate or inaccurate judgments.

Problem 4 illustrates the difference between an operation strategy and an inference strategy. Recall that problem 4 began with two unequal collections. Specifically, the child's collection contained 10 elements and the interviewer's collection contained 7 elements; then 2 elements were added to the interviewer's collection for a total of 9 elements. If the child judged the interviewer's collection to be more, then this judgment was categorized as an instance of an operation strategy. In contrast, if the child judged his or her collection to be more, then this judgment was categorized as an instance of an inference strategy.

Children's judgments on each arithmetic problem were categorized into four types of strategies: (1) operation, (2) accurate inference, (3) inaccurate inference, and (4) spatial alignment. The following procedure was used to determine children's strategies on the arithmetic problems. Since the standard inference question was a 3-choice question, three distinct judgments were possible on each problem: (1) the child's collection could be judged to be more, (2) the interviewer's collection could be judged to be more, or (3) both collections could be judged to be the same number. Each of these judgments corresponded to a particular strategy on a given problem. For example, on problem 4 a judgment that the child's collection had more corresponded to an accurate inference strategy, a judgment that the interviewer's collection had more corresponded to an operation strategy, and a judgment that both collections had the same corresponded to a spatial alignment strategy. The percentage of each type of strategy used on the standard inference question appears in Table 7. The strategy data are presented for each of the four problem types that were examined in the preceding multivariate analysis (see Table 6).

This analysis revealed three findings of interest. First, the use of an operation strategy accounted for the majority of inaccurate judgments on all problem types. Second, although children primarily relied on an operation strategy at levels I and II, their use of this strategy decreased at level III with the concomitant increase in the use of an inference strategy. Finally, the developmental pattern of children's errors on one-way function problems differed from that on two-way function problems. Children abandoned an operation strategy for an inference strategy at an earlier level with inequality one-way function problems than with inequality two-way function problems. This disparity was most evident at level III where an operation strategy was employed on 40% of the judgments with inequality two-way function problems as compared to only 5% of the judgments with inequality one-way function problems. Moreover, the shift from the use of an operation strategy to an inference strategy was more gradual with inequality two-way function problems.

Taken together, the results of the strategy analysis indicate that children's inaccurate judgments at levels I and II entailed the use of an arithmetic strategy, but one which was based on the operation

Table 7

Percentage of Each Type of Strategy on the Standard Inference Question in the Counting Experiment

		Type of Strategy					
		Arithmetic				Nonarithmetic	
			Inference				
Level	Problems[a]	Operation	Accurate	Inaccurate	Operation/ Acc. Inference[b]	Spatial Alignment	Other
I	One-Way, ≠	45	15	40	-	-	-
II		55	20	25	-	-	-
III		5	65	30	-	-	-
I	Two-Way, ≠	80	10	10	-	-	-
II		65	30	5	-	-	-
III		40	60	0	-	-	-
I	Size Axiom, =	-	-	-	65	20	15
II		-	-	-	100	0	0
III		-	-	-	100	0	0
I	Size Axiom, ≠	45	55	-	-	-	0
II		45	55	-	-	-	0
III		5	95	-	-	-	0

Note. Percentages were based on 120 judgments at each level.

[a] One-Way Function Inequality Problems included problems 4, 5; Two-Way Function Inequality Problems included problems 6, 8; Equality Size Axiom Problems included problems 9, 10; Inequality Size Axiom Problems included problems 11, 12.

[b] Operation/Accurate Inference.

alone. A developmental shift occurred between level II and level III such that children began to use an inference strategy (although frequently an inaccurate inference) in solving these arithmetic problems. However, the shift was completed earlier with inequality one-way function problems than with inequality two-way function problems. This pattern of more advanced performance on one-way function problems recurs in several analyses below. It provides further evidence for the influence of the logical form of the problem on children's ability to make an arithmetic inference.

Standard inference question: Explanations. Children's judgments on the arithmetic problems were accompanied by explanations. These explanations were analyzed as a source of evidence about how children arrived at their judgments. The focus of this analysis was on those explanations that reflected the process of making an arithmetic inference, that is, integrating numerical information about the initial collections with an arithmetic operation on these collections.

Some explanations were excluded form this analysis because they did not provide unambiguous evidence about the presence of an arithmetic inference. For example, children at level I frequently gave an explanation that referred only to the arithmetic operation performed on a collection, "'Cause you took 2 [dinosaurs] away from yours." However, two types of explanations were particularly informative about how children used or did not use their counting activity to make an arithmetic inference. Each type is defined below with sample explanations to illustrate it.

1. Mental Arithmetic. The child makes reference to an arithmetic computation, or to the sum/remainder of the final collections. This computation involves the use of verbal numerals.

S.T. (level II, age 5-5) on problem 4 made an incorrect judgment and gave an explanation that reflected an inaccurate computation: "'Cause you had 7 [dinosaurs] and I had 10, and [you] put 2 in yours and made it to 10."

S.T. (level II, age 5-5) on problem 8 made an incorrect judgment and gave an explanation that reflected an inaccurate computation: "'Cause I had 10 and you had 5; (child counts to herself) now I have 7 and you have 7."

A.S. (level III, age 5-11) on problem 7 made a correct judgment and gave an explanation that reflected an accurate computation: "'Cause you put 2 more in your cave; 8-9-10, the number '8' and you add 2 more makes 10."

B.W. (level III, age 6-5) on problem 12 made a correct judgment and gave an explanation that reflected an accurate computation: "'Cause 10 minus 2 equals 8, that's how I know I have 8, and you have 5 'cause 7 minus 2 equals 5."

2. Axiomatic Arithmetic. The child makes reference to the three terms of an inference (i.e., initial collections, arithmetic operations, and final collections) that corresponds to the Unique Sums law of equal quantities (equal size axiom) or the Monotonic law of unequal quantities (unequal size axiom) in a formal arithmetic system.

S.M. (level III, age 5-6) on problem 9 made a correct judgment and gave an explanation that reflected an axiomatic inference: "You put 2 [dinosaurs] in my cave and 2 in your cave, and we had the same number at the first, so we have the same."

J.K. (level III, age 6-1) on problem 11 made a correct judgment and gave an explanation that reflected an axiomatic inference: "'Cause you added 2 in both of them and I had more before, so I have more."

Mental arithmetic explanations were the most frequent type of explanation at all levels. In addition, there were a considerable number of axiomatic arithmetic explanations on size axiom problems. However, these explanations did not occur frequently until level III (43% axiomatic arithmetic explanations at level III). Therefore, the following discussion focuses on mental arithmetic explanations with reference to axiomatic arithmetic explanations at relevant points.

The frequency of mental arithmetic explanations is presented in Table 8. There are several noteworthy findings in this table. The first finding concerns the development of mental arithmetic explanations. Although the frequency of these explanations increased over levels, the <u>greatest</u> increase occurred from level I to level II. Recall that in children's explanations on the initial interview there was also a marked increase in the frequency of cardinal number explanations from level I to level II (see Table 3). The increase in both explanations at level II reflected a developmental change in children's understanding of their counting activity and how to use it to solve arithmetic problems. Thus, there was a relationship between representing the cardinal values of the collections during the initial interview, and manipulating the cardinal values of the collections through mental arithmetic during the final interview.

Table 8

Frequency of Mental Arithmetic Explanations on the Standard Inference Question in the Counting Experiment

	Inequality Problems[a]				Equality Problems[b]
Level	One-Way Function	Two-Way Function	Size Axiom	All Types	All Types
I	4 (3)	1 (1)	3 (2)	8 (6)	1 (1)
II	8 (4)	6 (5)	6 (4)	20 (13)	8 (8)
III	15 (11)	6 (6)	9 (9)	30 (26)	7 (7)

Note. Numbers in parentheses indicate the number of correct judgments that were accompanied by mental arithmetic explanations on the standard inference question.

[a] Frequencies were based on 20 explanations on each type of Inequality Problem at each level. One-Way Function Problems included problems 4, 5; Two-Way Function Problems included problems 6, 8; Size Axiom Problems included problems 11, 12.

[b] Frequencies were based on 60 explanations on all types of Equality Problems at each level. Equality Problems included problems 1, 2, 3, 7, 9, 10.

The second finding concerns the influence of the dimensions of the arithmetic problems on children's use of mental arithmetic. The first dimension to examine is the quantitative relationship between the initial collections of the problem. Mental arithmetic explanations occurred more frequently on the inequality problems than on the equality problems (see Table 8). One interpretation of this finding is that it was not necessary to employ an arithmetic computation to solve the equality problems. These problems could have been solved by using either a nonintegrative, operation strategy or an integrative, inference strategy. Nevertheless, some children occasionally gave mental arithmetic explanations on equality problems and, when they did, their judgments were always correct.

The next dimension to consider is the logical form of the problem. This dimension influenced children's use of mental arithmetic in two respects. The first respect concerned the types of problems on which children could have used mental arithmetic, but did not. Size axiom problems were solved primarily by axiomatic arithmetic, however, the ability to employ axiomatic arithmetic emerged over levels. Children at level I did not give any axiomatic arithmetic explanations on the size axiom problems (problems 9-12). But, children at levels II and III gave axiomatic arithmetic explanations on 18% and 43% of these problems, respectively.

The logical form of the arithmetic problem influenced children's use of mental arithmetic in a second respect that was consistent with previous analyses. It accounted for the differential frequency of mental arithmetic explanations on inequality one-way function problems as compared with inequality two-way function problems. Children gave mental arithmetic explanations more frequently on inequality one-way function problems than on inequality two-way function problems. This finding suggests that mental arithmetic was the procedure underlying the greater use of an inference strategy on inequality one-way function problems than on inequality two-way function problems. Moreover, if mental arithmetic mediated children's accurate inferences, then the use of mental arithmetic should be related to judgment accuracy on inequality problems. This hypothesis was supported by the data presented in Table 8.

It can be seen in Table 8 that the number of correct judgments on inequality problems which were accompanied by mental arithmetic explanations increased twofold from level II to level III. Specifically, the percentages of correct judgments accompanied by mental arithmetic explanations at level II and level III were 65% and 87%, respectively. The reason for this increase from level II to level III is revealed in the next table.

Table 9 presents the frequency of accurate computations in mental arithmetic explanations. This table reveals that children at level III made a greater percentage of accurate computations on inequality problems than children at level II. Specifically, there were 57% accurate computations at level III as compared to 35% accurate computations at level II. Moreover, even when children at level III made <u>inaccurate</u> computations, their errors were in the correct direction. That is, erroneous sums were greater than the initial collections, and erroneous remainders were less than the initial collections. This type of error is common in young children's arithmetic computations, and it is consistent with their knowledge of

Table 9

Frequency of Accurate Computations in Mental Arithmetic Explanations in the Counting Experiment

	Inequality Problems[a]				Equality Problems[b]
Level	One-Way Function	Two-Way Function	Size Axiom	All Types	All Types
I	3 (4)	1 (1)	2 (3)	6 (8)	1 (1)
II	4 (8)	3 (6)	0 (6)	7 (20)	4 (8)
III	10 (15)	4 (6)	3 (9)	17 (30)	5 (7)

Note. Numbers in parentheses indicate the number of mental arithmetic explanations.

[a] Frequencies were based on 20 explanations on each type of Inequality Problem at each level. The problems included under Inequality Problems are the same as those in Table 8.

[b] Frequencies were based on 60 explanations at each level. The problems included under Equality Problems are the same as those in Table 8.

the directional effects of addition and subtraction operations (Starkey & Gelman, 1982).

In conclusion, two implications can be drawn from the analysis of mental arithmetic explanations. First, the observed relationship between the frequency of correct judgments and the accuracy of arithmetic computations provides evidence that children used mental arithmetic to make inferences on this task. Second, the developmental trend in mental arithmetic explanations to increase over levels on *particular* problem types indicates that an understanding of both the arithmetic operations in the problems and the numerative activity of counting influenced how children arrived at their judgments.

Difference question. Mean accuracy on the difference question is presented in Table 10. The analysis and discussion of these data are combined with the data on the difference question in Experiment 2.

Anticipatory inference question. Mean accuracy on the anticipatory inference question is presented in Table 11. The analysis and discussion of these data are combined with the data on the anticipatory inference question in Experiment 2.

Inverse Task

The inverse task assessed children's ability to "undo" the outcome of an arithmetic operation. It was assumed that this ability reflected their knowledge of the relation between direct and inverse operations within an arithmetic system. The "inverse" operation is that operation which precisely negates or cancels another operation within an arithmetic system. Thus, with respect to this task, the inverse action of subtracting two objects from a given collection

Table 10

Mean Accuracy on the Difference Question for Each Problem in the Counting Experiment

	Problem[a]									
Level	1	2	3	4	5	6	7	8	11	12
I	.20	.30	.00	.00	.00	.00	.00	.00	.00	.00
II	.70	.40	.00	.20	.20	.00	.00	.10	.00	.00
III	.90	.70	.10	.70	.40	.20	.00	.40	.00	.10

Note. Mean Accuracy was based on the proportion of correct judgments on the difference question.

[a] Problems 9 and 10 do not appear in the table because the difference question was not applicable to them.

Table 11

Mean Accuracy on the Anticipatory Inference Question in the Counting Experiment

	Problem											
Level	1	2	3	4	5	6	7	8	9	10	11	12
I	1.00	1.00	1.00	1.00	.90	.90	1.00	1.00	.80	.80	.50	.20
II	1.00	1.00	1.00	1.00	.90	.90	1.00	1.00	1.00	1.00	.60	.50
III	1.00	1.00	1.00	1.00	.90	.90	1.00	.80	1.00	1.00	1.00	.90

Note. Mean Accuracy was based on the proportion of correct judgments on the anticipatory inference question.

precisely negates the direct action of adding two objects to the collection.

The unit of analysis in this task was children's actions on the collections. An action consisted of any addition, subtraction, or transfer operation that was performed on objects in the collections.[1] Children also engaged in a variety of empirical procedures (e.g., counting the objects in a collection) and repair processes (e.g., relocating objects as a consequence of previous actions). These actions were recorded in order to supplement the primary analyses. Children's actions on each problem of the inverse task were coded according to a set of categories that are defined in the coding manual. The coding manual and a sample coding form are given in the Appendix. Reliability for coding children's actions on this task was determined by the procedure described in the method section, and interobserver agreement was 94%.

A total of 635 actions was coded over all levels. The frequency of actions was examined by level in order to determine whether a developmental change in the number of actions was related to a change in the accuracy of these actions. Contrary to the expectation that the number of actions would decrease over levels as children performed <u>only</u> the inverse operation(s) on each problem, an inverted U-shaped function was found. The number of actions at levels I and III was approximately equivalent with a considerable increase at level II. In order, the number of actions at levels I, II, and III were 204, 231, and 200.[2] The significance of this inverted U-shaped function is discussed below in regard to the processes by which children solved the inverse task.

The first issue concerns the nature and extent of young children's knowledge of the inverse operation. What do young children know about the inverse, and how does their knowledge of the inverse develop?

This issue was addressed by examining the accuracy of children's actions. In order to evaluate both the process and the product of undoing arithmetic operations, children's actions on each problem were scored separately for operation accuracy (process measure) and for end state accuracy (product measure).

Operation accuracy assesses knowledge of the inverse operation. Operation accuracy is defined as actions that perform the inverse, and only the inverse, of the direct arithmetic operation performed by the interviewer. For example, operation accuracy on problem 1, where 2 objects had been added to the child's collection, consisted of subtracting some objects from the child's collection. Note that operation accuracy does not require the quantity of the inverse operation to be the same as the quantity of the direct operation. For problem 1, adding 2 objects might be undone by subtracting 1 object, rather than 2 objects. Thus, operation accuracy does not assess precise knowledge of the inverse operation.

In contrast, end state accuracy assesses knowledge of the initial numerical state of a problem. End state accuracy is defined as any actions that restore the collections of a problem to their initial numerical states. For example, end state accuracy on problem 1 consisted of restoring the child's collection to exactly 8 objects. Note that this definition of end state accuracy does not require the child to perform only the inverse of the direct arithmetic operation performed by the interviewer. For problem 1, adding 2 objects might be undone by subtracting 3 objects and then adding 1 object, rather than only subtracting 2 objects. Thus, end state accuracy does not assess the process of performing the inverse operation, but rather the product of that operation.

Table 12 presents the accuracy of children's attempts to undo the outcome of the arithmetic operations. A 3(level) x 2(screening) x 2(problem: one operation vs. two operations) multivariate analysis of variance (MANOVA) was performed on the operation accuracy data. The analysis revealed a significant effect of problem, with greater accuracy on one-operation problems ($M = 1.80$) than on two-operation problems ($M = 1.37$), $F(1,24)=19.66$, $p < .001$. No other effects were significant.

The same type of analysis was performed on the end state accuracy data (see Table 12). The analysis revealed significant effects of problem, $F(1,24)=12.01$, $p < .005$, and of level, $F(2,24)=5.64$, $p < .01$. There was no significant effect of screening. Children were more accurate on one-operation problems ($M = 1.88$) than on two-operation problems ($M = 1.63$), and their accuracy increased over levels. Mean end state accuracy scores for levels I, II, and III were 1.59, 1.72, and 1.97, respectively.

