Majority Logic and majority spaces in contrast with Ultrafilters

by

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Abstract

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In this thesis I will extend graded modal logic (**GML**) first discussed in [9, 10] to a logic that can capture the concept of majority. I will present the modal system **MJL** that will capture our intuition about majority and prove soundness and completeness for this system. I will also discuss May's theorem with infinite population. Graded modal logic, as presented in [7], extends propositional modal systems with a set of modal operators $\Diamond_n$ ($n \in \mathbb{N}$) that express "there are more than n accessible worlds such that...". I extend **GML** with a modal operator W that can express "there are at least half of the accessible worlds such that...". The semantics of W is straightforward provided that there are only finitely many accessible worlds; however if there are infinitely many accessible worlds the situation becomes much more complex. In order to deal with such situations, we introduce the notion of majority space. A majority space is a set W together with a collection of subsets of W intended to be the weak majority (at least half) subsets of W. We then extend standard Kripke structure with a function that assigns a majority space over the set of accessible states to each state. Given this extended Kripke semantics,

majority logic is proved sound and complete.

Part of this thesis is devoted to talk about May's theorem with infinite population. We will talk about three different kinds of anonymity: finite, bounded and infinite. We will compare all three of them together. Given an infinite subset A that we call majority, if we remove a finite set of elements of A then are we going to get a set that we still call majority?. This answer will be a generalized property of majority sets over finite spaces.

I will also talk about majority spaces in contrast with ultrafilters. I will present another way to construct ultrafilters and majority spaces by using the limits of sequences over that family of sets. I will also argue that ultrafilters have a dictatorship flavor while the majority spaces have a democracy flavor.

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To my parents, my brother Issa and his wife Jennifer, my sister Linda and her husband Ali, and to my sister Lina. Last but not least, to my niece Alaa and my nephew Hussein.

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Chapter 1

Overview

1.1 Overview

The 2000 presidential election highlighted the frailties and vicissitudes of the process of nominating and electing a president, beginning with state primaries and ending in the Electoral College. Logicians in coordination with others are trying to propose a good system that guarantees some of the conditions we expect.

The need to create a new logical system for reasoning about voting and majority is getting more crucial.

1.2 History

Normal modal logics like K, T, S4 and S5 have one modal operator, the necessity (or box) operator $\Box$. The possibility (or diamond) operator $\Diamond$ is defined as its dual. By definition,

$$\Diamond\phi \leftrightarrow \neg\Box\neg\phi$$

In 1970 Goble investigated modal logics with more than one modality. His logics have a fixed and finite number of modalities. Each modality represents a different grade of necessity. For example, the formula

$$N_m\phi \wedge N_n\psi$$

for positive integers $m < n$, says that ϕ is more necessary than ψ. Kit Fine 1969, 1972 generalizes this idea and introduces modal logics with numerical modalities. These are now commonly referred to as modal logics with graded modalities. In a series of papers, Fattorosi-Barnaba, de Caro and Cerrato [Fattorosi-Barnaba and de Caro, 1985; de Caro, 1988; Fattorosi-Barnaba and Cerrato, 1988; Cerrato, 1990] rediscover and analyze various modal logics of graded modalities. Recent investigations of graded modal logics are by van der Hoek in 1992. Together with de Rijke he applies graded modalities to linguistics and artificial intelligence. In 1991 they show that generalized quantifiers can be modeled with graded modalities. In 1992 they also show that certain numerical quantifier operations available in kl-one-based knowledge representation languages can be modeled with graded modalities. In a recent paper by Mark Fey "May's Theorem with an Infinite Population" they used an axiomatic approach and attempt to extend May's Theorem characterizing majority rule for an infinite population.

1.3 Introduction

The language of modal logic has long been used to model intensional notions such as knowledge, belief and obligation. In this thesis we present a new modal logic which models an agent's ability to reason about majorities. The concept of majority often plays an important role when an agent is faced with a decision in a social situation. For example, think of dinner with a group of friends. Chances are that many of the decisions, such as choice of restaurant, appetizers or wine, were based on the will of the majority. An extended example which illustrates this point is found in the next section. Of course, the concept of majority is integral to many voting systems. With these intuitions in mind, we propose a logic, **MJL**, in which the concept of majority is axiomatized.

Given a formula α, the language of propositional modal logic can express "α is true in *all* accessible worlds" ($\Box\alpha$), and "α is true in *at least one* accessible world" ($\Diamond\alpha$). But suppose that we want to express that α is true in at least *three* accessible worlds or that α is true in a *majority (more than half)* of the accessible worlds. The language of propositional modal logic cannot express such statements. The logic **MJL** presented in this paper will use modal operators that can specify exactly how many accessible worlds are of interest.

To start with, we add the graded modalities that were first discussed in [9, 10]. For each $n \in \mathbb{N}$, the formula $\Diamond_n\alpha$ is intended to mean that α is true in strictly more than n accessible worlds, and so its dual $\Box_n\alpha$ is intended to mean that $\neg\alpha$ is true in less than or equal to n accessible worlds. We may call $\Diamond_n\alpha$ an *at least* formula, since $\Diamond_n\alpha$ will be true precisely when α is true in *at least* $n+1$ accessible worlds. Similarly we may call $\Box_n\alpha$ *all but* formulas, since $\Box_n\alpha$ will be true precisely when α is true in *all but* n accessible worlds. For instance, if the formula $\Box_k\bot$ is true at some world w, then w has at most k accessible worlds. For simplicity we write $\Diamond\alpha$ instead of $\Diamond_0\alpha$ and $\Box\alpha$ instead of $\Box_0\alpha$.

We then extend the graded modal language with a new modal operator W, where $W\alpha$ is intended to mean that α is true in at least half of the accessible worlds. Hence its dual, $M\alpha$ will mean that α is true in strictly more than half of the accessible worlds. Thus, M represents strict Majority and W represents Weak majority. In what follows, when we use "majority", we mean weak majority (i.e. more than or equal to 50%).

Before proceeding we should check that we are in fact gaining expressive power with the new modal operators. To see this note that **MJL** does not obey bisimulation, i.e., bisimilar models need not validate the same formulas. We can easily find two bisimilar Kripke models where in one of them we have $W\alpha$ true at some state s and in the other $W\alpha$ not true at a bisimilar state. Since bisimilarity preserves traditional modal formulas, it follows that the operator W cannot be defined from the standard modal operators ($\Box$ and $\Diamond$). A similar argument shows that $\Diamond_n$ cannot be defined from the standard modal operators. For an extended discussion of this fact refer to [7].

Furthermore, we shall show that the modal operator M cannot be expressed with the graded modal operators[1]

As an example of the type of reasoning captured in our logic, consider the following variant of the well-known muddy children puzzle. Suppose that there are $n > 1$ children[2] who have been playing outside and $k > \lfloor n/2 \rfloor$ of them have mud on their forehead. After a while, the children's father comes outside and announces "A strict majority (strictly more than half) of you have mud on your forehead." The father then proceeds to ask the children to announce if they have dirt on their forehead. It is not too hard to see that the $(k - \lfloor \frac{n}{2} \rfloor)^{th}$ time the children are asked if they have mud on their forehead, the dirty children will correctly respond.

Given the intended interpretation of $W\alpha$, defining truth in a Kripke model is straightforward provided there are only finitely many accessible worlds. However, there are situations, such as in the canonical model, in which one cannot assume that the number of accessible worlds is finite. This leads us to the question "what is the majority of an infinite set?". The standard definition, i.e. more than half, no longer makes obvious sense. Should we consider the even numbers a weak majority of the natural numbers, and if

[1] In [6] de Rijke develops a notion of bisimulation (g-bisimulation) for graded modal logic. He then uses this notion to prove some model theoretic results such as the finite model property. So, we need to show that there are two models that are g-bisimilar but can be distinguished using the M operator. Of course, if the number of accessible worlds is fixed to be n then $M\alpha$ can be defined to be $\Diamond_{\lfloor n/2 \rfloor}\alpha$; however this definition only works if the number of accessible worlds is known.

[2] Of course, we assume that the children are perfect reasoners, honest, and cannot feel the mud on their forehead.

so what about the set that contains all the even numbers and the set $\{1, 3\}$? Mark Fey in [8] proposes a very interesting answer to this question. However, Fey's solution is not appropriate for our general framework and so we need another solution. We propose *majority spaces*, which generalize the concept of an ultrafilter, as a solution to the problem of defining a majority of an infinite set.

This thesis is organized as follows. The next chapter will be preliminaries. Chapter three reviews graded modal logic. In chapter four I talk about May's theorem with infinite population. In chapter five I talk about majority spaces in contrast with ultrafilters. Finally, in chapter six I describe the language of majority logic and offer a complete axiomatization.

Chapter 2

Preliminaries

2.1 Notation and Definitions

Consider the set of natural numbers $\mathbb{N} = \{1, 2, 3...\}$. Denote by $\mathcal{P}$ the power set of $\mathbb{N}$. Let $|A|$ denote the cardinality of A. Recall that $|A| = |B|$ if and only if there is a one to one mapping from A onto B. If A and B are infinite subsets of $\mathbb{N}$ then $|A| = |B| = |\mathbb{N}| = \aleph_0$. Denote the set of nonnegative integers by $\mathbb{N}* = \{0, 1, 2, ...\}$. For a set $A \subseteq \mathbb{N}$, let A^C denote the *complement* of A. If A^C is a finite set then we say A is cofinite. Let $\mathcal{I} \subset \mathcal{P}$ be the collection of infinite sets that have infinite complement. That is, $A \notin \mathcal{I}$ if and only if A is finite or cofinite.

There are two alternatives (or candidates), labeled 0 and 1. Under the assumption that the preferences of the voters are linear orders, a preference profile for the voters is given by a function $v : \mathbb{N} \to \{0, 1\}$. Alternatively, we can read v as a *ballot* cast by the voters with no abstentions. We can view v as an infinite binary sequence.

For our analysis, it will be useful to work with $V = v^{-1}(1)$, the set of all voters who prefer alternative 1 to alternative 0. Clearly, for every preference profile v, the corresponding set V is an element of $\mathcal{P}$ and visa versa.

Given v, there are three *outcomes* to consider: alternative 1 is selected, alternative 0 is selected or the alternatives are tied. An *aggregation rule* is a function $f : P(\mathbb{N}) \to \{0, 1/2, 1\}$ with the value 1/2 interpreted as a tie. An alternative interpretation is that f embodies the social preference of the voters, as a function of their individual preferences. Although we do not permit voters to be indifferent, we do allow social indifference.

The following conditions that will be imposed on aggregation rules are well known.

Definition 1 *An aggregation rule f satisfies* **neutrality** *if, for all $A \in P(\mathbb{N}), f(A^c) = 1 - f(A)$.*

Definition 2 *An aggregation rule f satisfies* **monotonicity** *if, for all $A, B \in P(\mathbb{N}), A \subseteq B$ implies $f(A) \leq f(B)$.*

Definition 3 *An aggregation rule f satisfies* **positive reponsiveness** *if, for all $A, B \in P(\mathbb{N}), A \subsetneq B$ and $f(A) \neq 0$ implies $f(B) = 1$.*

A neutral aggregation rule treats the two alternatives equally; if the preferences of the individuals in society are reversed , then the social preference is reversed. A rule is monotonic if increasing the support of the alternative does not lower the alternative in the group preference. Positive responsiveness strengthens monotonicity by imposing "fragility of ties." That is, if a configuration of preferences generates social indifference, changing (at least) one individual's preference gives strict social preference. Thus " breaking the tie." For clarity, we have defined both monotonicity and positive responsiveness with respect to alternative 1, but, in combination with the neutrality axiom, the corresponding condition applies to alternative 0.

A permutation on $\mathbb{N}$ is a mapping $\pi : \mathbb{N} \to \mathbb{N}$ that is one to one and onto. The image of a set A under a permutation π is denoted πA that is:

$$\pi A = \{\pi(j) | j \in A\}$$

A permutation $\pi \in \mathbb{G}$ in *finite* if there is an integer N such that for all $n > N$ implies $\pi(n) = n$. For $A, B \in \mathcal{I}$, the permutation *connecting* A and B, denoted by π_{AB}, is given as follows. Let $A = \{a_1, a_2, ...\}$, $A^C = \{a'_1, a'_2, ...\}$, $B = \{b_1, b_2, ...\}$ and $B^C = \{b'_1, b'_2, ...\}$, where each of these sets are listed in ascending order. Then the permutation π_{AB} is defined as $\pi_{AB}(a_i) = b_i$ and $\pi_{AB}(a'_i) = b'_i$.

The Levy group $\mathcal{G}$ is a proper subgroup of $\mathbb{G}$ defined as the group of permutations π for which

$$\lim_{n\to\infty} \frac{|\{k : k \leq n \leq \pi(k)\}|}{n} = 0$$

Permutations $\pi \in \mathcal{G}$ have been termed bounded by Lauwers([12], [13]), and we adopt the same terminology here. The permutation connecting the sets of odd and even numbers is an example of a permutation that is bounded, but not finite. On the other hand, the permutation connecting the sets $T = \{3, 6, 9, ...\}$ and $R = \mathbb{N} - T = \{1, 2, 4, 5, 7, 8, ...\}$ is an example of a permutation that is not bounded.

We will consider three different versions of anonymity in infinite societies. In general, anonymity requires that the aggregation rule be invariant under various rearrangements.

Definition 4 *An aggregation rule f satisfies*

1. *strong anonymity if for all $A \in P(\mathbb{N})$ and all π, $f(\pi A) = f(A)$*
2. *bounded anonymity if for all $A \in P(\mathbb{N})$ and all $\pi \in$ Levy group , $f(\pi A) = f(A)$*
3. *finite anonymity if for all $A \in P(\mathbb{N})$ and all finite permutation π, $f(\pi A) = f(A)$*

Thus, strong anonymity demands that the aggregation rule be invariant under all possible permutations. At the other extreme, finite anonymity requires invariance only under finite permutations. In between these two, bounded anonymity specifies invariance under not just finite permutations, but also infinite permutations with a certain limited scope of rearrangement.

2.2 Mathematical Preliminaries

In this section, we cover some important mathematical concepts. We begin with a definition of an *ultrafilter* on $\mathbb{N}$

Definition 5 *An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets that satisfies*

1. $\phi \notin \mathcal{U}$ *and* $\mathbb{N} \in \mathcal{U}$
2. *if* $A, B \in \mathcal{U}$, *then* $A \cap B \in \mathcal{U}$
3. *if* $A \subset B$ *and* $A \in \mathcal{U}$ *then* $B \in \mathcal{U}$ *and*
4. *for all* $A \in \mathbb{N}$ *either* $A \in \mathcal{U}$ *or* $A^C \in \mathcal{U}$

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ can be defined equivalently as a collection of subsets that satisfies:

1. $\phi \notin \mathcal{U}$ *and* $\mathbb{N} \in \mathcal{U}$
2. *if* $A \subset B$ *and* $A \in \mathcal{U}$ *then* $B \in \mathcal{U}$ *and*
3. *if* $A \cup B \in \mathcal{U}$ *and* $A \cap B = \varnothing$ *then either* $A \in \mathcal{U}$ *or* $B \in \mathcal{U}$

There are two different types of ultrafilters on $\mathbb{N}$, principal and non-principal. An ultrafilter is principal if each of its members contains a particular $k \in \mathbb{N}$. In particular, if an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is principal, then $\bigcap_{A \in \mathcal{U}} A = \{k\}$. Any ultrafilter which is not principal is called a non-principal ultrafilter. It can be shown that, an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is non-principal iff it has empty intersection; that is, if $\bigcap_{A \in \mathcal{U}} A = \varnothing$.

