Non-simple Closed Geodesics on 2-Orbifolds

by

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Abstract

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Given a Fuchsian group Γ, that is, a discrete subgroup of the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$, the quotient $\mathbb{H}/\Gamma$ is a 2-orbifold. If Γ contains torsion then the resulting 2-orbifold contains cone points corresponding to the elliptic fixed points. In this thesis we focus on minimal length non-simple closed geodesics on 2-orbifolds. Nakanishi, Pommerenke and Purzitsky discovered the shortest non-simple closed geodesic on a 2-orbifold, which passes through a cone point of the orbifold. This raises questions about minimal length non-simple closed geodesics disjoint from the cone points. We explore once self-intersecting closed geodesics disjoint from the cone points of the orbifold, called *figure eight geodesics*. Using fundamental domains and basic hyperbolic trigonometry we identify and classify all figure eight geodesics on triangle group orbifolds. This classification allows us to find the shortest figure eight geodesic on a triangle group orbifold. We then generalize to find the shortest figure eight geodesic on a 2-orbifold without cone points of order two.

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This dissertation is dedicated to my parents who always encouraged and believed in me, even when I did not believe in myself.

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Notation

$\mathbb{C}$: complex plane

$Re(z)$: real part of $z \in \mathbb{C}$

$Im(z)$: imaginary part of $z \in \mathbb{C}$

$\bar{z}$: complex conjugate of $z \in \mathbb{C}$

$\mathbb{U}$: upper half-plane model of the hyperbolic plane

$d_{\mathbb{U}}$: hyperbolic metric in $\mathbb{U}$

$\ell(\gamma)$: hyperbolic length of the path γ

$\mathbb{R}$: set of real numbers/real axis of the complex plane

$\mathcal{A}_g$: axis of the hyperbolic isometry g

$\mathcal{T}_g$: translation length of the hyperbolic isometry g

$\mathrm{SL}(2, \mathbb{Z})$: group of integer entry 2 x 2 matrices with determinant one

$\mathrm{SL}(2, \mathbb{R})$: group of real entry 2 x 2 matrices with determinant one

I: identity matrix

$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$

$tr(A)$: trace of the square matrix A

$|tr|$: absolute value of the trace function

$\mathbb{D}$: unit disk model of the hyperbolic plane

$d_{\mathbb{D}}$: hyperbolic metric in $\mathbb{D}$

S^1: unit circle

ρ: hyperbolic metric

$\mathbb{H}$: hyperbolic plane

$\partial\mathbb{H}$: boundary of the hyperbolic plane

$\overline{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$: closed hyperbolic plane

$Area_{\mathbb{H}}(E)$: hyperbolic area of a subset $E \subset \mathbb{H}$

$\mathrm{Isom}(\mathbb{H})$: group of isometries of $\mathbb{H}$

$\mathrm{Isom}^{+}(\mathbb{H})$: group of orientation-preserving isometries of $\mathbb{H}$

Γ: finitely generated Fuchsian group

Γ_x: stabilizer of the point $x \in \mathbb{H}$ in Γ

Γ_Y: stabilizer of a subset $Y \subset \mathbb{H}$ in Γ

$\Lambda = \Lambda(\Gamma)$: limit set of Γ

$\Omega = \Omega(\Gamma)$: ordinary set of Γ

$\Gamma(z)$: orbit of $z \in \mathbb{H}$ under Γ

$\mathbb{H}/\Gamma$: 2-dimensional orientable hyperbolic orbifold

N: Nielsen region of Γ

$\Pi_1(\mathbb{H}/\Gamma, p)$: orbifold fundamental group of $\mathbb{H}/\Gamma$ based at p

T: hyperbolic triangle

$\Gamma(p,q,r)$: (p,q,r)-triangle group

$\mathcal{O}(p,q,r)$: (p,q,r)-triangle group orbifold

$\pi(\mathcal{A}_g)$: closed geodesic corresponding to the hyperbolic isometry g

Chapter 1

Introduction

In this thesis we study the lengths of non-simple (self-intersecting) closed geodesics on a class of objects called 2-dimensional orientable hyperbolic orbifolds. Let Γ be a Fuchsian group, that is, a discrete subgroup of the group of orientation-preserving isometries of the hyperbolic plane, $\mathrm{Isom}^+(\mathbb{H})$. Consider the projection map $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$. If Γ is torsion-free then the quotient $\mathbb{H}/\Gamma$ is a 2-dimensional orientable hyperbolic manifold. However, if Γ contains torsion, then $\mathbb{H}/\Gamma$ contains *cone points* which are the projection under π of the fixed points of elliptic isometries in Γ. In this case the space $\mathbb{H}/\Gamma$ lies in a more general class called 2-dimensional orientable hyperbolic orbifolds. From now on, for simplicity, we will write 2-manifold and 2-orbifold, keeping in mind that the manifolds and orbifolds we consider here will always be orientable and hyperbolic.

If $g \in \Gamma$ is a hyperbolic isometry with axis $\mathcal{A}_g$, then the projection $\pi(\mathcal{A}_g)$ is a *closed geodesic* on $\mathbb{H}/\Gamma$. If $\pi(\mathcal{A}_g)$ passes through a cone point, then it is actually a piecewise closed geodesic, but we still refer to it as a closed geodesic. The length of a closed geodesic is the translation length of the corresponding hyperbolic isometry g. Given a Fuchsian group Γ, the *length spectrum* of Γ is the ordered set of translation lengths of conjugacy classes of hyperbolic isometries in Γ (including multiplicities): in other words, the ordered set of lengths of closed geodesics on $\mathbb{H}/\Gamma$.

The shortest non-simple closed geodesics lie on 2-orbifolds arising from hyperbolic triangles. A *triangle group* is the subgroup of orientation-preserving isometries of a discrete group generated by the reflections in the geodesic sides of a hyperbolic triangle. If the hyperbolic triangle has angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$, where $p, q, r \in \mathbb{N}$ and necessarily $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$, we call the resulting triangle group a (p, q, r)-triangle group. The quotient of $\mathbb{H}$ by a (p, q, r)-triangle group is called a (p, q, r)-triangle group orbifold, denoted $\mathcal{O}(p, q, r)$.

Yamada [25, 26] found that the shortest non-simple closed geodesics on a 2-manifold are the once self-intersecting closed geodesics on the thrice punctured sphere. Their length is $2\cosh^{-1}(3) = 4\ln(1+\sqrt{2}) = 3.52549\ldots$. Nakanishi [11] answered the same question but in the broader case of 2-orbifolds. He proved that

$$\ell_* = 2\cosh^{-1}\left[\cos\left(\frac{2\pi}{7}\right)+\frac{1}{2}\right] = 0.98399\ldots$$

is the shortest length of a non-simple closed geodesic on a 2-orbifold. Additionally, he showed that the closed geodesic γ_* pictured in Figure 5.14 has length ℓ_* and lies on $\mathcal{O}(2, 3, 7)$. Pommerenke and Purzitsky [16] gave a different (algebraic) proof that ℓ_* is the shortest length and furthermore, they showed that $\mathcal{O}(2, 3, 7)$ is the only 2-orbifold that has non-simple closed geodesics of length ℓ_*. Vogeler [24] computed the length spectrum for the (2,3,7)-triangle group and thereby showed that the shortest length is achieved uniquely by γ_*. Putting this all together, γ_* on $\mathcal{O}(2, 3, 7)$ is the shortest non-simple closed geodesic on a 2-orbifold.

The closed geodesic γ_* passes through a cone point on $\mathcal{O}(2,3,7)$. It is natural then to find the shortest non-simple closed geodesics on 2-orbifolds which are disjoint from the cone points. The results of this thesis are concerned with answering such questions. We study once self-intersecting closed geodesics disjoint from the cone points, which we call *figure eight geodesics*. By inspecting the fundamental domains of triangle groups we arrive at a classification of all figure eight geodesics on triangle group orbifolds (Theorem 5.1.4). It follows that with the exception of $\mathcal{O}(2,3,r)$ and $\mathcal{O}(2,4,r)$, all triangle group orbifolds contain a figure eight geodesic (Corollary 5.1.6). With this classification at our disposal, we find that the shortest figure eight geodesic on a triangle group orbifold is the unique one on $\mathcal{O}(3,3,4)$ whose component loops bound the order three cone points – its length is $2\cosh^{-1}\left(\frac{1+\sqrt{2}}{2}\right) = 1.26595\ldots$ (Theorem 5.1.13). From this result we look to generalize to finding the shortest figure eight geodesic on a 2-orbifold without cone points of order two. Observing that a figure eight geodesic on such a 2-orbifold lies on a type of generalized pair of pants (Proposition 5.2.3), coupled with the fact that interior lengths decrease as boundary components are shrunk down to punctures [23], we can reduce the problem to looking at triangle group orbifolds. As this was already established, we find that the shortest figure eight geodesic on a 2-orbifold without cone points of order two is the unique one on $\mathcal{O}(3,3,4)$ (Theorem 5.2.4).

In recent years, Philippe has written a series of related papers ([13], [14], [15]) in which he discusses the length spectrum of triangle groups. In [15], he used combinatorial methods to find the initial elements of the length spectrum of all triangle groups. He gave particular attention to the $(2,q,r)$-triangle groups in [14], including giving a geometric description of the shortest non-simple closed geodesic in the resulting triangle group orbifolds. In [13] he determined the length of the shortest non-simple closed geodesics on triangle group orbifolds without a cone point of order two, and found that the corresponding geodesics are in the free homotopy class of figure eight loops. It is left open to what the closed geodesics look like, and whether or not they pass through a cone point. These considerations are handled in this thesis in Theorem 5.1.3.

The thesis is organized as follows. Chapter 2 provides the basics of the hyperbolic plane, including three different classifications of the elements of $\mathrm{Isom}^+(\mathbb{H})$. In Chapter 3 we look at the structure and some properties of Fuchsian groups, as well as the resulting 2-orbifolds. The adjusted notions of paths and homotopy on 2-orbifolds are of particular importance and are usually missing from the literature (see Section 3.4). Chapter 4 contains some basic hyperbolic trigonometry followed by a detailed discussion of triangle groups and their fundamental domains, which are used to prove the classification theorem (Theorem 5.1.4). Finally, Chapter 5 includes the aforementioned results on minimal length non-simple closed geodesics, with particular focus on figure eight geodesics.

Chapter 2

The Hyperbolic Plane

2.1 The Upper Half-Plane Model

2.1.1 The Hyperbolic Metric

Let $\mathbb{U} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ and we will endow it with the metric $d_{\mathbb{U}}$ derived from the arc-length element $\frac{|dz|}{Im(z)}$, a scaling of the Euclidean arc-length element $|dz| = \sqrt{dRe(z)^2 + dIm(z)^2}$ on $\mathbb{C}$. We begin the construction of $d_{\mathbb{U}}$ in the standard way by defining the length of paths.

Suppose $\gamma : [0,1] \to \mathbb{U}$ is a piecewise C^1 (continuously differentiable) path in $\mathbb{U}$. The *hyperbolic length of* γ is defined to be

$$\ell(\gamma) = \int_\gamma \frac{|dz|}{Im(z)} = \int_0^1 \frac{|\gamma'(t)|}{Im(\gamma(t))} dt.$$

For any $z, w \in \mathbb{U}$, the *hyperbolic distance* between z and w is given by

$$d_{\mathbb{U}}(z, w) = \inf\{\ell(\gamma)\}, \tag{2.1}$$

where the infimum is taken over all piecewise C^1 paths connecting z and w. It follows that $d_{\mathbb{U}}$ is non-negative, symmetric and satisfies the triangle inequality, and thus $d_{\mathbb{U}}$ is a metric on $\mathbb{U}$. The hyperbolic distance $d_{\mathbb{U}}(\cdot,\cdot)$ satisfies the formula

$$\cosh d_{\mathbb{U}}(z,w) = 1 + \frac{|z-w|^2}{2Im(z)Im(w)},$$

where $z, w \in \mathbb{U}$ and $|\cdot|$ is the usual Euclidean norm. The metric space $(\mathbb{U}, d_{\mathbb{U}}) = \mathbb{U}$ is called the *upper half-plane model* of the *hyperbolic plane*. The boundary of $\mathbb{U}$, denoted $\partial\mathbb{U}$, is $\mathbb{R} \cup \{\infty\}$. Observe that $d_{\mathbb{U}}(i, it) = \left|\ln \frac{1}{t}\right| \to \infty$ as $t \to 0$ or $t \to \infty$. For this reason the boundary of the hyperbolic plane is referred to as the *boundary at infinity*. The union $\overline{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$ is the *closed hyperbolic plane*.

The infimum in (2.1) is realized by particular paths for any $z, w \in \mathbb{U}$. If $Re(z) = Re(w)$, then a vertical path connecting the points realizes the hyperbolic distance. However, if $Re(z) \neq Re(w)$, then there is a unique Euclidean semicircle orthogonal to $\mathbb{R}$ which passes through z and w; a path from z to w along this semicircle realizes the hyperbolic distance between the points. Therefore the *geodesics* of $\mathbb{U}$ are Euclidean vertical lines and Euclidean semicircles orthogonal to $\mathbb{R}$. The following is a list of some important properties of geodesics in the hyperbolic plane.

(1) There is a unique geodesic through any two distinct points of the hyperbolic plane.

(2) Two distinct geodesics intersect in at most one point in the closed hyperbolic plane.

(3) Given any two geodesics L_1 and L_2 which are disjoint in the closed hyperbolic plane, there is a unique geodesic which is orthogonal to both L_1 and L_2.

A *half-plane* in $\mathbb{U}$ is a connected component of the complement of a geodesic in $\mathbb{U}$. For example, the set $\{z \in \mathbb{U} \mid |z| > 1\}$ is a half-plane in $\mathbb{U}$. A *hyperbolic circle* in $\mathbb{U}$ is a set of the form $C = \{z \in \mathbb{U} \mid d_{\mathbb{U}}(x, z) = r\}$, where $x \in \mathbb{U}$ and $r > 0$ are fixed. We refer to x as the *hyperbolic center* of C and r as the *hyperbolic radius* of C. We remark that a hyperbolic circle in $\mathbb{U}$ is a Euclidean circle in $\mathbb{U}$ and vice versa, although the hyperbolic center and radius will in general be different from the Euclidean center and radius (see [1], for example).

The *angle* between two intersecting paths in $\mathbb{U}$ is defined in the same way as in $\mathbb{C}$, that is, the angle between the tangent vectors to the paths at the point of intersection. If E is a subset of $\mathbb{U}$, then we define the *hyperbolic area* of E to be

$$Area_{\mathbb{U}}(E) = \iint_E \frac{1}{y^2} dxdy,$$

where $z = x + iy$.

2.1.2 Classification of Isometries

An *isometry* of $\mathbb{U}$ is a bijective transformation $f : \mathbb{U} \to \mathbb{U}$ such that $d_{\mathbb{U}}(z, w) = d_{\mathbb{U}}(f(z), f(w))$, for all $z, w \in \mathbb{U}$. Consider the map

$$f(z) = \frac{az + b}{cz + d} \tag{2.2}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. It follows that f maps $\mathbb{U}$ onto itself and a simple calculation gives us

$$\frac{|f'(z)|}{Im(f(z))} = \frac{1}{Im(z)}$$

and hence

$$\ell(f\gamma) = \int_a^b \frac{|f'(\gamma(t))||\gamma'(t)|}{Im(f(\gamma(t)))} dt = \ell(\gamma).$$

From this it follows that f is an isometry of $\mathbb{U}$. Moreover, we note that f preserves angles between paths in $\mathbb{U}$.

The set of all isometries of $\mathbb{U}$, denoted $\mathrm{Isom}(\mathbb{U})$, form a group under composition and is generated by the set of maps of the form (2.2) along with the map $z \mapsto -\bar{z}$, which is reflection in the imaginary axis. The following are well-known transitivity properties of $\mathrm{Isom}(\mathbb{U})$.

(1) $\mathrm{Isom}(\mathbb{U})$ acts transitively on $\mathbb{U}$, that is, given any two points $z, w \in \mathbb{U}$ there exists some $f \in \mathrm{Isom}(\mathbb{U})$ such that $f(z) = w$.

(2) $\mathrm{Isom}(\mathbb{U})$ acts triply transitively on $\partial\mathbb{U}$, that is, given two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of distinct points of $\partial\mathbb{U}$, there exists some $f \in \mathrm{Isom}(\mathbb{U})$ such that $f(z_1) = w_1$, $f(z_2) = w_2$, and $f(z_3) = w_3$.

(3) $\mathrm{Isom}(\mathbb{U})$ acts transitively on the set of geodesics in $\mathbb{U}$, that is, given any two geodesics L_1 and L_2, there exists some $f \in \mathrm{Isom}(\mathbb{U})$ such that $f(L_1) = L_2$.

(4) $\mathrm{Isom}(\mathbb{U})$ acts transitively on the set of half-planes in $\mathbb{U}$.

The set of maps having the form (2.2) is the index two subgroup of orientation-preserving isometries of $\mathbb{U}$, denoted $\mathrm{Isom}^+(\mathbb{U})$. The non-trivial elements of $\mathrm{Isom}^+(\mathbb{U})$ can be classified into three (conjugation invariant in $\mathrm{Isom}(\mathbb{U})$) types based on the location and number of fixed points. We note that Brouwer's fixed point theorem guarantees that every isometry has a fixed point in $\overline{\mathbb{U}}$.

Proposition 2.1.1. *([1]). Suppose f is a non-trivial element of Isom*$^+(\mathbb{U})$. *Then, exactly one of the following holds.*

(1) f has exactly one fixed point in $\mathbb{U}$; in which case, f is called ***elliptic*** *and is conjugate in Isom$^+(\mathbb{U})$ to*

$$z \mapsto \frac{\cos(\theta)z + \sin(\theta)}{-\sin(\theta)z + \cos(\theta)}, \tag{2.3}$$

for some $\theta \in \mathbb{R}$.