In general, similar developmental patterns were obtained from the analyses of operation accuracy and end state accuracy. Accuracy on both one-operation problems and two-operation problems increased over levels. Moreover, a greater developmental increase in accuracy occurred on two-operation problems. These findings indicate that knowledge of the inverse operation undergoes more developmental change in respect to two-operation problems than one-operation problems.

Table 12

Mean Accuracy on the Inverse Task in the Counting Experiment

Level	Operation Accuracy[a]		End State Accuracy[b]	
	One-Operation Problems	Two-Operation Problems	One-Operation Problems	Two-Operation Problems
I	1.80	1.15	1.75	1.43
II	1.70	1.28	1.90	1.53
III	1.90	1.68	2.00	1.93

[a] Operation Accuracy scores ranged from 0-2 on each problem. Scores were assigned as follows: _0_ = not perform only the inverse operation on a collection; _1_ = perform the inverse operation with an incorrect quantity of elements on a collection; _2_ = perform the inverse operation with a correct quantity of elements on a collection.

[b] End State Accuracy scores ranged from 0-2 on each problem. Scores were assigned as follows: _0_ = not restore the initial numerical state of a collection; _2_ = restore the initial numerical state of a collection.

There was one notable difference between the patterns of operation accuracy and end state accuracy. The problem effect persisted at level III for operation accuracy but not for end state accuracy. Thus, although children at level III succeeded in restoring the collections of two-operation problems to their initial numerical states, they did not do so by performing _only_ the inverse of the direct arithmetic operation.

The next issue to be examined concerns the processes by which children solved the inverse task. In particular, an analysis of children's error patterns revealed the source of the problem effect at level III for operation accuracy. Children's errors were categorized into five principal types. Each type is defined below with an example from problem 1 (see Figure 1).

1. _Row_. The child performs operations on an irrelevant collection, or fails to perform operations on a relevant collection. For problem 1, subtracting (or adding) objects from the interviewer's collection constituted a row error.

2. _Operation_. The child performs operations on a relevant collection that are not the inverse of the direct operations performed by the interviewer. For problem 1, adding objects to the child's collection constituted an operation error.

3. _Inverse + Operation_. The child performs operations on a relevant collection that include the inverse, but not _only_ the inverse, of the direct operations performed by the interviewer. That is, the child performs both the inverse operation and an operation error on a particular collection. For problem 1, both subtracting

objects from the child's collection and adding objects to it constituted an inverse + operation error.

4. Quantity. The child performs inverse operations on a relevant collection that entail fewer or more than the precise quantity of objects entailed in the direct operations. For problem 1, subtracting 1 object, rather than subtracting 2 objects, from the child's collection constituted a quantity error.

5. Location. The child performs inverse operations at inaccurate places on a relevant collection. For problem 1, subtracting 2 objects from the right end of the child's collection, such that a rock is exposed under one of the objects, constituted a location error.

Table 13 presents the percentage of each type of error on the inverse task. This analysis revealed three developmental patterns of children's errors: (1) three error types that decreased over levels, (2) one error type that increased over levels, and (3) one error type that remained stable over levels.

The first pattern was comprised of row errors, operation errors, and quantity errors. All three error types decreased substantially over levels such that together they accounted for only 7% of the total errors at level III. The second pattern consisted of location errors. This error type increased markedly over levels. In contrast with the first pattern of errors, location errors accounted for 60% of the total errors at level III. Why was there an increase in location errors at level III? This increase in location errors can not be attributed to a decrement in children's knowledge of the inverse operation because both operation accuracy and end state accuracy increased, rather than decreased, at level III. A more plausible

Table 13

Percentage of Each Type of Error on the Inverse Task in the Counting Experiment

Level	Type of Error					
	Row	Operation	Inverse + Operation[a]	Quantity	Location	All Types[b]
I	13	14	26 (61)	14	32	100 (69)
II	13	13	35 (57)	15	25	100 (55)
III	0	5	33 (79)	2	60	100 (43)

[a] Numbers in parentheses indicate the percentage of Inverse + Operation errors which yielded end state accuracy on a collection.

[b] Numbers in parentheses indicate the total number of errors made by children at each level in the Counting Experiment.

explanation is that children represented the inverse operation more abstractly at level III than at previous levels. That is, they solved the inverse task by restoring problems to their initial numerical states without restoring them to their initial configurations.

The third pattern consisted of inverse + operation errors. Recall that these errors are due to performing both correct (i.e., inverse) and incorrect operations on a given collection. This error type remained stable over levels. Children frequently committed inverse + operation errors, a substantial proportion of which yielded end state accuracy (see Table 13). Moreover, there was an increase at level III in the proportion of inverse + operation errors that yielded correct end states. The source of this developmental increase is related to the analysis of mental arithmetic explanations on the judgment task (see Table 9). Children at level III made a greater percentage of accurate arithmetic computations than children at previous levels. Thus, converging evidence from the judgment task and the inverse task indicates that level III children used computational procedures more frequently and more accurately to solve these arithmetic problems.

The significance of this pattern of inverse + operation errors is that it reveals how children employed a trial-and-error process to solve the inverse task. Their objective was to restore the <u>counted number</u> of objects in the initial state of a collection, and they did so by performing a series of arithmetic operations that were often accompanied by a counting procedure. For example, on problem 1, the child might subtract 3 objects from his or her collection, count the collection (and presumably detect the error), add 1 object to the collection, and then count the collection again. Thus, children used a counting procedure to drive their repair of an arithmetic error (i.e., add 1 object) or to verify the outcome of their repair (i.e., the counted number of objects in the initial collection).

Further evidence that children employed a trial-and-error process was obtained from an analysis of their empirical procedures on the inverse problems (see Appendix for a complete description of all types of empirical procedures). Children at all levels used empirical procedures, predominantly counting, on more than one-third of the problems. Moreover, the frequency of counting was equivalent at levels I and III, but it increased 60% at level II. This finding suggests that level II children engaged in more trial-and-error solutions than children at the other two levels. Such a conclusion is consistent with the inverted U-shaped function for the total number of actions performed by children at each level.

Taken together, the error data indicate that children constructed a numerical representation of the initial collections through counting, and this numerical representation guided their solution process on the inverse task. Children attempted to restore the cardinal values of the initial collections rather than to perform the precise inverse of the direct arithmetic operation. Most importantly, children continued to construct numerical representations of the collections by counting them during the solution process in order to verify their arithmetic operations and to direct their repairs.

The final issue to be considered is whether young children's knowledge of the inverse operation is an isolated ability, or whether it is part of an early arithmetic system. Is children's knowledge of a direct arithmetic operation related to their knowledge of the inverse of that direct operation during the course of development? In order to answer this question, the consistency of children's performance between the judgment task (direct operation) and the inverse task (inverse operation) for a subset of problems was examined.[3]

The accuracy of children's judgments and inverse actions on each problem in this analysis was jointly categorized as consistent (i.e., pass both tasks or fail both tasks) or inconsistent (i.e., pass one task and fail one task). Table 14 presents the percentage of children at each level exhibiting consistent and inconsistent performance patterns on these two tasks. Table 15 presents the same data for the four types of problems over all levels. In general, the most striking finding in both tables is the high percentage of consistent performance at all levels and problem types. However, it can be seen in Table 14 that children at level III were more consistent in their performance on these two tasks than children at levels I and II. The reason for this trend was revealed in Table 15.

On problem pair 4 and 5 (inequality one-way function problems), 50% of the children exhibited an inconsistent pattern such that they passed the inverse task but failed the judgment task. In order to understand the basis of their inconsistent performance on problems 4 and 5, it is necessary to compare the demands of the judgment task and the inverse task for these problems. Recall that accuracy on the judgment task was significantly lower for inequality function problems than for the other types of problems because children at levels I and II frequently employed a nonintegrative, operation strategy on these problems. This finding indicates that inequality one-way function

Table 14

Percentage of Children at Each Level Exhibiting Consistent and Inconsistent Performance on the Judgment Task and the Inverse Task in the Counting Experiment

	Consistent Performance	Inconsistent Performance	
Level	Pass Judgment-Inverse Fail Judgment-Inverse	Pass Judgment-Fail Inverse	Fail Judgment-Pass Inverse
I	73	8	20
II	63	13	25
III	88	3	10

Note. Performance on the inverse task was based on end state accuracy scores for each problem.

[a] Problems 3, 7, 6, and 8 were not included in this analysis.

Table 15

Percentage of Children Exhibiting Consistent and Inconsistent Performance by Problems on the Judgment Task and the Inverse Task in the Counting Experiment

	Consistent Performance	Inconsistent Performance	
Problems[a]	Pass Judgment-Inverse Fail Judgment-Inverse	Pass Judgment-Fail Inverse	Fail Judgment-Pass Inverse
1 and 2	100	0	0
4 and 5	50	0	50
9 and 10	77	17	7
11 and 12	70	13	17
All Problems	74	8	18

Note. Performance on the inverse task was based on end state accuracy scores for each problem.

[a] Problems 3, 7, 6, and 8 were not included in this analysis.

problems were difficult on the judgment task because children were unable to integrate the inequality relation between the initial collections with the arithmetic operation on these collections. However, inequality one-way function problems did not pose the same difficulty on the inverse task. Children were simply required to undo one arithmetic operation (i.e., addition or subtraction) in order to be accurate on these problems, and this requirement was not incompatible with their nonintegrative, operation strategy. Therefore, given the different demands of the two tasks, it should not be surprising that children passed problems 4 and 5 on the inverse task but failed these problems on the judgment task.

In conclusion, children at all levels generally exhibited consistent performance between the judgment task and the inverse task for all problem types except one. This finding supports the hypothesis that children's knowledge of the inverse operation is not an isolated ability, but instead is coordinated with their knowledge of the direct operation within a developing arithmetic system.

Footnotes

[1] In this definition of an action, the term "operation" is used in a mathematical sense rather than a psychological sense. It refers to the addition or subtraction of objects in the collections. This definition is not intended to describe the psychological processes underlying children's actions on the inverse task.

[2] The minimum number of actions per level required to perform only the inverse operations on the set of 12 arithmetic problems was 180.

[3] Problems 3, 7, 6, and 8 (equality and inequality two-way function problems) were not included in this analysis because each of these problems entailed both a direct and an inverse operation (i.e., subtraction-addition transfer) on the judgment task. Therefore, it was not appropriate on these problems to examine consistency of performance between the judgment task and the inverse task as a measure of children's understanding of direct and inverse arithmetic operations.

4

FIGURAL CORRESPONDENCE EXPERIMENT

The second experiment provides a further examination of the role of numerative activities in the development of arithmetic reasoning. The focus of this experiment is on the developmental relations between understanding the numerative activity of establishing one-to-one correspondence between objects and understanding the logical forms of arithmetic operations. This numerative activity can be contrasted with the numerative activity of counting in order to address the general question of the study: How do young children use numerative activities to construct numerical representations of arithmetic problems and to reason about different logical forms of arithmetic operations on these problems?

An investigation of children's ability to use their one-to-one correspondence activity to reason about arithmetic operations is of interest for several reasons. First, an understanding of one-to-one correspondence between objects is an integral component of psychological theories of number development (e.g., Gelman, 1982; Gréco, 1962a; Piaget, 1965). Second, although recent studies have documented the invention of counting procedures (Fuson, 1982; Steffe, Thompson, & Richards, 1982) and body-part procedures (Saxe, 1981, 1982) to solve arithmetic problems, little is known about the invention of computational procedures that are based on one-to-one correspondence between objects. Finally, the use of one-to-one correspondence between objects as a numerative activity on arithmetic problems affords an opportunity to examine the influence of configurational properties on children's arithmetic reasoning.

It can be argued that the activity of establishing one-to-one correspondence between objects leads children to construct a form of numerical representation of the arithmetic problems which entails a spatial medium. Furthermore, reasoning about arithmetic problems that are represented in a spatial medium is more subject to the influence of configurational properties than reasoning about arithmetic problems that are represented in a nonspatial medium. This argument suggests a two-part hypothesis: (1) configurational properties of the final collections should influence children's arithmetic reasoning in the figural correspondence experiment but _not_ in the counting experiment, and (2) configurational properties should influence children's arithmetic reasoning in the figural correspondence experiment _only_ under post-transformation screening in which the final collections are perceived by the children.

The general procedure of the figural correspondence experiment was identical to that of the counting experiment. The same pretest was administered to all children in the present experiment. It assessed their understanding of counting and one-to-one correspondence between objects as numerative activities, and it also categorized their logical functioning at one of three levels of number development. The experimental interview which followed the pretest assessed children's arithmetic reasoning on the same set of problems in respect to the judgment task and the inverse task.

Children's arithmetic reasoning was analyzed with regard to four issues that parallel those of the counting experiment. First, does the numerative activity of establishing one-to-one correspondence between objects yield a particular form of numerical representation of the arithmetic problems, and does this form change over levels as children's understanding of one-to-one correspondence develops? Second, what types of arithmetic inferences are children able to make at different levels, and how do the dimensions of the arithmetic problems influence children's ability to make these inferences? Third, how do children at different levels use their one-to-one correspondence activity to make arithmetic inferences on the judgment task? Finally, what inverse operations are children able to perform at different levels, and how does the numerative activity of establishing one-to-one correspondence between objects influence children's solution processes on the inverse task?

Method

Subjects

Thirty preschool children participated in the experiment. Children ranged in age from 4 years-4 months to 6 years-6 months with a mean age of 5 years-5 months. Four additional children were excluded from this experiment. Three of them were excluded because their pretest performance was deficient. Specifically, two children (ages 4-2 and 5-0) were incorrect on the counting questions of the numerical comparison task, and one child (age 4-10) was incorrect on the comparison question of the same task. The fourth child (level I, age 4-5) had succeeded on the criterial pretest tasks but refused to complete the first arithmetic problem. All children attended preschools in the Eugene, Oregon community, and they were from predominantly white, middle-class backgrounds.

The pretest was administered to all children in the experiment. On the basis of their performance on the pretest, children were assigned to one of three levels of number development: Level I (mean age = 4-9; range = 4-4 to 5-3), Level II (mean age = 5-9; range = 4-11 to 6-5), and Level III (mean age = 5-10; range = 4-5 to 6-6). There were 10 children at each level equally divided between boys and girls. Analyses of the age differences between levels revealed that children at level II were significantly older (*M* = 5-9) than children at level I (*M* = 4-9), *t*(18) = 4.69, $p < .01$, and children at level III were also significantly older (*M* = 5-10) than children at level I (*M* = 4-9), *t*(18) = 5.05, $p < .01$. There was no significant age difference between children at level II and children at level III.

Design

The design and counterbalancing procedures of Experiment 2 (figural correspondence experiment) were identical to those of Experiment 1 (counting experiment).

Materials

The materials were the same as those in Experiment 1 with the exception of the boards on which the pair of initial collections comprising each problem were displayed. It was necessary to use another set of boards for the second experiment because the configurations of the arithmetic problems differed from the first experiment. Since the size and configuration of the initial collections varied across problems, each problem was presented on an unique board. Grey rectangular markers were affixed to each board according to the constraints described in Experiment 1. Recall that the interviewer placed dinosaurs in all locations specified by markers and, in the case of subtraction problems, the interviewer also placed dinosaurs in certain locations specified by lines drawn underneath the boards.

The principal consideration that determined the configurations of the arithmetic problems in this experiment was the salience of figural one-to-one correspondence between objects in the pairs of initial collections. Accordingly, the configurations were designed to facilitate the numerative activity of establishing one-to-one correspondence between objects by which children represented numerical information about the initial collections of dinosaurs. Two other considerations determined the configurations of the arithmetic problems. First, the configurations of the initial collections contained spatial alignment cues which supported accurate numerical comparisons of the initial collections. Second, the configurations of the final collections contained spatial alignment cues which conflicted with accurate arithmetic inferences. The purpose of these cues was to identify children's judgments that were based on the configurations of the final collections rather than on arithmetic inferences about these collections.