Several useful properties of ultrafilters are collected in the following lemma.

Lemma 1 *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Then the following hold:*

1. *if $A \cup B \in \mathcal{U}$ then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$*
2. *if $A \cup B \in \mathcal{U}$ and $A \cap B = \phi$ then exactly one of $A \in \mathcal{U}$ or $B \in \mathcal{U}$ holds*
3. *if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$ and A is finite, then $A \notin \mathcal{U}$*

One can show that non-principal ultrafilters on $\mathbb{N}$ exist, but the proof involves the axiom of choice in the form of Zorn's Lemma. So, an explicit example of a non-principal ultrafilter cannot be given. Nonetheless, almost all ultrafilters on $\mathbb{N}$ are non-principal.

A finite additive measure on $\mathbb{N}$ is a function $\mu : P(\mathbb{N}) \to [0,1]$ such that $\mu(\phi) = 0$, $\mu(\mathbb{N}) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for every two disjoint subsets A and B.

A pure finitely additive measure is a finitely additive measure μ such that $\mu(A) = 0$ for every finite set A.

Finally, a finitely additive measure μ is zero-one if $\mu(A) \in \{0,1\}$ for all $A \in \mathcal{P}$.

The set of zero-one finitely additive measures on $\mathbb{N}$ is equivalent to the set of ultrafilters on $\mathbb{N}$. For an ultrafilter $\mathcal{U}$, the corresponding zero-one finitely additive measure is given by:

$$\mu(A) = \begin{cases} 1 \text{ if } A \in \mathcal{U} \\ 0 \text{ if } A \notin \mathcal{U} \end{cases}$$

Moreover, the set of purely finitely additive zero-one measures is equivalent to the set of non-principal ultrafilters on $\mathbb{N}$.

2.2.1 Convergence up to an ultrafilter

Ultrafilters can also be used to generalize the notion of convergence of real sequence. Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded sequence $\{a_n\}$, we write $a = \lim_{\mathcal{U}} a_n$ if $\{j : |a_j - a| < \varepsilon\} \in \mathcal{U}$ for every positive ε.

Lemma 2 *Fix a non-principal ultrafilter $\mathcal{U}$. Then for any bounded sequence $\{a_n\}$ a unique a exists such that $a = \lim_{\mathcal{U}} a_n$.*

Proof Part 1 (Existence) Since $\{a_n\}$ is bounded then there is $M > 0$ such that $a_n < M$ for all n. Construct the intervals $I_n = [b_n, c_n]$ recursively so that $\{i : a_i \in [b_n, c_n]\} \in \mathcal{U}$ as follows: $I_0 = [-M, M] = [b_0, c_0]$ we have either $\{i : a_i \in [-M, 0]\} \in \mathcal{U}$ or $\{i : a_i \in [0, M]\} \in \mathcal{U}$. If $\{i : a_i \in [-M, 0]\} \in \mathcal{U}$ then set $I_1 = [-M, 0]$ and if $\{i : a_i \in [0, M]\} \in \mathcal{U}$ then set $I_1 = [0, M]$.

Suppose we constructed $I_n = [b_n, c_n]$. Let $d_n = (b_n + c_n)/2$. If $\{i : a_i \in [b_n, d_n]\} \in U$ the set $I_{n+1} = [b_n, d_n] = [b_{n+1}, c_{n+1}]$. If $\{i : a_i \in [d_n, c_n]\} \in U$ the set $I_{n+1} = [d_n, c_n] = [b_{n+1}, c_{n+1}]$. By the properties of ultrafilters we should have exactly one of the above cases.

These intervals are constructed so that $\{b_n\}$ and $\{c_n\}$ are Cauchy sequences. So one can show easily that there is a such that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = a$. We claim that $\lim_{\mathcal{U}} a_n = a$. Take any positive ε, then there is N such that $c_N - b_N < \varepsilon$. But if $a_i \in [b_N, c_N]$ then $|a_j - a| < \varepsilon$ and so $\{i : |a_i - a| < \varepsilon\} \in \mathcal{U}$.

Part 2 (Uniqueness) Suppose there is two distinct limits a and b. Let $\varepsilon = |a-b|/3$ then $A = \{i : |a_i - a| < \varepsilon\} \in \mathcal{U}$ and $B = \{i : |a_i - b| < \varepsilon\} \in \mathcal{U}$. Note that by our choice of ε we get $A \cap B = \varnothing$ and they are both in the ultrafilter which is a contradiction.

Note that if we fix a principal ultrafilter $\mathcal{U}$ generated by $\{k\}$, then for any sequence $\{a_n\}$ a unique limit exists such that $a_k = lim_{\mathcal{U}} a_n$.

Let A(n) be the number of elements of A that are less than are equal to n. Define the asymptotic density of A by

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n},$$

if this limit exists. You can see that the asymptotic density of any finite set is 0 and the asymptotic density of the set of even numbers is 1/2. The set $\{1, 2, 4, 8, 16, 32, ..., 2^k, ...\}$ is an example of a set in $\mathcal{I}$ with density 0.

Define the lower asymptotic density of A by :

$$\underline{d}(A) = \lim_{n\to\infty} \inf \frac{A(n)}{n}$$

and upper the asymptotic density if A by :

$$\overline{d}(A) = \lim_{n\to\infty} \sup \frac{A(n)}{n}.$$

One can check that $d(A)$ exists if and only if $\underline{d}(A) = d(A) = \overline{d}(A)$.

Asymptotic density does not exist for all subsets of $\mathbb{N}$. This mean that asymptotic density is not a measure on $\mathbb{N}$. In fact, if A and B are non-disjoint subsets of $\mathbb{N}$ and $d(A)$ and $d(B)$ exist, then $A \cup B$ may not have asymptotic density. Let $\mathcal{D} \subset \mathcal{P}$ be the collection of sets that have asymptotic density.

Lemma 3 *Let $A, B \in \mathcal{P}$. Then the following holds:*

1. *if $d(A)$ exists, the $d(A^C)$ exists and is equal to (1- $d(A)$), and*

2. *if A and B are disjoint and $d(A)$ and $d(B)$ exists, then $A \cup B$ has asymptotic density $d(A) + d(B)$.*

Since we need to get a measure for all the sets on $\mathbb{N}$ then it is natural to consider extensions of asymptotic density of all subsets of $\mathbb{N}$. A *density measure* is a finitely additive measure on $\mathbb{N}$ that extends density. That is, μ is a density measure if it is a finitely additive measure on $\mathbb{N}$ such that $\mu(A) = d(A)$ whenever $d(A)$ exists. It is well known ([14], [25]) that a density measure on $\mathbb{N}$ exists. Indeed, there are many such measures, as the extension is not unique. A widely used method of extension ([1], [11], [25], [12]) is to use the generalized limit operator $\lim_{\mathcal{U}}$, for some free ultrafilter $\mathcal{U}$, and define :

$$\mu(A) = \lim_{\mathcal{U}} \frac{A(n)}{n}.$$

Let $\mathcal{M}$ denote the set of all such density measures. It is easy to see that $\underline{d}(A) \leq \mu(A) \leq \overline{d}(A)$ for any $\mu \in \mathcal{M}$.

Chapter 3

Graded Modal Logic

3.1 Graded Modal Logic

In this section, I provide a brief overview of graded modal logic. Graded modal logic was first introduced in [9, 10]. It was then studied in [7, 18, 5, 6, 24] in which issues of axiomatization, completeness, decidability and translations into predicate logic are discussed. I briefly discuss the language of graded modal logic and state some of the main results found in the literature. All results and proofs can be found in [7] and [5].

Definition 6 *Given a countable set of atomic propositions* $\mathbb{P} = \{p_0, p_1, \ldots\}$*, a formula* α *of* ***GML*** *can have the following syntactic form:*

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha$$

where $p \in \mathbb{P}$ *and* $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, I define $\Box_n \alpha := \neg \Diamond_n \neg \alpha$, and $\Diamond !_n \alpha := \Diamond_{n-1} \alpha \wedge \neg \Diamond_n \alpha$ $(n \neq 0)$ where $\Diamond !_0 \alpha := \neg \Diamond_0 \alpha$. So $\Diamond !_n \alpha$ will have the intended meaning that α is true in *exactly* n accessible worlds. Let $\mathcal{L}_{\mathbf{GML}}$ be the set of all well-formed formulas of **GML**.

The following axiomatization was presented in [5].

G0 All tautologies in the language of **GML**

G1 $\Diamond_{n+1} \alpha \rightarrow \Diamond_n \alpha \quad (n \in \mathbb{N})$

G2 $\Box_0 (\alpha \rightarrow \beta) \rightarrow (\Diamond_n \alpha \rightarrow \Diamond_n \beta) \quad (n \in \mathbb{N})$

G3 $\Diamond !_0 (\alpha \wedge \beta) \rightarrow ((\Diamond !_{n_1} \alpha \wedge \Diamond !_{n_2} \beta) \rightarrow \Diamond !_{n_1+n_2} (\alpha \vee \beta)) \quad (n_1, n_2 \in \mathbb{N})$

GML is closed under modus ponens (MP) and necessitation (N), i.e., from $\vdash \alpha$ infer $\vdash \Box \alpha$. We write $\vdash_{\mathbf{GML}} \alpha$ if α can be deduced from $G0 - G1$ using the rules MP and N.

GML formulas are interpreted over Kripke structures. Let $\mathcal{M} = \langle S, R, V \rangle$ be a Kripke model, where S is a set of worlds, R is a binary relation over S and $V : \mathbb{P} \rightarrow 2^S$ is a valuation function. The boolean connectives and propositional variables are evaluated as usual. We will only show how the formula $\Diamond_n \alpha$ is evaluated at a world $s \in S$:

$$\mathcal{M}, s \models \Diamond_n \alpha \text{ iff } |\{t : sRt \text{ and } \mathcal{M}, t \models \alpha\}| > n$$

We say α is valid in $\mathcal{M}$ iff $\forall s \in S$, $\mathcal{M}, s \models \alpha$, and write $\mathcal{M} \models \alpha$. We write $\models \alpha$ if α is valid in all models (based on some class of frames[1]) We also make use of the following notation throughout this paper: $R(s) = \{t \mid sRt\}$ and for any formula α (of **MJL** or **GML**), $R_\alpha(s) = \{t \mid sRt \text{ and } t \models \alpha\}$. So, the above definition can be rewritten as

$$\mathcal{M}, s \models \Diamond_n \alpha \text{ iff } |R_\alpha(s)| > n$$

GML is shown in [7] to be sound and complete with respect to the class of all frames. Let $\mathfrak{F}$ be the class of all frames. It is easily verified that the axioms $G0 - G4$ are valid in any model based on $\mathfrak{F}$ and MP and N preserve validity. We state the completeness theorem below, but postpone discussion until section 6.3.

Theorem 4 (Soundness and Completeness of GML [7]) *For any formula α of **GML**, $\models \alpha$ iff $\vdash_{\textbf{GML}} \alpha$.*

In [5] **GML** is shown to be decidable by showing that **GML** has the finite model property. Maarten de Rijke [6] arrives at the same conclusion using an extended notion of bisimulation appropriate for a modal language with graded modalities. de Rijke also establishes invariance and definability results. Finally in [24], Tobies shows that the decidability problem for **GML** is in $PSPACE$.

[1] Unless otherwise stated we will assume that we are working with models based on the class of all frames. Refer to [2] for more information on frames.

Chapter 4

May's Theorem with infinite population

4.1 May's Theorem with infinite population

This chapter is based on a paper by Mark Fey "May's theorem with an Infinite Population". It describes three different kinds of anonymity. Fey argues that bounded anonymity is the best of them. In this chapter I argue that finite anonymity is the best of them.

4.2 Anonymity

We have three kinds of anonymity to be considered. We argue which one of these anonymities would be the best. We also will see that the majority spaces we will define in chapter 5 have the property of finite anonymity.

4.2.1 Strong Anonymity

We begin with the strong anonymity condition, which requires an aggregation rule to be invariant under any permutation of $\mathbb{N}$. The class of rules that satisfies this condition is the following:

Definition 7 *For $n \in \mathbb{N}* \cup \infty$ an aggregation rule f is an **n-rule** if*

$$f(V) = \begin{cases} 0 & \text{if } |V| < n \\ 1/2 & \text{if } |V| \geq n \text{ and } |V^C| \geq n \\ 1 & \text{if } |V^C| < n \end{cases}$$

for all $V \in \mathcal{P}$

So, an n-rule chooses 0 if fewer than n voters agree with the choice or 1 if fewer than n voters disagree with the choice, and otherwise chooses 1/2. Thus any preference profile in which both alternatives receives an infinite number of votes results in a tied outcome. We included in the above definition the possibility of 0-rule, which assigns value 1/2 to every subset of $\mathbb{N}$. An ∞-rule assigns value 0 to every finite set, 1 to every cofinite set and 1/2 to all other sets. The next theorem is due to Mark Fey:

Theorem 5 *An aggregation rule f satisfies neutrality, monotonicity and strong anonymity if and only if f is an n-rule for some $n \in \mathbb{N}* \cup \{\infty\}$.*

Proof The "if" statement is clearly true. To prove the converse, suppose that f satisfies neutrality, monotonicity and strong anonymity. Begin by considering a set $V \in \mathcal{I}$. Then $V^C \in \mathcal{I}$ and we can consider the permutation π_{VV^C} connecting V and V^C. By construction, $\pi_{VV^C}(V) = V^C$. So by strong anonymity, $f(V^C) = f(V)$. But neutrality requires that $f(V^C) = 1 - f(V)$, so it follows that f(V) = 1/2.

It remains to deal with the finite and cofinite subsets of $\mathbb{N}$. As each finite set is the complement of a cofinite set and vice versa, by neutrality it is sufficient to consider only the collection of finite subsets of $\mathbb{N}$. Let V be a finite set. It is easy to find an infinite superset of V with infinite complement. By monotonicity, $f(V) \leq 1/2$ for all finite sets.

If $f(V) = 1/2$ for all finite sets $V \in \mathcal{P}$, then f is a 0-rule. If f(V) = 0 for all finite sets $V \in \mathcal{P}$, then f is an ∞-rule. So assume that for some finite $W \in \mathcal{P}, f(W) = 1/2$ and for some finite $W' \in \mathcal{P}$. $f(W') = 0$. Let $n = \min\{|V| : f(V) = 1/2\}$. By strong anonymity, if $V \in \mathcal{P}$ is such that $|V| = n$, then $f(V) = 1/2$. If $Y \in \mathcal{P}$ is such that $Y > n$, then there is a subset of Y of size n. By monotonicity, $f(Y) = 1/2$. Thus f is an n-rule.

