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SOLUTION OF THE NEUMANN PROBLEM, FOR THE HELMHOLTZ EQUATION, IN THE EXTERIOR OF A SURFACE WITH CORNERS

City University of New York PH.D. 1981

University
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International 300 N. Zeeb Road, Ann Arbor, MI 48106

Solution of the Neumann problem, for the Helmholtz equation, in the exterior of a surface with corners.

by

Whitney S. Harris, Jr.

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1981

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

Solution of the Neumann problem,

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by

Whitney S. Harris, Jr.

Adviser: Professor Richard Sacksteder.

In R^3, let Σ be a C^2 manifold with corners. Let E denote the set of edges of Σ, and let $\dot{\Sigma} = \Sigma - E$ denote the set of open faces of Σ. The integral equation method is used to solve the Neumann problem for the Helmholtz equation in the exterior of Σ, under either of the following hypotheses:

1). Σ has the property that, near an edge or corner, Σ is part of a polyhedron having angle between faces at least 90° or is part of a right circular cylinder. The allowable Neumann data functions are in $C^0 \cap L^\infty(\dot{\Sigma})$.

2). All of Σ is smooth (C^2). The Neumann data functions $g(p)$ are in $C^0 \cap L^2(\dot{\Sigma})$. They are allowed to grow, in a controlled manner, as p approaches curves bounding regions on the surface (formerly edges and faces).

Proofs of lemmas about the behavior of single and double layer potentials are also given.

This dissertation is dedicated to

my wife, Joan A. Harris,

my mother, Anna M. Harris,

and to the memory of

my father, Whitney S. Harris.

Acknowledgements

It was through the C.U.N.Y. Preceptorial Program that I became acquainted with the City University Graduate Center. This program was led by Professor Louis Auslander who encouraged me to enter the Ph.D. program. I am grateful to him for this and for the wide applicability of what I learned from him.

I would also like to thank my mentor, Professor Richard Sacksteder, for a thesis topic that appealed to me and that I found rewarding. His erudition and consistent good humor were valuable assets to me. It was a privilege and a pleasure working with him.

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Chapter One

Preliminaries

1.1 Introduction.

In this work we are concerned with one of the three classical boundary value problems that have been investigated in Potential Theory ($\Delta u = 0$) and in the theory of the Helmholtz (or reduced wave) equation ($\Delta u + w^2 u = 0$, w real). Each problem depends on a given data function defined on points p of a closed bounded surface Σ in R^3. Specifically,

1) $u(p)$: Dirichlet problem.
2) Normal derivative, $\frac{\partial u}{\partial N}(p)$: Neumann Problem.
3) $\frac{\partial u}{\partial N} + \delta(p)u(p)$: Mixed problem.

The methods that we will use are applicable to all three problems but we will concentrate our attention on the exterior Neumann problem for the Helmholtz equation, that is, a solution is sought in the exterior of Σ, having the boundary value 2). For this type of problem the Sommerfeld (outward) radiation condition is also imposed, namely, $\frac{\partial u}{\partial r} - iwu = o(r^{-1})$, uniformly along all rays from the origin. Under these conditions the solution is unique [11], [15].

The integral equation method (Neumann method) of solving these problems depends on finding a solution which is a linear combination of a single layer potential

$$v(p) = \int_{\Sigma} g(q)Q(p,q)\, d\sigma(q)$$

and a double layer potential

$$u(p) = \int_{\Sigma} \phi(q)K(p,q)\, d\sigma(q) \quad .$$

g and ϕ are called density functions, N_q is the outward normal to the surface at $q \in \Sigma$, $r = |p - q|$, $Q(p,q) = e^{iwr}/r$, and $K(p,q) = \frac{\partial}{\partial N_q} Q(p,q)$.

