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The Arithmetic and Geometry of Bianchi Groups

by

Wei-Chen Yao

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1997

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Introduction

In this thesis, we discuss the arithmetic and geometric properties of Bianchi groups. These groups appeared first in the work of Dirichlet on quadratic forms with complex coefficients and in the work of Hermite on what we now call binary Hermitian forms. The first example of a Bianchi group was explicitly introduced and studied by Picard: it is the group of 2×2 invertible matrices with entries in the ring of Gaussian integers. Replacing the Gaussian integers by the ring of integers in an imaginary quadratic field, we obtain the standard Bianchi groups.

Poincaré, in his study of non-Euclidean geometry in dimension three, made a beautiful and important discovery concerning the group theoretic properties of discrete subgroups of $PSL_2(\mathbb{C})$, e.g., their presentations, and the geometric structure of their fundamental domains in hyperbolic upper 3-space. Following up on these ideas, Bianchi initiated at the end of the nineteenth century the arithmetic and geometric study of general Bianchi groups and the explicit description of their fundamental domains with a view towards

describing generators and relations. Early in this century Humbert studied further the ideas of Bianchi and made important arithmetic contributions to the subject, e.g., the calculation of the volume as a value of an L -function.

After Humbert, very little attention was paid to Bianchi groups until they reappeared as important objects in the work of Bass, Serre, and Milnor on the *congruence subgroup problem*. The first in-depth study of Bianchi groups was carried out by Swan [Sw1] and Woodruff [W] in late sixties. Swan gave a general algorithm to determine fundamental domains for the Bianchi groups and to describe the generators and relations for these groups. From Swan's description, the fundamental domain in the hyperbolic model of 3-space is bounded by four vertical planes and by finitely many geodesic hemispheres centered on the complex plane.

Throughout this thesis, we let $K = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field, \mathcal{O} be its ring of integers, and d_K be the discriminant of K . We have

$$d_K = \begin{cases} -4d & d \equiv 1 \text{ or } 2 \pmod{4} \\ -d & d \equiv 3 \pmod{4} \end{cases}.$$

The following are the contents of this thesis. In chapter 1, we follow the ideas of Bianchi, Humbert, and Swan to construct a specific fundamental domain of Bianchi group and give an improved bound for Swan's algorithm. Throughout this thesis, we make the convention that *the fundamental domain* refers to the fundamental domain constructed in chapter 1.

Theorem A. *If a geodesic hemisphere S meets the fundamental domain, then the radius of S is greater than $Cd^{-3.5}$ where C is an absolute constant.*

Remarks. (a). In Swan's paper, Swan has a bound $C'd^{-4.5}$ for the radius of geodesic hemispheres where C' is an absolute constant.

(b). Theorem A implies that the computational complexity of Swan's algorithm is a polynomial in d . Riley[R] developed a program called "Poincare File" to calculate fundamental domains and presentations of Bianchi groups. Riley derived presentations for Bianchi groups with d up to 99.

(c). In Riley's examples, there are several fundamental domains of Bianchi groups that can not be determined (e.g. $d = 33$, $d_K = 132$). On the basis of these examples, we believe the bound we obtained in Theorem A for Swan's algorithm is best possible, but this requires an argument which is not worked out in this thesis.

The method used in the proof of Theorem A is based on a complicated diophantine property which relates singular points (cusps) and the ideal class groups of K .

The results of chapter 1 have applications in the following three areas:

1. Reduction theory of binary Hermitian forms.

Let

$$F(X, Y) = AX\bar{X} + B\bar{X}Y + \bar{B}X\bar{Y} + CY\bar{Y}$$

be a positive definite Hermitian form (i.e. A, C are positive rational integers, B is integer in K and \bar{B} is the conjugate of B . The discriminant $D = AC - B\bar{B} > 0$). Two forms F, F' are called *equivalent* if there exists $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$ such that $\sigma \cdot F(X, Y) = F(\alpha X + \beta Y, \gamma X + \delta Y) = F'(X, Y)$.

Theorem B. *For any fixed discriminant D , there are at most a finite number of non-equivalent binary Hermitian forms. In fact, the number of non-equivalent binary Hermitian forms is less than $Cd^{10.5}$ where C is a constant depending only on D .*

This theorem is proved in chapter 2.

2. Reduction theory of binary quadratic forms.

Let $f(x, y) = ax^2 + 2bxy + cy^2$ be a binary quadratic form over \mathcal{O} (i.e. $a, b, c \in \mathcal{O}$) and let $D = b^2 - ac$ be its discriminant. Two forms f, f' are called *equivalent* if there exists $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$ such that $f(\alpha x + \beta y, \gamma x + \delta y) = f'(x, y)$.

Theorem C. *For any fixed discriminant D , there are at most a finite number of non-equivalent binary quadratic forms over the ring of integers of K . In fact, the number of non-equivalent binary quadratic forms is less than $C'd^{10.5}$ where C' is a constant depending only on D .*

Dirichlet described the relation between the set of non-equivalent binary quadratic forms and the ideal class groups of quadratic extensions of the Gauss field. J. Hurwitz and G.B. Mathews generalized the notion of continued fractions to the Gauss field in order to find the fundamental unit of the quadratic extension. We generalize the idea of Dirichlet and Mathews to any imaginary quadratic field with class number 1. This method is described in chapter 2.

The complexity of calculating a reduced set of binary Hermitian forms and binary quadratic forms is determined by the inequalities that define a fundamental domain for a Bianchi group. Conversely, for a fixed imaginary field, the determination of all reduced Hermitian forms for all discriminants is equivalent to the determination of the fundamental domain for the Bianchi group.

3. Dilogarithm and Bloch groups.

Following Thurston's idea, we can triangulate the fundamental domain as

union of tetrahedra. The volume of the fundamental domain of Bianchi group $PSL_2(\mathcal{O})$ is equal to $\sum n_i D_2(z_i)$ where D_2 is the Bloch-Wigner Dilogarithm function and $z_i \in \mathbf{C}$. From this operation, we get an element $a = \sum n_i [z_i] \in T(\mathbf{C})$ (the pre-Bloch group). We have the following questions:

- (1) Is $a \in B(\mathbf{C})$ (the Bloch group)?
- (2) Let F be a finite extension of an imaginary quadratic field K , and suppose $\sum n_i [z_i] \in B(F)$ has infinite order. Is there any map $\phi : B(F) \rightarrow B(K)$ such that $\phi(\sum n_i [z_i])$ has infinite order?

The above questions arise from Borel's theorem and Lichtenbaum's Conjecture. From Borel's theorem, we have $\text{rank}(B(F)) = r_2$ where F is an algebraic number field and r_2 is the number of complex archimedean valuations. In particular, if F is an imaginary quadratic field, then $\text{rank}(B(F)) = 1$. We can state Lichtenbaum's conjecture as follows: let K be an imaginary quadratic field and let \mathcal{O} be its ring of integers. We have

$$\#K_2(\mathcal{O}) = \frac{|d_K|^{3/2}}{2D_2(b)} L(2, \chi),$$

where b is a generator of $B(K)$, d_K is the discriminant of K , and χ is the character corresponding to K . If the answer of the above questions are positive, we obtain an element a in $B(K)$ for any given imaginary quadratic field K (i.e. $a = rb$ for some integer r). If a turns out to be a generator of $B(K)$, then Lichtenbaum's conjecture gives a method for calculating the

order of $\#K_2(\mathcal{O})$. As, yet we are unable to get b from rb .

Chapter 1

Fundamental Domains of Bianchi Groups

1.1 Introduction

In this chapter, we discuss the fundamental domains of Bianchi groups and an algorithm developed by Swan for their constructions. For this algorithm, we estimate the computational complexity and give an improved bound.

Let $K = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field where d is a positive square-free integer, \mathcal{O} be the ring of integers of K , and d_K be the discriminant of K . Since the Gauss field and the Eisenstein field had been studied by Picard, Klein, and Bianchi very clearly, we assume $d \geq 5$ in this thesis. Let \mathbf{H} be the upper space of $\mathbf{C} \times \mathbf{R}$ (i.e. the set $\{(z, \zeta) \mid z \in \mathbf{C}, \zeta > 0\}$). The Mobius transformation on \mathbf{H} is the Poincare extension of Mobius transformation on \mathbf{C} (i.e. $\sigma \cdot P = (\alpha P + \beta)(\gamma P + \delta)^{-1}$ where $P = (z, \zeta)$, $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathbf{C})$. Here we have written $P = x + iy + j\zeta$ as a pure quaternion.) We can identify

\mathbf{H} with the homogeneous space $SU_2(\mathbf{C}) \backslash SL_2(\mathbf{C})$.

1.2 Fundamental Domain (Geometric Aspects)

We recall some elementary and well known results from the literature which will be used constantly in this thesis.

Lemma 1.1. *If $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathbf{C})$, then the action of σ on \mathbf{H} is given by $\sigma(z, \zeta) = (z', \zeta')$ where*

$$\zeta' = \frac{\zeta}{|\gamma z + \delta|^2 + \zeta^2 |\gamma|^2}$$

$$z' = \frac{(\overline{\gamma z} + \overline{\delta})(\alpha z + \beta) + \zeta^2 \overline{\gamma} \alpha}{|\gamma z + \delta|^2 + \zeta^2 |\gamma|^2}.$$

Proof. Since $\sigma \cdot P = (\alpha P + \beta)(\gamma P + \delta)^{-1}$, we can write

$$\sigma \cdot P = (\alpha(z + j\zeta) + \beta) \frac{\overline{\gamma z} + \gamma(-j\zeta) + \overline{\delta}}{|\gamma z + \delta|^2 + |\gamma|^2 \zeta^2}.$$

It is easy to see that

$$(z', \zeta') = \left(\frac{(\overline{\gamma z} + \overline{\delta})(\alpha z + \beta) + \zeta^2 \overline{\gamma} \alpha}{|\gamma z + \delta|^2 + \zeta^2 |\gamma|^2}, \frac{\zeta}{|\gamma z + \delta|^2 + \zeta^2 |\gamma|^2} \right).$$

Lemma 1.2. *The Riemannian metric $ds^2 = \zeta^{-2}(dx^2 + dy^2 + d\zeta^2)$ on \mathbf{H} , where $z = x + iy$, is invariant under the action of $SL_2(\mathbf{C})$.*

Proof. $SL_2(\mathbf{C})$ is generated by $\sigma_1 = \begin{Bmatrix} 1 & t \\ 0 & 1 \end{Bmatrix}$, $\sigma_2 = \begin{Bmatrix} 0 & -1 \\ 1 & 0 \end{Bmatrix}$, $\sigma_3 = \begin{Bmatrix} a & 0 \\ 0 & a^{-1} \end{Bmatrix}$. By Lemma 1.1, we have $\sigma_1 \cdot (z, \zeta) = (z + t, \zeta)$, $\sigma_2 \cdot (z, \zeta) = (\frac{\bar{z}}{|z|^2 + \zeta^2}, \frac{\zeta}{|z|^2 + \zeta^2})$, and $\sigma_3 \cdot (z, \zeta) = (\frac{1}{a^2}z, |\frac{1}{a^2}\zeta|)$. It is sufficient to check that ds^2 is invariant under these three transformations. For instance, we verify that $d(\sigma_1 \cdot s)^2 = \zeta^{-2}(d|z + t|^2 + d\zeta^2) = \zeta^{-2}(dx^2 + dy^2 + d\zeta^2) = ds^2$ since t is a constant. We also see that $d(\sigma_2 \cdot s)^2 = \frac{(|z|^2 + \zeta^2)^2}{\zeta^2} (\frac{dx^2}{(|z|^2 + \zeta^2)^2} + \frac{dy^2}{(|z|^2 + \zeta^2)^2} + \frac{d\zeta^2}{(|z|^2 + \zeta^2)^2}) = ds^2$ and $d(\sigma_3 \cdot s)^2 = |a|^2 \zeta^{-2} \frac{1}{|a|^2} (dx^2 + dy^2 + d\zeta^2) = ds^2$.

The group $PSL_2(\mathbf{C}) = SL_2(\mathbf{C})/\{\pm I\}$ is the group of all orientation preserving isometries for the hyperbolic metric. We describe the elements of $PSL_2(\mathbf{C})$ by matrices $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathbf{C})$ identifying σ and $-\sigma$. Every discrete subgroup $\Gamma \in PSL_2(\mathbf{C})$ acts discontinuously on \mathbf{H} . In this thesis, we concern ourselves mainly with the Bianchi groups $PSL_2(\mathcal{O})$.

Definition. Let Γ be a discrete subgroup of $PSL_2(\mathbf{C})$. A fundamental domain \mathcal{D} for Γ is an open subset of \mathbf{H} satisfying the following conditions.

- (1) For every $P \in \mathbf{H}$, there is a $\sigma \in \Gamma$ with $\sigma \cdot P \in \overline{\mathcal{D}}$.
- (2) The members of $\{\sigma \cdot \mathcal{D} : \sigma \in \Gamma\}$ are mutually disjoint.

Let $\{1, w\}$ be a basis for \mathcal{O} as a \mathbf{Z} -module and write $w = w_1 + iw_2$. Let F be the set of all $z = x + iy$ with $-w_2/2 \leq y \leq w_2/2$ and $-1/2 \leq x \leq 1/2$. If $\mu, \lambda \in \mathcal{O}$, $(\mu, \lambda) = \mathcal{O}$, let $S_{\mu, \lambda}$ be the hemisphere in \mathbf{H} given by $|\mu z + \lambda|^2 + |\mu|^2 \zeta^2 = 1$. Let \mathcal{B} be the set of points in \mathbf{H} which lie above or on

all $S_{\mu,\lambda}$ (i.e. $\mathcal{B} = \{(z, \zeta) \in \mathbf{H} : |\mu z + \lambda|^2 + |\mu|^2 \zeta^2 \geq 1 \text{ for all } (\mu, \lambda) = \mathcal{O}\}$). We now introduce the fundamental domains of Bianchi groups. The structure of the fundamental domain was indicated by Bianchi and studied in greater detail by Swan.

Proposition 1.3. *The set $\mathcal{D} = \{(z, \zeta) : z \in F\} \cap \mathcal{B}$ is a fundamental domain of the Bianchi group $PSL_2(\mathcal{O})$.*

Remark. (a) Here, we abuse of language of fundamental domain, since in the last proposition we should say the closure of the fundamental domain.

(b) Throughout this thesis, we make the convention that *the fundamental domain* refers to \mathcal{D} .

Before we prove Proposition 1.3, we need the following Lemmas.

Lemma 1.4. *A point $(z, \zeta) \in \mathbf{H}$ lies in \mathcal{B} if and only if for every $\sigma \in SL_2(\mathcal{O})$ we have $\sigma \cdot (z, \zeta) = (z', \zeta')$ with $\zeta' \leq \zeta$.*

Proof. Let $(z, \zeta) \in \mathcal{B}$ and $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$. By Lemma 1.1, we can write $\zeta' = \frac{\zeta}{|\gamma z + \delta|^2 + \zeta^2 |\gamma|^2}$. Since $(z, \zeta) \in \mathcal{B}$, $|\gamma z + \delta|^2 + \zeta^2 |\gamma|^2 \geq 1$. We have $\zeta' \leq \zeta$.

Suppose $\zeta' \leq \zeta$ for all $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$, then $|\gamma z + \delta|^2 + \zeta^2 |\gamma|^2 \geq 1$ for all $(\gamma, \delta) = \mathcal{O}$. We have $(z, \zeta) \in \mathcal{B}$.

Lemma 1.5. Let $\sigma = \begin{Bmatrix} \alpha & \beta \\ \mu & \lambda \end{Bmatrix} \in SL_2(\mathcal{O})$ with $\mu \neq 0$. Then $\mathcal{B} \cap \sigma^{-1}\mathcal{B} = \mathcal{B} \cap S_{\mu,\lambda}$.

Proof. Let $(z, \zeta) \in \mathcal{B} \cap \sigma^{-1}\mathcal{B}$ and $(z', \zeta') = \sigma \cdot (z, \zeta)$. By Lemma 1.4, we have $\zeta' = \zeta$. So we have $|\mu z + \lambda|^2 + \zeta^2 |\mu|^2 = 1$ (i.e. $(z, \zeta) \in S_{\mu,\lambda}$).

Proof of Proposition 1.3. First, let $P = (z, \zeta)$ be any point on \mathbf{H} , and note that $|\mu P + \lambda|^2$ is a positive definite Hermitian form for $\mu, \lambda \in \mathcal{O}, (\mu, \lambda) = \mathcal{O}$. There are μ, λ such that $|\mu P + \lambda|^2$ is a minimum (i.e. $|\mu z + \lambda|^2 + |\mu|^2 \zeta^2$ is a minimum), hence there is a $\sigma = \begin{Bmatrix} \alpha & \beta \\ \mu & \lambda \end{Bmatrix} \in SL_2(\mathcal{O})$ such that $(z', \zeta') = \sigma \cdot (z, \zeta)$ with ζ' maximum. By Lemma 1.4, $(z', \zeta') \in \mathcal{B}$. Let $s \in \mathcal{O}$ be such that $z' - s = z'' \in F$ and $T = \begin{Bmatrix} 1 & -s \\ 0 & 1 \end{Bmatrix}$. We have $T \cdot (z', \zeta') = (z'', \zeta') \in \mathcal{D}$. From the above argument, it follows that for every point (z, ζ) in H , we can find an element $\sigma \in SL_2(\mathcal{O})$ such that $\sigma \cdot (z, \zeta) \in \mathcal{D}$.