Corollary 6 *There is no aggregation rule that satisfies neutrality, positive responsiveness and strong anonymity.*

This follows immediately from the fact that there are two sets $A, B \in \mathcal{I}$ such that $A \subsetneq B$. But the conditions of the theorem above requires $f(A) = f(B) = 1/2$ which is a contradiction to the positive responsiveness condition.

4.2.2 Finite Anonymity

In this part we consider a weakening of the anonymity axiom to permit only permutations that change only a finite number of voters.

We first show that a purely finitely additive zero-one measure satisfies finite anonymity and neutrality.

Theorem 7 *Suppose an aggregation rule f is a purely finitely additive zero-one measure. Then f satisfies neutrality, monotonicity and finite anonymity.*

Proof As f is a purely finitely additive zero-one measure, there exists a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that:

$$f(V) = \begin{cases} 1 & if\ V \in \mathcal{U} \\ 0 & if\ V \notin \mathcal{U} \end{cases}$$

By the definition of ultrafilters, f satisfies neutrality and monotonicity. We need to prove that f satisfies finite anonymity. Take $A \in \mathcal{P}$ and a finite permutation π and let $B = \pi A$. As π is finite, the set A and B differ by only a finite number of elements. Suppose $f(A) = 1$, so $A \in \mathcal{U}$. Let $C = A \cup B$ and $D = C - B$. As $A \subseteq C, C \in \mathcal{U}$. Because D is finite then $D \notin \mathcal{U}$. By construction, $D \cup B = C \in \mathcal{U}$ which gives us that $B \in \mathcal{U}$ and thus $f(\pi A) = 1$. A similar argument applies to the case when $f(A) = 0$.

Corollary 8 *There exists an aggregation rule f that satisfies neutrality, monotonicity and finite anonymity such that f(E) = 1 and f(O) = 0. (where E is the set of even numbers and O is the set of odd numbers).*

Proof Note that there exists a function that is purely finitely additive zero one measure with f(E) = 1 and F(O) = 0. By the previous theorem we get the corollary. theorem this

One can stipulate the even numbers to be a majority and the odd numbers not to be a majority.

Corollary 9 *For every $\epsilon > 0$, there exists an aggregation rule f that satisfies neutrality, monotonicity and finite anonymity such that $f(L) = 1$ for a set $L \in \mathcal{P}$ with $d(L) < \epsilon$.*

Proof Let $\mathcal{U}$ be an arbitrary free ultrafilter on $\mathbb{N}$ and pick $m \in \mathbb{N}$ such that $1/m < \epsilon$. Define

$$F_{i,m} = \{i + mj : j \in \mathbb{N}*\}.$$

Note that $d(F_{i,m}) = 1/m$ for $i = 1, ..., m$. Since $\bigcup_{i=1}^{m} F_{i,m} = \mathbb{N}$ then $F_{i,m} \in \mathcal{U}$ for some i. By setting $L = F_{i,m}$ and letting f be the purely finitely additive measure determined by $\mathcal{U}$, the result is established.

The importance of this corollary is that we can set L "arbitrarily small" in the sense of asymptotic density and yet the votes of the members of L still determine the outcome.

4.2.3 Bounded Anonymity

Definition 8 *An aggregation rule f is*

- *an open density measure q-rule (open density q-rule) for* $q \in [1/2, 1]$ *if there is a density measure* μ *such that, for all* $V \in \mathcal{P}(V \in \mathcal{D})$,

$$f(V) = \begin{cases} 0 & if\ \mu(V) < 1-q \\ 1/2 & if\ \mu(V) \in [1-q, q] \\ 1 & if\ \mu(V) > q, \end{cases}$$

- *a closed density measure q-rule (closed density q-rule) for* $q \in (1/2, 1]$ *if there is a density measure* $\mu \in \mathcal{M}(\mu = d)$ *such that for all* $V \in \mathcal{P}(V \in \mathcal{D})$,

$$f(V) = \begin{cases} 0 & if\ \mu(V) \leq 1-q \\ 1/2 & if\ \mu(V) \in (1-q, q) \\ 1 & if\ \mu(V) \geq q, \end{cases}$$

- *a density measure q-rule if it is either an open density measure q-rule or a closed density measure q-rule.*

- *a density q-rule if it is either an open density q-rule or closed density q-rule.*

Our interest in density measure q-rule is justified by the following theorem.

Theorem 10 *Each n-rule and each density measure q-rule satisfy neutrality, monotonicity and bounded anonymity.*

Proof Clearly, an n-rule satisfies the three axioms. A density, measure q-rule obviously satisfies neutrality and monotonicity. To show that it also satisfies bounded anonymity it suffices to show that if $\mu \in \mathcal{M}$ is a density measure and $\pi \in \mathcal{G}$, then $\mu(\pi V) = \mu(V)$ for all $V \in \mathcal{P}$. From Blülinger ([3]), $\pi \in \mathcal{G}$ implies that $\lim_{n\to\infty}(V(n) - \pi V(n))/n = 0$, for all $V \in \mathcal{P}$. Thus for every free ultrafilter $\mathcal{U}$, $\lim_{\mathcal{U}}(V(n) - \pi V(n))/n = 0$. So we get $\mu(\pi V) = \mu(V)$ for all $V \in \mathcal{P}$.

We present the following three lemmas, without proof, before addressing the converse question. For the proof refer to [8], [17]

Lemma 11 *For each* $B \in \mathcal{I}$ *there exists* $A \in \mathcal{I}$ *such that* $A \subseteq B$ *and* $d(A) = 0$*. In addition, for each* $B \in \mathcal{I} \cap \mathcal{D}$ *and for each* $\alpha \in [0, d(B)]$*, there exists* $A \in \mathcal{I}$ *such that* $A \subseteq B$ *and* $d(A) = \alpha$*.*

The next lemma states that, under bounded anonymity, sets with equal density must be assigned the same outcome.

Lemma 12 *Let* $A, B \in \mathcal{I} \cap \mathcal{D}$ *with* $d(A) = d(B)$*. If* f *satisfies bounded anonymity, then* $f(A) = f(B)$*.*

The above result is sharpened for the case of sets with asymptotic density 1/2 by the next lemma.

Lemma 13 *Let* $A \in \mathcal{I} \cap \mathcal{D}$ *with* $d(A) = 1/2$*. If* f *satisfies bounded anonymity and neutrality, then* $f(A) = 1/2$*.*

Chapter 5

Majority Spaces

5.1 Majority Spaces

A very interesting situation arises when a Kripke model is not finitely generated, that is when $R(s)$ may be infinite for some state $s \in S$. While the semantics of majority is very clear in the finite case, it is not clear what should constitute a majority when there are an infinite number of possibilities. We cannot for example stipulate that every infinite set is a (strict) majority. This would create the unsatisfactory situation where a set and its complement could both be majority sets.

Another natural choice would be to call a set $X \subseteq R(s)$ a majority if X^C is finite, i.e take the majority sets to be the co-finite sets. However, suppose that $R(s) = X_1 \cup X_2 \cup X_3$, where X_1, X_2, and X_3 are nonempty pairwise disjoint sets. Then one would expect that for some i and j where $i \neq j$,

$X_i \cup X_j$ would be a majority. This is certainly true in the finite case, and so one would expect it to be true in the infinite case. However, it is easy to come up with an example where all of the X_i are infinite; and so, none of the $X_i \cup X_j$ would be a majority.

Instead of trying to define a majority set as some special subset of $R(s)$, we will let a model stipulate which sets are to be considered a majority. In other words, at each state in the model, we attach a collection of subsets of $R(s)$ which will be called the "majority" sets. Thus a set X in this collection will be considered a (weak) majority of $R(s)$ at state s. Obviously, we do not want to allow *any* collection of subsets, but only those collections satisfying certain properties capturing our intuitions about majority.

Definition 9 *Let W be any set. We will call any set $\mathfrak{M} \subseteq 2^W$ a* **(weak) majority system** *if it satisfies the following properties.*

M1. *If $X \subseteq W$, then either $X \in \mathfrak{M}$ or $X^C \in \mathfrak{M}$.*

M2. *If $X \in \mathfrak{M}$, $Y \in \mathfrak{M}$ and $X \cap Y = \varnothing$, then $Y = X^C$.*

M3. *Suppose that $X \in \mathfrak{M}$ and $F \subseteq X$ is any finite set. If G is any set where $G \cap X = \varnothing$ and $|F| \leq |G|$, then $(X - F) \cup G \in \mathfrak{M}$.*

The pair $\langle W, \mathfrak{M} \rangle$ will be called a **weak majority space**. Given a set W, a set $X \subseteq W$ will be called a **strict majority** (with respect to $\mathfrak{M}$) if $X \in \mathfrak{M}$ and $X^C \notin \mathfrak{M}$. X will be called a **half majority** if $X \in \mathfrak{M}$ and $X^C \in \mathfrak{M}$. We need to check that the above properties correspond to our intuitions about majority sets. We will call any set $X \in \mathfrak{M}$ a **majority set**.

It is easy to see using M3 that majority spaces are closed under superset by simply taking $F = \varnothing$. We show that many of the intuitions we have about majority sets on a finite space remain true in a majority space. For example, we show that given any majority set X, if we add something new to X, then this new formed set will be a strict majority. We also show that if the set W is infinite, then all majority sets must also be infinite.

First, we will show that when W is finite, the sets that can be called a majority (i.e. satisfy properties M1-M3) are the sets that have more than or equal to half of the elements.

Proposition 14 *Suppose that* W *is a finite set and that* $\mathfrak{M}' = \{M \subseteq W \ : \ |M| \geq |W|/2\}$, *Then*

$$\langle W, \mathfrak{M}' \rangle \text{ is a majority space}$$

Furthermore, if $\langle W, \mathfrak{M} \rangle$ *is any other majority space then* $\mathfrak{M} = \mathfrak{M}'$.

Proof Suppose that W is a finite set and $\mathfrak{M}'$ is as defined above. We must first show that $\langle W, \mathfrak{M}' \rangle$ is a majority space. For any set, $X \subseteq W$, since $|X| + |X^C| = |W|$, either $|X| \geq |W|/2$ or $|X^C| \geq |W|/2$ and so either $X \in \mathfrak{M}'$ or $X^C \in \mathfrak{M}'$. Hence property M1 is satisfied. For property M2, suppose that $X, Y \in \mathfrak{M}'$, and $X \cap Y = \varnothing$. Since $|X| \geq |W|/2$ and $|Y| \geq |W|/2$, $|X| + |Y| \geq |W|$. But since $X \cup Y \subseteq W$, $|X \cup Y| \leq |W|$ and so $|X \cup Y| = |W|$.

Therefore, $X \cup Y = W$ (this follows since X and Y are assumed to be subsets of W). Since X and Y are disjoint and $X \cup Y = W$, then $Y = X^C$. Finally we need to show that property M3 is satisfied. Suppose that $X \in \mathfrak{M}'$. Then $|X| \geq |W|/2$. Suppose that $F \subseteq X$ and G is any finite set such that $|F| \leq |G|$ and $G \cap X = \varnothing$. Then

$$\begin{aligned}
|(X - F) \cup G| &= |(X - F)| + |G| - |(X - F) \cap G| \\
&= |X - F| + |G| \\
&\geq |X - F| + |F| \\
&= |X \cup F| = |X| \geq |W|/2
\end{aligned}$$

so, $(X - F) \cup G \in \mathfrak{M}'$.

Let $\langle W, \mathfrak{M} \rangle$ be any finite majority space and let $X \in \mathfrak{M}$. We must now show that $|X| \geq |W|/2$. Suppose not, that is suppose that $|X| < |W|/2$. Therefore $|X^C| > |X|$. Let $Y \subseteq X^C$ and $|Y| = |X|$ (such a set must exist since $|X^C| > |X|$). Then by property M3, since $|X| \leq |Y|$ and $Y \cap X = \varnothing$, $(X - X) \cup Y = Y \in \mathfrak{M}$. But by property M2, $Y = X^C$. But this is a contradiction, since $|X| < |W|/2$ and $|Y| < |W|/2$. Hence, $|X| \geq |W|/2$.

This last proposition demonstrates that our notion of an infinite majority is similar to the natural notion of a majority when we only have a finite number of elements. In other words, we showed that if W is a finite set, then the majority sets are the sets that have at least half of the elements.

Lemma 15 *If X is a weak majority and $F \neq \varnothing$ is a set such that $F \nsubseteq X$, then $X \cup F$ is a strict majority.*

Proof Suppose that X is a weak majority and $F \neq \varnothing$ is any set such that $F \nsubseteq X$. Notice first that since $X \in \mathfrak{M}$ and $X \subseteq X \cup F$, $X \cup F \in \mathfrak{M}$ by M3 (this is true for any set F). We need only show that $(X \cup F)^C \notin \mathfrak{M}$. Suppose that $(X \cup F)^C \in \mathfrak{M}$. By property M2, since $X \in \mathfrak{M}$, $(X \cup F)^C \in \mathfrak{M}$ and $X \cap (X \cup F)^C = \varnothing$, we must have $(X \cup F)^C = X^C$ which implies $F \subseteq X$. But this contradicts the assumption that $F \nsubseteq X$.

Lemma 16 *Suppose that $\langle W, \mathfrak{M} \rangle$ is a majority space and that W is infinite. If $X \in \mathfrak{M}$ then X is infinite.*

Proof Suppose that $\langle W, \mathfrak{M} \rangle$ is a majority space and W is infinite. Suppose that $X \subseteq W$ is finite and $X \in \mathfrak{M}$. Note that since X is finite, X^C is infinite. Take any finite set $G \subset X^C$, where $|X| \leq |G|$ (such a set must exist since W is infinite). Then by property M3, $(X - X) \cup G = G \in \mathfrak{M}$; and so, by property M2, $G = X^C$. But this is a contradiction since G is finite and X^C is infinite.

Lemma 17 *If W is infinite then the cofinite sets will be strict majority i.e. if X is a cofinite set then $X \in \mathfrak{M}$ and $X^C \notin \mathfrak{M}$.*

Proof Let X be a cofinite set. We need to prove that $X \in \mathfrak{M}$ and $X^C \notin \mathfrak{M}$.

1. Suppose $X \notin \mathfrak{M}$ then according to M1 $X^C \in \mathfrak{M}$. We have $X = (X^C - X^C) \cup X$ so according to M3 $X \in \mathfrak{M}$ which is a contradiction.

2. Suppose $X^C \in \mathfrak{M}$. Let $a \in X$ and $a \notin X^C$ and define $Y = X - \{a\}$. Note that $Y = (X^C - X^C) \cup Y$ and by M3 we get $Y \in \mathfrak{M}$. But $X^C \cap Y = \varnothing$ and yet $Y \neq (X^C)^C$ which is a contradiction with M2.

Finally, two points about majority spaces are worth discussing. The first is that the cofinite sets are strict majorities in any infinite majority space. Let $(W, \mathfrak{M})$ be a majority space (with W infinite) and $X \subseteq W$ a cofinite set. By property M1, to show that X is a strict majority, we need only show that $X^C \notin \mathfrak{M}$. But this follows directly from Lemma 16. The second point is about the connection between majority spaces and ultrafilters. This is the subject of the next section.