With these potentials we associate operators $Qg(p) = v(p)$, $K\phi(p) = u(p)$, and $K^t\phi(p) = \int_{\Sigma} \phi(q)K(q,p)\, d\sigma(q)$. For a closed smooth surface Σ and with the operators K, K^t defined on $C^o(\Sigma)$, Leis [7] solved problems 2) and 3). Solutions of these problems particularly suited to applications in Acoustics are given by Sacksteder [16].

From here on we will assume that the closed surface Σ is permitted to have edges and corners and is C^2 elsewhere. K^t now leads out of the space of continous functions (cf. Section 1.4). In Chapters 4 and 5 we solve the Neumann problem by the integral equation method with K, K^t defined on the Banach spaces $L^{\infty}(\Sigma), L^1(\Sigma)$ respectively. We will show compactness of an operator $K^t\pi_h$ that is near enough in norm to K^t to imply the Fredholm Alternative. This is the program that Leis used to solve 1) and 2) in two dimensional space for a boundary curve Σ allowing corners [8]. He required that Σ be linear in a neighborhood of each corner.

There are some distinguishing differences between the two and three dimensional problems. In two dimensions only two sides of the boundary meet at a corner whereas in three dimensions many faces may meet at a corner. Because of this our results are limited to a restricted class of surfaces (P.C. surfaces, Chapter 4), but these are sufficient for many important applications. An outline of a solution to the Dirichlet problem for a subclass of these surfaces was given by Leis in [9].

Another characteristic of the two dimensional case is that a certain criterion for compactness for sets of functions in $L^1(\Sigma)$, which depends on the concept of translation on the boundary curve Σ, can be used ([8] Lemma 3, [2], [13]).

It bears investigation as to whether there is a generalization of this notion that is applicable to a closed boundary surface Σ in R^3. Our treatment in Chapter 4 employs other methods to show compactness.

In another paper [10] Leis solves the Dirichlet problem in R^3 using concentric spheres S_1, S_2 with the property that $\Sigma \subset S_1 \subset S_2$. An interior solution for the region bounded by Σ, S_2 is joined to a solution in the exterior of S_1. The use of this method for the Neumann problem presents difficulties, and in any case it does not supply as explicit a solution as does the integral equation method.

Problems 1), 2), and 3) could be solved for general surfaces with corners if we could find a Banach space of functions on Σ with the property that K and K^t are compact operators on that space. K and K^t are bounded operators on the space $W(\Sigma)$ of Chapter 3. Whether these operators are compact on the space $W(\Sigma)$ or some other space is a matter for future investigation. However, we do show in Chapter 3 that K and K^t are compact operators on $W(\Sigma)$ provided all of Σ is smooth (C^2) . Based on this, using the integral equation method, in Chapters 3 and 5, we solve the Neumann problem for data functions in $W(\Sigma)$. Since the space of functions $W(\Sigma)$ is more extensive than $C^0(\Sigma)$, we have a new result. Also $W(\Sigma)$ contains functions not allowable as data in the solution for P.C. surfaces (particularized to smooth surfaces). That solution requires data functions in $L^\infty(\Sigma)$. The functions g(p) in $W(\Sigma)$ are allowed to grow in a controlled manner as p approaches curves bounding regions on the surface (formerly edges and faces).

The integral equation method of solution depends on a sizeable collection of lemmas about the behavior of the single and double layer potentials, v(p) and u(p), for p on or approaching the surface. We have included proofs of them in Chapter 2 for kernals Q, K, K^t that are solutions of the Helmholtz equation, for surfaces that have edges and corners, and for various classes of density functions including $L^1(\Sigma)$, $L^\infty(\Sigma)$, $W(\Sigma)$. The proofs are similar to those in the potential theory case (corresponding to $w = 0$). Good sources of these are [3] and [18]. For some of the lemmas a considerable amount of extra work is required for the Helmholtz version. Our proof of Lemma 2.24 is different from those in the potential theory sources. It is based on a theorem of Kellogg that is of interest in its own right.

1.2 Nature of the surface Σ.

We will now be more specific about the nature of the surface on which boundary values are defined.