The configurations of the initial and final collections for the arithmetic problems in this experiment are depicted in Figure 2. In view of the above considerations, the pairs of initial collections for equality problems were constructed such that all objects in one collection were in figural one-to-one correspondence with the objects in the other collection. The pairs of initial collections for inequality problems were constructed such that all objects were in

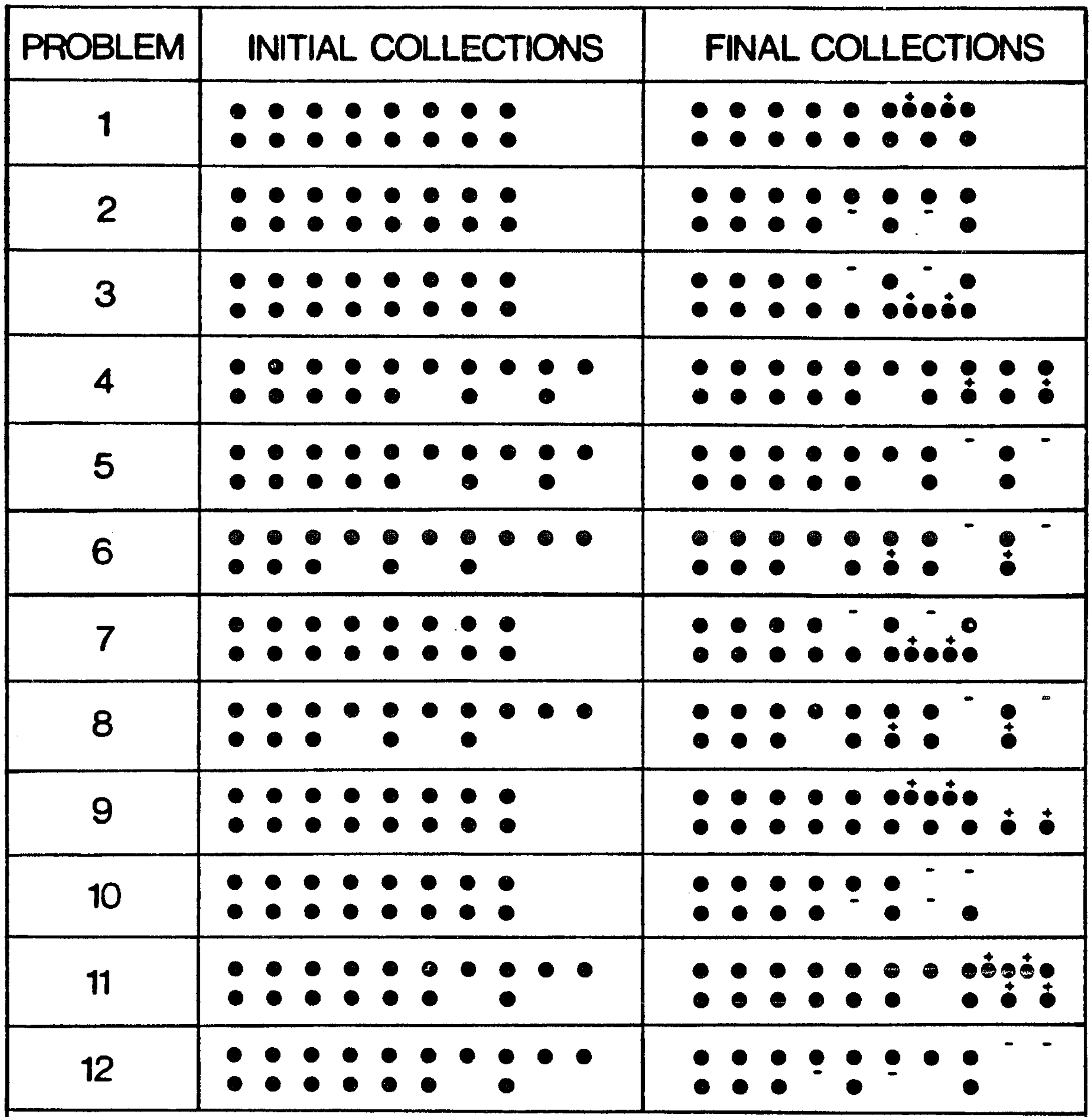

Figure 2. Configurations of the initial collections and final collections for the arithmetic problems in Experiment 2. (For each pair of collections, the top row denotes the child's collection and the bottom row denotes the interviewer's collection. + indicates the location of an addition operation on a collection, and - indicates the location of a subtraction operation on a collection.)

figural one-to-one correspondence except those objects comprising the numerical difference between the two collections. Moreover, spatial alignment was a consistent cue for making accurate numerical comparisons of the initial collections. For equality problems, the end-points of each pair of initial collections were aligned. For inequality problems, the end-points were not aligned, and the collection with more objects was longer than the collection with fewer objects.

In contrast, spatial alignment was a misleading cue with respect to making accurate arithmetic inferences about the final collections. When addition or subtraction operations produced equal collections (problems 9 and 10), the end-points of each pair of final collections were not aligned, and one collection was randomly chosen to be longer than the other (the particular collection chosen was counterbalanced within each pair across problems). Conversely, when addition or subtraction operations produced unequal collections (problems 1-8, 11, and 12), the end-points were aligned.

Procedure

All aspects of the general procedure, including the multiple testing sessions and the experimental interview administered in two phases (i.e., an initial interview and a final interview), were identical to those of Experiment 1. The same pretest was also administered to children in Experiment 2. The critical differences between the two experiments involved the initial interview on the addition and subtraction problems. The following section describes the experimental interview of Experiment 2, but only those procedures which differed from Experiment 1 are presented in detail.

Initial interview. The initial interview required the child to look at the pair of collections and then to compare them. The purpose of this interview was to direct the child to use figural one-to-one correspondence between objects to construct a numerical representation of the initial collections.

Two constraints were imposed on the child's performance during the initial interview. The first constraint was that the child was not permitted to count the initial collections. This was done in order to ensure the use of *different* numerative activities in the first and second experiments. If the child attempted to count any objects in a collection, the interviewer immediately screened the initial collections and explained that this game was played without counting. Then the interviewer removed the screens, and administered the initial interview on the problem. Only two children in the experiment attempted to count the objects in the initial collections, and neither persisted after the first problem.

The second constraint on the child's performance followed from the definition of an arithmetic inference. Recall that an arithmetic inference is defined as a numerical judgment about the final collections which is based on the integration of a numerical judgment about the initial collections with an addition or subtraction operation on these collections. As in Experiment 1, it was necessary to ensure that the child's judgment about the initial collections was accurate. Therefore, the configurations of the initial collections contained spatial alignment cues to facilitate accurate numerical comparisons. Moreover, if the child made an error on any initial interview question, the interviewer provided feedback and then repeated the question.

The initial interview began with instructions to observe the objects in each of the collections: (1) "Look carefully at the dinosaurs in your cave." (2) "Now, look carefully at the dinosaurs in my cave." Then the child was asked to make a numerical comparison of the initial collections, "Do we have the same number of dinosaurs, or do you have more dinosaurs, or do I have more dinosaurs?" Finally, an explanation was sought in order to determine the basis for the child's numerical comparison, "How do you know?" Note that the comparison and explanation questions were the same as those in the initial interview of Experiment 1.

Final interview. After completing the initial interview on a problem, the interviewer performed an addition or subtraction operation on one or both of the collections. The final interview examined the child's arithmetic reasoning on a judgment task and on an inverse task, and the judgment task preceded the inverse task on all problems.

The judgment task assessed the ability to make an inference about the outcome of an arithmetic operation. This task was identical in most respects to the judgment task in Experiment 1, but there was one difference in the procedure for the pre-transformation screening condition. Recall that in Experiment 1 two cardinality questions ("How many dinosaurs are there in your/my cave?") from the initial interview were repeated after the collections had been screened. These questions were repeated to ensure that the child had not forgotten his or her numerical information about the initial collections. However, in Experiment 2 the child did not possess cardinality information about the initial collections. Therefore, the interviewer repeated the numerical comparison question from the initial interview, "Did you say that we have the same number of dinosaurs, or do you have more dinosaurs, or do I have more dinosaurs?" If the child made an error on this question, the screens were removed and the initial interview on the problem was administered again. Only a few children made errors on the memory probe, and none of them required more than one repetition of the initial interview. All other aspects of the procedure for the judgment task and the inverse task were the same as in Experiment 1.

To summarize, the interview on the judgment task consisted of five questions. Specifically, it included the standard inference question with a request for an explanation, the difference question (if appropriate), and the anticipatory inference question with another request for an explanation. The interview on the inverse task consisted of two questions. It began with a request to "fix the dinosaurs to make it the way it was at the beginning of this dinosaur game." This question was followed by a request for an explanation in order to determine the basis for the child's actions on a problem.

No corrective feedback was provided to the child during the final interview. Each child's responses on the judgment task and the inverse task were recorded by a trained observer. In addition, the judgment task was recorded on audiotape in order to verify the observer's record. Reliability for coding actions on the inverse task was determined by the procedure described in Experiment 1. Interobserver agreement for Experiment 2 was 94%.

Results and Discussion

Initial Interview

In the initial interview on a problem, the child was required to look at the pair of collections and make a numerical comparison of the collections. Then the interviewer asked the child to provide an explanation for this judgment. Recall that if the child made an error on the numerical comparison question, it was repeated. However, few errors were expected on this question because all children in the experiment had passed the numerical equivalence task of the pretest. This expectation was supported by the lack of errors on the numerical comparison question.

Children's explanations for their numerical comparisons provided evidence about how they initially represented the arithmetic problems. Specifically, these explanations were examined with respect to three issues. First, does the numerative activity of establishing one-to-one correspondence between objects in the arithmetic problems result in a particular form of numerical representation of these problems? Second, if one-to-one correspondence between objects results in a particular form of numerical representation, does this form change over levels as children's understanding of one-to-one correspondence between objects develops? Third, how does the initial numerical representation constructed by children influence their subsequent reasoning on the judgment task and the inverse task?

Children's explanations were classified into seven types: (1) counting activity, (2) cardinal number, (3) spatial configuration, (4) correspondence relation, (5) numerical difference, (6) combined explanation, and (7) irrelevant or none. All types of explanations were the same as in Experiment 1, with the exception of the correspondence relation explanation which was more differentiated in Experiment 2.

A correspondence relation explanation is defined as a verbal and/or gestural reference to the relation between objects in one collection, and the presence or absence of objects in the other collection. Correspondence relation explanations can take two forms, and both forms occurred in this experiment.

Object-to-object correspondence explanations occurred only on equality problems. This type of explanation makes reference to the presence of objects in one collection that corresponds to the presence of objects in the other collection. "Because this one's straight behind that one..." (the child successively points to objects in the interviewer's collection and to corresponding objects in the child's collection).

Object-to-gap correspondence explanations occurred only on inequality problems. This type of explanation makes reference to the presence of objects in one collection that corresponds to the absence of objects in the other collection (i.e., a spatial "gap"). Four kinds of object-to-gap explanations were observed, the principal difference among them being the degree of quantification of the gaps. First, in global gap explanations the child treats the gap as a relatively undifferentiated space in a collection (e.g., "There's a big spot here and there's a big spot here"). Second, in explanations that entail a successive iteration of gaps, the child treats the gap as a discrete iterable unit. However the child does not perform any summation of the iterated gaps (e.g., "None there, there, there, there, there"). Third, in cardinal number gap explanations the child performs a summation of the iterable gaps (e.g., "I got all those and you got 3 spaces"). Finally, in explanations that combine successive iteration and cardinal number of gaps, the child iterates some gaps and performs a summation of the other iterable gaps (e.g., "This doesn't have 1 partner, this doesn't have 1 partner, and these 3 don't have partners").

The general pattern of results from children's explanations was similar to that of the counting experiment. It was revealed that establishing one-to-one correspondence between objects in the initial collections led children to construct a particular form of numerical representation of the arithmetic problems. Moreover, developmental changes in this form of numerical representation were also observed. By level II, the form of representation entailed a fundamental property of the one-to-one correspondence activity, namely the correspondence relation. Nevertheless, the form of the numerical representation in the figural correspondence experiment differed markedly from that in the counting experiment. These results are indicated in Table 16 which presents the percentage of each type of explanation on the initial interview.

There were two predominant types of explanations over all levels that accounted for 87% of the total number of explanations given on the initial interview. Spatial configuration explanations were most frequent at the first level, whereas correspondence relation explanations were most frequent at the second and third levels.

Table 16

Percentage of Each Type of Explanation

on the Initial Interview in the Figural Correspondence Experiment

	Type of Explanation							
				Correspondence Relation				
Level	Counting Activity	Cardinal Number	Spatial Conf.[a]	Object-Object	Object-Gap	Numerical Difference	Combined	Irrel. or none[b]
I	0	3	55	14	9	0	3	15
II	0	0	27	37	33	1	2	0
III	0	0	1	48	35	9	7	0
All Levels	0	1	28	33	26	4	4	5

Note. Percentages were based on 120 explanations at each level.

[a] Spatial Configuration

[b] Irrelevant or none

Children at level I referred to the spatial configuration of the collections in more than half of their explanations. Although they mentioned the correspondence relation between objects in 25% of the explanations at this level, it was not the preferred basis for children's numerical comparisons at level I. In contrast, the correspondence relation explanation became dominant at level II, and this type of explanation continued to increase in frequency through level III.[1] Thus, there was a developmental shift from spatial configuration explanations at level I to correspondence relation explanations at levels II and III, but the shift was more gradual than in the counting experiment.

This developmental shift in the types of explanations given on the initial interview reflected two trends in children's use of one-to-one correspondence between objects as a numerative activity: (1) increasing differentiation and coordination of the correspondence relation, and (2) increasing quantification of the correspondence relation. Each of these trends deserves comment because it indicates how the development of children's understanding of one-to-one correspondence between objects influenced their numerical representation of the arithmetic problems.

The first trend was an increasing differentiation and coordination of the correspondence relation in children's one-to-one correspondence activity. The shift in their explanations on inequality problems provided clear evidence for this trend. Children at level I gave spatial configuration explanations, children at level II mentioned the successive iteration of global or discrete gaps, and children at level III referred to a summation of discrete gaps. Thus, children's explanations reflected a differentiation of the correspondence relation from level I to level II, and then a coordination of the correspondence relation into a summation at level III.

The second trend was a greater quantification of the correspondence relation over levels. There was a steady increase in the proportion of object-to-gap explanations that referred to a partial or total summation of gaps in terms of a cardinal number, from 18% at level I to 33% at level II to 49% at level III. Moreover, numerical difference explanations that referred to the relation between two unequal collections in terms of a cardinal number of gaps were rare before level III (see Table 16).

Taken together, these trends indicate that children imposed a numerical structure on the configurations of the arithmetic problems through their one-to-one correspondence activity. However, the ability to do so emerged over levels. This evidence that children developed the ability to use one-to-one correspondence between objects as a numerative activity on the problems is significant for several reasons. First, it supports the view that children's understanding of one-to-one correspondence between objects influenced their numerical representation of the problems. Second, it demonstrates how children at different levels adapted their one-to-one correspondence activity to the problem-solving demands of the task. Finally, it suggests that the integration of counting and one-to-one correspondence activities, manifested in children's explanations at level III, reflected a more abstract understanding of number.

In conclusion, establishing one-to-one correspondence between objects in the initial collections led children to construct a form of numerical representation that was particular to this numerative activity. Children's explanations indicated that they represented the summation of objects in equal collections as object-to-object correspondence relations, and they represented the summation of objects in unequal collections as object-to-gap correspondence relations. Nevertheless, knowledge of these two forms of correspondence relations, which was evident in three-fourths of the explanations at level II and almost all of the explanations at level III, was evident in only one-fourth of the explanations at level I. Thus, the form of numerical representation that was revealed by these explanations changed over levels as children developed an understanding of the implications of their one-to-one correspondence activity.

Judgment Task

Standard inference question: Judgments. The judgment task assessed children's ability to make an inference about the outcome of an arithmetic operation. Children's judgments on this question were scored as accurate/inaccurate, and a set of analyses was conducted to determine the effect of the principal factors in the experiment on judgment accuracy. Each of these analyses is discussed below. The discussion addresses the types of arithmetic inferences that children make at different levels, and the strategies that underlie both their accurate and inaccurate inferences.

Table 17 presents the mean accuracy on the standard inference question for each problem in the experiment. The two patterns that

Table 17

Mean Accuracy on the Standard Inference Question for Each Problem in the Figural Correspondence Experiment

	Problem											
Level	1	2	3	4	5	6	7	8	9	10	11	12
I	1.00	1.00	.90	.50	.40	.30	.90	.30	.50	.80	.50	.30
II	1.00	1.00	1.00	.90	.60	.50	1.00	.70	.90	.70	1.00	.80
III	1.00	1.00	1.00	1.00	.80	.40	1.00	.60	1.00	.90	1.00	.90

Note. Mean Accuracy was based on the proportion of correct judgments on the standard inference question.

were noted in Experiment 1 (see Table 4) are also evident in this experiment. Specifically, the accuracy of children's judgments was comparable on pairs of problems which differed by the particular arithmetic operation performed within problem types. In addition, there was no difference in accuracy between the pairs of standard and control problems within problem type III (problems 7 and 3) and problem type IV (problems 8 and 6). Thus, performance was summed over individual problems comprising the same problem type in the following analyses.

A 3(level) x 2(screening) ANOVA on judgment accuracy over all problems revealed several results that were similar to those of Experiment 1. There was a highly significant effect for level, $\underline{F}(2,24)=13.78$, $\underline{p} < .001$, and no significant effect for screening condition. However, the pattern of the level effect was different in this experiment. Children at levels II and III made more correct judgments than children at level I. Mean proportions of correct judgments at levels I, II, and III were .62, .84, and .88, respectively. Thus, there was a significant improvement in judgment accuracy at level II in this experiment, whereas the significant improvement in judgment accuracy was at level III in the counting experiment. This difference in the pattern of the level effect between the two experiments was found in all of the accuracy analyses on the standard inference question for the judgment task.

The next two analyses examined the influence of the dimensions of the arithmetic problems on children's ability to make arithmetic inferences. The effect of the logical form of the problem on judgment accuracy was addressed in the first analysis. Table 18 reports the mean accuracy on the standard inference question for the three logical forms of problems. A 3(level) x 2(screening) x 3(problems: one-way function vs. two-way function vs. size axiom) MANOVA was performed on judgment accuracy with an arcsin transformation of the data. As in the counting experiment, control problems 3 and 6 under problem types III and IV, respectively, were not included in this analysis. The analysis revealed two significant effects, level and screening condition, but there was no significant effect for problems.