5.2 Majority spaces and ultrafilters

Majority spaces are closely related to ultrafilters. In fact in many papers on social choice theory on infinite populations, ultrafilters are used to capture the concept of "largeness", see [8, 23] for some examples. In this section, we study the connections between majority spaces and ultrafilters.

We first review some well-known definitions. Let W be a set. A non-empty collection $\mathcal{U} \subseteq 2^W$ is called a *filter* if $\mathcal{U}$ is closed under intersection and superset and does not contain the empty set. A filter is an *ultrafilter* if for all sets X either $X \in \mathcal{U}$ or $X^C \in \mathcal{U}$. Finally, $\mathcal{U}$ is *principal* if $\mathcal{U}$ contains a singleton, and $\mathcal{U}$ is non-principal if it is not principal. It is easy to check that a non-principal ultrafilter contains all cofinite sets. Given any infinite set, Zorn's lemma implies the existence of a non-principal ultrafilter.

Fix (an infinite) set W. At first ultrafilters seem to be a good candidate for the collection of majority subsets of W. Given any subset X, certainly either X or X^C should be considered a majority set; and majority sets are certainly closed under superset. However, it is easy to imagine a situation in which there are two majority sets whose intersection is not a majority.

We first show that every non-principal ultrafilter is a majority system.

Theorem 18 *Let W be an infinite set and $\mathcal{U}$ a non-principal ultrafilter over W. Then $\langle W,\mathcal{U}\rangle$ is a majority space.*

Proof Let W be an infinite set and $\mathcal{U}$ a nonprincipal ultrafilter over W. We need only show that $\mathcal{U}$ satisfies $M1, M2$ and $M3$. Obviously, $M1$ is satisfied. $M2$ is trivially satisfied since there are no sets $X, Y \in \mathcal{U}$ such that $X \cap Y = \emptyset$. We need only show that $M3$ is satisfied.

Let $Y = (X - F) \cup G$ where $X \in \mathcal{U}$, F is finite subset of X, $|F| \leq |G|$ and $X \cap G = \emptyset$. Since $\mathcal{U}$ is a non-principal ultrafilter then either $Y \in \mathcal{U}$ or $Y^C \in \mathcal{U}$. If $Y \in \mathcal{U}$ then we are done. Assume $Y^C \in \mathcal{U}$ then $Y^C \cap X \in \mathcal{U}$ and so $F \in \mathcal{U}$ which is a contradiction since F is finite.

Thus every ultrafilter is a majority system. We show below that the converse is not true. This is achieved by constructing an example of a majority system that is not an ultrafilter:

5.2.1 Example of a majority space:

In this section we provide concrete example of a majority space. Furthermore, we show that this majority space is not an ultrafilter.

Let X_1, X_2, X_3 be three disjoint infinite sets. Let $\mathfrak{U}_i$ be a non-principal ultrafilter over X_i for each $i = 1, 2, 3$. Now let $W = X_1 \cup X_2 \cup X_3$. Define

$$\mathfrak{M} = \{X | \exists i \neq j \text{ such that } X \cap X_i \in \mathfrak{U}_i \text{ and } X \cap X_j \in \mathfrak{U}_j\}$$

We claim that $(W, \mathfrak{M})$ is a majority space. We need only show that $\mathfrak{M}$ satisfies $M1 - M3$. First of all, notice that since the X_i are disjoint and each $\mathfrak{U}_i$ is an ultrafilter, for any set $Y \subseteq W$, either $Y \cap X_i \in \mathfrak{U}_i$ or $Y^C \cap X_i \in \mathfrak{U}_i$. Based on this observation, $M1$ follows easily:

Suppose that $X \subseteq W$. Assume $X \notin \mathfrak{M}$ then without loss of generality we can assume that $X \cap X_1 \notin \mathfrak{U}_1$ and $X \cap X_2 \notin \mathfrak{U}_2$. By the above observation, $X^C \cap X_1 \in \mathfrak{U}_1$ and $X^C \cap X_2 \in \mathfrak{U}_2$. So $X^C \in \mathfrak{M}$.

$M2$ is trivially true since the antecedent is always false. To see this, suppose that $X \in \mathfrak{M}, Y \in \mathfrak{M}$ and $X \cap Y = \emptyset$ Since the number of disjoint sets under consideration is odd, an easy application of the pigeon hole principle shows that there is an i such that $X \cap X_i \in \mathfrak{U}_i$ and $Y \cap X_i \in \mathfrak{U}_i$. So $(X \cap Y) \cap X_i \in \mathfrak{U}_i$ and thus $\emptyset \in \mathfrak{U}_i$ which is a contradiction.

Finally, $M3$ follows from the definition of ultrafilters and the following simple fact:

Fact Suppose $\mathfrak{U}$ is a nonprincipal ultrafilter and X any set. If $X \in \mathfrak{U}$ then for any finite set $F \subseteq X$, $X - F \in \mathfrak{U}$. Otherwise, $(X - F)^c \in \mathfrak{U}$ which implies $X^c \cup F \in \mathfrak{U}$. Therefore, $F = X \cap (X^c \cup F) \in \mathfrak{U}$, which is a contradiction since $\mathfrak{U}$ is a nonprincipal ultrafilter and so does not contain any finite sets.

Suppose that $X \in \mathfrak{M}$ and Let $Y = (X - F) \cup G$ where F is a finite subset of X, $X \cap G = \emptyset$ and $|F| \leq |G|$. Without loss of generality, we may assume that $X \cap X_1 \in \mathfrak{U}_1$ and $X \cap X_2 \in \mathfrak{U}_2$. Then, for each $i = 1, 2$, since F is a finite subset of X, $F \cap X_i$ is a finite subset of $X \cap X_i$. Therefore using the above fact, $X \cap X_i - (F \cap X_i) = (X - F) \cap X_i \in \mathfrak{U}_i$. Finally, since ultrafilters are closed under superset, for each $i = 1, 2$, $Y \cap X_i = ((X - F) \cup G) \cap X_i \in \mathfrak{U}_i$. Hence $Y \in \mathfrak{M}$.

Hence $(X, \mathfrak{M})$ is a majority space. Notice that $X_1 \cup X_2 \in \mathfrak{M}$ and $X_2 \cup X_3 \in \mathfrak{M}$ but their intersection $X_2 \notin \mathfrak{M}$. So $\mathfrak{M}$ is not an ultrafilter over X. It should be clear that this example can be generalized to any odd number of disjoint infinite sets.

Thus we have shown that every ultrafilter is a majority system, but not every majority system is an ultrafilter. In fact, if we add the following axiom to the axioms of a majority system then the resulting set of axioms become equivalent to an ultrafilter axioms.

M4 If $X, Y \in \mathfrak{M}$ then $X \cap Y \in \mathfrak{M}$.

Suppose that W is an infinite set, and $\mathfrak{M} \subseteq 2^W$ satisfies M1-M4. It is straightforward to check that $\mathfrak{M}$ is an ultrafilter. Notice that in the presence of M4, M2 is trivial. However, M2 and the fact that W has more than two elements is needed in order to show that $\emptyset \notin \mathfrak{M}$.

Another example of a majority space

In this section we will give another example of a majority space. We will construct this majority space over $\mathbb{N}$.

Consider subsets of $\mathbb{N}$: $O_1, O_2, ...$ where each O_i is a finite subset of $\mathbb{N}$ of odd size and $\bigcup_{i \in \mathbb{N}} O_i = \mathbb{N}$. Fix a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$.

Define

$$\mathfrak{M} = \{X | \{i : |X \cap O_i| \geq \lceil |O_i|/2 \rceil\} \in \mathcal{U}\}$$

We claim that $(\mathbb{N}, \mathfrak{M})$ is majority space. Here is the proof:

M1 Take $X \in \mathbb{N}$ and assume $X \notin \mathfrak{M}$, then $\{i : |X \cap O_i| \geq \lceil |O_i|/2 \rceil\} \notin \mathcal{U}\}$. But note that since O_i has an odd size then $\{i : |X^C \cap O_i| \geq \lceil |O_i|/2 \rceil\} \in \mathcal{U}\}$.

M2 Given $X, Y \in \mathfrak{M}$ then $\{i : |X \cap O_i| \geq \lceil |O_i|/2 \rceil\} \cap \{i : |Y \cap O_i| \geq \lceil |O_i|/2 \rceil\} \in \mathcal{U}$. Again since O_i has an odd size we get $X \cap Y \neq \varnothing$.

M3 Let $Y = (X - F)$ where $X \in \mathfrak{M}$ and F is a finite subset of X. We will prove that $Y \in \mathfrak{M}$. Since $X \in \mathfrak{M}$ then $I_X = \{i : |X \cap O_i| \geq \lceil |O_i|/2 \rceil\} \in \mathcal{U}\}$. We need to prove that $I_Y = \{i : |Y \cap O_i| \geq \lceil |O_i|/2 \rceil\} \in \mathcal{U}\}$. But note that $I_Y = I_X - T$ where T is some finite set of $\mathbb{N}$. So $I_Y \in \mathcal{U}$.

5.2.2 Strict Majority

A strict majority space $\mathfrak{M}$ of type 1 has the same properties as a majority space plus this additional one:

S_1 If $X \in \mathfrak{M}$ then $X^C \notin \mathfrak{M}$ for all X.

the strict majority space acts more like an ultrafilter than most majority spaces.

A strict majority space $\mathfrak{M}$ of type 2 has the same properties as a majority space plus this additional one:

S_2 If $X_1 \cup X_2$ is cofinite and $X_1 \cap X_2 = \varnothing$ then $X_1 \in \mathfrak{M}$ or $X_2 \in \mathfrak{M}$ but not both (for all X_1, X_2).

Definition 10 *A strict majority space $\mathfrak{M}$ of type $i > 1$ has the same properties of a majority space plus this additional one:*

S_i *If $\bigcup_{1 \le j \le i} X_j$ is cofinite and X_j's are pairwise disjoint then $X_j \in \mathfrak{M}$ for exactly one j where $1 \le j \le i$.*

Lemma 19 *A strict majority space $\mathfrak{M}$ of type $i \ge 3$ is a non-principal ultrafilter*

Proof We only need to prove this property:

$$X \cup Y \in \mathfrak{M} \text{ then } X \in \mathfrak{M} \text{ or } Y \in \mathfrak{M}$$

Assume $X \cup Y \in \mathfrak{M}$. Now, $W = X \cup Y \cup (X \cup Y)^C$ is cofinite and the three sets are disjoint so one of these sets will be in majority space of type i. Since $X \cup Y \in \mathfrak{M}$ we get $(X \cup Y)^C \notin \mathfrak{M}$ and so either $X \in \mathfrak{M}$ or $Y \in \mathfrak{M}$.

One can check easily that the above two example about majority spaces are in fact strict majority sets of type 2. From now on we will call majority spaces of type 2 as *comfortable majority spaces*. In a way, the comfortable majority spaces are more interesting than any other majority spaces. One important reason why they are interesting is that they are closer in definition to ultrafilters.

Another important note about the above two example is the choice of a non-principal ultrafilter. Note that if we replace non-principal ultrafilters with comfortable majorities we will still get the same results.

5.2.3 Relaxed majority spaces

M2 is a very strong condition. But it is M2 that generalize the majority properties in the finite case. In this section, we will introduce a new definition where we relaxed the condition M2.

Definition 11 *Let W be an infinite set. We call $\mathfrak{M}$ over W a relaxed majority system if it satisfies these three properties:*

M1 For every X, either $X \in \mathfrak{M}$ or $X^C \in \mathfrak{M}$

M2' For all $X, Y \in W$ if $X \cup Y$ is cofinite then $X \in \mathfrak{M}$ or $Y \in \mathfrak{M}$.

M2” for all X_1, X_2, X_3 that are pairwise disjoint there is $1 \leq i \leq 3$ such that $X_i \notin \mathfrak{M}$

M3 Suppose that $X \in \mathfrak{M}$ and $F \subseteq X$ is any finite set. If G is any set where $G \cap X = \varnothing$ and $|F| \leq |G|$, then $(X - F) \cup G \in \mathfrak{M}$.

In the definition up we gave up M2 and instead we replaced it by a weaker condition. Note that in M2’ we could have both $X, Y \in \mathfrak{M}$. The “or” in M2’ is not exclusive.

We will need the this definition to formulate the next theorem.

Definition 12 *A finitely additive measure on $\mathbb{N}$ is a function $\mu : 2^{\mathbb{N}} \to [0,1]$ such that $\mu(0) = 0$, $\mu(\mathbb{N}) = 1$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ for every two disjoint subsets A and B of $\mathbb{N}$.*

A purely finitely additive measure on $\mathbb{N}$ is a finitely additive measure μ such that $\mu(A) = 0$ for every finite set A.

Theorem 20 *Let μ be a purely finitely additive measure on $\mathbb{N}$ and let $\mathfrak{M} = \{X \in \mathbb{N} : \mu(X) \geq 1/2\}$. Then $\mathfrak{M}$ is a relaxed majority system over $\mathbb{N}$.*

Proof

- M1) Let $X \subseteq \mathbb{N}$. We have $\mu(X \cup X^C) = \mu(\mathbb{N}) = 1$. So either $\mu(X) \geq 1/2$ or $\mu(X^C) \geq 1/2$. Thus, either $X \in \mathfrak{M}$ or $X^C \in \mathfrak{M}$

- M2’) Let X, Y be two subsets of $\mathbb{N}$ such that $X \cup Y$ is cofinite. Since $\mu(F) = 0$ for every finite set F then $\mu(X \cup Y)^C = 0$. So $\mu(X \cup Y) \cup (X \cup Y)^C = \mu(\mathbb{N}) = 1$ and so $\mu(X \cup Y) = 1$. It is easy to verify that either $\mu(X) \geq 1/2$ or $\mu(Y) \geq 1/2$.

- M3) Let $X \in \mathfrak{M}$ and let $Y = X - F$ where F is finite subset of X. But $\mu(X \cup F) = \mu(Y)$ and since F is finite we get $\mu(X) = \mu(Y)$. Thus $Y \in \mathfrak{M}$

5.2.4 Vast Majority

We will talk about the properties of vast majority in this section.

M4 If $X, Y \in \mathfrak{M}$ then $X \cap Y \in \mathfrak{M}$.

If we add the property M4 to our system that will give us what we call "vast majority" and in this case the majority space will be defined by a non-principal ultrafilter.

One question to be asked is "what is the difference between ultrafilter and majority spaces and can we reduce one to the other".

5.3 Classification of majority spaces

In this section we will prove a general way to construct majority spaces.

Definition 13 *Let X be any infinite set, then $cof(X) = \{Y \subseteq X \mid Y^C$ is finite $\}$. So, $cof(X)$ is the set of co-finite subsets of X.*

Definition 14 *Let Y be any set and $X \subseteq 2^Y$. Then define*

$$X^f = \{A \mid \exists B \in X \text{ such that } A = (B - F) \cup G \text{ where } F \text{ is finite }, |F| \leq |G| \text{ and } X \cap G = \varnothing\}$$

So, X^f is X **closed under finite perturbations**. It is easy to see that $X \subseteq X^f$ (take F and G both to be empty).

