8212208

Penafiel-Nava, Victor Miguel

CANONICAL FORMALISM FOR RELATIVISTIC DYNAMICS

City University of New York PH.D. 1982

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

CANONICAL FORMALISM FOR RELATIVISTIC DYNAMICS

by

VICTOR MIGUEL PEÑAFIEL NAVA

A dissertation submitted to the Graduate Faculty
in Physics in partial fulfillment of the requirements
for the degree of Doctor of Philosophy, The City
University of New York.

1981

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

10/20/81 — date

Kenneth Rafanelli — Chairman of Examining Committee

12/22/81 — date

Frank Martino — Executive Officer

Arthur Komar

Joseph Klarfeld

Albert Fanelli

Supervisory Committee

Kenneth Rafanelli

The City University of New York

Abstract

CANONICAL FORMALISM FOR RELATIVISTIC DYNAMICS

by

Victor Miguel Penafiel Nava

Advisor: Professor Kenneth R. Rafanelli

The possibility of a canonical formalism appropriate for a dynamical theory of isolated relativistic multiparticle systems involving scalar interactions is studied. It is shown that a single time-parameter structure satisfying the requirements of Poincaré invariance and simultaneity of the constituents (global transversality) can not be derived from a homogeneous Lagrangian. The dynamics is deduced initially from a non-homogeneous but singular Lagrangian designed to accommodate the global transversality constraints with the equal-time plane associated to the total momentum of the system. An equivalent standard Lagrangian is used to generalize the parametrization procedure which is referred to an arbitrary geodesic in Minkowski space. The equations of motion and the definition of center of momentum are invariant with respect to the choice of geodesic and the entire formalism becomes separable. In the original 8N-dimensional phase-space, the symmetries of the Lagrangian give rise to a canonical realization of a fifteen-generator Lie algebra which is projected in the 6N dimensional hypersurface of dynamical motions. The time-component of the total momentum is thus reduced to a neutral element and the canonical Hamiltonian survives as the only generator for time-translations so that the no-interaction theorem becomes inapplicable.

ACKNOWLEDGEMENTS

I wish to express here my sincere gratitude to Dr. K. Rafanelli for the enormous patience and helpful attitude he showed as adviser of this thesis; to Drs. S. Orenstein, L. Lussana and H. Crater for many stimulating discussions and useful suggestions; to Mrs. Marie Rafanelli for the fine typing job and to the Department of Physics at Queens College as well as to the Research Foundation of C.U.N.Y. for their financial support.

V. Miguel Peñafiel N.
Queens, New York
September 1981

TABLE OF CONTENTS.

INTRODUCTION.

Non-relativistic or Newtonian dynamics is characterized by a well structured and elegant mathematical formalism and by an immediate physical interpretation in agreement with our common experiences and intuition. Here, the existence of a variational principle plus a general group of transformations (the Galilei group) are enough to provide us with all physically relevant information about the dynamical behaviour of isolated systems of material particles.

A rather different situation is found in relativistic dynamics, when the velocities of the particles can no longer be considered small compared with the velocity of light, which is taken as a universal constant. There are several points where the new physical reality puts us apart from the previous framework:

(a) The times of the constituents enter as extra coordinates in the theory, expanding the configuration space by one dimension per constituent.

(b) The time-parameter is no longer "absolute" and is, as a matter of fact, not even uniquely defined so there is not a precise identification of "the Lagrangian" for a variational principle.

(c) The transformation group among all inertial observers, the Poincare group, affects the time components as well as spatial coordinates, hence, the notion of the Hamiltonian as a generator for time-parameter translations becomes ambiguous.

This more complicated scheme still has a simple resolution when the system is composed of a single free particle, but when more constituents are introduced and interactions among them considered, it becomes apparent that a canonical transition from non-relativistic to relativistic dynamics is far from immediate. In fact, the problem of whether it is possible to construct a canonical formalism compatible with the theory of relativity for isolated systems involving scalar interactions has given rise to a remarkably abundant literature. In sharp contrast to what one is familiar with in Newtonian mechanics, many possibilities have been explored each one showing different degrees of success and different limitations. The main complication seems to arise from the fact that since the phase space has been enlarged to 8N dimensions while a physically meaningful theory requires only 6N degrees of freedom, a number of constraints have to be included to obtain a proper realization of the Poincare algebra and the form of these constraints are to be postulated in addition. The first step in that direction was given by Dirac[1] whose "Forms of Dynamics" are precisely such realizations with invariant hypersurfaces as constraints. The seemingly natural choice for a Hamiltonian, the time component of the total linear momentum (P_o), arises from the "instant form" which more recently was shown to be related with the no-interaction theorem[2,3,4].

Later, many other ways have been followed: theories without invariant world lines as in Bakamjian and Thomas[5,6], approximation methods like Foldy's[7] and multiparameter formalisms like Droz-Vincent's predictive mechanics[8,9,10,11]. Canonical formalisms without a

Lagrangian, velocity-dependent potentials and many body interactions add to an almost endless number of viable approaches to the problem[12-17].

Which of all these pathways is best seems to be a matter of choice, depending on what characteristics are to be emphasized. But the question of how close to the non-relativistic structure one can approach with relativistic formulation if one insists on a complete canonical formalism still remains unanswered. As an attempt to deal with this question, the consistency of a mathematical structure involving invariant world lines, a manifestly covariant variational principle, suitable constraints and canonical realizations of a maximal symmetry group is investigated in this work.

Although many parts of such a program have already been discussed in earlier papers by several authors[18-26], we take here a somewhat more intuitive point of view with respect to the geometry of the world lines in Minkowski space. So, while our general formulation is mostly in the spirit of the works of Rohrlich and of Mukunda & Sudarshan just cited, the main feature of the present models is the characterization of the particles by their own world lines which are regarded as invariant. A dynamical system of particles is, therefore, a collection of world lines in the Minkowski space and must admit a manifestly covariant description in terms of their canonical four-coordinates $\{q^{\mu}_{a}\}$. This being the case, a parametrization of those world lines with respect to any given equal-time plane should be possible; thus, the particles can be taken as simultaneous in any Lorentz frame but will be observed in different states of motion in each of those frames. As a consequence of this requirement, the norm of the four-momentum for every particle becomes a function of the chosen parameter and is to be determined only after the problem has been solved, i.e., when one has the functional expression for the world lines as a result of the equations of motion. Within the severe requirements imposed on the model, the no-interaction theorem can be avoided only if the overall structure is such that permits the introduction of a separate Hamiltonian as the operator for time-parameter translations instead of the time-component of the total momentum.

We find it convenient to make a brief exposition of the constraint formalism in Chapter I since the need for constraints to reduce the number of degrees of freedom calls for procedures intrinsic to non-standard Lagrangians, and because those procedures can also be extended to the treatment of the standard case subject to imposed constraints.

The basic framework is set in Chapter II through some general postulates; then several heuristic arguments and formal results are used to obtain the initial form of a manifestly covariant Lagrangian suitable for our purposes.

In Chapter III, an actual model is built up starting from a singular Lagrangian and equal-time plane constraints (global transversality conditions) with respect to the total momentum four-vector of the system. It is shown that consistency among the canonical variables, the manifestly covariant realization of the Poincare algebra and the constraints is possible in presence of scalar interactions and leads to consistent definitions of the "center of momentum" and "internal variables" in formal agreement with models based on expanded 8N + 8 dimensional phase-space.

Another important requirement, cluster decomposition, is discussed in Chapter IV. To achieve that property, the model has to be based on an equivalent standard Lagrangian and generalized global transversality conditions used to parametrize the world lines. The identification of the particles with their world lines and energies related to the equal-time plane employed for the parametrization becomes more apparent at this point.

Finally, it is shown in Chapter V that it is possible to derive the same dynamics from group theoretical considerations. A canonical representation of a fifteen parameter symmetry group up to neutral elements is shown to coincide with the equations generated by the previously established variational principle.

I. THE CONSTRAINT FORMALISM.

The canonical theory of constrained systems introduced by Dirac[27,28,29] has been exposed in various forms by different authors[30,31,32]. We quote here only those of its features relevant to our development.

Because our intention is to start from a variational principle, we insist on the existence of a Lagrangian as well as a Hamiltonian formalism.

In order to simplify the formulae, we shall adopt a full tensor notation over both, particle (Latin) and coordinate (Greek) indices. Hence, it is understood that sums are performed wherever repeated indices occur, one or more covariant and one or more contravariant, while no sum is carried out when the repeated indices are of the same kind. Exceptional cases will be explicitly indicated. Our metric tensor in the Minkowski space is

$$\delta_{\mu\nu} = \| \mathrm{Diag}(1,-1,-1,-1) \| .$$

1. Lagrangian Formalism.

Consider a closed system of N interacting particles in the Minkowski space. The degrees of freedom, whose number is initially 4N, are described by a set of independent variables $\{ q^{\mu}_{a} \}$ ($\mu = 0,1,2,3$; $a = 1 \ldots N$). But since a physically meaningful description is to be given only in terms of 3N degrees of freedom, a set of N constraints should be incorporated to the evolution of the system and required to be invariant throughout the motion. The equations of motion are supposed to follow from the variation of the action functional

$$A[s] = \int_1^2 \mathcal{L}(\dot{q}, q)\, ds \tag{1.1}$$

where $\mathcal{L}$ is the manifestly covariant Lagrangian, a function of the coordinates $\{q^\mu_a\}$ and velocities $\{\dot{q}^\mu_a\}$ taken as the total derivatives of the coordinates with respect to some arbitrary Lorentz parameter s.

In principle, the Lagrangian that appears in (1.1) may be non-standard, i.e., its Hessian matrix, defined by

$$W^{ab}_{\mu\nu} \equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^\mu_a \partial \dot{q}^\nu_b} \tag{1.2}$$

may be singular:

$$\mathrm{Det}(W) \equiv |W^{ab}_{\mu\nu}| = 0. \tag{1.3}$$

If such is the case, the constraints are obtained as follows: taking the two sets of variables $\{q\}$ and $\{\dot{q}\}$ as initially independent, the computed rank of the Hessian is less than 4N, say R = 4N - K, due to its singularity. So, there exists a set of K independent null eigenvectors $\lambda^{\mu k}_a$ (k = 1 ...K) such that

$$W^{ab}_{\mu\nu} \lambda^{\mu k}_a = 0. \tag{1.4}$$

On the other hand, the equations of motion following the variation of (1.1),

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^{\mu}_{a}} \right) = \frac{\partial \mathcal{L}}{\partial q^{\mu}_{a}} , \tag{1.5}$$

can be written in terms of the Hessian, (1.2), as

$$W^{ab}_{\mu\nu} \ddot{q}^{\nu}_{b} = \alpha^{a}_{\mu}(q, \dot{q}) \tag{1.6}$$

where

$$\alpha^{a}_{\mu}(q, \dot{q}) \equiv \frac{\partial \mathcal{L}}{\partial q^{\mu}_{a}} - \frac{\partial^2 \mathcal{L}}{\partial q^{\nu}_{b} \partial \dot{q}^{\mu}_{a}} \dot{q}^{\nu}_{b} \tag{1.7}$$

is a function of the coordinates and velocities. Because the Hessian is singular, we know that there are not enough equations of motion to determine all the accelerations $\{\ddot{q}\}$, but, according to (1.4), from (1.6) we obtain K additional relations

$$\gamma^{k}(q, \dot{q}) \equiv \alpha^{a}_{\mu} \lambda^{\mu k}_{a} = 0 \tag{1.8}$$

involving only the independent variables $\{q ; \dot{q}\}$. These relations are called "canonical constraint equations" or, when they do not identically vanish, "canonical constraints" and as such define a subspace S' of the original space S of $\{q\}$ <u>and</u> $\{\dot{q}\}$ to which the motion is confined. Although, in principle, the missing equations of motion can be generated from (1.8) by simple differentiation, still several possibilities may arise at this point:

(a) The constraint equations (1.8) may affect the rank of the Hessian and, hence, more null eigenvectors and more constraints may be found.

(b) A differentiation of (1.8) may lead to more constraints, since in order to satisfy the stability of the constraints along the evolution of the system one must have also

$$\dot{\gamma}^k(q,\dot{q}) = 0 \tag{1.9}$$

If the resulting expressions (1.9) do not involve the accelerations and can not be derived from (1.6) and (1.8) by algebraic operations alone, then they constitute new constraints.

(c) It is possible that the number of constraints is not large enough to confine the motion to a subspace S' with the required dimensionality. Then, additional constraints can be imposed from outside; those will be called "subsidiary constraints" and treated in exactly the same way as the canonical constraints when consistent with the equations of motion.

Once all possible constraints have been found by an iterative procedure or imposed, the general situation should involve 4N independent equations of motion and 2N independent constraints whose differentiations will produce either constraints or equations of motion that are already contained in the set.

If the Lagrangian is invariant or quasi-invariant (see appendix B) under gauge transformations such that the variations

$$\delta q_a^\mu = \epsilon f_{\overset{\circ}{i}}(s)\, \lambda_a^{\overset{\circ}{i}\mu} \tag{1.10a}$$

(i = 1 ...I) involve a number I of arbitrary functions of the parameter and the same number of null eigenvectors, then the corresponding canonical constraint equations will be identically satisfied, i.e.,

$$\gamma^{\overset{\circ}{i}}(q,\dot{q}) \equiv \alpha_\mu^a \lambda_a^{\overset{\circ}{i}\mu} \equiv 0 \tag{1.10b}$$

leaving I unsolved accelerations (without equations of motion). Those gauges must also be fixed by subsidiary constraints as in (c).

If we choose to keep the gauge invariances until all other constraints are determined, our final scheme will be specified by a set of R = 4N - I independent equations of motion

$$W'^{ab}_{\mu\nu}\, \ddot{q}_b^\nu = \alpha'^a_\mu(\dot{q},q) \tag{1.11}$$

defined in an 8N-dimensional space S of $\{q;\dot{q}\}$ and a set of K constraints

$$\gamma^k(q,\dot{q}) = 0 \tag{1.12a}$$

whose derived equations of motion

$$\dot{\gamma}^k = \frac{\partial\gamma^k}{\partial\dot{q}_a^\mu}\ddot{q}_a^\mu + \frac{\partial\gamma^k}{\partial q_a^\mu}\dot{q}_a^\mu = 0 \tag{1.12b}$$

are already implicitly contained in (1.11) and (1.12a); therefore, if equations (1.12a) are used to specify initial conditions, those will automatically be preserved along the motion. Relations (1.12a) define an 8N-K dimensional subspace of S on which the actual evolution of the system takes place. The only unsolved degrees of freedom are those corresponding to the I gauge invariances; manifest covariance implies that K + 2I = 2N.

It is worth pointing out that if some of the constraints depend only on the coordinates,

$$\gamma(q) = 0,$$

because they produce two additional sets of relations:

$$\dot{\gamma} = \frac{\partial \gamma}{\partial q_a^\mu} \dot{q}_a^\mu = 0$$

and

$$\ddot{\gamma} = \frac{\partial \gamma}{\partial \dot{q}_a^\mu} \ddot{q}_a^\mu - \frac{\partial^2 \gamma}{\partial q_a^\mu \partial q_b^\nu} \dot{q}_a^\mu \dot{q}_b^\nu = 0,$$

they can be used to actually reduce the number of independent variables and, hence, the dimensionality of the original space S itself.[30]

2. Hamiltonian Formalism.

The generalized momenta are defined, as usual, by the derivatives

$$p^a_\mu(q,\dot q) \equiv \frac{\partial \mathcal{L}}{\partial \dot q^\mu_a} \; . \tag{1.13}$$

As a consequence of the singularity of the Hessian (1.2), definitions (1.13) can not be used here to express the velocities in terms of the variables $\{ q^\mu_a \}$ and $\{ p^a_\mu \}$. Instead, since relations (1.13) are not all independent, a number K of relations among those variables alone should arise without the use of the equations of motion (1.5); such relations are called "primary constraints" and can be represented by a set of "weak equalities"

$$\varphi^k(q,p) \approx 0 \tag{1.14}$$

where the weak equality symbol ($\approx$) is used to stress the fact that they are valid only on their self-defined subspace of the total phase-space $\{ q\,; p \}$, so that their Poisson brackets (P.b.) with the canonical variables may be different from zero. Obviously, the number of primary constraints must be the same as the number of null eigenvectors of the Hessian and, hence, equal to the number of canonical constraints (1.8). So, after taking the derivative of (1.14) with respect to the parameter and using the equations of motion to eliminate the parameter derivatives of the momenta, we must have relation

$$\alpha^a_\mu \lambda^{k\mu}_a \equiv \gamma^k(q,\dot q) = \frac{\partial \varphi^k}{\partial q^\mu_a} \dot q^\mu_a + \frac{\partial \varphi^k}{\partial p^a_\mu} \frac{\partial \mathcal{L}}{\partial q^\mu_a} \approx 0 \; . \tag{1.15}$$

The transition from the Lagrangian to the Hamiltonian formalisms, therefore, must take into account the invariance of the subspace defined by the primary constraints. So, if the "canonical Hamiltonian" is defined by

$$H_c = p^a_\mu \dot{q}^\mu_a - \mathcal{L}(q, \dot{q}) \tag{1.16}$$

then the "unsolved" velocities can be included in a set of arbitrary coefficients for a linear combination of the primary constraints that is to be added to the canonical Hamiltonian H_c to form the total Hamiltonian,

$$\mathcal{H} = H_c + v_k \, \varphi^k(q, p) \approx H_c , \tag{1.17}$$

that leaves the subspace, defined by these constraints, invariant. Now, in terms of the P.b.,

$$\{A, B\} = \frac{\partial A}{\partial q^\sigma_c} \frac{\partial B}{\partial p^c_\sigma} - \frac{\partial B}{\partial q^\sigma_c} \frac{\partial A}{\partial p^c_\sigma} , \tag{1.18}$$

total derivatives with respect to the parameter (s) are expressed in terms of (1.17) through the general equation of motion

$$\dot{A} = \{A, \mathcal{H}\} + \frac{\partial A}{\partial s} . \tag{1.19a}$$

In particular, the equations of motion for the canonical variables become

$$\dot{q}^{\mu}_{a} = \{ q^{\mu}_{a}, \mathcal{H} \} \approx \frac{\partial H_c}{\partial p^{a}_{\mu}} + v_k \frac{\partial \varphi^k}{\partial p^{a}_{\mu}} \tag{1.19b}$$

$$\dot{p}^{a}_{\mu} = \{ p^{a}_{\mu}, \mathcal{H} \} \approx -\frac{\partial H_c}{\partial q^{\mu}_{a}} - v_k \frac{\partial \varphi^k}{\partial q^{\mu}_{a}} \tag{1.19c}$$

and are equivalent to those following from the variational principle when expressed in terms of the canonical Hamiltonian (1.16) and subject to the existence of the constraints φ^k.

A completely consistent formalism requires that the constraints (1.14) themselves be preserved along the evolution of the system. Therefore, we still must require the s-derivatives of the constraints to vanish, i.e.,

$$\dot{\varphi}^k = \{ \varphi^k, \mathcal{H} \} = \{ \varphi^k, H_c \} + v_\ell \{ \varphi^k, \varphi^\ell \} \approx 0 \tag{1.20}$$

(see equation (1.15)). At this point the situation is entirely similar to that encountered in the Lagrangian formalism:

(a) It may happen that equations (1.20) give rise to more relations among the independent variables $\{ q ; p \}$ that <u>do not involve the coefficients v_k</u>.

(b) One may need to increase the number of constraints to account for further requirements of the model, in order to reduce the invariant subspace to an even smaller dimensionality.

In either case, the linear combination that appears in the Hamiltonian (1.17) will be expanded to include all the "secondary constraints" obtained from (a), the "subsidiary constraints" imposed in (b), if any, and all subsequent independent constraints that may arise by again requiring the stability of the new constraints. This iterative procedure ends up with perhaps a larger set of stable constraints that can be divided into "first class constraints" or those that commute (have vanishing P.b.) with all other constraints and "second class constraints" which have at least one nonvanishing P.b. So, if the final set of constraints is complete and self-consistent, the new equations (1.20) will contain only the second class constraints φ^{r} whose number must be even in order to admit a solution for the coefficients v_r since the antisymmetric matrix

$$C^{rr'} \equiv \| \{ \varphi^{r}, \varphi^{r'} \} \| \tag{1.21}$$

($r, r' = 1 \ldots R$) is singular when its dimensionality is odd. The coefficients v_r are then

$$v_r = \{ H_c, \varphi^{r'} \} C^{-1}_{r'r} \tag{1.22}$$

and the Hamiltonian (1.17) becomes

$$\mathcal{H} = H_c + \{ H_c, \varphi^{\tau'} \} C^{-1}_{\tau'\tau} \varphi^{\tau} + v_i \varphi^i \tag{1.23}$$

where the last term corresponds to a linear combination of the first class constraints (i = 1 ...I) with I unsolved coefficients to account for the I gauge invariances of the Lagrangian.