Now we claim, $\sigma\mathcal{B} = \mathcal{B}$ if and only if $\sigma \in \Phi = \left\{ \begin{Bmatrix} \alpha & \beta \\ 0 & \delta \end{Bmatrix} \in SL_2(\mathcal{O}) \right\}$ and if $\sigma \notin \Phi$, then $\text{int}\mathcal{B} \cap \sigma \cdot \text{int}\mathcal{B} = \emptyset$.

We can easily see that if $\sigma \in \Phi$, then $\sigma = \begin{Bmatrix} 1 & s \\ 0 & 1 \end{Bmatrix}$ or $\begin{Bmatrix} -1 & s \\ 0 & -1 \end{Bmatrix}$ for some $s \in \mathcal{O}$. So, $\sigma \cdot S_{\mu,\lambda} = S_{\mu,\lambda+s\mu}$ only permutes the equations defining \mathcal{B} . If $\sigma = \begin{Bmatrix} \alpha & \beta \\ \mu & \lambda \end{Bmatrix} \notin \Phi$, then $\mathcal{B} \cap \sigma^{-1}\mathcal{B} = \mathcal{B} \cap S_{\mu,\lambda}$ by Lemma 1.5. This lies on the boundary of \mathcal{B} .

From the above argument, we can say \mathcal{D} is a fundamental domain of the Bianchi group.

1.3 Fundamental Domain(Arithmetic Aspects)

In this section, we study some properties of $PSL_2(\mathcal{O})$ which depend heavily on the arithmetic of the ring \mathcal{O} .

Definition. We call $s \in \mathbb{C}$ a singular point of the fundamental domain if we cannot find $\mu, \lambda \in \mathcal{O}$ with $(\mu, \lambda) = \mathcal{O}$ and $|\mu s - \lambda| < 1$ (i.e. there is no $S_{\mu, \lambda}$ such that s is lower than $S_{\mu, \lambda}$).

Definition. Let \mathcal{A} be any ideal of \mathcal{O} . We say an element $\beta \in \mathcal{A}$ is minimal if $|\beta|$ is minimal among all $|\alpha|$ for $\alpha \in \mathcal{A}, \alpha \neq 0$.

Proposition 1.6. Let $\mathcal{A}_0 = \mathcal{O}, \mathcal{A}_1, \dots, \mathcal{A}_{h-1}$ represent the h ideal classes of K . For each $i = 1, \dots, h-1$, write $\mathcal{A}_i = (\beta, \alpha)$ in all possible ways with β minimal in \mathcal{A}_i . The singular points are of the form α/β . In this way we get all possible singular points.

Proof. If $z \in \mathbb{C} \setminus K$, then there are infinitely many $\lambda, \mu \in \mathcal{O}$ with $(\lambda, \mu) = \mathcal{O}$ such that

$$\left| z - \frac{\lambda}{\mu} \right| \leq \frac{c}{|\mu|^2}$$

where c is a constant only depending on K [Sw1]. So, z is not a singular point.

Suppose now $s \in K$. Write $s = \alpha/\beta$ with $\alpha, \beta \in \mathcal{O}$. Let $\mathcal{A} = (\alpha, \beta)$. If $(\gamma, \delta) = \mathcal{A}$, then there are $\mu, \lambda \in \mathcal{O}$ with $(\lambda, \mu) = \mathcal{O}$ such that $\mu\alpha - \lambda\beta = \gamma$

(see Lemma 1.16). Therefore $|\mu s - \lambda| = |\gamma|/|\beta|$. If $|\beta|$ is not minimal, then s is not a singular point.

If $(\alpha, \beta) = \mathcal{A}_0$, then $|\beta \frac{\alpha}{\beta} - \alpha| = 0 < 1$. So, $\frac{\alpha}{\beta}$ is not a singular point. Let $(\beta, \alpha) = \mathcal{A}_i$ for $i > 0$. If $(\mu, \lambda) = \mathcal{O}$, then $|\mu \frac{\alpha}{\beta} - \lambda| = \frac{|\mu\alpha - \lambda\beta|}{|\beta|} \geq 1$. Because $\mu\alpha - \lambda\beta \in \mathcal{A}_i$ and β is minimal. Note that $\mu\alpha - \lambda\beta \neq 0$ (otherwise $\alpha = t\lambda, \beta = t\mu$ for some $t \in K$. Then $\mathcal{A}_i = t\mathcal{O}$, a contradiction.).

Now we check the uniqueness. Suppose $\frac{\alpha}{\beta} = \frac{\gamma}{\delta}, (\beta, \alpha) = \mathcal{A}_i, (\delta, \gamma) = \mathcal{A}_j$ with $i, j \neq 0$ and β, δ minimal in $\mathcal{A}_i, \mathcal{A}_j$ respectively, then there is $t \in K$ such that $\gamma = t\alpha, \delta = t\beta$. Therefore $\mathcal{A}_i = t\mathcal{A}_j$. We have $i = j$ and t is a unit. Since we only consider $d \geq 5, t = \pm 1$.

For enumerating all singular points, we state the following well-known theorem about the representatives of ideal classes. Let $\mathcal{O} = \mathbf{Z} + \mathbf{Z}w$.

Definition. Let I be the set of all pairs (q, a) with $q, a \in \mathbf{Z}, q > 0, -q/2 < a \leq q/2, q^2 \leq N(a+w)$ and $q \mid N(a+w)$. For $(q, a) \in I$, let $\mathcal{A}_{q,a} = (q, a+w)$.

Theorem 1.7. (1) Every ideal class is represented by some $\mathcal{A}_{q,a}$ for $(q, a) \in I$ and the ideal class group is in one-to-one correspondence with the $\mathcal{A}_{q,a}$, except for the identification $\mathcal{A}_{q,a} \approx \mathcal{A}_{q,b}$ which occurs if $q^2 = N(a+w)$ and $b = -a$ for $d \equiv 1$ or $2 \pmod{4}, b = 1 - a$ for $d \equiv 3 \pmod{4}$.

(2) The minimal elements of $\mathcal{A}_{q,a}$ are just $\pm q$ unless $q^2 = N(a+w)$ when they are $\pm q$ and $\pm(a+w)$. In fact, $\mathcal{A}_{q,a} = \mathbf{Z}q + \mathbf{Z}(a+w)$ as a \mathbf{Z} module.

Remark. Theorem 1.7 states the connection between the ideal classes of a quadratic field K and a set of reduced binary quadratic forms of discriminant d_K . The calculation of ideal classes based on this relation can be found in [He §53].

Corollary 1.8. *A complete set of singular points of K is given by the set of all $p(a+w)/q + \gamma$, where $(q, a) \in I$, $q \neq 1, \gamma \in \mathcal{O}$, and p runs over a set of representatives, modulo q of the integers (in \mathbf{Z}) relatively prime to q .*

Proof. By (2) of Theorem 1.7, $\pm q, \pm(a+w)$ are the only minimal element of $\mathcal{A}_{q,a}$. The singular points are of the form $\frac{\alpha}{q}, \frac{\alpha}{(a+w)}$ for some $\alpha \in \mathcal{A}_{q,a}$. From Theorem 1.7, we can write $\alpha = xq + y(a+w)$ where $x, y \in \mathbf{Z}$. For the first form, we have $\frac{p(a+w)}{q} + x + y'(a+w)$. In the second form, we have $y + \frac{xq}{(a+w)} = y + \frac{xq(a-w)}{q^2} = y' + x'(a+w) + \frac{p(a+w)}{q}$.

We now check that the points obtained are all distinct mod \mathcal{O} . Suppose $\frac{p(a+w)}{q}, \frac{p'(a'+w)}{q'}$ are two points. If $\frac{p(a+w)}{q} = \frac{p'(a'+w)}{q'} + x + yw, x, y \in \mathbf{Z}$, then $\frac{p'}{q'} \equiv \frac{p}{q} \pmod{1}$. Since $(p, q) = 1 = (p', q')$, we have $q' = q$ and $p' \equiv p \pmod{q}$. Also $\frac{p'a'}{q'} \equiv \frac{pa}{q} \pmod{1}$ imply that $p'a' \equiv pa \pmod{q}$ or $pa' \equiv pa \pmod{q}$. But $(p, q) = 1$, we have $a' \equiv a \pmod{q}$ (i.e. $a' = a, p' = p$).

Lemma 1.9. (1) *If $s \in \mathbf{C}$ is not a singular point, there is a neighborhood U of s and $\epsilon > 0$ so that every $(z, \zeta) \in \mathcal{B}$ with $z \in U$ has $\zeta > \epsilon$.*

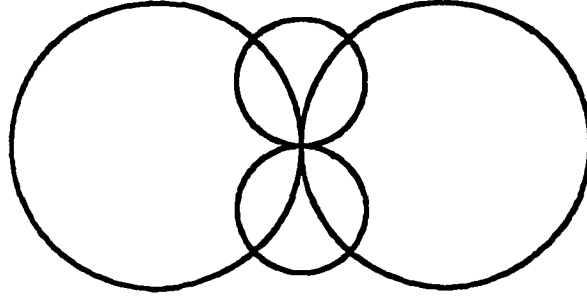
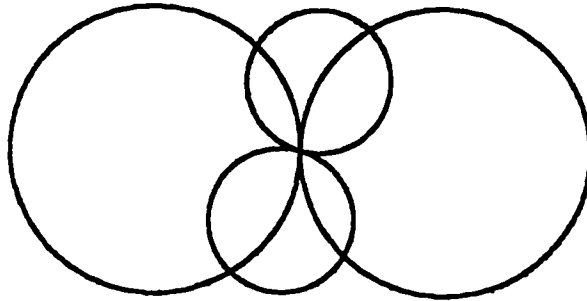
(2) *If $s \in \mathbf{C}$ is a singular point, there is neighborhood U of s and $\epsilon > 0$ so that every $(z, \zeta) \in \mathcal{B}$ with $z \in U$ has $\zeta > \epsilon |z - s|^{1/2}$.*

Proof. We follow Swan's method, except for part (2) we use the form of singular points as given in Corollary 1.8 to get a better bound.

(1) Choose $\mu, \lambda \in \mathcal{O}$ so $|\mu s - \lambda| < 1$ and $(\mu, \lambda) = \mathcal{O}$. Let U be a neighborhood of s and let $0 < \theta < 1$ so that for $z \in U$ we have $|\mu z - \lambda| < \theta$. Since all points of \mathcal{B} satisfy $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geq 1$ we can take $\epsilon = |\mu|^{-1}(1 - \theta^2)^{1/2}$.

(2) Suppose now $s = \frac{\alpha}{\beta}$ is singular. Let $1, w$ be a basis for \mathcal{O} . Then we have $|\mu s - \lambda| = 1$ and $(\mu, \lambda) = \mathcal{O}$ for $\mu = \beta^2, \lambda = \alpha\beta \pm 1$ and for $\mu = w\beta^2, \lambda = w\alpha\beta \pm 1$. The two circles $|\beta^2 z - \alpha\beta \pm 1| = 1$ pass through $z = s$ and are tangent along the line $\Re(\beta^2 z - \alpha\beta) = 0$. The two circles $|w\beta^2 z - w\alpha\beta \pm 1| = 1$ also pass through $z = s$ and are tangent along the line $\Re(w\beta^2 z - w\alpha\beta) = 0$. Since the lines are distinct, each of the first pair of circles overlaps each of the second pair. Therefore a neighborhood of s may be divided into 4 parts each lying inside one of circles and bounded by two secants passing through s (Figure 1.1 and Figure 1.2). We consider the following two cases.

(i) For $d \equiv 1$ or $2 \pmod{4}$, $w = \sqrt{-d}$. By Corollary 3.4, we can write

Figure 1.1: $d \equiv 1$ or $2 \pmod{4}$ Figure 1.2: $d \equiv 3 \pmod{4}$

$s = \frac{p(a+w)}{q}$. The four circles have the equations

$$\left(x - \frac{pqa \pm 1}{q^2}\right)^2 + \left(y - \frac{p\sqrt{d}}{q}\right)^2 = \frac{1}{q^4} \text{ and } \left(x - \frac{pa}{q}\right)^2 + \left(y - \frac{pqd \pm 1}{q^2\sqrt{d}}\right)^2 = \frac{1}{dq^4}.$$

(ii) For $d \equiv 3 \pmod{4}$, $w = \frac{1}{2}(-1 + \sqrt{-d})$. As in (i), we can write $s = \frac{p(a+w)}{q}$. The four circles have the equations

$$\left(x - \frac{2pqa - pq \pm 2}{2q^2}\right)^2 + \left(y - \frac{p\sqrt{d}}{2q}\right)^2 = \frac{1}{q^4}$$

and

$$\left(x - \frac{(1+d)pq(2a-1) \mp 4}{2q^2(1+d)}\right)^2 + \left(y - \frac{[(1+d)pq \mp 4]\sqrt{d}}{2q^2(1+d)}\right)^2 = \frac{4}{q^4(1+d)}.$$

For each of these circles, choose coordinates in the z plane so that the circle has the equation $(x - r)^2 + y^2 = r^2$, the point s having the coordinate $(0, 0)$. The secants will have equations $y = m_1x$ and $y = m_2x$. Let $M = \max\{|m_1|, |m_2|\}$. The inequality $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geq 1$ becomes $\zeta^2 \geq 2rx - x^2 - y^2$. So, $\zeta^2 \geq 2rx - x^2 - M^2x^2 > \epsilon^2(x^2 + y^2)^{1/2} = \epsilon^2|z|$ for $|x|$ very small, since $|y| \leq M|x|$ are satisfied by the points between the two secants.

For $d \equiv 1$ or $2 \pmod{4}$, we put

$$U = \{z \in \mathbf{C} \mid |z - s| < \frac{2d + 1}{(d + 1)^{3/2}q^2}\},$$

then $\epsilon = 1/q(1 + d)^{3/4}$.

For $d \equiv 3 \pmod{4}$, we put

$$U = \{z \in \mathbf{C} \mid |z - s| < \frac{4(2d - 1)}{q^2(d^2 + 10d + 9)}\},$$

then $\epsilon = \sqrt{2}/qd^{1/4}(d^2 + 10d + 9)^{1/4}$.

Remark. The idea of the proof of Lemma 1.9 is to pick spheres that pass through singular points. This choice of spheres is the best possible we can make.

In addition to the four vertical planes which enclose the fundamental domain \mathcal{D} , the lower boundary is covered by a finite number of geodesic hemispheres $S_{\mu,\lambda}$. This is the content the following result which in addition provides an estimate for the number of the spheres.

Theorem 1.10. *There are only finitely many $\mu, \lambda \in \mathcal{O}$ with $(\mu, \lambda) = \mathcal{O}$ such that $\mathcal{D} \cap S_{\mu, \lambda} \neq \emptyset$.*

Proof. (1) For a given μ , $\mathcal{D} \cap S_{\mu, \lambda} = \emptyset$ unless the distance from $\frac{\lambda}{\mu}$ to $F \leq \frac{1}{|\mu|}$. Therefore if \mathbf{d} is the diameter of F , $\mathcal{D} \cap S_{\mu, \lambda} = \emptyset$ unless $\frac{|\lambda|}{|\mu|} \leq \mathbf{d} + \frac{1}{|\mu|}$. It is easy to get $\mathbf{d} = \sqrt{1 + d_K/4}$. Hence there are only finite number of λ satisfying this.

(2) Let s_1, \dots, s_n be the singular points in F . Let U_i, ϵ_i satisfying (2) of Lemma 1.7. Let $\epsilon = \min \epsilon_i$ and $V_i = \{z : |z - s_i| < \frac{\epsilon_i^2}{2}\}$.

(a) Since $F \setminus \cup V_i$ compact, Lemma 1.7 (1) implies that there is $\eta > 0$ such that $(z, \zeta) \in \mathcal{D}, z \in F \setminus \cup V_i$ then $\zeta > \eta$. If $|\mu| > \eta^{-1}$, then the highest point of $S_{\mu, \lambda}$ has $\zeta = \frac{1}{|\mu|} < \eta^{-1}$.

(b) Suppose $z \in V_i$. Then $\zeta > \epsilon|z - s_i|^{1/2}$. If $(z, \zeta) \in S_{\mu, \lambda}$, then $|z + \frac{\lambda}{\mu}|^2 + \zeta^2 = \frac{1}{|\mu|^2}$. Therefore $|z + \frac{\lambda}{\mu}|^2 + \epsilon^2|z - s_i| < \frac{1}{|\mu|^2}$. In particular, $|z + \frac{\lambda}{\mu}|^2 < \frac{1}{|\mu|^2}$. Since s_i is a singular point, $|\mu s_i + \lambda| \geq 1$ (i.e. $|s_i + \frac{\lambda}{\mu}| \geq \frac{1}{|\mu|}$). But $|s_i + \frac{\lambda}{\mu}| \leq |z + \frac{\lambda}{\mu}| + |z - s_i|$, and we have

$$\begin{aligned} \frac{1}{|\mu|^2} &\leq (|z + \frac{\lambda}{\mu}| + |z - s_i|)^2 = |z + \frac{\lambda}{\mu}|^2 + 2|z - s_i||z + \frac{\lambda}{\mu}| + |z - s_i|^2 \\ &< \frac{1}{|\mu|^2} - \epsilon^2|z - s_i| + 2|z - s_i|\frac{1}{|\mu|} + |z - s_i|^2 \end{aligned}$$

or

$$0 < -\epsilon^2|z - s_i| + 2\frac{1}{|\mu|}|z - s_i| + |z - s_i|^2.$$

¹The effective estimate of η will be obtained in Theorem 1.17.

So $|z - s_i| \neq 0$ and $0 < -\epsilon^2 + 2\frac{1}{|\mu|} + |z - s_i|$. Since $z \in V_i$, $|z - s_i| < \frac{\epsilon^2}{2}$. Therefore $|\mu| < 4\epsilon^{-2}$.