(2) f has exactly one fixed point in $\partial\mathbb{U}$; in which case, f is called ***parabolic*** *and is conjugate in Isom$(\mathbb{U})$ to $z \mapsto z + 1$.*

(3) f has exactly two fixed points in $\partial\mathbb{U}$; in which case f is called ***hyperbolic*** *and is conjugate in Isom$^+(\mathbb{U})$ to $z \mapsto \lambda z$, for some $\lambda > 0, \lambda \neq 1$.*

Corollary 2.1.2. *Parabolic and hyperbolic isometries have infinite order, while elliptic isometries may have finite or infinite order.*

Suppose f is an elliptic isometry with fixed point $x \in \mathbb{U}$. Then any hyperbolic circle C with hyperbolic center x is kept invariant by f, that is, $f(C) = C$. Now by Proposition 2.1.1, f is conjugate to a map of the form (2.3), for some $\theta \in \mathbb{R}$. It follows that given any geodesic L passing through x, the geodesic $f(L)$ also passes through x and makes an angle of 2θ with L. For the above reasons, the action of f on $\mathbb{U}$ is considered a (non-Euclidean) rotation by 2θ about x.

A *horodisc* Σ in $\mathbb{U}$ is an open Euclidean disc in $\mathbb{U}$ which is tangent to $\partial\mathbb{U}$. If the point of tangency is v, we say that Σ is *based* at v. The boundary $\partial\Sigma$ of Σ is called a *horocircle*. A horodisc based at ∞ is a set of the form $\{z \in \mathbb{U} \mid Im(z) > t\}$, where $t > 0$. If we consider the parabolic isometry $f(z) = z + 1$, then it is clear that f keeps invariant any horodisc based at ∞, the fixed point of f. Since any parabolic isometry g is conjugate in Isom$(\mathbb{U})$ to f and isometries map horodiscs to horodiscs, it follows that g

keeps invariant any horodisc based at its fixed point.

If f is a hyperbolic isometry, then the unique geodesic joining the two fixed points of f is called the *axis* of f, denoted $\mathcal{A}_f$. The axis $\mathcal{A}_f$ is the unique geodesic in $\mathbb{U}$ invariant under f, and moreover f acts as translation along $\mathcal{A}_f \cap \mathbb{U}$. If $z \in \mathcal{A}_f$, then $\mathcal{T}_f = d_{\mathbb{U}}(z, f(z))$ is called the *translation length* of f. We note that translation length is invariant under conjugation.

For computational reasons, it is often useful to view an orientation-preserving isometry as an equivalence class of matrices. Denote by $\mathrm{SL}(2,\mathbb{R})$ the group of real entry 2×2 matrices with determinant one. The map

$$\phi : \mathrm{Isom}^+(\mathbb{U}) \to \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\},$$

defined by

$$\left\{ f(z) = \frac{az+b}{cz+d}, \text{for } a,b,c,d \in \mathbb{R}, ad-bc=1 \right\} \mapsto \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right) \right\}$$

is an isomorphism. Recall that the *trace* of a square matrix M, denoted $tr(M)$, is the sum of the diagonal entries in M. Now, consider the well-defined absolute value of the trace function, $|tr| : \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}_+ \cup \{0\}$, where if $M = \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] \in \mathrm{PSL}(2,\mathbb{R})$, then $|tr(M)| = |a+d|$. We can use this function to establish another classification of the orientation-preserving isometries. If $f \in \mathrm{Isom}^+(\mathbb{U})$, we use the notation $|tr(f)|$ to denote the absolute value of the trace of the element $\phi(f) \in \mathrm{PSL}(2,\mathbb{R})$.

Theorem 2.1.3. *([2]). Let f be a non-trivial element in Isom$^+(\mathbb{U})$. Then*

(i) f is elliptic $\Leftrightarrow |tr(f)| < 2$;

(ii) f is parabolic $\Leftrightarrow |tr(f)| = 2$;

(iii) f is hyperbolic $\Leftrightarrow |tr(f)| > 2$.

Remark 2.1.4. Suppose f is a hyperbolic isometry of $\mathbb{U}$. We seek a relationship between $\mathcal{T}_f$ and $|tr(f)|$. Since both quantities are invariant under conjugation, we may assume that $f(z) = \lambda z$, for some $\lambda > 0, \lambda \neq 1$. Rewriting f as $f(z) = \frac{\sqrt{\lambda} z}{\frac{1}{\sqrt{\lambda}}}$ and noting that $\mathcal{T}_f = \ln(\lambda)$, we get

$$|tr(f)| = 2\cosh\left(\frac{\mathcal{T}_f}{2}\right). \tag{2.4}$$

2.2 The Unit Disk Model

The map

$$g(z) = \frac{z-i}{z+i}$$

is a conformal isometry from $\mathbb{U}$ onto $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. From this arises the *unit disk model* of the hyperbolic plane, $(\mathbb{D}, d_{\mathbb{D}}) = \mathbb{D}$, where $d_{\mathbb{D}}$ is the metric derived from the arc-length element $\frac{2|dz|}{1-|z|^2}$. This leads to the following distance formula:

$$\cosh^2\left[\frac{1}{2} d_{\mathbb{D}}(z, w)\right] = \frac{|1 - z\bar{w}|^2}{(1-|z|^2)(1-|w|^2)},$$

where $z, w \in \mathbb{D}$ and $|\cdot|$ is the usual Euclidean norm.

The boundary at infinity $\partial\mathbb{D}$ of $\mathbb{D}$ is the unit circle, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The geodesics in this model are the Euclidean diameters of S^1 and the intersection of Euclidean circles with $\mathbb{D}$ which are orthogonal to S^1. Isom($\mathbb{D}$) has the same transitivity properties as Isom($\mathbb{U}$) and the subgroup of orientation-preserving isometries of $\mathbb{D}$ is:

$$\text{Isom}^+(\mathbb{D}) = \left\{ z \mapsto \frac{az + \bar{c}}{cz + \bar{a}} : a, b, c, d \in \mathbb{C}, |a|^2 - |c|^2 = 1 \right\}.$$

The presence of an isometry between $\mathbb{U}$ and $\mathbb{D}$ allows us to move from one model to another when needed. One advantage to the unit disk model is that an elliptic isometry can be conjugated to have the form, $z \mapsto e^{i2\theta}z$, for some $\theta \in \mathbb{R}$, which is of course a Euclidean rotation by 2θ about the origin.

For the remainder of the discussion we use ρ for both the metric in $\mathbb{U}$ and in $\mathbb{D}$. If the particular model is not important or understood, we will denote the hyperbolic plane and the closed hyperbolic plane by $\mathbb{H}$ and $\overline{\mathbb{H}}$, respectively.

2.3 Pencils

We give one more (geometric) way to classify the orientation-preserving isometries of the hyperbolic plane, which begins with the fact that the set of reflections in geodesics in $\mathbb{H}$ generates the group $\text{Isom}(\mathbb{H})$. In fact, we will see that any orientation-preserving isometry of $\mathbb{H}$ can be written as a composition of two reflections in geodesics. Note that when composing functions we apply the functions from right to left.

We start by giving explicit formulas for the reflection in a geodesic in $\mathbb{U}$, and similar formulas can be developed in $\mathbb{D}$. As mentioned above, if $\mathcal{I}$ is the positive imaginary axis, then $\gamma(z) = -\bar{z}$ is the reflection in $\mathcal{I}$. If L is an arbitrary geodesic in $\mathbb{U}$, then the transitivity of $\text{Isom}(\mathbb{U})$ guarantees the existence of an isometry f mapping $\mathcal{I}$ to L. We define *reflection in the geodesic* L to be the composition $f\gamma f^{-1}$. We note that this does not depend on the choice of f and so reflection in a geodesic is well-defined. If $L(t) = \{z \in \mathbb{U} \mid Re(z) = t\}$, then reflection in the vertical geodesic $L(t)$ is given by

$$\delta(z) = -\bar{z} + 2t.$$

If $S(c, r)$ is a Euclidean semicircle in $\mathbb{U}$ with Euclidean center $c \in \mathbb{R}$ and Euclidean radius $r > 0$, then reflection in $S(c, r)$ is

$$\psi(z) = c + \frac{r^2}{\bar{z} - c}.$$

In the special case of $S(0, 1)$, the unit Euclidean semicircle in $\mathbb{U}$, this reduces to $\psi(z) = \frac{1}{\bar{z}}$.

Each pair of geodesics, L and L', lie in a one-parameter family $\mathcal{P}$ of geodesics called the *pencil* determined by L and L'. There are three cases to consider based on the intersection of L and L'. Here, we follow the exposition of Beardon [2].

First, if L and L' are geodesics which intersect at a point $w \in \partial\mathbb{H}$, we define the pencil $\mathcal{P}$ to be the family of all geodesics with end point w. Such a pencil will be called a *parabolic pencil*. If we use the model $\mathbb{U}$ with $w = \infty$, the geodesics in $\mathcal{P}$ are simply the vertical lines. An isometry g is parabolic if and only if it can be represented as $g = \sigma_2\sigma_1$ where σ_i is the reflection in L_i, and L_1 and L_2 lie in a parabolic pencil. This can be verified by considering the case where $g(z) = z + 1$. Here we may choose L_1 to be the imaginary axis and L_2 to be the vertical line $\{z \in \mathbb{U} \mid Re(z) = \frac{1}{2}\}$. In general, given a parabolic isometry g, the associated parabolic pencil is the pencil containing all geodesics which end at the fixed point of g. Also, L_1 (or L_2) can be chosen arbitrarily from this pencil and the other L_j is uniquely determined by g.

If L and L' are geodesics which intersect at a point $w \in \mathbb{H}$, we define the pencil $\mathcal{P}$ to be the family of all geodesics through w. Such a pencil will be called an *elliptic pencil*. Here, if we choose the model $\mathbb{D}$ with $w = 0$, the geodesics in $\mathcal{P}$ are the Euclidean diameters of S^1. An isometry g is elliptic if and only if it can be represented as $g = \sigma_2\sigma_1$ where σ_i is the reflection in L_i, and L_1 and L_2 lie in an elliptic pencil. To see this we consider the elliptic isometry $g(z) = e^{i2\theta}z$, for some $\theta \in \mathbb{R}$. Here we may let L_1 be the diameter of $\mathbb{D}$ with end points ± 1, while L_2 is the diameter with end points $\pm e^{i\theta}$. Given an elliptic isometry g, the associated elliptic pencil is the pencil containing all geodesics which pass through the fixed point of g. Also, L_1 (or L_2) can be chosen arbitrarily from this pencil and the other L_j is uniquely determined by g.

Finally, suppose L and L' are geodesics disjoint in $\overline{\mathbb{H}}$. Let L_0 be the unique common orthogonal geodesic to L and L'. We define the pencil $\mathcal{P}$ to be the family of all geodesics which are orthogonal to L_0. Such a pencil will be called a *hyperbolic pencil*. If we use the model $\mathbb{U}$ and L_0 is the imaginary axis, then the geodesics in $\mathcal{P}$ are all those of the form $\{z \in \mathbb{U} \mid |z| = \text{constant}\}$. An isometry g is hyperbolic if and only if it can be represented as $g = \sigma_2\sigma_1$ where σ_i is the reflection in L_i, and L_1 and L_2 lie in a hyperbolic pencil. The axis $\mathcal{A}_g$ is the unique geodesic orthogonal to all geodesics in the pencil. Suppose $g(z) = \lambda z$, for $\lambda > 1$. Here, $\mathcal{A}_g$ is the imaginary axis and we can let $L_1 = \{z \in \mathbb{U} \mid |z| = 1\}$ and $L_2 = \{z \in \mathbb{U} \mid |z| = \sqrt{\lambda}\}$. We note that $\rho(L_1, L_2) = \inf\{\rho(z_1, z_2) \mid z_1 \in L_1, z_2 \in L_2\} = \frac{1}{2}\mathcal{T}_g$. Given a hyperbolic isometry g, the associated hyperbolic pencil is the pencil containing all geodesics orthogonal to $\mathcal{A}_g$. As in the other cases, we can choose L_1 (or L_2) arbitrarily from this pencil and the other L_j is determined by g.

There is another important way of representing a hyperbolic isometry which ultimately reduces to a composition of two reflections. An isometry g is hyperbolic if and only if it can be represented as $g = \xi_2\xi_1$ where each ξ_i is an elliptic isometry of order two with fixed point $x_i \in \mathcal{A}_g$. Here, x_1

(or x_2) can be chosen arbitrarily and the other x_i is determined by g. If $g(z) = \lambda z$, for some $\lambda > 1$, then we can write $g = \xi_2\xi_1$, where $\xi_1 = -\frac{1}{z}$ and $\xi_2 = -\frac{\lambda}{z}$. Note that the corresponding fixed points of ξ_1 and ξ_2 are, respectively, $x_1 = i$ and $x_2 = \sqrt{\lambda}i$, and observe that $\rho(x_1, x_2) = \frac{1}{2}\mathcal{T}_g$. From above, we can represent ξ_1 and ξ_2 each as a composition of two reflections in geodesics. Let $L_1 = \{z \in \mathbb{U} \mid |z| = 1\}, L_2 = \{z \in \mathbb{U} \mid Re(z) = 0\}$, and $L_3 = \{z \in \mathbb{U} \mid |z| = \sqrt{\lambda}\}$. If σ_i denotes reflection in L_i, then $\xi_1 = \sigma_2\sigma_1$ and $\xi_2 = \sigma_3\sigma_2$. Hence, $g = \xi_2\xi_1 = \sigma_3\sigma_2\sigma_2\sigma_1 = \sigma_3\sigma_1$, a composition of two reflections.

Chapter 3

Fuchsian Groups and Orbifolds

3.1 Partition of $\overline{\mathbb{H}}$

A *topological group* G is both a group and a topological space, in which the two structures are related by the requirement that the maps $g \mapsto g^{-1}$ (of G onto G) and $(g, h) \mapsto gh$ (of $G \times G$ onto G) are continuous. A topological group G is *discrete* if the topology on G is the discrete topology. Equivalently, G is discrete if the identity element is an isolated point in G.

$\mathrm{Isom}^+(\mathbb{H})$ is a topological group and we give it the compact-open topology. This topology is equivalent to uniform convergence on compact subsets of $\mathbb{H}$. A discrete subgroup Γ of $\mathrm{Isom}^+(\mathbb{H})$ is called a *Fuchsian group.*

Let G be a subgroup of $\mathrm{Isom}^+(\mathbb{H})$. We say that G *acts properly discontinuously in* $\mathbb{H}$ if and only if for every compact set K of $\mathbb{H}$, $g(K) \cap K = \varnothing$, for all but finitely many $g \in G$. It follows that G acts properly discontinuously in $\mathbb{H}$ if and only if G is a Fuchsian group. Discreteness has a major influence on the behavior and types of elements in a group. For example, there can be no infinite order elliptic isometries in a Fuchsian group. The following proposition is an important well-known fact about common fixed points in a Fuchsian group (see [2] or [10], for example). We will use it to explore some of the structure of Fuchsian groups.

Proposition 3.1.1. *Suppose $g, h \in Isom^+(\mathbb{H})$ where g is hyperbolic, and g and h have exactly one fixed point in common. Then the group $\langle g, h \rangle$ is not a Fuchsian group.*

Let Γ' be a non-trivial Fuchsian group all of whose elements fix the point $x \in \overline{\mathbb{H}}$. Recall that elliptic isometries fix points in $\mathbb{H}$, while parabolic and hyperbolic isometries fix points in $\partial\mathbb{H}$. Adding to this the fact that hyperbolic and parabolic isometries cannot share a fixed point in a Fuchsian group (Proposition 3.1.1) , we have that Γ' must contain isometries of one type only. In fact, with this result plus discreteness, we find that given any point $x \in \overline{\mathbb{H}}$, the *stabilizer* of x in a Fuchsian group Γ,

$$\Gamma_x = \{g \in \Gamma \mid g(x) = x\}$$

is a cyclic subgroup of Γ.

If Γ is a Fuchsian group all of whose elements fix the point $x \in \partial\mathbb{H}$ and each element of Γ is parabolic, we call Γ a *parabolic group*. Similarly, if Γ is a non-trivial Fuchsian group in which every element fixes $x \in \mathbb{H}$, then each element of Γ is elliptic and we say Γ is an *elliptic group*.

Let G be a parabolic subgroup of the Fuchsian group Γ. So each element of G fixes some element, say $x \in \partial\mathbb{H}$, and thus G is a subgroup of Γ_x. It follows that Γ_x is a parabolic cyclic subgroup of Γ. In fact, Γ_x is a maximal parabolic cyclic subgroup of Γ and every parabolic subgroup of Γ lies in a unique maximal parabolic cyclic subgroup of Γ. The same discussion and results are true for elliptic subgroups.

If Γ is a Fuchsian group all of whose elements fix the point $x \in \partial\mathbb{H}$ and each element of Γ is hyperbolic, we say Γ is a *hyperbolic group.* In such a group Γ, the point x is an end point of the axis of each element of Γ. It follows by Proposition 3.1.1 that all the elements of Γ have the same axis, say $\mathcal{A}$, and we call $\mathcal{A}$ the *axis* of the hyperbolic group Γ.

Let G be a hyperbolic subgroup of the Fuchsian group Γ and suppose $\mathcal{A}$ is the axis of G. Furthermore, suppose Γ does not contain any elliptic isometries of order two (which may interchange the end points of the axis of $\mathcal{A}$). Well G is a subgroup of $\Gamma_{\mathcal{A}} = \{f \in \Gamma \mid f(\mathcal{A}) = \mathcal{A})\}$. It follows that $\Gamma_{\mathcal{A}}$ is a hyperbolic cyclic subgroup of Γ. In fact, $\Gamma_{\mathcal{A}}$ is a maximal hyperbolic cyclic subgroup of Γ, and moreover, every hyperbolic subgroup of Γ lies in a unique maximal hyperbolic cyclic subgroup of Γ.

The following lemma pertains to restrictions on the elements of a Fuchsian group containing the parabolic isometry $z \mapsto z+1$.