Definition 15 *Let Y be any set and $X \subseteq 2^Y$, then define*

$$\overline{X} = \{A \ : \ A \notin X \text{ and } A^C \in X\}$$

Now we will define what do we mean by pre-majority:

Definition 16 *Let W be any set. We will call $\mathfrak{M} \subseteq 2^W$ a pre-majority if it is not empty and it satisfies conditions M2 and M3:*

M2. If $X \in \mathfrak{M}$, $Y \in \mathfrak{M}$ and $X \cap Y = \varnothing$, then $Y = X^C$.

M3. Suppose that $X \in \mathfrak{M}$ and $F \subseteq X$ is any finite set. If G is any set where $G \cap X = \varnothing$ and $|F| \leq |G|$, then $(X - F) \cup G \in \mathfrak{M}$.

As an example if W is infinite then the $cof(W)$ will form a pre-majority and it will be the smallest one. We will assume W is infinite for the rest of the section.

Definition 17 *Let $\mathfrak{M}$ be a collection of subsets over W and X be a subset of W. We say that X has the non-empty intersection property with $\mathfrak{M}$ if there is no $Y \in \mathfrak{M}$ such that $X \cap Y = \varnothing$ and $X \cup Y \neq W$.*

Lemma 21 *Let $\mathfrak{M}$ be a pre-majority over W. Take $X \in 2^W$ then either X has the non-intersection property with $\mathfrak{M}$ or X^C has the non-intersection property with $\mathfrak{M}$*

Proof Assume that X and X^C don't have that property. Then there are $Y, Z \in \mathfrak{M}$ such that:

$$X \cap Y = \varnothing \text{ and } X \cup Y \neq W$$

$$X^C \cap Z = \varnothing \text{ and } X^C \cup Z \neq W$$

In this case we get $Y \cap Z = \varnothing$ and $Y \cup Z \neq W$ which is a contradiction since $\mathfrak{M}$ is a pre-majority.

Theorem 22 *Let $\mathfrak{M}$ be a pre-majority over W. Take $X \in 2^W$ then either $(\mathfrak{M} \cup X^f)$ or $(\mathfrak{M} \cup (X^C)^f)$ is a pre-majority.*

Proof If $X \in \mathfrak{M}$ already then it is clear by M3 that $X^f \subseteq \mathfrak{M}$. Assume without loss of generality that X has the non-intersection property with $\mathfrak{M}$. We will prove that $\mathfrak{M} \cup X^f \in \mathfrak{M}$. It is clearly satisfies M2 by the non-intersection property. It is also easy to show that this new set satisfies M3.

One can easily give an alternate proof of the existence of a majority spaces, by using the above lemma and Zorn's lemma. From here one can prove the existence of weak majority spaces. Let X be an infinite subset of W such that it is not cofinite and take $\mathfrak{M} = (X^f) \cup (X^C)^f$. This $\mathfrak{M}$ can be proven to be a pre-majority and so it can be extended to a weak majority space.

Lemma 23 *Let $\mathfrak{M}$ be a pre-majority space and let X be any set. Assume* *that $\mathfrak{M} \cup X^f$ is pre-majority and $\mathfrak{M} \cup (X^C)^f$ is also a pre-majority. Then $\mathfrak{M} \cup (X^f \cup (X^C)^f)$ is a pre-majority.*

Lemma 24 *Let $(\mathfrak{M}, W)$ and $(\mathfrak{M}', W')$ be two majority spaces where $W \cap W' = \varnothing$. Then $(\mathfrak{M} \sqcup \mathfrak{M}', W \cup W')$ is a pre-majority space where $\mathfrak{M} \sqcup \mathfrak{M}' = \{X : X = Y \cup Z \text{ and } Y \in \mathfrak{M}, Z \in \mathfrak{M}'\}$*

5.4 May's Theorem and majority spaces

5.4.1 Overview

Of course, the question still remains as to whether our definition of a majority subset of an infinite set is "correct". The results and discussion of the previous two sections are meant to demonstrate that we have correctly generalized the concept of a majority set to the infinite case. However, there is another direction we could go. May's celebrated theorem ([15, 16] completely characterizes simple majority rule for a finite set of individuals. In [8], Mark Fey generalizes this theorem to a countable set of individuals. This section discusses how majority spaces are related to Fey's framework and the result of May.

We first need some definitions. Fix an infinite set W. Suppose that there are two alternatives, x and y, under consideration. Elements of W are called voters. We assume that each voter has a linear preference over x and y, so for each $w \in W$, either w prefers x to y or y to x, but not both. Assume that a subset $X \subseteq W$, represents the set of all voters that prefer x to y. Thus X represents the outcome of a particular vote.

There are three possible outcomes to consider: 0 means that alternative y was chosen, $\frac{1}{2}$ means the vote was a tie, and 1 means that alternative x was chosen. An *aggregation function* is a function $f : 2^W \to \{0, \frac{1}{2}, 1\}$. Intuitively for a set $X \subseteq W$, $f(X)$ represents the social preference of the group W ($\frac{1}{2}$ is interpreted as a tie).

In [15, 16], May was concerned with which conditions on an aggregation function f force f to be equivalent to a simple majority decision.

5.4.2 Aggregation rule properties

The last condition that May considers is **anonymity**[1]. Anonymity essentially says that it is the *number* of votes that counts when determining the outcome, not *who* voted for what. When W is finite, this condition is straightforward to impose. Fix an arbitrary order on W, then each subset of W can be represented by a finite sequence of 1s and 0s. Then f satisfies anonymity if f is symmetric in this sequence of 1s and 0s. We will talk more about generalizing this condition below.

May showed that when W is finite, the conditions neutrality, positive responsiveness and anonymity[2] completely characterize the simple majority decision rule. Our goal in this section is to generalize this result to the infinite

[1]May calls this condition equality

[2]May has a fourth condition which essentially says that the aggregation function f is actually a function.

case using majority spaces. A formal comparison between Fey's framework and our framework would take us to far afield, and so will be reserved for a later paper.

Given any aggregation rule, define the collection $\mathfrak{M}_f$ of subsets of W as follows:

$$\mathfrak{M}_f = \{X \mid f(X) \geq \frac{1}{2}\}$$

One result that we are after is that for any aggregation function satisfying neutrality, positive responsiveness and (an appropriate form) of anonymity, $(W, \mathfrak{M}_f)$ is a majority space. Conversely, given an majority space $(W, \mathfrak{M})$, we can construct an aggregation function $f_{\mathfrak{M}}$ as follows for each $X \subseteq W$,

$$f_{\mathfrak{M}}(X) = \begin{cases} 1 & \text{if } X^C \notin \mathfrak{M} \\ \frac{1}{2} & \text{if } X, X^C \in \mathfrak{M} \\ 0 & \text{if } X \notin \mathfrak{M} \end{cases}$$

Our desired result is that if $(W, \mathfrak{M})$ is a majority space, then $f_{\mathfrak{M}}$ satisfies May's three conditions. Of course, the appropriate generalization of May's theorem depends on an appropriate generalization of the definition of neutrality. Indeed, this constitutes the major portion of Fey's paper. We first deal with neutrality and positive responsiveness.

Theorem 25

- *If W is infinite and $(W, \mathfrak{M})$ is a majority space, then $f_{\mathfrak{M}}$ satisfies neutrality and positive responsiveness.*

- *If f satisfies neutrality, then $\mathfrak{M}_f$ satisfies property $M1$.*
- *If f satisfies positive responsiveness and neutrality, then $\mathfrak{M}_f$ satisfies property $M2$.*

Proof

- Neutrality is a consequence of $M1$ and the definition of $f_{\mathfrak{M}}$. Positive responsiveness is a consequence of Lemma 15.
- Suppose $\mathfrak{M}_f$ does not satisfies M1, then there is X such that $X \notin \mathfrak{M}_f$ and $X^C \notin \mathfrak{M}_f$. So $f(X) = f(X^C) = 0$. By neutrality, $f(X^C) = 1 - f(X) = 1 - 0 = 1$. Contradiction.
- Suppose that $X \cap Y = \emptyset$ and $X, Y \in \mathfrak{M}_f$. We must show that $X = Y^C$. Suppose not. Then since $X \cap Y = \emptyset$, $X \subseteq Y^C$ and so $Y^C \nsubseteq X$. Hence, it must be the case that $X \subsetneq Y^C$. By positive responsiveness, since $f(X) \geq \frac{1}{2}$, $f(Y^C) = 1$. But this implies by neutrality that $f(Y) = 0$, contradicting the fact that $f(Y) \geq \frac{1}{2}$.

We now turn to the subtle issue of generalizing May's anonymity condition. This condition says that an aggregation rule f should use the *size* of a set, not its contents. The problem, as shown by Cantor, is that every infinite subset of a (countably) infinite set has the same "size". And so, the intuition behind anonymity seems to imply that f should assign the same value to *every* infinite set.

Following [8], for the rest of this section only we assume that W is a countably infinite set. Fey's approach is to look to the set of permutations over W, i.e., the automorphism group over W. Recall that π a permutation if it is a 1-1 function from W to W. Then for any set $X \subseteq W$ and any given permutation π, define $\pi[X] = \{\pi(v) | v \in X\}$. Then we say that f satisfies **anonymity** provided $f(X) = f(\pi[X])$ for each permutation π. Clearly this condition is much too strong, since for any two infinite subsets of W in $\mathcal{I}$ we can find a permutation π such that $\pi[X] = Y$; and so if f satisfies anonymity, then every infinite subset in $\mathcal{I}$ must be assigned the same value. Fey considers two possible ways of restricting the set of permutation: finite permuations and bounded permutations. For a complete discussion of bounded permuations, the reader is referred to [8] and the references therein. We only discuss finite permutations.

Definition 18 *A permutation $\pi : W \to W$ is* **finite** *provided that there is a finite set $F \subseteq W$ such that $\pi(w) = w$ for each $w \in W - F$.*

We say that an aggregation function satisfies **finite anonymity** provided $f(X) = f(\pi[X])$ for every finite permutation π. The last two observations will show that condition $M3$ corresponds to imposing finite anonymity, thus completing our generalization of May's theorem using majority spaces.

Theorem 26

- *If f satisfies finite anonymity and positive responsiveness, then $\mathfrak{M}_f$ satisfies $M3$.*
- *If $\mathfrak{M}$ satisfies $M3$, then $f_{\mathfrak{M}}$ satisfies finite anonymity.*

Proof

- Suppose that f satisfies finite anonymity, $X \in \mathfrak{M}_f$, F is a finite subset of X and G is any set such that $X \cap G = \emptyset$ and $|F| \leq |G|$.

We must show that $(X-F)\cup G \in \mathfrak{M}_f$. Let $G' \subseteq G$ be any subset of G with $|F| = |G'|$ (such a set exists since $|F| \leq |G|$). Let π be the following permutation on W, $\pi(w) = w$ for each $w \in W - (F \cup G')$, and on $F \cup G'$ arrange it so that $\pi[F] = G'$. It is clear, that π is a finite permutation. Since f satisfies finite anonymity and $X \in \mathfrak{M}_f$, $f(\pi[X]) = f(X) \geq \frac{1}{2}$. If $(X-F)\cup G' = (X-F)\cup G$ then we are done, otherwise, $(X-F)\cup G' \subsetneq (X-F)\cup G$ and using positive responsiveness, $f((X-F)\cup G) = 1 \geq \frac{1}{2}$, and so $(X-F)\cup G \in \mathfrak{M}_f$.

- Suppose that $\mathfrak{M}$ satisfies $M3$ and π is any finite permutation. We must show that $f_{\mathfrak{M}}(\pi[X]) = f_{\mathfrak{M}}(X)$ for each $X \subseteq W$. Given any set $X \subseteq W$, let $F_X = \{w \in X \mid \pi(w) \neq w\}$. Since π is a finite permutation, F_X is finite for every set X. It is easy to see that for any set $X \subseteq W$, $\pi[X] = (X - F_X) \cup \pi[F_X]$ and $X = (\pi[X] - \pi[F_X]) \cup F_X$. Hence, by property $M3$, $X \in \mathfrak{M}$ iff $\pi[X] \in \mathfrak{M}$. Using this fact, it is easy to check that $f_{\mathfrak{M}}$ satisfies finite anonymity. There are three cases to consider: $f_{\mathfrak{M}}(X) = 0$, $f_{\mathfrak{M}}(X) = \frac{1}{2}$ and $f_{\mathfrak{M}}(X) = 1$. The proofs are analogous, so only one case will be checked. The others are left to the reader. Suppose that $f_{\mathfrak{M}}(X) = 0$. We must show that $f_{\mathfrak{M}}(\pi[X]) = 0$. Suppose not, i.e., $f_{\mathfrak{M}}(\pi[X]) \geq \frac{1}{2}$. Then by construction of $f_{\mathfrak{M}}$, $\pi[X] \in \mathfrak{M}$, and so by the above discussion, $X \in \mathfrak{M}$. By the construction of $f_{\mathfrak{M}}$, this implies that $f_{\mathfrak{M}}(X) \geq \frac{1}{2}$, which contradicts the assumption that $f(X) = 0$.

Theorem 27 *Given an aggregation rule f on a set W. Let* $M = \{A : f(A) \geq 1/2\}$. *if f satisfies neutrality, monotonicity, positive responsiveness and finite anonymity then* $< W, M >$ *is a majority space.*

Proof We will prove that $< W, M >$ satisfies the three conditions of majority space:

M1 Let $X \subseteq W$ by neutrality we have $f(X) = 1 - f(X^C)$. From here we get either $X \in M$ or $X^C \in M$

M2 Let $X, Y \in M$ and $X \cap Y = \varnothing$. First note that $f(X) \geq 1/2$ and $f(Y) \geq 1/2$.
Assume $X \neq Y^C$ then there a $Z \neq \varnothing$ such that $W = X \cup Y \cup Z$. $X \subset X \cup Z$ so by positive responsiveness we have $f(X \cup Z) = 1$ and so $f(Y) = 0$ which is a contradiction.

M3 Let $Y = (X - F) \cup G$ where $X \in M$ and F is a finite subset of X and $|F| \leq |G|$ and $X \cap G = \varnothing$. Assume $f(Y) = 0$.
Since $|F| \leq |G|$ then there is $F' \subseteq G$ such that $|F'| = |F|$. Let $Z = (Y - F') \cup F$ then by finite anonymity we have $f(Z) = 0$ but $X \subseteq Z$ so by monotonicity $f(X) = 0$ which a contradiction.

5.5 Characterization of ultrafilters and majority spaces

Give a set A of $\mathbb{N}$ we will use the notation A_n to mean $A_n = |A \cap \{1, 2, ..., n\}|$.

Given a family $\mathcal{A}$ of sets over $\mathbb{N}$. For any sequence $\{a_n\}$ define $a = lim_{\mathcal{A}} a_n$ if $\{j : |a_j - a| < \varepsilon\} \in \mathcal{A}$ for every positive ε.

Fix a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. For a bounded sequence $\{a_n\}$, we write $a = lim_{\mathcal{U}} a_n$ if $\{j : |a_j - a| < \varepsilon\} \in \mathcal{U}$ for every positive ε. We showed in chapter 2 that a unique such a exists and it is an accumulation point of a_n. In this way, a free ultrafilter generalizes the standard limit operation.