The remaining coefficients v_i can be fixed by imposing I subsidiary constraints involving the parameter S,

$$\chi_i(q, p, s) \approx 0 . \tag{1.24}$$

The stability conditions now read

$$\dot{\chi}_i = \{ \chi_i, \mathcal{H} \} + \frac{\partial \chi_i}{\partial s} \approx 0 \tag{1.25a}$$

and

$$\dot{\varphi}^i = \{ \varphi^i, \mathcal{H} \} \approx 0 \tag{1.25b}$$

where the total Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} = H + v_i \varphi^i + u^i \chi_i \tag{1.25c}$$

with

$$H = H_c + \{H_c, \varphi^{\tau'}\} C^{-1}_{\tau'\tau} \varphi^{\tau} \qquad (1.25d)$$

The equations for the coefficients will be, therefore,

$$\begin{pmatrix} \{\varphi^i, \varphi^j\} & \{\varphi^i, \chi_j\} \\ \{\chi_i, \varphi^j\} & \{\chi_i, \chi_j\} \end{pmatrix} \begin{pmatrix} v_j \\ u^j \end{pmatrix} \approx \begin{pmatrix} \{H, \varphi^i\} \\ \{H, \chi_i\} - \frac{\partial \chi_i}{\partial s} \end{pmatrix} \qquad (1.26)$$

but the set $\{\varphi^i\}$ is initially first class, so we have the vanishing brackets

$$\{\varphi^i, \varphi^j\} \approx 0 \quad ; \quad \{\varphi^i, H\} \approx 0$$

with which (1.26) reduces immediately to

$$u^j \approx 0 \quad ; \quad \{\chi_i, \varphi^j\} v_j \approx \{H, \chi_i\} - \frac{\partial \chi_i}{\partial s} \quad ; \qquad (1.27a)$$

then, the subsidiary constraints do not appear in the final form of the Hamiltonian, while the coefficients of the first class constraints are

$$v_j = \left(\{H, \chi_i\} - \frac{\partial \chi_i}{\partial s} \right) I^{-1\,i}{}_j \qquad (1.27b)$$

where the matrix $I^{-1i}{}_{j}$ is the inverse of

$$I^{j}{}_{i} = \| \{ \chi_i , \varphi^j \} \| \qquad (1.27c)$$

Once all coefficients have been determined as before, the total Hamiltonian is given, now without arbitrary elements, by

$$\mathcal{H} = H + \left(\{ H , \chi_j \} - \frac{\partial \chi_j}{\partial s} \right) I^{-1j}{}_{i}\, \varphi^{i} \qquad (1.28)$$

where H is just (1.25d).

To simplify the notation, let us write the complete set of constraints, which in our case must be 2N in number, as $\{ \varphi^K \}$ (K = 1 ...2N). The matrix formed by their P.b. is

$$C^{K}{}_{L} = \| \{ \varphi^K , \varphi_L \} \| \qquad (1.29a)$$

and we can write the Hamiltonian in a more general form as

$$\mathcal{H} = H_c + \left(\{ H_c , \varphi_L \} - \frac{\partial \varphi_L}{\partial s} \right) C^{-1L}{}_{K}\, \varphi^K \qquad (1.29b)$$

where only those constraints with explicit parameter dependence will survive the partial derivative inside the parenthesis.

The Dirac brackets (D.b.) are defined by

$$\{ A , B \}^{*} = \{ A , B \} - \{ A , \varphi_L \}\, C^{-1L}{}_{K} \{ \varphi^K , B \} \qquad (1.30)$$

so that the D.b. of any variable with a constraint will vanish and, hence, if we choose to work only with them instead of the P.b., then all constraints can be replaced by strong equalities, i.e., treated as identities and we confine ourselves to the invariant subspace in which the motion takes place. Eventually, one can use only the minimum number of variables to describe the dynamics.

The equation for the total derivatives with respect to the time-parameter,

$$\dot{A} = \{A, \mathcal{H}\} + \frac{\partial A}{\partial s}, \tag{1.31}$$

when expressed in terms of the D.b. are

$$\dot{A} = \{A, \mathcal{H}\}^* + \frac{\partial^* A}{\partial s} \tag{1.32a}$$

with

$$\frac{\partial^* A}{\partial s} \equiv \frac{\partial A}{\partial s} - \{A, \varphi_L\} \overset{-1}{C}{}^{L}{}_{K} \frac{\partial \varphi^K}{\partial s} \tag{1.32b}$$

since, using (1.30), we have, in general,

$$\{A, \mathcal{H}\}^* = \{A, \mathcal{H}\} + \{A, \varphi_L\} \overset{-1}{C}{}^{L}{}_{K} \frac{\partial \varphi^K}{\partial s}$$

because

$$\dot{\varphi}^K = \{\varphi^K, \mathcal{H}\} + \frac{\partial \varphi^K}{\partial s} \approx 0 .$$

Naturally, only the constraints with explicit time-parameter dependence will contribute to the final result. In terms of equations (1.27), for instance, it is

$$\frac{\partial^* A}{\partial s} = \frac{\partial A}{\partial s} + \{A, \varphi_i\} I^{-1 i}{}_j \frac{\partial \chi^j}{\partial s} \, . \qquad (1.32c)$$

In particular, the equations of motion

$$\dot{q}^\mu_a = \{ q^\mu_a, \mathcal{H} \} \quad ; \quad \dot{p}^a_\mu = \{ p^a_\mu, \mathcal{H} \} \qquad (1.33)$$

with $\mathcal{H}$ given by (1.29) will read

$$\dot{q}^\mu_a = \{ q^\mu_a, \mathcal{H} \}^* + \frac{\partial^* q^\mu_a}{\partial s} \qquad (1.34a)$$

$$\dot{p}^a_\mu = \{ p^a_\mu, \mathcal{H} \}^* + \frac{\partial^* p^a_\mu}{\partial s} \qquad (1.34b)$$

and the expressions

$$\{ q^\mu_a, p^b_\nu \}^* = \delta^\mu_\nu \delta^b_a - \{ q^\mu_a, \varphi_L \} C^{-1 L}{}_K \{ \varphi^K, p^b_\nu \} \qquad (1.35)$$

define the new canonical brackets with the variables confined to the invariant dynamical subspace.

II. CANONICAL SYSTEMS WITH THE GLOBAL TRANSVERSALITY CONDITION.

While the last chapter was an outline of how to handle constrained systems in general, we intend here to be more specific about the kind of constraints that should be expected in a relativistic manifestly covariant multiparticle scalar dynamics, and to analyze some of their formal consequences.

1. Basic Postulates.

(i) As stated previously, we insist from the outset that the theory be based on the existence of a variational principle with an action integral of the form (1.1),

$$A[s] = \int \mathcal{L}(\dot{q}, q)\, ds, \qquad (2.1)$$

which is, of course, required to be invariant under transformations belonging to the Poincare group ($\Lambda^{\mu}_{\nu}, a^{\mu}$). The parameter (S) is supposed to be a Lorentz invariant scalar and is being kept arbitrary during the initial deductive stages. Reparametrization invariance of (2.1) is a desirable feature, but following Rohrlich[18], we shall not make a necessary requirement of it. Thus, at this point, the Lagrangian may or may not be homogeneous.

(ii) The variables $\{q^{\mu}_{a}\}$ appearing in the Lagrangian are the coordinates of the particles (a = 1 ...N) in a Minkowski space. The configuration space is, therefore, the cartesian product of N

Minkowski spaces where the evolution of the system will trace an N-world-line whose parametric equations are

$$q_a^\mu = q_a^\mu (s) \tag{2.2}$$

In the phase-space, such a set of variables will be augmented by a conjugate set $\{ p_\mu^a \}$ defined by (1.13):

$$p_\mu^a \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_a^\mu} \tag{2.3}$$

where $\dot{q}_a^\mu \equiv d q_\alpha^\mu / ds$. With those definitions, the <u>manifest covariance</u> requirement of the theory can be simply fulfilled if the Poincare generators P_μ , $J_{\mu\nu}$ are given in terms of the particle variables by

$$P_\mu = \sum_a p_\mu^a \ ; \ J_{\mu\nu} = \sum_a (q_\mu^a p_\nu^a - q_\nu^a p_\mu^a) \tag{2.4}$$

and the manifestly covariant realization of the Poincare algebra

$$\{ J_{\mu\nu}, J_{\rho\sigma} \} = C_{\mu\nu\rho\sigma}^{\alpha\beta} J_{\alpha\beta} \ ; \ \{ J_{\mu\nu}, P_\mu \} = C_{\mu\nu\rho}^{\alpha} P_\alpha$$

$$\{ P_\mu , P_\nu \} = 0 \tag{2.5a}$$

with the structure constants

$$C^{\alpha\beta}_{\mu\nu\rho\sigma} \equiv \delta_{\mu\rho}\delta^{\alpha}_{\nu}\delta^{\beta}_{\sigma} + \delta_{\nu\sigma}\delta^{\alpha}_{\mu}\delta^{\beta}_{\rho} - \delta_{\mu\sigma}\delta^{\alpha}_{\nu}\delta^{\beta}_{\rho} - \delta_{\nu\rho}\delta^{\alpha}_{\mu}\delta^{\beta}_{\sigma}$$

$$C^{\alpha}_{\mu\nu\rho} \equiv \delta_{\mu\rho}\delta^{\alpha}_{\nu} - \delta_{\mu\nu}\delta^{\alpha}_{\rho} \tag{2.5b}$$

is satisfied as a consequence of the fundamental Poisson brackets

$$\{q^{\mu}_{a}, p^{b}_{\nu}\} = \delta^{b}_{a}\delta^{\mu}_{\nu} \; ; \; \{q^{\mu}_{a}, q^{\nu}_{b}\} = \{p^{a}_{\mu}, p^{b}_{\nu}\} = 0 \tag{2.6}$$

(iii) In order to insure the existence of the independent world lines for each of the particles in the system, the treatment should be centered around a set of equal-time-plane equations[11,22] that initially are set as

$$P_{\mu}\,\xi^{\mu}_{ab} = 0 \tag{2.7}$$

were

$$\xi^{\mu}_{ab} \equiv q^{\mu}_{a} - q^{\mu}_{b} \tag{2.8}$$

and P_{μ} is the total momentum (2.4). The condition stated by equations (2.7), valid for all particle indices, will be referred hereafter as Global Transversality (G.T.) providing us with N-1 independent constraints which guarantee that the interparticle separations (2.8) remain spacelike throughout the motion in all reference frames since their positions in the momentum rest frame ($\underline{P} = 0$) are all simultaneous, i.e., $q_a^o - q_b^o = 0$. This G.T. condition should arise, therefore, at some point of the calculations either as a set of canonical or subsidiary constraints.

The question of separability (or Cluster Decomposition) will be considered later when a explicit model is studied. This condition requires that after division of the system into, say, two clusters of N_1 and N_2 particles which are sufficiently far removed from each other so that the inter-cluster interactions can be neglected, the entire formalism: Lagrangian, equations of motion and constraints automatically become applicable to the two independent subsystems.

2. General Dynamical Implications.

The equations of motion are, hence, the Euler-Lagrange equations (1.5) derived from the variational principle. If one uses definition (2.3), those can be formally written as

$$\dot{p}^a_\mu(q,\dot{q}) = \frac{\partial \mathcal{L}}{\partial q^\mu_a} \qquad (2.9)$$

Now, since the subsequent dynamics should correspond to a closed system of particles in mutual scalar interaction, the Poincare invariance of the action integral (2.1) must lead us inmediately to the conservation of the total linear and angular momenta, i.e., one expects that, for the quantities defined in (2.4), the equations of motion (2.9) imply

$$\dot{P}_\mu = 0 \quad ; \quad \dot{J}_{\mu\nu} = 0 . \qquad (2.10)$$

Both of these results are obtained by choosing a Lagrangian which: (a) depends only on the internal interparticle sepatations through

$$\mathcal{L} = \mathcal{L}(\dot{q}, \rho) \qquad (2.11)$$

with

$$\rho_{ab} = \xi^{\mu}_{ab} \, \xi_{ab\,\mu} \qquad (2.12)$$

and (b) is such that the momenta are expressible as linear combinations of the velocities

$$p^a_\mu(q,\dot{q}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu_a} = B^a{}_b(q,\dot{q}) \, \dot{q}^b_\mu \qquad (2.13)$$

where the $B^a{}_b$ are some symmetric coefficients (which can not be tensors in the coordinate indices μ, ν since only scalar interactions are allowed). $\mathcal{L}$ will depend, therefore, on quadratic functions of the velocities.

Indeed, with (2.11), the equations of motion can be expressed by

$$\dot{p}^a_\mu = \Lambda^a{}_b \, \xi^{ab}_\mu \qquad (2.14a)$$

since

$$\frac{\partial \mathcal{L}}{\partial q^\mu_a} = \frac{\partial \mathcal{L}}{\partial \rho_{ab}} \frac{\partial \rho_{ab}}{\partial q^\mu_a} = 4 \frac{\partial \mathcal{L}}{\partial \rho_a{}^b} \xi^{ab}_\mu \equiv \Lambda^a{}_b \xi^{ab}_\mu \qquad (2.14b)$$

Then, with $\Lambda^a{}_b$ symmetric ($\rho_{ab} = \rho_{ba}$) while ξ^{ab} is antisymmetric in a,b, adding (2.14a) over index a, one has

$$\dot{P}_\mu = \sum_a \dot{P}^a_\mu = \Lambda_{(ab)} \xi^{[ab]}_\mu = 0 \qquad (2.15a)$$

which is the first of the relations (2.10); therefore, we have

$$P^\mu P_\mu = M_0^2 = \text{constant} \qquad (2.15b)$$

where M_0 represents, naturally, the total mass of the system.

Also, from (2.4) it follows that

$$\dot{J}_{\mu\nu} = (q_{a\mu} \dot{P}^a_\nu - q_{a\nu} \dot{P}^a_\mu) + (\dot{q}_{a\mu} P^a_\nu - \dot{q}_{a\nu} P^a_\mu) ;$$

the first parenthesis vanishes as a consequence of (2.14), for,

$$q_{a\mu} \dot{P}^a_\nu - q_{a\nu} \dot{P}^a_\mu = \Lambda^a{}_b (q_{a\mu} \xi^{ab}_\nu - q_{a\nu} \xi^{ab}_\mu) \equiv 0,$$

while the second term vanishes becouse of (2.13), i.e.,

$$\dot{q}_{a\mu} P^a_\nu - \dot{q}_{a\nu} P^a_\mu = B^a{}_b (\dot{q}_{a\mu} \dot{q}^b_\nu - \dot{q}_{a\nu} \dot{q}^b_\mu) = B_{(ab)} \dot{q}^{[a}_\mu \dot{q}^{b]}_\nu \equiv 0$$

and the second of the relations (2.10) is satisfied as well.

A further, and very important, restriction on the choice of possible Lagrangians comes from postulate (iii). If the G.T. conditions are to be integrated somehow into the dynamics, it is required that the particle energies in the momentum rest frame (MRF) be constants of the motion. Indeed, contracting both sides of equations (2.15) with the total momentum vector, we have

$$\mathcal{P}^\mu \dot{p}^a_\mu = \Lambda^a{}_b \mathcal{P}^\mu \xi^{ab}_\mu ; \qquad (2.16)$$

thus, if G.T. (2.7) holds. the right hand side vanishes for all values of a leaving

$$\mathcal{P}^\mu \dot{p}^a_\mu = 0 , \qquad (2.17a)$$

but, due to the constancy of $\mathcal{P}_\mu$, this equation is entirely equivalent to

$$\mathcal{P}^\mu p^a_\mu = k^a = \text{constant.} \qquad (2.17b)$$

The converse is also true provided all particles are indeed interacting. If we start from (2.17), the equations (2.16) can be written in terms of the set of N-1 antisymmetric quantities

$$\chi^{ab} = \mathcal{P}^\mu \xi^{ab}_\mu \qquad (2.18a)$$

as

$$\Lambda^a{}_b \chi^{ab} = 0 \; ; \qquad (2.18b)$$

but not all χ^{ab} are independent. For example, we can set

$$\chi^{ab} = \chi^{a1} - \chi^{b1} \qquad (2.18c)$$

transforming (2.19b) into

$$(\Lambda^a \delta^a{}_b - \Lambda^a{}_b)\chi^{b1} = 0 \qquad (2.19a)$$

with

$$\Lambda^a \equiv \sum_c \Lambda^a{}_c \qquad (2.19b)$$

then, if no other special condition is imposed on the elements $\Lambda^a{}_c$, the matrix

$$S^a{}_b \equiv \Lambda^a \delta^a{}_b - \Lambda^a{}_b \qquad (2.19c)$$

is singular and admits only one null eigenvector, namely, the vector whose elements are all ones, i.e.,

$$\delta_a \equiv (1, \ldots, 1) \tag{2.19d}$$

since

$$\delta_a S^a{}_b = \sum_a S^a{}_b = 0 .$$

The rank of $(S^a{}_b)$ is, therefore, N-1 so that the only solution for the set of N-1 independent equations in N-1 unknowns, χ^{a1}, is the trivial one: $\chi^{a1} = 0$, which, becouse of (2.19c) and (2.19a) is the G.T. statement (2.7).

When (2.17b) is expressed in the MRF (which is inertial be-couse $\mathcal{P}_\mu$ is constant) defined by

$$\mathcal{P}^\mu = M_0 \delta^\mu_0 \; ; \tag{2.20a}$$

we find that

$$P^a_0 = \frac{k^a}{M_0} \equiv M^a = \text{const.}, \tag{2.20b}$$

where the c-number M^a represents the time component of the a-th particle's momentum, which is the energy in the MRF, and equation (2.17b) can now be written

$$\mathcal{P}^{\mu} p^{a}_{\mu} = M_o M^a . \tag{2.21}$$

Summing over index a and comparing with (2.15b) one sees that

$$M_o = \sum_a M^a \tag{2.22}$$

The consequence of the above results in connection with the development given in (Ch. I, #2) is that if G.T. is to constitute a set of canonical constraints, then equations (2.21) must necessarily be the corresponding primary constraints according to equation (1.15),

3. Homogeneous Lagrangians.

By a "homogeneous Lagrangian" it is understood a Lagrangian whose dependence on the velocities is such that for any real positive quantity α,

$$\mathcal{L}(\alpha \dot{q}, \rho) = \alpha \mathcal{L}(\dot{q}, \rho) \; ; \tag{2.23}$$

i.e., $\mathcal{L}$ possesses a first order homogeneity in the velocities. It is well known that this property is directly related with reparametrization invariance of the action integral and, consequently, that this type of Lagrangian must involve interaction potentials only in a multiplicative form.

However, the conclusions of the last section rule out the possibility of having a homogeneous Lagrangian for (2.11) in the general case of a system with more than two particles!

To prove this statement, let us consider two special cases, remembering that the dependence on the velocities must also be quadratic. For a Lagrangian of the form

$$\mathcal{L} = \mathcal{U}_a \mathcal{g}^a \tag{2.24a}$$

with

$$\mathcal{U}_a = \mathcal{U}_a(\rho) \quad \text{and} \quad \mathcal{g}^a \equiv \sqrt{\dot{q}^\mu_a \dot{q}_{a\mu}} \tag{2.24b}$$

which is the natural generalization of the two particle Lagrangian of the Gomis model[22], the momenta are

$$p^a_\mu = \frac{\mathcal{U}^a}{\mathcal{g}_a} \dot{q}^a_\mu \tag{2.24c}$$

and give N primary mass-shell type constraints

$$p^a_\mu p^{a\mu} = (\mathcal{U}^a)^2 \tag{2.24d}$$

which can not be made compatible with (2.21) without introducing more conditions which overconstrain the system, viz

$$\sum_b \frac{\mathcal{U}^b(\rho)}{g^b} \frac{\mathcal{U}^a(\rho)}{g_a} \dot{q}^\mu_b \dot{q}^a_\mu = M_o M^a . \qquad (2.24e)$$

The exceptional case of N=2 is discussed in appendix A. The only other choice consistent with (2.13) is

$$\mathcal{L} = [\ C^a{}_b(\rho)\, \dot{q}^\mu_a \dot{q}^b_\mu\ ]^{1/2} \qquad (2.25a)$$

The momenta (2.3) are now

$$p^a_\mu = \frac{1}{\mathcal{L}} C^a_b \dot{q}^b_\mu \qquad (2.25b)$$

thus,

$$\mathcal{P}_\mu = \frac{1}{\mathcal{L}} \sum_a C^a_b \dot{q}^b_\mu \qquad (2.25c)$$

and we can seek the structure of the $C^a{}_b$'s such that the condition (2.21) is fulfilled, i.e.,

$$\mathcal{P}^\mu p^a_\mu = \frac{1}{\mathcal{L}^2} \sum_e C^e_c C^a_b \dot{q}^c_\mu \dot{q}^{b\mu} = M_o M^a$$

or, equivalently,

$$\sum_e C^e_c C^a_b \dot{q}^c_\mu \dot{q}^{b\mu} = M_o M^a \mathcal{L}^2 \tag{2.25d}$$

Substituting for $\mathcal{L}^2$ its explicit form (2.25a) we obtain the bilinear equality

$$\left(\sum_e C^c_e C^a_b - M_o M^a C^c_b\right) \dot{q}^\mu_c \dot{q}^b_\mu = 0 \tag{2.25e}$$

that can be satisfied, without leading to more constraints, only if the expression inside the parenthesis vanishes identically, i.e., if

$$\sum_e C^c_e C^a{}_b = M_o M^a C^c_b \quad . \tag{2.25f}$$

This reduces the coefficients $C^a b$ to plain c- numbers since the only possible structure consistent with (2.25f) is

$$C^a{}_b = M^a M_b \quad . \tag{2.25g}$$

The Lagrangian (2.25a) is, therefore, reduced to the form

$$\mathcal{L} = \left(M^a M_b \dot{q}^\mu_a \dot{q}^b_\mu\right)^{1/2} . \tag{2.26}$$

The preceding arguments can be extended, in fact, to any Lagrangian and, although a formal proof would be rather cumbersome, the evidence that Lagrangian (2.26) is the only one which admits equations (2.21) as its primary constraints can be seen from the fact that since (2.13) is the general form for the momenta, to satisfy (2.21) one has to reduce the equivalent expression

$$\sum_e B^c_e B^a_b \dot{q}^\mu_c \dot{q}^b_\mu = M_0 M^a \tag{2.27}$$

to an identity without any further requirement. After an inspection of the coefficients $B^a b$, one is forced to admit that the only possibility to achieve that goal is precisely by deriving those coefficients from the Lagrangian (2.26).