First, the effect for level revealed that children at levels II and III made more correct judgments on the problems than did children at level I, $F(2,24)=14.37$, $p < .001$. Mean proportions of correct judgments for levels I, II, and III were .62, .86, and .90, respectively. Second, there was an effect for screening condition, $F(1,24)=4.37$, $p < .05$. Children made more correct judgments under the pre-transformation screening condition ($M = .84$) than under the post-transformation screening condition ($M = .74$). However, an inspection of the regression weights for the pre-transformation and post-transformation screening conditions at each level revealed that the screening effect was limited to levels I and II. Mean proportions of correct judgments under pre-transformation screening and post-transformation screening, respectively, were .72 and .52 for level I, and .93 and .78 for level II.

The screening effect obtained in this experiment, but not in the counting experiment, supported a principal hypothesis of the study: Perceiving the configurations of the final collections, which contain misleading spatial alignment cues, should influence the accuracy of children's judgments in the figural correspondence experiment but <u>not</u>

Table 18

Mean Accuracy on the Standard Inference Question

for One-Way Function, Two-Way Function, and Size Axiom Problems

in the Figural Correspondence Experiment

	Logical Form of Problem							
	One-Way Function[a]		Two-Way Function[b]				Size Axiom[c]	
Level	Equality	Inequality	Equality-S	Equality-C	Inequality-S	Inequality-C	Equality	Inequality
I	1.00	.45	.90	.90	.30	.30	.65	.40
II	1.00	.75	1.00	1.00	.70	.50	.80	.90
III	1.00	.90	1.00	1.00	.60	.40	.95	.95

Note. Mean Accuracy was based on the proportion of correct judgments on the standard inference question.

[a] One-Way Function Problems included Equality problems 1, 2 and Inequality problems 4, 5.

[b] S = standard form of problem; C = control form of problem. Two-Way Function Problems included Equality-S problem 7, Equality-C problem 3, Inequality-S problem 8, and Inequality-C problem 6.

[c] Size Axiom Problems included Equality problems 9, 10 and Inequality problems 11, 12.

in the counting experiment. This hypothesis is based on the argument that the activity of establishing one-to-one correspondence between objects results in a form of numerical representation which entails a spatial medium. Reasoning about arithmetic problems that are represented in a spatial medium is subject to the influence of configurational properties of the final collections such as misleading spatial alignment cues. In contrast, the activity of counting objects results in a form of numerical representation which entails a linguistic medium. Thus, reasoning about arithmetic problems that are represented in a nonspatial medium is less subject to the influence of configurational properties of the final collections.

In summary, the screening effect was consistent with two predictions that follow from the preceding argument. First, the screening effect was obtained in the figural correspondence experiment but not in the counting experiment. Second, the direction of the screening effect (i.e., accuracy was greater under pre-transformation screening than under post-transformation screening) indicated that perceiving the configurations of the final collections reduced the accuracy of children's judgments at levels I and II. Thus, converging evidence from the screening effect on the judgment task and the spatial configuration explanations on the initial interview supports the view that the activity of establishing one-to-one correspondence between objects results in a form of numerical representation which entails a spatial medium.

The first analysis of the dimensions of the arithmetic problems did not reveal any significant effect for the logical form of the problem. Recall that the same result was obtained in the counting experiment, and it was interpreted that the logical dimension alone can not account for children's accuracy on the different types of problems. Accordingly, the joint effect of the logical form of the problem and the quantitative relationship between the initial collections of the problem was examined in the next analysis.

A 3(level) x 2(screening) x 4(problems: inequality one-way function vs. inequality two-way function vs. equality size axiom vs. inequality size axiom) MANOVA was performed on judgment accuracy. Table 19 presents the data in this analysis. Note that equality one-way function problems and equality two-way function problems were not included in this MANOVA because children's performance was at ceiling on these problems.

This analysis yielded both significant simple effects and interaction effects. Consider the simple effects first. There were highly significant effects for level, $F(2,24)=10.84$, $p < .001$, and for problems, Hotelling T^2 (critical value=9.9799) =30.14, $p < .001$. The interpretation of these effects was straightforward. With respect to the effect for problems, children's accuracy was greater for both size axiom problems and inequality one-way function problems than for inequality two-way function problems. In order, mean proportions of correct judgments for equality size axiom, inequality size axiom, inequality one-way function, and inequality two-way function problems were .80, .75, .70, and .47. The effect for level was the same as in the preceding analyses. Children at level II (M = .76) and level III (M = .85) were more accurate than children at level I (M = .45).

Table 19

Mean Accuracy on the Standard Inference Question under the Pre-Transformation and Post-Transformation Screening Conditions in the Figural Correspondence Experiment

		Logical Form of Problem			
		One-Way Function[a]	Two-Way Function[b]	Size Axiom[c]	
Level	Screen Condition	Inequality	Inequality	Equality	Inequality
I	Pre	.50	.50	1.00	.40
	Post	.40	.10	.30	.40
II	Pre	.70	.70	1.00	.90
	Post	.80	.50	.60	.90
III	Pre	.90	.20	1.00	1.00
	Post	.90	.80	.90	.90

Note. Mean Accuracy was based on the proportion of correct judgments on the standard inference question.

[a] One-Way Function Problems included Inequality problems 4, 5; Equality problems 1, 2 do not appear in the table because they were not included in the multivariate analysis.

[b] Two-Way Function Problems included Inequality problems 6, 8; Equality problems 3, 7 do not appear in the table because they were not included in the multivariate analysis.

[c] Size Axiom Problems included Equality problems 9, 10 and Inequality problems 11, 12.

In addition, there were two interaction effects that bear on the interpretation of the simple effect for problems. The first interaction was a significant effect for Screening Condition x Problems, Hotelling T^2 (critical value = 9.9799) = 16.56, $p < .05$. It can be seen in Table 19 that the screening condition influenced children's ability to make arithmetic inferences on particular problem types. Specifically, judgment accuracy for equality size axiom problems over all levels was greater under the pre-transformation screening condition (M = 1.00) than under the post-transformation screening condition (M = .60). The screening condition also influenced judgment accuracy for inequality two-way function problems, but the direction of the screening effect was not the same at all levels. This finding was clarified by examining the second interaction effect.

A significant effect was obtained for Level x Screening Condition x Problems, Largest Root Criterion (S = 2.0, M = 0.0, N = 10.0, critical value = 7.1626) = 10.08, $p < .05$. This effect is evident in Table 19. The general finding for both equality size axiom problems and inequality two-way function problems was that children made more correct judgments under the pre-transformation screening condition than under the post-transformation screening condition at levels I and II, but not at level III.

The particular pattern of this effect differed for these two problem types, however. For equality size axiom problems, there was a substantial difference in children's accuracy under the two screening conditions at level I. This difference decreased at level II, and it disappeared by level III. In comparison, for inequality two-way function problems there was also a large difference in children's accuracy under the two screening conditions at level I. The magnitude of this difference decreased at level II, and the direction of this difference was reversed at level III. That is, children at level III made more correct judgments under the post-transformation screening condition than under the pre-transformation screening condition. Thus, the basis of this interaction effect for both problem types was children's developing ability to make arithmetic inferences under the post-transformation screening condition.

To summarize the results of the judgment analyses, children's ability to make an arithmetic inference emerged over levels with the greatest improvement in accuracy at level II. However, their ability was not uniform for all types of arithmetic problems. Children's performance was at ceiling for the equality one-way function problems and the equality two-way function problems. Among the remaining four problem types, accuracy was greater for both size axiom problems and inequality one-way function problems than for inequality two-way function problems.

Children's ability to make an arithmetic inference was also influenced by the screening condition in this experiment. In general, children at levels I and II made more correct judgments under the pre-transformation screening condition than under the post-transformation screening condition. A more precise picture of the influence of these three factors -- level, screening condition, and problems -- on judgment accuracy was revealed by the complex interaction effect. The screening condition only affected judgment accuracy for particular problem types, equality size axiom problems and inequality two-way function problems, and the direction of the screening effect varied over levels. Thus, at level I judgment accuracy for both problem types was reduced under the post-transformation screening condition. In contrast, at level III this screening condition did not differentially affect judgment accuracy for equality size axiom problems, and it even facilitated judgment accuracy for inequality two-way function problems.

The principal conclusion to be drawn from these findings is that the influence of perceiving the final collections on the ability to make an arithmetic inference was jointly determined by two factors. The first factor was understanding the arithmetic operations performed on the collections, and the second was utilizing the configurational properties of the final collections to solve the judgment task. Each of these factors contributed to the interaction effect that was obtained in this experiment.

The role of the first factor, understanding the arithmetic operations, can be illustrated by children's performance on equality size axiom problems. Recall that at levels I and II children's accuracy on these problems was at ceiling under the pre-transformation screening condition, but it was significantly lower under the post-transformation screening condition (see Table 19). However, at level III children's accuracy on these problems was at ceiling under both screening conditions.

This developmental trend indicates that children at levels I and II did not fully understand the logical form of the arithmetic operations in equality size axiom problems (i.e., an equivalence relation between two collections that remains invariant under the transfer of equal numbers of elements to/from the two collections). Thus, their arithmetic inferences on these problems were easily disrupted by the presence of misleading configurational cues in the final collections, resulting in lower accuracy under the post-transformation screening condition. In contrast, children at level III demonstrated a full understanding of the arithmetic operations in equality size axiom problems by their ability to make accurate arithmetic inferences despite misleading configurational cues. Taken together, this evidence on the development of size axiom inferences for equivalence relations is consistent with the research on elementary recurrential reasoning (Inhelder & Piaget, 1963) as well as with the screening studies on early conservation inferences (Bruner, Olver, & Greenfield, 1966; Inhelder et al., 1974).

The second factor that determined children's ability to make an arithmetic inference under the post-transformation screening condition involved utilizing the configurational properties of the final collections to solve the task. Children's performance on inequality two-way function problems illustrates this factor. It was revealed at level I that although children's accuracy on these problems was only moderate under the pre-transformation screening condition, their accuracy was still significantly lower under the post-transformation screening condition (see Table 19). This difference in performance decreased at level II such that children's accuracy was moderate under both screening conditions. However, the most intriguing finding was a reversal of the direction of this difference at level III. Children's accuracy on these problems approached ceiling under the post-transformation screening condition, but it was significantly lower under the pre-transformation screening condition.

The interpretation of this developmental trend is twofold. First, children did not fully understand the logical form of the arithmetic operations in inequality two-way function problems (i.e., a dependence relation between the number of elements transferred from/to two unequal collections and the resulting difference between the two collections). Unlike their performance on equality size axiom problems, children were unable to make accurate arithmetic inferences on these problems for both screening conditions at any level.

Second, the ability to utilize the configurational properties of figural correspondence between objects emerged over levels. On the one hand, children at level I did not use figural correspondence information about the final collections to solve inequality two-way function problems. Evidence from an analysis of their errors on the judgment task (discussed below) indicated that it was not the case that they simply used this information ineffectively. Rather, level I children consistently employed an operation strategy on these problems under the post-transformation screening condition. Thus, the use of an operation strategy led to their lower accuracy in this screening condition. On the other hand, children at level III used figural correspondence information about the final collections very effectively to solve these problems. Although they did not fully understand the arithmetic operations in inequality two-way function problems, perceiving the final collections allowed level III children to discover the location of noncorresponding relations between the collections (i.e., to determine which collection contained "gaps"), and thereby to achieve an empirical solution to these problems.

The preceding results of the judgment analyses raise a question about the basis of children's inaccurate judgments, particularly inaccurate judgments on equality size axiom problems and inequality two-way function problems. In order to address this question, children's strategies for making judgments on this task were analyzed. The strategy analysis entailed the same procedure as in Experiment 1. Children's judgments on each arithmetic problem were categorized into one of four types of strategies: (1) operation, (2) accurate inference, (3) inaccurate inference, and (4) spatial alignment. Recall that each judgment on a given problem (i.e., child's collection has more, interviewer's collection has more, both collections have the same) corresponded to a particular strategy. Table 20 contains the percentage of each type of strategy used on the standard inference question. The strategy data for only the four problem types examined in the multivariate analysis (see Table 19) are presented in this table.

The strategy analysis revealed several interesting findings. In the following discussion, these findings are contrasted with those of the counting experiment to indicate points of similarity and difference between the two experiments.

First, an operation strategy was used substantially less often and on fewer problem types in this experiment. It occurred on inequality size axiom problems at level I, and on inequality two-way function problems at all levels. Second, a nonarithmetic strategy of spatial alignment was employed on equality size axiom problems and on

Table 20

Percentage of Each Type of Strategy on the Standard Inference Question in the Figural Correspondence Experiment

		Type of Strategy					
		Arithmetic				Nonarithmetic	
			Inference				
Level	Problems[a]	Operation	Accurate	Inaccurate	Operation/ Acc. Inference[b]	Spatial Alignment	Other
I	One-Way, ≠	15	45	-	-	40	-
II		0	75	25	-	-	-
III		0	90	10	-	-	-
I	Two-Way, ≠	65	30	-	-	5	-
II		25	60	15	-	-	-
III		30	50	20	-	-	-
I	Size Axiom, =	-	-	-	65	30	5
II		-	-	-	80	15	5
III		-	-	-	95	5	0
I	Size Axiom, ≠	60	40	-	-	-	0
II		10	90	-	-	-	0
III		5	95	-	-	-	0

Note. Percentages were based on 120 judgments at each level.

[a] One-Way Function Inequality Problems included problems 4, 5; Two-Way Function Inequality Problems included problems 6, 8; Equality Size Axiom Problems included problems 9, 10; Inequality Size Axiom Problems included problems 11, 12.

[b] Operation/Accurate Inference.

inequality one-way function problems at level I and, to some extent, at level II. By comparison, this strategy was employed infrequently in the counting experiment. Third, there was a developmental shift in the predominant types of strategies, from nonarithmetic and operation strategies at level I to accurate and inaccurate inference strategies at levels II and III. Although a similar developmental shift from an operation strategy to inference strategies was observed in the counting experiment, the shift occurred between levels II and III.

Finally, the pattern of children's strategies on one-way function problems differed from that on two-way function problems. Children abandoned noninference strategies (i.e., spatial alignment and operation strategies) for inference strategies earlier and more completely with inequality one-way function problems than with inequality two-way function problems. For example, even at level III children still employed an operation strategy on 30% of their judgments with inequality two-way function problems. This finding of more advanced performance on one-way function problems was consistent with the results of the accuracy analysis.

In conclusion, the preceding strategy analysis revealed both similarities and differences between the two experiments. On the one hand, consider the following similarities. Children's inaccurate judgments at level I entailed the use of noninference strategies. Children also exhibited more advanced performance on one-way function problems than on two-way function problems. Thus, there was converging evidence from both experiments that the logical form of the problem influenced the accuracy of children's judgments and their strategies for making judgments on this task.

On the other hand, there were important differences in children's strategies between the two experiments. The first difference was the use of a spatial alignment strategy on several problem types in the figural correspondence experiment but not in the counting experiment. In particular, a spatial alignment strategy was used on equality size axiom problems at level I and, to a lesser extent, at level II. The use of a spatial alignment strategy was consistent with the effect of the screening condition on judgment accuracy for these problems. It indicated that the presence of misleading configurational cues in the final collections led children at levels I and II to adopt a spatial alignment strategy. This strategy resulted in lower accuracy for equality size axiom problems under the post-transformation screening condition. Moreover, the use of a spatial alignment strategy in the figural correspondence experiment provided further support for the view that the activity of establishing one-to-one correspondence between objects led children to construct a particular form of numerical representation which entailed a spatial medium.

The second difference in children's strategies between the two experiments was the earlier shift from a noninference strategy to an inference strategy in the figural correspondence experiment. This developmental shift occurred earlier (between levels I and II) on more problem types in the figural correspondence experiment than the comparable shift in the counting experiment. Thus, the adoption of an inference strategy at an earlier level in the figural correspondence experiment revealed that children's one-to-one correspondence activities also influenced how they solved the judgment task.

Standard inference question: Explanations. Children's explanations accompanying their judgments were analyzed as a source of evidence about how they arrived at their judgments. As in Experiment 1, the focus of this analysis was on those explanations that reflected the process of making an arithmetic inference (i.e., integrating numerical information about the initial collections with an arithmetic operation on these collections). Explanations were excluded from this analysis if they did not provide unambiguous evidence of making an arithmetic inference (see Experiment 1 for an example of an explanation that did not reflect an arithmetic inference). There were four types of explanations that were informative about how children used or did not use their one-to-one correspondence activity to make an arithmetic inference. Each type is defined below with sample explanations to illustrate it.

1. Spatial Alignment. The child makes reference to the spatial alignment or nonalignment of the collections.

L.B. (Level I, age 4-11) on problem 4 made an incorrect judgment and gave an explanation that reflected the use of spatial alignment cues: "I got the longest [collection], and you've got the shortest."

2. Configurational Gap. The child makes reference to the correspondence relation between the presence of objects in one collection and the absence of objects in the other collection (i.e., a spatial gap in the collection). This correspondence relation is expressed either as a quantified or an unquantified number of gaps in a collection.