Fix a strict majority space $\mathcal{M}$ over $\mathbb{N}$. For a bounded sequence $\{a_n\}$, we write $a = lim_{\mathcal{M}} a_n$ if $\{j : |a_j - a| < \varepsilon\} \in \mathcal{M}$ for every positive ε. It can be shown that if a exists and then it unique and it is an accumulation point of a_n.

The next theorem was suggested by Rohit Parikh.

Theorem 28 *Given a family $\mathcal{A}$ of sets over $\mathbb{N}$. If for every bounded sequence $\{a_n\}$ there is a unique a such that $a = lim_{\mathcal{A}} a_n$ then $\mathcal{A}$ forms an ultrafilter.*

Proof I will prove that this family $\mathcal{A}$ will satisfy these three properties of an ultrafilter:

- $\mathbb{N} \in A$. Any constant sequence is bounded. So it has a unique limit. This will give you $\mathbb{N} \in \mathcal{A}$

- Either X or $X^C \in \mathcal{A}$ (exclusive or). Consider this sequence $\{a_n\}$ defined by

$$a_n = 1 \text{ if } n \in X$$

$$a_n = 0 \text{ if } n \in X^C$$

Take $\varepsilon = 1/2$ then you get either $a = 1$ and $X \in \mathcal{A}$, or $a = 0$ and $X^C \in \mathcal{A}$

- $X \cup Y \in \mathcal{A}$ then $X \in \mathcal{A}$ or $Y \in \mathcal{A}$.Assume $X \cup Y \in \mathcal{A}$. Define a sequence $\{a_n\}$ by:

$$a_n = 1 \text{ if } n \in X$$

$$a_n = 0 \text{ if } n \in Y$$

$$a_n = -1 \text{ otherwise}$$

Take $\varepsilon = 1/2$ then you get either $X \in \mathcal{A}$ or $Y \in \mathcal{A}$ or $(X \cup Y)^C \in \mathcal{A}$. But by case 2 above we can't have $(X \cup Y)^C \in \mathcal{A}$ so we get $X \in \mathcal{A}$ or $Y \in \mathcal{A}$.

Theorem 29 *Given a family $\mathcal{A}$ of sets over $\mathbb{N}$. If for every bounded sequence $\{a_n\}$ with no more than two accumulation points there is a unique a such that $a = lim_{\mathcal{A}} a_n$ then $\mathcal{A}$ forms a strict majority set of type 2.*

Proof I will prove that this family $\mathcal{A}$ will satisfy the three properties of a strict majority set of type 2:

- $\mathbb{N} \in A$. Any constant sequence is bounded. So it has a unique limit. This will give you $\mathbb{N} \in \mathcal{A}$

- if $X \cup Y$ is cofinite and X, Y are disjoint, then either $X \in \mathcal{A}$ or $Y \in \mathcal{A}$

(Exclusive or). Consider this sequence $\{a_n\}$ defined by

$$a_n = 1 \text{ if } n \in X$$

$$a_n = 0 \text{ if } n \in Y$$

$$a_n = -1 \text{ otherwise}$$

This sequence is bounded with no more than two accumulation points. Take $\varepsilon = 1/2$ then you get either $X \in \mathcal{A}$ or $Y \in \mathcal{A}$

- $X \in \mathcal{A}$ and $Y = (X - F)$ where F is finite subset of X then $Y \in \mathcal{A}$. Assume $X - F \notin \mathcal{A}$. Since $X \in \mathcal{A}$ we get by the above case that $X^C \notin \mathcal{A}$. Define a sequence $\{a_n\}$ as:

$$a_n = 0 \text{ if } n \in X^C$$

$$a_n = 1 \text{ if } n \in (X - F)$$

$$a_n = -1 \text{ if } n \in F$$

In this case the limit is either 1 or 0. Take $\varepsilon = 1/2$ then we get a contradiction since both sets X^C and $X - F$ are not in $\mathcal{A}$.

5.6 Application to probability

Fix a non-principal ultrafilter $\mathcal{U}$.

5.6.1 Measure theory

A finitely additive measure on $\mathbb{N}$ is a function $\mu : P(\mathbb{N}) \to [0,1]$ such that $\mu(\phi) = 0,\ \mu(\mathbb{N}) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for every two disjoint subsets A and B.

A pure finitely additive measure is a finitely additive measure μ such that $\mu(A) = 0$ for every finite set A.

A finitely additive measure μ is zero-one if $\mu(A) \in \{0,1\}$ for all A.

The set of zero-one finitely additive measures on $\mathbb{N}$ is exactly the set of ultrafilters on $\mathbb{N}$.

Let A(n) be the number of elements of A that are less than are equal to n.

Define the asymptotic density of A by

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n},$$

if this limit exists.

Theorem 30 *Suppose an aggregation rule f is a purely finitely additive zero-one measure. Then f satisfies neutrality, monotonicity, and finite anonymity.*

Proof For the proof refer to [8].

Define $\mu_{\mathcal{U}}(A) = lim_{\mathcal{U}} \frac{A_n}{n}$. (this limit exists and is unique for every A)

Corollary 31 *There exists an aggregation rule f that satisfies neutrality, monotonicity and finite anonymity such that f(E) = 1 and f(O) = 0. (where E is the set of even numbers and O is the set of odd numbers).*

5.6.2 Probability over $\mathbb{N}$

Now define $P_{\mathcal{U}}(A, B) = lim_{\mathcal{U}}(A \cap B)_n/B_n$ where $A, B \subseteq \mathbb{N}$ and $B \neq \varnothing$. And define $P_{\mathcal{M}}(A, B) = lim_{\mathcal{M}}(A \cap B)_n/B_n$ if the limit exists.

Theorem 32 *Given $A, B \subseteq \mathbb{N}$ and $B \neq \varnothing$ then $P_{\mathcal{U}}$ satisfies the following properties:*

- $0 \leq P_{\mathcal{U}}(A, B) \leq 1$
- $P_{\mathcal{U}}(A, B) = P_{\mathcal{U}}(A \cap B, B)$
- $P_{\mathcal{U}}(B, B) = 1$
- $\mathcal{P}(A \cup C, B) = \mathcal{P}(A, B) + \mathcal{P}(C, B)$ *if* $A \cap C = \varnothing$
- $\mathcal{P}(A, B) = 0$ *if A is finite and B is not.*

Proof

- Let $A, B \subseteq \mathbb{N}$ the note that $0 \leq (A \cap B)_n/B_n \leq 1$ and so since the limit up to an ultrafilter is one of the accumulation points then $0 \leq lim_{\mathcal{U}}(A \cap B)_n/B_n \leq 1$.
- This is straight forward since $(A \cap B)_n/B_n = ((A \cap B) \cap B)_n/B_n$.
- we have $(B \cap B)_n/B_n = 1$ for all n and so $P_{\mathcal{U}}(B, B) = 1$.

Define a generalized asymptotic density $\mu_{\mathcal{U}}(A) = lim_{\mathcal{U}}\frac{A_n}{n}$. This limit exist and unique for all A.

Theorem 33 *if $\mu(B) \neq 0$ then $P_{\mathcal{U}}(A, B) = \mu(A \cap B)/\mu(B)$.*

Proof The proof is easy and straight forward.

Define $A^n = \{x^n : x \in A\}$ and
$n.A = \{n.x : x \in A\}$

Proposition 34 *We have the following:*

- *if* $\mu(A) = 1$ *then* $P_{\mathcal{U}}(n.A, A) = 1/n$ *for all* $n \geq 1$.
- *if* $P_{\mathcal{U}}(n.\mathbb{N}, A) = 0$ *then* $P_{\mathcal{U}}(n.A, A) = 0$.
- *if* $P_{\mathcal{U}}(n.\mathbb{N}, A) = 1$ *then* $P_{\mathcal{U}}(n.A, A) = 1$.
- $P_{\mathcal{U}}(A^n, A) = 0$ *for all* $n > 1$ *and* $\mu(A) = 1$

Chapter 6

Majority Logic

6.1 Majority Logic: Syntax

We extend the graded modal language with a new modal operator W where W is interpreted as "weak majority." First we will talk about the syntax and axiomatization and some properties of these axioms. Then we will talk about Semantics by using majority spaces and we will prove completeness.

Definition 19 *Given a countable set of atomic propositions* $\mathbb{P} = \{p_0, p_1, \ldots\}$*, a formula* α *of* ***MJL*** *can have the following syntactic form:*

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha \mid W\alpha$$

where $p \in \mathbb{P}$ *and* $n \in \mathbb{N}$.

Let $\mathcal{L}_{\mathbf{MJL}}$ be the set of all well-formed formulas of the majority modal logic. Define $M\alpha := \neg W \neg \alpha$. So, **MJL** takes the language of **GML** and closes under the operator W. Notice in particular that there are an infinite number of modal operators, one for each natural number plus the majority operators.

6.1.1 Axiomatization

We propose the following axiomatization of **MJL**. Since **MJL** extends graded modal logic, we will include the axiom schemes $G1, G2$ and $G3$. These axioms capture our intuitions when we can count accessible worlds. But what axioms shall we adopt to reason about "majority"? The following discussion will motivate the proposed axiomatization which can be found at the end of the discussion.

Suppose a group of friends are trying to decide where to go for dinner. As is common in most social situations, the goal is to keep as many people happy as possible. If more than half of the people like Indian for dinner and more than half like Italian for dinner, then there must be someone who likes both Italian and Indian. This is easy to see if we consider a specific example.This reasoning is captured by the following axiom scheme

$$M\alpha \wedge M\beta \to \Diamond(\alpha \wedge \beta)$$

Now, suppose that more than half of the friends like Italian for dinner. Also, suppose that every one in the group group who likes Italian food, likes wine to be served. We can conclude that a majority of the friends like wine with dinner. And so, we will include the following axiom scheme

$$M\alpha \wedge \Box(\alpha \rightarrow \beta) \rightarrow M\beta$$

Suppose that you are put in charge of making dinner reservations for the group of 10 people. Given that 5 people like Italian and 5 people like Indian, what can you conclude if you are given additional information that more that 3 people do not like Italian and do not like Indian. The natural conclusion to draw is that more than 3 people like both Indian and Italian food. Otherwise, say you conclude that only two people like both Indian and Italian food. This would mean that 3 people like Italian but not Indian, 3 people like Indian but not Italian and (more than) 3 people like neither Indian nor Italian. Since these sets are disjoint, the total sum of people is 11 or more, and so it must be the case that more than 3 people like Indian and Italian. This line of reasoning is captured by the following axiom scheme

$$W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \rightarrow \Diamond_n(\alpha \wedge \beta) \qquad (n \in \mathbb{N})$$

The final situation is similar to the above situation. except suppose that a strict majority of the people like Italian.

$$W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \rightarrow \Diamond_{n+1}(\alpha \wedge \beta) \qquad (n \in \mathbb{N})$$

The preceding discussion is summarized by the following list of axioms and rules.

Axiom 1 *Classical propositional tautologies*

Axiom 2 $\Diamond_{n+1}\alpha \rightarrow \Diamond_n\alpha \quad (n \in \mathbb{N})$

Axiom 3 $\Box(\alpha \rightarrow \beta) \rightarrow (\Diamond_n\alpha \rightarrow \Diamond_n\beta) \quad (n \in \mathbb{N})$

Axiom 4 $\Diamond!_0(\alpha \wedge \beta) \rightarrow ((\Diamond!_{n_1}\alpha \wedge \Diamond!_{n_2}\beta) \rightarrow \Diamond!_{n_1+n_2}(\alpha \vee \beta)) \quad (n_1, n_2 \in \mathbb{N})$

Axiom 5 $M\alpha \wedge M\beta \rightarrow \Diamond(\alpha \wedge \beta)$

Axiom 6 $M\alpha \wedge \Box(\alpha \rightarrow \beta) \rightarrow M\beta$

Axiom 7 $W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \rightarrow \Diamond_n(\alpha \wedge \beta) \quad (n \in \mathbb{N})$

Axiom 8 $W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \rightarrow \Diamond_{n+1}(\alpha \wedge \beta) \quad (n \in \mathbb{N})$

MP From α and $\alpha \rightarrow \beta$ derive β.

N From α derive $\Box\alpha$.

We write $\vdash_{\mathbf{MJL}} \alpha$ if α can be deduced from Axioms 1 - 8 using the rules MP and N. If it is clear from context, we may write $\vdash \alpha$ instead of $\vdash_{\mathbf{MJL}} \alpha$.

6.1.2 Properties of the Axioms

We now discuss some of the properties of the axioms proposed in the previous section. Some of these properties turn out to be useful in the completeness proof and others are natural properties of majorities and weak majorities.

This first lemma gives some consequences of the proposed axioms. The lemma also shows that our axiomatization captures many natural properties of "majority" and "weak majority". Part (i) shows that M and W are both normal modal operators. Part (ii) is equivalent to saying that given any subset X then either X or its complement (or both) constitutes weak majority. (iii) are obvious properties of majority and weak majority sets.

Lemma 35 *Suppose that α and β are arbitrary formulas of **MJL**. Then*

i. If $\vdash \alpha \to \beta$ then $\vdash M\alpha \to M\beta$ and $\vdash W\alpha \to W\beta$.

ii. $\vdash W\alpha \vee W\neg\alpha$

iii. $\vdash M\alpha \to W\alpha$ and $\vdash M\alpha \to \Diamond\alpha$

Proof Suppose that α and β are any formulas of **MJL**.

i. Suppose that $\vdash \alpha \to \beta$. We will show $\vdash M\alpha \to M\beta$ and $\vdash W\alpha \to W\beta$. By the NEC, $\vdash \Box(\alpha \to \beta)$, and so by modus ponens and Axiom 6, $\vdash M\alpha \to M\beta$. $\vdash W\alpha \to W\beta$ follows easily using contraposition.

ii. $\vdash \neg\Diamond(\alpha \wedge \neg\alpha) \to \neg(M\neg\alpha \wedge M\alpha)$ is an instance of Axiom 5. Hence $\vdash \Box\top \to (W\alpha \vee W\neg\alpha)$. By NEC $\vdash \Box\top$. Therefore by MP, $\vdash W\alpha \vee W\neg\alpha$.

iii. $\vdash M\alpha \rightarrow W\alpha$ follows from (ii), and $\vdash M\alpha \rightarrow \Diamond\alpha$ is an instance of Axiom 5.

Using the language of graded modal logic, we can find a formula that expresses exactly how many worlds are accessible at any given state. For any $n \in \mathbb{N}$, the formula $\Diamond_n!\top$ will be true at some world w iff there are exactly n accessible worlds. Similarly, the formulas $\Box_n\bot$ will be true at some world w iff there are *at most* n accessible worlds. We will define $\mathbf{A_n} := \Diamond_n!\top$, $\mathbf{A_{\leq n}} := \Box_n\bot$ and $\mathbf{A_{>n}} := \Diamond_n\top$. So, $\mathbf{A_n}$ is true at a state s if there are exactly n accessible worlds. The following lemma shows what happens when we know the exact number of accessible worlds.