But it is not difficult to see that (2.26) and the subsequent equations of motion describe a system with only three degrees of freedom. Indeed, its Hessian

$$W^{ab}_{\mu\nu} = \frac{M^a M^b}{\mathcal{L}} \left(\delta_{\mu\nu} - \frac{\dot{P}_\mu \dot{P}_\nu}{M_0^2} \right) \tag{2.28}$$

admits 4N-3 independent null eigenvectors, namely, N null eigenvectors of the form

$$\lambda^c_a{}^\mu = \delta^c_a P^\mu \tag{2.29a}$$

and 3(N-1) of the form

$$\lambda^{c\ \mu}_{a\ n} = \left(\delta^c_a M^c - \delta^c_a M_a \right) X^\mu_n \tag{2.29b}$$

where the three four-vectors X^{μ}_{n} (n=1,2,3) are orthogonal to the total momentum,

$$P_{\mu} X^{\mu}_{n} = 0 . \tag{2.29c}$$

The Hessian (2.28) is, hence, of rank three. Furthermore, the momenta for this Lagrangian,

$$p^{a}_{\mu} = \frac{M^{a}}{\mathcal{L}} M_{b} \dot{q}^{b}_{\mu} \tag{2.30a}$$

can be expressed in terms of the total momentum (see eq. (2.22)),

$$P_{\mu} = \frac{M_{0}}{\mathcal{L}} M_{b} \dot{q}^{b}_{\mu} \tag{2.30b}$$

as

$$p^{a}_{\mu} = \frac{M^{a}}{M_{0}} P_{\mu} \tag{2.30c}$$

and the equations of motion reduce to

$$\dot{p}^{a}_{\mu} = 0 \qquad ; \qquad \dot{P}_{\mu} = 0 . \tag{2.30d}$$

Consequently, Lagrangian (2.26) can be interpreted as appropriate to describe the motion of the center of momentum

$$Q^{\mu} = \frac{M^{a}}{M_{0}} q^{\mu}_{a} \tag{2.31}$$

of a rigid system of particles whose rest masses, because those constituents have no internal motion, become in this case identical to their energies M^a.

We conclude, therefore, that G.T. conditions can not be expected to constitute a set of canonical constraints for systems with more than two particles but, rather, they must be imposed as subsidiary constraints in a way that becomes consistent with the equations of motion if internal motions can be included somehow in (2.11).

4. Non-homogeneous Lagrangians.

From the results of the last section, it is apparent that in order to include G.T. condition in a manifestly covariant canonical formalism we have to relax reparametrization invariance and turn our attention to non-homogeneous Lagrangians which, singular or standard, must be compatible with G.T. as a set of subsidiary constraints.

One example of that kind of Lagrangian has been proposed by Rohrlich[18] in an initially expanded configuration space of 4 (N + 1) coordinates $\{Q^\mu; \xi^\mu_\alpha\}$ (the C.M. plus internal variables). We shall show in this section and next chapters that similar, but not equivalent, approaches are possible on the original configuration space of 4N coordinates $\{q^\mu_\alpha\}$.

Actually, a model consistent with G.T. can be built up by adding terms to (2.26), allowing for relative constituent motions and interactions, in such a way that the last 3N-3 null eigenvectors (2.29b) of the Hessian and N-1 of those in (2.29a) are eliminated from the analysis. To this end, we consider a Lagrangian with the following general structure

$$\mathcal{L} = \mathcal{L}_H + \mathcal{L}_o(\xi, \rho) \tag{2.32a}$$

where $\mathcal{L}_H$ is the homogeneous form (2.26) and $\mathcal{L}_o$ a quadratic function of the relative velocities $\dot{\xi}^{ab}_\mu = \dot{q}^a_\mu - \dot{q}^b_\mu$ through

$$\xi^{ab} \equiv \dot{\xi}^{ab}_\mu \dot{\xi}^{ab\mu} \tag{2.32b}$$

The momenta are here

$$p^a_\mu = \frac{M^a M_b}{\mathcal{L}_H} \dot{q}^b_\mu + D^a{}_b \dot{\xi}^{ab}_\mu \tag{2.33a}$$

with

$$D^a{}_c \equiv 4 \frac{\partial \mathcal{L}_o}{\partial \xi^c_a} \tag{2.33b}$$

some symmetric coefficients. Since

$$D_{(ab)} \dot{\xi}^{[ab]}_\mu \equiv 0 \tag{2.33c}$$

by symmetry arguments; the total momentum is

$$P_\mu = \frac{M_o M_b}{\mathcal{L}_H} \dot{q}^b_\mu \,. \tag{2.34}$$

From this equation and (2.26) we get immediately

$$P^{\mu} P_{\mu} = M_0^2 \tag{2.35a}$$

as a primary constraint. Because the G.T. conditions and their derivatives already give a set of 2N-2 constraints on the variables $\{q;\dot{q}\}$, we expect (2.35a) to constitute a first class constraint, which is to say that there must be only one null eigenvector for the Hessian and it must not lead to any canonical constraint; therefore, $\mathcal{L}_0$ can not be homogeneous and we should make

$$\frac{\partial D^a_c}{\partial \dot{q}^{\nu}_b} = 0 \tag{2.35b}$$

so that the Hessian becomes

$$W^{ab}_{\mu\nu} = \frac{M^a M^b}{\mathcal{L}_H}\left(\delta_{\mu\nu} - \frac{P_\mu P_\nu}{M_0^2}\right) + \left(D^a \delta^{ab} - D^{ab}\right)\delta_{\mu\nu} \tag{2.35c}$$

$\left(D^a \equiv \sum_b D^a{}_b \right)$ whose only null eigenvector is, as required,

$$\lambda^{\mu}_a = \delta_a P^{\mu}, \tag{2.35d}$$

where δ_a is the vector defined in (2.19d). The equations of motion are

$$W^{ab}_{\mu\nu} \ddot{q}^{\nu}_b = \alpha^a_{\mu}(q,\dot{q}) \tag{2.36a}$$

where, in this case, the function (1.7) is

$$\alpha^a_\mu(q,\dot{q}) = \Lambda^a{}_b\, \xi^{ab}_\mu - \frac{\partial D^a_c\, \dot{\xi}^{ac}_\mu}{\partial q^\nu_b}\, \dot{q}^\nu_b \tag{2.36b}$$

The fact that there are no canonical constraints following from these equations of motion can be seen immediately, as a result of summations performed over objects with opposite symmetries, when a contraction is carried out between (2.35d) and (2.36a) obtaining

$$\alpha^a_\mu\, \delta_a \mathcal{P}^\mu = \left(\Lambda_{(ab)}\, \chi^{[ab]} - \frac{\partial D_{(ac)}}{\partial q^\nu_b}\, \dot{\chi}^{[ac]}\, \dot{q}^\nu_b\right) \equiv 0 \tag{2.36c}$$

as in (2.15a) and (2.33c); hence

$$\alpha^a_\mu \lambda^\mu_a \equiv 0 \; . \tag{2.36d}$$

this is an indication that the Lagrangian (2.32) is at least quasi-invariant under a gauge transformation of the form (1.10):

$$\delta q^\mu_a = \epsilon f(s)\, \lambda^\mu_a \tag{2.37}$$

and that the only primary constraint (2.35a) will be first class.

The subsidiary constraints to be imposed come naturally if we transform the identity (2.36d) into a stronger set of requirements:

$$\alpha^a_\mu \lambda^{a\mu} = 0 \tag{2.38a}$$

whose satisfaction implies simultaneously the G.T. condition and its derivative,

$$\mathcal{P}^{\mu} \xi^{ab}_{\mu} = 0 \quad ; \quad \mathcal{P}^{\mu} \dot{\xi}^{ab}_{\mu} = 0 . \tag{2.38b}$$

Those equations constitute the only set of constraints in the theory so far, because a differentiation of the second one gives

$$\mathcal{P}^{\mu} \ddot{\xi}^{ab}_{\mu} = 0 \tag{2.38c}$$

a set of N-1 equations for the accelerations which are already contained in the equations of motion (2.36a) implicitly. The proof of this statement follows from the fact that (2.38c) can be deduced from (2.36a) and (2.38b) by algebraic operations alone. Indeed, if we take the internal product of (2.36a) with $\mathcal{P}_{\mu}$ and use (2.38a) we are left with

$$\left(D^{a} \delta^{ab} - D^{ab} \right) \mathcal{P}_{\mu} \ddot{q}^{\mu}_{b} = 0 \tag{2.39a}$$

which can be rewritten as

$$D^{b}_{a} \mathcal{P}_{\mu} \ddot{\xi}^{\mu}_{ab} \equiv D^{b}_{a} \ddot{\chi}_{ab} = 0 \; ; \tag{2.39b}$$

by arguments similar to those that led to equations (2.19e), equations (2.39b) constitute a set of N-1 independent equations in N-1 independent variables $\ddot{\chi}_{1a}$ whose only solution is the trivial one, so

$$\ddot{\chi}_{ab} = P_\mu \ddot{\xi}^\mu_{ab} = 0$$

as in (2.38c).

We have then 4N-1 independent equations of motion (2.36a) which leave the 2N-2 constraints (2.38b) invariant in $\{q; \dot{q}\}$. Our effective number of degrees of freedom in the configuration space is, therefore, 3N + 1; the remaining one degree of freedom corresponds to the arbitrariness of time-parameter and can be determined by a gauge-fixing constraint.

One can notice that, using (2.34), the momenta can be expressed in the form

$$p^a_\mu = \frac{M^a}{M_0} P_\mu + \pi^a_\mu \tag{2.40a}$$

where the "internal momenta" π^a_μ are defined as

$$\pi^a_\mu \equiv D^a_c \dot{\xi}^{ac}_\mu . \tag{2.40b}$$

Now, equations (2.38b) are equivalent to

$$P^\mu \pi^a_\mu = 0 \tag{2.40c}$$

and one finds from (2.40a) that equations (2.21) will indeed play the role of "primary constraints" when the passage to the Hamiltonian formalism is done.

The Lagrangian, according to the above analysis, must have the general structure

$$\mathcal{L} = \left(M_a M^b \dot{q}^a_\mu \dot{q}^\mu_b\right)^{1/2} + \frac{1}{4} D_{ab}(\rho)\, \zeta^{ab} + \mathcal{U}(\rho) \,. \tag{2.41}$$

In the next chapter we shall consider in detail a model based on the simplest explicit choice, i.e.,

$$D_{ab} = \frac{M_a M_b}{M_0} \tag{2.42}$$

leaving only the additive term for the interaction potentials.

III. MULTIPARTICLE DYNAMICAL MODEL BASED ON A SINGULAR LAGRANGIAN.

From the analysis carried out in the previous chapters we see that the possible choices for a specific Lagrangian structure have been reduced greatly as a result of the G.T. requirement. Since the impossibility of having such condition as a set of canonical constraints makes completely unnecessary the requirement of homogeneity of the Lagrangian and, thence, the choice of multiplicative potentials, the form (2.41) with (2.42) as its coefficients becomes a sufficient starting point for a canonical formalism.

1. Lagrangian Formalism.

Let us consider the following Lagrangian:

$$\mathcal{L} = \left(M_a M^b \dot{q}^a_\mu \dot{q}^\mu_b \right)^{1/2} + \frac{M_a M_b}{4 M_0} \zeta^{ab} - \mathcal{U}(\rho) \qquad (3.1)$$

with all quantities defined previously. The nature of the c-numbers M^a is still undetermined and the explicit form of $\mathcal{U}(\rho)$ which contains the interactions is also arbitrary at this point. Of course, in spite of the non homogeneity of the last two terms, the Lagrangian (3.1) is singular and the results of (Ch. II, #4) are all applicable. The Hessian (2.35c) becomes

$$W^{ab}_{\mu\nu} = \frac{M^a M^b}{\mathcal{L}_H} \left(\delta_{\mu\nu} - \frac{P_\mu P_\nu}{M_0^2} \right) + \left(M^b \zeta^{ab} - \frac{M^a M^b}{M_0} \right) \delta_{\mu\nu} \qquad (3.2)$$

whose only null eigenvector is (2.35d), while the function α^a_μ

is here specifically

$$\alpha^a_\mu = \Lambda^a{}_b \, \dot{\xi}^{ab}_\mu \tag{3.3a}$$

with

$$\Lambda^a{}_b \equiv -4 \frac{\partial \mathcal{U}}{\partial \rho_a{}^b} \tag{3.3b}$$

thus, $\alpha^a_\mu \lambda^\mu_a \equiv 0$ as in (2.36d). According to equation (B.7) of Appendix B, this means that under a gauge transformation of the form (2.37) the variation of the Lagrangian is

$$\delta \mathcal{L} = \epsilon \, \frac{d}{ds}(F) \tag{3.4a}$$

with

$$F = f(s)\, P_\mu P^\mu \; . \tag{3.4b}$$

The Lagrangian (3.1) is, therefore, quasi-invariant under such transformation and the associated conserved identity is, as we already know,

$$\varphi \equiv P^2 - M_0^2 = 0 \; . \tag{3.4c}$$

We have, then, that (3.1) gives rise to 4N-1 independent equations to motion

$$W^{ab}_{\mu\nu}\,\ddot{q}^{\nu}_{b} = \alpha^{a}_{\mu}(q) \tag{3.5}$$

supplemented and compatible with a set of 2N-2 independent constraints given by the G.T. conditions and their derivatives, (2.38b), imposed as the subsidiary constraints (2.38a). Because of the relationship of G.T. with equations (2.21), the identification of the constants M^a with the M.R.F. energies of the particles appears at this point. Only the gauge remains to be fixed but we shall postpone that until after the Hamiltonian has been found.

2. Hamiltonian Formalism.

According to equation (2.33a), the momenta are

$$p^{a}_{\mu} = \frac{M^a}{M_o}\mathcal{P}_\mu + \frac{M^a M_b}{M_o}\dot{\xi}^{ab}_{\mu} \tag{3.6a}$$

where

$$\mathcal{P}_\mu = \frac{M_a M_b}{\mathcal{L}_H}\dot{q}^{b}_{\mu} \,. \tag{3.6b}$$

The canonical Hamiltonian is defined as usual by

$$H_c = p^{a}_{\mu}\dot{q}^{\mu}_{a} - \mathcal{L} \,. \tag{3.7}$$

Because the Lagrangian yields one primary constraint, (3.4c), the Hamiltonian for the system will be weakly equal to the canonical Hamiltonian modulo φ :

$$H = H_c + v\,(\mathcal{P}^2 - M_o^2) \tag{3.8}$$

The explicit form of H_c is easily determined if we use the relation

$$\dot{\xi}^{ab} = \frac{p_\mu^a}{M_a} - \frac{p_\nu^b}{M_b} \tag{3.9}$$

which follows directly from (3.6) through

$$\dot{q}_\mu^a = \frac{1}{M_a}\, p_\mu^a - \frac{1}{M_o}\left(1 - \frac{\mathcal{L}_H}{M_o}\right)\mathcal{P}_\mu \,, \tag{3.10}$$

(3.7) then gives

$$H_c = \frac{M^a M_b}{M_o}\left(\frac{p_\mu^a}{M_a} - \frac{p_\mu^b}{M_b}\right)\left[\frac{p_a^\mu}{M^a} - \left(\frac{1}{M_o} - \frac{\mathcal{L}_H}{M_o^2}\right)\mathcal{P}^\mu\right] -$$
$$- \frac{M_a M_b}{4 M_o}\left(\frac{p_\mu^a}{M_a} - \frac{p_\mu^b}{M_b}\right)\left(\frac{p^{a\mu}}{M_a} - \frac{p^{b\mu}}{H_b}\right) + \mathcal{U}(\rho)\,;$$

after expanding the terms and performing the sums this reduces to

$$H_c = \frac{p_\mu^a p_a^\mu}{2 M_a} - \frac{\mathcal{P}_\mu \mathcal{P}^\mu}{2 M_o} + \mathcal{U}(\rho) \tag{3.11}$$

an expression that, by virtue of (3.4c), is weakly equal to

$$H_o = \frac{p^a_\mu p^\mu_a}{2M_a} + \left(\mathcal{U}(\rho) - \frac{M_o}{2}\right). \qquad (3.12)$$

Therefore, (3.8) becomes

$$H = H_o + v\,(\mathcal{P}^2 - M_o^2)\,. \qquad (3.13)$$

The stability of (3.4c) is automatically satisfied since commutes with H_0 :

$$\dot{\varphi} = \{\varphi, H\} = \{\varphi, H_o\} \approx 0\,,$$

as should be the case since φ is first class. But (3.13) is not yet complete; one has to impose now the subsidiary constraints (2.21). The set

$$\varphi^a \equiv \mathcal{P}^\mu p^a_\mu - M_o M^a \approx 0 \qquad (3.14)$$

contains only N-1 independent constraints because they are connected with (3.4c) through

$$\varphi = \sum_a \varphi^a \approx 0 \qquad (3.15)$$

and, involving only the momenta, they obviously commute with φ :

$$\{ \varphi, \varphi^a \} \approx 0 . \qquad (3.16)$$

Consequently, stability conditions on the φ^a give rise to a second set of N-1 independent constraints,

$$\dot{\varphi}^a = \{ \varphi^a, H \} = \{ \varphi^a, H_o \} = -\frac{\partial H_o}{\partial q_c^\mu} \left(\delta_c P^{a\mu} + \delta_c^a P^\mu \right) \approx 0 \qquad (3.17a)$$

where, from (3.12) and (3.3b),

$$-\frac{\partial H_o}{\partial q_c^\mu} = \Lambda_e^c \; \xi_\mu^{ec} . \qquad (3.17b)$$

Further, using (2.15a), one reduces (3.17a) to

$$\dot{\varphi}^a = \{ \varphi^a, H_o \} = \Lambda_e^a \, P^\mu \xi_\mu^{ae} \approx 0 \qquad (3.17c)$$

which is satisfied if G.T. holds:

$$\chi_{ab} \equiv P_\mu \, \xi_{ab}^\mu \approx 0 \qquad (3.18)$$

Now, it can be readily verified that the χ_{ab} commute with φ and that stability conditions for these constraints are satisfied modulo (3.14), i.e.,

$$\{\chi_{ab}, H\} = \mathcal{P}_\mu \left(\frac{\partial H_o}{\partial p^a_\mu} - \frac{\partial H_o}{\partial p^b_\mu}\right) = \mathcal{P}_\mu \left(\frac{p^\mu_a}{M^a} - \frac{p^\mu_b}{M^b}\right) \approx$$

$$\approx \left(\frac{M_o M_a}{M^a} - \frac{M_o M_b}{M^b}\right) = 0$$

then

$$\dot{\chi}_{ae} \approx 0 \ . \tag{3.19}$$

Now, if we rewrite the complete set of 2N-2 subsidiary constraints as

$$\varphi^{a'} \equiv \mathcal{P}^\mu p^{a'}_\mu - M_o M^{a'} \approx 0 \tag{3.20a}$$

$$(a' = 2 \ldots N)$$

$$\chi_{a'} \equiv \mathcal{P}_\mu \, \xi^\mu_{1a'} \approx 0 \tag{3.20b}$$

the total Hamiltonian, $\mathcal{H}$, becomes

$$\mathcal{H} = H_o + v\varphi + v_{a'}\varphi^{a'} + u^{a'}\chi_{a'} \ ; \tag{3.21}$$

in terms of this Hamiltonian, since all subsidiary constraints commute with φ , equations (1.20) become, after some rearrangement,

$$\begin{pmatrix} \{\varphi^{a'}, \varphi^{c'}\} & \{\varphi^{a'}, \chi_{c'}\} \\ \{\chi_{a'}, \varphi^{c'}\} & \{\chi_{a'}, \chi_{c'}\} \end{pmatrix} \begin{pmatrix} v_{c'} \\ u^{c'} \end{pmatrix} \approx - \begin{pmatrix} \{\varphi^{a'}, H_o\} \\ \{\chi_{a'}, H_o\} \end{pmatrix} \tag{3.22}$$

where the matrix of P.b. among the subsidiary constraints, C^{A}_{c}, is nonsingular. But, because of (3.17c) and (3.19), the right hand side of equation (3.22) vanishes weakly giving as a result the trivial solution

$$v_{c'} \approx 0 \qquad ; \quad u^{c'} \approx 0 \qquad (3.23)$$

for the coefficients in (3.21). Thus, the total Hamiltonian is strongly equal to (3.13),

$$\mathcal{H} \equiv H = H_0 + v\varphi \;, \qquad (3.24)$$

and only the gauge is left to be determined. The P.b. equations of motion are

$$\dot{q}^{\mu}_{a} = \{ q^{\mu}_{a}, \mathcal{H} \} = \frac{p^{\mu}_{a}}{M^{a}} + 2v\delta_{a} P^{\mu} \qquad (3.25a)$$

$$\dot{p}^{a}_{\mu} = \{ p^{a}_{\mu}, \mathcal{H} \} = -\frac{\partial \mathcal{U}}{\partial q^{\mu}_{a}} = \Lambda^{a}{}_{b}\, \xi^{ab}_{\mu} \qquad (3.25b)$$

3. Gauge Fixing Constraint.

The Hamiltonian (3.24) and equations (3.26) describe a system with 6N + 1 degrees of freedom. In order to obtain a 6N dimensional description one can apply formulae (1.24) to (1.27) once the last subsidiary gauge fixing constraint is chosen. Although such a constraint has an arbitrary structure in principle, our model requires it to be manifestly covariant.