L.B. (Level II, age 6-4) on problem 6 made a correct judgment and gave an explanation that reflected the relation between objects in one collection and an unquantified number of gaps in the other collection: "There's no dinosaurs in this place (child points to screen covering a location in the interviewer's collection), but I have some there."

R.W. (Level II, 5-11) on problem 4 made a correct judgment and gave an explanation that reflected the relation between objects in one collection and a quantified number of gaps in the other collection: "'Cause I have 1 [dinosaur] right here, and you don't have 1 there."

3. Configurational Arithmetic. The child makes reference to an arithmetic computation, or to the sum/remainder obtained by the computation. This computation involves the use of a quantified number of gaps in a collection. There were two fundamental differences between a configurational gap explanation and a configurational arithmetic explanation. First, the correspondence relation was expressed either as a quantified or an unquantified number of gaps in the configurational gap explanation. However, this relation was expressed exclusively as a quantified number of gaps in the configurational arithmetic explanation. Second, in the configurational arithmetic explanation the quantified number of gaps was used to compute the sum/remainder resulting from an addition/subtraction operation on the collections. In contrast, in the configurational gap explanation the quantified or unquantified number of gaps was not used in any arithmetic computation. If a given explanation did not refer unambiguously to an arithmetic computation that involved a quantified number of gaps, then it was categorized as a configurational gap explanation.

I.E. (Level II, 6-5) on problem 8 made a correct judgment and gave an explanation that reflected an accurate computation: "I had 3 more, then you took 2, so I still have more."

M.A. (Level III, 6-0) on problem 4 made a correct judgment and gave an explanation that reflected an accurate computation: "You only took 2 [dinosaurs] out of the water hole and put them into yours; there were 3 places that were empty."

4. Axiomatic Arithmetic. The child makes reference to the three terms of an inference (i.e., initial collections, arithmetic operation, and final collections) that corresponds to the Unique Sums law of equal quantities (equality size axiom) or the Monotonic law of unequal quantities (inequality size axiom) in a formal arithmetic system.

S.B. (Level III, age 6-4) on problem 10 made a correct judgment and gave an explanation that reflected an axiomatic inference: "2 [dinosaurs] left mine [my collection] and 2 left yours too, and at the start we had the same [number], so we have the same."

L.T. (Level III, age 5-8) on problem 12 made a correct judgment and gave an explanation that reflected an axiomatic inference: "There was more here (child points to her collection) and less there (child points to interviewer's collection), and you took 2 and 2, so now I still have more."

Figural explanations were predominant at all levels. There were three principal types of figural explanations: (1) spatial alignment, (2) configurational gap, and (3) configurational arithmetic. These three types of explanations occurred with differential frequency at each level. In addition, there was a considerable number of axiomatic arithmetic explanations on size axiom problems. However, these explanations did not occur frequently until level III (53% axiomatic arithmetic explanations at level III). Therefore, the focus of the following discussion is on the different types of figural explanations with reference to axiomatic arithmetic explanations at relevant points.

The type of figural explanation that revealed the most sophisticated understanding of arithmetic operations was configurational arithmetic. The frequency of configurational arithmetic explanations is presented in Table 21. There are several noteworthy findings in this table. As with the preceding analyses, these findings are compared with those of the counting experiment so as to clarify points of similarity and difference between the two experiments.

The first finding concerns the development of configurational arithmetic explanations. In contrast to mental arithmetic explanations, which steadily increased in frequency over levels, configurational arithmetic explanations were rare before level III. In fact, only one explanation of this type occurred at level II. Recall that in children's explanations on the initial interview of this experiment there was a parallel increase in the frequency of quantified object-to-gap correspondence explanations at level III (see Table 16). This increase in both types of explanations at level III reflected a developmental change in children's understanding of their one-to-one correspondence activity and how to use this activity to solve arithmetic problems. Thus, there was a relationship between representing the cardinal number of gaps between two unequal

Table 21

Frequency of Configurational Arithmetic Explanations on the Standard Inference Question in the Figural Correspondence Experiment

	Inequality Problems[a]				Equality Problems[b]
Level	One-Way Function	Two-Way Function	Size Axiom	All Types	All Types
I	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
II	0 (0)	1 (1)	0 (0)	1 (1)	0 (0)
III	12 (11)	5 (5)	0 (0)	17 (16)	2 (2)

Note. Numbers in parentheses indicate the number of correct judgments that were accompanied by configurational arithmetic explanations on the standard inference question.

[a] Frequencies were based on 20 explanations on each type of Inequality Problem at each level. One-Way Function Problems included problems 4, 5; Two-Way Function Problems included problems 6, 8; Size Axiom Problems included problems 11, 12.

[b] Frequencies were based on 60 explanations on all types of Equality Problems at each level. Equality Problems included problems 1, 2, 3, 7, 9, 10.

collections during the initial interview, and manipulating the cardinal number of gaps through configurational arithmetic during the final interview. This evidence of a relationship between the initial numerical representation of an arithmetic problem and the subsequent process of making an arithmetic inference constitutes an important point of similarity between the two experiments.

The second finding of interest concerns the influence of the dimensions of the arithmetic problems on children's use of configurational arithmetic. The first dimension to consider is the quantitative relationship between the initial collections of the problem. Configurational arithmetic explanations occurred more frequently on the inequality problems than on the equality problems (see Table 21). Recall that a similar pattern was found with mental arithmetic explanations in the counting experiment, but this pattern was even more striking in the figural correspondence experiment. A plausible interpretation of the findings from both experiments is that it was not necessary to make an arithmetic computation to solve the equality problems. These problems could have been solved by using either a nonintegrative, operation strategy or an integrative, inference strategy. Therefore, on the basis of the preceding strategy analysis, it is reasonable to assume that children at level I used an operation strategy and children at levels II and III used an inference strategy, but neither strategy required an arithmetic computation on equality problems.

The next dimension to examine is the logical form of the arithmetic problem. The influence of this dimension on children's use of configurational arithmetic was evident in two respects: (1) lack of configurational arithmetic explanations on size axiom problems, and (2) fewer configurational arithmetic explanations on two-way function problems than on one-way function problems.

The first respect was that children did not use configurational arithmetic to solve size axiom problems at any level (see Table 21). Size axiom problems were solved exclusively by axiomatic arithmetic. However, the ability to employ axiomatic arithmetic in making inferences on these problems developed over levels. Children at level I did not give any axiomatic arithmetic explanations on size axiom problems (problems 9-12). In contrast, children at levels II and III gave axiomatic arithmetic explanations on 18% and 53% of these problems, respectively. Note that the percentage of axiomatic arithmetic explanations at level III was slightly higher in this experiment than in the counting experiment. This was due to the occurrence of some mental arithmetic as well as axiomatic arithmetic explanations on size axiom problems in the counting experiment. Nevertheless, the findings on the development of axiomatic arithmetic were generally comparable in the two experiments.

The second respect in which the logical form of the problem influenced children's use of configurational arithmetic was that the frequency of configurational arithmetic explanations differed on one-way function problems as compared with two-way function problems. Children gave these explanations more frequently on inequality one-way function problems than on inequality two-way function problems (see Table 21). Recall that the same pattern was found with mental arithmetic explanations in the counting experiment (i.e., children gave more mental arithmetic explanations on inequality one-way function problems than on inequality two-way function problems). In the figural correspondence experiment, this finding suggests that the greater use of configurational arithmetic on inequality one-way function problems led children to make more accurate inferences on these problems than on inequality two-way function problems. If this were indeed the case, then the use of configurational arithmetic should be related to the accuracy of children's judgments. This hypothesis was supported by the following analysis.

The final finding to be noted in Table 21 involves the number of correct judgments that were accompanied by configurational arithmetic explanations. It can be seen in Table 21 that the combined percentage of correct judgments on inequality one-way function and inequality two-way function problems that were accompanied by configurational arithmetic explanations was 94% at level III. Even the one explanation of this type at level II (see explanation given by I. E. on problem 8) resulted in a correct judgment. Thus, these data support the view that the use of configurational arithmetic mediated children's accurate judgments.

In summary, the results at level III were similar in the two experiments. The combined percentage of correct judgments on inequality one-way function and inequality two-way function problems that were accompanied by mental arithmetic explanations in Experiment 1 was 81%; the combined percentage of these judgments that were accompanied by configurational arithmetic explanations in Experiment 2 was 94%. Note that the above percentage of correct judgments was somewhat lower in Experiment 1. Since the values (i.e., the cardinal numbers of objects in the collections) manipulated in mental arithmetic computations were larger than the values (i.e., the cardinal numbers of gaps between unequal collections) manipulated in configurational arithmetic computations, it is possible that children made more computational errors in Experiment 1 than in Experiment 2. Accordingly, the greater frequency of inaccurate computations would account for the lower percentage of correct judgments in Experiment 1. Nevertheless, the results of both experiments indicate that the use of the computational procedures of mental arithmetic and configurational arithmetic facilitated the accuracy of children's judgments.

A comparison can be made between the use of configurational arithmetic in this experiment and the use of mental arithmetic in the counting experiment. On the one hand, the same general pattern of findings was observed for both kinds of explanations with respect to the influence of the problem dimensions and the relationship of these computational procedures to judgment accuracy. Children gave both kinds of explanations on inequality problems more frequently than on equality problems, and on one-way function problems more frequently than on two-way function problems; these explanations occurred infrequently, if at all, on size axiom problems. Moreover, the percentage of correct judgments accompanied by configurational arithmetic or mental arithmetic explanations was close to ceiling in the two experiments.

On the other hand, two important differences emerged from the analyses of configurational arithmetic in Experiment 2 and mental arithmetic in Experiment 1. First, mental arithmetic explanations occurred frequently at levels II and III, whereas configurational arithmetic explanations did not occur until level III. Second, the total frequency of mental arithmetic explanations was greater than that of configurational arithmetic explanations. In the following discussion, it is argued that both the later development and the less frequent use of configurational arithmetic reflect an underlying change in children's understanding of number.

The comparison between configurational arithmetic in Experiment 2 and mental arithmetic in Experiment 1 raised a fundamental question: Why did configurational arithmetic develop later than mental arithmetic? Three possible answers to this question are considered, and the adequacy of each answer is evaluated in respect to the relevant evidence from both experiments.

The first answer attributes the later use of configurational arithmetic to a *computational deficit*. That is, children were unable to manipulate their initial numerical representations of inequality problems by arithmetic computations to obtain sums or remainders for the problems. The principal difficulty with this explanation is that in Experiment 1 mental arithmetic, which entails the same type of computational procedures as configurational arithmetic, was used frequently at level II. Therefore, a computational deficit is an inadequate answer to the question of why configurational arithmetic emerged later than mental arithmetic.

The second answer explains the later use of configurational arithmetic in terms of a *representational deficit*. This explanation holds that children were unable to employ their one-to-one correspondence activity to construct numerical representations of the problems, specifically, of the object-to-gap correspondence relations in inequality problems. Since the computational procedures of configurational arithmetic manipulate object-to-gap correspondence relations, a failure to represent these relations would preclude the use of configurational arithmetic. However, the evidence from Experiment 2 was not consistent with this explanation. It was found that as early as level II children were able to employ one-to-one correspondence between objects to construct numerical representations of inequality problems. Children at level II gave correspondence relation explanations on the initial interview and configurational gap explanations on the final interview (see Table 22 below). Both types of explanations made reference to the object-to-gap correspondence relations in inequality problems. Therefore, the second answer is also inadequate because a representational deficit can not account for the emergence of configurational arithmetic at level III.

The final answer under consideration holds that the later use of configurational arithmetic is due to a transformational deficit. According to this explanation, children were unable to transform their initial form of numerical representation of inequality problems into an alternate form of numerical representation that could be manipulated by arithmetic computations. This explanation assumes that the form of numerical representation constructed through one-to-one correspondence between objects (i.e., object-to-gap correspondence relations) could not be utilized by the computational procedures of children at level II and level III. Thus, it was possible for children at level II to be able to use one-to-one correspondence between objects to construct numerical representations of inequality problems and to be able to perform the requisite computational procedures. Their deficit was an inability to *transform* a numerical representation of the object-to-gap correspondence relations into a numerical representation of the cardinal number of gaps between two unequal collections.

In conclusion, the answer that a transformational deficit accounts for the later emergence of configurational arithmetic constitutes an adequate explanation in two respects. First, this answer was consistent with the representational and computational abilities of children at level II who did not yet use configurational arithmetic. Second, it was also consistent with the transformational ability of children at level III who used configurational arithmetic to make their judgments on inequality problems. Recall that level III children gave explanations on the initial interview which reflected a quantification of the correspondence relation. Specifically, they imposed a numerical structure on the initial configurations of inequality problems by performing a partial or total summation of the object-to-gap correspondence relations and by referring to the cardinal number of gaps between two unequal collections. Thus, performing a summation enabled level III children to transform a numerical representation based on correspondence relations into a numerical representation based on cardinal values.

This conclusion raises the question of why the ability to transform one form of numerical representation into another form emerged at level III. It can be argued that children at this level developed a more abstract understanding of number which entailed knowledge of the reciprocal relationship between one-to-one correspondence and counting as numerative activities.

The preceding analysis revealed that children did not use configurational arithmetic until level III. This finding bears on the issue of how children at levels I and II arrived at their judgments on the arithmetic problems. Children at level II are of particular interest because their judgment accuracy did not differ significantly from the judgment accuracy of children at level III (see Tables 17 and 18). In order to address this issue, the frequency of different types of figural explanations at each level was examined.

Table 22 presents the frequency of the three types of figural explanations on the standard inference question. It can be seen in this table that these explanations occurred with differential frequency over levels. Moreover, there was a developmental shift at level II, and again at level III, in the predominant type of explanation. Children referred to the spatial alignment of the collections at level I, they referred to configurational gaps at level II, and they referred to configurational arithmetic at level III. Although the frequency of these explanations was not high, the percentage of children who gave at least one instance of the predominant explanation at each level was consistently high. Specifically, the percentages of children at levels I, II, and III were 70%, 90%, and 80%, respectively (see Table 22). Thus, these findings suggest that children at different levels used different processes to make their judgments on the arithmetic problems, but they did not refer to these processes on every problem.

This analysis of figural explanations, together with the preceding analyses of initial explanations and judgment accuracy, provide a consistent picture of how children at levels I and II solved

Table 22

Frequency of Types of Figural Explanations on the Standard Inference Question in the Figural Correspondence Experiment

	Type of Figural Explanation		
Level	Spatial Alignment	Gap	Configurational Arithmetic
I	15 (7)	2 (1)	0 (0)
II	5 (5)	25 (9)	1 (1)
III	1 (1)	11 (4)	19 (8)

Note. Frequencies were based on 120 explanations at each level. Numbers in parentheses indicate the number of children who gave each type of explanation.

the judgment task. Consider the findings for children at level I. These children gave spatial configuration explanations on the initial interview. They also employed a nonarithmetic, spatial alignment strategy on several problem types and gave spatial alignment explanations on the final interview. Moreover, the analyses of judgment accuracy revealed that level I children adopted a spatial alignment strategy under the post-transformation screening condition; this strategy led to inaccurate judgments on particular arithmetic problems. Thus, it can be concluded that children at level I frequently relied on spatial alignment cues in both the initial and final collections to make their judgments on the arithmetic problems.

In contrast, children at level II relied on figural correspondence relations in the initial and final collections to make their judgments. These children gave correspondence relation explanations on the initial interview that referred to the successive iteration of global gaps (i.e., undifferentiated correspondence relations) or discrete gaps (i.e., differentiated, object-to-gap correspondence relations). Moreover, they employed arithmetic strategies (predominantly accurate and inaccurate inference strategies) on the problem types, and gave configurational gap explanations on the final interview. Finally, the greatest improvement in judgment accuracy occurred at level II, and the screening condition did not affect judgment accuracy except on equality size axiom problems. It is apparent that although both level I children and level II children based their judgments on configurational properties of the arithmetic problems, only children at level II used figural correspondence relations between objects to construct numerical representations of these problems. Thus, children at level II made arithmetic inferences by integrating numerical information about the object-to-object or object-to-gap correspondence relations in the initial collections with an addition or subtraction operation on these collections.

Several conclusions can be drawn from the preceding analyses of figural explanations. First, children at level II did not use a computational procedure to solve the judgment task. Instead, they relied on their numerical representations of figural correspondence relations in the problems to make arithmetic inferences. Second, the observed relationship between the use of configurational arithmetic and the frequency of correct judgments indicated that children at level III used a computational procedure, configurational arithmetic, to make inferences on inequality problems. Finally, children's ability to transform one form of numerical representation based on correspondence relations into another form of numerical representation based on cardinal values emerged at level III. The development of this transformational ability reflected a more abstract understanding of the relationship between these two numerative activities - counting and one-to-one correspondence between objects.