Lemma 36 *Suppose that $\mathbf{A_n}$ is the formula defined above. Then*

i. $\vdash \mathbf{A_n} \rightarrow (\Box_{\lfloor n/2 \rfloor}\alpha \vee \Box_{\lfloor n/2 \rfloor}\neg\alpha)$ *For all* $n \in \mathbb{N}$

ii. $\vdash \mathbf{A_n} \rightarrow (\Box_{\lfloor n/2 \rfloor - 1}\alpha \rightarrow \Diamond_{\lfloor n/2 \rfloor}\alpha)$ *For all* $n > 2$

iii. For all $n \in \mathbb{N}$, $\vdash \mathbf{A_n} \rightarrow (M\alpha \leftrightarrow \Diamond_{\lfloor n/2 \rfloor}\alpha)$.

Proof Part (i) and (ii) are statements of graded modal logic, and given the completeness and soundness proofs in [5, 7], follow easily from semantic arguments. We first note the following properties which are instances of Axioms 8 and 7 respectively (let $\beta = \alpha$).

1. $\vdash M\alpha \wedge \Diamond_n\neg\alpha \rightarrow \Diamond_{n+1}\alpha$

2. $\vdash W\alpha \wedge \Diamond_n\neg\alpha \rightarrow \Diamond_n\alpha$

We need only show property (*iii*).

- Suppose that $\vdash \mathbf{A_n} \wedge \Diamond_{\lfloor n/2 \rfloor}\alpha$. By part (i) of this lemma we get $\vdash \mathbf{A_n} \wedge \Diamond_{\lfloor n/2 \rfloor}\alpha \rightarrow \Box_{\lfloor n/2 \rfloor}\alpha$ and by (2) we get $\vdash \mathbf{A_n} \wedge \Diamond_{\lfloor n/2 \rfloor}\alpha \wedge \Box_{\lfloor n/2 \rfloor}\alpha \rightarrow M\alpha$
- Suppose that $\vdash \mathbf{A_n} \wedge M\alpha$. If $n = 0$ or $n = 1$ then we have $M\alpha \rightarrow \Diamond\alpha$ so we get $\Diamond_{\lfloor n/2 \rfloor}\alpha$. Assume $n > 2$ by (1) we have $M\alpha \rightarrow (\Diamond_{\lfloor n/2 \rfloor}\alpha \vee \Box_{\lfloor n/2 \rfloor - 1}\alpha)$. Using part (ii) of this lemma we get $\vdash \mathbf{A_n} \wedge M\alpha \rightarrow \Diamond_{\lfloor n/2 \rfloor}\alpha$

6.2 Majority Logic: Semantics

In this section we will present the semantics for **MJL**. The semantics will be an extension of the usual Kripke semantics. The formula $W\alpha$ will be true provided that the set of all accessible worlds in which α is true is a majority of the set of all accessible worlds. The definition makes sense only if there are *finitely* many accessible worlds. But what constitutes a weak majority of an infinite set? The following section offers a solution to this question.

Recall that if S is any set of states and R a binary relation on S, then $R(s) = \{t \mid sRt\}$ and for any formula α (of **MJL** or **GML**), $R_\alpha(s) = \{t \mid sRt \text{ and } t \models \alpha\}$. This definition of course depends on the definition of truth in a model which is given below.

6.2.1 Majority Models

In this section we will extend the definition of a Kripke model in order to define the truth of a majority logic formula.

Definition 20 *A* **majority model** *is a tuple* $\mathcal{M} = \langle S, R, m, V \rangle$*. Here* S *is any set of states,* R *is an accessibility relation and* V *is the valuation function* $V : \mathbb{P} \to 2^S$*, and* $m : S \to 2^{2^S}$ *is a* **majority function** *such that for each* $s \in S$*,* $\langle R(s), m(s) \rangle$ *is a majority space.*

So, m assigns a majority space to each state. Let $s \in S$ be any state. We will define the truth of a formula α at state s in model $\mathcal{M}$ as follows:

1. $\mathcal{M}, s \models p$ iff $s \in V(p)$, where $p \in \mathbb{P}$
2. $\mathcal{M}, s \models \neg\alpha$ iff $\mathcal{M}, s \not\models \alpha$
3. $\mathcal{M}, s \models \alpha \vee \beta$ iff $\mathcal{M}, s \models \alpha$ or $\mathcal{M}, s \models \beta$
4. $\mathcal{M}, s \models \Diamond_n \alpha$ iff $|R_\alpha(s)| > n \qquad (n \in \mathbb{N})$
5. $\mathcal{M}, s \models W\alpha$ iff $R_\alpha(s) \in m(s)$

And so $\mathcal{M}, s \models M\alpha$ iff $R_{\neg\alpha}(s) \notin m(s)$. First notice that if $R(s)$ is finite for some $s \in S$, then by proposition 14, then $\mathcal{M}, s \models W\alpha$ iff $|R_\alpha(s)| \geq |R(s)|/2$. We will now show that the axioms of majority logic are valid in all majority models.

Theorem 37 ***MJL*** *is sound with respect to the class of all majority models.*

Proof Soundness was shown in [7] for axioms 1 - 4, MP, and Nec. Let $\mathcal{M} = \langle S, R, V, m \rangle$ be any majority model and $s \in S$. We will show Axiom 5 - 8 are true at state s. Since s is arbitrary, each axiom will be valid in $\mathcal{M}$; and hence, the axioms are sound. All of the proofs are straightforward and are left to the reader. As an example, we show the result holds for Axiom 7 and 8.

Axiom 7: Assume $s \models W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta)$ so we have $R_\alpha(s) \in m(s), R_\beta(s) \in m(s)$ and $|R_{\neg\alpha\wedge\neg\beta}(s)| > n$ we need to prove that $|R_{\alpha\wedge\beta}(s)| > n$. Assume $|R_{\alpha\wedge\beta}(s)| \leq n$. Let $X \subset R_{\neg\alpha\wedge\neg\beta}(s)$ where $|X| = n$ (X is a proper subset). Let $Y = (R_\beta(s) - R_{\alpha\wedge\beta}(s)) \cup X$ according to M3 $Y \in m(s)$ and we have $Y \cap R_\alpha(s) = \varnothing$ and $Y \neq R^C_\alpha(s)$ which is a contradiction with M2. So $|R_{\alpha\wedge\beta}(s)| > n$ and thus $s \models \Diamond_n(\alpha \wedge \beta)$

Axiom 8: Assume $s \models W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta)$ so we have $R_\alpha(s) \in m(s), R_{\neg\beta}(s) \notin m(s)$ and $R_{\neg\alpha\wedge\neg\beta}(s) > n$ we need to prove that $|R_{\alpha\wedge\beta}(s)| > n+1$. Assume $|R_{\alpha\wedge\beta}(s)| \leq n+1$. But $R_{\neg\beta}(s) = (R_\alpha(s) - R_{\alpha\wedge\beta}(s)) \cup R_{\neg\alpha\wedge\neg\beta}(s)$ and by M3 we get $R_{\neg\beta}(s) \in m(s)$ which is a contradiction.

6.3 Completeness

We first discuss completeness of graded modal logic. We will then adapt the proof to show completeness for **MJL**.

6.3.1 Completeness of Graded Modal Logic

Given any consistent set of formulas of majority logic, Γ, using Lindenbaum's Lemma, we can construct a maximally consistent superset of Γ. As usual, the states of our canonical model will be maximally consistent sets. In what follows, Γ will always be assumed to be a maximally consistent set of formulas.

When constructing a canonical model for a graded modal logic, it is necessary to control the number of worlds accessible from any given state. Given any state, i.e. maximally consistent set, Γ, our goal is to construct $R(\Gamma)$ such that

$$\Diamond_n \alpha \in \Gamma \text{ iff } |\{\Gamma' \in R(\Gamma) \mid \alpha \in \Gamma'\}| > n$$

Following [7], a satisfying family for each Γ, denoted by $SF(\Gamma)$ is constructed so that we may define $R(\Gamma) = SF(\Gamma)$ and then R will satisfy the above property. To this end we will present the following definitions and lemmas from [7]. Recall that ω is the first countable ordinal, and that $\omega+1 = \omega \cup \{\omega\}$. Let Φ be the set of all maximally consistent sets.

Definition 21 *The function $\mu : \Phi \times \Phi \to \omega + 1$ is defined as follows: for every $\Gamma_1, \Gamma_2 \in \Phi$*

$$\begin{aligned}
\mu(\Gamma_1, \Gamma_2) &= \omega \text{ \textit{if for every} } \alpha \in \Gamma_2,\ \Diamond_n \alpha \in \Gamma_1 \text{ \textit{for all} } n \in \mathbb{N} \\
\mu(\Gamma_1, \Gamma_2) &= \min\{n \in \mathbb{N} \ : \ \Diamond!_n \alpha \in \Gamma \text{ \textit{and} } \alpha \in \Gamma_2\} \text{ \textit{otherwise}}
\end{aligned}$$

That the function μ is well defined and for more properties of μ, the reader is referred to [5, 7]. The main idea is that μ will tell us how many accessible worlds are needed. Given two maximally consistent sets, Γ_1, Γ_2, $\mu(\Gamma_1, \Gamma_2)$ tells us the minimum number of copies of Γ_2 that are needed to be accessible from Γ_1.

The following lemma is an easy consequence of definition 21

Lemma 38 *Let $\Gamma_1, \Gamma_2 \in \Phi$. The following conditions are equivalent:*

1. $\mu(\Gamma_1, \Gamma_2) \neq 0$
2. *For every α, if $\alpha \in \Gamma_2$ then $\Diamond_0 \alpha \in \Gamma_1$.*
3. *For every α, if $\Box_0 \alpha \in \Gamma_1$ then $\alpha \in \Gamma_2$.*

The following lemma shows that μ works as we expect.

Lemma 39 *Let $\Gamma_1 \in \Phi$ and α be any formula*

1. *If $\Diamond_0 \alpha \in \Gamma_1$ then there exists $\Gamma_2 \in \Phi$ such that $\alpha \in \Gamma_2$ and $\mu(\Gamma_1, \Gamma_2) \neq 0$.*
2. *If $\Diamond_n \alpha \in \Gamma_1$ for every $n \in \mathbb{N}$, then there exists $\Gamma_2 \in \Phi$ such that $\alpha \in \Gamma_2$ and $\mu(\Gamma_1, \Gamma_2) = \omega$*

Refer to [7] for a proof. We are now ready to define the **satisfying family** of a maximally consistent set Γ_0.

Definition 22 *Let* $\Gamma_0 \in \Phi$. *The set*

$$SF(\Gamma_0) = \{\langle \Gamma, m\rangle | m < \mu(\Gamma_0, \Gamma) \text{ and } \Gamma \in \Phi\}$$

will be called the satisfying family of Γ_0.

An element of $SF(\Gamma_0)$ is of the form $\langle \Gamma, n\rangle$ where $n < \mu(\Gamma_0, \Gamma)$, therefore we shall think of $SF(\Gamma_0)$ as made up of $\mu(\Gamma_0, \Gamma)$ ordered copies of Γ, for every $\Gamma \in \Phi$ such that $\mu(\Gamma_0, \Gamma) \neq 0$.

The following theorem is the main theorem from [7].

Theorem 40 ([7]) *For any* α *and any* $n \in \mathbb{N}$,

$$\Diamond_n \alpha \in \Gamma_0 \text{ iff } |\{\Gamma \in SF(\Gamma_0) : \alpha \in \Gamma\}| > n$$

where to simplify notation, we identify a pair $\langle \Gamma, n\rangle$ *(*$n < \mu(\Gamma_0, \Gamma)$*) with its first component.*

6.3.2 Canonical Models for MJL

In this section we will define a canonical model for majority logic $\mathcal{M}^* = \langle S^*, R^*, V^*, m^*\rangle$ for **MJL** as follows: First of all, let

$$\mu(\Gamma) = \sup\{\mu(\Gamma', \Gamma) \mid \Gamma' \in \Phi\}$$

So $\mu(\Gamma)$ gives the maximum number of copies of Γ that will be needed in the canonical model. Define

$$S^* = \{\langle \Gamma, m\rangle | \ m < \mu(\Gamma) \text{ and } \Gamma \in \Phi\} \cup \{\langle \Gamma, 0\rangle \mid \mu(\Gamma) = 0\}$$

So we may think of S^* as made up of $\mu(\Gamma)$ copies of Γ if $\mu(\Gamma) \neq 0$, and by one copy of Γ if $\mu(\Gamma) = 0$, for any maximally consistent set Γ.

For each $\langle \Gamma, i\rangle \in S^*$ define,

$$R^*(\langle \Gamma, i\rangle) = SF(\Gamma)$$

and for every proposition p and every $\langle \Gamma, i\rangle \in S^*$ we set:

$$V^*(p) = \{\langle \Gamma, i\rangle \mid p \in \Gamma\}$$

It only remains to define a majority function $m^* : S^* \to 2^{2^{S^*}}$. In what follows we will write $\Gamma \in S^*$ instead of $\langle \Gamma, i\rangle$. This abuse of notation should not cause any confusion and so will be used to simplify the presentation.

Let

$$R^*_\alpha(\Gamma) = SF_\alpha(\Gamma) = \{\Gamma' : \Gamma' \in SF(\Gamma) \text{ and } \alpha \in \Gamma'\}.$$

We are ready to define $m^*(\Gamma)$ so that $\langle R^*(\Gamma), m^*(\Gamma)\rangle$ is a majority space.

Given any maximally consistent set Γ, it is easy to see that exactly one of the following cases must be true:

Case 1: $\Diamond!_n\top \in \Gamma$ for some $n \in \mathbb{N}$

Case 2: $\Diamond_n\top \in \Gamma\ \forall n \in \mathbb{N}$

If we are in Case 1, then $|SF(\Gamma)| = n$, and so we can define

$$m^*(\Gamma) = \{X \ : \ X \subseteq SF(\Gamma) \text{ and } |X| \geq \lceil |SF(\Gamma)|/2 \rceil\}$$

By Proposition 14, $\langle R(\Gamma), m^*(\Gamma)\rangle$ is a majority space. So suppose that we are in case 2, that is for all $n \in \mathbb{N}$, $\Diamond_n\top \in \Gamma$. We first need some definitions.

Definition 23 *Let X be any set, then $cof(X) = \{Y \subseteq X \mid Y^C$ is finite $\}$. So, $cof(X)$ is the set of co-finite subsets of X.*

Definition 24 *Let Y be any set and $X \subseteq 2^Y$. Then define*

$$X^f = \{A \mid \exists B \in X \textit{ such that } A = (B - F) \cup G \textit{ where } F$$

$$\textit{is finite, } |F| \leq |G| \textit{ and } X \cap G = \varnothing\}$$

So, X^f is X **closed under finite perturbations**. It is easy to see that $X \subseteq X^f$ (take F and G both to be empty).

Definition 25 *Let Y be any set and $X \subseteq 2^Y$, then define*

$$\overline{X} = \{A \ : \ A \notin X \textit{ and } A^C \in X\}$$

Note that if $A \in X \cup \overline{X}$, then $A^C \in X \cup \overline{X}$; and hence if $A \notin (X \cup \overline{X})$, then $A^C \notin (X \cup \overline{X})$.