Because the form of the only primary constraint corresponds to the mass-shell condition for a free single particle, it seems quite evident that the unsolved velocity, contained in the coefficient $\mho$, should be related to the velocity of the center of momentum. The immediate choice for the constraint in question is, therefore,

$$\chi \equiv P_\mu Q^\mu - M_0 s \approx 0 \tag{3.26}$$

where s represents hereafter the proper time (or world line length) of the C.M. which also requires a manifestly covariant definition.

Since none of the C.M. definitions listed by Pryce[33] seem to fit properly in the present scheme, we shall adopt the following:

$$Q^\mu = \eta^a q^\mu_a \tag{3.27a}$$

with

$$\eta^a = \frac{M^a}{M_0} \quad ; \quad \sum_a \eta^a = 1 \; . \tag{3.27b}$$

It can be readily verified that (3.27) satisfies the correct P.b. with P_μ, and only when restricted to the subspace determined by (3.14) and (3.4c) is consistent with the weak definition

$$Q^{\mu} \approx \frac{\mathcal{P}^{\nu} p_{\nu}^{a}}{\mathcal{P}_{\nu} \mathcal{P}^{\nu}} q_{a}^{\mu} \quad . \tag{3.28}$$

The total Hamiltonian is now complete and given by

$$\mathcal{H} = H_0 + v\varphi + u\chi \tag{3.29}$$

To determine the coefficients, we apply here the solutions (1.26) taking into account that

$$\{\chi, \varphi\} = 2\mathcal{P}^2 \tag{3.30a}$$

while

$$\{\chi, H_0\} \approx \eta^a \frac{p_a^{\mu}}{M^a} \mathcal{P}_{\mu} \approx \frac{\mathcal{P}^2}{M_0} \tag{3.30b}$$

and

$$\frac{\partial \chi}{\partial s} = M_0 \quad . \tag{3.30c}$$

Consequently,

$$\begin{pmatrix} 0 & -2\mathcal{P}^2 \\ 2\mathcal{P}^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} \approx 0 \tag{3.31a}$$

so that

$$\mathcal{V} \approx 0 \quad ; \quad \mathcal{U} \approx 0 \ . \tag{3.31b}$$

This reduces the total Hamiltonian to an expression strongly equal to H_0, i.e.,

$$\mathcal{H} \equiv H_o = \frac{p_\mu^a \, p_a^\mu}{2 M_o} + U(\rho) \tag{3.32}$$

with

$$U(\rho) = \mathcal{U}(\rho) - \frac{M_o}{2} \tag{3.33}$$

and the equations of motion (3.17) assume their final form

$$\dot{q}_a^\mu = \{ q_a^\mu , \mathcal{H} \} = \frac{p_a^\mu}{M^a} \tag{3.34a}$$

$$\dot{p}_\mu^a = \{ p_\mu^a , \mathcal{H} \} = - \frac{\partial U}{\partial q_a^\mu} \tag{3.34b}$$

leaving invariant the subspace of the phase-space defined by equations (3.4c), (3.20), (3.26) which are 2N in total.

Finally, if we compare (3.10b) with (3.25a), we get

$$\mathcal{V} = \frac{1}{2} \left(\frac{\mathcal{L}_H}{M_o^2} - \frac{1}{M_o} \right) \tag{3.35}$$

so, at the Lagrangian level, the solution (3.31b) corresponds to

$$\mathcal{L}_H = M_o \, , \tag{3.36}$$

consistent with definition (3.27): if we adopt that definition from the outset, the Lagrangian descriptions can be given by

$$\mathcal{L} = M_o \left(\dot{Q}^\mu \dot{Q}_\mu\right)^{1/2} + \frac{M_a M_b}{4 M_o} \zeta^{ab} - \mathcal{U}(\rho) \tag{3.37a}$$

plus the subsidiary constraints

$$\mathcal{P}_\mu \, \xi^\mu_{ab} = 0 \quad ; \quad \dot{Q}^\mu \dot{Q}_\mu = 1 \, . \tag{3.37b}$$

4. Internal Variables.

The model, as developed thus far, permits consistent definitions of the C.M. variables $\{Q^\mu ; P_\mu\}$ in terms of the canonical set $\{q^\mu_a ; p^a_\mu\}$. It therefore permits explicit realization of the "internal variables", i.e., those relative to the C.M., as well. Such variables $\{\xi^\mu_a ; \pi^a_\mu\}$ have the familiar definition[18,20,25,26]:

$$q^\mu_a = Q^\mu + \xi^\mu_a \tag{3.38a}$$

$$p^a_\mu = \eta^a \mathcal{P}_\mu + \pi^a_\mu \quad . \tag{3.38b}$$

One then finds, recalling (3.6a) and (3.27), that

$$\xi^{\mu}_{a} = \Delta^{c}_{a}\, q^{\mu}_{c} \quad ; \quad \pi^{a}_{\mu} = \Delta^{a}{}_{c}\, p^{c}_{\mu} \tag{3.39a}$$

with

$$\Delta^{a}{}_{b} \equiv \delta^{a}{}_{b} - \eta^{a}\delta_{b} \tag{3.39b}$$

Further, substitution of (3.38) into (3.32) allows separation of the Hamiltonian into a part for the C.M. and a remaining part which describes the internal dynamics of the particles relative to the C.M., i.e.,

$$\mathcal{H} = \mathcal{H}_{CM} + \mathcal{H}_{I} \tag{3.40a}$$

where

$$\mathcal{H}_{CM} = \frac{1}{2M_o}\left(P^2 - M_o^2\right) \tag{3.40b}$$

and

$$\mathcal{H}_{I} = \frac{\pi^{a}_{\mu}\pi^{\mu}_{a}}{2M_a} + \mathcal{U}(\rho)\,. \tag{3.40c}$$

Therefore, the equations of motion (3.34) become

$$\dot{\xi}^{\mu}_{a} = \{\xi^{\mu}_{a}, \mathcal{H}\} = \frac{\pi^{\mu}_{a}}{M^{a}} \tag{3.41a}$$

$$\dot{\pi}^a_\mu = \{ \pi^a_\mu, \mathcal{H} \} = - \frac{\partial \mathcal{U}}{\partial \xi^\mu_a} \qquad (3.41b)$$

and

$$\dot{Q}^\mu = \{ Q^\mu, \mathcal{H} \} = \frac{P^\mu}{M_o} \; ; \; \dot{P}^\mu = \{ P^\mu, \mathcal{H} \} = 0 \qquad (3.41c)$$

Clearly, the solution of the last set of equations is as for free particle motion

$$Q^\mu = \frac{P^\mu}{M_o} s + Q^\mu(o) \;, \qquad (3.42)$$

since $Q^\mu(o)$ should be consistent with constraints (3.4c), (3.20) and (3.26), in terms of all the constants of the motion we can write (3.42) as a weak equality

$$Q^\mu \approx \frac{P^\mu}{M_o} s + \frac{J^{\mu\nu} P_\nu}{M_o^2} \qquad (3.43)$$

whose P.b.'s

$$\{ Q^\mu, Q^\nu \} = \frac{-1}{P^2} J^{\mu\nu} \; ; \; \{ Q^\mu, P_\nu \} \approx \delta^\mu_\nu - \frac{P^\mu P_\nu}{P^2} \qquad (3.44)$$

(valid only in the subspace of the constraints) are identical, as we shall see, with the corresponding Dirac brackets.

The internal motion of the particles will be determined, of course, from equations (3.41) once the explicit form of the potential $\mathcal{U}(\rho)$ is given; except for its dependence on the relative separations alone, we have made no other assumption on the possible structure of that term, so there is still a good deal of flexibility in the model. If $\mathcal{U}(\rho)$ does not depend explicitly on the time-parameter, the Hamiltonian itself becomes a constant of the motion:

$$\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} + \frac{\partial \mathcal{H}}{\partial s} = 0 ,$$

then, from (3.40) we have that

$$\mathcal{H}_T = \frac{\Pi^a_\mu \Pi^\mu_a}{2M_a} + \mathcal{U}(\rho) = \text{constant} \qquad (3.45)$$

5. Dirac Bracket Realization.

An alternative, equivalent and more symmetrical way to express the 2N constraints (3.4c), (3.20) and (3.26) is as the set of 2N weak equalities

$$\varphi^a \equiv P^\mu p^a_\mu - M_0 M^a \approx 0 \qquad (3.46a)$$

$$\chi_a \equiv P_\mu q^\mu_a - M_0 s \approx 0 \qquad (3.46b)$$

It can be verified that (3.4c) and (3.26) are consequences of (3.46) since

$$\varphi = \sum_a \varphi^a \quad ; \quad \chi = \eta^a \chi_a . \tag{3.47}$$

Using (3.46) and (3.47) one can also write

$$\varphi^a = P^\mu \left(p^a_\mu - \eta^a P_\mu \right) = P^\mu \pi^a_\mu \approx 0 \tag{3.48a}$$

$$\chi_a = P_\mu \left(q^\mu_a - Q^\mu \right) = P_\mu \xi^\mu_a \approx 0 \tag{3.48b}$$

for the internal variables (3.39).

The Hamiltonian (3.32) remains, of course, unchanged and the stability conditions on (3.46) are automatically satisfied

$$\{ \varphi^a , \mathcal{H} \} \approx 0 \quad ; \quad \{ \chi_a , \mathcal{H} \} + \frac{\partial \chi_a}{\partial s} \approx 0 . \tag{3.49}$$

Now, the matrix (1.28) of P.b.'s among those constraints is

$$C^A{}_B = P^2 \begin{pmatrix} 0 & (\delta^a{}_b + \eta^a \delta_c) \\ (\delta^a_b + \delta^a \eta_b) & 0 \end{pmatrix} \tag{3.50a}$$

whose inverse is

$$C^{-1 A}{}_B = \frac{1}{2P^2} \begin{pmatrix} 0 & (2\delta^a{}_b - \delta^a \eta_b) \\ (2\delta^a_b - \delta_b \eta^a) & 0 \end{pmatrix} \tag{3.50b}$$

For any given operator, X , let us adopt the notation:

$$(X, \varphi^A) \equiv \begin{pmatrix} \{X, \varphi^a\} \\ \{X, \chi_a\} \end{pmatrix} , \qquad (3.51)$$

and we find by direct calculation the following:

$$(J_{\mu\nu}, \varphi^A) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad (P_\mu, \varphi^A) = \begin{pmatrix} 0 \\ -\delta^a P_\mu \end{pmatrix} \qquad (3.52a)$$

$$(Q^\mu, \varphi^A) = \begin{pmatrix} \eta^a P^\mu + p^{a\mu} \\ q^\mu_a \end{pmatrix} ; \quad (\mathcal{H}, \varphi^A) = \begin{pmatrix} 0 \\ -M_0 \end{pmatrix} \qquad (3.52b)$$

$$(p^a_\mu, \varphi^c) = \begin{pmatrix} 0 \\ -\delta^{ac} P_\mu \end{pmatrix} ; \quad (q^\mu_a, \varphi^c) = \begin{pmatrix} \delta^c_a P^\mu - \delta_a p^{c\mu} \\ \delta_a q^{c\mu} \end{pmatrix} \qquad (3.52c)$$

From this and (1.30) we have

$$\{J_{\mu\nu}, J_{\mu\nu}\}^* = C^{\alpha\beta}_{\mu\nu\rho\sigma} J_{\alpha\beta} \; ; \; \{J_{\mu\nu}, P_\rho\}^* = C^{\alpha}_{\mu\nu\rho} P_\nu \; ; \; \{P_\mu, P_\nu\}^* = 0 \qquad (3.53a)$$

$$\{J_{\mu\nu}, Q_\rho\}^* = C^{\alpha}_{\mu\nu\rho} Q_\alpha \; ; \; \{Q^\mu, P_\nu\}^* = \Theta^\mu_\nu \; ; \; \{Q^\mu, Q^\nu\}^* = \frac{-1}{P^2} J^{\mu\nu} \qquad (3.53b)$$

$$\{J_{\mu\nu}, \mathcal{H}\}^* = 0 \; ; \; \{P_\mu, \mathcal{H}\}^* = 0 \; ; \; \{Q^\mu, \mathcal{H}\}^* = 0 \qquad (3.53c)$$

with

$$\Theta^\mu_\nu \equiv \delta^\mu_\nu - \frac{P^\mu P_\nu}{P^2} \qquad (3.54)$$

The remaining D.b. are

$$\{q^\mu_a, q^\nu_b\}^* = -\frac{1}{P^2}\{(\delta_a q^\mu_b P^\nu - \delta_b q^\nu_a P^\mu) - -\delta_a\delta_b(Q^\mu P^\nu - Q^\nu P^\mu) + \delta_a\delta_b J^{\mu\nu}\} \qquad (3.55a)$$

$$\{q^\mu_a, p^b_\nu\}^* = \Theta^\mu_\nu - \frac{\delta_a}{P^2}(p^{b\mu} - \eta^b P^\mu)P_\nu \; ; \; \{p^a_\mu, p^b_\nu\}^* = 0 \qquad (3.55b)$$

$$\{q^\mu_a, \mathcal{H}\}^* = \frac{p^\mu_a}{M^a} - \frac{P^\mu}{M_0} \; ; \; \{p^a_\mu, \mathcal{H}\}^* = -\frac{\partial U}{\partial q^\mu_a} \qquad (3.55c)$$

We see that the Poincare algebra (3.53a) is preserved when confined to the dynamical subspace. The equations of motion are given here by (1.32)

$$\dot{A} = \{A, \mathcal{H}\}^* + \frac{\partial^* A}{\partial s}$$

and they coincide, as they should, with (3.34). Or, if internal variables are introduced, with (3.40) and (3.41). In terms of those internal variables, incidentally, we can replace the D.b.'s (3.55) by the simpler relations

$$\{\xi^\mu_a, \xi^\nu_b\}^* = 0 \quad ; \quad \{Q^\mu, \xi^\nu_a\}^* = -\frac{1}{P^2}\xi^\mu_a P^\nu \tag{3.56a}$$

$$\{\xi^\mu_a, \pi^b_\nu\}^* = \Delta^b_a \Theta^\mu_\nu \quad ; \quad \{\xi^\mu_a, P_\nu\}^* = 0 \tag{3.56b}$$

$$\{\pi^a_\mu, \pi^b_\nu\}^* = \{P_\mu, \pi^a_\nu\}^* = 0 \quad ; \quad \{Q^\mu, \pi^a_\nu\}^* = \frac{-1}{P^2}\pi^{a\mu}P_\nu \tag{3.56c}$$

which coincide with algebraic structures given elsewhere[18,20,26].

Questions related with whether we can systematically use time-parameters other than the C.M.'s and whether we can achieve cluster decomposition will lead us to look at the problem from a different point of view in the next chapter.

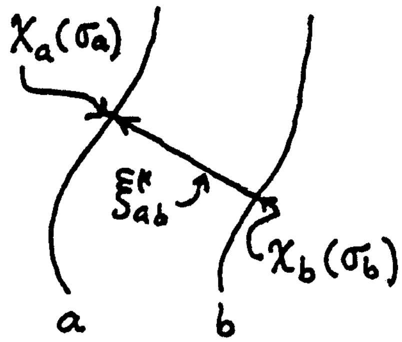
IV. MULTIPARTICLE DYNAMICAL MODEL BASED ON A STANDARD LAGRANGIAN.

The result that G.T. conditions can not follow, as a set of canonical constraints, from a Lagrangian treatment has lead us to the model based on the singular Lagrangian considered in the preceding chapter and explicitly adapted to accommodate G.T. as a set of subsidiary constraints. However, the main properties of the formalism developed before can also be obtained starting from a standard Lagrangian with the additional advantage that the cluster decomposition property can be made possible after a reinterpretation of the constraints. This calls for a more detailed discussion of postulate (iii) in (Ch. II, #1).

1. World Lines and Parametrization.

Since we are working with a manifestly covariant formalism in which the particles are defined by their invariant world lines in Minkowski space, these world lines must also be independent of the parameter used to describe them, i.e., invariant under a change of time-parameter in their differential equations, provided a clear specification of such parameters, and the procedure for replacement, is made. Because little is known at this point about the behavior of an interacting multiparticle system, while the free particle case is quite familiar, it is important to clarify the nature of the generalizations introduced.

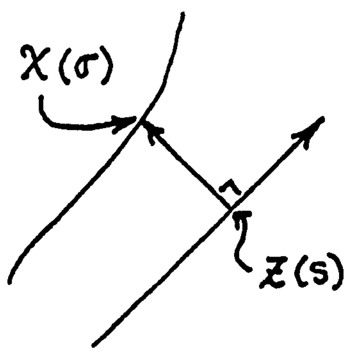
From a purely geometrical point of view, it is noticeable that in order to introduce a common parametrization to a given set of world lines, one needs of a fixed operational rule such as the concept of <u>distance</u>[34]: we have called "separation" between two given world lines to the four vector



$$\xi^\mu_{ab} = \chi^\mu_a(\sigma_a) - \chi^\mu_b(\sigma_b)\,; \qquad (4.1)$$

now, if we take separations with respect to some geodesic (straight world line) given by

$$Z^\mu(s) = k^\mu s \qquad (4.2a)$$



(only the direction is important, so $Z(0) = 0$ is taken for simplicity) with

$$k^\mu k_\mu = 1\,, \qquad (4.2b)$$

then, for any arbitrary world line, $\chi^\mu(\sigma)$, the separation

$$\xi^\mu = \chi^\mu(\sigma) - k^\mu s \qquad (4.3)$$

can be used to define the "distance" from the world line χ to the geodesic Z at the point s through an additional restriction, namely,

$$k_\mu \xi^\mu = 0 \qquad (4.4a)$$

hence

$$D^2 = \xi^\mu \xi_\mu \tag{4.4b}$$

represents that distance. This procedure assigns automatically the value s to the point σ in χ, i.e.,

$$\chi^\mu(\sigma(s)) \rightarrow k_\mu \chi^\mu(s) = s \, . \tag{4.5}$$

So, if (4.4a) holds for every world line in the set, then the separation (4.1) is orthogonal to k^μ :

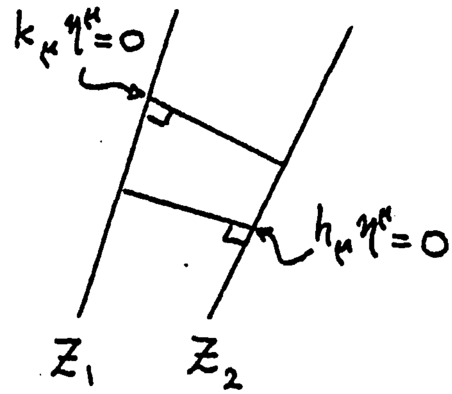
$$k_\mu \xi^\mu_{ab} = 0 \tag{4.6a}$$

and we can take

$$\rho_{ab} = \xi^\mu_{ab} \xi_{ab\mu} \tag{4.6b}$$

as the distance between world lines a,b with respect to $Z^\mu(s)$. Equation (4.6a) is a G.T. condition with respect to an arbitrary equal-time place perpendicular to the four vector k^μ. Since geodesics are well determined in direction, any other geodesic can be used to parametrize the whole set of world lines.