Lemma 3.1.2. *(Shimizu-Leutbecher) Let Γ be a Fuchsian group containing the parabolic isometry $z \mapsto z+1$. Then for every $g(z) = \frac{az+b}{cz+d}$ in Γ, either $c = 0$ or $|c| \geq 1$.*

Let Γ be a Fuchsian group. A point $x \in \overline{\mathbb{H}}$ is called a *limit point* for Γ if there exists an element $z \in \overline{\mathbb{H}}$ and a sequence of distinct isometries $\{g_n\}$ in Γ such that $g_n(z) \to x$. The set of all limit points of Γ is called the *limit set* of Γ, denoted $\Lambda = \Lambda(\Gamma)$. It is a closed, Γ-invariant set which is nowhere dense in $\overline{\mathbb{H}}$. If $|\Lambda| \leq 2$, then Γ is called an *elementary group.* However, if $|\Lambda| > 2$, then Γ is called *non-elementary* and Λ is perfect, that is, every point of Λ is an accumulation point of other points in Λ. A perfect set in

$\overline{\mathbb{H}}$ is uncountable, and it follows that $|\Lambda| = 0, 1, 2$ or ∞. It is important to note that if $g \in \Gamma$ and g is not elliptic, then the fixed points of g are in Λ.

The limit set Λ of a Fuchsian group is contained in $\partial\mathbb{H}$. We say that a Fuchsian group is of the *first kind* if $\Lambda = \partial\mathbb{H}$ and of the *second kind* if $\Lambda \subsetneq \partial\mathbb{H}$.

The complement of Γ in $\overline{\mathbb{H}}$ is called the *ordinary set*, denoted $\Omega = \Omega(\Gamma)$. This set is open in $\overline{\mathbb{H}}$ and is Γ-invariant. Most importantly, Γ acts properly discontinuously in Ω. We will need the following definition to state an important property possessed by an element of Ω.

Definition 3.1.3. Let G be a subgroup of the Fuchsian group Γ. A subset Y of $\mathbb{H}$ is said to be *precisely invariant* under G in Γ if

(1) $g(Y) = Y$, for all $g \in G$, and

(2) $f(Y) \cap Y = \varnothing$, for all $f \in \Gamma - G$.

Proposition 3.1.4. *([10]). Let Γ be a Fuchsian group. Then $x \in \Omega$ if and only if*

(1) Γ_x is finite cyclic, and

(2) there exists an open neighborhood U of x which is precisely invariant under Γ_x in Γ.

It follows that if g is an elliptic isometry of a Fuchsian group Γ with fixed point x, then $x \in \Omega$ and the precisely invariant neighborhood is a sufficiently small hyperbolic disc centered at x.

Example 3.1.5. Consider the group $\Gamma = \langle z \mapsto z + 1 \rangle$ acting in $\mathbb{U}$. This is an elementary Fuchsian group of the second kind. In particular, $\Omega = \mathbb{U} \cup \mathbb{R}$ and $\Lambda = \infty$.

Example 3.1.6. The group $\Gamma = \langle z \mapsto e^{\frac{2\pi i}{n}} z \rangle$, for some integer $|n| > 1$, acting in $\mathbb{D}$, is also an elementary Fuchsian group of the second kind. Here, $\Omega = \overline{\mathbb{D}}$ and $\Lambda = \varnothing$.

All elementary Fuchsian groups are of the second kind. In fact, elementary Fuchsian groups are either cyclic or conjugate to some group $\langle g, h \rangle$, where $g(z) = \lambda z$ $(\lambda > 1)$ and $h(z) = -\frac{1}{z}$.

Example 3.1.7. The Modular group $\Gamma = \langle z \mapsto z + 1, z \mapsto -\frac{1}{z} \rangle = \mathrm{SL}(2, \mathbb{Z})$ acting in $\mathbb{U}$ is a non-elementary Fuchsian group of the first kind. It falls into a class of Fuchsian groups called triangle groups, which will be a main focal point in later chapters.

3.2 Hyperbolic Orbifolds

Let Γ be a Fuchsian group. Recall that the *orbit* of a point $z \in \mathbb{H}$ under Γ is the set $\Gamma(z) = \{g(z) \in \mathbb{H} \mid g \in \Gamma\}$. The group Γ induces the continuous, open projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$, where $\pi(z) = \Gamma(z)$. Our ultimate goal is to discuss particular loops on $\mathbb{H}/\Gamma$ called closed geodesics and to begin this journey we need to further explore the space $\mathbb{H}/\Gamma$. We will need the following definition for our description. If G is a group acting on a set X, we say G acts *freely* on X if the only element of G fixing a point in X is the identity element. If Γ is a torsion-free Fuchsian group then Γ acts both properly discontinuously and freely on $\mathbb{H}$. By basic covering space theory, it follows that π is a universal covering map with deck group isomorphic to Γ. (Recall that the deck group consists of *deck transformations* which are self homeomorphisms of the covering space which after post composing with the covering map is equal to the covering map.) Furthermore, $\mathbb{H}/\Gamma$ is a *2-*

dimensional orientable hyperbolic manifold with constant curvature -1 and it comes equipped with a hyperbolic metric endowed by the local isometry π [19].

However, if Γ contains torsion, then Γ does not act freely on $\mathbb{H}$ and thus π cannot be a covering map. Such a map π falls into a different class of covering maps, which we now define in a slightly more general setting.

Let $\widetilde{X}$ be a topological space and G a group of homeomorphisms acting on X. Set $X = \widetilde{X}/G$, and let $\pi : \widetilde{X} \to X$ be the natural projection; we endow X with the usual identification topology, so that π is both open and continuous. Assume that G acts freely except at a discrete set of points $\{\widetilde{x_m}\}$. The map $\pi : \widetilde{X} \to X$ is a *branched covering map* with *branch points* $\{\widetilde{x_m}\}$ and *ramification points* $\{x_m\}$, if the following hold.

(i) $\pi : \widetilde{X} - \{\widetilde{x_m}\} \to X - \{x_m\}$ is a covering with deck group G;

(ii) For every branch point $\widetilde{x}$, $G_{\widetilde{x}}$ is finite, and there is a topological disc U containing $\widetilde{x}$, called a *nice neighborhood* of $\widetilde{x}$, so that U is precisely invariant under $G_{\widetilde{x}}$ in G.

For each branch point $\widetilde{x}$, the *branch number*, or *order* at $\widetilde{x}$ is the order v of $G_{\widetilde{x}}$. This number v is also called the *ramification number* or *order* at $x = \pi(\widetilde{x})$.

It follows that $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is a branched regular covering map if Γ is a Fuchsian group containing elliptic isometries. In this context the branch points are the elliptic fixed points and the ramification points are called *cone points*. The nice neighborhood about each elliptic fixed point is some hyperbolic disc centered at the fixed point with small enough radius (guaranteed by Proposition 3.1.4). The quotient $\mathbb{H}/\Gamma$ fails to be a manifold at the cone points and it falls into a broader class called *2-dimensional orientable* *hyperbolic orbifolds.* We note that any 2-dimensional orientable hyperbolic manifold or orbifold can be realized as $\mathbb{H}/\Gamma$, for some Fuchsian group Γ. From now on, for simplicity, we will write 2-manifold and 2-orbifold, keeping in mind that the manifolds and orbifolds we consider here will always be orientable and hyperbolic.

3.3 Fundamental Domains

In order to better understand the space $\mathbb{H}/\Gamma$, we look at fundamental domains for the action of the Fuchsian group Γ on $\mathbb{H}$. In our preliminary discussion of fundamental domains we generalize and consider discrete subgroups of Isom($\mathbb{H}$), that is, we allow the discrete groups to have orientation-reversing isometries. In this section we follow the exposition of Beardon [2].

A *fundamental set* for a discrete subgroup G of Isom($\mathbb{H}$) is a subset $F \subset \mathbb{H}$ which contains exactly one point from each orbit in $\mathbb{H}$. It follows that

$$\bigcup_{f \in \Gamma} f(F) = \mathbb{H}.$$

A subset D of $\mathbb{H}$ is a *fundamental domain* for a discrete subgroup G of Isom($\mathbb{H}$) if and only if

(1) D is open and connected;

(2) there is some fundamental set F of G with $D \subset F \subset \bar{D}$;

(3) $Area_{\mathbb{H}}(\partial D) = 0$.

Example 3.3.1. A fundamental domain for the Modular group $\mathrm{SL}(2,\mathbb{Z})$ is the intersection of the following three regions in $\mathbb{U}$: $Re(z) > -\frac{1}{2}, Re(z) < \frac{1}{2}$, and $|z| > 1$.

If D is a fundamental domain for a discrete subgroup G of $\mathrm{Isom}(\mathbb{H})$, then for all non-trivial elements $g \in G$, $g(D) \cap D = \varnothing$, and $\bigcup_{f \in G} f(\bar{D}) = \mathbb{H}$. These facts say that D is precisely invariant under $\{id\}$ in G and that $\bar{D}$ and its images tesselate the hyperbolic plane. The area of a fundamental domain D for G depends only on the group G and not on the choice of D. Furthermore, we have the following result about fundamental domains of subgroups (see [2]).

Theorem 3.3.2. *Suppose G is a discrete subgroup of Isom$(\mathbb{H})$ with fundamental domain D, and let G^* be an index n subgroup in G. If we decompose G into cosets, say $G = \bigcup_n G^* g_n$, then $D^* = \bigcup_n g_n(D)$ is a fundamental domain for G^*.*

Now, let D be a fundamental domain for the Fuchsian group Γ. From above we know that the group Γ induces the projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$, where $\pi(z) = \Gamma(z)$. We may also use Γ to induce an equivalence relation on $\bar{D}$ by identifying equivalent points on ∂D. Thus, with $\bar{D}/\Gamma$ inheriting the quotient topology, we have another continuous projection $\bar{\pi} : \bar{D} \to \bar{D}/\Gamma$, where $\bar{\pi}(z) = \bar{D} \cap \Gamma(z)$. Let $\varepsilon : \bar{D} \to \mathbb{H}$ be the inclusion map. We may now define a map $\theta : \bar{D}/\Gamma \to \mathbb{H}/\Gamma$ by $\theta(\bar{D} \cap \Gamma(z)) = \Gamma(z)$, and it follows that $\theta\bar{\pi} = \pi\varepsilon$. In general, the map θ is not a homeomorphism. However, the following condition will guarantee that θ be a homeomorphism.

A fundamental domain D for a Fuchsian group Γ is said to be *locally finite* if and only if each compact subset of $\mathbb{H}$ meets only finitely many Γ-images of $\bar{D}$.

Theorem 3.3.3. *([2]). The map θ is a homeomorphism between $\bar{D}/\Gamma$ and $\mathbb{H}/\Gamma$ if and only if D is a locally finite fundamental domain for Γ.*

The important consequence of this theorem and of fundamental domains in general is that we can understand the quotient space $\mathbb{H}/\Gamma$ via the action of Γ on a locally finite fundamental domain, which often times is easier to visualize and understand then $\mathbb{H}/\Gamma$ itself.

Example 3.3.4. A locally finite fundamental domain for the group $\Gamma = \langle z \mapsto z+1 \rangle$ is the strip $D = \{z \in \mathbb{U} \mid 0 < Re(z) < 1\}$. The resulting 2-manifold $\mathbb{H}/\Gamma \cong \bar{D}/\Gamma \cong S^1 \times \mathbb{R}$. An open neighborhood of $\Gamma(\infty)$ in $\mathbb{H}/\Gamma$ is called a *cusp* and it is hyperbolically a punctured disc.

Example 3.3.5. Using polar coordinates $z = re^{i\theta} \in \mathbb{D}$, the group $\Gamma = \langle z \mapsto e^{\frac{2\pi i}{n}} z \rangle$, for some integer $|n| > 1$, has locally finite fundamental domain, the sector $D = \{z \in \mathbb{D} \mid 0 < \theta < \frac{2\pi}{n}\}$. The resulting 2-orbifold $\mathbb{H}/\Gamma \cong \bar{D}/\Gamma$ is a *hyperbolic cone.*

Let $E \subset \mathbb{H}$ be precisely invariant under the subgroup H in the Fuchsian group Γ. We consider the quotient space E/H, which in general, is easier to discuss then the projection of E under the quotient map $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$. The two spaces E/H and $\pi(E)$ will not always be homeomorphic as the next example shows.

Example 3.3.6. Let $E = \{z \in \mathbb{H} \mid 0 \leq Re(z) < 1\}$ and let $\Gamma = \langle z \mapsto z+1 \rangle$. Then E is precisely invariant under $H = \{id\}$ and so $E/H \cong E$, which is simply connected. However, $\pi(E)$ contains a puncture and so is not simply connected.

The following theorem yields sufficient conditions for the two spaces to be homeomorphic.

Theorem 3.3.7. *([2]). Suppose $E \subset \mathbb{H}$ is precisely invariant under H in the Fuchsian group Γ. If either*

(i) E is open in $\mathbb{H}$, or

(ii) E/H is compact

then E/H (with the quotient topology) and $\pi(E)$ (with the subspace topology of $\mathbb{H}/\Gamma$) are homeomorphic.

We give two examples to demonstrate the usefulness of this theorem.

Example 3.3.8. Let Γ be a Fuchsian group containing the parabolic isometry $f(z) = z + 1$ and further suppose $\Gamma_\infty = \langle f \rangle$. Consider the horodisc $E = \{z \in \mathbb{H} \mid Im(z) > 1\}$. It follows by an application of Lemma 3.1.2 that E is precisely invariant under $\langle f \rangle$ in Γ. Hence, $E/\langle f \rangle$ is a cusp and since E is open in $\mathbb{H}$, it follows that $\pi(E)$ is a cusp in $\mathbb{H}/\Gamma$. In fact, it follows that for every conjugacy class of a maximal parabolic cyclic subgroup of Γ there exists a cusp in $\mathbb{H}/\Gamma$.

Example 3.3.9. Let g be a hyperbolic isometry in the Fuchsian group Γ having axis $\mathcal{A}_g$. Suppose $\mathcal{A}_g$ is precisely invariant under $\Gamma_{\mathcal{A}_g}$ in Γ. Further assume that $\Gamma_{\mathcal{A}_g}$ does not contain elliptic isometries of order two. Thus $\Gamma_{\mathcal{A}_g}$ is the maximal cyclic subgroup containing g. Assume that $\Gamma_{\mathcal{A}_g} = \langle g \rangle$. Now $\mathcal{A}_g/\langle g \rangle$ is compact and, in fact, a simple closed curve. Thus by Theorem 3.3.7 the projection $\pi(\mathcal{A}_g)$ in $\mathbb{H}/\Gamma$ is also a simple closed curve.

A *boundary hyperbolic isometry* of a Fuchsian group Γ has axis that bounds an interval of the ordinary set $\Omega(\Gamma)$ on $\partial\mathbb{H}$. It follows that boundary hyperbolic isometries only exist in Fuchsian groups of the second kind. In Example 3.3.9, if g is a boundary hyperbolic isometry, then the resulting simple closed curve $\pi(\mathcal{A}_g)$ in $\mathbb{H}/\Gamma$ is called a *boundary component* of $\mathbb{H}/\Gamma$.

Let Γ be a finitely generated Fuchsian group. Then it follows that Γ will have finitely many conjugacy classes of maximal parabolic, elliptic, and boundary hyperbolic cyclic subgroups. Suppose $\mathbb{H}/\Gamma$ has genus g, and Γ contains c conjugacy classes of maximal elliptic cyclic subgroups of orders $m_1, \ldots, m_c \geq 2$, n conjugacy classes of maximal parabolic cyclic subgroups, and b conjugacy classes of maximal boundary hyperbolic cyclic subgroups. Then, we say Γ has *signature*

$$(g : m_1, \ldots, m_c; n; b). \tag{3.1}$$

The import is that if we suppose Γ has the signature (3.1), then the resulting 2-orbifold $\mathbb{H}/\Gamma$ has genus g, c cone points of orders $m_1, \ldots, m_c$, n punctures, and b boundary components.

Let Γ be a non-elementary finitely generated Fuchsian group. If Γ is of the first kind, we let $N = \mathbb{H}$. Suppose Γ is of the second kind. Then $\partial\mathbb{H}$ is the disjoint union of $\Lambda(\Gamma)$ and a countable union of disjoint open intervals I_j. The end points of each I_j are also the end points of the axis $\mathcal{A}_j$ of some boundary hyperbolic subgroup Γ_j in Γ. Let H_j be the half-plane bounded by $\mathcal{A}_j$ and separated from I_j by $\mathcal{A}_j$. Let $N = \bigcap_j H_j$. Then in both cases we call N the *Nielsen region* of Γ, and it follows that N is the smallest non-empty Γ-invariant open convex subset of $\mathbb{H}$.

Theorem 3.3.10. *([2]). Let Γ be a non-elementary finitely generated Fuch-*

sian group of signature (3.1) with Nielsen region N. Then

$$Area_{\mathbb{H}}(N/\Gamma) = 2\pi \left\{ 2g - 2 + n + b + \sum_{j=1}^{c} \left(1 - \frac{1}{m_j}\right) \right\}. \tag{3.2}$$

Given a Fuchsian group Γ with Nielsen region N, we define the *area* of $\mathbb{H}/\Gamma$ to be $Area_{\mathbb{H}}(N/\Gamma)$. The next theorem states exactly which signatures of non-elementary Fuchsian groups are possible.

Theorem 3.3.11. *([2]). There is a non-elementary finitely generated Fuchsian group with the signature (3.1) if and only if*

$$2g - 2 + n + b + \sum_{j=1}^{c} \left(1 - \frac{1}{m_j}\right) > 0. \tag{3.3}$$

The forward implication follows from Theorem 3.2, while the reverse implication involves an application of Poincaré's polygon theorem [2] – a theorem which given a polygon P with a set of transformations $\{s_i\}$ pairing the sides of P, gives conditions for the group generated by the s_i to be discrete with fundamental domain P.

3.4 Paths and Homotopy on Orbifolds

Suppose Γ is a Fuchsian group and let $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ be the projection map. Recall that if Γ contains elliptic isometries then π is a branched covering map. The projection under π of the elliptic fixed points correspond to cone points on $\mathbb{H}/\Gamma$. A point on $\mathbb{H}/\Gamma$ which is not a cone point will be called a *regular point*.