Let Γ be any maximally consistent set. We will now construct m^*:

1. Define $\mathfrak{M}_0(\Gamma) = \{SF_\alpha(\Gamma) \mid W\alpha \in \Gamma\}$

2. Define $\mathfrak{M}_1(\Gamma) = (\mathfrak{M}_0(\Gamma))^f$. That is take $\mathfrak{M}_0(\Gamma)$ and close off under finite perturbations.

3. Let $\mathcal{O} = 2^{SF(\Gamma)} - (\mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)})$. The set $\mathcal{O}$ contains the "other" sets. That is the sets X such that neither X nor X^C have made it into $\mathfrak{M}_1(\Gamma)$. In order to satisfy $M1$, we must pick one of X or X^C to be elements of $\mathfrak{M}_1$. These choices must be made in a way that is consistent with the properties $M1 - M4$. Let $\mathcal{U}$ be any non-principal ultrafilter over $SF(\Gamma)$. Define

$$m^*(\Gamma) = \mathfrak{M}_1(\Gamma) \cup (\mathcal{O} \cap \mathcal{U})$$

Before proving that $m^*(\Gamma)$ is in fact a majority system, we need a lemma.

Lemma 41 *Let Γ be any maximally consistent set. Suppose that $X, Y \in m^*(\Gamma)$ and $X \cap Y = \varnothing$. Then $X, Y \in \mathfrak{M}_1(\Gamma)$.*

Proof Let Γ be a maximally consistent set, and suppose that $X, Y \in m^*(\Gamma)$. Then by construction, there are four cases. If $X, Y \in \mathfrak{M}_1(\Gamma)$ then we are done. We need only show that the other three cases lead to a contradiction.

Suppose that $X \in \mathcal{O} \cap \mathcal{U}$ and $Y \in \mathcal{O} \cap \mathcal{U}$. Then $X \cap Y \in \mathcal{U}$, which implies $\varnothing \in \mathcal{U}$. But this contradicts the fact the $\mathcal{U}$ is non-principal. Thus both X and Y cannot be elements of $\mathcal{O} \cap \mathcal{U}$. Suppose that $X \in (\mathcal{O} \cap \mathcal{U})$ and $Y \in \mathfrak{M}_1(\Gamma)$. Since $X \cap Y = \varnothing$, $Y \subseteq X^C$ so $X^C = Y \cup G$ for some set G. Therefore, $X^C \in \mathfrak{M}_1(\Gamma)$ which implies $X \in (\mathfrak{M}_1 \cup \overline{\mathfrak{M}_1})$. But this contradicts the assumption that $X \in \mathcal{O}$. Similarly we can show that $X \in \mathfrak{M}_1(\Gamma)$ and $Y \in \mathcal{O} \cap \mathcal{U}$ leads to a contradiction.

Lemma 42 *Given any maximally consistent set* Γ, $\langle R^*(\Gamma), m^*(\Gamma)\rangle$ *is a majority space.*

Proof Let Γ be any maximally consistent set. Then we are either in case 1 or case 2 (as stated above). If we are in case 1, then $\langle R^*(\Gamma), m^*(\Gamma)\rangle$ is a majority space by Proposition 14. Thus we may assume that we are in case 2, and so $R^*(\Gamma)$ is infinite.

We must show $m^*(\Gamma)$ satisfies properties $M1, M2$, and $M3$.

(M1) Let $X \subseteq R^*(\Gamma)$. We must show that either $X \in m^*(\Gamma)$ or $X^C \in m^*(\Gamma)$. By construction, either $X \in \mathcal{O}$ or $X \in \mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)}$. Suppose that $X \in \mathcal{O}$. Then $X^C \in \mathcal{O}$. By the definition of an ultrafilter, either $X \in \mathcal{U}$ or $X^C \in \mathcal{U}$. Say $X \in \mathcal{U}$. Then by construction, $X \in m^*(\Gamma)$. The result is similar if $X^C \in \mathcal{U}$. Suppose that $X \in \mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)}$. Then either $X \in \mathfrak{M}_1(\Gamma)$ or $X \in \overline{\mathfrak{M}_1(\Gamma)}$. If $X \in \mathfrak{M}_1(\Gamma)$ then we are done. If $X \in \overline{\mathfrak{M}_1(\Gamma)}$, then $X^C \in m^*(\Gamma)$. In either case either $X \in m^*(\Gamma)$ or $X^C \in m^*(\Gamma)$.

(M2) Suppose that $X, Y \in m^*(\Gamma)$ and $X \cap Y = \varnothing$ we must show $X = Y^C$. Since $X, Y \in m^*(\Gamma)$ and $X \cap Y = \varnothing$, by Lemma 41, both X and Y are elements of $\mathfrak{M}_1(\Gamma)$.

Suppose that $X^C \neq Y$. By construction there are sets $Z_1, Z_2 \in \mathfrak{M}_0(\Gamma)$, finite sets $F_1 \subseteq Z_1, F_2 \subseteq Z_2$ and sets G_1, G_2 such that $|F_1| \leq |G_1|$, $|F_2| \leq |G_2|$, $G_1 \cap X = \varnothing$, $G_2 \cap Y = \varnothing$ and

$$X = (Z_1 - F_1) \cup G_1$$

and

$$Y = (Z_2 - F_2) \cup G_2$$

Let

$$a = |Z_1 \cap Z_2| = |F_1 \cap F_2| \text{ since } X \cap Y = \varnothing$$

$$b = |F_1 \cap G_2|$$

and

$$c = |F_2 \cap G_1|$$

We have $(G_1 - F_2) \cup (G_2 - F_1) \subseteq Z_1^C \cap Z_2^C$

- if a = 0 then $Z_1 = Z_2^C$. $X \cup Y = (Z_1 - F_1) \cup G_1 \cup (Z_2 - F_2) \cup G_2 = ((Z_1 \cup Z_2) - (F_1 \cup F_2)) \cup G_1 \cup G_2 = (W - F') \cup G'$ where $F' = F_1 \cup F_2$ which is finite subset of W and $G' = G_1 \cup G_2$. So we get $F_1 \cup F_2 = G_1 \cup G_2$ and so $X = Y^C$ which contradict the assumption.

- if $a \neq 0$ then $|Z_1^C \cap Z_2^C| \geq |G_1| - b + |G_2| - c \geq |F_1| + |F_2| - (b+c) \geq a + b + a + c - (b + c) = 2 \times a > a$ (since $Z_1 \cap Z_2 \neq \varnothing$) .So $|Z_1 \cap Z_2| < |Z_1^C \cap Z_2^C|$ which contradicts Axiom 7.

(M3) Let $X \in m^*(\Gamma)$ and $Y = (X - F) \cup G$ where F is finite subset of X, $|F| \leq |G|$ and $X \cap G = \varnothing$. We need to prove that $Y \in m^*(\Gamma)$

- If $X \in \mathfrak{M}_1(\Gamma)$, then there is a $Z \in \mathfrak{M}_0(\Gamma)$ such that $X = (Z - F') \cup G'$, where F' is a finite subset of Z, $|F'| \leq |G'|$. We will prove that $Y = (Z - F'') \cup G''$ where F'' is a finite subset of Y and $|F''| \leq |G''|$ and $G'' \cap Z = \varnothing$. We have, $Y = (Z - ((F' - G) \cup (F - G'))) \cup (G - F') \cup (G' - F)$ which implies that $Y \in \mathfrak{M}_1$.
- if $X \in (\mathcal{O} \cap \mathcal{U})$ then $Y \in \mathcal{U}$. If $Y \in \mathcal{O}$ then $Y \in \mathcal{O} \cap \mathcal{U}$. Assume $Y \notin \mathcal{O}$ then $Y \in (\mathfrak{M}_1 \cup \overline{\mathfrak{M}_1})$. $Y \in \mathfrak{M}_1$ or $Y^C \in \mathfrak{M}_1$ if $Y \in \mathfrak{M}_1$ then $Y \in \mathfrak{M}$ if $Y^C \in \mathfrak{M}_1$ then $X^C = ((Y^C - F) \cup G) \in \mathfrak{M}_1$ which is a contradiction since we assume $X \in (\mathcal{O} \cap \mathcal{U})$.

Theorem 43 *For any maximally consistent set* Γ *and any formula* α *of* ***MJL****,*

$$R^*_\alpha(\Gamma) \in m^*(\Gamma) \textit{ iff } W\alpha \in \Gamma$$

Proof Let Γ be a maximally consistent set and α any formula of **MJL**.

($\Leftarrow$) Suppose $W\alpha \in \Gamma$ then $R^*_\alpha(\Gamma) \in \mathfrak{M}_O(\Gamma) \subseteq m^*(\Gamma)$. Thus, $R^*_\alpha(\Gamma) \in m^*(\Gamma)$.

(⇒) Suppose $R^*_\alpha(\Gamma) \in m^*(\Gamma)$. Since Γ is maximally consistent, by Lemma 35 part 2, $W\alpha \vee W\neg\alpha \in \Gamma$. Therefore, we need only show that $W\neg\alpha \in \Gamma$ and $W\alpha \notin \Gamma$ leads to a contradiction.

Suppose $W\neg\alpha \in \Gamma$ and $W\alpha \notin \Gamma$, by construction $R^*_{\neg\alpha}(\Gamma) \in \mathfrak{M}_0(\Gamma)$.

Since $W\alpha \notin \Gamma$, $R^*_\alpha(\Gamma) \in \mathfrak{M}_1(\Gamma)$. Therefore there is a set $Z \in \mathfrak{M}_0(\Gamma)$, a finite set $F \subseteq R^*_\alpha(\Gamma)$ and set $G \subseteq R^*(\Gamma)$ such that $|F| \leq |G|$ and

$$R^*_\alpha(\Gamma) = (Z - F) \cup G \qquad (*)$$

Thus, there is some formula β such that $W\beta \in \Gamma$ and $Z = R^*_\beta(\Gamma)$. Suppose that $|F| = k$ for some integer k. Using Lemma 40 and $(*)$, $\Diamond!_k(\neg\alpha\wedge\beta) \in \Gamma$. Hence (1) $\Diamond_{k-1}(\neg\alpha\wedge\beta) \in \Gamma$ and (2) $\neg\Diamond_k(\neg\alpha\wedge\beta) \in \Gamma$. Since $G \cap Z = \varnothing$ and $|F| \leq |G|$, $\Diamond_{k-1}(\alpha \wedge \neg\beta) \in \Gamma$. Since $W\alpha \notin \Gamma$ and Γ is maximally consistent, $M\neg\alpha \in \Gamma$. Thus by axiom 8, since $W\beta \in \Gamma$, $M\neg\alpha \in \Gamma$ and $\Diamond_{k-1}(\alpha \wedge \neg\beta) \in \Gamma$, $\Diamond_k(\neg\alpha \wedge \beta) \in \Gamma$. But this contradicts (2). Therefore it cannot be the case that $W\neg\alpha \in \Gamma$ and $W\alpha \notin \Gamma$. Therefore, $W\alpha \in \Gamma$. Note that we need not consider the case when $R^*_\alpha(\Gamma) \in \mathcal{O}$, since we are assuming $W\neg\alpha \in \Gamma$ which implies $R^*_{\neg\alpha}(\Gamma) \in \mathfrak{M}_0(\Gamma)$ and so $R^*_\alpha(\Gamma) \notin \mathcal{O}$.

Given the previous lemmas, completeness is straightforward. We give some of the details.

Lemma 44 (Truth Lemma) *For any formula α and any $\Gamma \in S^*$ we have*

$$\mathcal{M}^*, \Gamma \models \alpha \textit{ iff } \alpha \in \Gamma$$

Proof The proof is by induction on the complexity of α. The proof is trivial for the base case and boolean connectives. We will show the result for the modal formulas.

1. Suppose $\alpha = \Diamond_n \beta$. $\mathcal{M}^*, \Gamma \models \Diamond_n \beta$ iff by definition of truth in a model $|R^*_\beta(\Gamma)| > n$ iff by Theorem 40 $\Diamond_n \beta \in \Gamma$

2. Suppose $\alpha = W\beta$. $\mathcal{M}^*, \Gamma \models W\beta$ iff (by the induction hypothesis) $R^*_\beta(\Gamma) \in m^*(\Gamma)$ iff (by Theorem 43) $W\beta \in \Gamma$

Given the truth lemma for **GML** and **MJL**, the completeness theorem follows using a standard argument.

Theorem 45 (Canonical Model Theorem for MJL) *Let $\mathcal{M}^*$ be the canonical model described above, then for any formula α of majority logic, $\vdash_{\boldsymbol{MJL}} \alpha$ iff α is valid in $\mathcal{M}^*$.*

Chapter 7

Conclusion

7.1 Conclusion and Future Work

We have extended graded modal logic with an operator W that can express the concept of weak majority. In order to interpret W in a Kripke structure, we defined a majority space. A majority space extends the well-defined concept of a majority of a finite set to an infinite set. An axiom system was presented and shown to be both sound and complete.

Along the way, we looked at how to define the majority of an infinite set. Instead of trying to find a naturally occurring definition, we define a majority space which gives a lot of room in the definition of a majority subset of an infinite set. So if asked if the even numbers ($\mathbb{E}$) are a strict majority or a weak majority of the natural numbers ($\mathbb{N}$), we would answer that it depends on what is being modeled. On the one hand, it seems reasonable that $\mathbb{E}$

is a weak majority of $\mathbb{N}$. However, consider the following sequence of sets: $\{0, 2, 1\}, \{0, 2, 4, 1, 3\}, \{0, 2, 4, 6, 1, 3, 5\}, \ldots$. The first set has a strict majority of even numbers, and since each new set adds only one even number and one odd number, every element of this sequence has a strict majority of even numbers. The limit of this sequence is $\mathbb{N}$; and so if we think of $\mathbb{N}$ as being "constructed" by this sequence of sets, one would expect that $\mathbb{E}$ is a *strict* majority. Essentially, this implies that which sets constitutes a majority of natural numbers depends on the which well-ordering of the natural numbers is assumed.

The main technical question is the decidability of **MJL**. Since it was shown in [5] the graded modal logic has the finite model property, we expect that **MJL** will share this property.

We also point out that we cannot express the statement "among the worlds in which α is true, β is a majority " in our language. Such statements are often used when reasoning about candidates in an election. For example, among the Democratic registered voters, Kerry has the majority of their votes. We would like to extend the language of majority logic with an operator that can express such statements. A step in this direction would be to introduce a binary modality $\leq$, in which the intended meaning of $\alpha \leq \beta$ is α is true in "fewer" states than β.

We presented a detailed comparison between the concept of ultrafilters and the concept of majority. Ultrafilters are being used in a wide variety of applications especially in mathematics. One reason why we need the majority spaces is they have the democracy flavor in them, while ultrafilters have this dictatorship flavor.

Finally, we point to some possible applications of our logic. Although, the primary interest of this paper is technical, we feel that our framework can be used to reason about social software (see [21] for more information) such as voting systems [4]. In particular, This line of research will be pursued in a different paper.

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