Suppose, for instance, that we have two geodesics given by the equations



$$Z_1^\mu = k^\mu s + a^\mu \qquad (4.7a)$$

$$Z_2^\mu = h^\mu \tau + b^\mu \qquad (4.7b)$$

$(k_\mu k^\mu = 1 ; h_\mu h^\mu = 1)$; then, if the separation

$$\eta_{12}^\mu = Z_1^\mu - Z_2^\mu \qquad (4.8a)$$

is made orthogonal (spacelike) to Z_1,

$$k^\mu \eta_\mu^{12} = 0, \qquad (4.8b)$$

we have

$$s + k_\mu a^\mu - k_\mu h^\mu \tau - k_\mu b^\mu = 0 \qquad (4.8c)$$

and, hence,

$$\tau = \alpha s + \beta_1 , \qquad (4.9a)$$

with

$$\alpha = (k_\mu h^\mu)^{-1} \quad ; \quad \beta_1 = \alpha k_\mu (b^\mu - a^\mu) \qquad (4.9b)$$

therefore Z_2 parametrized with respect to Z_1 becomes

$$Z_2^{\mu} = \grave{h}^{\mu} s + \grave{b}^{\mu} \tag{4.10a}$$

where

$$\grave{h}^{\mu} = \alpha h^{\mu} \quad ; \quad \grave{b}^{\mu} = \beta_1 h^{\mu} + b^{\mu} \tag{4.10b}$$

Conversely, if the separation (4.8a) is made orthogonal to Z_2,

$$h_{\mu} \eta_{12}^{\mu} = 0 \quad , \tag{4.11a}$$

instead of (4.8c) we get

$$\tau + h_{\mu} b^{\mu} - h_{\mu} k^{\mu} s - h_{\mu} a^{\mu} = 0 \, . \tag{4.11b}$$

Then

$$s = \alpha \tau + \beta_2 \tag{4.12a}$$

with

$$\alpha = (h_{\mu} k^{\mu})^{-1} \quad ; \quad \beta_2 = \alpha h_{\mu} (b^{\mu} - a^{\mu}) \tag{4.12b}$$

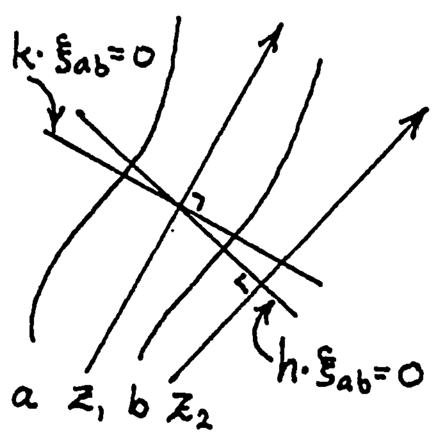
and Z_1 parametrized with respect to Z_2 is, analogously,

$$z_1^{\mu} = \grave{k}^{\mu}\tau + \grave{a}^{\mu} \qquad (4.13a)$$

where

$$\grave{k}^{\mu} = \alpha k^{\mu} \quad ; \quad \grave{a}^{\mu} = \beta_2 k^{\mu} + a^{\mu} \qquad (4.13b)$$

Obviously (4.10) and (4.13) represent the same world lines as (4.7). We see, therefore, that the G.T. condition (4.6a) remains valid even when all world lines are geodesics and, hence, it should not be regarded as a dynamical relation but simply as a technique to associate a single time-parameter to the entire set, in the sense that it will not affect the motion of the particles (the configuration of the world lines) but just their description. It is in this respect that it is always permissible to change the time-parameter by referring all world lines to a different equal-time plane (4.6a), i.e., by specifying



a new set of separations (4.3). Suppose the world lines are parametrized with respect to Z_1, the separations are

$$\xi_a^{\mu}(s) = q_a^{\mu} - k^{\mu}s \qquad (4.14a)$$

(it is clear that only the direction of the geodesics is important and the last terms, a^μ, b^μ, in (4.7) can be neglected). Now setting

$$k_\mu \xi_a^\mu = 0 \tag{4.14b}$$

is equivalent to parametrize the coordinates q_a^μ by s, i.e.,

$$k_\mu q_a^\mu(s) = s\,. \tag{4.14c}$$

Another description of the same set of world lines is obtained when the parametrization is changed to be with respect to Z_2. The separations are now

$$\grave{\xi}_a^\mu(\tau) = q_a^\mu - h^\mu \tau \tag{4.15a}$$

so

$$h_\mu \grave{\xi}_a^\mu = 0 \tag{4.15b}$$

is equivalent to parametrize the coordinates q_a^μ by τ, i.e.,

$$h_\mu q_a^\mu(\tau) = \tau\,. \tag{4.15c}$$

The outcome of the dynamical equations for a multiparticle system will be a collection of invariant world lines described with respect to a geodesic if the corresponding differential equations are covariant and compatible with a set of relations like (4.14c),

$$k_\mu q^\mu_a = s \tag{4.16a}$$

which imply the G.T. condition

$$k_\mu \xi^\mu_{ab} = 0 \,, \tag{4.16b}$$

Thus, if we restrict the family of possible time-parameters to those associated with geodesics in Minkowski space, permitting only standard clocks associated with inertial observers, the parametrizations will be specified as in (4.14c), (4.15c) and the time-parameters will be related to each other by affine transformations of the form (4.9a), (4.12a).

Since the procedure refers the same set of world lines to different equal-time-planes, we find that the particles can be taken as simultaneous in any reference frame but are observed in different states of motion in each of them. The parametrization with respect to the total momentum four vector as in (Ch. III) is, then, convenient but not necessary and its applicability is related to the fact that P^μ is known to be a constant of the motion from the outset.

2. Standard Lagrangian for a Closed Multiparticle System.

Let us consider the manifestly covariant Lagrangian

$$\mathcal{L} = \frac{1}{2} E_a \dot{q}^\mu_a \dot{q}^a_\mu - \mathcal{U}(\rho) \tag{4.17}$$

where the c- numbers E_a are still undetermined and $\mathcal{U}(\rho)$ represents the additive potential, a function only of the inter-particle separations.

The generalized momenta (2.3) are obtained as usual,

$$p^a_\mu = E^a \dot{q}^a_\mu \tag{4.18}$$

and the equations of motion, (2.14), become

$$E^a \ddot{q}^a_\mu = \Lambda^a{}_b \, \xi^{ab}_\mu \tag{4.19}$$

These equations (4N in total) are consistent with the imposition of N subsidiary constraints

$$k_\mu q^\mu_a = s \quad , \tag{4.20}$$

through which the parameter s is fixed; k^μ must be specified at this point in a practical situation. In general, it represents a momentum-like constant four vector such that

$$k^\mu k_\mu = 1 \tag{4.21}$$

and only the direction of the associated geodesic is considered for simplicity. Since the variables $\{q\}$ and $\{\dot{q}\}$ are initially independent, a second set of constraints arise from the differentiation of (4.20),

$$k_\mu \dot{q}_a^\mu = 1 \quad . \tag{4.22}$$

A further differentiation does not provide additional information since

$$k_\mu \ddot{q}_a^\mu = 0 \tag{4.23a}$$

can be obtained algebraically from the equations of motion and (4.20); indeed, (4.20) implies the G.T. conditions

$$k^\mu \xi_\mu^{ab} = 0 \;, \tag{4.23b}$$

hence, the contraction of equations (4.19) with k^μ gives

$$E^a k^\mu \ddot{q}_\mu^a = \Lambda^a{}_b k^\mu \xi_\mu^{ab} = 0 ,$$

as (4.23a). We have, therefore, the correct number of degrees of freedom.

The total momentum

$$\mathcal{P}_\mu = E_a \dot{q}_\mu^a = \sum_a p_\mu^a \tag{4.24}$$

is obviously conserved and the definition (3.27) for the C.M. can be introduced at once, so that

$$\mathcal{P}_{\mu} = E\,\dot{Q}_{\mu} \qquad ; \quad \mathcal{P}_{\mu}\mathcal{P}^{\mu} = E^2 G^2 = M_o^2 \tag{4.25a}$$

with

$$E = \sum_a E^a \qquad ; \quad G^2 \equiv \dot{Q}^{\mu}\dot{Q}_{\mu} \tag{4.25b}$$

are simple consequences of the equations of motion (4.19) and M_o stands for the rest mass of the whole system.

3. Hamiltonian Formalism.

Using definition (3.7) and equations (4.18), the canonical Hamiltonian becomes simply

$$H_c = \frac{p^a_{\mu}\, p^{\mu}_a}{2E_a} + \mathcal{U}(\rho) \tag{4.26}$$

while from (4.22) and (4.18) one has that the N subsidiary constraints are

$$\varphi \equiv k^{\mu} p^a_{\mu} - E^a \approx 0\,. \tag{4.27}$$

The stability of these constraints,

$$\dot{\varphi}^a = \{\varphi^a, H_c\} = -k^{\mu}\frac{\partial \mathcal{U}(\rho)}{\partial q^{\mu}_a} = \Lambda^a{}_b\, k^{\mu}\, \xi^{ab}_{\mu} \approx 0$$

requires

$$k^{\mu}\,\xi^{ab}_{\mu}\approx 0 \tag{4.28}$$

which is satisfied with the existence of N additional "gauge-fixing" constraints

$$\chi_a \equiv k_\mu\, q^\mu_a - s \approx 0 \tag{4.29}$$

which are preserved already as a result of (4.27),

$$\dot{\chi}_a = \{\chi_a, H_c\} + \frac{\partial \chi_a}{\partial s} = k_\mu \frac{P^\mu_a}{E^a} - 1 \approx 0 .$$

The total Hamiltonian is, thus,

$$\mathcal{H} = H_c + v_a \varphi^a + u^a \chi_a \tag{4.30}$$

But equations (1.20), (1.26) reduce in this case to

$$\begin{pmatrix} 0 & -\delta^a{}_b \\ \delta^b{}_a & 0 \end{pmatrix} \begin{pmatrix} v_b \\ u^b \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \tag{4.31a}$$

therefore

$$v_b \approx 0 \quad ; \quad u^b \approx 0 \tag{4.31b}$$

and the total Hamiltonian becomes strongly equal to the canonical

Hamiltonian H_c:

$$\mathcal{H} \equiv \frac{p^a_\mu p^\mu_a}{2E_a} + \mathcal{U}(\rho) \tag{4.32}$$

with the subsequent equations of motion

$$\dot{q}^\mu = \{q^\mu_a, \mathcal{H}\} = \frac{p^\mu_a}{E^a} \; ; \; \dot{p}^a_\mu = \{p^a_\mu, \mathcal{H}\} = -\frac{\partial \mathcal{U}}{\partial q^\mu_a} \tag{4.33}$$

The final Hamiltonian formalism reduces, then, to a structure identical to that obtained in (Ch. III, #4) for the singular Lagrangian (3.1). It is clear also that the definition of internal variables $\{\xi^\mu_a ; \Pi^a_\mu\}$ can be maintained and it leads to essentially the same equations of motion (3.41) (even in the new parametrization). Such results are not really unexpected since Lagrangian (3.1) contains mainly the same characteristics as (4.17). In fact, one can obtain the latter from the former by elimination of the singularity in (3.1). To see this, one expands (3.1) and substitutes (4.25b) to get

$$\mathcal{L} = M_0 (G - \tfrac{1}{2} G^2) + \frac{M_a}{2} \dot{q}^a_\mu \dot{q}^\mu_a - \mathcal{U}(\rho) .$$

Now, because G = constant, the first term is a constant altogether and does not affect the dynamics of the system, hence

$$\mathcal{L} = \frac{M_a}{2} \dot{q}^a_\mu \dot{q}^\mu_a - \mathcal{U}(\rho)$$

as (4.17). The substitution of G = constant into the Lagrangian eliminates $P^2 - M_o^2 = 0$ as the only existing primary constraint and the Lagrangian has been standardized.

4. Reparamatrization and Cluster Decomposition.

Lagrangian (4.17), as written, contains 8N independent variables $\{\dot{\xi};\xi\}$ and N arbitrary c-numbers E_a. We have seen that a formalism based on this standard structure is relativistically valid if

(i) the c-numbers E_a do not represent the rest masses of the particles, i.e., if we relinquish the mass as the primary concept for the identification of a particle. "A particle" means here "a world-line".

(ii) While the world lines are time-like, their mutual separations must be space-like. Therefore, the definition of distance between world lines implies the specification of a common parameter through constraints which determine the simultaneity of the particles in a given frame of reference. Since the allowed frames are all inertial, we are restricting the set of possible time-parameters to proper times measured by inertial standard clocks. The parametrization is then referred to equal-time planes orthogonal to geodesics in the Minkowski flat space. The transformations among time-parameters in the set are thus always linear.

Consequently, a change in parameter of the form (4.12a) in the action integral (2.1), will alter the Lagrangian only by a multiplicative constant:

$$A[s]=\int_1^2 \mathcal{L}\,ds=\int_1^2 \mathcal{L}\frac{ds}{d\tau}\,d\tau=\int_1^2 \alpha\mathcal{L}\,d\tau=\int_1^2 \mathcal{L}'\,d\tau=A[\tau] \tag{4.34a}$$

where

$$\mathcal{L}'=\alpha\left(\frac{E^a}{2}\,\dot{q}^\mu_a\,\dot{q}^a_\mu-\mathcal{U}(\rho)\right) \tag{4.34b}$$

but, if we put

$$\mathring{q}^\mu_a\equiv dq^\mu_a/d\tau\ , \tag{4.34c}$$

we have

$$\dot{q}^\mu_a=\frac{1}{\alpha}\,\mathring{q}^\mu_a \tag{4.34d}$$

hence,

$$\mathcal{L}'=\frac{E^a}{2\alpha}\,\mathring{q}^\mu_a\,\mathring{q}^a_\mu-\alpha\,\mathcal{U}(\rho)\,. \tag{4.34e}$$

Now, <u>the form</u> of the Lagrangian remains the same if we introduce the potentials so that

$$\mathcal{U}(\rho)=\frac{1}{2E_a}\left(V_a(\rho)-m_a^2\right) \tag{4.35}$$

(the rest masses m_a are introduced for convenience and obviously do not play a role in the equations of motion). The new Lagrangian is now

$$\mathcal{L}' = \frac{E'^a}{2}\, \overset{\circ}{q}{}^{\mu}_{a}\, \overset{\circ}{q}{}^{a}_{\mu} - \mathcal{U}'(\rho) \tag{3.36a}$$

with

$$\mathcal{U}'(\rho) = \frac{1}{2E'_a}\left(V_a(\rho) - m_a^2 \right) \tag{3.36b}$$

Relative to the new geodesic, the constraints are

$$h_\mu \, \overset{\circ}{q}{}^{\mu}_{a} - \tau = 0 \tag{3.37a}$$

the rest of the analysis being completely equivalent to that previously performed and the c- numbers E'_a come to be determined by

$$h^\mu p^a_\mu - E'^a \approx 0 \tag{3.37b}$$

at the Hamiltonian level. Both descriptions are connected through a transformation (4.12) and

$$E'^a = E^a/\alpha \,, \tag{3.38}$$

so the equations of motion will have the same solutions except for these constants, i.e., the reparametrization has an effect only on the initial conditions (3.37). The C.M. also remains unchanged since

$$Q^{\mu} = \frac{\grave{E}^{a}}{\grave{E}} q^{\mu}_{a} = \frac{E^{a}}{E} q^{\mu}_{a} = \frac{M^{a}}{M_o} q^{\mu}_{a} \qquad (3.39a)$$

from (3.38), with

$$\grave{E} = \sum_{a} \grave{E}^{a} \; ; \; E = \sum_{a} E^{a} \; ; \; M_o = \sum_{a} M^{a} \qquad (3.39b)$$

As a result of the above analysis, we find that if k^{μ} and s are arbitrary in the sense that no particular clock or reference frame is preferred to any other to parametrize and describe the system as long as we deal with inertial observers, then the model based on Lagrangian (4.17) plus constraints (4.20) can be made completely separable if the potentials in (4.35) are chosen so that

$$V_a(\rho) = \sum_{b}{}' V_{ab}(\rho_{ab}) \qquad (4.40)$$

i.e., if they include only binary interactions[(1)]. Indeed, when

(1) Recent papers[17,35] have shown that, in theories based on the Todorov-Komar model, cluster decomposition can not be achieved with pairwise interactions like (4.40) in the general N-particle case. The reason behind this conclusion is that, in such theories, the mass-shell type constraints are required to be first class which introduces more conditions on the potentials than the simpler relation (4.40).

the system is divided into two clusters with labeling $a_1 = 1\ldots N_1$ and $a_2 = N_1 + 1 \ldots N$; the Lagrangian (4.17) is written as

$$\mathcal{L} = \frac{E^{a_1}}{2}\dot{q}^{\mu}_{a_1}\dot{q}^{a_1}_{\mu} - \frac{1}{2E_{a_1}}(V_{a_1} - m^2_{a_1}) + \\ + \frac{E^{a_2}}{2}\dot{q}^{\mu}_{a_2}\dot{q}^{a_2}_{\mu} - \frac{1}{2E_{a_2}}(V_{a_2} - m^2_{a_2}) \\ - \frac{1}{2E_{a_1}}\sum_{a_2} V_{a_1a_2} - \frac{1}{2E_{a_2}}\sum_{a_1} V_{a_2a_1} \quad (4.41a)$$

with

$$V_{a_1} = {\sum_{a_1}}' V_{a_1b_1}(\rho_{a_1b_1}) \; ; \; V_{a_2} = {\sum_{b_2}}' V_{a_2b_2}(\rho_{a_2b_2}) . \quad (4.41b)$$

So, when the two clusters are brought far apart, the last two terms in (4.41a) can be assumed to vanish and the Lagrangian is separated into the two independent terms

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \quad (4.42a)$$

with separate respective sets of constraints (4.20)

$$k_\mu q^\mu_{a_1} - s = 0 \quad ; \quad k_\mu q^\mu_{a_2} - s = 0 \quad (4.42b)$$

since k^μ does not necessarily belong to the original system. The Hamiltonian is then similarly separable,

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \qquad (4.42c)$$

and constraints (4.27) give two separate sets

$$\varphi^{a_1} \equiv k^\mu p^{a_1}_\mu - E^{a_1} \approx 0 \; ; \; \varphi^{a_2} \equiv k^\mu p^{a_2}_\mu - E^{a_2} \approx 0 \qquad (4.42d)$$

while the C.M.'s become, respectively,

$$Q^\mu_1 = \frac{E^{a_1}}{E_1} q^\mu_{a_1} \quad ; \quad Q^\mu_2 = \frac{E^{a_2}}{E_2} q^\mu_{a_2} \qquad (4.43a)$$

where

$$E_1 = \sum_{a_1} E^{a_1} \quad ; \quad E_2 = \sum_{a_2} E^{a_2} \quad , \qquad (4.43b)$$

plus corresponding equations of motion for each cluster. Clearly, the definition of internal variables can be reintroduced in terms of (4.43) after the separation is performed leading to basically the same structures except for index adjustments.

5. Systems of Free Particles.

For a system of particles without mutual interaction, the Hamiltonian (4.32) with potentials (4.35) reduces to the familiar form

$$\mathcal{H} = \frac{1}{2E_a}\left(p^a_\mu p^\mu_a - m^2_a \right) \tag{4.44}$$

The solutions of the equations of motion are in this case

$$q^\mu_a = \alpha^\mu_a s + \beta^\mu_a \tag{4.45a}$$

where α^μ_a and β^μ_a are constants subject to the conditions

$$k_\mu \alpha^\mu_a = 1 \quad ; \quad k_\mu \beta^\mu_a = 0 \tag{4.45b}$$

obtained from the constraints (4.20). Thus only 6N initial values remain to be specified, in accordance with the correct number of degrees of freedom. All the world lines are, of course, geodesics. The rest mass of each particle is determined in terms of the c-numbers E_a by

$$m_a = E_a g_a = E_a \left(\alpha^\mu_a \alpha_{a\mu} \right)^{1/2} \tag{4.45c}$$

and the total rest mass for the system is

$$M_o = E\,G = \left(E_a E^b \alpha^a_\mu \alpha^\mu_b \right)^{1/2}. \tag{4.45d}$$

Since the momenta become

$$p^a_\mu = E^a \alpha^a_\mu \quad ; \quad P_\mu = E_a \alpha^a_\mu , \tag{4.46a}$$

one can write (4.45c) and (4.45d) as

$$p_a^2 - m_a^2 = 0 \quad ; \quad P^2 - M_o^2 = 0 \tag{4.46b}$$

while the proper time of each particle,

$$d\sigma_a^2 = dq^\mu_a \, dq_{a\mu} , \tag{4.47a}$$

satisfies the relation

$$\frac{d\sigma_a}{ds} = g_a = \frac{m_a}{E^a} \tag{4.47b}$$

so, in terms of the common time-parameter, we have

$$\sigma_a(s) = \frac{m_a}{E^a} s + \text{constant} \tag{4.47c}$$

as in (4.9), (4.12); hence

$$k^\mu p^a_\mu = E^a ,$$

as should be. This shows that G.T. condition persists and is needed to give a complete description of the world lines in terms of a common time-parameter even when the particles are free.