Let Σ be a compact oriented two manifold. It will be assumed that Σ has the structure of a C^k manifold with corners, that is:

1) Σ has a decomposition $\Sigma = \bigcup_{i=1}^{n} \Sigma_i$, where n is a positive integer, each Σ_i is closed, and there is a homeomorphism $h_i: \Sigma_i \to T_i$.
2) Each set T_i is a compact subset of the plane bounded by a finite number of line segments and T_i is the closure of the interior of T_i .
3) For every pair i,j $\Sigma_i \cap \Sigma_j$ is either empty, the image under h_i^{-1} of a vertex of T_i (and similarly of T_j) or the image

under h_i^{-1} of a side of T_i (similarly for T_j).

4) Where the third possibility in 3) holds, the mapping $h_j \circ h_i^{-1}$: $h_i(\Sigma_i \cap \Sigma_j) \longrightarrow h_j(\Sigma_i \cap \Sigma_j)$ is of class C^k .

Note: The definition just given is equivalent to the apparently more general one in which the sides of T_i are replaced by images of C^k imbeddings of line segments into E^3 .

A homeomorphism $f: \Sigma \longrightarrow E^3$ will be called a C^k imbedding if for every $i = 1,2,\ldots,n$, $f \circ h_i^{-1}$: $T_i \longrightarrow E^3$ is a regular C^k map, that is, a C^k map $T_i \longrightarrow E^3$ such that the tangent map is everywhere of rank 2.

The surface Σ on which boundary data is given is understood from here on to be the image in E^3 of a C^2 imbedding of a C^2 manifold with corners.

1.3 <u>Notation and definitions</u>.

The sets Σ_i in the definition of Σ will be called the closed faces of Σ . The interior of these sets denoted by $\overset{\circ}{\Sigma}_i$ are called the (open) faces of Σ. Let $\overset{\circ}{\Sigma} = \bigcup_{i=1}^{n} \overset{\circ}{\Sigma}_i$. This is the same as $\Sigma - E$ where E is the set of edges of Σ .

Denote by Ω the compact region of R^3 bounded by Σ.

Let Ω_- (or Σ_-) denote the interior of Ω and Ω_+ (or Σ_+) the exterior of Ω. We will also feel free to use a subscripted sign to indicate for example a point p_+ in Ω_+ or a limit $F(p)_+$ of a function $F(p_+)$ as $p_+ \to p \in \Sigma$.

We will use the notation $\iint_{\Omega} f(p,q)\, d\Omega(q)$ for volume integrals and $\int_{\Sigma} f(p,q)\, d\sigma(q)$ for surface integrals.

To clarify and emphasize calculus operations with respect to a certain variable (in this case q) we will use notations like $\frac{\partial}{\partial N_q} f(p,q)$ or $\Delta_q f(p,q)$.

We will make repeated use of the following definitions in Chapter 2.

Definition 1.1: Tangent - normal coordinate system at $p_o \in \Sigma$.

At p_o consider a rectilinear coordinate system with x,y plane the same as the tangent plane at p_o and the positive z axis in the direction of the outward normal to Σ at p_o. This will be called a tangent - normal coordinate system at p_o.

Definition 1.2: An element of surface E of radius ϵ at $p_o \in \Sigma$.

Let F be the face containing p_o and l the normal line to Σ at p_o . Let C be a right circular cylinder with axis l and radius ϵ . Call the portion of the face F locally cut out by C , an element of surface of radius ϵ at p_o and denote it by E . Here, ϵ is assumed small enough so that E is described by a single valued function $z = q(x,y)$ in the tangent - normal coordinate system at p_o . Note that E may be truncated if p is closer than ϵ to the edge of F .

Definition 1.3: Local cylindrical region (surface) at p_o.