From (a) and (b), there are only finitely many choices for μ ².

Theorem 1.11. *The fundamental domain is symmetric about x, y axis.*

Proof. Let (z, ζ) be a point in the fundamental domain.

(a) If (\bar{z}, ζ) is not in the fundamental domain then there is a hemisphere $S_{\mu, \lambda}$ such that (\bar{z}, ζ) is lower than the sphere (i.e. $|\bar{z} + \frac{\lambda}{\mu}|^2 + \zeta^2 < \frac{1}{|\mu|^2}$). So we have the inequality, $|z + \frac{\bar{\lambda}}{\mu}|^2 + \zeta^2 < \frac{1}{|\mu|^2}$. (i.e. (z, ζ) is lower than the hemisphere $S_{-\mu, \bar{\lambda}}$). This contradicts our assumption, hence the fundamental domain is symmetric about x-axis.

(b) If $(-z, \zeta)$ is not in the fundamental domain then there is a hemisphere $S_{\mu, \lambda}$ such that $(-z, \zeta)$ is lower the sphere (i.e. $|z - \frac{\lambda}{\mu}|^2 + \zeta^2 < \frac{1}{|\mu|^2}$). So we can say that (z, ζ) is lower than $S_{-\mu, \lambda}$. This contradicts our assumption, and the fundamental domain is symmetric about the line $\{(x, y, z) | x = y = 0\}$. From (a) and (b) we can say that the fundamental domain is symmetric about y-axis.

By Theorem 1.11, we only need to consider $\frac{1}{4}$ of the fundamental domain instead of the whole domain.

Before we begin to sketch Swan's algorithm. We will review methods

²This estimate will be given in part (c) of Swan's algorithm.

used by Woodruff in his thesis [W]. Let h be the class number of K and let $\mathcal{A}_i = (\gamma_i, \delta_i), i = 1, \dots, h$, be a set of representatives for these classes where we may assume that $\gamma_1 = 0$ and $\delta_1 = 1$. Let α_i and β_i be chosen from K so that $\sigma_i = \begin{Bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{Bmatrix}, \det(\sigma_i) = 1$, with $\sigma_1 = I_2$. Let $L = \cup \sigma_i \cdot SL_2(\mathcal{O})$.

Lemma 1.12. *If μ and λ are integers of \mathcal{O} with $(\mu, \lambda) \neq (0, 0)$, then there is a transformation σ_0 in L such that $\sigma_0 = \begin{Bmatrix} \alpha_0 & \beta_0 \\ \mu_0 & \lambda_0 \end{Bmatrix}, \mu_0 = n\mu, \lambda_0 = n\lambda, |n| \leq q$ where the constant q depends only on K .*

Proof. The ideal (μ, λ) is equivalent to \mathcal{A}_i for some i . Thus there are two non-zero integers n_1, n_2 in K such that $(n_1)(\mu, \lambda) = (n_2)(\gamma_i, \delta_i)$. Hence $N(n_1)N(\mu, \lambda) = N(n_2)N(\mathcal{A}_i)$. So we have

$$\frac{N(n_1)}{N(n_2)} = \frac{N(\mathcal{A}_i)}{N(\mu, \lambda)} \leq N(\mathcal{A}_i).$$

Let $q = \max N(\mathcal{A}_i)$, we have $|\frac{n_1}{n_2}| \leq N(\mathcal{A}_i) \leq q$. By Lemma 1.16, there is a transformation $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$ such that

$$(n_1\mu, n_1\lambda) = \sigma \cdot (n_2\gamma_i, n_2\delta_i)$$

$$\text{i.e.} \quad \begin{cases} n_1\mu = \alpha n_2\gamma_i + \gamma n_2\delta_i \\ n_1\lambda = \beta n_2\gamma_i + \delta n_2\delta_i \end{cases}.$$

Let $\sigma_0 = \sigma_i\sigma$ so that we have

$$\begin{cases} \mu_0 = \alpha\gamma_i + \gamma\delta_i \\ \lambda_0 = \beta\gamma_i + \delta\delta_i \end{cases}.$$

Let $n = \frac{n_1}{n_2}$, we note that $\mu_0 = n\mu, \lambda_0 = n\lambda$ where $|n| \leq q$.

Lemma 1.13. *To each (z, ζ) in \mathbf{H} , there is a pair of integers $(\mu, \lambda) \neq (0, 0)$ in K for which the inequality*

$$|\mu|^2 \zeta^2 + |\mu z + \lambda|^2 \leq \frac{\sqrt{|d_K|} \zeta}{\sqrt{2}}$$

holds.

Proof. Let $\{1, w\}$ be a basis for \mathcal{O} . We write $\mu = \mu_1 + \mu_2 w$, $\lambda = \mu_3 + \mu_4 w$ where μ_i are rational integers. Then $\phi(\mu_1, \mu_2, \mu_3, \mu_4) = |\mu|^2 \zeta^2 + |\mu z + \lambda|^2 = |\mu|^2(|z|^2 + \zeta^2) + \mu \bar{\lambda} z + \lambda \bar{\mu} \bar{z} + |\lambda|^2$ is a positive definite quadratic form in μ_i . Let $a_{i,j}$ be the coefficient of $\mu_i \mu_j$ and let A be the corresponding matrix of this quadratic form (i.e. $A = (b_{i,j})$, $b_{i,i} = a_{i,i}$ and $b_{i,j} = a_{i,j}/2$ for $i \neq j$). In particular, we can write

$$A = \left\{ \begin{array}{cccc} \rho^2 & 0 & x & \sqrt{dy} \\ 0 & d\rho^2 & -\sqrt{dy} & dx \\ x & -\sqrt{dy} & 1 & 0 \\ \sqrt{dy} & dx & 0 & d \end{array} \right\} \text{ for } d \equiv 1 \text{ or } 2 \pmod{4},$$

and

$$A = \left\{ \begin{array}{cccc} \rho^2 & \rho^2/2 & x & (x + \sqrt{dy})/2 \\ \rho^2/2 & \rho^2(1+d)/4 & (x - \sqrt{dy})/2 & x(1+d)/4 \\ x & (x - \sqrt{dy})/2 & 1 & 1/2 \\ (x + \sqrt{dy})/2 & x(1+d)/4 & 1/2 & (1+d)/4 \end{array} \right\}$$

for $d \equiv 3 \pmod{4}$ where $z = x + iy$ and $\rho^2 = x^2 + y^2 + \zeta^2$. It is easy to see that $\det(A) = \frac{|d_K|^2 \zeta^4}{16}$.

The theorem of Korkine and Zolotareff ([K-Z], page 583) can be stated as follows: *Let $\phi(x_1, x_2, x_3, x_4) = \sum a_{i,j} x_i x_j$ be a positive definite quadratic form*

with the corresponding matrix A where x_i are integers. There are n_1, n_2, n_3, n_4 such that $\phi(n_1, n_2, n_3, n_4) \leq \sqrt[4]{4\det(A)}$.

From this theorem, we can find μ_i such that $\phi(\mu_1, \mu_2, \mu_3, \mu_4) \leq \sqrt[4]{4\det(A)}$.

In the other word, we can find μ, λ such that

$$0 < |\mu|^2\zeta^2 + |\mu z + \lambda|^2 \leq \frac{\sqrt{|d_K|\zeta}}{\sqrt{2}}.$$

Remark. In Woodruff's paper, the inequality is

$$0 < |\mu|^2\zeta^2 + |\mu z + \lambda|^2 \leq \frac{|d_K|\zeta}{\sqrt{2}}.$$

Note that we have reduced the exponent of d_K to $1/2$.

Proposition 1.14. For each $(z, \zeta) \in \mathbf{H}$, there is a $\sigma \in L$ such that $(z', \zeta') = \sigma \cdot (z, \zeta)$ with $\zeta' \geq \sqrt{2}/(q^2\sqrt{|d_K|})$ where $q = \max\{\sqrt{\text{Norm}(\mathcal{A}_i)}\}$.

Proof. By Lemma 1.13, there is a pair (μ, λ) such that

$$|\mu|^2\zeta^2 + |\mu z + \lambda|^2 \leq \frac{\sqrt{|d_K|\zeta}}{\sqrt{2}}.$$

By Lemma 1.12, there is a $\sigma_0 = \begin{Bmatrix} \alpha_0 & \beta_0 \\ \mu_0 & \lambda_0 \end{Bmatrix} \in L$ such that $\mu_0 = n\mu, \lambda_0 = n\lambda, |n| < q$. So, we have the following inequality

$$\zeta' = \frac{\zeta}{|\mu_0|^2\zeta^2 + |\mu_0 z + \lambda_0|^2} \geq \frac{\zeta}{q^2(|\mu|^2\zeta^2 + |\mu z + \lambda|^2)} \geq \frac{\sqrt{2}}{q^2\sqrt{|d_K|}}.$$

1.4 Swan's Algorithm

We begin to describe the algorithm for determining the fundamental domain for given d . Suppose we are given a set $\alpha_i = \lambda_i/\mu_i \in K$ with $(\mu_i, \lambda_i) = \mathcal{O}$, $i = 1, \dots, n$. Let $S(\alpha_i) = S_{\mu_i, \lambda_i}$ and $\mathcal{B}(\alpha_1, \dots, \alpha_n) = \{(z, \zeta) \in \mathbf{H} \mid (z, \zeta) \text{ lying above or on all } S(\alpha_i + \gamma) \text{ for } \gamma \in \mathcal{O}\}$. As in proof of Proposition 1.3, we have $\mathcal{B}(\alpha_1, \dots, \alpha_k) = \mathcal{B}$ for some finite number k . Swan showed that $\mathcal{B}(\alpha_1, \dots, \alpha_k) = \mathcal{B}$ if and only if no vertex of $\partial\mathcal{B}(\alpha_1, \dots, \alpha_k)$ can be strictly below any $S_{\mu, \lambda}$ with $(\mu, \lambda) = \mathcal{O}$. [Sw1, Proposition 8.4] Hence the only question left is to find a bound for $|\mu_i|$ and $|\lambda_i|$.

(a) As proof of Theorem 1.10, $|\lambda|$ is bounded for any giving μ .

(b) Let s_1, \dots, s_n be the singular points in F . By Corollary 1.8, every singular point s_i has the form $\frac{p(a_i+w)}{q_i}$. Let

$$V_i = \{z \in \mathbf{C} \mid |z - s_i| < \frac{1}{2q_i^2(1+d)^{3/2}}\} \text{ for } d \equiv 1 \text{ or } 2 \pmod{4}$$

and

$$V_i = \{z \in \mathbf{C} \mid |z - s_i| < \frac{1}{q_i^2 \sqrt{d} \sqrt{d^2 + 10d + 9}}\} \text{ for } d \equiv 3 \pmod{4}.$$

By Theorem 10, if $S_{\mu, \lambda} \cap \{(z, \zeta) \mid z \in V_i\} \neq \emptyset$ then $|\mu| < 16q^2(1+d)$ for $d \equiv 1$ or $2 \pmod{4}$ and $|\mu| < 2q^2 \sqrt{d^2 + 10d + 9}$ for $d \equiv 3 \pmod{4}$ where $q = \max q_i$.

We state the following results before we finish the algorithm.

Lemma 1.15. *If z is any complex number not in K and M is a positive integer, there are $\mu, \lambda \in \mathcal{O}$ such that $|\mu z - \lambda| \leq \mathbf{d}M^{-1}$ with $|\mu| \leq AM$ where A, \mathbf{d} are constants depending only on K .*

Proof. Let $\{1, w\}$ be a basis for \mathcal{O} as a module over \mathbf{Z} and $P = \{x + yw | 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Any complex number is congruent, mod \mathcal{O} , to a number in P . Divide P into M^2 similar regions P_k by dividing the intervals $0 \leq x \leq 1, 0 \leq y \leq 1$ into M parts each. These small P_k have diameter $\mathbf{d}M^{-1}$ where \mathbf{d} is the diameter of P . Let $\mu = r + sw$ where $0 \leq r, s \leq M$ are integers. There are $(M+1)^2$ values of μ . We can find two distinct μ_1, μ_2 with $\mu_1 z, \mu_2 z \pmod{\mathcal{O}}$ lying in the same P_k . Let $\mu = \mu_1 - \mu_2$, then $|\mu z - \lambda| \leq \mathbf{d}M^{-1}$ for some $\lambda \in \mathcal{O}$. Also $|\mu| = |\mu_1 - \mu_2| \leq |1 + w|M$.

Lemma 1.16. *Let $\mathcal{A} = (\alpha, \beta)$ be a fractional ideal of \mathcal{O} and let $\gamma \neq 0 \in \mathcal{A}$. There are $\mu, \lambda \in \mathcal{O}$ such that $\mu\alpha + \lambda\beta = \gamma$ and $|\mu| \leq C|\beta|$ where C is a constant depending only on \mathcal{A} and γ .*

Proof. We can assume \mathcal{A} is an integral ideal by replacing \mathcal{A} by $t\mathcal{A}$ for some $t \in \mathcal{O}$. Since \mathcal{O} is a Dedekind domain, we can find $\delta \in \mathcal{A}$ such that $\mathcal{A} = (\gamma, \delta)$. Let $\gamma', \delta' \in \mathcal{A}^{-1}$ be such that $\gamma\gamma' + \delta\delta' = 1$. Let $\alpha', \beta' \in \mathcal{A}^{-1}$ such that $\alpha\alpha' + \beta\beta' = 1$. Let $r \in \mathcal{O}$ be such that $|\frac{\alpha}{\beta} - r| \leq \mathbf{d}$ where \mathbf{d} is a constant depending only on K (as in Lemma 1.15). Replace α', β' by $\alpha' - r\beta, \beta' + r\alpha$, we can assume $|\alpha'| \leq \mathbf{d}|\beta|$. Let $\sigma = \begin{Bmatrix} \alpha' & -\beta \\ \beta' & \alpha \end{Bmatrix}$ and $\tau = \begin{Bmatrix} \gamma & \delta \\ -\delta' & \gamma' \end{Bmatrix}$,

then $\det\sigma = \det\tau = 1$ and $\sigma\tau = \begin{Bmatrix} \alpha'\gamma + \beta\delta' & * \\ \beta'\gamma - \alpha\delta' & * \end{Bmatrix} \in SL_2(\mathcal{O})$. We have $(\alpha, \beta)\sigma\tau = (\gamma, \delta)$. Let $\mu = \alpha'\gamma - \beta\delta'$ and $\lambda = \beta'\gamma - \alpha\delta'$. We have $\mu\alpha + \lambda\beta = \gamma$ and $|\mu| \leq |\gamma||\alpha'| + |\delta'||\beta| \leq (|\gamma|d + |\delta'|)|\beta|$.

Theorem 1.17. *If (z, ζ) is a point in the fundamental domain such that $(z, \zeta) \notin V_i$ for all i , then $\zeta \geq Cd^{-3.5}$ where C is an absolute constant.*

Proof. (i) Let

$$W_i = \left\{ z \in \mathbb{C} \mid \frac{1}{2q_i^2(1+d)^{3/2}} \leq |z - s_i| < \frac{2d+1}{q_i^2(1+d)^{3/2}} \right\} \text{ for } d \equiv 1 \text{ or } 2 \pmod{4}$$

and

$$W_i = \left\{ z \in \mathbb{C} \mid \frac{1}{q_i^2\sqrt{d}\sqrt{d^2+10d+9}} \leq |z - s_i| < \frac{4d-1}{q_i^2\sqrt{d}\sqrt{d^2+10d+9}} \right\}$$

for $d \equiv 3 \pmod{4}$. We can easily see if (z, ζ) in fundamental domain and $z \in W_i$ then

$$\zeta \geq \frac{\sqrt{4d+3}}{2q_i^2(1+d)^{3/2}} \text{ for } d \equiv 1 \text{ or } 2 \pmod{4}$$

and

$$\zeta \geq \frac{\sqrt{4d-1}}{q_i^2\sqrt{d}\sqrt{d^2+10d+9}} \text{ for } d \equiv 3 \pmod{4}.$$

(ii) Let (z, ζ) be a point in fundamental domain with $z \notin W_i \cup V_i$ for any i . For $z \in \mathbb{C} \setminus K$, there exist $\alpha, \beta \in \mathcal{O}$ such that $|\beta z - \alpha| \leq dM^{-1}$ and $\beta \leq AM$ for any positive integer M where $A = |1+w|$. By theorem 1.7, we choose $\mathcal{A}_i = (q_i, a_i + w)$ as representative of the ideal classes of K where $(q_i, a_i) \in I$; furthermore for any \mathcal{A}_i , we put the constant $C_i = (q_i d + |a_i + w|)$.