Note that for the internal momenta

$$\Pi^a_\mu = P^a_\mu - \frac{E^a}{E} P_\mu , \qquad (4.48a)$$

the relations equivalent to (4.46b) are

$$\Pi^a_\mu \Pi^{a\mu} = (m^a)^2 - (E^a)^2 ; \qquad (4.48b)$$

therefore, with the rest masses defined as in (4.45c), the Hamiltonians (3.40c) and (4.45) can be set equal to zero initially and that value will be kept fixed for the rest of the evolution because they are constants of the motion.

6. Nonrelativistic Limit.

When all speeds become small compared with the speed of light, c, the Lorentz boosts become negligible, so the nonrelativistic limit is obtained by equating all times and retaining only the first power in c^{-1} of the usual series expansion.

In order to obtain the correct signs in this limit, we should actually write the Lagrangian as

$$\mathcal{L} = -\left[\frac{E^a}{2} \dot{q}^\mu_a \dot{q}^a_\mu + \mathcal{U}(\rho) \right] \qquad (4.49)$$

Since

$$\gamma^a d\sigma^a = c\, dt^a \quad ; \quad \gamma\, ds = c\, dt \qquad (4.50a)$$

we have

$$\dot{q}^{\mu}_{a} = \frac{\gamma}{c} \frac{d q^{\mu}_{a}}{dt} \tag{4.50b}$$

with

$$\gamma^{a} = \left(1 - \frac{\vec{v}_a^{\,2}}{c^2}\right)^{-1/2} \quad ; \quad \gamma = \left(1 - \frac{\vec{v}^{\,2}}{c^2}\right)^{-1/2} \tag{4.50c}$$

and

$$\vec{v}_a = d\vec{q}_a / dt \quad ; \quad \vec{v} = c\,\vec{k} / k^{\circ} \tag{4.50d}$$

where $\vec{v}$ is written in terms of the vector k^{μ}. The action integral (2.1) is, thus,

$$A[s] = \int_1^2 \mathcal{L}(\dot{q},q)\, ds = \int_1^2 \gamma^{-1} c\, \mathcal{L}\, d\sigma = \int_1^2 \mathcal{L}'(\vec{v},\vec{q},t)\, dt = A[t] \tag{4.51a}$$

with

$$\mathcal{L}' = -\left\{ \frac{\gamma E^{a}}{2c}\left[c^2\left(\frac{dt_a}{dt}\right)^2 - \vec{v}_a^{\,2}\right] + \frac{c}{2\gamma E_a}\left[V_a(\rho) - m_a^2 c^2\right] \right\} \tag{4.51b}$$

One can now use the constraints (4.20) in the k-rest frame where

$$\vec{v} = 0 \quad ; \quad \gamma = 1 \quad ; \quad t^{a} = t \tag{4.51c}$$

to write the Lagrangian (4.51b) in its reduced form

$$\mathcal{L} = \frac{E^a}{2c}\,\vec{v}_a^{\,2} - \frac{c}{2E_a}\left[V_a(r^2) - m_a^2 c^2\right] - \frac{Ec}{2}\,. \tag{4.51d}$$

On the other hand, if we define the "canonical mass", $m^a(s)$, as the norm of the momentum four-vector,

$$m^a(s) = \left(p_\mu^a\, p^{a\mu}\right)^{1/2}, \tag{4.52}$$

from (4.18) one has that

$$p_\mu^a\, p^{a\mu} = \left(E^a g^a\right)^2 = \left[m^a(s)\right]^2 \tag{4.53}$$

so, in the k-rest frame and restoring the familiar units, the relation (4.52) becomes

$$m^a(t)\,c = E^a(\gamma^a)^{-1} \tag{4.54}$$

In the nonrelativistic limit, when

$$\gamma_a^{-1} \simeq 1 + \frac{1}{2}\frac{v_a^2}{c^2}\,, \tag{4.55a}$$

we find

$$m^a(t) \simeq \frac{E^a}{c} + \frac{1}{2}\frac{v_a^2}{c^3} \tag{4.55b}$$

Therefore, retaining only the first term of the expansion

($\beta^a = v^a/c \ll 1$), we can make

$$m^a(t) = m^a = \frac{E^a}{c} = \text{constant} \tag{4.56}$$

The nonrelativistic potential now turns out to be

$$\mathcal{U}_{nr.} = \sum_a \frac{V_a(r^2)}{2\, m_a} \tag{4.57}$$

while the last two terms in (4.51d) cancel each other. Consequently, the nonrelativistic Lagrangian is

$$\mathcal{L} = \frac{m^a}{2} \vec{v}_a^{\,2} - \mathcal{U}_{n.r.}(r^2) \tag{4.58}$$

as expected.

V. CANONICAL REALIZATIONS OF THE RELATIVISTIC SYMMETRY GROUP.

The Hamiltonian (4.26) can also be obtained from group theoretical considerations after we recognize the enlarged symmetries of the Lagrangian (4.17), namely a quasi-invariance under replacement of the coordinates by spacelike variables. As will be seen, that invariance is at the root of the center of momentum definition (3.27).

1. Transformations to Spacelike Variables.

From the discussion given in the last chapter concerning the invariance of the world lines and the separability of the Lagrangian, we see that the equations of motion can be written down in terms of separations like (4.3) instead of using the coordinates $\{ q_a^\mu \}$. In fact, the Lagrangian (4.17) itself can be expressed as a function of these separations without alteration of the dynamics. So, adopting the notation of (Ch. IV, #1), if we substitute in (4.17) the relation

$$q_a^\mu(s) = \grave{\xi}_a^\mu + \grave{h}^\mu s \tag{5.1}$$

where $\grave{h}^\mu$ is the direction vector of the geodesic Z_2 parametrized with respect to Z_1, we find

$$\mathcal{L} = \frac{E^a}{2}\left(\dot{\grave{\xi}}_a^\mu + \grave{h}^\mu\right)\left(\dot{\xi}_\mu^a + \grave{h}_\mu\right) - \mathcal{U}(\rho)$$

$$= \frac{E^a}{2} \dot{\grave{\xi}}^{\mu}_{a} \dot{\grave{\xi}}^{a}_{\mu} - \mathcal{U}(\rho) + E^a \dot{\grave{\xi}}^{\mu}_{a} \grave{h}_{\mu} + \frac{E}{2} \grave{h}^{\mu} \grave{h}_{\mu},$$

but

$$E_a \dot{\grave{\xi}}^{a}_{\mu} = E_a \dot{q}^{a}_{\mu} - E \grave{h}_{\mu} = E(\dot{Q}_{\mu} - \grave{h}_{\mu})$$

so that

$$\mathcal{L} = \frac{E^a}{2} \dot{\grave{\xi}}^{\mu}_{a} \grave{\xi}^{a}_{\mu} - \mathcal{U}(\rho) + E(\dot{Q}_{\mu} \grave{h}^{\mu} - \tfrac{1}{2} \grave{h}^{\mu} \grave{h}_{\mu}). \tag{5.2a}$$

Since the last term is an exact differential of the function

$$F = E(Q^{\mu} - \tfrac{1}{2} \grave{h}^{\mu} s) \grave{h}_{\mu}, \tag{5.2b}$$

it can be dropped out, leaving

$$\mathcal{L} = \frac{E^a}{2} \dot{\grave{\xi}}^{\mu}_{a} \dot{\grave{\xi}}^{a}_{\mu} - \mathcal{U}(\rho). \tag{5.3}$$

Only the constraints can tell us if we are dealing with time-like or space-like variables; in this last case, (4.20) becomes

$$k_{\mu} \grave{\xi}^{\mu}_{a} = 0. \tag{5.4}$$

As we see from (5.2b), a transformation like the one just performed is possible only because of the existence of a definition for the C.M., (3.27), whose conjugate momentum $\mathcal{P}_\mu$ is a constant of the motion. Aside from the constraints, therefore, those two types of variables are not discriminated by the canonical equations and a transformation like (5.1) constitute a symmetry transformation for the Lagrangian. Furthermore, since the direction of the geodesics is well determined, it is always possible to find a four vector $\grave{\ell}^\mu$ such that the spacelike variables

$$\xi^\mu_a = q^\mu_a - k^\mu s \quad ; \quad \grave{\xi}^\mu_a = q^\mu_a - \grave{h}^\mu s \tag{5.5a}$$

become related to each other by

$$\grave{\xi}^\mu_a = \xi^\mu_a - \grave{\ell}^\mu s \tag{5.5b}$$

(e.g., $\grave{\ell}^\mu = \grave{h}^\mu - \grave{k}^\mu$) hence, in a given parametrization, the transformations (5.5) or (5.1) form a group. The world lines are thus equally well described using any geodesic as a "guide" to the separation variables. The decomposition of the equations of motion into two sets, one for the C.M. geodesic and one for the internal variables as in (3.41) is a good example of that situation.

2. The Lie Algebra of the Complete Symmetry Group.

If we postpone the specification of the time-parameter until the complete algebra has been deduced, and we incorporate also a translation in the parameter itself of the form

$$\grave{s} = s - \theta \quad , \tag{5.6}$$

then the complete group of transformations that leaves the world lines invariant can be written

$$\grave{\xi}^{\mu}(\grave{s}) = \Lambda^{\mu}_{1\,\nu}\left(x^{\nu} - k_1^{\nu} s\right) + a_1^{\mu} \quad ; \quad \grave{s} = s - \theta_1 \qquad (5.7)$$

where all relations are assumed to hold, initially, in a unconstrained four dimensional space but with the explicit supposition that all world lines can be referred to a common time-parameter. Hence, the vectors k^{μ} are all constants but not necessarily normalized to unity.

When a second transformation (5.7) is performed, one has

$$\xi''^{\rho}(s'') = \Lambda^{\rho}_{2\,\mu}\left(\grave{\xi}^{\mu}(\grave{s}) - k_2^{\mu}\grave{s}\right) + a_2^{\rho} \,;\quad s'' = \grave{s} - \theta_2 \qquad (5.8a)$$

$$= \Lambda^{\rho}_{3\,\nu}\left(x^{\nu} - k_3^{\nu} s\right) + a_3^{\rho} \quad ; \quad s'' = s - \theta_3 \qquad (5.8b)$$

with

$$\Lambda^{\rho}_{3\,\nu} = \Lambda^{\rho}_{2\,\mu}\Lambda^{\mu}_{1\,\nu} \quad ; \quad k_3^{\nu} = k_1^{\nu} + \Lambda^{-1}_{1}{}^{\nu}{}_{\sigma}\, k_2^{\sigma}$$

$$a_3^{\rho} = \Lambda^{\rho}_{2\,\mu}\left(a_1^{\mu} + k_2^{\mu}\theta_1\right) + a_2^{\rho} \quad ; \quad \theta_3 = \theta_1 + \theta_2 \qquad (5.8c)$$

We observe at this point that although the quantities involved in this transformation have a different meaning and function, the structure (5.8c) bears a close resemblance with that of the Galilei group for transformations in the Euclidean three dimensional space[30] so we can proceed by analogy to deduce the respective Lie algebra and its canonical representation.

Let us construct the operators

$$T_l(\lambda) \equiv e^{-\frac{1}{2}\lambda^{\mu\nu} l_{\mu\nu}} \quad ; \quad T_g(k) \equiv e^{k_\mu g^\mu}$$
$$T_h(\theta) \equiv e^{\theta h} \quad ; \quad T_d(a) \equiv e^{a^\mu d_\mu} \tag{5.9a}$$

Then, the complete action of the abstract group can be represented by

$$\mathbb{T}(\lambda, k, \theta, a) = T_d\, T_h\, T_g\, T_l \tag{5.9b}$$

and

$$\mathbb{T}(\lambda_2, k_2, \theta_2, a_2)\, \mathbb{T}(\lambda_1, k_1, \theta_1, a_1) =$$
$$= \mathbb{T}(\lambda_1\lambda_2\,,\ \lambda_2 k_1 + k_2\,,\ \theta_1 + \theta_2\,,\ \lambda_2 a_1 + k_2\theta_1 + a_2)$$
$$= \mathbb{T}(\lambda_3, k_3, \theta_3, a_3)\,. \tag{5.9c}$$

The Lie algebra among the operators l,g,h and d in (5.9) is easily found by observing that the factors T_d and T_1, corresponding to the Poincare group, will yield the known commutation relations among their respective generators. These commute with h of T_h (θ) because the Poincare group leaves the parameter, s, invariant. It also follows that the transformations $T_g(k)$ and Lorentz transformations $T_1(\lambda)$ form a semidirect product of the same kind as the translations $T_d(a)$ with $T_1(\lambda)$, hence, the brackets between $l_{\mu\nu}$ and g^{μ} have the same structure as those between $l_{\mu\nu}$ and d_{μ}. Also, since T_d and T_g form abelian subgroups and commute with each other, their generators will all commute in the same way. Finally, the bracket between h and g^{μ} can be calculated by observing that because $T_h(\theta)$ and $T_d(a)$ commute, the relation

$$\mathbb{T}(0,k_2,0,0)\,\mathbb{T}(0,0,\theta_1,0)=\mathbb{T}(0,k_2,\theta_1,k_2\theta_1)$$

can be written as

$$\mathbb{T}(0,0,-\theta_1,0)\mathbb{T}(0,k_2,0,0)\mathbb{T}(0,0,\theta_1,0)=\mathbb{T}(0,k_2,\theta_1,k_2\theta_1)$$

or

$$e^{-\theta_1 h}\,e^{k_{2\mu}g^{\mu}}e^{\theta_1 h} = e^{k_{2\mu}(\theta_1 d^{\mu}+g^{\mu})}$$

expanding both sides up to the first order in θ_1 one gets

$$k_{2\mu}\,(g^{\mu}+\theta_1[g^{\mu},h]) = k_{2\mu}\,(g^{\mu}+\theta_1 d^{\mu})$$

and, thence,

$$[g^{\mu},h]= d^{\mu}.$$

The complete Lie algebra for the group (5.8) is, therefore, as follows:

$$[l_{\mu\nu}, l_{\rho\sigma}] = \delta_{\mu\sigma} l_{\nu\rho} - \delta_{\nu\sigma} l_{\mu\rho} + \delta_{\mu\rho} l_{\sigma\nu} - \delta_{\nu\rho} l_{\sigma\mu}$$

$$[l_{\mu\nu}, d_\rho] = \delta_{\mu\rho} d_\nu - \delta_{\nu\rho} d_\mu \; ; \; [d_\mu, d_\nu] = 0 \tag{5.10a}$$

$$\left.\begin{array}{c} [l_{\mu\nu}, g_\rho] = \delta_{\mu\rho} g_\nu - \delta_{\rho\nu} g_\mu \; ; \; [g^\mu, g^\nu] = 0 \\ [l_{\mu\nu}, h] = 0 \; ; [d_\mu, h] = 0 \; ; [d_\mu, g^\nu] = 0 \\ [g^\mu, h] = d^\mu \end{array}\right\} \tag{5.10b}$$

Furthermore, because of the existence of a faithful canonical realization of the Poincare algebra, (2.5), the fact that the generators G^μ (the canonical representatives of g^μ) have to transform in the same way as P_μ under Lorentz transformations and still working by analogy with the Galilean case, we find that the canonical realization of (5.10) will retain the structure of that abstract algebra except for a neutral element (a c-number) for the bracket between the representatives of d^μ and g^μ, i.e.,

$$[J_{\mu\nu}, J_{\rho\sigma}] = C^{\alpha\beta}_{\mu\nu\rho\sigma} J_{\alpha\beta} \; ; [J_{\mu\nu}, P_\rho] = C^{\alpha}_{\mu\nu\rho} P_\alpha ; [P_\mu, P_\nu] = 0 \tag{5.11a}$$

$$[J_{\mu\nu}, G_\rho] = C^{\alpha}_{\mu\nu\rho} G_\alpha \; ; [G^\mu, P_\nu] = E\, \delta^\mu_\nu \; ; [G^\mu, G^\nu] = 0 \tag{5.11b}$$

$$[J_{\mu\nu}, \mathcal{H}] = 0 \quad ; \quad [\mathcal{P}_\mu, \mathcal{H}] = 0 \quad ; \quad [G_\mu, \mathcal{H}] = \mathcal{P}_\mu \tag{5.11c}$$

Although realizations with E = 0 can be here acceptable, we shall take E ≠ 0 from now on. This canonical realization of the fifteen-parameter symmetry group (5.7) replaces (2.5) as the point of departure for a relativistic dynamics. Further realizations in terms of Dirac brackets can be constructed as soon as one identifies the c-number E and the parameter s by means of the imposition of subsidiary constraints which are to be treated as additional neutral elements for the algebra that can break some symmetries reducing the number of degrees of freedom so that the actual physical dynamics is obtained. Let us consider the simplest case first.

2. Single Particle Systems.

For a closed system consisting only of a single particle, the choice

$$J_{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu \quad ; \quad \mathcal{P}_\mu = p_\mu \tag{5.12a}$$

$$G^\mu = E\, q^\mu \quad ; \quad \mathcal{H} = \frac{1}{2E}\left(p_\mu p^\mu - m^2\right) \tag{5.12b}$$

satisfies the algebra (5.11) using the fundamental P.b.'s

$$\{ q^\mu, p_\nu \} = \delta^\mu{}_\nu \quad . \tag{5.13}$$

Several realizations with the proper number of degrees of freedom can be obtained from (5.11) and (5.12) according to the form of neutral elements involving dynamical variables (the constraints) that are imposed. The choice of such subsidiary conditions, is in principle, a matter of convenience depending on the physical information one wishes to extract from the abstract general form.

For instance, the most common realizations assume that the mass of the particle determine the Hamiltonian operator in (5.12b) through the constraint

$$\varphi \equiv p^2 - m^2 \approx 0 \tag{5.14a}$$

so that

$$\mathcal{H} = \lambda \varphi \tag{5.14b}$$

while the remaining condition, which is here a gauge-fixing constraint,

$$\chi(q, p, s) \approx 0 \tag{5.14c}$$

is chosen as one of the following:

$$\text{(a)}\ q^0 \approx 0 \ ; \ \text{(b)}\ q^0 - t \approx 0 \ ; \ \text{(c)}\ p_\mu q^\mu - ms \approx 0 \tag{5.14d}$$

The choice (a) involves only a function of the coordinates, meaning that the subsidiary condition fixing the time-parameter, whose translations are generated by H, is used, in fact, to reduce the phase space. The time-parameter comes to be one of the coordinates and H ceases to exist since translations of the coordinates are generated by P_μ of the Poincare algebra alone. This is precisely the kind of constraint that leads to Dirac's "instant form"[1] and, for multi-particle systems, to the no-interaction theorem. Indeed, a direct application of the procedure given in (Ch. I, #2) shows that the stability of the constraints are satisfied, in this case, only with vanishing coefficients

$$\mathcal{H} = \lambda \varphi + \lambda' \chi \approx 0$$

$$\dot{\varphi} \approx 0 \; ; \; \dot{\chi} \approx 0 \;\Rightarrow\; \lambda = \lambda' \approx 0 \, ,$$

therefore, H becomes the product of two weakly vanishing quantities and itself vanishes strongly (identically) $\mathcal{H} \equiv 0$. The algebra (5.11) in terms of D.b.'s (1.30) with

$$C^{-1}_{AB} = \frac{1}{2P_0} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.15a}$$

reduces to

$$\{J_{\mu\nu}, J_{\rho\sigma}\}^* = C^{\alpha\beta}_{\mu\nu\rho\sigma} J_{\alpha\beta} \; ; \; \{J_{\mu\nu}, P_\rho\}^* = C^{\alpha}_{\mu\nu\rho} P_\alpha$$

$$\{P_\mu, P_\nu\}^* = 0 \; ; \; \{q^\mu, q^\nu\}^* = 0 ; \; \{q^\mu, P_\nu\}^* = \delta^\mu_\nu - \delta^0_\nu \frac{P^\mu}{P_0} \tag{5.15b}$$

From this, the number of variables can effectively be reduced since P_0 comes to play the role of a Hamiltonian; so if we make $H \equiv P_0$, all other operators are expressed as functions of the six phase-space variables ($q^0 = 0$):

$$J_k \equiv J_{ij} = q_i q_j - q_j q_i \quad ; \quad \vec{\mathcal{P}} = \vec{p}$$

$$H = (\vec{p}^{\,2} + m^2)^{1/2} \quad ; \quad K^i \equiv J^{i0} = q^i (\vec{p}^{\,2} + m^2)^{1/2} \tag{5.15c}$$

and the algebra (5.15b) coincides with the usual P.b. realization

$$\{J_i, J_j\} = \epsilon_{ijk} J_k \; ; \; \{J_i, P_j\} = \epsilon_{ijk} P_k \; ; \; \{J_i, K_j\} = \epsilon_{ijk} K_k$$

$$\{K_i, K_j\} = -\epsilon_{ijk} J_k \; ; \; \{K_i, P_j\} = \delta_{ij} H \; ; \; \{K_i, q_j\} = q_i \{H, q_j\}$$

$$\{P_i, P_j\} = 0 \quad ; \quad \{q^i, P_j\} = \delta^i_j \quad ; \quad \{q^i, q^j\} = 0 \tag{5.15d}$$

$$\{J_i, H\} = \{P_i, H\} = 0 \; ; \; \{q^i, H\} = \frac{p^i}{(p^2 + m^2)^{1/2}} \; ; \; \{K_i, H\} = P_i$$

which, as is well known, can not be extended to multiparticle systems involving interactions[3,4].