Now with a better understanding of the space $\mathbb{H}/\Gamma$, we would like to establish an association between loops on $\mathbb{H}/\Gamma$ and the Fuchsian group Γ. The first step is to define the notion of a path on $\mathbb{H}/\Gamma$. It will be crucial that

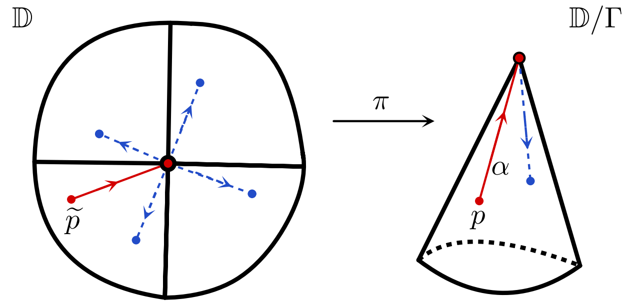

Figure 3.1: A path passing through a cone point of order four and its lifts

we have the uniqueness of path liftings property from covering space theory. Fix a regular point $p \in \mathbb{H}/\Gamma$ and some lift $\widetilde{p} \in \mathbb{H}$ of p to be base points. Note that we cannot simply deem a continuous map $\alpha : [0,1] \to \mathbb{H}/\Gamma$, $\alpha(0) = p$, to be a path on $\mathbb{H}/\Gamma$ since if the path passes through a cone point we would not have a unique lift of α to $\mathbb{H}$ starting at $\widetilde{p}$. For example, in Figure 3.1 we have the group $\Gamma = \langle z \mapsto iz \rangle$ acting on $\mathbb{D}$. The path α passes through the cone point of order four and we see the four possible continuous lifts to $\mathbb{D}$. In general, if α passes through a cone point of order n, then there are n choices of a continuous lift of α starting at $\widetilde{p}$.

Since we would like our lifts to be unique, when specifying a path in $\mathbb{H}/\Gamma$ which passes through a cone point we must specify a lift as well. We therefore make the following definition.

Definition 3.4.1. A *path* in $\mathbb{H}/\Gamma$ consists of a pair $(\alpha, \widetilde{\alpha})$, in which

(1) $\alpha : [0,1] \to \mathbb{H}/\Gamma$, is a continuous map with $\alpha(0) = p$, and there are at most finitely many $t \in [0,1]$ such that $\alpha(t)$ is a cone point; and

(2) $\widetilde{\alpha} : [0,1] \to \mathbb{H}$ is some continuous lift of α to $\mathbb{H}$ with $\widetilde{\alpha}(0) = \widetilde{p}$.

If the context is clear we will denote the pair as simply α.

Remark 3.4.2. Note that if α does not pass through a cone point then $\widetilde{\alpha}$ is uniquely defined since the map π is a covering map in the complement of the elliptic fixed points, and thus has the uniqueness of path liftings property.

Definition 3.4.3. Let α and β be two paths in $\mathbb{H}/\Gamma$. We say that α and β are $\mathbb{H}/\Gamma$-*homotopic*, denoted $\alpha \sim \beta$, if their respective lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ have the same end point.

Ultimately $\mathbb{H}/\Gamma$-homotopy is the usual homotopy in the complement of the cone points, but we need to take a closer look at what happens to a path as it is $\mathbb{H}/\Gamma$-homotoped passed a cone point. We will consider an example for an elliptic isometry of order three, but it generalizes to an elliptic isometry of any finite order (see [6]). As the definition indicates homotopic paths in $\mathbb{H}$ project to $\mathbb{H}/\Gamma$-homotopic paths in $\mathbb{H}/\Gamma$. Thus, the top row of Figure 3.2 represents three homotopic paths in $\mathbb{D}$ with end points on the boundary of a hyperbolic disc centered at the origin, which we normalize to be the fixed point of an elliptic isometry of order three. Below each figure is an aerial view of the corresponding $\mathbb{D}/\Gamma$-homotopic paths on the resulting cone. Hence, we see that a path going around the cone point of order three once in one direction is $\mathbb{D}/\Gamma$-homotopic to a path going around the cone point twice in the opposite direction. This observation is closely related to the order of the cone point and will become more transparent as we establish the relationship between loops and the Fuchsian group.

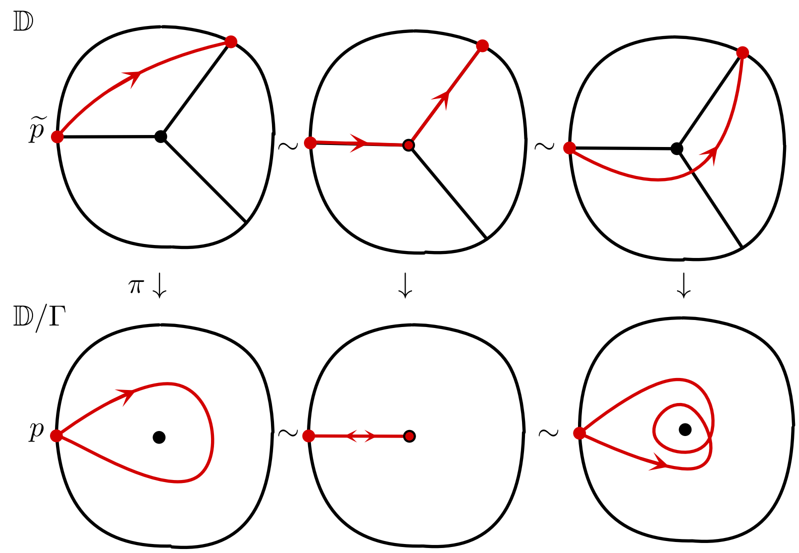

Figure 3.2: $\mathbb{D}/\Gamma$-homotopic paths about a cone point of order three

Our primary concern is closed paths. A *loop* in $\mathbb{H}/\Gamma$ based at p is a path α in $\mathbb{H}/\Gamma$ such that $\alpha(0) = \alpha(1) = p$.

Remark 3.4.4. If α is a loop based at p, then the end point of its lift $\widetilde{\alpha}$ is $g(\widetilde{p})$, for some $g \in \Gamma$. In fact, g is unique since we chose p to be a regular point.

$\mathbb{H}/\Gamma$-homotopy induces an equivalence relation on the set of loops in $\mathbb{H}/\Gamma$ based at p. Suppose $(\alpha, \widetilde{\alpha})$ and $(\beta, \widetilde{\beta})$ are two loops in $\mathbb{H}/\Gamma$ based at p and furthermore, suppose the end point of $\widetilde{\alpha}$ is $g(\widetilde{p})$, for some $g \in \Gamma$. The *concatenation* of $(\alpha, \widetilde{\alpha})$ with $(\beta, \widetilde{\beta})$, denoted $(\alpha, \widetilde{\alpha}) * (\beta, \widetilde{\beta})$ is defined to be the loop $(\alpha * \beta, \widetilde{\alpha} * g\widetilde{\beta})$, where

$$\alpha * \beta = \begin{cases} \alpha(2t) & ,0 \leq t \leq 1/2 \\ \beta(2t-1) & ,1/2 \leq t \leq 1 \end{cases}$$

and similarly,

$$\widetilde{\alpha} * g\widetilde{\beta} = \begin{cases} \widetilde{\alpha}(2t) & ,0 \leq t \leq 1/2 \\ g\widetilde{\beta}(2t-1) & ,1/2 \leq t \leq 1 \end{cases}$$

We remark that we have defined concatenation of loops to be performed from left to right, and recall that function composition was defined from right to left. Concatenation of loops respects $\mathbb{H}/\Gamma$-homotopy, that is, if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha * \beta \sim \alpha' * \beta'$. Hence we define the *orbifold fundamental group* of $\mathbb{H}/\Gamma$ based at p, denoted $\Pi_1(\mathbb{H}/\Gamma, p)$, to be the set of $\mathbb{H}/\Gamma$-homotopy classes of loops based at p under the operation of concatenation. It can be shown that $\Pi_1(\mathbb{H}/\Gamma, p)$ is indeed a group. We now come to the main purpose for the above definitions.

Theorem 3.4.5. *Suppose Γ is a Fuchsian group and $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is the projection map. Fix a regular point $p \in \mathbb{H}/\Gamma$ and a lift $\widetilde{p} \in \pi^{-1}(p)$. Then $\Pi_1(\mathbb{H}/\Gamma, p)$ is isomorphic to Γ.*

Proof. Let α be a loop in $\mathbb{H}/\Gamma$ based at p. Since α is a loop the end point of the lift $\widetilde{\alpha}$ is $g(\widetilde{p})$, for some $g \in \Gamma$. Let the map

$$\Phi_{p,\widetilde{p}} : \Pi_1(\mathbb{H}/\Gamma, p) \to \Gamma$$

be defined by $\Phi_{p,\widetilde{p}}([\alpha]) = g$. The subscripts on Φ are to suggest that the map depends on the choice of base points, but we will suppress these in the rest of the proof as the base points are fixed. We now show that Φ is an isomorphism.

First we must prove that Φ is well-defined. Suppose α and β are $\mathbb{H}/\Gamma$-homotopic loops based at p. By definition this means that the lifts of α and β starting at $\widetilde{p}$ have the same end point, say $g(\widetilde{p})$, for some $g \in \Gamma$. Thus

$\Phi([\alpha]) = \Phi([\beta]) = g$ so Φ is well-defined.

Let $g \in \Gamma$. Since $\mathbb{H}$ is path connected there exists a path $\widetilde{\alpha}$ from $\widetilde{p}$ to $g(\widetilde{p})$ and moreover it can be made to pass through at most finitely many elliptic fixed points. Thus $\pi(\widetilde{\alpha}) = \alpha$ is a loop based at p such that $\Phi([\alpha]) = g$ and so Φ is surjective.

Suppose $[\alpha], [\beta] \in \Pi_1(\mathbb{H}/\Gamma, p)$ such that $\Phi([\alpha]) = \Phi([\beta]) = g$. It follows that the lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ both end at $g(\widetilde{p})$. Thus by definition the loops α and β are $\mathbb{H}/\Gamma$-homotopic, and so Φ is injective.

Finally we show that Φ is a homomorphism, that is, $\Phi([\alpha * \beta]) = \Phi([\alpha])\Phi([\beta])$. Suppose $\Phi([\alpha]) = g$ and $\Phi([\beta]) = h$. Then the end points of the lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ starting at $\widetilde{p}$ are $g(\widetilde{p})$ and $h(\widetilde{p})$, respectively. Thus, we need to show that $\Phi([\alpha * \beta]) = gh$. In other words, we want the end point of the lift $\widetilde{\alpha} * g\widetilde{\beta}$ starting at $\widetilde{p}$ to have end point $gh(\widetilde{p})$. Note that $g\widetilde{\beta}$ is a path starting at $g(\widetilde{p})$ and ending at $gh(\widetilde{p})$. It follows that the concatenation $\widetilde{\alpha} * g\widetilde{\beta}$ starts at $\widetilde{p}$ and ends at $gh(\widetilde{p})$, as desired. □

The power of this theorem is that with base points chosen if working with an $\mathbb{H}/\Gamma$-homotopy class of loops on $\mathbb{H}/\Gamma$ we can instead look to the associated isometry of the Fuchsian group Γ when needed. This will be a main tool in proving theorems in the final chapter.

3.5 Closed Geodesics

Now with a better understanding of loops and homotopy on $\mathbb{H}/\Gamma$ we define our main objects of study, closed geodesics.

Definition 3.5.1. Let g be a hyperbolic isometry in the Fuchsian group Γ with axis $\mathcal{A}_g$. If $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is the projection map, then $\pi(\mathcal{A}_g)$ is a *closed* *geodesic* on $\mathbb{H}/\Gamma$.

We remark that in the case that $\pi(\mathcal{A}_g)$ passes through a cone point, the resulting geodesic is a piecewise closed geodesic with smoothness lacking at the cone points. For our purposes we will still refer to such phenomena as closed geodesics. Furthermore, unless otherwise stated we will assume all closed geodesics are primitive.

Fix regular points p and q, as well as lifts $\widetilde{p}$ and $\widetilde{q}$ of p and q, respectively. Suppose α and β are loops on $\mathbb{H}/\Gamma$ where α is based at p and β is based at q. We say that α is *freely* $\mathbb{H}/\Gamma$*-homotopic* to β if and only if there exists a path δ on $\mathbb{H}/\Gamma$ which starts at p, ends at q, and is disjoint from the cone points of $\mathbb{H}/\Gamma$, so that α is $\mathbb{H}/\Gamma$-homotopic to $\delta * \beta * \delta^{-1}$, that is, the end points of the lifts of α and $\delta * \beta * \delta^{-1}$ starting at $\widetilde{p}$ coincide.

A loop α on $\mathbb{H}/\Gamma$ is called *essential* if it does not bound a topological disc on $\mathbb{H}/\Gamma$.

Theorem 3.5.2. *Suppose Γ is a Fuchsian group and $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is the projection map. Fix a regular point $p \in \mathbb{H}/\Gamma$ and a lift $\widetilde{p}$ of p. If α is an essential loop on $\mathbb{H}/\Gamma$ based at p, then exactly one of the following holds.*

(1) There exists a unique closed geodesic in the free $\mathbb{H}/\Gamma$-homotopy class of α.

(2) There is no closed geodesic in the free $\mathbb{H}/\Gamma$-homotopy class of α; furthermore, there is a loop in the free $\mathbb{H}/\Gamma$-homotopy class of α that bounds a puncture.

(3) There is no closed geodesic in the free $\mathbb{H}/\Gamma$-homotopy class of α; furthermore, there is a loop in the free $\mathbb{H}/\Gamma$-homotopy class of α that bounds a cone point.

Proof. Let $\Phi_{p,\widetilde{p}} : \Pi_1(\mathbb{H}/\Gamma, p) \to \Gamma$ be the corresponding isomorphism and suppose $\Phi_{p,\widetilde{p}}([\alpha]) = g$. The three cases arise respectively from whether g is hyperbolic, parabolic, or elliptic. As the proof is very similar to the 2-manifold case, we only verify the existence part of (1). So suppose g is hyperbolic. Choose a path $\widetilde{\delta}$ in $\mathbb{H}$ which starts at $\widetilde{p}$, ends at some point $\widetilde{q} \in \mathcal{A}_g$, and does not pass through any elliptic fixed points. Let $\widetilde{\gamma}$ be the path on $\mathcal{A}_g$ from $\widetilde{q}$ to $g(\widetilde{q})$. It follows that $g(\widetilde{\delta^{-1}})$ is a path from $g(\widetilde{q})$ to $g(\widetilde{p})$. Thus, the paths $\widetilde{\alpha}$ and $\widetilde{\delta} * \widetilde{\gamma} * g(\widetilde{\delta^{-1}})$ are homotopic paths in $\mathbb{H}$. Letting $\delta = \pi(\widetilde{\delta})$ and $\gamma = \pi(\widetilde{\gamma}) = \pi(\mathcal{A}_g)$, we have that α is $\mathbb{H}/\Gamma$-homotopic to $\delta * \gamma * \delta^{-1}$. Thus, α is freely $\mathbb{H}/\Gamma$-homotopic to the closed geodesic γ.

□

Definition 3.5.3. Let Γ be a Fuchsian group and suppose γ is a closed geodesic on $\mathbb{H}/\Gamma$. Then the lift $\widetilde{\gamma}$ of γ is a geodesic segment lying on the axis $\mathcal{A}_g$, of some hyperbolic isometry $g \in \Gamma$. Now consider all axes $\{\mathcal{A}_{h_i}\}$ of hyperbolic isometries h_i conjugate to g in Γ which intersect $\widetilde{\gamma}$ transversely. We define the *self-intersection number* of γ to be the positive integer $|\{\mathcal{A}_{h_i}\}|$. Suppose γ has self-intersection number n. If $n = 0$, then we call γ a *simple closed geodesic*, while if $n > 0$, then we call γ a *non-simple closed geodesic.*

Definition 3.5.4. Let Γ be a Fuchsian group. A non-simple closed geodesic on $\mathbb{H}/\Gamma$ which is disjoint from the cone points and has one self-intersection is called a *figure eight geodesic.*

Chapter 4

Triangle Groups

4.1 Hyperbolic Trigonometry

4.1.1 Area of a Hyperbolic Triangle

Many of the results of this thesis involve the study of triangle groups, a particular class of Fuchsian groups which are intimately connected to hyperbolic triangles. Therefore, it is important for us to develop some basic facts of hyperbolic trigonometry. We begin with the definition of a hyperbolic triangle.

A *hyperbolic triangle* in $\overline{\mathbb{H}}$ is the analogue of a Euclidean triangle in $\mathbb{C}$ except that the sides of a hyperbolic triangle are geodesic segments in $\overline{\mathbb{H}}$. We adopt the following notation for hyperbolic triangles. A hyperbolic triangle T will have vertices labelled v_a, v_b, and v_c; the sides opposite these vertices will have hyperbolic lengths a, b, and c respectively and the interior angles at the vertices will be α, β, and γ. We allow the vertices of a hyperbolic triangle to be in $\partial\mathbb{H}$. If, for example, we are in the model $\mathbb{U}$ and v_c is at ∞, then $\gamma = 0$ and $a = b = \infty$.

Our first striking result about hyperbolic triangles is the area formula. The result is well-known and we refer the reader to [1] or [2] for a proof.

Theorem 4.1.1. *For any hyperbolic triangle T with angles α, β, and γ,*

$$Area_{\mathbb{H}}(T) = \pi - (\alpha + \beta + \gamma).$$

Thus, the area of a hyperbolic triangle depends solely on the angles of the triangle. This theorem also yields the following surprising fact.

Corollary 4.1.2. *The angle sum of a hyperbolic triangle is less than π.*

Even more can be said about the angles of a hyperbolic triangle.

Theorem 4.1.3. *Given any three real numbers $\alpha, \beta, \gamma \in [0, \pi)$ such that $\alpha + \beta + \gamma < \pi$, there exists a hyperbolic triangle T in $\overline{\mathbb{H}}$ having angles α, β, and γ.*

Proof. We use the unit disk model $\mathbb{D}$. Let L be the geodesic segment from 0 to 1. Take L' to be the geodesic segment from 0 to the point on S^1 so that L' makes angle of α with L. Thus both L and L' are Euclidean line segments. Let $p \in L$ and consider the geodesic segment M starting at p, ending on S^1, and making angle β with L (see Figure 4.1). If $M \cap L' \neq \varnothing$, call the resulting interior angle γ_p. Observe that as $p \to 0$ (along L), we have that $\gamma_p \to \pi - (\alpha + \beta)$ since M is limiting to a straight Euclidean line segment and thus the resulting triangle is limiting to a Euclidean triangle. Also, as $p \to 1$ (along L), we get $\gamma_p \to 0$. Since the map $p \mapsto \gamma_p$ is continuous, the Intermediate Value Theorem implies the existence of some $p \in L$ where $\gamma_p = \gamma$.