Let $q = \max\{q_i\}$ and $C = \max\{C_i\}$. For each $|\beta| < \frac{3}{2}q$ and each nonsingular $\frac{\alpha}{\beta}$, there are $\mu, \lambda \in \mathcal{O}, (\mu, \lambda) = \mathcal{O}$ such that $|\mu\frac{\alpha}{\beta} - \lambda| = |\frac{\gamma}{\beta}| < 1$ where $\gamma = \mu\alpha - \lambda\beta$ (Since β is not minimal). So we have $N(\gamma) \leq N(\beta) - 1$ or $|\frac{\gamma}{\beta}|^2 \leq 1 - \frac{1}{|\beta|^2} \leq 1 - \frac{4}{9q^2}$. By Lemma 1.16, we can take $|\mu| \leq C|\beta| \leq \frac{3}{2}qC$, therefore we have

$$|\frac{\alpha}{\beta} - \frac{\lambda}{\mu}| \leq \frac{1}{|\mu|^2} |\frac{\gamma}{\beta}|^2 \leq \frac{1}{|\mu|^2} - \frac{16}{81} \frac{1}{q^4 C^2}.$$

Let $\theta = \frac{8}{81} \frac{1}{q^4 C^2}$ and $\eta = \frac{2}{9} \frac{1}{q^2 C}$. If $(z, \zeta) \in \mathcal{D}$ and $z \in U_{\beta, \alpha} = \{z \in \mathbb{C} \mid |z - \frac{\alpha}{\beta}| < \theta\}$, then $\zeta \geq \eta$. As in Lemma 1.9, we have $(\beta^2, \alpha\beta \pm 1) = (w\beta^2, w\alpha\beta \pm 1) = \mathcal{O}$. The spheres $|\beta^2 z - \alpha\beta \pm 1|^2 + |\beta|^4 \zeta^2 = 1$ pass through $\frac{\alpha}{\beta}$ and tangent along the plane $\Re(\beta^2 z - \alpha\beta) = 0$. The spheres $|w\beta^2 z - w\alpha\beta \pm 1|^2 + |w|^2 |\beta|^4 \zeta^2 = 1$ pass through $\frac{\alpha}{\beta}$ and tangent along the plane $\Re(w\beta^2 z - w\alpha\beta) = 0$. Since the planes are different, each of the first pair of spheres overlaps each of the second pair. we can choose ρ, η' as in (i) and let $U'_{\beta, \alpha} = \{z : \theta \leq |z - \frac{\alpha}{\beta}| < \rho\}$. Therefore if (z, ζ) is in the fundamental domain and $z \in U'_{\beta, \alpha}, \zeta \geq \eta'$. We can get easily that $\rho = O(d^{-3/2})$ and $\eta' = O(d^{-5/2})$.

Let $\epsilon = \min\{\epsilon_i, \rho\}$ where ϵ_i is the radius of $V_i \cup W_i$ and \mathcal{U} be the union of $V_i, W_i, U_{\beta, \alpha}$, and $U'_{\beta, \alpha}$ with $\beta \leq 3/2q$. Now choose M such that $Cd/M < 1/4$ and $d/M < \epsilon$. If $z \in \mathbb{C} \setminus K$ and $z \notin \mathcal{U}$. We can find $\beta, \alpha \in \mathcal{O}$ so that $|\beta| < AM, |\beta z - \alpha| < d/M < \epsilon$. Since $\beta \in \mathcal{O}$, we have $|z - \frac{\alpha}{\beta}| < \epsilon$. Since $z \notin \mathcal{U}$, $|\beta| \geq 3/2q$. Now find $\mu, \lambda \in \mathcal{O}$ with $(\mu, \lambda) = \mathcal{O}$ such that

$|\mu\alpha - \lambda\beta| = |\gamma| \leq q$ and $|\mu| \leq C|\beta| \leq CAM$. Then

$$\begin{aligned} \left|z - \frac{\lambda}{\mu}\right| &\leq \left|z - \frac{\alpha}{\beta}\right| + \left|\frac{\alpha}{\beta} - \frac{\lambda}{\mu}\right| \leq \frac{d}{|\beta|M} + \frac{q}{|\beta||\mu|} \leq \\ &\frac{dC}{M|\mu|} + \frac{2q}{3q|\mu|} \leq \frac{1}{4|\mu|} + \frac{2}{3|\mu|} \leq \frac{11}{12|\mu|}. \end{aligned}$$

So, we have

$$\zeta^2 \geq \frac{1}{|\mu|^2} - \left|z - \frac{\lambda}{\mu}\right|^2 \geq \frac{1}{|\mu|^2} - \frac{121}{144|\mu|^2} = \frac{23}{144|\mu|^2}.$$

Therefore

$$\zeta \geq \frac{\sqrt{23}}{12|\mu|} \geq \frac{\sqrt{23}}{12CAM}.$$

We now return to the algorithm.

(c) By Theorem 1.17, if $S_{\mu,\lambda}$ meets the fundamental domain, the radius of $S_{\mu,\lambda}$, $\frac{1}{|\mu|}$, must be

$$\geq \zeta \geq Cd^{-3.5} \text{ (i.e. } |\mu| \leq \frac{d^{3.5}}{C}\text{)}.$$

Remarks. (a) In [Sw1], the bound obtained in Swan's algorithm is $C'd^{-4.5}$ where C' is a constant.

(b) If K has class number 1, then $L = SL_2(\mathcal{O})$. By Proposition 1.14, we have $|\mu| \leq \sqrt{|d_K|}/\sqrt{2}$.

Chapter 2

Binary Hermitian Forms and Binary Quadratic Forms

2.1 Introduction

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with $a, b, c \in \mathbf{Z}$. The reduction theory of binary quadratic forms was studied by many mathematicians. In *History of the Theory of Numbers*, Dickson described the development of relations between reduction theory and the fundamental domain of $PSL_2(\mathbf{Z})$.

"In 1851, Hermite developed his fundamental method of continual reduction. Closely related to it is the geometrical theory introduced by Smith in 1876 and applied by him to elliptic modular functions, later simplified by Hurwitz in 1894, by Klein in 1890, 1896, and by Humbert in 1916, 1917. A like goal was reached by Dedekind in 1877 and Hurwitz in 1881, both by means of equivalence of complex numbers.

H. Minkowski proved that if $f(x, y) = ax^2 + 2bxy + cy^2$ is a positive form with $\delta = b^2 - ac < 0, a > 0$, we can assign integral values not both zero to x and y such that $f \leq 2\sqrt{\delta/3}$. The equality sign holds only when f is equivalent to $\sqrt{\delta}(x^2 + xy + y^2)$. He gave a simple geometrical interpretation by means of the thickest packing of circles." [D]

Hermite first introduced the form $F(X, Y) = A|X|^2 + B\bar{X}Y + \bar{B}X\bar{Y} + C|Y|^2$ where $A < C \in \mathbf{Z}$ and $B \in \mathbf{Z}[i]$ the so-called Hermitian form. Bianchi and Humbert generalized Hermite's idea to study the form $F(X, Y) = A|X|^2 + B\bar{X}Y + \bar{B}X\bar{Y} + C|Y|^2$ where $A < C \in \mathbf{Z}$ and $B \in \mathcal{O}$, the ring of integers of an imaginary quadratic field. They described the relations between fundamental domains of Bianchi groups and reduced sets of binary Hermitian forms and implied that the reduced sets of binary Hermitian forms are decided by the fundamental domains of Bianchi groups.

Dirichlet followed Gauss' method to consider the binary quadratic forms $f(x, y) = ax^2 + 2bxy + cy^2$ with $a, b, c \in \mathbf{Z}[i]$. Hurwitz, Fricke, and Klein described those forms that have geometric meaning similar to the classical case.

In this chapter, we discuss the reduction theory of binary Hermitian forms and binary quadratic forms with coefficients in the ring of integers of an imaginary quadratic field.

2.2 Binary Hermitian Forms

Throughout this chapter $K = \mathbf{Q}(\sqrt{-d})$ is an imaginary quadratic field where d is a positive square free integer and \mathcal{O} is the ring of integers of K . Let

$$F(X, Y) = AX\bar{X} + B\bar{X}Y + \bar{B}X\bar{Y} + CY\bar{Y}$$

be a positive definite Hermitian form (i.e. A, C are positive rational integers, B is an integer in K and \bar{B} is the conjugate of B . The discriminant $D = AC - B\bar{B} > 0$). The form is called a *proper primitive form* if $(A, C) = 1$ and A, B, \bar{B}, C have no common real divisor. Two forms F, F' are called *equivalent* if there exists $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$ such that $\sigma \cdot F(X, Y) = F(\alpha X + \beta Y, \gamma X + \delta Y) = F'(X, Y) = A'X\bar{X} + B'\bar{X}Y + \bar{B}'X\bar{Y} + C'Y\bar{Y}$ where

$$A' = F(\alpha, \gamma) = A\alpha\bar{\alpha} + B\bar{\alpha}\gamma + \bar{B}\alpha\bar{\gamma} + C\gamma\bar{\gamma},$$

$$C' = F(\beta, \delta) = A\beta\bar{\beta} + B\bar{\beta}\delta + \bar{B}\beta\bar{\delta} + C\delta\bar{\delta},$$

$$B' = A\bar{\alpha}\beta + B\beta\bar{\gamma} + \bar{B}\bar{\alpha}\delta + C\bar{\gamma}\delta.$$

It follows that

$$A'C' - B\bar{B}' = (\alpha\delta - \beta\gamma)(\overline{\alpha\delta - \beta\gamma})(AC - B\bar{B}) = D,$$

so that the discriminant D is an invariant.

We also have a matrix definition of Hermitian form. We can write

$$F(X, Y) = \{X, Y\} \begin{Bmatrix} A & B \\ \bar{B} & C \end{Bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix},$$

$$\sigma \cdot F(X, Y) = \{X, Y\} \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \begin{Bmatrix} A & B \\ \bar{B} & C \end{Bmatrix} \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix}^{-1} \begin{Bmatrix} X \\ Y \end{Bmatrix},$$

$$\text{and } D = \det \begin{Bmatrix} A & B \\ \bar{B} & C \end{Bmatrix}.$$

For the binary Hermitian form $F(X, Y) = AX\bar{X} + B\bar{X}Y + \bar{B}X\bar{Y} + CY\bar{Y}$,
let

$$P_F = \frac{B}{A} + j \frac{\sqrt{D}}{A} \in \mathbf{H}$$

be its *representative point*.

Definition. A positive Hermitian form $F(X, Y) = AX\bar{X} + B\bar{X}Y + \bar{B}X\bar{Y} + CY\bar{Y}$ is called *semi-reduced* if P_F lies in the fundamental domain \mathcal{D} of the Bianchi group $PSL_2(\mathcal{O})$.

Remark. Following chapter 1, \mathcal{D} is equal to $\{(z, \zeta) | z \in F\} \cap \mathcal{B}$. The coefficients of the reduced Hermitian form satisfy $2b_1 \leq A \leq C$, $2b_2 \leq A \leq C$ and some other finite inequalities corresponding to hemispheres on the lower boundary of the fundamental domain (e.g. when $d = 5$, we have $|b_1| + 4|b_2| \leq A + C$ and $5|b_2| \leq A + C$) where $B = b_1 + \sqrt{-d}b_2$.

If P_F is the point corresponding to an arbitrary positive Hermitian form $F(X, Y) = AX\bar{X} + B\bar{X}Y + \bar{B}X\bar{Y} + CY\bar{Y}$ of discriminant D , we can find $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$ such that $\sigma \cdot P_F$ lies inside the fundamental domain. We can easily check that $\sigma' \cdot F(X, Y)$ is a semi-reduced form where $\sigma' =$

$$\begin{Bmatrix} \delta & \beta \\ \gamma & \alpha \end{Bmatrix} \in SL_2(\mathcal{O}).$$

Theorem 2.1. *For any fixed discriminant D , there are only finitely many non-equivalent Hermitian Forms. In fact, the number of non-equivalent binary Hermitian forms is less than $Cd^{10.5}$ where C is a constant depending only on D .*

Proof. We only need to prove that there are only finitely many semi-reduced positive Hermitian forms for fixed discriminant D . Let $F(X, Y) = AX\bar{X} + B\bar{X}Y + \overline{B}X\bar{Y} + CY\bar{Y}$ be a semi-reduced Hermitian form; then $P_F \in \mathbf{C}^2 \times \mathbf{R}^+$ lies in the fundamental domain. We separate the following three cases :

(a) $(\frac{B}{A}, 0)$ is a singular point. By Corollary 1.8, the singular points are of the form $\frac{p(a+w)}{q}$ where $(q, a+w)$ are representatives of ideal classes of \mathcal{O} and $p \bmod q$. We have $q \mid N(a+w)$ and $q^2 \leq a^2 + d$. We can write that $A = qr, B = p(a+w)r$, where r is a positive integer. Since $C = \frac{|B|^2 + D}{A}$ is an integer, we have $A \mid (|B|^2 + D)$. (i.e. $qr \mid p^2 r^2 N(a+w) + D$). Since $q \mid N(a+w)$, $qr \mid D$ (i.e. $A \mid D$).

(b) $\frac{B}{A} \in V_i = \{z \in \mathbf{C} \mid |z - s_i| < \epsilon_i\}$ as described in Theorem 1.10 for some i where s_i is a singular point and $\frac{B}{A} \neq s_i$. By Lemma 1.9, we have

$$\frac{D}{A^2} > \frac{1}{q^2(1+d)^{3/2}} \left| \frac{B}{A} - \frac{a+w}{q} \right| \text{ for } d \equiv 1 \text{ or } 2 \pmod{4},$$

and

$$\frac{D}{A^2} > \frac{2}{q^2 d^{1/2} (d^2 + 10d + 9)^{1/2}} \left| \frac{B}{A} - \frac{a+w}{q} \right| \text{ for } d \equiv 3 \pmod{4}.$$

It easily follows that

$$\frac{D}{A^2} > \frac{1}{q^2(1+d)^{3/2}} \left| \frac{qB - A(a+w)}{Aq} \right| \geq \frac{1}{Aq^3(1+d)^{3/2}} \text{ for } d \equiv 1 \text{ or } 2 \pmod{4},$$

and

$$\frac{D}{A^2} > \frac{2}{q^2d^{1/2}(d^2 + 10d + 9)^{1/2}} \left| \frac{qB - A(a+w)}{Aq} \right| \geq \frac{2}{Aq^3d^{1/2}(d^2 + 10d + 9)^{1/2}}$$

for $d \equiv 3 \pmod{4}$. Hence

$$A < q^3D(1+d)^{3/2} \text{ for } d \equiv 1 \text{ or } 2 \pmod{4},$$

and

$$A < q^3Dd^{1/2}(d^2 + 10d + 9)^{1/2} \text{ for } d \equiv 3 \pmod{4}.$$

(c) If $\frac{B}{A} \notin V_i$, then

$$\frac{\sqrt{D}}{A} \geq Cd^{-3.5}$$

by Theorem 1.17. Hence

$$A \leq \frac{\sqrt{D}d^{3.5}}{C}.$$

From the above, we can say that A is bounded. There are only finitely many semi-reduced positive Hermitian form.

Remarks. (a) From the proof of Theorem 2.1, we can easily find semi-reduced Hermitian forms for given discriminant D .

(b) From the proof of Theorem 2.1, given a discriminant D , the number of non-equivalent binary Hermitian forms is less than MD where M is a constant depending only on K .

Example. Let $K = \mathbf{Q}(\sqrt{-5})$. Following Theorem 2.1, if $F(X, Y) = AX\bar{X} + B\bar{X}Y + \overline{B}X\bar{Y} + CY\bar{Y}$ is a semi-reduced form, we have $A \leq Dq(1 + d)^{3/2}$ where D is the discriminant of F . Since the representative point P_F lies inside of the fundamental domain of $PSL_2(\mathbf{Z}[\sqrt{-5}])$, we have the following inequalities

$$-\frac{1}{2} \leq \frac{b_1}{A} \leq \frac{1}{2}, \quad -\frac{1}{2} \leq \frac{b_2}{A} \leq \frac{1}{2}$$

$$A \leq C, \quad |b_1| + 4|b_2| \leq A + C, \quad 5|b_2| \leq A + C$$

where $B = b_1 + \sqrt{-5}b_2$.

For $D = 6$, we have the set of semi-reduced Hermitian forms as follows:

$$(1, 0, 6), (2, 0, 3), (2, \pm 1 \pm w, 6),$$

$$(3, \pm 1 \pm w, 4), (5, \pm 2 \pm 2w, 6), (6, \pm 3 \pm 3w, 10),$$

$$(12, \pm 5 \pm 5w, 13), (15, \pm 7 \pm 7w, 20)$$

where $w = \sqrt{-5}$ and we represent the reduced form $F(X, Y) = AX\bar{X} + B\bar{X}Y + \overline{B}X\bar{Y} + CY\bar{Y}$ as (A, B, C) .

2.3 Binary Quadratic(Dirichlet) Forms

Let $f(x, y) = ax^2 + 2bxy + cy^2$ be a binary quadratic form over \mathcal{O} (i.e. $a, b, c \in \mathcal{O}$). Two forms f, f' are called *equivalent* if there exists $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in SL_2(\mathcal{O})$ such that $f(\alpha x + \beta y, \gamma x + \delta y) = f'(x, y)$. As above, the discriminant

$D = b^2 - ac$ is an invariant. For any binary quadratic form, let

$$Q_f = \frac{b}{a} + j \frac{\sqrt{|D|}}{|a|} \in \mathbf{H}$$

be its representative point.

Definition. A binary Dirichlet form $f(x, y) = ax^2 + 2bxy + cy^2$ is called reduced if Q_f lies in the fundamental domain \mathcal{D} of the Bianchi group $PSL_2(\mathcal{O})$.

Remark. As above, the fundamental domain provides several inequalities, including the following obvious ones :

$$\Re\left\{\frac{|b|}{|a|}\right\} \leq \frac{1}{2}, \Im\left\{\frac{|b|}{|a|}\right\} \leq \frac{w_2}{2}, |c| \geq |a|$$

Theorem 2.2. For any fixed discriminant D , there are only finitely many non-equivalent binary quadratic forms. In fact, the number of non-equivalent binary quadratic forms is less than $C'd^{10.5}$ where C' is a constant depending only on D .