Difference question. The difference question assessed children's ability to infer the numerical difference between a pair of final collections in which one collection had been judged to have more objects on the preceding standard inference question. Mean accuracy on the difference question is presented in Table 23. A 2(experiment) x 2(level) x 2(sex) ANOVA was performed on the difference question data for Experiment 1 and Experiment 2. The analysis revealed a

Table 23

Mean Accuracy on the Difference Question for Each Problem in the Figural Correspondence Experiment

	Problem[a]									
Level	1	2	3	4	5	6	7	8	11	12
I	.80	.30	.00	.00	.00	.00	.00	.00	.00	.00
II	.90	.60	.00	.30	.00	.10	.00	.00	.00	.00
III	1.00	1.00	.30	.60	.60	.40	.00	.00	.20	.30

Note. Mean Accuracy was based on the proportion of correct judgments on the difference question.

[a] Problems 9 and 10 do not appear in the table because the difference question was not applicable to them.

significant effect for level, $\underline{F}(2,48)=22.46$, $\underline{p} < .001$. Children at level III gave more correct answers to the difference question than children at levels I or II. Mean proportions of correct answers for levels I, II, and III were .08, .18, and .40, respectively. There were no other significant effects.

The accuracy analysis of the difference question yielded converging results for Experiment 1 and Experiment 2. It can be seen in Table 10 and Table 23 that children at levels I and II rarely answered the difference question correctly. The only problems on which mean accuracy exceeded .50 were equality one-way function problems. However, accuracy on the difference question increased over levels. Children at level III gave more correct answers on both equality and inequality one-way function problems (problems 1, 2, 4, and 5) than children at the previous levels.

These results indicate that the ability to infer the numerical difference between two collections emerged later than the ability to infer the relational outcome (i.e., same number or more) of an arithmetic operation. Moreover, there was evidence that the logical form of the problem influenced accuracy on the difference question. Children gave more correct answers to the difference question on one-way function problems than on two-way function problems or size axiom problems. Note that this finding was consistent with the analyses of the standard inference question in which an effect for the logical form of the problem was also obtained (see Table 5 and Table 18 above).

On the one hand, the difficulty of the difference question at level III was surprising because children's performance on the standard inference question was close to ceiling on all problems except inequality two-way function problems. Consider the contrast in performance between equality one-way function problems (problems 1 and 2) and equality two-way function problems (problems 3 and 7) as a case in point. Children at level III were equally accurate on both types of problems with respect to the standard inference question (mean accuracy over both experiments was 1.00 and 1.00 for equality one-way function problems and equality two-way function problems, respectively). However, they were considerably more accurate on equality one-way function problems (mean accuracy was .90) than on equality two-way function problems (mean accuracy was .10) with respect to the difference question. Thus, children were better able to make an inference about the outcome of a particular arithmetic operation on a pair of collections than to make an inference about the numerical difference between the collections following an arithmetic operation.

On the other hand, the results of this analysis were consistent with two lines of research that have examined young children's ability to determine the numerical difference between unequal collections. The first line of research has dealt with the logical demands of the problems. Evidence that the logical form of the problem influenced children's accuracy on the difference question supported findings by Piaget and his collaborators (Piaget, 1980; Piaget et al., 1977). Specifically, their research examined children's reasoning about particular forms of logical functions expressed by arithmetic transfers of objects from one collection to another. It was found that children from 3 years to 7 years of age were able to determine the numerical difference between collections for transfers in the form of a one-way function, but not in the form of a two-way function. Moreover, they erred on two-way function transfers by giving difference values that were accurate for one-way function transfers. These errors indicate that children interpreted two-way function transfers as one-way function transfers.

Similar results were obtained in the present study. Children at level III generally gave correct answers to the difference question on equality and inequality one-way function problems (mean proportions of correct answers over both experiments were .90 and .58, respectively). In comparison, these children predominantly gave incorrect answers on equality and inequality two-way function problems (mean proportions of correct answers over both experiments were .10 and .25, respectively). Furthermore, a large proportion of children's errors on two-way function problems (1.00 and .67 on equality and inequality two-way function problems, respectively) were consistent with their interpretation of these problems as one-way function problems.

One explanation for the greater difficulty of two-way function problems than one-way function problems is a difference in the logical properties of these problem types (see Chapter 2). Recall that two-way function problems entail a transformation of the number of transferred objects in order to determine the difference between collections, whereas one-way function problems do not entail this transformation. Thus, it can be argued that children were unable to perform the transformation required to answer the difference question correctly on two-way function problems and, consequently, their accuracy on these problems was substantially lower than on one-way function problems.

The second line of research has been concerned with the task demands of the problems. Hudson (1983) conducted a series of experiments to identify the source of young children's difficulty on arithmetic word problems that involved comparing disjoint sets and determining the numerical difference between them. The objective of his investigation was to establish whether children's difficulty on these problems reflected a lack of knowledge about one-to-one correspondences and numerical differences between disjoint sets or, alternatively, a misinterpretation of the comparative construction ("How many[comparative term]...than...?") employed in the problems.

In order to test these alternative explanations, Hudson examined children's understanding of numerical differences between disjoint sets under two different question formats. Children from 4 years to 6 years of age were shown a series of drawings of two unequal sets of items (e.g., birds and worms) that were <u>not</u> arranged in figural one-to-one correspondence. For each drawing, children were asked two different questions in separate testing sessions: (1)"How many more birds than worms are there?" and (2)"Will every bird get a worm? How many birds won't get a worm?" It was found that young children were significantly more accurate in answering "Won't get" questions than "How many more" questions. Hudson concluded that the source of children's difficulty on numerical difference problems was not a lack of correspondence knowledge, but rather a misinterpretation of the comparative question employed in these problems.

Riley, Greeno, and Heller (1983) interpreted these findings within an information-processing analysis of improvement in arithmetic problem-solving skill. They argued that conceptual knowledge of the semantic structure of a problem was the major factor in improving children's ability to solve arithmetic word problems. According to their analysis, Hudson's evidence that young children were more successful on numerical difference problems in answer to "Won't get" questions than "How many more" questions demonstrated the role of conceptual knowledge in the development of problem-solving skill. Rewording facilitated children's performance by improving their understanding of the relationships among quantities in arithmetic word problems. Children's improved understanding of these problems enabled them to utilize their available solution procedures.

The results of the difference question in the present study are similar to Hudson's results on one principal point. The difficulty of the difference question ("How many more dinosaurs do you [I] have?") for children at all levels was consistent with Hudson's finding that young children exhibited poor performance in answer to "How many more" questions. Nevertheless, there are both empirical and theoretical points of difference between these two studies.

Consider first an empirical point of difference that the logical form of the problem affected children's accuracy in the present study. Children gave more correct answers to the difference question on one-way function problems than on two-way function problems. Since Hudson's study only examined children's performance on numerical difference problems under static conditions, and <u>not</u> under transformational conditions (e.g., addition or subtraction), it was not possible to reveal the effect of a logical factor in his study. Thus, Hudson's analysis of numerical difference problems was at best incomplete.

Consider next a theoretical point of difference. Hudson claims that the source of children's difficulty on numerical difference problems is linguistic, specifically, a misinterpretation of the comparative construction employed in these problems. Although this is an adequate description of the data, it is an inadequate explanation for two reasons. First, linguistic difficulty does not explain the greater accuracy in answering "How many more" questions on particular logical problem types in the present study. Second, it does not specify how the two different questions led to different understandings of the relationships among quantities in the problems such that they facilitated or hindered children's performance. Riley, Greeno, and Heller's analysis of compare (numerical difference) problems is more adequate in this respect. Their account specifies the conceptual knowledge and procedural knowledge involved in both successful and unsuccessful solutions to these problems. In view of the results of the present study, the principal shortcoming of their account is a failure to consider the logical form of arithmetic operations as a determinant of arithmetic problem-solving skill.

In conclusion, children's ability to determine the numerical difference between unequal collections emerged gradually during the preschool and primary-school years. The results of the studies reviewed here indicate that the development of this ability entailed the coordination of several types of knowledge: quantifying the nonequivalence relation between two collections in a problem, understanding the logical form of the arithmetic operation(s) performed in a problem, and interpreting the semantic structure expressed by the wording of a problem.

Anticipatory inference question. The anticipatory inference question assessed children's ability to infer the outcome of an imaginary arithmetic operation. Mean accuracy on the anticipatory inference question is presented in Table 24. A 2(experiment) x 2(level) x 2(sex) ANOVA was performed on the anticipatory inference question data for Experiment 1 and Experiment 2. The analysis revealed significant effects for experiment, $\underline{F}(1,48)=14.01$, $\underline{p} < .001$, and for level, $\underline{F}(2,48)=22.19$, $\underline{p} < .001$. There was no significant effect for sex. Although accuracy on the anticipatory inference question was generally high in both experiments, children in Experiment 1 were consistently more accurate ($\underline{M} = .90$) than children in Experiment 2 ($\underline{M} = .82$). Moreover, accuracy increased over levels in both experiments. Mean proportions of correct answers for levels I, II, and III were .77, .87, and .95, respectively.

The accuracy analysis of the anticipatory inference question yielded somewhat different results for Experiment 1 and Experiment 2. In general, at level I children in Experiment 1 were more accurate than children in Experiment 2. It can be seen in Table 11 that level I children in Experiment 1 exhibited considerable accuracy on all problems except inequality size axiom problems (problems 11 and 12). In contrast, Table 24 reveals that level I children in Experiment 2 were less accurate on inequality one-way function problems (problems 4 and 5) as well as inequality size axiom problems. Nevertheless, this difference in accuracy declined over levels because children's

Table 24

Mean Accuracy on the Anticipatory Inference Question in the Figural Correspondence Experiment

	Problem											
Level	1	2	3	4	5	6	7	8	9	10	11	12
I	1.00	1.00	.90	.40	.50	.90	1.00	.70	.70	.90	.20	.20
II	1.00	1.00	1.00	.40	.60	.90	1.00	.70	.90	.90	.90	.70
III	1.00	1.00	1.00	.70	.90	1.00	1.00	.70	1.00	1.00	1.00	.90

Note. Mean Accuracy was based on the proportion of correct judgments on the anticipatory inference question.

performance approached ceiling on all problems in both experiments. These results raise two related issues. First, why did children make more correct judgments on the anticipatory inference question than on the standard inference question in both experiments? Second, why did they make particular error patterns on the anticipatory inference question in each experiment? In order to address these issues, the basis of children's correct and incorrect anticipatory judgments were examined.

Contrary to the expectation that it would be more difficult to make an inference about an imaginary arithmetic operation than a concrete one, children actually employed the same strategy to judge both concrete and imaginary arithmetic operations. However, the particular strategies employed, especially at level I, differed between the two experiments (compare Table 7 and Table 20). Recall that in Experiment 1 children at level I used an operation strategy to some extent on all problems. In contrast, in Experiment 2 children at the same level adopted an operation strategy on specific problems (e.g., inequality size axiom problems) and a spatial alignment strategy on other problems (e.g., inequality one-way function problems). This use of the same strategy to answer both the standard inference question and the anticipatory inference question for each experiment had several implications for children's performance.

First, it resulted in children's greater accuracy on the anticipatory inference question than on the standard inference question, particularly in Experiment 1. This was due to the effect of performing two arithmetic operations (a concrete operation, and then an imaginary operation) on the collections of certain problems

(problems 4, 5, 6, and 8). A noninference strategy such as an operation strategy led to an incorrect judgment about the first operation (concrete operation) on these problems, but the same strategy led to a *correct* judgment about the second operation (imaginary operation). For example, problem 4 began with 10 elements in the child's collection and 7 elements in the interviewer's collection. The first arithmetic operation entailed adding 2 elements to the interviewer's collection for a total of 9 elements, and the second arithmetic operation entailed *pretending* to add 2 elements to the interviewer's collection for an "imaginary" total of 11 elements. If children relied on an operation strategy to solve problem 4, this strategy led to an incorrect judgment on the standard inference question (10 vs. 9 elements) and a correct judgment on the anticipatory inference question (10 vs. 11 elements). In fact, the outcome of using an operation strategy on all problems except size axiom problems was greater accuracy on the anticipatory inference question.

Second, the use of the same strategy to answer both questions resulted in the greater accuracy of level I children in Experiment 1 than in Experiment 2. This was due to the differential use of an operation strategy in the two experiments. Recall that an operation strategy was the predominant strategy at level I in Experiment 1, whereas it occurred to a lesser extent in Experiment 2. Since an operation strategy led to a correct judgment on the anticipatory inference question for all problems except size axiom problems (problems 9-12), children in Experiment 1 were consistently more accurate than children in Experiment 2.

Finally, the use of the same strategy on both questions yielded particular error patterns on the anticipatory inference question in each experiment. In Experiment 1, children's reliance on an operation strategy at level I led to incorrect judgments on inequality size axiom problems. In Experiment 2, children employed different strategies on different problems. Their use of a spatial alignment strategy as well as an operation strategy at level I led to incorrect judgments on inequality one-way function problems and inequality size axiom problems, respectively. In summary, children's use of the same strategy to answer both the standard inference question and the anticipatory inference question accounted for the two principal findings of this analysis: (1) children in Experiment 1 were more accurate on the anticipatory inference question than children in Experiment 2, and (2) children in *both* experiments were more accurate on the anticipatory inference question about an imaginary operation than on the standard inference question about a concrete operation.

Inverse Task

The inverse task assessed children's ability to "undo" the outcome of an arithmetic operation. This ability was assumed to reflect their knowledge of the relation between direct and inverse operations within an arithmetic system. The unit of analysis was children's actions on the collections. Recall that an action is defined in this task as any addition, subtraction, or transfer operation that was performed on objects in the collections (see Chapter 3, Footnote 1). The categories for coding children's actions on each problem of the inverse task are specified in the coding manual (see Appendix). Reliability for coding children's actions was determined by the procedure described in Experiment 1, and interobserver agreement was 94%.

A total of 587 actions was coded over all levels. The frequency of actions was examined by level in order to determine whether a developmental change in the number of actions was related to a change in the accuracy of these actions. As in the counting experiment, the number of actions did not directly decrease as accuracy increased over levels. Instead, the developmental pattern described an inverted U-shaped function, with an equivalent number of actions at levels I and III and a considerable increase at level II. In order, the number of actions at levels I, II, and III was 186, 215, and 186 (see Chapter 3, Footnote 2). An interpretation of this inverted U-shaped function is provided below in regard to the processes by which children solved the inverse task.

The first issue to be addressed concerns the nature and extent of young children's knowledge of the inverse operation. This issue was examined by analyzing the accuracy of children's actions on the inverse task. As in the counting experiment, children's actions on each problem were scored separately for the process measure of operation accuracy and for the product measure of end state accuracy. Recall that operation accuracy is defined as actions that perform the inverse, and _only_ the inverse, of the direct arithmetic operation performed by the interviewer. In contrast, end state accuracy is defined as _any_ actions that restore the collections of a problem to their initial numerical states.

Table 25 presents the accuracy of children's attempts to undo the outcome of the arithmetic operations. A 3(level) x 2(screening) x 2(problem: one operation vs. two operations) multivariate analysis of variance (MANOVA) was performed on the operation accuracy data. The analysis revealed a significant effect of problem, with greater accuracy on one-operation problems ($\underline{M}$ = 1.68) than on two-operation problems ($\underline{M}$ = 1.36), $\underline{F}(1,24)=12.66$, $\underline{p} < .005$. There were no other significant effects.

The same type of analysis was performed on the end state accuracy data (see Table 25). This analysis revealed significant effects of problem, $\underline{F}(1,24)=15.87$, $\underline{p} < .001$, and of level, $\underline{F}(2,24)=6.96$, $\underline{p} < .005$. There was no significant effect of screening. Children were more accurate on one-operation problems ($\underline{M}$ = 1.80) than on two-operation problems ($\underline{M}$ = 1.56), and their accuracy increased over levels. Mean end state accuracy scores for levels I, II, and III were 1.43, 1.77, and 1.85, respectively.

In general, the analyses of operation accuracy and end state accuracy yielded similar developmemtal patterns. Accuracy on one-operation problems was greater than on two-operation problems, and accuracy on both one-operation problems and two-operation problems increased over levels. However, there was one notable difference between the patterns of operation accuracy and end state accuracy. As in the counting experiment, the problem effect persisted at level III for operation accuracy but not for end state accuracy. This finding indicates that children at level III succeeded in restoring the collections of two-operation problems to their initial numerical

Table 25

Mean Accuracy on the Inverse Task in the Figural Correspondence Experiment

	Operation Accuracy[a]		End State Accuracy[b]	
Level	One-Operation Problems	Two-Operation Problems	One-Operation Problems	Two-Operation Problems
I	1.50	1.21	1.60	1.25
II	1.55	1.43	1.85	1.68
III	2.00	1.45	1.95	1.75

[a] Operation Accuracy scores ranged from 0-2 on each problem. Scores were assigned as follows: *0* = not perform only the inverse operation on a collection; *1* = perform the inverse operation with an incorrect quantity of elements on a collection; *2* = perform the inverse operation with a correct quantity of elements on a collection.

[b] End State Accuracy scores ranged from 0-2 on each problem. Scores were assigned as follows: *0* = not restore the initial numerical state of a collection; *2* = restore the initial numerical state of a collection.

states, but they did not do so by performing only the inverse of the direct arithmetic operation.

The next issue to be examined concerns the processes by which children solved the inverse task. In particular, an analysis of children's error patterns revealed that the source of the problem effect at level III for operation accuracy differed from that of the counting experiment. Although level III children in both experiments committed operation errors on two-operation problems, their errors were due to different processes for achieving end state accuracy on these problems.