□

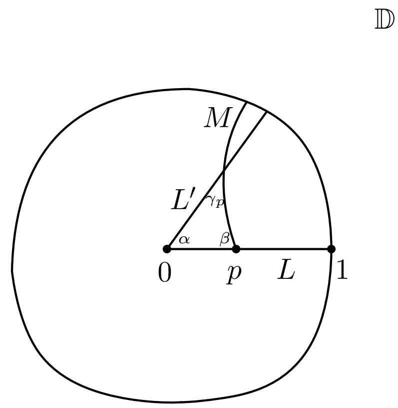

Figure 4.1: A hyperbolic triangle in $\mathbb{D}$ with angles α, β, γ_p

We remark that the above three results generalize to hyperbolic polygons with n sides. We only state one of these generalizations and the proof follows by subdividing the polygon into $(n-2)$ hyperbolic triangles.

Theorem 4.1.4. *The angle sum of a hyperbolic polygon with* n *sides is less than* $(n-2)\pi$.

4.1.2 Law of Sines and Cosines

In this section we assume that the angles of the hyperbolic triangle T, namely α, β and γ, are positive so that a, b, and c are finite. The following three identities hold for such hyperbolic triangles, the first two of which are analogues to the Euclidean law of sines and cosines, respectively. See [1] or [2], for example, for the proofs.

Hyperbolic Law of Sines:

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

Hyperbolic Law of Cosines I:

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$$

Hyperbolic Law of Cosines II:

$$\cosh c = \frac{\cos \gamma + \cos \alpha \cos \beta}{\sin \alpha \sin \beta} \tag{4.1}$$

The existence of the second law of cosines demonstrates another difference between hyperbolic geometry and Euclidean geometry. This law implies that the angles of a triangle determine its side lengths. In other words there is no notion of similar triangles in hyperbolic geometry. Moreover, it leads to the following important result.

Theorem 4.1.5. *There exists an isometry of $\mathbb{H}$ mapping a hyperbolic triangle T onto a hyperbolic triangle $\widetilde{T}$ if and only if T and $\widetilde{T}$ have the same angles (or sides of the same lengths).*

Proof. The forward implication follows from the fact that isometries of the hyperbolic plane preserve angles and length. For the reverse implication we will use the second law of cosines. Let $\widetilde{T}$ be the hyperbolic triangle in $\mathbb{D}$ with vertices $\widetilde{v_a} = 0, \widetilde{v_b} = r > 0$, and $\widetilde{v_c} = se^{i\alpha}$, for some $s > 0$, and corresponding angles, α, β, and γ. The existence of such a triangle was seen in the proof of Theorem 4.1.3. Given an arbitrary triangle T in $\mathbb{D}$ with vertices v_a, v_b, and v_c and corresponding angles α, β, and γ, we will produce an isometry of $\mathbb{D}$ mapping T onto $\widetilde{T}$. First, map the geodesic containing the vertices v_a and v_b of T to the diameter D of $\mathbb{D}$ connecting -1 and 1. Then use a hyperbolic isometry whose axis is D to translate v_a to 0. At this point, v_b is either r or $-r$ (by the second law of cosines). If $v_b = -r$, just apply the reflection in the diameter connecting $-i$ and i. This guarantees $v_b = r$. Finally, by the second law of cosines v_c is either $se^{i\alpha}$ or $se^{-i\alpha}$. In the latter case simply reflect in D. Thus, we have mapped T onto $\widetilde{T}$ using isometries of $\mathbb{D}$. $\square$

4.2 Triangle Groups

Suppose p, q, and r are positive integers satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Then let T be a hyperbolic triangle in $\mathbb{H}$ with interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$. Consider the group G generated by the reflections in the geodesics containing the sides of T. By Poincaré's polygon theorem [2], G is a discrete group with fundamental domain T. The subgroup of orientation-preserving isometries of G, denoted $\Gamma(p, q, r)$, is referred to as a (p, q, r)-*triangle group.* We call a Fuchsian group Γ a *triangle group* if it is a (p, q, r)-triangle group for some integers p, q and r.

In order to obtain a group presentation of $\Gamma(p, q, r)$ which will make computations easier, we normalize the triangle T, as well as use the same notation, as in [11]. We work in $\mathbb{U}$ and take T to be the triangle in the left half of Figure 4.2. Two vertices of T are i and $\lambda^{-1}i$, for some determined $\lambda = \lambda(p, q, r) > 1$, and T has sides labeled s_1, s_2, s_3. Let σ_i denote the reflection in the geodesic containing side s_i, for $i = 1, 2, 3$. Thus $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and the subgroup $\Gamma(p, q, r)$ consists of all the even length elements in the σ_i. It follows that $\Gamma(p, q, r)$ is index two in G and so we may decompose G into two cosets, say $G = \Gamma(p, q, r) \cup \Gamma(p, q, r)\sigma_1$. Therefore, by Theorem 3.3.2, $T^* = T \cup \sigma_1(T)$ is a fundamental domain for $\Gamma(p, q, r)$.

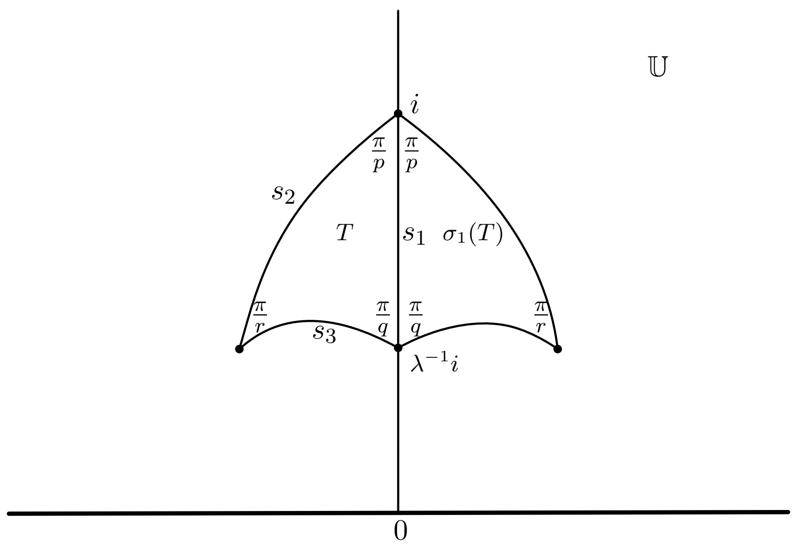

Figure 4.2: Fundamental domain T^* for $\Gamma(p, q, r)$

The group $\Gamma(p, q, r)$ can be generated by the two elements $A = \sigma_2\sigma_1$ and $B = \sigma_1\sigma_3$. Thus from Figure 4.2 we see that A is an elliptic isometry rotating clockwise by $\frac{2\pi}{p}$ about its fixed point i and B is elliptic rotating clockwise by $\frac{2\pi}{q}$ about its fixed point $\lambda^{-1}i$. Both A and B can be expressed as elements of $\mathrm{SL}(2, \mathbb{R})$:

$$A = \begin{pmatrix} \cos(\frac{\pi}{p}) & -\sin(\frac{\pi}{p}) \\ \sin(\frac{\pi}{p}) & \cos(\frac{\pi}{p}) \end{pmatrix}, \quad B = \begin{pmatrix} \cos(\frac{\pi}{q}) & -\lambda^{-1}\sin(\frac{\pi}{q}) \\ \lambda\sin(\frac{\pi}{q}) & \cos(\frac{\pi}{q}) \end{pmatrix}$$

Let $C = B^{-1}A^{-1} = \sigma_3\sigma_1\sigma_1\sigma_2 = \sigma_3\sigma_2$. Then C is an elliptic isometry of order r whose fixed point is the vertex of T^* to the left of the imaginary axis. Using the fundamental domain in Figure 4.2 we can write down the following presentation for $\Gamma(p, q, r)$:

$$\Gamma(p, q, r) = \langle A, B \mid A^p = B^q = (B^{-1}A^{-1})^r = I \rangle \qquad (4.2)$$

It follows that the quotient $\mathbb{H}/\Gamma(p,q,r) = \mathcal{O}(p,q,r)$ is a sphere with three cone points of orders p,q and r. We refer to $\mathcal{O}(p,q,r)$ as a (p,q,r)-*triangle group orbifold.* If Γ is a triangle group, we call $\mathbb{H}/\Gamma$ a *triangle group orbifold.*

Our next goal is to determine λ. Using either triangle in Figure 4.2, the hyperbolic law of cosines II (4.1) yields the following equation.

$$\cosh(\ln\lambda) = \frac{\cos\left(\frac{\pi}{r}\right) + \cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right)}$$

Hence, we get

$$\lambda + \lambda^{-1} = \frac{2E}{\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right)}, \tag{4.3}$$

where

$$E = \cos\left(\frac{\pi}{r}\right) + \cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right). \tag{4.4}$$

Solving (4.3) for λ we obtain

$$\lambda = \lambda(p,q,r) = \frac{E + \left(E^2 - \sin^2\left(\frac{\pi}{p}\right)\sin^2\left(\frac{\pi}{q}\right)\right)^{\frac{1}{2}}}{\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right)}.$$

Another fact we will need later is that $tr(C) < 0$. We see this by multiplying $B^{-1} \cdot A^{-1}$ and using (4.3):

$$tr(C) = 2\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right) - (\lambda + \lambda^{-1})\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right) = -2\cos\left(\frac{\pi}{r}\right) \tag{4.5}$$

In our discussion we allow triangle groups to contain parabolic isometries. Under appropriate normalizations we may view a parabolic isometry as the limit of a sequence of elliptic isometries $\{f_n\}$ whose rotation angles approach zero and whose fixed points approach $\partial\mathbb{H}$ as $n \to \infty$. This situation occurs if we use the presentation (4.2) and consider the elliptic isometry C while letting $r \to \infty$. In the limit C is a parabolic isometry, and if we apply this limit ($r \to \infty$) considering the entire group $\Gamma(p, q, r)$ from (4.2), we get the following group:

$$\Gamma(p, q, \infty) = \langle A, B \mid A^p = B^q = I \rangle \tag{4.6}$$

, where

$$A = \begin{pmatrix} \cos(\frac{\pi}{p}) & -\sin(\frac{\pi}{p}) \\ \sin(\frac{\pi}{p}) & \cos(\frac{\pi}{p}) \end{pmatrix}, \quad B = \begin{pmatrix} \cos(\frac{\pi}{q}) & -\lambda_{p,q}^{-1}\sin(\frac{\pi}{q}) \\ \lambda_{p,q}\sin(\frac{\pi}{q}) & \cos(\frac{\pi}{q}) \end{pmatrix},$$

$$\lambda_{p,q} = \frac{E_{p,q} + \left(E_{p,q}^2 - \sin^2\left(\frac{\pi}{p}\right)\sin^2\left(\frac{\pi}{q}\right)\right)^{\frac{1}{2}}}{\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right)},$$

$E_{p,q} = 1 + \cos(\frac{\pi}{p})\cos(\frac{\pi}{q})$, and ∞ represents the conjugacy class of the maximal parabolic cyclic subgroup $\langle C \rangle$. A fundamental domain for $\Gamma(p, q, \infty)$ can be attained in an analagous way as was done above for $\Gamma(p, q, r)$, where $p, q, r < \infty$. The difference will be that the fundamental domain for $\Gamma(p, q, \infty)$ will have two vertices on $\mathbb{R}$. It follows that $\mathcal{O}(p, q, \infty)$ is a sphere with a puncture and two cone points having orders p and q.

Example 4.2.1. The Modular group $\mathrm{SL}(2, \mathbb{Z})$ is a $(2, 3, \infty)$-triangle group.

Again, start with the presentation (4.2) and this time let $q, r \to \infty$. Our group now becomes:

$$\Gamma(p, \infty, \infty) = \langle A, B \mid A^p = I \rangle \tag{4.7}$$

, where

$$A = \begin{pmatrix} \cos(\frac{\pi}{p}) & -\sin(\frac{\pi}{p}) \\ \sin(\frac{\pi}{p}) & \cos(\frac{\pi}{p}) \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \frac{2E_p}{\sin(\frac{\pi}{p})} & 1 \end{pmatrix},$$

and $E_p = 1 + \cos(\frac{\pi}{p})$.

Finally, we consider $\Gamma(\infty, \infty, \infty)$. We treat this case independently from the others since $A \to I$ as $p \to \infty$, and I is not a parabolic isometry. So, let

$$\Gamma(\infty, \infty, \infty) = \langle A, B \rangle \tag{4.8}$$

, where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$

One must take caution and check that the element $C = B^{-1}A^{-1}$ is parabolic in order for the resulting orbifold to have three punctures. Here,

$$C = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$$

and thus C is parabolic since $|tr(C)| = 2$ and $C \neq I$. We note that $\mathcal{O}(\infty, \infty, \infty)$ is the only triangle group orbifold which is a hyperbolic surface, the thrice punctured sphere.

Remark 4.2.2. $\Gamma(p, q, r)$ is conjugate to $\Gamma(p', q', r')$ in $\text{Isom}(\mathbb{H})$ if and only if (p', q', r') is a permutation of (p, q, r). This follows from the fact that there exists an isometry of $\mathbb{H}$ taking one triangle to another if and only if they have the same angles (Theorem 4.1.5). Thus, without loss of generality we will assume throughout that $2 \leq p \leq q \leq r \leq \infty$.

Chapter 5

Figure Eight Geodesics

5.1 Figure Eight Geodesics on Triangle Group Orbifolds

Let $\Gamma(p,q,r)$ be the normalized triangle group as in Section 4.2, and furthermore, assume that $2 \leq p \leq q \leq r \leq \infty$. Let $\pi : \mathbb{H} \to \mathcal{O}(p,q,r) = \mathbb{H}/\Gamma(p,q,r)$ be the projection map. We remind the reader that concatenation of paths is performed from left to right, while function composition is from right to left.

First, cut $\mathcal{O}(p,q,r)$ into two isometric triangles by drawing in the geodesic paths between each cone point (the black segments in Figure 5.1). Orient the "front" triangle clockwise and similarly orient the two triangles in the lift of $\mathcal{O}(p,q,r)$ pictured in Figure 4.2 to be clockwise. This establishes an orientation on $\mathcal{O}(p,q,r)$. Let x in Figure 5.1 be the base point of $\mathcal{O}(p,q,r)$ and choose the base point $\widetilde{x} \in \mathbb{U}$ to be the lift of x which is on the imaginary axis between i and $\lambda^{-1}i$. In Figure 5.1 consider the three oriented loops α, β, and γ on $\mathcal{O}(p,q,r)$ based at x. Letting $\Phi : \Pi_1(\mathcal{O}(p,q,r),x) \to \Gamma(p,q,r)$ be the corresponding isomorphism, it follows that $\Phi([\alpha]) = A$, $\Phi([\beta]) = B$, and $\Phi([\gamma]) = B^{-1}A^{-1} = C$, where A, B and C are as in Section 4.2. Now consider the following three loops on $\mathcal{O}(p,q,r)$: $\mathcal{C}_{pq} = \beta * \alpha^{-1}$, $\mathcal{C}_{qr} = \beta * \gamma^{-1}$, and $\mathcal{C}_{pr} = \alpha * \gamma^{-1}$. Hence, $\Phi([\mathcal{C}_{pq}]) = BA^{-1}$, $\Phi([\mathcal{C}_{qr}]) = BC^{-1}$, and $\Phi([\mathcal{C}_{pr}]) = AC^{-1}$. Then the loops $\mathcal{C}_{pq}, \mathcal{C}_{qr}$, and $\mathcal{C}_{pr}$ are respectively, freely $\mathcal{O}(p,q,r)$-homotopic to either a loop bounding the projection under π of the fixed point (if elliptic or parabolic) or the projection of the axis (if hyperbolic) of BA^{-1}, BC^{-1}, and AC^{-1}.

The main theorem of this section explicity describes what these projections look like in the hyperbolic case, with the goal of identifying all figure eight geodesics on triangle group orbifolds. Recall that a figure eight geodesic is a non-simple closed geodesic which is disjoint from the cone points and has one self-intersection. The key fact to observe is that a figure eight geodesic on a triangle group orbifold must be freely $\mathcal{O}(p,q,r)$-homotopic to at least one of the loops $\mathcal{C}_{pq}, \mathcal{C}_{qr}$, $\mathcal{C}_{pr}$, or these loops with the opposite orientation. Before stating and proving the theorem we prove two lemmas. The first is a general fact about axes of hyperbolic isometries in a Fuchsian group while the second is a result about a hyperbolic isometry in a particular class of triangle groups which will be used in the proof of the main theorem.