Proof. Let $f(x, y) = ax^2 + 2bxy + cy^2$ be an arbitrary binary quadratic form. There is a $\delta \in \mathcal{O}$ such that $b = a\delta + b_1$ and $\frac{b_1}{a} \in F$. Let $f_1(x, y) = \begin{Bmatrix} 1 & \delta \\ 0 & 1 \end{Bmatrix} \cdot f(x, y) = ax^2 + 2b_1xy + c_1y^2$. If Q_{f_1} does not lie in the fundamental domain, then there is a S_{μ_1, λ_1} such that $(\frac{b_1}{a}, \frac{\sqrt{|D|}}{|a|})$ lies lower than S_{μ_1, λ_1} (i.e. $|\frac{b_1}{a}\mu_1 + \lambda_1|^2 + |\mu_1|^2 \frac{|D|}{|a|^2} < 1$). Let $\sigma_1 = \begin{Bmatrix} \lambda_1 & \beta_1 \\ \mu_1 & \delta_1 \end{Bmatrix} \in SL_2(\mathcal{O})$ and $f_2(x, y) = \sigma_1 \cdot f_1(x, y) = a_1x^2 + 2b_2xy + c_2y^2$ where $a_1 = a\lambda_1^2 + 2b_1\lambda_1\mu_1 + c\mu_1^2$.

We have

$$\frac{\sqrt{|D|}}{|a_1|} = \frac{\sqrt{|D|}}{|a\lambda_1^2 + 2b_1\lambda_1\mu_1 + c\mu_1^2|} \geq \frac{\frac{\sqrt{|D|}}{|a|}}{|\mu_1\frac{b}{a} + \lambda_1|^2 + |\mu_1|^2\frac{|D|}{|a|^2}} > \frac{\sqrt{|D|}}{|a|}.$$

(i.e. $|a_1| < |a|$)

Let $\delta_1 \in \mathcal{O}$ be such that $b_2 = a_1\delta + b_3$ and $\frac{b_3}{a_1} \in F$. Let $f_3(x, y) = \begin{Bmatrix} 1 & \delta_1 \\ 0 & 1 \end{Bmatrix} \cdot f_2(x, y) = a_1x^2 + 2b_3xy + c_2y^2$. If Q_{f_3} does not lie in the fundamental domain, then there is a S_{μ_2, λ_2} such that $(\frac{b_3}{a_1}, \frac{\sqrt{|D|}}{|a_1|})$ lies lower than S_{μ_2, λ_2} . Let $\sigma_2 = \begin{Bmatrix} \lambda_2 & \beta_2 \\ \mu_2 & \delta_2 \end{Bmatrix} \in SL_2(\mathcal{O})$ and $f_4(x, y) = \sigma_2 \cdot f_3(x, y) = a_2x^2 + 2b_4xy + c_4y^2$. We have

$$\frac{\sqrt{|D|}}{|a_2|} > \frac{\sqrt{|D|}}{|a_1|} \quad (\text{i.e. } |a_2| < |a_1|).$$

Now consider the sequence of forms

$$f_2, f_4, f_6, \dots$$

We have

$$\frac{\sqrt{|D|}}{|a|} < \frac{\sqrt{|D|}}{|a_1|} < \frac{\sqrt{|D|}}{|a_2|} < \dots,$$

where a, a_1, a_2, \dots are the first coefficients of f, f_2, f_4, \dots . Hence $\{|a_i|\}$ is a diminishing sequence of positive real integers which must eventually stop. Let $f_{2n}(x, y) = a_nx^2 + 2b_{2n}xy + c_{2n}y^2$ be the last form at which we arrive; then there is no sphere higher than $(\frac{b_{2n}}{a_n} + j\frac{\sqrt{|D|}}{|a_n|})$. So, $f_{2n}(x, y)$ is a reduced form.

To complete the proof, we need to prove that there are only finitely many reduced forms.

Let $f(x, y) = ax^2 + 2bxy + cy^2$ be a reduced form and Q_f be its representative point. Since Q_f lies in the fundamental domain, $\frac{|b|}{|a|} \leq \frac{4+d_K}{16}$ where d_K is the discriminant of K . We consider the following cases:

(a) If $\frac{b}{a}$ is a singular point of the fundamental domain, then $\frac{b}{a} = \frac{p(a+w)}{q}$. We may write $b = np(a+w)$, $a = nq$ for some integer n . Since $D = b^2 - ac$, we can write $D = n^2p^2(a+w)^2 - nqc = n(np^2(a+w)^2 - qc)$. That is, $|D| = |n| \cdot |m|$ for some integer m . Since $|m| \geq 1$, $|n| \leq |D|$ (i.e. $|a| \leq |D|q$).

(b) If $\frac{b}{a} \in V_i = \{z \in \mathbf{C} \mid |z - s_i| < \epsilon_i\}$ as described in Theorem 1.10 for some i where s_i is a singular point and $\frac{b}{a} \neq s_i$, Then by Lemma 1.9, we have

$$\frac{|D|}{|a|^2} > \frac{1}{q^2(1+d)^{3/2}} \left| \frac{b}{a} - \frac{a+w}{q} \right| \geq \frac{1}{|a|q^3(1+d)^{3/2}} \text{ for } d \equiv 1 \text{ or } 2 \pmod{4}$$

and

$$\frac{|D|}{|a|^2} > \frac{2}{q^2\sqrt{d}\sqrt{d^2+10d+9}} \left| \frac{b}{a} - \frac{a+w}{q} \right| \geq \frac{2}{|a|q^3\sqrt{d}\sqrt{d^2+10d+9}}$$

for $d \equiv 3 \pmod{4}$.

Thus,

$$|a| < q^3(1+d)^{3/2}|D| \text{ for } d \equiv 1 \text{ or } 2 \pmod{4}$$

and

$$|a| < \frac{q^3\sqrt{d}\sqrt{d^2+10d+9}|D|}{2} \text{ for } d \equiv 3 \pmod{4}.$$

(c) If $\frac{b}{a} \notin V_i$, then

$$\frac{\sqrt{|D|}}{|a|} \geq Cd^{-3.5}$$

by Theorem 1.17. Hence

$$|a| \leq \frac{\sqrt{|D|}d^{3.5}}{C}.$$

From the above argument, there are only finitely many reduced forms.

Remarks. (a) For any given discriminant, the complexity of calculation of reduced sets of binary quadratic forms and binary Hermitian forms is decided by the fundamental domains of Bianchi groups. On the other hand, not obviously, the determination of all reduced Hermitian forms for all discriminants implies the determining of the fundamental domains of Bianchi groups.

(b) From the proof of Theorem 2.2, given a discriminant D , the number of non-equivalent binary quadratic forms is less than $M'|D|$ where M' is a constant depending only on K .

J. Hurwitz and G. B. Mathews [Ma] generalized the continued fraction to the Gauss field to calculate the class number and the fundamental unit of biquadratic fields. Mathews' idea is as follows: Let $K = \mathbf{Q}(\sqrt{-1})$ be the Gauss field, Then

$$z = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots a_k + \frac{1}{z_k}}}}$$

where z is a complex number and a_i are Gaussian integers. We can write $z = \sigma_1 \tau \sigma_2 \tau \cdots \tau z_k$ where $\sigma_i = \begin{Bmatrix} 1 & a_i \\ 0 & 1 \end{Bmatrix}$ and $\tau = \begin{Bmatrix} 0 & -1 \\ 1 & 0 \end{Bmatrix}$ and the action $\sigma \cdot z = \frac{az+b}{cz+d}$ for $\sigma = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$. We generalize Mathews' idea to imaginary quadratic fields $K = \mathbf{Q}(\sqrt{-d})$ with class number 1 as follows: Let

S_1, S_2, \dots, S_n be the boundary hemispheres of the fundamental domain of the Bianchi group and let A_i be the corresponding generator of S_i (see chapter 3). For any complex number z , there is an integer $a_1 \in \mathcal{O}$ such that $z - a_1 \in F$ where F is defined as in chapter 1. We can write $z = \sigma_1 z'$ where $\sigma_1 = \begin{Bmatrix} 1 & a_1 \\ 0 & 1 \end{Bmatrix}$ and $z' \in F$. If S_i is the highest sphere that covers z' , we write $z = \sigma_1 \tau_1 z_1$ where $\tau_1 = A_i$. Let $a_2 \in \mathcal{O}$ be such that $z_1 - a_2 \in F$, and let $\sigma_2 = \begin{Bmatrix} 1 & a_2 \\ 0 & 1 \end{Bmatrix}$. Then $z = \sigma_1 \tau_1 \sigma_2 z'_1$. Like $z', z'_1 \in F$. Let S_j be the highest sphere that covers z'_1 , then $z = \sigma_1 \tau_1 \sigma_2 \tau_2 z_2$ where $\tau_2 = A_j$. Continuing in this manner, we can write $z = \sigma_1 \tau_1 \cdots \sigma_k \tau_k z_k$ (i.e.

$$z = \frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}$$

where $\sigma_1 \tau_1 \cdots \sigma_k \tau_k = \begin{Bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{Bmatrix}$). It is easy to see that if the generalized continued fraction is periodic, then z is of the form $a + b\sqrt{c}$ where $a, b, c \in K$.

We have the following conjecture:

Conjecture: If z is of the form $a + b\sqrt{c}$ where $a, b, c \in K$, then the generalized continued fraction of z is periodic.

If the conjecture is true, we can follow the classical method to determine the fundamental units and ideal class groups of complex biquadratic fields.

Chapter 3

Poincare's Polyhedron Theorem and Swan's Theorem

Poincare gave a method for extending the action of $SL_2(\mathbf{C})$ to the upper 3-space \mathbf{H} . This action extends the transformation of $SL_2(\mathbf{C})$ from \mathbf{C} to \mathbf{H} . He also gave a famous theorem relating the presentation of a discrete subgroup Γ for $SL_2(\mathbf{C})$ with the fundamental domain of Γ . Bianchi and Humbert used these ideas of Poincare in their work (see [B], [Hu2]). Following Bianchi and Humbert, R. Swan gave an algorithm to determine the fundamental domain and together with a theorem of Macbeath to describe the presentations of Bianchi groups. In this chapter, we will discuss Poincare's theorem and Swan's theorem from the point of view of the arithmetic structure of Bianchi groups.

3.1 Hyperbolic 3-space and Polyhedra

Let $\mathbf{H} = \{(z, \zeta) | z \in \mathbf{C}, \zeta > 0\}$ be the upper half space. Consider every point P of \mathbf{H} as a quaternion $P = x + iy + j\zeta$. As in Chapter 1, we define the action of $PSL_2(\mathbf{C})$ on \mathbf{H} as follows: Let $\sigma = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \in PSL_2(\mathbf{C})$, $\sigma \cdot P = (\alpha P + \beta)(\gamma P + \delta)^{-1}$ (This is called the Poincare extension of σ). Let Γ be a discrete subgroup of $PSL_2(\mathbf{C})$. It is well known that Γ has a fundamental domain[Mas1]. For example, the fundamental domain of $\Gamma = PSL_2(\mathbf{Z}[\sqrt{-2}])$ is given Figure 3.1.

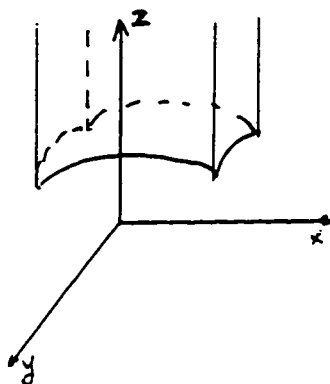


Figure 3.1:

Definition. A (convex) polyhedron \mathcal{P} in \mathbf{H} is a nonempty, open, (convex) subset of \mathbf{H} such that the collection \mathcal{S} of its sides is locally finite in \mathbf{H} .

The polyhedra with which we are concerned in this chapter are typically bounded by four vertical planes and several geodesic hemi-spheres (e.g. Figure 3.1).

3.2 Poincare's Polyhedron Theorem

We now proceed to describe the facts needed to obtain a presentation of Γ . They include the seven conditions given below. Let \mathcal{P} be a polyhedron. An *identification* on the polyhedron \mathcal{P} is a map which assigns, to each side(face) s , a side s' not necessarily distinct from s and an isometry $\sigma(s, s') \in \Gamma$, satisfying the following conditions:

- (1) $\sigma(s, s')$ maps s onto s' .
- (2) $(s')' = s$ and $\sigma(s', s) = (\sigma(s, s'))^{-1}$
- (3) For each side s , there is a neighborhood V of s such that $\sigma(V \cap \mathcal{P}) \cap \mathcal{P} = \emptyset$ where $\sigma = \sigma(s, s')$.

The elements $\sigma(s, s')$ which identify the sides of the polyhedron \mathcal{P} are called *generators*. Let Γ be the group generated by those generators. If $s = s'$, then condition (2) implies that the corresponding generator $\sigma = \sigma(s, s)$ is of order 2. In fact, σ is a reflection in the side s ; the relations of the form $\sigma^2 = 1$ are called *reflection relations*.

Let \mathcal{P} be a polyhedron with an identification; then there is a \mathcal{P}^* obtained by identifying the sides of \mathcal{P} (i.e. there is a surjective map $\pi : \overline{\mathcal{P}} \rightarrow \mathcal{P}^*$ where $\pi(x) = \pi(x')$ if there is a generator σ with $\sigma(x) = x'$ for $x, x' \in \mathcal{P}^*$). We set $d^*(x, x') = \inf \sum_{i=1}^n d(z_i, z'_i)$ where the infimum is taken over all finite sets of points $\{z_1, z'_1, \dots, z_n, z'_n\}$ with $\pi(z_1) = x, \pi(z'_i) = \pi(z_{i+1}),$ and $\pi(z'_n) = x'$ and d is the hyperbolic metric defined in chapter 1.

(4) For each $x \in \mathcal{P}^*$, $\pi^{-1}(x)$ is a finite set, in which case d^* defines a metric on \mathcal{P}^* .

(5) \mathcal{P}^* is complete in this metric.

The basic property of the metric d^* is $d^*(\pi(z), \pi(z')) \leq d(z, z')$ so that $\pi : \overline{\mathcal{P}} \rightarrow \mathcal{P}^*$ is continuous. We call \mathcal{P} *complete* if (4), (5) are satisfied.

Let e_1 be some edge of \mathcal{P} . There are precisely 2 sides of \mathcal{P} which meet at e_1 ; call one of them s_1 . Then there is a side s'_1 , and there is a side transformation $\sigma_1 = \sigma(s, s')$. Set $e_2 = \sigma_1(e_1)$. Like e_1 , e_2 lies on the boundary of exactly two sides. One of them is s'_1 ; call the other s_2 . Again there is a side s'_2 , and a side pairing $\sigma_2(s_2) = s'_2$. Continuing in this manner, we have a sequence $\{e_i\}$ of edges, a sequence $\{\sigma_i\}$ of side pairing transformations, and a sequence $\{(s_i, s'_i)\}$ of pairs of sides. Since each point of e_1 is equivalent to at most finitely many other points of $\overline{\mathcal{P}}$ (condition (4)), the sequences are periodic. Let k be the least period so that all three sequences are periodic with period k . The cyclically ordered sequence of edges $\{e_1, \dots, e_k\}$ is called a *cycle* of edges, and k is the period of the cycle. Observe that $\sigma_k \circ \dots \circ \sigma_1(e_1) = e_1$; $\tau = \sigma_k \circ \dots \circ \sigma_1$ is called the *cycle transformation* at e_1 .

(6) For each edge e , there is a positive integer n such that $\tau^n = 1$.

The relations in Γ of the form $\tau^n = 1$ are called the *cycle relations*. Let $\alpha(e)$ be the angle measured from inside \mathcal{P} at the edge e , we require

$$(7) \sum_{i=1}^k \alpha(e_i) = \frac{2\pi}{n}.$$

We are now ready to state Poincare's Theorem.

Theorem 3.1. (Poincare) *Let \mathcal{P} be a polyhedron satisfying conditions (1) through (7), and let Γ be the group generated by the identifying side pairing isometries. Then Γ is discrete, \mathcal{P} is a fundamental polyhedron¹ for Γ , and the cycle relations together with the reflection relations form a complete set of relations for Γ .*

3.3 Swan's Theorem

Poincare's Theorem applies to fairly general polyhedra in hyperbolic 3-space. In order to bring out the arithmetic structure of Bianchi groups, one must use more specific information about the fundamental domains. We now describe Swan's version of Poincare's Theorem.

Let $K = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field and let \mathcal{O} be its ring of integers. The Bianchi group $PSL_2(\mathcal{O})$ is a discrete subgroup of $PSL_2(\mathbf{C})$. Following Poincare's theorem, we can determine the presentation of $PSL_2(\mathcal{O})$ by determining the fundamental domain. The determination of a presentation of a Bianchi group is in general a difficult problem. Swan gave a method that exploits the arithmetic structure of the group. We now describe Swan's method.

¹ \mathcal{P} is a fundamental domain of Γ and is also a polyhedron.

Swan generalized Macbeath's result [Mac] on groups of transformations of a topological space. In particular, let X be a pathwise connected topological space, and let G act on X as a group of homeomorphisms of X . Suppose V is a pathwise connected open subset of X such that $G \cdot V = X$. Let $S = \{\sigma \in G : \sigma V \cap V \neq \emptyset\}$. Let Γ be a group generated by the symbol $[\sigma]$ if $\sigma \in S$ and with relations $[\sigma][\tau] = [\sigma\tau]$ for each $\sigma, \tau \in S$ with $V \cap \sigma V \cap \tau V \neq \emptyset$. Define $\psi : \Gamma \rightarrow G$ by $\psi([\sigma]) = \sigma$. Since X is connected, ψ is onto. We can now state Swan's results[Sw1].