Children's errors were categorized into the five types defined in Experiment 1: (1) row, (2) operation, (3) inverse + operation, (4) quantity, and (5) location. Table 26 presents the percentage of each type of error on the inverse task. This analysis revealed three developmental patterns of children's errors: (1) three error types that remained stable over levels, (2) one error type that exhibited an U-shaped function over levels, and (3) one error type that exhibited an inverted U-shaped function over levels. It is evident that these error patterns differed from those in the counting experiment. Most notably, no patterns simply decreased or increased over levels, and patterns 2 and 3 were inversely related to each other. Each of these patterns is discussed below.

The first pattern was comprised of row errors, quantity errors, and location errors. Although the individual frequencies of these error types were not equivalent (i.e., location errors were more frequent than row errors and quantity errors), all three error types remained stable over levels. In comparison with the counting

Table 26

Percentage of Each Type of Error on the Inverse Task in the Figural Correspondence Experiment

Level	Type of Error Row	Operation	Inverse + Operation[a]	Quantity	Location	All Types[b]
I	14	23	19 (15)	13	30	100 (69)
II	10	4	47 (54)	10	29	100 (51)
III	8	35	19 (71)	11	27	100 (37)

[a] Numbers in parentheses indicate the percentage of Inverse + Operation errors which yielded end state accuracy on a collection.

[b] Numbers in parentheses indicate the total number of errors made by children at each level in the Figural Correspondence Experiment.

experiment, children at level III continued to make a small number of row errors and quantity errors on two-operation problems. Moreover, their location errors did not increase in frequency at level III. Thus, several conclusions can be drawn from this error pattern. First, the occurrence of row errors and quantity errors at level III indicated that children's knowledge of the inverse operation was still fragile or incompletely constructed in respect to two-operation problems. Recall that in the counting experiment level III children systematically eliminated their row errors and quantity errors through a self-correction process. However, in this experiment, level III children did not always attempt to correct their errors. Second, the evidence of a stable percentage of location errors over levels indicated that children did not reproduce the precise configurations of the initial collections in solving the inverse task.

The second pattern consisted of operation errors. This error type exhibited an U-shaped developmental function over levels. Operation errors were higher in frequency at levels I and III, but they decreased markedly at level II. The third pattern consisted of inverse + operation errors. In contrast with the second pattern, this error type exhibited an inverted U-shaped developmental function over levels. Inverse + operation errors were lower in frequency at levels I and III, but they increased at level II. Thus, these two error types were inversely related to each other over levels. In order to determine the basis of this inverse relationship at each level, the developmental patterns of operation errors and of inverse + operation errors were examined together.

It can be seen in Table 26 that operation errors occurred frequently at level I. However, these errors decreased substantially at level II such that they accounted for only 4% of the total errors at this level. The reason for this decrease in operation errors was evident from an inspection of inverse + operation errors at level II. Recall that this error type was important in the counting experiment because it revealed how children used a trial-and-error process to solve the inverse task. Inverse + operation errors occurred frequently at all levels, and a large proportion of these errors yielded correct end states. In contrast, in the present experiment, inverse + operation errors occurred frequently only at level II. Nevertheless, many of these errors at level II also yielded correct end states (see Table 26).

Taken together, the evidence on the frequency and accuracy of inverse + operation errors indicates that children at level II engaged in trial-and-error solutions to the inverse task. Although such solutions occurred much less frequently at level III, they generally led to end state accuracy on the problems (see Table 26). Thus, both the decrease in operation errors and the increase in inverse+operation errors from level I to level II reflected children's use of a trial-and-error solution process.

The significance of the increase in inverse + operation errors at level II is that it indicates children's use of a trial-and-error process to solve the inverse task. However, the way in which this process was used to achieve end state accuracy on the inverse problems differed in two respects from the way it was used in the counting experiment. First, the apparent objective in the present experiment was to restore the figural correspondence relations between pairs of initial collections. Unlike children in the counting experiment, these children did not attempt to restore the cardinal values of the initial collections nor did they employ a counting procedure to drive or to verify their solution processes. Second, empirical procedures, such as counting, were rarely used in the present experiment (see Appendix for a description of all empirical procedures). In fact, children in the counting experiment employed empirical procedures on the inverse problems three times more frequently than children in the figural correspondence experiment.

These findings reveal that children in the present experiment relied on a trial-and-error solution process. However, this solution process was not driven by their knowledge of the cardinal values of the collections, it was not accompanied by empirical procedures and, most notably, it was not used by children all levels. Recall that the frequency of inverse + operation errors increased more than twofold at level II, and then decreased by the same proportion at level III. It can be concluded that reliance on a trial-and-error process was limited to level II, and therefore, constituted a transitional phase in children's solutions to the inverse task. This conclusion is consistent with the finding that the frequency of children's actions on the inverse task described an inverted U-shaped function over levels, with a marked increase in frequency at level II. Thus, this increase in the frequency of children's actions provided further evidence for the use of a trial-and-error process to solve the inverse task at level II.

The preceding discussion focused on the inverse relationship between operation errors and inverse + operation errors from level I to level II. However, in order to complete the interpretation of this inverse relationship, it is necessary to examine the changes in frequency of these two error types from level II to level III. Table 26 reveals that the frequency of inverse + operation errors decreased from level II to level III, whereas the frequency of operation errors increased over these levels.

On the one hand, the reason for the decrease in inverse+operation errors is straightforward. Children at level III did not employ a trial-and-error process very often to solve the inverse task. Unlike children at level II, they tended to perform a single operation, rather than a series of operations, to undo the outcome of a direct arithmetic operation on a collection.

On the other hand, the reason for the increase in operation errors is not as clear. Although children at level III often performed a single operation that was <u>not</u> the inverse of the direct operation performed by the interviewer, these operation errors were not due to their lack of knowledge of the inverse operation or their inability to employ a trial-and-error process. Consider the following characteristics of children's operation errors at level III.

First, all but one of these errors occurred on two-way function problems. Second, the majority (85%) of these errors were <u>equivalent operation</u> errors. That is, children performed an operation that was mathematically equivalent to the inverse operation on a particular problem, but it was not the physical inverse of the direct operation on that problem. For example, in problem 3 the direct operations entailed subtracting 2 elements from the child's collection and placing them into the reserve collection, and then taking 2 other elements from the reserve collection and adding them to the interviewer's collection. Instead of undoing these external arithmetic operations, children who committed an equivalent operation error on this problem performed internal arithmetic operations. Specifically, they transferred 2 elements from the interviewer's collection to the child's collection, and they did not utilize the reserve collection at all! Finally, all of the equivalent operation errors at level III yielded end state accuracy for the collections on which they occurred.

Thus, the reason for both the increased frequency of operation errors and the decreased frequency of inverse + operation errors at level III was different from that at level I. Children at level I who committed operation errors either did not know that their solutions were incorrect, or they were unable to repair their errors. In contrast, children at level III did not attempt to repair their (equivalent) operation errors because these errors generally restored the collections to their initial numerical states.

In summary, the error data reveal two principal similarities between the present experiment and the counting experiment with respect to the processes by which children solved the inverse task. First, the use of a trial-and-error process at level II in the present experiment indicated that children relied on a numerical representation of the initial collections of a problem to guide their solutions to the inverse task. Children constructed a numerical representation of the initial collections by establishing one-to-one correspondence between objects, and they sought to restore the figural correspondence relations in the initial collections on the inverse task. Second, the frequency of equivalent operation errors at level III in the present experiment indicated a more abstract representation, rather than a lack of knowledge, of the inverse operation. Children at level III performed their inverse actions more flexibly than children at previous levels. Their inverse actions preserved the mathematical identity, but not the physical identity, of the direct actions performed by the interviewer. Note that a similar conclusion was reached for level III children in the counting experiment. It was argued that due to their more abstract representation of the inverse operation, children at level III disregarded the precise configurations of the initial collections and committed an increased number of location errors.

Thus, converging evidence from both experiments indicates that children relied on their knowledge of the inverse of an arithmetic operation and on their numerical representation of the initial collections of a problem to solve the inverse task. Moreover, children's errors at level III reflected a more abstract knowledge of the inverse operation. There was one fundamental difference in children's solution processes between the two experiments. In the figural correspondence experiment, the use of a trial-and-error process was limited primarily to children at level II. By comparison, in the counting experiment, a trial-and-error process was used frequently by children at all levels. This difference in solution processes between the two experiments provided evidence that children's numerative activities, counting or establishing one-to-one correspondence between objects, influenced how they solved the inverse task.

The final issue to be considered is whether young children's knowledge of the inverse operation is an isolated ability, or whether it is part of an early arithmetic system. Specifically, what is the relationship between children's knowledge of a direct arithmetic operation and their knowledge of the inverse of that direct operation over the course of development? As in the counting experiment, this question was answered by analyzing the consistency of children's performance between the judgment task (direct operation) and the inverse task (inverse operation) for a subset of problems (see Chapter 3, Footnote 3).

The accuracy of children's judgments and inverse actions on each problem in this analysis was jointly categorized as consistent (i.e., pass both tasks or fail both tasks) or inconsistent (i.e., pass one task and fail one task). Table 27 presents the percentage of children at each level exhibiting consistent and inconsistent performance patterns on these two tasks. Table 28 presents the same data for the four types of problems over all levels. This consistency analysis revealed two main findings of interest. In the following discussion, these findings are compared with those of the counting experiment to note points of convergence between the two experiments.

First, it is evident in both tables that a high percentage of children exhibited consistent performance at all levels and over all problems. Similar findings were obtained from the consistency analysis in the counting experiment. The only difference was the slightly higher percentage of consistent performance in the figural

Table 27

Percentage of Children at Each Level Exhibiting Consistent and Inconsistent Performance on the Judgment Task and the Inverse Task in the Figural Correspondence Experiment

Level	Consistent Performance	Inconsistent Performance	
	Pass Judgment-Inverse Fail Judgment-Inverse	Pass Judgment-Fail Inverse	Fail Judgment-Pass Inverse
I	75	13	13
II	78	15	8
III	90	8	3

Note. Performance on the inverse task was based on end state accuracy scores for each problem.

[a] Problems 3, 7, 6, and 8 were not included in this analysis.

Table 28

Percentage of Children Exhibiting Consistent and Inconsistent Performance by Problems on the Judgment Task and the Inverse Task in the Figural Correspondence Experiment

	Consistent Performance	Inconsistent Performance	
Problems[a]	Pass Judgment-Inverse Fail Judgment-Inverse	Pass Judgment-Fail Inverse	Fail Judgment-Pass Inverse
1 and 2	93	7	0
4 and 5	77	17	7
9 and 10	83	3	13
11 and 12	70	20	10
All Problems	81	12	8

Note. Performance on the inverse task was based on end state accuracy scores for each problem.

[a] Problems 3, 7, 6, and 8 were not included in this analysis.

correspondence experiment (81% in Table 28 as compared with 74% in Table 15).

Second, it can be seen in Table 27 that there was a trend toward more consistent performance over levels. Children at level III were more consistent than children at levels I and II. This developmental trend was also found in the counting experiment, however, the basis of this trend differed in the two experiments. Recall that in the counting experiment the inconsistent performance at levels I and II was primarily restricted to problems 4 and 5 (inequality one-way function problems). Specifically, half of the children exhibited an inconsistent pattern on these problems whereby they failed the judgment task but passed the inverse task. In contrast, in the present experiment the inconsistent performance at levels I and II was *not* restricted to particular problems, but instead it was distributed about equally between the two inconsistent patterns over all problems. Therefore, the inconsistency in children's performance at levels I and II in the figural correspondence experiment can be attributed to the unstable nature of their emerging knowledge of arithmetic operations.

The principal point to be made in regard to this analysis is that children at all levels generally exhibited consistent performance between the judgment task and the inverse task. Taken together, the findings from the consistency analyses of both experiments support the hypothesis that children's knowledge of the inverse operation is coordinated with their knowledge of the direct operation within a developing arithmetic system.

Footnote

[1] Correspondence relation explanations included object-to-object explanations on equality problems, and object-to-gap and numerical difference explanations on inequality problems. It can be seen in Table 16 that the percentages of correspondence relation explanations on equality problems and on inequality problems (based on the total number of object-to-gap and numerical difference explanations) were approximately equivalent over all levels.

5

GENERAL DISCUSSION AND CONCLUSIONS

This study examined the early development of arithmetic reasoning in respect to children's understanding of numerative activities and logical operations. The general question of interest was how children use different numerative activities, counting objects as contrasted with establishing one-to-one correspondence between objects, in reasoning about different logical forms of arithmetic operations. In the introductory chapter, a theoretical approach to this study was formulated and a set of research questions was posed. The theoretical approach entailed two proposals about children's arithmetic reasoning. First, the ability to solve an arithmetic problem is influenced by both the numerative activities that construct a numerical representation of an arithmetic problem, on the one hand, and the logical operations of addition and subtraction that transform this numerical representation, on the other hand. Second, children's knowledge of addition and subtraction is structured as a system, albeit an incomplete system, of interrelated operations during the course of arithmetic development.

This chapter reconsiders these theoretical proposals in view of the results of the two experiments presented in the preceding chapters. The initial sections are devoted to the research questions and the principal findings that bear on each of them. Then the final section discusses the implications of these findings for an account of the development of arithmetic reasoning.

Research Questions Reconsidered

Numerative Activities

One of the fundamental questions addressed by this study concerns the role of numerative activities in the development of arithmetic reasoning. To preview the conclusions, it is evident from the counting experiment and the figural correspondence experiment that children constructed different forms of numerical representation of the arithmetic problems through their activities of counting objects as opposed to establishing one-to-one correspondence between objects. Furthermore, these different numerative activities influenced how children solved the arithmetic problems in the two experiments.

Consider first the numerative activity of counting objects and its influence on children's arithmetic reasoning in the counting experiment. The analyses of children's explanations on the initial interview and on the judgment task as well as their errors on the inverse task provide a consistent picture of the role of counting in solving the arithmetic problems. Children's explanations on the initial interview revealed that counting the collections of objects led them to construct a form of numerical representation of the problems which had particular properties. That is, at levels II and III their explanations referred to the cardinal values of the initial collections of objects, and their representations of the cardinal values entailed a linguistic or verbal medium.

These properties of children's initial numerical representations of the arithmetic problems clearly influenced their reasoning on both the judgment task and the inverse task. With respect to the judgment task, mental arithmetic explanations increased in frequency at level II and remained frequent thereafter. Recall that a parallel pattern was observed with cardinal number explanations on the initial interview. Therefore, it can be argued from these convergent findings that representing the cardinal values of the collections on the initial interview was related to manipulating these cardinal values through mental arithmetic on the judgment task.

Further evidence that children's initial numerical representations of the problems influenced their arithmetic reasoning came from their errors and use of empirical procedures on the inverse task. The frequency of inverse + operation errors that yielded correct end states at all levels indicated that children employed a trial-and-error process to solve the inverse task. Their apparent objective was to restore the cardinal values of the initial collections of objects. They did not perform the precise inverse of the direct arithmetic operation, but rather engaged in a series of additions to and subtractions from the collections. Moreover, this trial-and-error process was often accompanied by counting procedures that served to direct and to verify their repairs of the collections. Thus, children constructed numerical representations of the arithmetic problems through their counting activity, and they used the initial numerical representations of the problems to guide their solution process on the inverse task.

A similar pattern of results was obtained with the numerative activity of establishing one-to-one correspondence between objects. This numerative activity led children to construct a particular form of numerical representation of the arithmetic problems, and it also influenced their arithmetic reasoning in the figural correspondence experiment. However, the form of the numerical representation in the figural correspondence experiment differed from that in the counting experiment. Children's explanations on the initial interview revealed that several properties of their numerical representations of the problems were particular to the one-to-one correspondence activity. Specifically, children at levels II and III referred to the correspondence relation between pairs of initial collections, and their representations of the correspondence relation entailed a spatial medium. This property of children's representations of the problems was evident even in their spatial configuration explanations at level I.

The influence of children's initial numerical representations of the problems on their arithmetic reasoning was demonstrated in both the judgment task and the inverse task. Children gave explanations on the judgment task that entailed the same distinctive properties as their initial numerical representations of the problems. Note the parallel at level II between global or discrete gap explanations on the initial interview and configurational gap explanations on the judgment task. Similarly, there was a parallel at level III between quantified gap explanations on the initial interview and configurational arithmetic explanations on the judgment task. Children at this advanced level represented summations of discrete gaps in terms of cardinal values, and then manipulated these cardinal values through configurational arithmetic in solving the judgment task. One additional finding is germane to the argument that children's initial numerical representations of the problems influenced their reasoning on the judgment task. The significant effect for screening condition that was obtained in this experiment, but not in the counting experiment, attests to the spatial medium of the form of numerical representation constructed through children's one-to-one correspondence activity.

Children's errors on the inverse task also reflected their initial numerical representations of the problems. In particular, the increase in inverse + operation errors that yielded correct end states at level II revealed children's reliance on a trial-and-error process to solve the inverse task. This solution process differed, however, from that in the counting experiment. It was not driven by knowledge of the cardinal values of the initial collections, it was not generally accompanied by empirical procedures, and it was not frequent at all levels. Instead, children at level II in this experiment used a trial-and-error process to restore the <u>figural correspondence relations</u> between pairs of initial collections. Furthermore, their use of a trial-and-error process was limited primarily to level II, and thereby served as a transitional procedure for solving the inverse task.