Lemma 5.1.1. *Suppose g is a hyperbolic isometry represented by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in$ $\mathrm{PSL}(2,\mathbb{R})$ where $bc > 0$. Then its axis $\mathcal{A}_g$ intersects the imaginary axis at the point $i\sqrt{\frac{b}{c}}$.*

Proof. The fixed points of g are $\frac{a-d\pm\sqrt{(a+d)^2-4}}{2c}$. So the geodesic $\mathcal{A}_g$ has corresponding equation $\left|z - \frac{a-d}{2c}\right| = \frac{\sqrt{(a+d)^2-4}}{2c}$, where $z \in \mathbb{U}$. Letting $z = iy$ in this equation and solving for y we get:

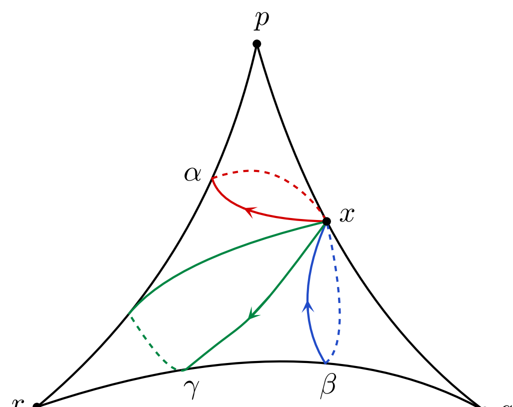

Figure 5.1: Loops on $\mathcal{O}(p, q, r)$ based at x

$$\begin{aligned}
\left|iy - \frac{a-d}{2c}\right|^2 &= \frac{(a+d)^2 - 4}{4c^2} \\
y^2 + \frac{(a-d)^2}{4c^2} &= \frac{(a+d)^2 - 4}{4c^2} \\
y^2 &= \frac{(a+d)^2 - 4 - (a-d)^2}{4c^2} \\
y^2 &= \frac{4ad - 4}{4c^2} = \frac{ad-1}{c^2} = \frac{bc}{c^2} = \frac{b}{c} \\
y &= \sqrt{\frac{b}{c}}
\end{aligned}$$

Since $bc > 0$, b and c have the same sign and thus $\sqrt{\frac{b}{c}}$ is a real number. So $\mathcal{A}_g$ intersects the imaginary axis at the point $i\sqrt{\frac{b}{c}}$. □

Lemma 5.1.2. *The element $BA^{-1} \in \Gamma(3, q, r)$ for $q \geq 3, r \geq 4$, is hyperbolic and its axis intersects the imaginary axis at the point ti, where $t \leq 1$. Moreover, $t = 1$ if and only if $q = r$.*

Proof. To show that BA^{-1} is hyperbolic we will show that its trace is greater than two. We follow the development in [18] which yields a useful trace identity. By the Cayley-Hamilton Theorem we have the following identity which holds for determinant one matrices.

$$A^{-1} = tr(A) \cdot I - A$$

Hence,

$$A^{-1}B = tr(A) \cdot B - AB$$

and so,

$$tr(A^{-1}B) = tr(A)tr(B) - tr(AB). \tag{5.1}$$

Considering $A, B, C \in \Gamma(p, q, r)$ where $\Gamma(p, q, r)$ has presentation (4.2), we see that $tr(A) = 2\cos(\frac{\pi}{p})$, $tr(B) = 2\cos(\frac{\pi}{q})$ and we have $tr(AB) = tr(C^{-1}) = tr(C) = -2\cos(\frac{\pi}{r})$ from (4.5). Thus, with (5.1) and since we are considering the case where $p = 3$, $q \geq 3$, and $r \geq 4$ we get

$$\begin{aligned} tr(BA^{-1}) &= tr(A^{-1}B) \\ &= tr(A)tr(B) - tr(AB) \\ &= 2\left[2\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right) + \cos\left(\frac{\pi}{r}\right)\right] \\ &\geq 2\left[\frac{1}{2} + \frac{\sqrt{2}}{2}\right] = 1 + \sqrt{2} > 2 \end{aligned}$$

Thus, $BA^{-1} \in \Gamma(3, q, r)$ (for $q \geq 3, r \geq 4$) is hyperbolic. Using (4.2), we have

$$BA^{-1} = \begin{pmatrix} \frac{1}{2}\cos(\frac{\pi}{q}) + \frac{\sqrt{3}}{2}\lambda^{-1}\sin(\frac{\pi}{q}) & \frac{\sqrt{3}}{2}\cos(\frac{\pi}{q}) - \frac{1}{2}\lambda^{-1}\sin(\frac{\pi}{q}) \\ \frac{1}{2}\lambda\sin(\frac{\pi}{q}) - \frac{\sqrt{3}}{2}\cos(\frac{\pi}{q}) & \frac{\sqrt{3}}{2}\lambda\sin(\frac{\pi}{q}) + \frac{1}{2}\cos(\frac{\pi}{q}) \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In light of Lemma 5.1.1, it suffices to show that $b, c > 0$ and $\sqrt{\frac{b}{c}} \leq 1$, with equality holding if and only if $q = r$. We first show that $b > 0$.

$$
\begin{aligned}
b > 0 \quad &\Leftrightarrow \quad \frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{q}\right) - \frac{1}{2}\lambda^{-1}\sin\left(\frac{\pi}{q}\right) > 0 \\
&\Leftrightarrow \quad \sqrt{3}\cos\left(\frac{\pi}{q}\right) > \lambda^{-1}\sin\left(\frac{\pi}{q}\right) \\
&\Leftrightarrow \quad \sqrt{3}\lambda\cos\left(\frac{\pi}{q}\right) > \sin\left(\frac{\pi}{q}\right)
\end{aligned}
$$

This last inequality clearly holds for $q \geq 4$, since $\sqrt{3}, \lambda > 1$ and $\cos(\frac{\pi}{q}) \geq \sin(\frac{\pi}{q})$. If $q = 3$, the last inequality reduces to $\lambda > 1$, which is true. Next, we show that $b \leq c$, and since we have established that $b > 0$, we will have $c > 0$.

$$
\begin{aligned}
b \leq c \quad &\Leftrightarrow \quad \frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{q}\right) - \frac{1}{2}\lambda^{-1}\sin\left(\frac{\pi}{q}\right) \leq \frac{1}{2}\lambda\sin\left(\frac{\pi}{q}\right) - \frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{q}\right) \\
&\Leftrightarrow \quad \sqrt{3}\cos\left(\frac{\pi}{q}\right) \leq \frac{1}{2}(\lambda + \lambda^{-1})\sin\left(\frac{\pi}{q}\right) \\
&\underset{(4.3)}{\Leftrightarrow} \quad \sqrt{3}\cos\left(\frac{\pi}{q}\right) \leq \frac{1}{2} \cdot \frac{2E}{\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{q}\right)} \cdot \sin\left(\frac{\pi}{q}\right) \\
&\underset{(4.4)}{\Leftrightarrow} \quad \frac{3}{2}\cos\left(\frac{\pi}{q}\right) \leq \cos\left(\frac{\pi}{r}\right) + \frac{1}{2}\cos\left(\frac{\pi}{q}\right) \\
&\Leftrightarrow \quad \cos\left(\frac{\pi}{q}\right) \leq \cos\left(\frac{\pi}{r}\right)
\end{aligned}
$$

This last inequality holds as $3 \leq q \leq r$, and it is an equality if and only if $q = r$. This finishes the proof since $0 < b \leq c$ implies that $\sqrt{\frac{b}{c}} \leq 1$. □

Among the elements BA^{-1}, BC^{-1}, and AC^{-1} in $\Gamma(p, q, r)$ which are hyperbolic, we want to know what the projection of their axes look like. To achieve this we choose appropriate fundamental domains for each of the elements (Figures 5.2-5.4). The choices were made in order to represent the elements as a product of two reflections in geodesics containing sides of the fundamental quadrilateral. In each figure, and for all figures in the proof of the main theorem, we label the relevant vertices of the fundamental domain of $\Gamma(p,q,r)$ with the elliptic isometry fixing that point. Thus, the label A is at the point i and the label B is at the point $\lambda^{-1}i$. For Figures 5.2-5.4, let σ_i be the reflection in the geodesic L_i containing the side s_i, for $i = 1, 2, 3$. In Figure 5.2, we can write $B = \sigma_1\sigma_2$ and $A^{-1} = \sigma_2\sigma_3$, and so $BA^{-1} = \sigma_1\sigma_2\sigma_2\sigma_3 = \sigma_1\sigma_3$. Hence, if L_1 and L_3 are disjoint in $\overline{\mathbb{H}}$, then BA^{-1} is hyperbolic and $\mathcal{A}_{BA^{-1}}$ is the common orthogonal to L_1 and L_3 (see Section 2.3). Similarly, in Figure 5.3, $B = \sigma_1\sigma_2$, $C^{-1} = \sigma_2\sigma_3$, and thus $BC^{-1} = \sigma_1\sigma_3$. Finally, in Figure 5.4, $A = \sigma_1\sigma_2$, $C^{-1} = \sigma_2\sigma_3$, and thus $AC^{-1} = \sigma_1\sigma_3$. The key idea in all three figures was to choose L_2 to be the unique geodesic which passes through the fixed points of the two elliptic isometries in the composition. For example for BA^{-1} we chose L_2 to be the imaginary axis which is the unique geodesic passing through the points i and $\lambda^{-1}i$, the fixed points of A^{-1} and B, respectively.

In order to state and prove the main theorem in a less cumbersome way, we adopt the following notation. If f is conjugate to g in $\Gamma(p,q,r)$, we denote this by $f \sim g$. Also, we interpret an order ∞ cone point as a puncture.

Theorem 5.1.3. *Let* $\pi : \mathbb{H} \to \mathcal{O}(p,q,r) = \mathbb{H}/\Gamma(p,q,r)$ *be the projection map and assume* $2 \leq p \leq q \leq r \leq \infty$.

(i) In $\Gamma(2,q,r)$ *for* $q \geq 3$*:*

- BA^{-1} *is elliptic of order* r *(parabolic if* $r = \infty$*).*
- *If* $q = 3$*, then* BC^{-1} *is elliptic of order* r *(parabolic if* $r = \infty$*); if* $q = 4$*, then* BC^{-1} *is hyperbolic and* $\pi(\mathcal{A}_{BC^{-1}})$ *is the geodesic path*

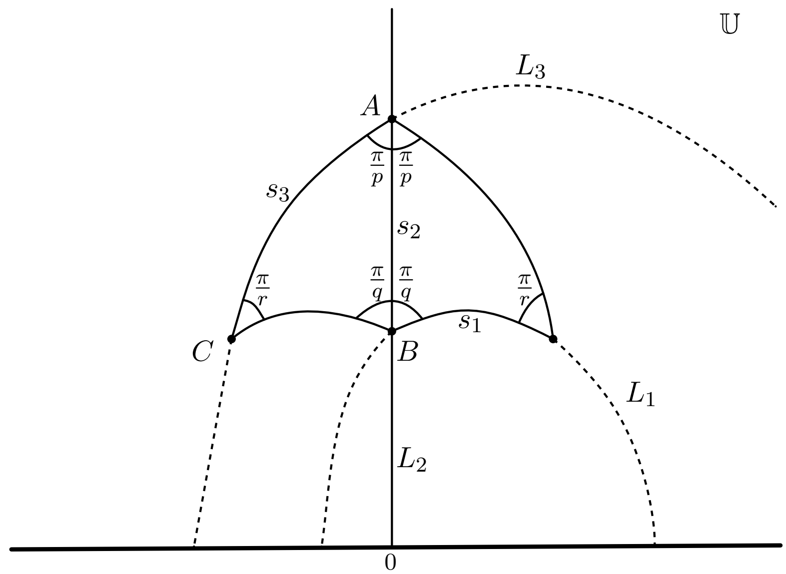

Figure 5.2: Fundamental domain of $\Gamma(p, q, r)$ chosen for BA^{-1}

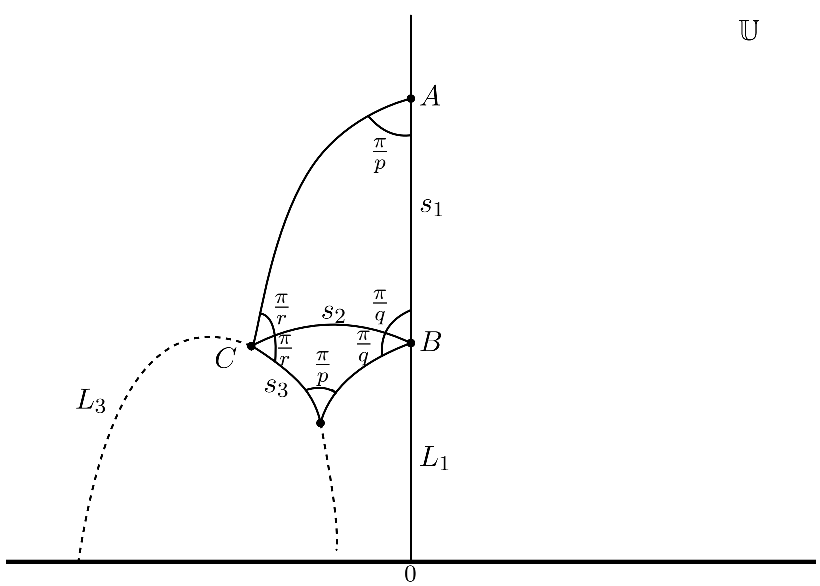

Figure 5.3: Fundamental domain of $\Gamma(p, q, r)$ chosen for BC^{-1}

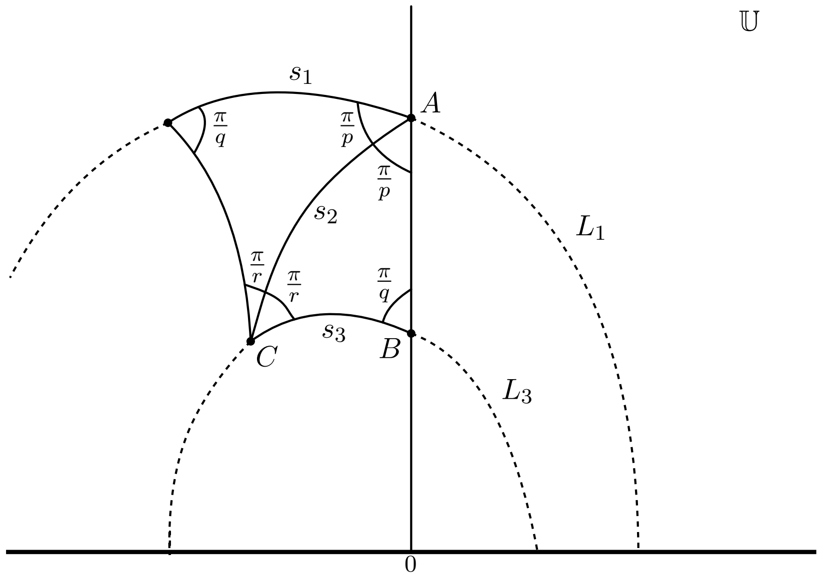

Figure 5.4: Fundamental domain of $\Gamma(p, q, r)$ chosen for AC^{-1}

from the cone point of order two to the cone point of order four and back again; if $q \geq 5$, then BC^{-1} is hyperbolic and $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of orders q and r.

- *AC^{-1} is elliptic of order q (parabolic if $q = \infty$).*

(ii) In $\Gamma(3, q, r)$ for $q \geq 3, r \geq 4$:

- *$BA^{-1} \sim AC^{-1}$ and they are hyperbolic. If $q = r$, then $\pi(\mathcal{A}_{BA^{-1}}) = \pi(\mathcal{A}_{AC^{-1}})$ is the geodesic path passing through the cone point of order three pictured in Figure 5.9; if $q \neq r$, then $\pi(\mathcal{A}_{BA^{-1}}) = \pi(\mathcal{A}_{AC^{-1}})$ is the figure eight geodesic bounding the cone points of orders three and q.*
- *If $q = 3$, then $BC^{-1} \sim (BA^{-1})^{-1}$ and they are hyperbolic. Thus, $\pi(\mathcal{A}_{BC^{-1}}) = \pi\left(\mathcal{A}_{(BA^{-1})^{-1}}\right)$, which is simply $\pi(\mathcal{A}_{BA^{-1}})$ but with*

opposite orientation. However, if $q \geq 4$, then BC^{-1} is hyperbolic and $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of orders q and r.

(iii) In $\Gamma(p,q,r)$ for $p,q,r \geq 4$:

- *BA^{-1} is hyperbolic and $\pi(\mathcal{A}_{BA^{-1}})$ is the figure eight geodesic bounding the cone points of orders p and q.*
- *BC^{-1} is hyperbolic and $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of orders q and r.*
- *AC^{-1} is hyperbolic and $\pi(\mathcal{A}_{AC^{-1}})$ is the figure eight geodesic bounding the cone points of orders p and r.*

Proof. We note that in this proof all of the figures are drawn for the cases when p, q, and r are finite, but the figures and corresponding arguments can be easily adjusted in the infinite cases.

Case (i): $\Gamma(2,q,r)$ for $q \geq 3$. First note that $BA^{-1} = BA \sim AB = C^{-1}$, an elliptic isometry of order r (parabolic if $r = \infty$). Thus, BA^{-1} is elliptic of order r (parabolic if $r = \infty$).

Now, if $q = 3$, we see that $BC^{-1} = BAB \sim B^2A = B^{-1}A^{-1} = C$, an elliptic isometry of order r (parabolic if $r = \infty$). Next, if $q = 4$, $BC^{-1} = BAB \sim B^2A$, which is the product of two non-conjugate order two elliptic isometries. It follows that BC^{-1} is hyperbolic and its axis projects to the geodesic path from the order two cone point to the order four cone point and back again. If $q \geq 5$, we work with the fundamental domain in Figure 5.5, which is for $\Gamma(2,q,r)$, where $q, r \geq 5$. As was done above in Figure 5.3, here in Figure 5.5 we express BC^{-1} as the composition of reflection in the imaginary axis with reflection in the dashed geodesic, L. Note that since

$q, r \geq 5$, we get the lower bound $\frac{3\pi}{5}$ for the angles so indicated in the figure. It is clear then that L and the imaginary axis do no intersect in $\overline{\mathbb{H}}$ because otherwise a triangle is formed whose sum of interior angles is greater than π. So indeed, BC^{-1} is hyperbolic and $\mathcal{A}_{BC^{-1}}$ is the common orthogonal of L and the imaginary axis. Furthermore, we can see that $\mathcal{A}_{BC^{-1}}$ must pass through the sides of the fundamental quadrilateral, disjoint from the vertices (the red geodesic segment), since all other possibilities result in a polygon whose interior angle sum is too large to be hyperbolic (see Theorem 4.1.4). The red segment of $\mathcal{A}_{BC^{-1}}$ represents only half of the translation length, but a symmetric copy lies on the other side of the imaginary axis. When translating that other segment into the original fundamental quadrilateral by B^{-1} (Figure 5.6) and then passing to the quotient, we get the figure eight geodesic whose component loops bound the cone points of orders q and r (Figure 5.7).

Finally, $AC^{-1} = A^2B = B$, an order q elliptic isometry (parabolic if $q = \infty$).