Theorem 3.2. *Let X, V, G, Γ be as above. Then there exists an exact sequence*

$$1 \rightarrow N \rightarrow \pi_1(X) \xrightarrow{\theta} \Gamma \xrightarrow{\psi} G$$

where N is a normal subgroup of G and is generated by all elements represented by the loops in at least one of the sets $\sigma V \cap \tau V$ with $\sigma, \tau \in G$. The map θ is defined by subdividing a loop λ into small paths contained successively in the sets $\sigma_0 V, \sigma_1 V, \dots, \sigma_n V, \sigma_0 = \sigma_n$ and let $\theta(\lambda) = [\sigma_0^{-1} \sigma_1][\sigma_1^{-1} \sigma_2] \cdots [\sigma_{n-1}^{-1} \sigma_n]$.

Let $\lambda_1, \dots, \lambda_n \in \pi_1(X)$. We say these elements generate $\pi_1(X)$ normally over G if the smallest G -stable normal subgroup of $\pi_1(X)$ containing $\lambda_1, \dots, \lambda_n$ is $\pi_1(X)$ itself.

Corollary 3.3. *Let $\lambda_i \in \pi_1(X)$ be elements which generate $\pi_1(X)$ normally over G . Then $\ker(\psi)$ is the smallest normal subgroup of Γ containing the elements $\theta(\lambda_i)$.*

Since we discussed the closure of the fundamental domains of Bianchi groups in the first chapter, we need discuss closed subsets of X rather than open subsets of X . A subset W with $GW = X$ is called *G -inflatable* if there exists an open $V \supset W$ such that $\sigma_1 V \cap \cdots \sigma_n V \neq \emptyset$ if and only if $\sigma_1 W \cap \cdots \sigma_n W \neq \emptyset$.

Definition. *A loop $\lambda : S^1 \rightarrow X$ is called well-behaved relative to W if there is a subdivision of S^1 into arcs I_0, \dots, I_n such that for each i there exists $\sigma_i \in G$ with $\lambda(I_i) \subset \sigma_i W$. For such a loop, $\theta(\lambda)$ is conjugate to $[\sigma_0^{-1} \sigma_1][\sigma_1^{-1} \sigma_2] \cdots [\sigma_{n-1}^{-1} \sigma_n]$.*

Corollary 3.4. *Let G, X be as above and W a G -inflatable subset with $GW = X$. Let $\alpha_i \in \pi_1(X)$ be elements which generate $\pi_1(X)$ normally over G , and let λ be a well-behaved loop relative to W which represents each α_i up to conjugacy. Let $S = \{\sigma \in G, \sigma W \cap W \neq \emptyset\}$. Then G has a representation with generators $[\sigma]$ for $\sigma \in S$ and relations $[\sigma][\tau] = [\sigma\tau]$ for $\sigma, \tau \in S$ where $W \cap \sigma W \cap \sigma\tau W \neq \emptyset$ and also the relations $\theta(\lambda_i) = 1$.*

We are going to use Swan's results to give an arithmetic description of the structure of Bianchi groups.

Now, we recall that

$$\mathcal{B} = \{(z, \zeta) \in \mathbf{H} : |\mu z + \lambda|^2 + |\mu|^2 \zeta^2 \geq 1 \text{ for all } (\mu, \lambda) = \mathcal{O}\},$$

$$F = \{z = x + iy \mid -1/2 \leq x \leq 1/2, -w_2/2 \leq y \leq w_2/2\},$$

$\mathcal{D} = \mathcal{B} \cap \{(z, \zeta) : z \in F\}$, and $S_{\mu, \lambda} = \{(z, \zeta) \in \mathbf{H} : |\mu z + \lambda|^2 + |\mu|^2 \zeta^2 = 1\}$ where $1, w = w_1 + iw_2$ is a basis of \mathcal{O} . The boundary of \mathcal{B} has a regular cell decomposition with 2-cells of the form $\mathcal{B} \cap S_{\mu, \lambda}$. Let E_0 be the 1-skeleton of the cell decomposition and $E = \cup_{\sigma \in SL_2(\mathcal{O})} \sigma E_0$. Then E is a 1-complex which is stable under $SL_2(\mathcal{O})$ and E_0 is the subcomplex of E . Corollary 3.4 can be applied to $G = SL_2(\mathcal{O})$, $X = \mathbf{H} \setminus E$, and $W = \mathcal{B} \setminus E = \mathcal{B} \setminus E_0$. Since \mathcal{D} is $SL_2(\mathcal{O})$ -normal (i.e. each point of \mathbf{H} has a neighborhood meeting $\sigma \mathcal{D}$ for only a finite number of $\sigma \in SL_2(\mathcal{O})$). Following Macbeath's theorem [Mac, Theorem 2], \mathcal{D} is G -inflatable. Then $\mathcal{D}' = \mathcal{D} \setminus E$ is G -inflatable. Since $W = \Phi \mathcal{D}'$ where $\Phi = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in SL_2(\mathcal{O}) \right\}$, W is G -inflatable [Sw1, Lemma 1.8]. Let $S = \{\sigma \in SL_2(\mathcal{O}), \sigma W \cap W \neq \emptyset\}$. If $\sigma = \begin{pmatrix} \alpha & \beta \\ \mu & \lambda \end{pmatrix}$, we have $\sigma W \cap W \neq \emptyset$ if and only if $\sigma \mathcal{B} \cap \mathcal{B} \not\subset E$ or $\mathcal{B} \cap \sigma^{-1} \mathcal{B} = \mathcal{B} \cap S_{\mu, \lambda} \not\subset E$. Therefore S is the set of $\begin{pmatrix} \alpha & \beta \\ \mu & \lambda \end{pmatrix} \in SL_2(\mathcal{O})$ such that either $\mu = 0$ or $\mathcal{B} \cap S_{\mu, -\lambda}$ is a 2-cell of $\partial \mathcal{B}$. Let $S_{\mu_1, \lambda_1}, \dots, S_{\mu_n, \lambda_n}$ be the boundary spheres of the fundamental domain. Since $(\mu_i, \lambda_i) = \mathcal{O}$, there exist α_i, β_i such that $\alpha_i \lambda_i - \beta_i \mu_i = 1$. We call $\sigma_i = \begin{pmatrix} \alpha_i & \beta_i \\ \mu_i & -\lambda_i \end{pmatrix} \in SL_2(\mathcal{O})$ the corresponding transformations of S_{μ_i, λ_i} .

Definition. If $\sigma \in S$ and $\sigma \notin \Phi$, we can write $\sigma = \phi\sigma_i\phi'$ where $\phi, \phi' \in \Phi$. We refer to this as the canonical form for σ and denote the formal word $\phi\sigma_i\phi'$ by $[\sigma]$.

The above definition gives the relation $\sigma_i^{-1} = [\sigma_i^{-1}]$ which stands for $[\sigma_i^{-1}] = \phi\sigma_j\phi'$ where $\phi, \phi' \in \Phi$. Suppose e is an edge of $\partial\mathcal{B}$ and let S_{μ_i, λ_i} be the distinct $S_{\mu, \lambda}$ where e lies on $S_{\mu, \lambda}$. Then $z_i = \frac{-\lambda_i}{\mu_i}$ all lie on a line in \mathbb{C} . Order these in some linear order z_1, \dots, z_s and choose matrices $\sigma_i = \begin{Bmatrix} \alpha_i & \beta_i \\ \mu_i & -\lambda_i \end{Bmatrix} \in SL_2(\mathcal{O})$ with $z_i = \lambda_i/\mu_i$. Then the formal word $[\sigma_i\sigma_{i+1}^{-1}]$ is defined for all i ; let $R(e) = [\sigma_1^{-1}][\sigma_1\sigma_2^{-1}]\cdots[\sigma_s]$. In $SL_2(\mathcal{O})$, we have $R(e) = 1$. We give Swan's theorem now.

Theorem 3.5. (Swan) $SL_2(\mathcal{O})$ is generated by $J = \begin{Bmatrix} -1 & 0 \\ 0 & -1 \end{Bmatrix}$, $T = \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}$, $U = \begin{Bmatrix} 1 & w \\ 0 & 1 \end{Bmatrix}$, $\sigma_1, \dots, \sigma_n$ with the following relations where σ_i are the corresponding transformations of the boundary spheres.

- (1) J central, $J^2 = 1$.
- (2) $TU = UT$.
- (3) $\sigma_i^{-1} = [\sigma_i^{-1}]$
- (4) The edge reflection relations $R(e) = 1$, where e runs over a set of representatives for edges of $\partial\mathcal{B}$ modulo the action of $SL_2(\mathcal{O})$.

3.4 Elliptic Elements

The torsion elements determine the edge relations of Bianchi groups. In this section, we discuss the elliptic elements (torsion elements) of Bianchi groups.

We start with a well-known theorem. [Si]

THEOREM 3.6. *Let A be a finite subgroup of $PGL_2(\mathbb{C})$.*

(1) *The group A is isomorphic to one of the following groups:*

$A \cong C_n =$ *cycle group of order n , $n \geq 1$.*

$A \cong D_{2n} =$ *dihedral group of order $2n$, $n \geq 2$.*

$A \cong U_4 =$ *tetrahedral group = alternating group of order 12.*

$A \cong G_4 =$ *octahedral group = symmetric group of order 24.*

$A \cong U_5 =$ *icosahedral group = alternating group of order 60.*

(2) *More precisely, if we let ζ_n denote a primitive n th root of unity, then A is linearly conjugate to one of the following subgroups of PGL_2 :*

$$C_n = \langle \zeta_n z \rangle, \quad D_{2n} = \langle \zeta_n z, \frac{1}{z} \rangle, \quad U_4 = \langle -z, \frac{1}{z}, i \frac{z+1}{z-1} \rangle,$$

$$G_4 = \langle iz, \frac{1}{z}, i \frac{z+1}{z-1} \rangle, \quad U_5 = \langle \zeta_5, -\frac{1}{z}, \frac{(\zeta_5 + \zeta_5^{-1})z + 1}{1 - (\zeta_5 + \zeta_5^{-1})z} \rangle.$$

(Here, $\langle f_1, \dots, f_n \rangle$ denotes the group generated by f_1, \dots, f_n .) In particular, A is linearly conjugate to a subgroup of PGL_2 which is defined over \mathbb{C} .

By Theorem 3.6, we can easily say that the order of the torsion elements in Bianchi Groups are 2, 3, 4, and 5. Now we study separately each case.

(A) Torsion elements of order 2.

Since $d \geq 5$, the units of \mathcal{O} are only ± 1 . $PSL_2(\mathcal{O}) = SL_2(\mathcal{O})/(\pm 1)$. Let $T = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix}$ be an element of order 2 in $PSL_2(\mathcal{O})$, then $T^2 = I_2$ or $-I_2$. If $T^2 = I_2$, $T = I_2$ or $T = -I_2$ since the only units of \mathcal{O} are ± 1 . For $T^2 = -I_2$, we have $\begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} = \begin{Bmatrix} -\delta & \beta \\ \gamma & -\alpha \end{Bmatrix}$ (i.e., $\alpha = -\delta$). Since $T \in SL_2(\mathcal{O})$, we have $\alpha\delta - \beta\gamma = 1$. So we have the diophantine equation $\alpha^2 + \beta\gamma = -1$.

(B) Torsion elements of order 3.

Let $T = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix}$ be an element of order 3 in $PSL_2(\mathcal{O})$. Then $T^3 = I_2$ or $-I_2$.

If $T^3 = I_2$, we have $T^2 = T^{-1}$ (i.e., $\begin{Bmatrix} \alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \alpha\gamma + \gamma\delta & \beta\gamma + \delta^2 \end{Bmatrix} = \begin{Bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{Bmatrix}$). If β or γ is equal 0, we have $\alpha^2 = \delta$ and $\delta^2 = \alpha$. So $\alpha = \delta = 1$ and $T = I_2$, contradicting our assumption. For β and γ both not equal to 0, $\alpha + \delta = -1$. Then $\alpha^2 + \beta\gamma = \delta$, $\delta^2 + \beta\gamma = \alpha$ and $\alpha\delta - \beta\gamma = 1$ are the same diophantine equations.

If $T^3 = -I_2$, we have $T^2 = -T^{-1}$. So we have the diophantine equation $\alpha^2 + \beta\gamma = \alpha - 1$.

(C) Torsion elements of order 4.

Let $T = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix}$ be an element of order 4 in $PSL_2(\mathcal{O})$. Then $T^4 = -I_2$

(If $T^4 = I_2$, T is order 2 in $PGL_2(\mathcal{O})$). We have $T^2 = -T^{-2}$ (i.e.,

$$\begin{Bmatrix} \alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \alpha\gamma + \gamma\delta & \beta\gamma + \delta^2 \end{Bmatrix} = \begin{Bmatrix} -\delta^2 - \beta\gamma & \beta\delta + \alpha\beta \\ \gamma\delta + \alpha\gamma & -\alpha^2 - \beta\gamma \end{Bmatrix}.$$

From the matrix, we have $\alpha^2 + \beta\gamma = -\delta^2 - \beta\gamma$. Since $\alpha\delta - \beta\gamma = 1$, we have $(\alpha + \delta)^2 = 2$. This is impossible for $d \geq 5$. There is no element of order 4 in the Bianchi Groups.

(D) Torsion elements of order 5.

Let $T = \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix}$ be an element of order 5 in $PSL_2(\mathcal{O})$. Then $T^5 = I_2$ or $-I_2$. We have $T^3 = T^{-2}$ or $T^3 = -T^{-2}$ (i.e.,

$$\begin{Bmatrix} \alpha^3 + 2\alpha\beta\gamma + \beta\gamma\delta & \alpha^2\beta + \beta^2\delta + \alpha\beta\delta + \beta\delta^2 \\ \alpha^2\gamma + \alpha\gamma\delta + \alpha\gamma^2 + \gamma^2\delta & \alpha\beta\gamma + 2\beta\gamma\delta + \delta^3 \end{Bmatrix} = \begin{Bmatrix} \delta^2 + \beta\gamma & -\beta\delta - \alpha\beta \\ -\gamma\delta - \alpha\gamma & \alpha^2 + \beta\gamma \end{Bmatrix}$$

or

$$\begin{Bmatrix} \alpha^3 + 2\alpha\beta\gamma + \beta\gamma\delta & \alpha^2\beta + \beta^2\delta + \alpha\beta\delta + \beta\delta^2 \\ \alpha^2\gamma + \alpha\gamma\delta + \alpha\gamma^2 + \gamma^2\delta & \alpha\beta\gamma + 2\beta\gamma\delta + \delta^3 \end{Bmatrix} = \begin{Bmatrix} -\delta^2 - \beta\gamma & \beta\delta + \alpha\beta \\ \gamma\delta + \alpha\gamma & -\alpha^2 - \beta\gamma \end{Bmatrix}.$$

From the matrix, we have four equations together with $\alpha\delta - \beta\gamma = 1$. By an easy calculation, we know that there is no solution for the equations. There is no element of order 5 in the Bianchi groups.

Remark. From the above discussion, the torsion elements of Bianchi groups only have order 2 or 3.

3.5 Poincare's Theorem for Bianchi Groups

In this section, we use algebraic properties of Bianchi groups to explicate Poincare's Theorem. If $S_{\mu,\lambda} \cap \mathcal{B}$ is a side(face) of the fundamental domain, then let $\sigma = \begin{Bmatrix} \alpha & \beta \\ \mu & -\lambda \end{Bmatrix}$ be the corresponding transformation of $S_{\mu,\lambda}$. Let $(z, \zeta) \in S_{\mu,\lambda} \cap \mathcal{B}$. By Lemma 1.5, $\sigma \cdot (z, \zeta) \in S_{\mu,\alpha} \cap \mathcal{B}$. Replacing α, β by $\alpha - r\mu, \beta + r\lambda$ respectively if necessary, we can assume that $S_{\mu,\alpha} \cap \mathcal{B}$ is a side of the fundamental domain. For any side $s = S_{\mu,\lambda} \cap \mathcal{B}$, we have $s' = S_{\mu,\alpha} \cap \mathcal{B}$ and $\sigma(s, s') = \begin{Bmatrix} \alpha & \beta \\ \mu & -\lambda \end{Bmatrix}$. We can state Poincare's theorem for Bianchi groups as follows.

Theorem 3.7. $SL_2(\mathcal{O})$ is generated by $J = \begin{Bmatrix} -1 & 0 \\ 0 & -1 \end{Bmatrix}, T = \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}, U = \begin{Bmatrix} 1 & w \\ 0 & 1 \end{Bmatrix}, \sigma_1, \dots, \sigma_n$ where σ_i are the corresponding transformations of the boundary spheres and the cycle relations together with the reflection relations form a complete set of relations for $SL_2(\mathcal{O})$.

Remark. From the above construction, we know that s and s' have the same radius.

Example. Let $K = \mathbf{Q}(\sqrt{-2})$. The fundamental domain of the Bianchi group $PSL_2(\mathbf{Z}[\sqrt{-2}])$ is described in section 1 (Figure 3.1). The four vertical planes are

$$S_1 = \{(x, y, z) : x = \frac{1}{2}\}, S_2 = \{(x, y, z) : x = -\frac{1}{2}\},$$

$$S_3 = \{(x, y, z) : y = \frac{\sqrt{2}}{2}\}, S_4 = \{(x, y, z) : y = -\frac{\sqrt{2}}{2}\},$$

and the geodesic sphere is $S_5 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. The four vertices are

$$p_1 = (1/2, \sqrt{2}/2, 1/2), p_2 = (-1/2, \sqrt{2}/2, 1/4),$$

$$p_3 = (-1/2, -\sqrt{2}/2, 1/4), p_4 = (1/2, -\sqrt{2}/2, 1/4).$$

By Theorem 3.7, we see that the generators of $SL_2(\mathbf{Z}[\sqrt{-2}])$ are J, T, U , and $A = \begin{Bmatrix} 0 & -1 \\ 1 & 0 \end{Bmatrix}$ with $J^2 = I$ and J central. We describe the relations as follows.