The choice (b) in (5.14d) is only slightly different from the previous one. In this case the Hamiltonian operator vanishes only weakly

$$\mathcal{H} = \frac{1}{2P_0} (\vec{p}^{\,2} - m^2) \approx 0 \tag{5.16a}$$

and with (5.15a) the algebra in terms of D.b. is the same as (5.15b) and (5.15c) but one has to add the following brackets

$$\{J_{\mu\nu}, q_\rho\}^* = C^{\alpha}_{\mu\nu\rho}\, q_\alpha \quad ; \quad \{J_{\mu\nu}, \mathcal{H}\}^* = 0$$
$$\{P_\mu, \mathcal{H}\}^* = 0 \quad ; \quad \{q^\mu, \mathcal{H}\}^* = 0 \tag{5.16b}$$

while the equations of motion have to be evaluated according to formulae (1.32) and are, as expected,

$$\dot{q}^\mu = \{q^\mu, \mathcal{H}\}^* + \frac{\partial^* q^\mu}{\partial t} = \frac{p^\mu}{p_0} \quad ; \quad \dot{p}^\mu = \{p^\mu, \mathcal{H}\}^* = 0\,. \tag{5.16c}$$

Generators G^μ are

$$G^\mu = P_0\, q^\mu \tag{5.16d}$$

This type of constraint still destroys manifest covariance and that will be carried over to any generalization to many particle systems.

With choice (c) we get a more symmetrical realization due to its manifestly covariant nature. In this case, the standard procedure yields

$$\mathcal{H} = \frac{1}{2m}\,(p^2 - m^2) \approx 0 \tag{5.17a}$$

so with

$$C^{-1}_{AB} = \frac{1}{2p^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (5.17b)$$

the D.b. algebra for the Poincare group is left unchanged as in (5.15b); the remaining brackets are

$$\{J_{\mu\nu}, q_\rho\}^* = C^{\alpha}_{\mu\nu\rho}\, q_\alpha \quad ; \quad \{q^\mu, p_\nu\}^* = \delta^\mu_\nu - \frac{p^\mu p_\nu}{p^2}$$
$$\{q^\mu, q^\nu\}^* = -\frac{J^{\mu\nu}}{p^2} \quad ; \{J_{\mu\nu}, \mathcal{H}\}^* = \{p_\mu, \mathcal{H}\}^* = \{q^\mu, \mathcal{H}\}^* = 0 \qquad (5.17c)$$

from (1.32), the equations of motion give

$$\dot{q}^\mu = \{q^\mu, \mathcal{H}\}^* + \frac{\partial^* q^\mu}{\partial s} = \frac{p^\mu}{m} \quad ; \quad \dot{p}_\mu = \{p_\mu, \mathcal{H}\}^* = 0 \qquad (5.17d)$$

and the generators G^μ become

$$G^\mu = m\, q^\mu. \qquad (5.17e)$$

Finally, still another realization can be obtained from which the Hamiltonian does not vanish as a result of the constraints and is manifestly covariant. This realization is obtained by relaxing the mass-shell constraint (5.14a) and replacing it by

$$\varphi \equiv k^\mu p_\mu - E \approx 0 \qquad (5.18a)$$

which give a different meaning to the neutral element E. Instead of (5.14d), the time parameter is also specified by the vector k_μ

(k_μ $k^\mu = 1$) through the additional constraint

$$\chi \equiv k_\mu q^\mu - s \approx 0 \; . \tag{5.18b}$$

The algebra (5.11) with operators (5.12) and

$$C^{-1}_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.18c}$$

then gives the D.b. realization

$$\{J_{\mu\nu}, J_{\rho\sigma}\}^* = K^{\alpha\beta}_{\mu\nu\rho\sigma} J_{\alpha\beta} \; ; \{J_{\mu\nu}, P_\mu\}^* = K^{\alpha}_{\mu\nu\rho} P_\alpha \; ; \{P_\mu, P_\nu\}^* = 0 \tag{5.19a}$$

$$\{J_{\mu\nu}, q_\mu\}^* = K^{\alpha}_{\mu\nu\rho} q_\alpha \quad ; \; \{q^\mu, P_\nu\}^* = \varkappa^\mu{}_\nu \; ; \; \{q^\mu, q^\nu\}^* = 0 \tag{5.19b}$$

$$\{J_{\mu\nu}, \mathcal{H}\}^* = k_\mu P_\nu - k_\nu P_\mu \; ; \{P_\mu, \mathcal{H}\}^* = 0 \; ; \{q^\mu, \mathcal{H}\}^* = \frac{P_\mu}{E} - k_\mu \tag{5.19c}$$

where k_μ is the vector given in (5.18), $\varkappa^\mu{}_\nu$ the constants

$$\varkappa^\mu{}_\nu \equiv \delta^\mu_\nu - k^\mu k_\nu \tag{5.19d}$$

and

$$K^{\alpha\beta}_{\mu\nu\rho\sigma} \equiv \varkappa_{\mu\rho} \delta^\alpha_\nu \delta^\beta_\sigma + \varkappa_{\nu\sigma} \delta^\alpha_\mu \delta^\beta_\rho - \varkappa_{\mu\sigma} \delta^\alpha_\nu \delta^\beta_\rho - \varkappa_{\nu\rho} \delta^\alpha_\mu \delta^\beta_\sigma \tag{5.19e}$$

$$K_{\mu\nu\rho}^{\ \ \ \alpha} \equiv \kappa_{\mu\rho}\delta_{\nu}^{\alpha} - \kappa_{\mu\nu}\delta_{\rho}^{\alpha} . \qquad (5.19f)$$

In spite of the brackets (5.19c) we obtain the correct equations of motion using (1.32), i.e.,

$$\dot{J}_{\mu\nu} = \{J_{\mu\nu}, \mathcal{H}\}^{*} + \frac{\partial^{*} J_{\mu\nu}}{\partial s} = 0 \ ; \ \dot{P}_{\mu} = \{P_{\mu}, \mathcal{H}\}^{*} = 0 \qquad (5.20a)$$

$$\dot{q}^{\mu} = \{q^{\mu}, \mathcal{H}\}^{*} + \frac{\partial^{*} q^{\mu}}{\partial s} = \frac{P^{\mu}}{E} \qquad (5.20b)$$

Because the Hamiltonian itself is a constant of the motion, one can always choose it equal to zero by adjusting the constant m in (5.12b) so that it becomes exactly equal to the final rest mass of the particle which must be defined here by

$$P_{\mu} P^{\mu} = m^{2} \quad \Rightarrow \quad m = E\left(\dot{q}^{\mu}\dot{q}_{\mu}\right)^{1/2} = E\, g \qquad (5.20c)$$

Of course, for a single particle all those realizations are finally equivalent in their physical content and interpretation, but the generalization to multiparticle systems is greatly simplified if we use the last one.

3. Multiparticle Systems.

A generalization of the last structure to systems involving more than one particle and scalar interactions is now straight forward, regarding the new realization as the direct product of N single particle realizations where N is the number of particles.

Then, all operators of the algebra become direct sums of their single particle counterparts as do the neutral elements. Consequently, the algebra (5.11) is satisfied by the fifteen generators

$$J_{\mu\nu} = q^a_\mu p_{a\nu} - q^a_\nu p_{a\mu} \; ; \; P_\mu = \sum_a p^a_\mu \tag{5.20a}$$

$$G^\mu = E^a q^\mu_a \; ; \; \mathcal{H} = \frac{p^a_\mu p^\mu_a}{2E_a} + \mathcal{U}(\rho) \tag{5.20b}$$

provided that

$$E = \sum_a E^a , \tag{5.20c}$$

and $\mathcal{U}(\rho)$ defined as in (4.40). The c-numbers E^a, on the other hand, will acquire a physical meaning by the generalization of (5.18a) while the time-parameter is specified in accordance to (5.18b). One has, then, a set of 2N weak equalities

$$\varphi^a \equiv k^\mu p^a_\mu - E^a \approx 0 \tag{5.21a}$$

$$\chi_a \equiv k_\mu q^\mu_a - s \approx 0 \quad , \tag{5.21b}$$

with k^μ some arbitrary constant momentumlike vector ($k_\mu k^\mu = 1$). A plain repetition of the calculations performed in (Ch. III, #5) using k^μ instead of P^μ shows that with

$$C^{-1}_{AB} = \begin{pmatrix} 0 & \delta_{ab} \\ -\delta_{ab} & 0 \end{pmatrix} \tag{5.22a}$$

and

$$(J_{\mu\nu}, \varphi^A) = \begin{pmatrix} k_\mu p^a_\nu - k_\nu p^a_\mu \\ k_\mu q^a_\nu - k_\nu q^a_\mu \end{pmatrix} ; \; (P_\mu, \varphi^A) = \begin{pmatrix} 0 \\ -\delta^a k_\mu \end{pmatrix}$$

$$(Q_\mu, \varphi^A) = \begin{pmatrix} \eta^a k_\mu \\ 0 \end{pmatrix} \; ; \; (\mathcal{H}, \varphi^A) = \begin{pmatrix} 0 \\ -\delta^a \end{pmatrix} \tag{5.22b}$$

$$(p^a_\mu, \varphi^c) = \begin{pmatrix} 0 \\ -\delta^{ac} k_\mu \end{pmatrix} ; \; (q^\mu_a, \varphi^c) = \begin{pmatrix} \delta^c_a k_\mu \\ 0 \end{pmatrix}$$

we obtain the D.b. algebra

$$\{J_{\mu\nu}, J_{\rho\sigma}\}^* = K^{\alpha\beta}_{\mu\nu\rho\sigma} J_{\alpha\beta} \; ; \; \{J_{\mu\nu}, P_\rho\}^* = K^{\alpha}_{\mu\nu\rho} P_\alpha \; ; \; \{P_\mu, P_\nu\}^* = 0 \tag{5.23a}$$

$$\{J_{\mu\nu}, Q_\rho\}^* = K^{\alpha}_{\mu\nu\rho} Q_\alpha \; ; \; \{Q^\mu, P_\nu\}^* = \kappa^\mu_\nu \; ; \; \{Q^\mu, Q^\nu\}^* = 0 \tag{5.23b}$$

$$\{J_{\mu\nu}, \mathcal{H}\}^* = k_\mu P_\nu - k_\nu P_\mu \; ; \; \{P_\mu, \mathcal{H}\}^* = 0 \; ; \; \{Q^\mu, \mathcal{H}\}^* = \frac{P^\mu}{E} - k^\mu \tag{5.23c}$$

where the constants $K^{\alpha\beta}_{\mu\nu\rho\sigma}$, $K^{\alpha}_{\mu\nu\rho}$, κ^μ_ν are defined exactly as in (5.19) and the C.M. variable is given by

$$Q^\mu = \frac{G^\mu}{E} . \tag{5.24}$$

Again, the equations of motion give the correct results

$$\dot{J}_{\mu\nu} = \{J_{\mu\nu}, \mathcal{H}\}^* + \frac{\partial^* J_{\mu\nu}}{\partial s} = 0 \tag{5.25a}$$

$$\dot{Q}^\mu = \{Q^\mu, \mathcal{H}\}^* + \frac{\partial^* Q^\mu}{\partial s} = \frac{P^\mu}{E} \; ; \; \dot{P}_\mu = \{P_\mu, \mathcal{H}\}^* - \frac{\partial^* P_\mu}{\partial s} = 0 \tag{5.25b}$$

while the rest of the brackets are

$$\{q^\mu_a, p^b_\nu\}^* = \kappa^\mu_\nu \delta^b_a \; ; \; \{q^\mu_a, q^\nu_b\}^* = \{p^a_\mu, p^b_\nu\}^* = 0 \tag{5.26a}$$

$$\{q_a^\mu, \mathcal{H}\}^* = \frac{p_a^\mu}{E^a} - k^\mu \; ; \; \{p_\mu^a, \mathcal{H}\}^* = -\frac{\partial \mathcal{U}(\rho)}{\partial q_a^\mu} \tag{5.26b}$$

Thus, the individual particle equations of motion become

$$\dot{q}_a^\mu = \{q_a^\mu, \mathcal{H}\}^* + \frac{\partial^* q_a^\mu}{\partial s} = \frac{p_a^\mu}{E^a} \tag{5.27a}$$

$$\dot{p}_\mu^a = \{p_\mu^a, \mathcal{H}\}^* + \frac{\partial^* p^a}{\partial s} = -\frac{\partial \mathcal{U}(\rho)}{\partial q_a^\mu} \tag{5.27b}$$

as before. The use of P_μ instead of the arbitrary vector k_μ is, of course, permissible because of (5.25b); in this case the D.b. algebra coincides entirely with (3.53) and (3.55).

4. Reduction of Variables.

It is now possible, using the constraint equations, to reduce the algebra just obtained so that it becomes expressed solely in terms of the independent variables. Constraints (5.21b) are the manifestly covariant counterparts of (5.14b) (b). Hence, when one passes to the k-rest frame of reference, a generalization of algebra (5.15e) is obtained for the multiparticle case with the exceptions that the rest masses do not enter explicitly in the formalism and the time component of the total momentum reduces to a c-number. Apparently, these are the facts that permit the presence of interaction potentials thus sidestepping the no-interaction theorem.

Using (5.21) we have

$$q_a^0 = k_0^{-1} s + \vec{u} \cdot \vec{q}_a \quad ; \quad P_0^a = k_0^{-1} E^a + \vec{u} \cdot \vec{P}^a \tag{5.28a}$$

with

$$\vec{u} = \vec{k} / k_0 \; . \tag{5.28b}$$

The operators (5.20 a,b) become

$$\vec{J} = \vec{q}_a \wedge \vec{P}^a \quad ; \quad \vec{\mathcal{P}} = \sum_a \vec{P}^a \quad ; \quad \vec{Q} = \eta^a \vec{q}_a \tag{5.29a}$$

$$\mathcal{H} = - \frac{\vec{P}_a \cdot \vec{P}^a}{2 E_a} + \mathcal{U}\left([\vec{u} \cdot \vec{r}_{ab}]^2 - \vec{r}_{ab}^{\,2} \right) + \\ + \frac{(\vec{u} \cdot \vec{P}_a)^2}{2 E_a} - \frac{\vec{u} \cdot \vec{\mathcal{P}}}{k_0} - \frac{E}{2 k_0^2} \tag{5.29b}$$

$$K_i \equiv J_{ai} = \frac{\mathcal{P}_i}{k_0} s - \frac{E}{k_0} Q_i + u^j J_{ji} \tag{5.29c}$$

$$\mathcal{P}_0 = k_0^{-1} E + \vec{u} \cdot \vec{\mathcal{P}} \quad ; \quad Q^0 = k_0^{-1} s + \vec{u} \cdot \vec{Q} \tag{5.29d}$$

The complete algebra for these operators is given in appendix C. Here we are interested only in pointing out that in the k-rest frame, i.e., when $\vec{k} = 0$ and $k_0 = 1$, P_0 becomes a neutral element of the algebra while the Hamiltonian is left as the only generator for time translations. This is something expected since, in that particular reference frame, the common time-parameter coincides with the time coordinate for all particles so P_0 and H have to coincide or one of them must be eliminated from the algebra. Then with

$$P_0 = E \quad ; \quad Q^0 = t \quad ; \quad \vec{K} = \vec{P} t - E \vec{Q} , \tag{5.30a}$$

the rest of the operators reduce to

$$\vec{J} = \vec{q}_a \wedge \vec{p}^a \quad ; \quad \vec{P} = \sum_a \vec{p}^a \quad ; \quad \vec{Q} = \eta^a \vec{q}_a \tag{5.30b}$$

$$\mathcal{H} = - \frac{\vec{p}_a \cdot \vec{p}^a}{2 E_a} + \mathcal{U}(r^2) + \frac{E}{2} \tag{5.30c}$$

and, because K in (5.30a) is just a linear combination of P and Q in (5.30b), the only meaningful P.b. of the algebra are

$$\{ J_i , J_j \} = \epsilon_{ijk} J_k \quad ; \quad \{ J_i , P_j \} = \epsilon_{ijk} P_k$$

$$\{ J_i , Q_j \} = \epsilon_{ijk} Q_k \quad ; \quad \{ P_i , P_j \} = 0$$

$$\{Q^i, Q^j\} = 0 \quad ; \; \{Q^i, P_j\} = \delta^i{}_j$$

$$\{J_i, \mathcal{H}\} = \{P_i, \mathcal{H}\} = 0 \; ; \; \{Q^i, \mathcal{H}\} = \frac{P^i}{E} \tag{5.31}$$

and the equations of motion have the expected form,

$$\dot{q}^i_a = \{q^i_a, \mathcal{H}\} = \frac{p^i_a}{E^a} \quad ; \quad \dot{p}^a_i = \{p^a_i, \mathcal{H}\} = -\frac{\partial U}{\partial q^i_a} \tag{5.32}$$

The bare rest masses of the particles are not defined but they can be assumed to be implicitly included in the definition of the "canonical mass" (4.52),

$$(p^a_\mu p^{a\mu})^{1/2} = m^a(s) \tag{5.33}$$

then, from the first of the equations of motion (5.32) and since (5.28a) implies

$$\beta^0_a \equiv \dot{q}^0_a = 1 \tag{5.34}$$

when s = t, equation (5.33) can be written as

$$m^a(t) = E^a\left[1 - (\vec{\beta}^a)^2\right]^{1/2} = E^a \gamma_a^{-1}(t)$$

i.e.,

$$E^a = \gamma^a(t)\, m^a(t) \,. \qquad (5.35)$$

Consequently, if the energy E^a is positive, the particle moves in its world line in such a way that when its velocity increases the "canonical mass" decreases accordingly and vice versa.

CONCLUSIONS.

The no-interaction theorem has placed severe restrictions on the extension of the theory of relativity to the description of isolated systems with more than one particle involving direct interactions. Nevertheless, several starting points have proven useful in the development of a consistent solution to the problem. Theories without invariant world lines, multiparameter Hamiltonian formalisms and canonical theories based on generalized mass-shell type constraints, among others, are still in the process of being completed.

In the present work, an attempt has been made to analyze the most immediate mathematical implications of a set of postulates chosen to constitute a direct generalization of the non-relativistic case. The postulation of a variational principle involving individual canonical coordinates and subject to equal time-plane constraints (global transversality conditions) has led us to the introduction of arbitrary c-numbers (other than the rest masses of the particles) into the manifestly covariant Lagrangian. The arbitrariness of such constants is removed by the constraints and they soon become identified with the energies of the particles relative to the constraining plane. With the abandonment of the mass-shell type constraints, the need for a homogeneous Lagrangian disappears, as well as the existence of a vanishing canonical Hamiltonian. The insertion of arbitrary constants in the formalism permits the definition of a center of momentum (C.M.) coordinate satisfying the correct Poisson brackets and the norm of its canonically conjugate momentum coincides with the rest-mass of the system as a whole. But the invariance of the individual world lines prevents the norm of the individual momenta from being constants of the motion, in fact, they must remain indeterminate until the solution of the equations of motion is eventually found.

Further, because the Lagrangian is not invariant under arbitrary reparametrization in our case, one has to restrict the set of admissible time-parameters to be proper times associated with inertial observers, i.e., only arc-lengths of Minkowski flat-space geodesics can be used in the formulation of the variational principle and constraints. Since those parameters are related to one another by affine transformations, we have the important consequence that the Lagrangian remains formally unchanged under a parameter transformation if the structure of the interaction elements are such that the coefficients associated with the transformation can be absorbed into the c-numbers. These constants are finally identified when the geodesic involved in the parametrization is specified via the constraints. A Lagrangian which clearly admits the new requirement and is also expressible in terms of spacelike variables related with the chosen geodesic, turns out to be of a simple standard form. The interpretation of the parametrization process can be given in purely geometrical arguments and is valid even when the particles are free; therefore the G.T. conditions affect the description of the world lines but not the dynamics of the system.