Case (ii): $\Gamma(3, q, r)$ *for* $q \geq 3, r \geq 4$. Well, $BA^{-1} \sim A^{-1}B = A^2B = AC^{-1}$, and we choose to consider BA^{-1}. The corresponding fundamental quadrilateral for $\Gamma(3, q, r)$ for $q \geq 3, r \geq 4$ is in Figure 5.8. The two dashed geodesics in the figure cannot intersect and so BA^{-1} is hyperbolic. After ruling out potential geodesic segments of $\mathcal{A}_{BA^{-1}}$ which create polygons whose interior angle sums are too large to be hyperbolic, we are left with the three geodesic segments labeled α, β, and γ. Here α is representing a geodesic segment with end points on the sides s_1 and s_2 of the quadrilateral (disjoint from the vertices), while β has an end point at i and an end point on the side s_2 (disjoint from the vertices), and finally γ has an end point on the dashed

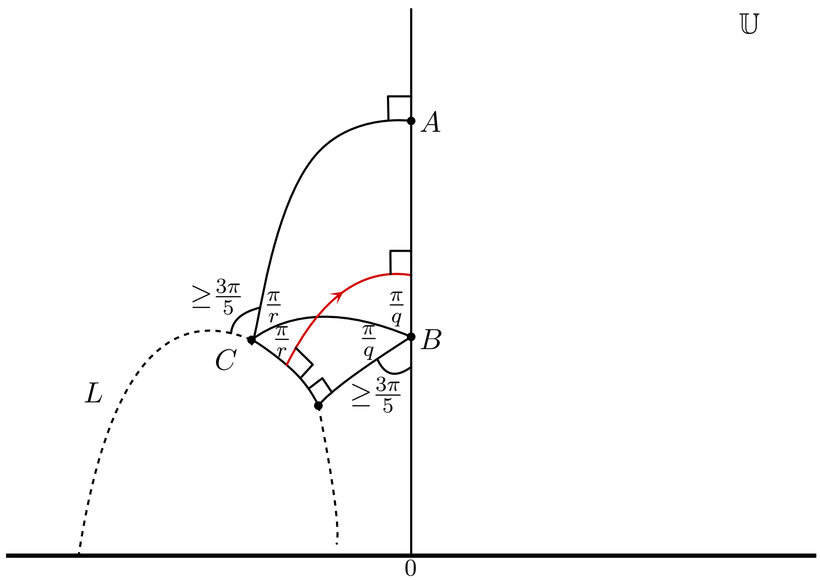

Figure 5.5: A segment of $\mathcal{A}_{BC^{-1}}$ where $BC^{-1} \in \Gamma(2, q, r)$ for $q, r \geq 5$

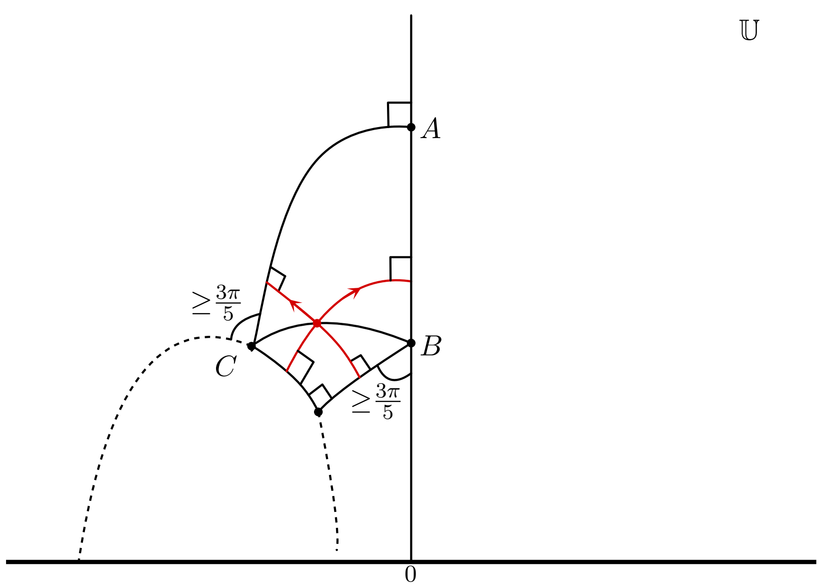

Figure 5.6: Full translation length of $BC^{-1} \in \Gamma(2, q, r)$ for $q, r \geq 5$

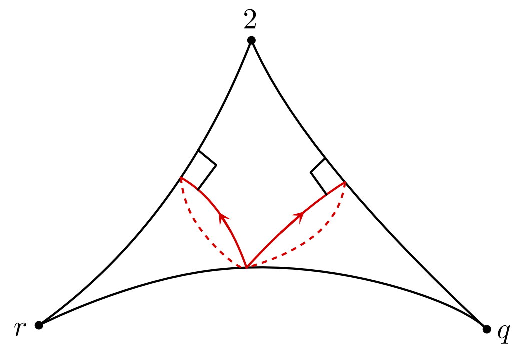

Figure 5.7: $\pi(\mathcal{A}_{BC^{-1}}) \subset \mathcal{O}(2, q, r)$ for $q, r \geq 5$

geodesic segment u and the side s_2 (disjoint from the vertices). By Lemma 5.1.2, we know that if $q = r$, $\mathcal{A}_{BA^{-1}}$ intersects the imaginary axis at i and thus β is the correct segment. It represents half of the translation length of BA^{-1}. The other half is just the reflection of β in the imaginary axis with opposite orientation. In the quotient, we will have a geodesic loop passing through the order three cone point (see Figure 5.9). If $q \neq r$, then the correct geodesic segment in Figure 5.8 is α, since by Lemma 5.1.2, $\mathcal{A}_{BA^{-1}}$ must pass through the imaginary axis strictly below i. (The segment γ cannot intersect the imaginary axis below i because it would yield a triangle whose interior angles sum to at least π.) The segment α together with its reflection in the imaginary axis with opposite orientation gives a full translation length of BA^{-1} and so $\mathcal{A}_{BA^{-1}}$ projects to the figure eight geodesic bounding the cone points of orders three and q.

If $q = 3$, $BC^{-1} = BAB \sim B^2A = B^{-1}A \sim AB^{-1} = (BA^{-1})^{-1}$, which

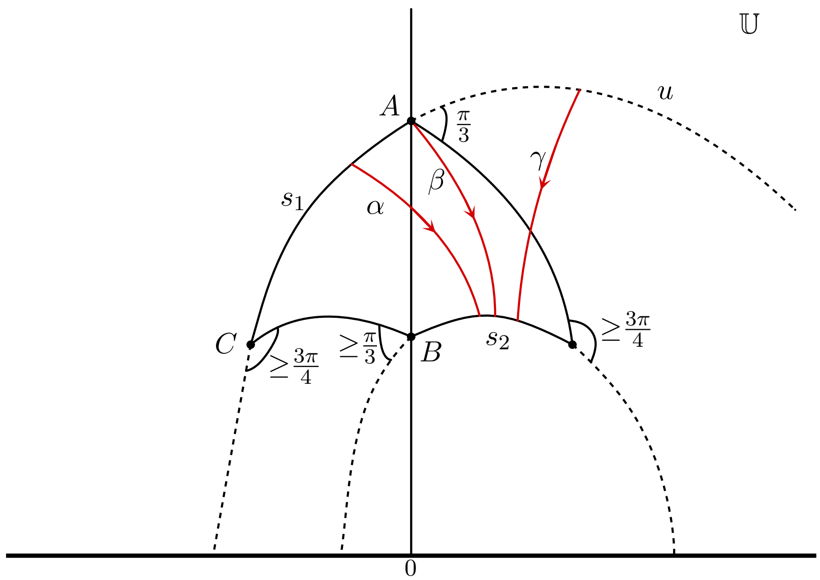

Figure 5.8: Possible segments of $\mathcal{A}_{BA^{-1}}$ for $BA^{-1} \in \Gamma(3, q, r)$, $q \geq 3, r \geq 4$

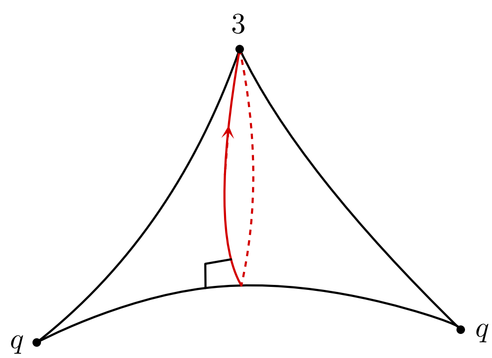

Figure 5.9: $\pi(\mathcal{A}_{BA^{-1}}) \subset \mathcal{O}(3, q, q)$ for $q \geq 4$

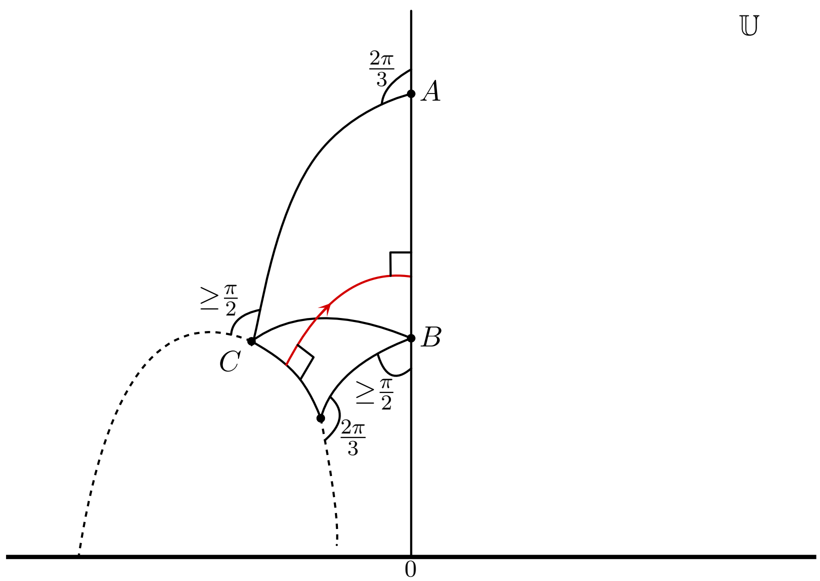

Figure 5.10: A segment of $\mathcal{A}_{BC^{-1}}$ where $BC^{-1} \in \Gamma(3, q, r)$ for $q, r \geq 4$

is hyperbolic from above. Thus, BC^{-1} is hyperbolic and $\pi(\mathcal{A}_{BC^{-1}})$ is also a figure eight geodesic bounding the cone points of order three, but with opposite orientation as $\pi(\mathcal{A}_{BA^{-1}})$. If $q \geq 4$, then necessarily $r \geq 4$, and a similar analysis to cases above (e.g. $BC^{-1} \in \Gamma(2, q, r)$ for $q, r \geq 5$) shows that BC^{-1} is hyperbolic and $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of orders q and r (see Figure 5.10).

Case (iii): $\Gamma(p, q, r)$ *for* $p, q, r \geq 4$. By choosing the appropriate fundamental domains for each of the three isometries (Figures 5.11-5.13), one sees that each isometry is hyperbolic and the axis of each isometry must pass through opposite sides of the quadrilateral disjoint from the vertices (the red curves, again, are segments of the axes). As in above cases, this is done by showing any other proposed common orthogonal will form a polygon whose interior

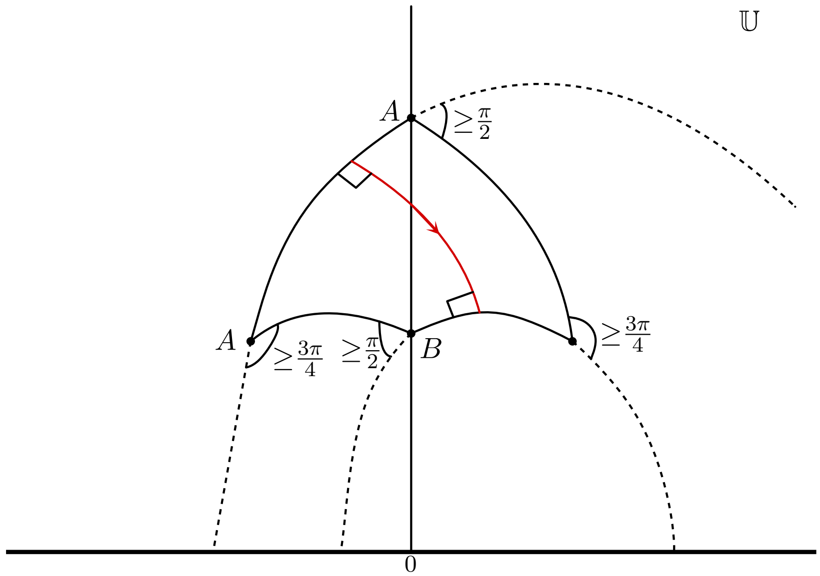

Figure 5.11: A segment of $\mathcal{A}_{BA^{-1}}$ where $BA^{-1} \in \Gamma(p, q, r)$ for $p, q, r \geq 4$

angle sum is too large.

□

Theorem 5.1.3 yields the following classification theorem.

Theorem 5.1.4. *The following is a list of the figure eight geodesics on triangle group orbifolds.*

(i) On $\mathcal{O}(2, q, r)$ for $q, r \geq 5$, $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of orders q and r.

(ii) On $\mathcal{O}(3, q, q)$ for $q \geq 4$, $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of order q.

(iii) On $\mathcal{O}(3, q, r)$ for $q \neq r$, $\pi(\mathcal{A}_{BA^{-1}}) = \pi(\mathcal{A}_{AC^{-1}})$ is the figure eight geodesic bounding the cone points of orders three and q. If $q = 3$,

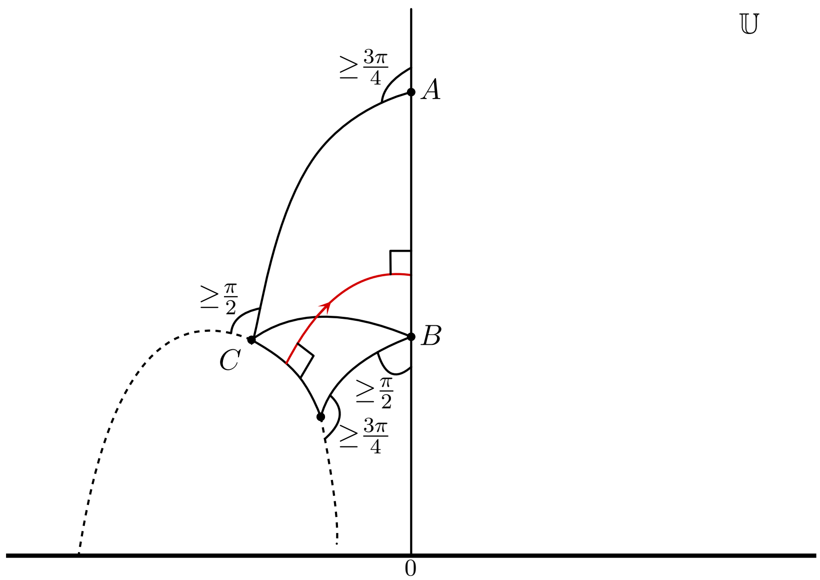

Figure 5.12: A segment of $\mathcal{A}_{BC^{-1}}$ where $BC^{-1} \in \Gamma(p,q,r)$ for $p,q,r \geq 4$

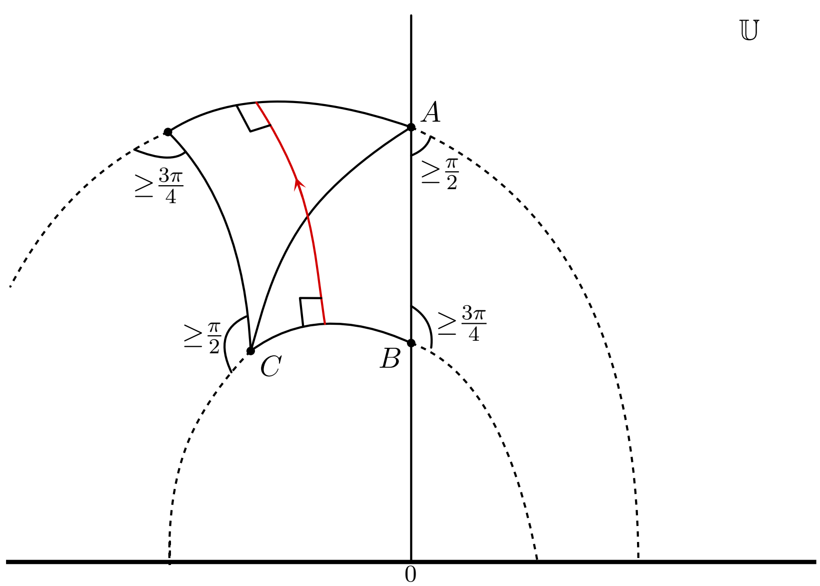

Figure 5.13: A segment of $\mathcal{A}_{AC^{-1}}$ where $AC^{-1} \in \Gamma(p,q,r)$ for $p,q,r \geq 4$

then $\pi(\mathcal{A}_{BC^{-1}})$ is the same figure eight geodesic, but with opposite orientation. However, if $q > 3$, then $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of orders q and r.

(iv) On $\mathcal{O}(p, q, r)$ for $p, q, r \geq 4$, $\pi(\mathcal{A}_{BA^{-1}})$ is the figure eight geodesic bounding the cone points of orders p and q, $\pi(\mathcal{A}_{BC^{-1}})$ is the figure eight geodesic bounding the cone points of orders q and r, and $\pi(\mathcal{A}_{AC^{-1}})$ is the figure eight geodesic bounding the cone points of orders p and r.

Definition 5.1.5. A triangle group is called *exceptional* if it has the form $\Gamma(2, 3, r)$ or $\Gamma(2, 4, r)$. Note that $\Gamma(2, 3, r)$ is a triangle group if and only if $r \geq 7$ and $\Gamma(2, 4, r)$ is a triangle group if and only if $r \geq 5$.