In the context and notation of Def. 1.2, let $z=q(x,y)$ describe the surface locally in a tangent - normal coordinate system at $p_o \in \Sigma$. Let M be the minimum value of z on E . Let D be the disc cut out from the plane $z = M - \epsilon$ by the cylinder C . Let Ω be the three dimensional region bounded by the disc D , part of the cylinder C and the surface element E . Ω will be called a local cylinderical region at p_o . Its boundary, $\partial\Omega$, will be called a local cylindrical surface at p_o .

Definition 1.4: Extension of a function $\phi(q)$ defined on a surface element E , to a function F(q) defined in (on) the corresponding local cylindrical region (surface).

In the context and notation of Defs. 1.2 and 1.3, the given function $\phi(q)$ can be expressed in the tangent - normal coordinate system as $\phi(q) = \overline{\phi}(x,y)$. Ω is the local cylindrical region corresponding to E . We define an extension $F(q)$ of $\phi(q)$ to Ω ($\partial\Omega$) as follows. At a point $q \in \Omega(\partial\Omega)$, let $F(q)$ have the same value that ϕ has at the orthogonal projection of q on the surface element E, that is $F(x,y,z) = \hat{\phi}(x,y)$.

1.4 Example

There follows below, an example of a surface with edges, for which, with density function $f(p) \equiv 1$, $K^t f(p)$ is logarithmically singular.

Consider a half cylinder, $H = \{(x,y,z) \in R^3 \mid y \geq 0, x^2 + y^2 = 1, 0 \leq z \leq 1\}$. Let p be the point $(0,0,t)$ and let F be the face containing p. Let B be that face of H on which $z = 0$ and C the curved face, $\{(x,y,z) \in R^3 \mid x^2 + y^2 = 1, 0 \leq z \leq 1\}$.

$$K^t f(p) = \int_F K(q,p)\, d\sigma(q) + \int_C K(q,p)\, d\sigma(q) + \int_B K(q,p) d\sigma(q).$$

Let $A(r) = e^{iwr}(iwr - 1)$, and for $p,q \in \Sigma$ let A be a bound for $|A(r)|$. By Lemma 2.2, $K(q,p) = \frac{A(r)}{r^2} \frac{\overrightarrow{p-q}}{r} \cdot \overrightarrow{N_p}$.

Since p is in F, the first integral is zero, and since for q on C, $r \geq 1$, a bound on the second is A times the area of C. For the third integral, $q = (x,y,0)$, $N_p = (0,-1,0)$, $p-q = (-x,-y,t)$, and $r^2 = \rho^2+t^2$ where $\rho^2 = x^2+y^2$.

In polar coordinates,

$$\int_B K(p,q)\,d\sigma(q) = \int_0^{\pi}\int_0^1 \frac{\rho \sin\theta}{(\rho^2+t^2)^{3/2}}\,\rho\, d\rho\, d\theta$$

$$= \int_0^{\pi} \left[\frac{-\rho}{\sqrt{\rho^2+t^2}} + \ln(\sqrt{\rho^2+t^2}+\rho)\right]_0^1 \sin\theta\, d\theta$$

$$= 2\left[\frac{-1}{\sqrt{1+t^2}} + \ln(\sqrt{1+t^2}+1) - \ln t\right].$$

As $t \to 0$, the first two terms approach limits, and the third term manifests the singular behavior.

Chapter Two

Behavior of Potentials

In this chapter we will investigate the properties of the single and double layer potentials. Since this is lengthy, the reader who desires an overview and summary of these results, before proceeding, is advised to read sections 5.1 and 5.2 .

Many of the results of this chapter can be shown for surfaces which are only required to satisfy a Hölder condition. Schauder [18] does this in the potential theory case for surfaces without edges. Our requirements for the surface are stated in Section 1.2 .

2.1 Elementary estimates

Lemma 2.1 Let p_0 be a point in $\dot{\Sigma}$. Suppose Σ is of class C^2 in a neighborhood $N(p_0)$ of p_0 , and in $N(p_0)$ the function $z = q(x,y)$ describes Σ in a tangent - normal coordinate system with origin at p_0 . Let $\rho = \sqrt{x^2+y^2}$, then independent of p_0 ,

$$z = O(\rho^2) \ , \quad z_x = O(\rho) \ , \quad z_y = O(\rho) \ .$$

The proof follows from Taylor's formula with remainder.