(a). Since the corresponding transformation of S_5 is A and $A \cdot S_5 = S_5$, A has a reflection relation (i.e. $A^2 = J$).

(b). Let $e_1 = \{(x, y, z) : x = -1/2, y = -\sqrt{2}/2\}$. We have $e_1 = S_2 \cap S_4$. Let $s_1 = S_2, \sigma_1 = T$. Then $e_2 = \{(x, y, z) : x = 1/2, y = -\sqrt{2}/2\}, s'_1 = S_1$. Let $s_2 = S_2, \sigma_2 = U$. Then $e_3 = \{(x, y, z) : x = 1/2, y = \sqrt{2}/2\}, s'_2 = S_3$. Let $s_3 = S_1, \sigma_3 = T^{-1}$, then $e_4 = \{(x, y, z) : x = -1/2, y = \sqrt{2}/2\}, s'_3 = S_4$. Let $s_4 = S_3, \sigma_4 = U^{-1}$. Then $e_5 = e_1$. Let $\tau = \sigma_4 \sigma_3 \sigma_2 \sigma_1$. We have $\tau = I$ (i.e. $TU = UT$).

(c). Let $e_1 = \text{arc}(p_1 p_4)$. We have $e_1 = S_1 \cap S_5$. Let $s_1 = S_5, \sigma_1 = A$. Then $e_2 = \text{arc}(p_2 p_3), s'_1 = S_5$. Let $s_2 = S_2, \sigma_2 = T$. Then $e_3 = e_1$. Let $\tau = \sigma_2 \sigma_1$. We have $\tau^3 = J$ (i.e. $(TA)^3 = J$).

(d). Let $e_1 = \text{arc}(p_3 p_4)$. We have $e_1 = S_4 \cap S_5$. Let $s_1 = S_4, \sigma = U$. Then $e_2 = \text{arc}(p_2 p_1), s'_1 = S_3$. Let $s_2 = S_5, \sigma_2 = A$. Then $e_3 = \text{arc}(p_1 p_2), s'_2 = S_5$.

Let $s_3 = S_3, \sigma_3 = U^{-1}$. Then $e_3 = \text{arc}(p_4 p_3), s'_3 = S_4$. Let $s_4 = S_5, \sigma_4 = A$. Then $e_5 = e_1$. Let $\tau = \sigma_4 \sigma_3 \sigma_2 \sigma_1$. We have $\tau^2 = J$ (i.e., $(AU^{-1}AU)^2 = J$).

Theorem 3.8 $SL_2(\mathbb{Z}[\sqrt{-2}])$ is generated by T, U, A, J with the following relations:

- (a). $J^2 = 1$,
- (b). J is central,
- (c). $TU = UT$,
- (d). $(AT)^3 = J$,
- (e). $(AU^{-1}AU)^2 = J$.

Chapter 4

Bloch Group and Dilogarithms

4.1 Introduction

Following Thurston's idea, we can triangulate the fundamental domain of Bianchi group as a union of ideal tetrahedra $\Delta_1, \dots, \Delta_k$ parametrized by complex numbers z_1, \dots, z_k with multiplities n_i . The volume of the fundamental domain for the Bianchi group $PSL_2(\mathcal{O})$ is equal to $\sum n_i D_2(z_i)$ where D_2 is the Bloch-Wigner Dilogarithm. These parameters define an element $a = \sum_i^k n_i [z_i] \in T(\mathbf{C})$ (the pre-Bloch group). In this chapter, we give two algebraic methods to determine the element a in the Bloch group. Since this element a is constructed from the fundamental domain of the Bianchi group, we would like to understand the Bloch group and relating topics (e.g. Lichtenbaum's conjecture) from this direction. As yet, we are unable to resolve all the questions in this area.

4.2 Basic Definitions

For a field F , let $A(F)$ be the free abelian group generated by z , where $z \in F, z \neq 0, 1$; and let $\lambda : A(F) \rightarrow F^* \wedge F^* = F^* \otimes F^* / S^2 A$ be the homomorphism given by $z \mapsto z \wedge (1-z)$ where $S^2 A$ is the subgroup generated by elements of the form $(a \otimes b + b \otimes a)$. The properties of \wedge may be found in *Linear Algebra* by K. Hoffman and R. Kunze. Let

$$R(x, y) = x - y + \frac{y}{x} + \frac{1 - x^{-1}}{1 - y^{-1}} + \frac{1 - x}{1 - y} \text{ for } x, y \in F \setminus \{0, 1\} \text{ and } x \neq y.$$

It is easy to show that the induced homomorphism $A(F) \rightarrow F^* \wedge F^*$ is trivial on elements of the form $R(x, y)$. We denote by $T(F)$ the factor group of $A(F)$ by the subgroup generated by the above elements.

Definition. Define $T(F) = A(F) / \langle R(x, y) | x, y \in F \setminus \{0, 1\} \text{ and } x \neq y \rangle$ and call $T(F)$ the pre-Bloch group of F . Let $B(F)$ be the kernel of the induced homomorphism $T(F) \rightarrow F^* \wedge F^*$ and call $B(F)$ the Bloch group of F .

From the above definition, we have the following exact sequence:

$$0 \rightarrow B(F) \rightarrow T(F) \rightarrow F^* \wedge F^* \rightarrow K_2 F \rightarrow 0.$$

We now give an alternative definition for Bloch group. For an arbitrary field F , define an abelian group $\mathcal{P}(F)$ generated by all 4-tuples of distinct points of the projective line $\mathbf{P}^1(F)$ which satisfies the following relations:

$$(\sigma z_0, \sigma z_1, \sigma z_2, \sigma z_3) = (z_0, z_1, z_2, z_3),$$

for $\sigma \in PGL_2(F)$, z_j distinct in $\mathbf{P}^1(F)$ for $0 \leq j \leq 3$ and

$$\sum_{i=0}^4 (-1)^i (z_0, \dots, \hat{z}_i, \dots, z_4) = 0, \quad z_i \text{ distinct in } \mathbf{P}^1(F)$$

where \hat{z}_i means that z_i is omitted. Recall the definition of the cross ratio:

$$\{z_0, z_1, z_2, z_3\} = \frac{z_0 - z_2}{z_0 - z_3} \frac{z_1 - z_3}{z_1 - z_2} \in F \setminus \{0, 1\}$$

for four different points $z_0, z_1, z_2, z_3 \in \mathbf{P}^1(F)$. Our definition is chosen so that the cross ratio of $\infty, 0, 1, z$ is just z . We have the following properties.

Lemma 4.1. (1) $PGL_2(F)$ acts exactly 3-transitively on $\mathbf{P}^1(F)$.

(2) For two quadruples (x_0, x_1, x_2, x_3) and (x'_0, x'_1, x'_2, x'_3) of distinct points there exists $\sigma \in PGL_2(F)$ with $(\sigma x_0, \sigma x_1, \sigma x_2, \sigma x_3) = (x'_0, x'_1, x'_2, x'_3)$ if and only if

$$\{x_0, x_1, x_2, x_3\} = \{x'_0, x'_1, x'_2, x'_3\}.$$

Proof. (1) For any $z_1, z_2, z_3 \in \mathbf{P}^1(F)$ with $z_i \neq z_j$, we define $\phi(z) = \{z, z_1, z_2, z_3\}$. It is easy to see that $\phi(z_1) = 1, \phi(z_2) = 0, \phi(z_3) = \infty$. Let $\sigma = \begin{Bmatrix} z_1 - z_2 & -z_2(z_1 - z_3) \\ z_1 - z_3 & -z_3(z_1 - z_3) \end{Bmatrix} \in GL_2(F)$. We can see that $\sigma \cdot z = \phi(z)$. For any two 3-tuples (z_1, z_2, z_3) and (z'_1, z'_2, z'_3) , we have $(\sigma z_1, \sigma z_2, \sigma z_3) = (1, 0, \infty)$, and $(\sigma' z'_1, \sigma' z'_2, \sigma' z'_3) = (1, 0, \infty)$ for some $\sigma, \sigma' \in GL_2(F)$. Let $\tau = \sigma'^{-1} \sigma$. Then $(\tau z_1, \tau z_2, \tau z_3) = (z'_1, z'_2, z'_3)$.

(2) If $(\sigma x_0, \sigma x_1, \sigma x_2, \sigma x_3) = (x'_0, x'_1, x'_2, x'_3)$ for some $\sigma \in PGL_2(F)$, then $\{x_0, x_1, x_2, x_3\} = \{\sigma x_0, \sigma x_1, \sigma x_2, \sigma x_3\} = \{x'_0, x'_1, x'_2, x'_3\}$.

Conversely, from (1), we can find $\sigma \in PGL_2(F)$ such that $(\sigma x_1, \sigma x_2, \sigma x_3) = (x'_1, x'_2, x'_3)$. Since $\{\sigma x_0, \sigma x_1, \sigma x_2, \sigma x_3\} = \{x_0, x_1, x_2, x_3\} = \{x'_0, x'_1, x'_2, x'_3\} = \{x'_0, \sigma x_1, \sigma x_2, \sigma x_3\}$, it follows that $\sigma x_0 = x'_0$.

Definition. Let $T'(F)$ be the abelian group generated by the symbols $[z] = (\infty, 0, 1, z)$, $z \in \mathbb{C} \setminus \{0, 1\}$ with the relations

$$\sum_{i=0}^4 (-1)^i [\{x_0, \dots, \hat{x}_i, \dots, x_4\}] = 0, \quad x_i \text{ distinct in } \mathbf{P}^1(F).$$

From Lemma 4.1, we have the following result:

Proposition 4.2. $\mathcal{P}(F)$ is isomorphic to $T'(F)$.

Proof. Let $\phi : \mathcal{P}(F) \rightarrow T'(F)$ be defined by $(z_0, z_1, z_2, z_3) \mapsto \{z_0, z_1, z_2, z_3\}$. Since $\phi((\infty, 1, 0, z)) = [z]$ for any $z \in F$, we can say the map is onto. By Lemma 4.1, it is easily seen that the map is injective.

In the relation of Proposition 4.2, any term involving a cross-ratio of non-distinct points is interpreted as zero. Thus at least four elements among x_0, \dots, x_4 must be distinct. By Proposition 4.1, three of them can be taken to be $\infty, 0, 1$ and we have the following lemma:

Lemma 4.3. *The relations in Proposition 4.2 can be written as*

$$[x] - [y] + \left[\frac{y}{x}\right] + \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0$$

where $x, y \in F \setminus \{0, 1\}$ and $x \neq y$.

Proof. Let $x_0, x_1, x_2, x_3, x_4 = \infty, 0, 1, x, y$ respectively. We can easily see that the relation in Proposition 4.2 can be written in the above form.

Denote $F^* \otimes F^* / \tilde{S}^2 A$ by $F^* \hat{\wedge} F^*$ where $\tilde{S}^2 A = \{x \in F^* \otimes F^* | 2x \in S^2 A\}$ and let $\lambda' : T'(F) \rightarrow F^* \hat{\wedge} F^*$ be defined by $[z] \mapsto z \wedge (1 - z)$. Let $g : F^* \wedge F^* \rightarrow F^* \hat{\wedge} F^*$ be defined by $x \wedge y \mapsto 2x \hat{\wedge} y$. The kernel of g is annihilated by multiplication by 2. This provides another definition for the Bloch group:

Definition. *Let $B'(F)$ be the group $\text{Ker}(\lambda')$.*

We now investigate the difference between these two definitions of Bloch group. Let $f : T(F) \rightarrow T'(F)$ be defined by $[z] \mapsto [z]$. The following diagram is commutative.

$$\begin{array}{ccc} T(F) & \rightarrow & F^* \wedge F^* \\ \downarrow & & \downarrow \\ T'(F) & \rightarrow & F^* \hat{\wedge} F^* \end{array}$$

Let $f' : B(F) \rightarrow B'(F)$ be the induced map. We have the following results:

Proposition 4.4. (1) *Coker(f') is annihilated by multiplication by 2.*

(2) *Ker(f') is annihilated by multiplication by 6.*

Proof. Part (1) follows from the definition of $B(F)$ and $B'(F)$.

It was shown in [Su2] that the following relations hold in $B(F)$:

$$6([x] + [1 - x]) = 0, \quad 2([x] + [x^{-1}]) = 0.$$

By the definition of $B(F)$ and $B'(F)$, we get (2).

Dilogarithms have been studied for a long time[Lel] and can be defined from the series

$$Li_2(z) = \sum_1^{\infty} \frac{z^n}{n^2}, \quad |z| \leq 1$$

or the integral

$$Li_2(z) = - \int_0^z \frac{\log(1-u)}{u} du \text{ for } z \in \mathbb{C} \setminus \{0, 1\}.$$

The latter extends the domain outside the unit circle, and the function, as thus defined, has real values for real z in the range $-\infty < z \leq 1$. S. Bloch and D. Wigner introduced[B12] the single-valued dilogarithm as follows:

Definition. *The real-valued function*

$$D_2(z) = \Im(Li_2(z)) + \arg(1-z) \cdot \log |z|$$

is called Bloch-Wigner dilogarithm.

The Bloch-Wigner dilogarithm has the following properties:

Lemma 4.5. (1) $-D_2(z) = D_2(\bar{z}) = D_2(1/z) = D_2(1 - z)$.

(2) $D_2(z^n) = n \sum_{\mu^n=1} D_2(\mu z)$.

(3) $D_2(z_1) - D_2(z_2) + D_2(z_1/z_2) - D_2((1 - z_1)/(1 - z_2)) + D_2((1 - z_2)z_1/(1 - z_1)z_2) = 0$, for $z_1, z_2 \in \mathbb{C} \setminus \{0, 1\}$, $z_1 \neq z_2$.

Proof. The relations of (1) come from $\arg(1/z) = \arg(\bar{z}) = -\arg(z)$ and $Li_2(z) + Li_2(1 - z) = \frac{\pi^2}{6} - \log z \cdot \log(1 - z)$ [Le1, p.5]. (2) and (3) come from the properties of Li_2 [Le1, p.6 and p.9].

The function $D_2(z)$ can be extended to the group $A(\mathbb{C})$ by linearity. From (c) of Lemma 4.5, we can define D_2 as a function from $T(\mathbb{C})$ to \mathbb{R} .

4.3 The Known Results

In this section, we will state several well-known theorems of Bloch groups and Hyperbolic Geometry. The motivation for the results stated in this section comes from the Dirichlet class number formula which states as follows: Let $F = \mathbb{Q}(\xi)$ be a cyclotomic field with discriminant d_F where ξ is an f th root of unity and let h be the class number of F . We have

$$\frac{(2\pi)^{r_2} h R}{w |d_F|^{1/2}} = \prod_{\chi \neq 1} L(1, \chi)$$

where r_2 is the number of complex valuations of F , w is the number of roots of unity in F , R is the regulator of F , and χ run through all the Dirichlet

characters of conductor f . The L – series is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

([L], page 77)

The followings are the results relating this topic.

Humbert’s Formula for the volume of the fundamental domain of the Bianchi group. *Let K be an imaginary quadratic field. The volume of the fundamental domain of the Bianchi group $PSL_2(\mathcal{O})$ is equal to*

$$\frac{|d_K|^{3/2}}{4\pi^2} \zeta_K(2)$$

where ζ_K is the Dedekind zeta function associates to K and d_K is the discriminant of K . [El-G-M]

The most important Theorem of this section is the Milnor’s formulas for the volume of the hyperbolic 3-mainfold. We state two important Milnor’s formulas.

(1) Consider an ideal tetrahedron, that is a tetrahedron Δ with all four vertices on the sphere of points at infinity. If α, β, γ are the dihedral angles along three edges meeting at a common vertex, then $\alpha + \beta + \gamma = \pi$ (e.g. Figure 4.1). The first formula of Milnor can stated as follows:

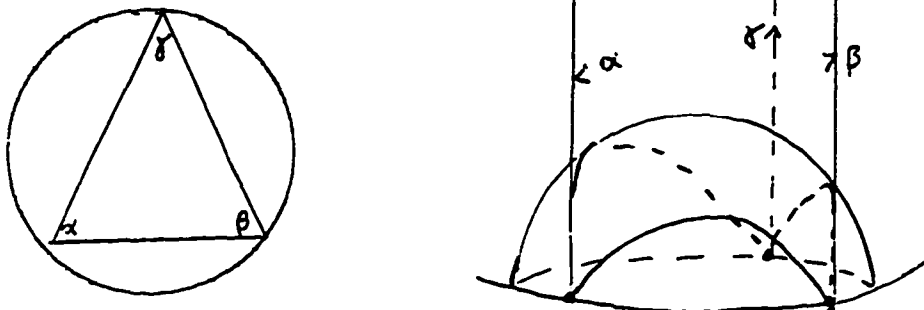


Figure 4.1:

Milnor's Formula for the volume of Hyperbolic 3-manifold (I). Let Δ be an ideal tetrahedron with dihedral angles α, β, γ . The volume of Δ is given by

$$V = \Omega(\alpha) + \Omega(\beta) + \Omega(\gamma)$$

where

$$\Omega(\theta) = \frac{1}{2} \sum_1^{\infty} \frac{\sin 2n\theta}{n^2} = - \int_0^{\theta} \log |2 \sin t| dt$$

is the Lobachevsky function. [Mi2]

Let $\Delta(z)$ be the tetrahedron with four vertices $1, 0, \infty, z$. (see figure 4.2)

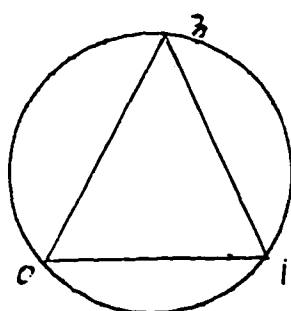


Figure 4.2:

This tetrahedron is an ideal tetrahedron. By Milnor's formula, the volume of $\Delta(z)$ is equal to $\Omega(\alpha) + \Omega(\beta) + \Omega(\gamma)$ where α, β, γ are angles of the triangle $(1, 0, z)$. We can write $\alpha = \arg(z)$, $\beta = \arg(1 - 1/z)$, $\gamma = \arg(1/(1 - z))$. Since $D_2(e^{2i\theta}) = 2\Omega(\theta)$, we obtain

$$\text{Vol}(\Delta(z)) = (D_2(\frac{z}{\bar{z}}) + D_2(\frac{1 - 1/z}{1 - 1/\bar{z}}) + D_2(\frac{1 - \bar{z}}{1 - z}))/2.$$

By (1) and (2) of Lemma 4.5, we get $\text{Vol}(\Delta(z)) = D_2(z)$.