Corollary 5.1.6. *If $\Gamma(p, q, r)$ is not an exceptional triangle group, then $\mathcal{O}(p, q, r)$ contains a figure eight geodesic.*

Corollary 5.1.7. *Let $\mathcal{C}_{pq}, \mathcal{C}_{qr}$, and $\mathcal{C}_{pr}$ be the three loops on $\mathcal{O}(p, q, r)$ described at the beginning of this section. If such a loop is freely $\mathcal{O}(p, q, r)$-homotopic to a closed geodesic, then that geodesic will have one of the following three forms:*

(i) a figure eight geodesic

(ii) the geodesic path from a cone point of order two to a cone point of order four and back again.

(iii) the geodesic passing through the order three cone point pictured in Figure 5.9.

5.1.1 The Shortest Figure Eight Geodesics

Now that we have identified the figure eight geodesics on triangle group orbifolds we set out to find the shortest one. Suppose Γ is a Fuchsian group and let $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ be the projection map. If $g \in \Gamma$ is hyperbolic then the *length of the closed geodesic* $\pi(\mathcal{A}_g)$ on $\mathbb{H}/\Gamma$, denoted $\ell(\pi(\mathcal{A}_g))$, is the translation length $\mathcal{T}_g$ of g. Note that we have already established the relationship $|tr(g)| = 2\cosh(\mathcal{T}_g/2)$ in (2.4). Thus, $|tr(g)|$ and $\ell(\pi(\mathcal{A}_g))$ have a direct relationship. We will use the absolute value of the trace in proofs, but we state our results in terms of hyperbolic length. From now on we will write |trace| in place of the phrase "absolute value of the trace." In all of the proofs below we use the normalized triangle group $\Gamma(p,q,r)$ as in Section 4.2 and we assume $2 \leq p \leq q \leq r \leq \infty$.

Lemma 5.1.8. *The shortest figure eight geodesic on a triangle group orbifold without punctures is the unique one on* $\mathcal{O}(3,3,4)$ *which bounds the cone points of order three and has length*

$$2\cosh^{-1}\left(\frac{1+\sqrt{2}}{2}\right) = 1.26595\ldots.$$

Proof. We need only consider those elements BA^{-1}, BC^{-1}, and AC^{-1} in $\Gamma(p,q,r)$ which have an axis that projects to a figure eight geodesic (see Theorem 5.1.4). Recalling the trace identity derived from the Cayley-Hamilton Theorem above, (5.1), we have:

$$\begin{aligned} |tr(BA^{-1})| &= |tr(A)tr(B)| + |tr(C)| && (5.2)\\ &= 2\left[2\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right) + \cos\left(\frac{\pi}{r}\right)\right] && (5.3)\end{aligned}$$

Similar computations yield

$$|tr(BC^{-1})| = 2\left[2\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right) + \cos\left(\frac{\pi}{p}\right)\right] \tag{5.4}$$

$$|tr(AC^{-1})| = 2\left[2\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{r}\right) + \cos\left(\frac{\pi}{q}\right)\right]. \tag{5.5}$$

To find the smallest |trace| we consider two cases. First, for each triangle group of the form $\Gamma(2,q,r)$, the only element which has an axis projecting to a figure eight geodesic is BC^{-1}, and only if $q,r \geq 5$. From (5.4) we see that $|tr(BC^{-1})| = 4\cos(\frac{\pi}{q})\cos(\frac{\pi}{r})$ where $BC^{-1} \in \Gamma(2,q,r)$ for $q,r \geq 5$, and thus $|tr(BC^{-1})|$ will increase as q and/or r increase. It follows that $BC^{-1} \in \Gamma(2,5,5)$ has the smallest |trace|, namely $4\cos^2(\frac{\pi}{5}) = 2.618\ldots$.

Next, we look at triangle groups of the form $\Gamma(p,q,r)$ for $p,q \geq 3$ and $r \geq 4$. This includes all remaining triangle groups with elements having axes projecting to figure eight geodesics. We note that all three elements (BA^{-1}, BC^{-1}, and AC^{-1}) in $\Gamma(3,3,4)$ have the same |trace|, $1+\sqrt{2} = 2.414\ldots$. Observing (5.3)-(5.5), we see that as any of p, q, and/or r increase, the |trace| will increase. Thus $1+\sqrt{2}$ is the smallest |trace| in this case.

Comparing the results of the two cases we find that the unique figure eight geodesic on $\mathcal{O}(3,3,4)$ is the shortest. Using that the |trace| of the corresponding hyperbolic elements is $1+\sqrt{2}$, we get the resulting hyperbolic length of the figure eight geodesic using (2.4). □

Lemma 5.1.9. *The shortest figure eight geodesic on a triangle group orbifold with one puncture is the unique one on $\mathcal{O}(3,3,\infty)$ which bounds the cone points of order three and has length*

$$2\cosh^{-1}\left(\frac{3}{2}\right) = 1.92485\ldots.$$

Proof. Using the matrices in the presentation (4.6) for triangle groups with one parabolic generator, or equivalently letting $r \to \infty$ in (5.3)-(5.5), we get the following:

$$|tr(BA^{-1})| = 2\left[2\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)+1\right] \quad (5.6)$$

$$|tr(BC^{-1})| = 2\left[2\cos\left(\frac{\pi}{q}\right)+\cos\left(\frac{\pi}{p}\right)\right] \quad (5.7)$$

$$|tr(AC^{-1})| = 2\left[2\cos\left(\frac{\pi}{p}\right)+\cos\left(\frac{\pi}{q}\right)\right]. \quad (5.8)$$

To find the smallest |trace| we consider two cases. First, for each triangle group of the form $\Gamma(2,q,\infty)$, the only element which has an axis projecting to a figure eight geodesic is BC^{-1}, and only if $q \geq 5$. From (5.7) we see that $|tr(BC^{-1})| = 4\cos(\frac{\pi}{q})$ where $BC^{-1} \in \Gamma(2,q,\infty)$ for $q \geq 5$, and thus $|tr(BC^{-1})|$ will increase as q increases. It follows that $BC^{-1} \in \Gamma(2,5,\infty)$ has the smallest |trace|, namely $4\cos(\frac{\pi}{5}) = 3.236\ldots$.

Next, we look at triangle groups of the form $\Gamma(p,q,\infty)$ for $p,q \geq 3$. This includes all remaining triangle groups with a single puncture which contain elements having axes projecting to figure eight geodesics. We note that all three elements (BA^{-1}, BC^{-1}, and AC^{-1}) in $\Gamma(3,3,\infty)$ have the same |trace|, 3. Observing (5.6)-(5.8), we see that as p and/or q increase, the |trace| will increase. Thus 3 is the smallest |trace| in this case.

Comparing the results of the two cases we find that the unique figure eight geodesic on $\mathcal{O}(3,3,\infty)$ is the shortest. Using that the |trace| of the corresponding hyperbolic elements is 3, we get the resulting hyperbolic length of the figure eight geodesic using (2.4). □

Lemma 5.1.10. *The shortest figure eight geodesic on a triangle group orbifold with two punctures is the unique one on* $\mathcal{O}(2,\infty,\infty)$ *which bounds the punctures and has length*

$$2\cosh^{-1}(2) = 2.63392\ldots.$$

Proof. Letting $q \to \infty$ in (5.6)-(5.8), we get:

$$\begin{aligned} |tr(BA^{-1})| &= 2\left[2\cos\left(\frac{\pi}{p}\right)+1\right] \\ |tr(BC^{-1})| &= 2\left[2+\cos\left(\frac{\pi}{p}\right)\right] \\ |tr(AC^{-1})| &= 2\left[2\cos\left(\frac{\pi}{p}\right)+1\right]. \end{aligned}$$

From Theorem 5.1.4 we find that every element we are considering in this case yields a figure eight geodesic except for BA^{-1} and AC^{-1} in both $\Gamma(2,\infty,\infty)$ and $\Gamma(3,\infty,\infty)$. Observing the above |trace| formulas for the remaining elements, we get that $BC^{-1} \in \Gamma(2,\infty,\infty)$ has the smallest |trace|, namely 4.

□

Lemma 5.1.11. *$\mathcal{O}(\infty,\infty,\infty)$ contains three figure eight geodesics all having length*

$$2\cosh^{-1}(3) = 4\ln(1+\sqrt{2}) = 3.52549\ldots.$$

Proof. Using the presentation (4.8) we find that $|tr(BA^{-1})| = |tr(BC^{-1})| = |tr(AC^{-1})| = 6$. □

Theorem 5.1.12. *The shortest figure eight geodesic on a triangle group orbifold containing at least one puncture is the unique one on $\mathcal{O}(3,3,\infty)$ which bounds the cone points of order three and has length*

$$2\cosh^{-1}\left(\frac{3}{2}\right) = 1.92485\ldots.$$

Theorem 5.1.13. *The shortest figure eight geodesic on* ***any*** *triangle group orbifold is the unique one on $\mathcal{O}(3,3,4)$ which bounds the cone points of order* *three and has length*

$$2\cosh^{-1}\left(\frac{1+\sqrt{2}}{2}\right) = 1.26595\ldots.$$

Remark 5.1.14. Among those triangle groups containing figure eight geodesics, the one containing the shortest figure eight geodesic, $\mathcal{O}(3,3,4)$, has the smallest area: $\frac{\pi}{6}$.

5.2 Figure Eight Geodesics on 2-Orbifolds

With Theorem 5.1.13 established we look to generalize by finding the shortest figure eight geodesic on any (orientable hyperbolic) 2-orbifold without cone points of order two. We begin with a lemma and proof from [10] which highlights one of the reasons order two cone points are exceptional.

Lemma 5.2.1. *Let Γ be a Fuchsian group and let $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ be the projection map. Suppose w is an essential simple loop on $\mathbb{H}/\Gamma$ which is disjoint from the cone points and is freely $\mathbb{H}/\Gamma$-homotopic to a closed geodesic. If w does not bound a disc containing exactly two cone points both of order two, then the closed geodesic corresponding to w is also simple. If w does bound a disc containing exactly two cone points of order two, then the corresponding closed geodesic is the path which runs from one cone point of order two to the other cone point of order two, and back again.*

Proof. Let W be a connected component of $\pi^{-1}(w)$. W can be obtained by lifting w to a path W_0 and then iterating, that is, lift w again starting at the end point of W_0, then lift w again, and so on; at the same time lift w^{-1}, starting at the beginning point of W_0, then lift w^{-1} again, and so on.

The path W is invariant under $\langle g = \Phi([w])\rangle$, where Φ is the isomorphism between $\Pi_1(\mathbb{H}/\Gamma)$ and Γ. Hence W has well-defined end points: the fixed points of the hyperbolic isometry g. Since w is simple, no translate of W under Γ crosses W. Thus, the fixed points of g do not separate the fixed points of any conjugate of g. Then no translate of $\mathcal{A}_g$ crosses $\mathcal{A}_g$.

Let the stabilizer of $\mathcal{A}_g$ be denoted by $\Gamma_{\mathcal{A}_g}$. If $\Gamma_{\mathcal{A}_g} = \langle g\rangle$, then we have shown that $\mathcal{A}_g/\langle g\rangle$, the geodesic in the free $\mathbb{H}/\Gamma$-homotopy class of w, is simple. However, if $\Gamma_{\mathcal{A}_g}$ is not cyclic, then there is an elliptic isometry h of order two in $\Gamma_{\mathcal{A}_g}$ which interchanges the fixed points of g. Since the stabilizer is elementary and not cyclic, it must be the group $\langle g, h\rangle$. It is easy to see that the element $g \circ h$ is also elliptic of order 2 with fixed point on $\mathcal{A}_g$. Thus, $\Gamma_{\mathcal{A}_g} = \langle h, g \circ h\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$. In this case $\mathcal{A}_g/\langle g, h\rangle$ is a path from one cone point of order two to another cone point of order two, and then back again. Note that $W \cup h(W)$ is the boundary of a regular neighborhood of $\mathcal{A}_g$. Hence the original loop w is a simple loop that surrounds these two cone points. □

Definition 5.2.2. Let Γ be a Fuchsian group. The 2-orbifold $\mathbb{H}/\Gamma$ is called a *generalized pair of pants* if Γ has signature $(0 : m_1, \ldots, m_c; n; b)$ satisfying (3.3) and $c + n + b = 3$.

Thus triangle group orbifolds, as well as a usual pair of pants (a sphere with three boundary components), are examples of generalized pairs of pants.

Proposition 5.2.3. *A figure eight geodesic on a 2-orbifold without cone points of order two is contained in a generalized pair of pants.*

Proof. Suppose Γ is a Fuchsian group and let γ be a figure eight geodesic on $\mathbb{H}/\Gamma$. Choose the point of self-intersection of γ, say x, to be the base point of $\mathbb{H}/\Gamma$. Then we can write $\gamma = \alpha * \beta$, where α and β are the two component simple loops of γ based at x. Each of α and β either bound a cone point, bound a puncture, or are freely $\mathbb{H}/\Gamma$-homotopic to a simple closed geodesic (Theorem 3.5.2). Note that neither α nor β can be freely $\mathbb{H}/\Gamma$-homotopic to the geodesic path between two cone points of order two (see Lemma 5.2.1) because $\mathbb{H}/\Gamma$ does not contain any cone points of order two. Let $\Phi : \Pi_1(\mathbb{H}/\Gamma, x) \to \Gamma$ be the corresponding isomorphism from Theorem 3.4.5 and assume $\Phi([\alpha]) = f$ and $\Phi([\beta]) = g$. Thus, $\Phi([\gamma]) = fg$. Then, the conjugacy class of fg^{-1} represents a free $\mathbb{H}/\Gamma$-homotopy class containing a simple loop, which either bounds a cone point, bounds a puncture, or is freely $\mathbb{H}/\Gamma$-homotopic to a simple closed geodesic (based on whether fg^{-1} is respectively, elliptic, parabolic, or hyperbolic). Thus γ is contained in a generalized pair of pants with the three components being the projections of the fixed point (if elliptic or parabolic) or the axis (if hyperbolic) of f, g, and fg^{-1}. $\square$

Theorem 5.2.4. *The shortest figure eight geodesic on a 2-orbifold without cone points of order two is the unique one on $\mathcal{O}(3, 3, 4)$ which bounds the cone points of order three and has length*

$$2\cosh^{-1}\left(\frac{1+\sqrt{2}}{2}\right) = 1.26595\ldots.$$

Proof. Suppose Γ is a Fuchsian group and let γ be a figure eight geodesic on $\mathbb{H}/\Gamma$. By Proposition 5.2.3, γ is contained in a generalized pair of pants P within $\mathbb{H}/\Gamma$. If P has no boundary components, then $P = \mathbb{H}/\Gamma$ is a triangle group orbifold and so the result follows from Theorem 5.1.13. So suppose P

contains at least one boundary component. As we decrease the length of the boundary components of P so they become punctures, the lengths of loops on P will decrease (see [23], Lemma 3.4). Thus, the smallest figure eight geodesic cannot be contained in a generalized pair of pants with a boundary component. It follows that it must lie on a triangle group orbifold and so again the result follows from Theorem 5.1.13. □

Remark 5.2.5. Yamada [25, 26] found that the shortest non-simple closed geodesics on a 2-manifold are the figure eight geodesics on the thrice punctured sphere, $\mathcal{O}(\infty, \infty, \infty)$. As we saw above their length is

$$2\cosh^{-1}(3) = 4\ln(1+\sqrt{2}) = 3.52549\ldots.$$

It was shown through the works of Nakanishi [11] and Pommerenke and Purzitsky [16] that the shortest non-simple closed geodesic on a 2-orbifold is on $\mathcal{O}(2, 3, 7)$ and has length

$$\ell_* = 2\cosh^{-1}\left[\cos\left(\frac{2\pi}{7}\right) + \frac{1}{2}\right] = 0.98399\ldots.$$

Since $\Gamma(2, 3, 7)$ is an exceptional triangle group we know that a closed geodesic on $\mathcal{O}(2, 3, 7)$ with length ℓ_* cannot be a figure eight geodesic. Nakanishi [11] found that the non-simple closed geodesic γ_* pictured in Figure 5.14 has length ℓ_* and we observe that it passes through the cone point of order two. Vogeler's computation of the length spectrum of $\Gamma(2, 3, 7)$ [24] shows that γ_* is the unique non-simple closed geodesic on $\mathcal{O}(2, 3, 7)$ with length ℓ_*. Thus, γ_* is the shortest non-simple closed geodesic on a 2-orbifold.

The following table compares the results in Remark 5.2.5 with the main

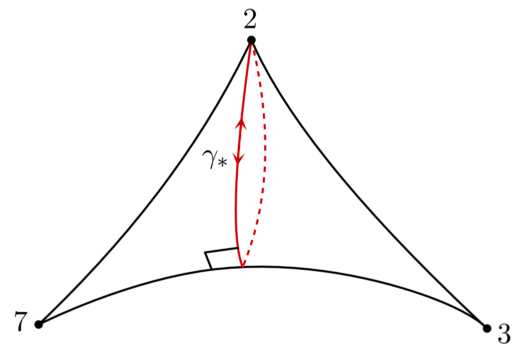

Figure 5.14: Shortest non-simple closed geodesic on a 2-orbifold

result of this thesis, Theorem 5.2.4.

Table 5.1: Minimal length non-simple closed geodesics

Classification	Closed Geodesic(s)	|trace|	Hyperbolic length	Orbifold	Area of Orbifold
Shortest non-simple closed geodesic on a 2-orbifold ([11, 16, 24])	Passes through order two cone point (Figure 5.14)	$2\cos(\frac{2\pi}{7}) + 1 = 2.24698\ldots$	$0.98399\ldots$	$\mathcal{O}(2,3,7)$	$\frac{\pi}{21}$
Shortest figure eight geodesic on a 2-orbifold without order two cone points (Theorem 5.2.4)	Figure eight geodesic bounding cone points of order three	$1+\sqrt{2} = 2.41421\ldots$	$1.26595\ldots$	$\mathcal{O}(3,3,4)$	$\frac{\pi}{6}$
Shortest non-simple closed geodesic on a 2-manifold ([25, 26])	All three figure eight geodesics	6	$3.52549\ldots$	$\mathcal{O}(\infty,\infty,\infty)$	2π

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