On a number of occasions we will use local polar coordinates. Their use is justified by the following discussion. Under the hypotheses of Lemma 2.1 we introduce polar coordinates ρ, Θ in the tangent plane. The position vector to a point on the surface is $X = (x,y,z)$ and as usual $x = \rho \cos \Theta$, $y = \rho \sin \Theta$. Let $z = q(x,y) = \hat{q}(\rho,\Theta)$. We calculate the first fundamental form using $z_x = O(\rho)$ and $z_y = O(\rho)$ from Lemma 2.1:

$$EF - G^2 = (X_\rho \cdot X_\rho)(X_\Theta \cdot X_\Theta) - (X_\rho \cdot X_\Theta)^2$$
$$= \rho^2 + O(\rho^4) .$$

The Binomial Theorem then yields

$$\sqrt{EF - G^2} = \rho + O(\rho^3) .$$

Therefore, an integral over an element of surface E of sufficiently small radius ϵ at p_0 can be expressed as

$$\int_E f(q)d\sigma(q) = \int_0^{2\pi}\int_0^{\epsilon} f(\hat{q}(\rho,\Theta))\left[\rho + O(\rho^3)\right] d\rho d\Theta .$$

For ϵ small, ρ dominates terms of higher order in ρ hence,

$$\int_E |f(q)|\, d\sigma(q) \leq c\int_0^{2\pi}\int_0^{\epsilon} |f(\hat{q}(\rho,\Theta))|\, \rho\, d\rho d\Theta ,$$

where c is a positive constant.

Lemma 2.2 Consider a point p in Σ and a ball B of radius r_0 centered at p. For $p,q \in \Sigma$, let A be a bound of $|e^{iwr}(iwr-1)|$. Then, for $\phi \in L^1(\Sigma)$ and $\|\phi\|$ the L^1 norm,

$$\int_{\Sigma - B} |\phi(q)|\, |K(p,q)|\, d\sigma(q) \leq \frac{A}{r_0^2} \|\phi\| .$$

The above is also true if we replace K by K^t.

<u>Proof</u>. Adopt the following notation: $p = (X,Y,Z)$, $q = (x,y,z)$, $r = |p-q| = \sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}$. The exterior normal vectors at q and p are respectively,

$$N_q = (N_q^x, N_q^y, N_q^z) \quad , \quad N_p = (N_p^x, N_p^y, N_p^z) \quad .$$

We organize calculations for use here as well as for future reference.

(1)
$$\nabla_q r = \frac{1}{r}(x-X, y-Y, z-Z) = \frac{q-p}{r}$$

(2)
$$\nabla_p r = \frac{1}{r}(X-x, Y-y, Z-z) = \frac{p-q}{r}$$

$$K(p,q) = \frac{\partial}{\partial N_q} Q(p,q) = \nabla_q Q \cdot N_q = \frac{\partial Q}{\partial r} \nabla_q r \cdot N_q$$

(3)
$$K(p,q) = \frac{e^{iwr}(iwr-1)}{r^2} \frac{(q-p)}{r} \cdot N_q$$

(4)
$$K(p,q) = \frac{e^{iwr}(iwr-1)}{r^3} \left[(x-X)N_q^x + (y-Y)N_q^y + (z-Z)N_q^z\right].$$

Similarly,

$$K^t(p,q) = K(q,p) = \nabla_p Q \cdot N_p = \frac{\partial Q}{\partial r} \nabla_p r \cdot N_p$$

(5)
$$K^t(p,q) = \frac{e^{iwr}(iwr-1)}{r^2} \frac{(p-q)}{r} \cdot N_p$$

(6)
$$K^t(p,q) = \frac{e^{iwr}(iwr-1)}{r^3} \left[(X-x)N_p^x + (Y-y)N_p^y + (Z-z)N_p^z\right].$$