(2) The second formula of Milnor relates the volume of the fundamental domain of the Bainchi group. Let K be an imaginary quadratic field. By Humbert's formula, the volume of the fundamental domain of the Bianchi group $PSL_2(\mathcal{O})$ is equal to

$$\frac{|d_K|^{3/2}}{4\pi^2} \zeta_K(2)$$

where ζ_K is the Dedekind zeta function associates to K and d_K is the discriminant of K . Using the standard identity

$$\zeta_K(s) = \zeta_{\mathbf{Q}}(s)L(s, \chi)$$

where χ is the character corresponding to K (i.e. $\chi(a) = (\frac{d_K}{a})$). We can express the number $L(2, \chi)$ by the fomula

$$L(2, \chi) = |d_K|^{-1/2} \sum_{t=1}^{|d_K|-1} \chi(t) D_2(e^{2\pi t/|d_K|}). \quad ([\text{Mi2}], \text{page 21})$$

We now state the second formula of Milnor.

Milnor's Formula for the volume of Hyperbolic 3-manifold (II). Let K be an imaginary quadratic field. By Humbert's formula, the volume of the fundamental domain of the Bianchi group $PSL_2(\mathcal{O})$ is equal to

$$\frac{|d_K|}{24} \sum_{t=1}^{|d_K|-1} \chi(t) D_2(e^{2\pi t/|d_K|})$$

where d_K is the discriminant of K . [Mi2]

Bloch has shown that $\text{rank}(B(F)) = \text{rank}(K_3(F))$ [Bl2, G1] where F is an algebraic number field. In order to calculate the rank of Bloch group, we now introduce Borel's Theorem.

Borel's Theorem. For a number field F and $n \geq 0$,

$$\text{rank}(K_n(\mathcal{O}_F)) = \begin{cases} 0, & \text{if } n \text{ is even, } n > 0, \\ r_2, & \text{if } n \equiv 3 \pmod{4}, \\ r_1 + r_2, & \text{if } n \equiv 1 \pmod{4}, n > 1, \\ r_1 + r_2 - 1, & \text{if } n = 1, \\ 1, & \text{if } n = 0, \end{cases}$$

where r_1 (resp. r_2) is the number of real (resp. complex) valuations of F and \mathcal{O}_F is the ring of integers of F . Moreover $\text{rank}(K_n(F)) = \text{rank}(K_n(\mathcal{O}_F))$, for $n > 1$. [Bo1, Bro2]

Remark. Let K be an imaginary quadratic field. Following Borel's theorem, we have $\text{rank}(B(K)) = \text{rank}(K_3(K)) = 1$.

The general frame work for understanding these formulas is the Conjecture of Lichtenbaum. Because we mainly interest in the Bianchi groups, we

only consider the Lichtenbaum's conjecture for imaginary quadratic fields.

Lichtenbaum's Conjecture. *Let K be an imaginary quadratic field and let \mathcal{O} be its ring of integers. Then we have the formula*

$$\#K_2(\mathcal{O}) = \frac{|d_K|^{3/2}}{2D_2(b)} L(2, \chi)$$

where d_K is the discriminant of K , b is a generator of $B(F)$ and χ is the character corresponding to K (i.e. $\chi(a) = \left(\frac{d_K}{a}\right)$). [Bro1]

The last result relating this topic is Thurston's Theorem. Here we are mainly concerned with the fundamental domain of the Bianchi group. We state Thurston's Theorem in this special case.

Thurston's Theorem. *Let \mathcal{D} be a fundamental domain of the Bianchi group $PSL_2(\mathcal{O})$. Then \mathcal{D} can be triangulated as a union of ideal tetrahedra. ([Z1], page 288 or [Z2], page 242)*

4.4 The New Results

Let $K = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field where d is a square-free positive integer and let \mathcal{O} be its ring of integers. From chapter 1, we have a fundamental domain of the Bianchi group $PSL_2(\mathcal{O})$ which is bounded by four vertical planes and finitely many geodesic spheres. Following Thurston's idea, we can triangulate the fundamental domain of Bianchi group as a union of ideal tetrahedra $\Delta_1, \dots, \Delta_k$ parametrized by complex numbers z_1, \dots, z_k with

multiplities n_i . The volume of the fundamental domain for the Bianchi group $PSL_2(\mathcal{O})$ is equal to $\sum n_i D_2(z_i)$ which D_2 is the Bloch-Wigner Dilogarithm. These parameters define an element $a = \sum_i^k n_i [z_i] \in T(\mathbb{C})$ (the pre-Bloch group). D. Zagier and W. Neumann has shown that a is an element in the Bloch group $B(\mathbb{C})$. [N-Y] In this section, we introduce two algebraic methods to get the element a from the fundamental domain of the Bianchi group.

(i) For an arbitray imaginary quadratic field, we can cut the fundamental domain in such a way that the fundamental domain is a union of finitely many tetrahedra(i.e. bounded by three vertical planes and a geodesic sphere, e.g. Figure 4.3). For any tetrahedron Δ , we can follow Milnor's idea [Mi2] to

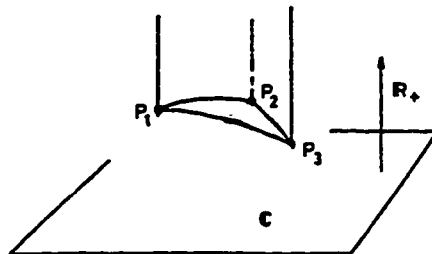


Figure 4.3:

calculate the Volume of Δ as follows: As in Figure 4.4, looking down from infinity, we see a triangle and a point P ; draw the straight lines from P to the vertices and perpendiculars from P to the sides of this triangle. In this way, we can cut the triangle into six right triangles and the tetrahedron into six tetrahedra of the kind shown in Figure 4.5. The volume of the tetrahedron

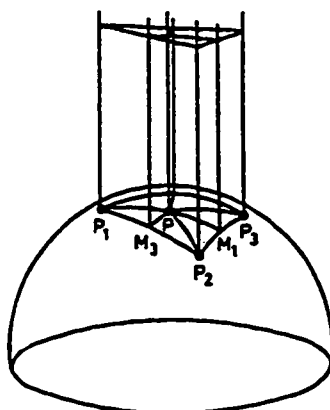


Figure 4.4:

of Figure 4.5 is given by the formula

$$\text{Vol}(\Delta_{\alpha,\gamma}) = \frac{1}{4}(\Omega(\alpha + \gamma) + \Omega(\alpha - \gamma) + 2\Omega(\frac{\pi}{2} - \alpha))$$

where Ω is the Lobachevsky function.

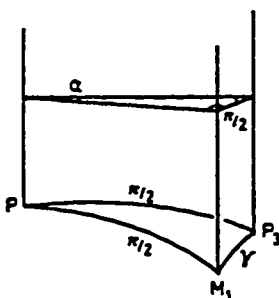


Figure 4.5:

After necessary transformations, we can assume that the tetrahedron lies above the unit sphere $S_{1,0}$ and has vertices $(0, 1), \infty, (x, \zeta), (x + iy, \zeta')$ where $x, y \in \mathbb{R}$. Hence we can write $\cos \gamma = x, \sin \gamma = \sqrt{1 - x^2}$ and $\cos \alpha =$

$x/\sqrt{x^2+y^2}$, $\sin \alpha = y/\sqrt{x^2+y^2}$. Since $D_2(e^{2i\theta}) = 2\Omega(\theta)$, we can write

$$\text{Vol}(\Delta_{\alpha,\gamma}) = \frac{1}{8}(D_2(e^{2(\alpha+\gamma)i}) + D_2(e^{2(\alpha-\gamma)i}) + 2D_2(e^{(\pi-2\alpha)i})).$$

Hence

$$\text{Vol}(\Delta_{\alpha,\gamma}) = \frac{1}{8}(D_2(z_1) + D_2(z_2) - 2D_2(z_3))$$

where $z_1 = (x^2 - y\sqrt{1-x^2} + i(xy + x\sqrt{1-x^2}))^2/(x^2 + y^2)$, $z_2 = (x^2 + y\sqrt{1-x^2} + i(xy - x\sqrt{1-x^2}))^2/(x^2 + y^2)$, and $z_3 = -(x + iy)^2/(x^2 + y^2)$.

From this construction, we have that the volume of the fundamental domain is equal to $\sum k_i D_2(z_i)$ where $k \in \mathbf{Q}$. By the choice of x, y we can assume z_i is of degree at most 4 over K . Multiplying by a suitable constant, we can get an element $\sum n_i[z_i] \in T(F)$ where $[F : K] < \infty$.

Examples. (a) $K = \mathbf{Q}(\sqrt{-2})$.

In chapter 1, we show the fundamental domain is symmetric about x, y -planes. We only need to consider the $\frac{1}{4}$ of the fundamental domain in the first quadrant. The dotted edges are spurious ones. We can cut Q as in figure 4.6. The volume of (I) is

$$\frac{1}{4}\left(\frac{1}{2}D_2(e^{2(\alpha+\gamma)i}) + \frac{1}{2}D_2(e^{2(\alpha-\gamma)i}) + D_2(e^{2(\pi/2-\alpha)i})\right)$$

where

$$\alpha = \arccos\left(\frac{1/2}{\sqrt{3}/2}\right) = \arccos\left(\frac{1}{\sqrt{3}}\right), \gamma = \arccos\left(\frac{1}{2}\right).$$

We can write $\text{Volume(I)} =$

$$\frac{1}{8}\left[D_2\left(\frac{1-2\sqrt{6}}{6} - i\left(\frac{2\sqrt{2}+\sqrt{3}}{6}\right)\right) + D_2\left(\frac{1+2\sqrt{6}}{6} + i\left(\frac{\sqrt{3}-2\sqrt{2}}{6}\right)\right) + 2D_2\left(\frac{1}{3} + i\frac{2\sqrt{2}}{3}\right)\right].$$

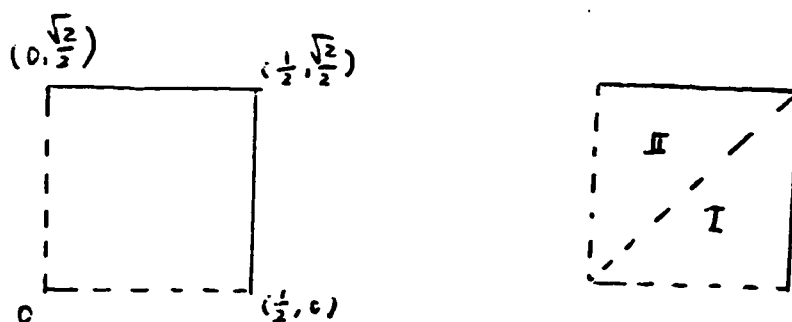


Figure 4.6:

By the same calculation, we can get $\text{Volume(II)} =$

$$\frac{1}{8} \left[D_2\left(\frac{-2\sqrt{2}}{3} + \frac{i}{3}\right) + D_2\left(\frac{2\sqrt{2}}{3}\right) + 2D_2\left(\frac{-1}{3} + i\frac{2\sqrt{2}}{3}\right) \right].$$

Let $z = (1+2\sqrt{-2})/3$ and $w = (-1+i\sqrt{3})/2$. The volume of the fundamental domain is given by

$$\begin{aligned} V &= D_2(z) + D_2(-\bar{z}) + \frac{1}{2}D_2(iz) + \frac{1}{2}D_2(-iz) + \frac{1}{2}D_2(-w\bar{z}) + \frac{1}{2}D_2(\bar{w}z) \\ &= D_2(z) + \frac{1}{2}D_2(-\bar{z}) + \frac{1}{4}D_2(-z^2) + \frac{1}{6}D_2(-\bar{z}^3). \end{aligned}$$

The second equality follows from Lemma 4.5. Let

$$a = 12\left([z] + \frac{1}{2}[\bar{z}] + \frac{1}{4}[-z^2] + \frac{1}{6}[-\bar{z}^3]\right) = 12[z] + 6[\bar{z}] + 3[-z^2] + 2[-\bar{z}^3].$$

Since $\#K_2(\mathcal{O}) = 1$ [Bro1], following Lichtenbaum's conjecture, a is a generator of the Bloch group $B(K)$.

(b) $K = \mathbf{Q}(\sqrt{-7})$

As in (a), we only need to consider $\frac{1}{4}$ of the fundamental domain. We cut Q into four parts (Figure 4.7). As in (a), we can compute the volume of the

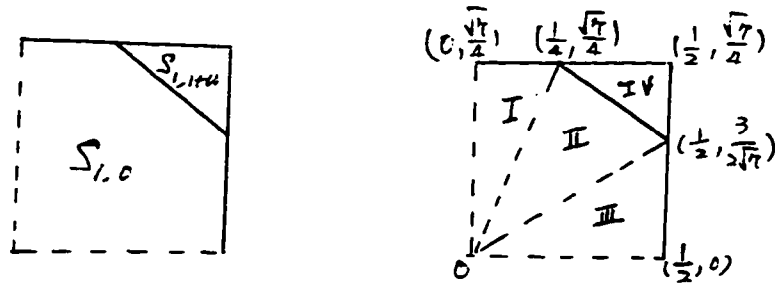


Figure 4.7:

fundamental domain as follows:

$$\begin{aligned}
 V = & 2D_2\left(\frac{-3 + \sqrt{7}i}{4}\right) + D_2\left(\frac{-\sqrt{7} + 3i}{4}\right) - D_2\left(\frac{\sqrt{7} + 3i}{4}\right) + D_2\left(\frac{1 + 3\sqrt{7}i}{8}\right) + \\
 & D_2\left(\frac{\sqrt{7} - 3\sqrt{3}}{8} + \frac{(3 + \sqrt{21})i}{8}\right) + D_2\left(\frac{3\sqrt{3} - \sqrt{7}}{8} - \frac{(3 + \sqrt{21})i}{8}\right) - \\
 & D_2\left(\frac{\sqrt{7} + 3\sqrt{3}}{8} + \frac{(\sqrt{21} - 3)i}{8}\right) - D_2\left(\frac{-\sqrt{7} - 3\sqrt{3}}{8} + \frac{(3 - \sqrt{21})i}{8}\right).
 \end{aligned}$$

Let $z = (3 + \sqrt{7}i)/4$ and $w = (-1 + i\sqrt{3})/2$, the cube root of unity. By Lemma 4.5, we can write the volume of the fundamental domain as follows:

$$V = 2D_2(z) + D_2(-z^2) - \frac{1}{6}D_2(-z^6).$$

Let $a = 6(2[z] + [-z^2] - \frac{1}{6}[-z^6]) = 12[z] + 6[-z^2] - [-z^6]$. We get an element of $B(K)$. This element coincides with an example of Browkin[Brol]. Using similar reasoning as above, a is a generator of $B(K)$.

(ii) The second method comes from how the evaluation of the L -series of an imaginary quadratic field. Following the second formula of Milnor, the volume of the fundamental domain of the Bianchi group

$$V = \frac{|d_K|}{24} \sum_{t=1}^{|d_K|-1} \chi(t) D_2(e^{2\pi t/|d_K|}).$$

Let $a' = |d_K| \sum_{t=1}^{|d_K|-1} \chi(t) [e^{2\pi t/|d_K|}]$ be an element in $T(\mathbf{C})$. It is known that $a' \in B(\mathbf{C})$ ([Gro] page 13, [Brol] page 31).

Example

This example comes from Zagier [Z2]. $K = \mathbf{Q}(\sqrt{-7})$. Since

$$\zeta_K(2) = \frac{\pi^2}{3\sqrt{7}} (D_2(e^{2\pi i/7}) + D_2(e^{4\pi i/7}) - D_2(e^{6\pi i/7}))$$

where χ is the character corresponding to K , we have the element $a = 7([e^{2\pi i/7}] + [e^{4\pi i/7}] - [e^{6\pi i/7}]) \in B(\mathbf{C})$.

4.5 Conclusion

From the last section, we can produce an element of the pre-Bloch group from the fundamental domain of the Bianchi group. In order to evaluate $\#K_2(\mathcal{O})$ from the Lichtenbaum's conjecture, we need to find a generator of the Bloch group. The following questions need to be answered.

Question 1. Is the element a that we produced in that last section belonging to the Bloch group $B(K)$?

Question 2. Let F be a finite extension of an imaginary quadratic field K and $\sum n_i [z_i] \in B(F)$ has infinite order, is there any map $\phi : B(F) \rightarrow B(K)$ such that $\phi(\sum n_i [z_i])$ has infinite order?

Question 3. Given an element a in the Bloch group $B(K)$, Can we determine a generator b of the Bloch group from this element a ?

If the answer to the above questions are positive, then we can calculate $\#K_2\mathcal{O}$ from Lichtenbaum's conjecture. As yet, we are unable to answer the questions.

Table of Symbols

Symbols	pages
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