Taken together, the evidence from children's explanations on the initial interview and from their trial-and-error solutions on the inverse task indicates how the one-to-one correspondence activity influenced children's arithmetic reasoning on the inverse task. Children at level II constructed numerical representations of the arithmetic problems that entailed the correspondence relation, and they relied on this property of the initial numerical representations of the problems to guide their solution process on the inverse task.

Several conclusions can be drawn regarding the role of numerative activities in the development of arithmetic reasoning. First, the different numerative activities of counting objects as contrasted with establishing one-to-one correspondence between objects resulted in different forms of numerical representation of the arithmetic problems. Moreover, the properties of each form of numerical representation influenced how children solved the judgment task and the inverse task in the particular experiment. Thus, the process of arithmetic reasoning clearly differs for problems represented in a linguistic medium and problems represented in a spatial medium. The next section considers whether the ability to reason about different types of arithmetic problems is also affected by the properties of these numerative activities.

Types of Arithmetic Problems

A second fundamental question of this study concerns the influence of particular dimensions of the arithmetic problems on children's ability to reason about these problems. The dimensions of interest were the logical form of the arithmetic problems, and the quantitative relationship between the pairs of initial collections in the problems. Each of these dimensions dealt with a central issue in the study. The logical form dimension addressed the role of logical operations in children's arithmetic reasoning. Specifically, the study examined three types of arithmetic problems which differed in their logical properties -- one-way functions, two-way functions, and size axioms. The quantitative relationship dimension addressed an unresolved issue in the literature on early arithmetic reasoning. Young children demonstrate an understanding of arithmetic problems that require only directional reasoning, but their understanding of problems that require the integration of directional reasoning with an inequality relationship between two initial collections is very limited. Therefore, the set of arithmetic problems in the present study included both inequality problems that required this integration process to achieve correct solutions, and equality problems that did not require this process.

There are two general points to be made about the development of arithmetic reasoning for different types of problems. First, despite the differences in children's problem-solving process in the counting experiment as opposed to the figural correspondence experiment, there were marked similarities in their ability to solve the logical forms of the arithmetic problems across experiments. Second, there was considerable consistency in children's performance between the judgment task and the inverse task of both experiments, supporting the proposal that children's understanding of addition and subtraction constitutes an arithmetic system of interrelated operations. The principal results of the judgment task and the inverse task that bear on these points are discussed below.

In general, converging evidence from both experiments regarding children's ability to make an arithmetic inference on the judgment task supported the initial hypotheses about the two dimensions of the problem types. Consider the quantitative relationship dimension. As predicted, children solved equality problems earlier than inequality problems within each logical form of the arithmetic problems. The results for the logical form dimension were somewhat more complex. The most consistent finding was that children solved one-way function problems earlier than two-way function problems (whose logical form entailed a transformation). Even at level III, children's accuracy for two-way function problems did not exceed 60% in either the counting experiment or the figural correspondence experiment. Moreover, strategy analyses in both experiments revealed that the logical form of the problems also influenced the developmental pattern of children's strategies on this task. Children shifted from the use of noninference strategies to inference strategies earlier with inequality one-way function problems than with inequality two-way function problems. The last finding that is relevant to the logical form dimension concerns size axiom problems. Although children began to solve these problems at level II, they did not make axiomatic inferences (e.g., if equals are added to equals, the sums are equal) until level III. Thus, size axiom problems were intermediate in difficulty between one-way function problems and two-way function problems.

The only exception to the preceding generalizations was the greater accuracy for size axiom problems than for inequality one-way function problems in the counting experiment. This effect was primarily due to children's difficulty in integrating directional reasoning about addition or subtraction with an inequality relationship between two initial collections. It can be argued that children at level II in the counting experiment represented the initial collections of the problems in terms of cardinal values, but they were unable to manipulate these cardinal values accurately through mental arithmetic until level III.

In contrast, children at level II in the figural correspondence experiment demonstrated substantial accuracy on inequality one-way function problems. Recall that these children represented the initial collections of the problems in terms of figural correspondence relations between objects. Thus, children in the figural correspondence experiment were able to integrate their initial numerical representations of the problems with directional reasoning about addition or subtraction in order to solve inequality one-way function problems. The significance of this finding is that it indicates how different numerative activities influenced children's ability to reason about particular types of arithmetic problems in the present study. Counting objects constrained children's early arithmetic reasoning about inequality one-way function problems, whereas establishing one-to-one correspondence between objects facilitated it.

The inverse task also revealed convergence between the two experiments regarding children's ability to "undo" the outcome of an arithmetic operation. Children demonstrated some knowledge of the inverse operation as early as level I by their ability to solve one-operation problems (i.e., one-way function problems), but their ability to solve two-operation problems (i.e., two-way function problems and size axiom problems) was still limited at this level. In respect to this point, knowledge of the inverse for two-operation problems developed substantially over levels. Moreover, the general consistency in performance between the judgment task and the inverse task advanced the argument that children's knowledge of the inverse operation is coordinated with their knowledge of the direct operation within a developing arithmetic system.

In conclusion, the evidence from the judgment task and the inverse task regarding children's ability to reason about different types of arithmetic problems is consistent with the existing literature in several respects. First, young children have some understanding of addition and subtraction operations, but it is primarily limited to directional reasoning problems (Beilin, 1968). An understanding of integrative reasoning problems (Brush, 1978) and reciprocal reasoning problems (Piaget et al., 1977) occurs later in arithmetic development.

The most intriguing of these findings is the difficulty of integrative reasoning problems such as inequality one-way function problems. Children were not able to integrate their numerical representations of the inequality relationship between two collections with their directional reasoning about addition or subtraction until level II in the figural correspondence experiment, and not until level III in the counting experiment. Recent research on children's early classification (Sugarman, 1983) has revealed the development of a similar ability, that is, the ability to coordinate two conceptual relations simultaneously. Although this coordination ability in classification emerged earlier than the integration ability in arithmetic reasoning, evidence of these related abilities in different domains of cognitive functioning indicates that this may be a general property of a logico-mathematical system.

Finally, young children possess an understanding of the inverse operation for directional reasoning problems, and they often employ a trial-and-error process to "undo" the outcome of an arithmetic operation (Gelman, 1977). The present study extends this picture, however. It reveals how different numerative activities and logical forms of arithmetic operations influence children's ability to solve inverse problems.

Level of Number Development

The final research question to be considered concerns the role of the developmental factor, level of number development, in children's arithmetic reasoning. Recall that level was defined in this study as children's figurational and operational functioning within the number domain. Accordingly, children's level of number development was determined by a pretest that assessed their numerative activities of counting and establishing one-to-one correspondence between objects and their logical operations with respect to the number concept.

Consistent with this theoretical approach, numerous findings in both experiments indicated developmental effects for children's understanding of different numerative activities and types of arithmetic problems. On the one hand, developmental changes at level II in children's understanding of counting and one-to-one correspondence between objects influenced the respective forms of numerical representation of the problems. Moreover, these developmental changes also had implications for how children used different numerative activities to reason about the problems. For example, children used mental arithmetic as a computational procedure in the counting experiment, whereas they used configurational arithmetic in the figural correspondence experiment. On the other hand, developmental changes in children's understanding of addition and subtraction operations influenced their ability to reason about different logical forms of arithmetic operations, and to integrate the form of their arithmetic reasoning with the quantitative relationship between two collections. At level I correct solutions primarily occurred on problems that required only directional reasoning. By comparison, at level III correct solutions occurred on problems that required _integrative_ directional reasoning (e.g., inequality one-way function problems) as well as on problems that required conservation reasoning (e.g., size axiom problems).

Taken together, the preceding evidence indicates that children's arithmetic reasoning underwent considerable changes between level I and level III, both in their understanding of different numerative activities and in their ability to solve different types of arithmetic problems. It is argued here that these developmental changes in arithmetic reasoning reflect the qualitative restructuring of children's logico-mathematical thought.

The Development of Arithmetic Reasoning

Young children possess considerable arithmetic knowledge. Nonetheless, the present study has demonstrated that children's reasoning about arithmetic operations changes qualitatively during the preschool years. Moreover, as proposed in the theoretical approach to this study, children's numerative activities and logical operations differentially influence their ability to solve an arithmetic problem. Developmental changes in both of these factors contribute to the greater accuracy and flexibility of children's arithmetic reasoning at level III in the counting experiment and the figural correspondence experiment.

On the one hand, children's abilities are no longer limited to directional reasoning problems, but extend to more complex logical forms of arithmetic problems that entail integrative reasoning. Their knowledge of the inverse operation is also more abstract at level III. They perform inverse actions that preserve the mathematical identity, but not the physical identity, of a direct arithmetic operation.

On the other hand, children's understanding of the numerative activities of counting and establishing one-to-one correspondence between objects enables them to construct a particular form of numerical representation of the arithmetic problems at level II, and to transform one form of numerical representation into another form at level III. Furthermore, the properties of each form of numerical representation influence children's reasoning about arithmetic problems. A fruitful direction for future research will be to examine how different forms of representation facilitate as well as constrain reasoning in other domains of mathematical cognition.

An adequate account of the development of arithmetic reasoning must consider not only the abilities that comprise children's arithmetic knowledge, but also how those abilities are structured at different developmental points. Contrary to the skills view of independent arithmetic abilities, it has been argued in this study that young children's knowledge of the direct operation is related to their knowledge of the inverse operation within an emerging arithmetic system. Furthermore, consistent effects for the logical form of arithmetic operations on children's reasoning in both experiments supports the view that the development of an arithmetic system is constrained, in part, by a more general system of logico-mathematical thought.

A note of caution must be added to these conclusions about an emerging arithmetic system. Such an arithmetic system is still incomplete, and the reasoning that is based on this system tends to be inductive in nature. Nonetheless, understanding that addition and subtraction are interrelated operations is a significant insight. It indicates that children's arithmetic knowledge is structured as a system early in development and, thus, constitutes a precursor to a deductive arithmetic system.

Appendix

Coding Manual For The Inverse Task

This manual defines the categories for coding children's actions on each problem of the inverse task. A sample coding form for the inverse task is presented at the end of the manual.

I. ARITHMETIC OPERATIONS

A. CHILD'S ROW (C - ROW)

1. Addition (+) : Code total number of addition operations
2. Subtraction (-) : Code total number of subtraction operations
3. Transfer (T) : Code total number of subtraction - addition transfer operations

B. EXPERIMENTER'S ROW (E - ROW)

1. Addition (+) : Code total number of addition operations
2. Subtraction (-) : Code total number of subtraction operations
3. Transfer (T) : Code total number of subtraction - addition transfer operations

II. OPERATION ACCURACY

A. CHILD'S ROW (C - ROW)

1. Row (R) : Code row on which each operation was performed as one of the following :
    a. accurate
    b. inaccurate - omission error
    c. inaccurate - commission error
    d. not apply

2. Operation (O) : Code each operation that was performed on an accurate row (code transfer operation in child's row and in experimenter's row) as one of the following :
    - a. accurate - quantity precise
    - b. accurate - quantity under
    - c. accurate - quantity over
    - d. inaccurate
3. Location (L) : Code location at which each accurate operation was performed as one of the following :
    - a. accurate
    - b. inaccurate

B. EXPERIMENTER'S ROW (E - ROW)

1. Row (R) : Code row on which each operation was performed as one of the following :
    - a. accurate
    - b. inaccurate - omission error
    - c. inaccurate - commission error
    - d. not apply
2. Operation (O) : Code each operation that was performed on an accurate row as one of the following :
    - a. accurate - quantity precise
    - b. accurate - quantity under
    - c. accurate - quantity over
    - d. inaccurate

3. Location (L) : Code location at which each accurate operation was performed as one of the following :
    a. accurate
    b. inaccurate

III. TYPE OF ERROR

A. ROW

1. Centration (C) : Code total number of omission errors
2. Extension (E) : Code total number of commission errors
3. Omission + Commission (O/C) : Code total number of combined omission and commission errors

B. OPERATION

1. Same (S) : Code total number of operations that were the same as the arithmetic operations performed by the experimenter
2. Equivalent - Same (ES) : Code total number of operations that were equivalent to the same arithmetic operations performed by the experimenter
3. Equivalent - Inverse (EI) : Code total number of operations that were equivalent to the inverse of the arithmetic operations performed by the experimenter

4. Inverse + Operation (I+O) : Code total number of types of operations that combined accurate inverse operations with operation errors (i.e., categories 1 - 3)

C. QUANTITY (QTY)

1. Under (U) : Code total number of accurate operations that involved fewer than the precise quantity of objects

2. Over (O) : Code total number of accurate operations that involved more than the precise quantity of objects

D. LOCATION (LOC)

1. Place (P) : Code total number of accurate addition operations that were performed at inaccurate places in one or both rows

2. Rock (RK) : Code total number of accurate subtraction operations that were performed at inaccurate places such that rocks were exposed in one or both rows

IV. EMPIRICAL PROCEDURES

A. COUNT (C)

1. Drive (D) : Code total number of occurrences of counting that preceded any actions on a row or were followed by actions on a row

2. Verify (V) : Code total number of occurrences of counting that followed a sequence of actions such that no other actions were performed on a row

B. ROCK-CHECK (RK-CK)

1. Drive (D) : Code total number of occurrences of rock-checking (defined as sequentially lifting 2 or more dinosaurs in a row to check for rocks under the dinosaurs) that constituted drive procedures (see definition above)

2. Verify (V) : Code total number of occurrences of rock-checking that constituted verify procedures (see definition above)

C. ONE-TO-ONE CORRESPONDENCE (1-1)

1. Drive (D) : Code total number of occurrences of establishing one-to-one correspondence between objects (manifested in gestural or visual actions) that constituted drive procedures (see definition above)

2. Verify (V) : Code total number of occurrences of establishing one-to-one correspondence between objects that constituted verify procedures (see definition above)

V. <u>SOLUTION PROCESS</u>

A. NOT REPAIR (NR) : Code to indicate that no actions were performed to correct anticipated or completed actions on a problem

B. REPAIR

1. Anticipated Action (ANTIC.) : Code all actions that were performed to correct anticipated actions on a problem. Code total number of actions in the following categories :

   a. count (drive)
   b. rock-check (drive)
   c. one-to-one correspondence (drive)
   d. lift and replace dinosaur

2. Completed Action (COMPL.) : Code all actions that were performed to correct completed actions on a problem. Code total number of actions in the following categories :

   a. rock exposed - repaired by +, -, T, or Relocate
   b. addition - repaired by -, T, or Relocate
   c. subtraction - repaired by +, T, Relocate
   d. transfer - repaired by +, -, T, or Relocate

VI. <u>END STATE</u>

A. NUMBER (NUM)

1. Child's Row (C) : Code total number of dinosaurs in the end state of the child's row

2. Experimenter's Row (E) : Code total number of dinosaurs in the end state of the experimenter's row

B. LENGTH (LTH)

1. Child's Row (C) : Code length of child's row relative to length of experimenter's row as one of the following :

    a. equal (E)

    b. longer (L)

    c. shorter (S)

2. Experimenter's Row (E) : Code length of experimenter's row relative to length of child's row as one of the following :

    a. equal (E)

    b. longer (L)

    c. shorter (S)

VII. <u>END STATE ACCURACY</u>

A. CHILD'S ROW (C - ROW)

1. Relative Quantity (RQ) : Code quantity of dinosaurs in child's row relative to experimenter's row as accurate or inaccurate

2. Number (NM) : Code absolute number of dinosaurs in child's row as accurate or inaccurate

3. Relative Length (RL) : Code length of child's row relative to experimenter's row as accurate or inaccurate

4. Configuration (CF) : Code spatial configuration of dinosaurs in child's row as accurate or inaccurate

B. EXPERIMENTER'S ROW (E - ROW)

1. Relative Quantity (RQ) : Code quantity of dinosaurs in experimenter's row relative to child's row as accurate or inaccurate

2. Number (NM) : Code absolute number of dinosaurs in experimenter's row as accurate or inaccurate

3. Relative Length (RL) : Code length of experimenter's row relative to child's row as accurate or inaccurate

4. Configuration (CF) : Code spatial configuration of dinosaurs in experimenter's row as accurate or inaccurate

Coding Form for the Inverse Task

EXPERIMENT LEVEL ____

PROBLEM ____ (C)
(E)

SUBJECT NO.	ARITHMETIC OPERATIONS						OPERATION ACCURACY						TYPE OF ERROR											EMPIRICAL PROCEDURES			SOLUTION PROCESS			END STATE				END STATE ACCURACY							
	C-ROW			E-ROW			C-ROW			E-ROW			ROW			OPERATION				QTY		LOC		C	RK-CK	1-1	NR	REPAIR		NUM		LTH		C-ROW				E-ROW			
	+	-	T	+	-	T	R	O	L	R	O	L	C	E	O/C	S	ES	EI	I+O	U	O	P	RK					ANTIC.	COMPL.	C	E	C	E	RQ	NM	RL	CF	RQ	NM	RL	CF

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