This being the case, cluster decomposition can be achieved here with binary interactions.

All those features appear as a consequence of the fact that although transformations among inertial observers are carried out by operators belonging to the Poincare group, still the parametrization process introduces additional symmetries in the description of the world lines so that the C.M. and the Hamiltonian can be accepted as independent additional operators. Then a canonical realization of the complete fifteen operator algebra modulo the c-numbers and constraints as neutral elements yields the same dynamics that follow from the variational principle. When Dirac brackets are calculated, one can, in fact, reduce the number of variables and a further Lorentz transformation reduces the time component of the total linear momentum to a neutral element so the Hamiltonian is left as the only time-translation operator.

For the models proposed in this work, the nonrelativistic limit is obtained by identifying the observable constant mass of every particle with its total energy divided by the square of the speed of light when the usual limit for that speed approaching infinity is applied.

Although this work deals only with the classical formulation of the problem, we think that the theory can be quantized(*) and,

(*) A first attempt in that direction is the work of Crater[36] who obtained a Hamiltonian similar to (5.20b) for a three-body system. Although his derivation is more directly related to the Todorov-Komar model, some of his final results can be linked to ours and subject, eventually, to equivalent analysis.

eventually, extended to include particles with nonzero spin. Realizations of the fifteen operator algebra (5.11) with null c-number neutral elements, $E = 0$, may be also interesting and worthwhile to study. Also, some velocity-dependent potentials can be included in this formalism without major changes but whether the electromagnetic interactions can fit in the present scheme is not yet clear.

APPENDIX A

The Two-Body Case.

It has been shown that, for a system composed of just two particles, the homogeneous Lagrangian (2.24a) gives the G.T. as its canonical constraints[22]. This apparently contradicts the conclusions arrived at before equation (2.24d), i.e., that (2.21) and (2.24c) will overconstrain the system.

While such a conclusion is true in general, for this particular system with two constituents we find a circumstantial relation between those two sets of constraints. Indeed, here the primary constraints (2.24d) can be written

$$p^1_\mu p^{1\mu} = (\mathcal{U}^1)^2 \quad ; \quad p^2_\mu p^{2\mu} = (\mathcal{U}^2)^2 \tag{A.1}$$

while from the conservation of the total linear momentum, $P^2 = M_o^2$, one has

$$p^1_\mu p^{1\mu} + 2\, p^1_\mu p^{2\mu} + p^2_\mu p^{2\mu} = M_o^2 \quad ; \tag{A.2}$$

therefore,

$$\mathcal{P}^\mu p^1_\mu = \tfrac{1}{2}\left[M_o^2 + (\mathcal{U}^1)^2 - (\mathcal{U}^2)^2 \right] \tag{A.3a}$$

$$\mathcal{P}^\mu p^2_\mu = \tfrac{1}{2}\left[M_o^2 + (\mathcal{U}^2)^2 - (\mathcal{U}^1)^2 \right] . \tag{A.3b}$$

Now, for a two body interaction, $(U^1)^2$ and $(U^2)^2$ can differ only by constants; for instance, in the Gomis model

$$(U^a)^2 = m^2 - V(\rho) \tag{A.4}$$

(a = 1,2). So, (A.3) becomes

$$\begin{aligned} P^\mu p^1_\mu &= \frac{1}{2}\left(M_0^2 + m_1^2 - m_1^2\right) = \text{const.} \\ P^\mu p^2_\mu &= \frac{1}{2}\left(M_0^2 + m_2^2 - m_1^2\right) = \text{const.} \end{aligned} \tag{A.5}$$

Equations (2.24d) and (2.21) are thus satisfied without overconstraining the system. Obviously, the critical relation (A.4) can not hold for N greater than two, nor can the procedure followed above be generalized since the number of mixed products like in (A.2) increases too rapidly with N.

APPENDIX B

Gauge Transformations.

An arbitrary variation of the Lagrangian in (1.1) is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\dot{q}^{\mu}_{a}}\,\delta\dot{q}^{\mu}_{a} + \frac{\partial\mathcal{L}}{\partial q^{\mu}_{a}}\,\delta q^{\mu}_{a}\,. \tag{B.1}$$

In particular, let us consider a gauge transformation which produces the variations (1.10) on the coordinates with

$$\delta\dot{q}^{\mu}_{a} = \frac{d}{ds}(\delta q^{\mu}_{a}) \tag{B.2}$$

the respective variations on the velocities. If we introduce the functions

$$p^{a}_{\mu}(q,\dot{q}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}^{\mu}_{a}}\quad, \tag{B.3}$$

one has for (B.1),

$$\delta\mathcal{L} = p^{a}_{\mu}\frac{d}{ds}(\delta q^{\mu}_{a}) + \frac{\partial\mathcal{L}}{\partial q^{\mu}_{a}}\,\delta q^{\mu}_{a} \tag{B.4a}$$

which, by adding and subtracting the quantity

$$\dot{p}^{a}_{\mu}\,\delta q^{\mu}_{a} \equiv \frac{d}{ds}(p^{a}_{\mu})\,\delta q^{\mu}_{a} \tag{B.4b}$$

is transformed into

$$\delta \mathcal{L} = \frac{d}{ds} \left(p^a_\mu \, \delta q^\mu_a \right) - \left(\dot{p}^a_\mu - \frac{\partial \mathcal{L}}{\partial q^\mu_a} \right) \delta q^\mu_a \qquad \text{(B.4c)}$$

Using (1.10) and the definition

$$L^a_\mu \equiv \dot{p}^a_\mu - \frac{\partial \mathcal{L}}{\partial q^\mu_a} \qquad \text{(B.5)}$$

we find

$$\delta \mathcal{L} = \epsilon \left\{ \frac{d}{ds} \left(p^a_\mu \, f_b \, \lambda^{b\mu}_{\ \ a} \right) - L^a_\mu \, f_b \, \lambda^{\mu}_{a} \right\} ; \qquad \text{(B.6)}$$

then, if the last term vanishes identically (i.e., without requiring that (B.5) must vanish),

$$L^a_\mu \, f_b \, \lambda^{b\mu}_{\ \ a} \equiv 0 , \qquad \text{(B.7)}$$

the total variation of the Lagrangian becomes equal to the total derivative of a given function F:

$$\delta \mathcal{L} = \epsilon \, \frac{d}{ds} (F) \qquad \text{(B.8a)}$$

with

$$F \equiv p^a_\mu \, f_b \, \lambda^{b\mu}_{\ \ a} \qquad \text{(B.8b)}$$

and we say that the Lagrangian is quasi-invariant[30], under the transformations considered. The Lagrangian is said invariant when $\delta \mathcal{L} = 0$ under the same transformation.

Since the $\lambda^{b\,\mu}_{a}$ are the null eigenvectors of the Hessian, equation (1.4) holds automatically and (B.7) is identical to the requirement that the constraint equations (1.8) be identically satisfied and no constraints will actually appear.

APPENDIX C

P.B. Algebra in the Reduced Phase-Space.

With operators (5.29) one obtains, from the D.b. expressions (5.23), or, by direct calculation the following structure, valid in any Lorentz frame:

$$\kappa^{\mu}_{\nu} = \delta^{\mu}_{\nu} - k^{\mu} k_{\nu} \Rightarrow \begin{cases} \kappa^{i}{}_{j} = \delta^{i}_{j} - k^{i}{}_{j} \\ \kappa^{0}{}_{j} = -k^{0} k_{j} \\ \kappa^{0}{}_{0} = 1 - k_{0}^{2} \end{cases}$$

$$\{J_{ij}, J_{rs}\} = \kappa_{ir} J_{js} - \kappa_{ij} J_{jr} - \kappa_{jr} J_{is} - \kappa_{js} J_{ir}$$

$$\{J_{0i}, J_{rs}\} = -k^{0}(k_{r} J_{is} + k_{s} J_{ir}) - \kappa_{ir} J_{0s} - \kappa_{is} J_{0r}$$

$$\{J_{00}, J_{ij}\} = k^{0}(k_{j} J_{0i} + k_{i} J_{0j}) + (1-k_{0}^{2}) J_{ij}$$

$$\{J_{ij}, P_{r}\} = \kappa_{ir} P_{j} - \kappa_{jr} P_{i}$$

$$\{J_{ij}, P_{0}\} = -k^{0}(k_{i} P_{j} - k_{j} P_{i})$$

$$\{J_{0i}, P_{j}\} = -(k_{0} k_{j} P_{i} + \kappa_{ij} P_{0})$$

$$\{J_{0i}, P_{0}\} = (1-k_{0}^{2}) P_{i} + k_{0} k_{i} P_{0}$$

$$\{J_{ij}, Q_r\} = \kappa_{ij} Q_j - \kappa_{jr} Q_i$$

$$\{J_{ij}, Q_0\} = -k^0 (k_i Q_j - k_j Q_i)$$

$$\{J_{0i}, Q_j\} = -(k_0 k_j Q_i + \kappa_{ij} Q_0)$$

$$\{J_{0i}, Q_0\} = (1-k_0^2) Q_i + k_0 k_i Q_0$$

$$\{P_i, P_j\} = \{P_0, P_i\} = \{P_0, P_0\} = 0$$

$$\{Q_i, Q_j\} = \{Q_0, Q_i\} = \{Q_0, Q_0\} = 0$$

$$\{Q^i, P_j\} = \kappa^i{}_j \;;\; \{Q^0, P_j\} = -k^0 k_j \;;\; \{Q^i, P_0\} = -k^0 k^i$$

$$\{J_{ij}, \mathcal{H}\} = k_i P_j - k_j P_i \;;\; \{J_{0i}, \mathcal{H}\} = k_0 P_i - k_i P_0$$

$$\{P_i, \mathcal{H}\} = 0 \quad ; \quad \{P_0, \mathcal{H}\} = 0$$

$$\{Q^i, \mathcal{H}\} = \frac{P^i}{E} - k^i \quad ; \quad \{Q^0, \mathcal{H}\} = \frac{P^0}{E} - k^0$$

A Lorentz transformation to the k-rest frame where $k_i = 0$, $k_0 = 1$ gives us (5.31).

APPENDIX D

Poincare-Invariant
D.b. Realizations.

The Dirac-bracket realizations (5.19) and (5.23) do not correspond to the familiar form of the Poincare algebra; that is due to the fact that, strictly speaking, one should include the parametrizing geodesic into the system as an arbitrary freely moving particle. Indeed, if the operators (5.20) are consequently modified, we have that

$$\mathcal{P}'_\mu = \mathcal{P}_\mu + k_\mu \quad ; \quad J'_{\mu\nu} = J_{\mu\nu} + l_{\mu\nu}$$
$$\mathcal{Q}'^\mu = \frac{1}{E+1}\left(E\mathcal{Q}^\mu + \chi^\mu\right) ; \quad \mathcal{H}' = \mathcal{H} + \tfrac{1}{2}(k^2 - 1) \qquad \text{(D.1)}$$

with

$$l_{\mu\nu} = \chi_\mu k_\nu - \chi_\nu k_\mu ,$$

still satisfy algebra (5.11) in terms of the P.b. brackets

$$\{A, B\}' \equiv \frac{\partial A}{\partial q^\sigma_c}\frac{\partial B}{\partial p^c_\sigma} - \frac{\partial B}{\partial q^\sigma_c}\frac{\partial A}{\partial p^c_\sigma} + \frac{\partial A}{\partial \chi^\sigma}\frac{\partial B}{\partial k_\sigma} - \frac{\partial B}{\partial \chi^\sigma}\frac{\partial A}{\partial k_\sigma} \qquad \text{(D.2)}$$

the complete set of constraints is now

$$\varphi^a \equiv k^\mu p^a_\mu - E^a \approx 0 \quad ; \quad \chi_a \equiv k_\mu q^\mu_a - s \approx 0$$
$$\varphi' \equiv k_\mu k^\mu - 1 \approx 0 \quad ; \quad \chi' \equiv k_\mu \chi^\mu - s \approx 0 \qquad \text{(D.3)}$$

so that the matrix (1.29a) becomes

$$C^A{}_B \approx \left(\begin{array}{cc|cc} 0 & -\delta^a{}_b & 0 & -E^a \\ \delta^a{}_b & 0 & 0 & -s\delta^a \\ \hline 0 & 0 & 0 & -2 \\ E_b & s\delta_b & 2 & 0 \end{array}\right) \tag{D.4}$$

its inverse being

$$C^{-1\,A}{}_B \approx \left(\begin{array}{cc|cc} 0 & \delta^a{}_b & -\frac{s}{2}\delta^a & 0 \\ -\delta^a{}_b & 0 & \frac{1}{2}E^a & 0 \\ \hline \frac{s}{2}\delta_b & -\frac{1}{2}E_b & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{array}\right) \tag{D.5}$$

Instead of the relations (5.22b) one obtains

$$(J'_{\mu\nu}, \varphi^A)' = \left(\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right) \quad ; \quad (P'_\mu, \varphi^A)' = \left(\begin{array}{c} 0 \\ -\delta^a k_\mu \\ \hline 0 \\ -k_\mu \end{array}\right)$$

$$(Q'_\mu, \varphi^A)' = \left(\begin{array}{c} \eta'_a k^\mu + \eta_0 p^\mu_a \\ \eta'_0 q^\mu_a \\ \hline 2\eta'_0 k^\mu \\ \eta'_0 \chi^\mu \end{array}\right) ; \quad (\mathcal{H}', \varphi^A)' = \left(\begin{array}{c} 0 \\ -\delta^a \\ \hline 0 \\ -1 \end{array}\right) \tag{D.6}$$

$$(q^\mu_a, \varphi^c)' = \left(\begin{array}{c} \delta^c_a k^\mu \\ 0 \\ \hline 0 \\ 0 \end{array}\right) \quad ; \quad (p^a_\mu, \varphi^c)' = \left(\begin{array}{c} 0 \\ -\delta^{ac} k_\mu \\ \hline 0 \\ 0 \end{array}\right)$$

$$(\chi^\mu, \varphi^c)' = \left(\begin{array}{c} p^{c\mu} \\ q^{c\mu} \\ \hdashline 2k^\mu \\ x^\mu \end{array} \right) \quad ; \quad (k_\mu, \varphi^c)' = \left(\begin{array}{c} 0 \\ 0 \\ \hdashline 0 \\ -k_\mu \end{array} \right)$$

where

$$\eta'_a = \frac{E a}{E+1} \quad ; \quad \eta'_0 = \frac{1}{E+1} \quad . \tag{D.7}$$

Using (D.5) and (D.6) we find that the complete D.b. structure

$$\{ J'_{\mu\nu}, J'_{\rho\sigma} \}'^{*} = C^{\alpha\beta}_{\mu\nu\rho\sigma} J'_{\alpha\beta} \; ; \; \{ J'_{\mu\nu}, P'_{\rho} \}'^{*} = C^{\alpha}_{\mu\nu\rho} P'_{\alpha} \tag{D.8a}$$

$$\{ P'_\mu, P'_\nu \}'^{*} = 0$$

8a)

$$\{ J'_{\mu\nu}, Q'_{\rho} \}'^{*} = C^{\alpha}_{\mu\nu\rho} Q'_{\alpha} \; ; \; \{ Q'^{\mu}, P'_{\nu} \}'^{*} = \delta^{\mu}_{\nu} - \eta'_0 P'^{\mu} k_{\nu}$$

$$\{ J'_{\mu\nu}, Q'_{\rho} \}'^{*} = C^{\alpha}_{\mu\nu\rho} Q'_{\alpha} \; ; \; \{ Q'^{\mu}, P'_{\nu} \}'^{*} = \delta^{\mu}_{\nu} - \eta'_0 P'^{\mu} k_{\nu}$$

$$\{ Q'^{\mu}, Q'^{\nu} \}'^{*} = -\eta'^{2}_0 \{ J'^{\mu\nu} + s (P'^{\mu} k^{\nu} - P'^{\nu} k^{\mu}) \} \tag{D.8b}$$

$$\{ J'^{\mu\nu}, \mathcal{H}' \}'^{*} = \{ P'_{\mu}, \mathcal{H}' \}'^{*} = 0 \; ; \; \{ Q'^{\mu}, \mathcal{H}' \}'^{*} = 0$$

Also, since

$$\frac{\partial^{*} J'^{\mu\nu}}{\partial s} = 0 \; ; \; \frac{\partial^{*} P'_{\mu}}{\partial s} = 0 \quad ; \quad \frac{\partial^{*} Q'^{\mu}}{\partial s} = \eta'_0 P'^{\mu} \tag{D.9}$$

one has, according to (1.32a),

$$\dot{J}'_{\mu\nu} = 0 \quad ; \quad \dot{P}'_{\mu} = 0 \quad ; \quad \dot{Q}'^{\mu} = \eta'_0 \dot{P}^{\mu} \qquad \text{(D.10)}$$

Obviously, the insertion of the geodesic into the formalism has no effect in the equations of motion or the cluster decomposition process since the addition of independent terms to the Lagrangian and to the Hamiltonian is permissible.

REFERENCES.

1. P.A.M. Dirac, Rev. Mod. Phys. 21, 392 (1949)

2. D.G. Currie, T.F. Jordan and E.C.G. Sudarshan, Rev. Mod.Phys. 35, 350 (1965)

3. H. Leutwyler, Nuov. Cim. 37, 556 (1965)

4. S. Orenstein and K. Rafanelli, Nuov. Cim. 23, 93 (1978)

5. L.H. Thomas, Phys. Rev. 85, 868 (1952)

6. B. Bakamjian and L.H. Thomas, Phys. Rev. 92, 1300 (1953)

7. L.L. Foldy, Phys. Rev. 122, 275 (1961)

8. D.G. Currie, Phys. Rev. 142, 817 (1966)

9. R.N. Hill, Journ. Math. Phys. 8, 201 (1967)

10. P.H. Droz-Vincent, Am. Inst. H. Poincare, 27, 407 (1977)

11. P.H. Droz-Vincent, "N-Body Relativistic Systems", College de France preprint (1979).

12. I.T. Todorov, "Dynamics of Relativistic Point Particles as a Problem with Constraints", Commum. of the JINR EZ-10125, Dubna (1976)

13. A. Komar, Phys. Rev. D 18, 2872 (1978)

14. A. Komar, Phys. Rev. D 18, 1881 (1978)

15. A. Komar, "Relativistic Action-at-a-Distance and Quasi-Separability" Yeshiva University preprint (1979)

16. M. King and F. Rohrlich, "Dynamic Confinement from Velocity Dependent Interactions" Syracuse University preprint (1979)

17. H. Sazdjian, "Separable Interactions in Classical Relativistic Hamiltonian Mechanics" Institute de Physique Nucleaire preprint (1980)

18. F. Rohrlich, Annals of Physics 117, 292 (1979)

19. F. Rohrlich, Physics 96A, 290 (1979)

20. N. Mukunda and E.C.G. Sudarshan, "Forms of Relativistic Dynamics with world Lines" University of Texas preprint DOE 401 (1980)

21. N. Mukunda et.al."Constraint Dynamics of Particle World Lines" University of Texas report DOE 400 (1980)

22. D. Dominici, J. Gomis and G. Longhi, Nuov. Cim. 48B, 152 (1978)

23. D. Dominici, J. Gomis and G. Longhi, Nuov. Cim. 48A, 257 (1978)

24. P. Bergmann and A. Komar, "The Hamiltonian in Relativistic Systems of Interacting Particles" Y.U. preprint (1980)

25. K. Rafanelli and S. Orenstein, Nuov. Cim. 48A, 227 (1978)

26. K. Rafanelli and S. Orenstein, Nuov. Cim. 50A, 496 (1979)

27. P.A.M. Dirac, Can. J. Math. 2, 129 (1950)

28. P.A.M. Dirac, Proc. Roy. Soc. A 246, 326 (1958)

29. P.A.M. Dirac, "Lectures on Quantum Mechanics", Yeshiva University, N.Y. (1964)

30. E.C.G. Sudarshan and N. Mukunda, "Classical Dynamics A Modern Perspective" John Wiley, N.Y. (1974)

31. A.J. Hanson, T. Regge and C. Teiterboim, "Constrained Dynamical Systems", Academia Nationale dei Lincei Rome (1976)

32. S. Shanmugadhasan, J. Math. Phys. 14, 677 (1973)

33. M.H.L. Pryce, Proc. Roy. Soc. A 195, 62 (1948)

34. J.L. Synge, "Relativity: The Special Theory" North Holland (1965)

35. F. Rohrlich, Phys. Rev. 23 1305 (1981)

36. H.W. Crater, Phys. Rev. D. 18 1881 (1978)