Therefore,

(7) $$|K(p,q)| \leq \frac{A}{r^2} \quad , \quad |K^t(p,q)| \leq \frac{A}{r^2} \quad ;$$

and for $q \in \Sigma - B$, $r \geq r_o$, hence

$$\int_{\Sigma-B} |\phi(q)|\,|K(p,q)|\,d\sigma(q) \leq \frac{A}{r_o^2}\int_{\Sigma-B} |\phi(q)|\,d\sigma(q)$$
$$\leq \frac{A}{r_o^2}\,\|\phi\| \quad .$$

The same holds for K^t in place of K .

Lemma 2.3 With respect to a tangent - normal coordinate system at the fixed point $p \in \Sigma$, (Def. 1.1), the orders of magnitude of the components of the normal at point q , in the same face as p , are given by

(1) $$N_q = (O(\rho), O(\rho), O(1)) \quad ,$$

and the change with respect to the normal vector at the origin is

(2) $$N_q - N_p = (O(\rho), O(\rho), O(\rho^2)) \quad .$$

Proof. The Calculus formulas for the components of N_q are

$$N_q^x = \frac{-z_x}{D} \quad , \quad N_q^y = \frac{-z_y}{D} \quad , \quad N_q^z = \frac{1}{D}$$

where $D = \sqrt{1+z_x^2+z_y^2}$.

Substituting orders of magnitude from Lemma 2.1 we obtain

$$N_q^x = O(\rho) \quad , \quad N_q^y = O(\rho) \quad , \quad N_q^z = O(1) \quad .$$

$N_p = (0,0,1)$, so clearly the first two components of $N_q - N_p$ are correct. The third component is

$$\frac{1}{D} - 1 = \frac{1-D}{D} = \frac{1-\sqrt{1+z_x^2+z_y^2}}{\sqrt{1+z_x^2+z_x^2}} \quad .$$

The Binomial Theorem applied to the numerator shows that it has $O(z_x^2+z_y^2)$ which by Lemma 2.1 equals $O(\rho^2)$. The denominator has $O(1)$, hence the third component has $O(\rho^2)$ as claimed.

<u>Definition 2.1</u>: L.B. denotes the set of integrable functions ϕ on Σ that are <u>locally essentially bounded</u> on $\dot{\Sigma}$, that is, given any point $p \in \dot{\Sigma}$, there is a neighborhood of p , $N(p)$ with the property that ϕ is essentially bounded in $N(p)$. This is true for functions of $W(\Sigma)$ (Chapter 3) or $L^{\infty}(\Sigma)$.

<u>Lemma 2.4</u> For any point q in a surface element E of sufficiently small radius ϵ , centered at a fixed point $p \in \dot{\Sigma}$ and $\phi \in$ L.B.

$$|K(p,q)| = O(r^{-1})$$

and

$$\int_E |\phi(q)| \, |K(p,q)| \, d\sigma(q) \leq a\epsilon \operatorname*{ess\,sup}_{q \in E} |\phi(q)|$$

where a is a positive constant independent of p . The above estimates also hold for K replaced by K^t .

<u>Proof</u>. In a tangent - normal coordinate system at p, $p = (X,Y,Z) = (0,0,0)$. From (4) of Lemma 2.2,

$$|K(p,q)| \leq \left|\frac{A}{r^3}(xN_q^x + yN_q^y + zN_q^z)\right| .$$

Substituting orders of magnitude from Lemmas 2.1 and 2.3, the factor in parentheses is seen to have $O(\rho^2)$. That, together with $r \geq \rho$, implies

$$|K(p,q)| = O(r^{-1}) = O(\rho^{-1}) .$$

From (6) of Lemma 2.2,

$$|K^t(p,q)| \leq \left|\frac{A}{r^3}(xN_p^x + yN_p^y + zN_p^z)\right| .$$

Using $N_p = (0,0,1)$, Lemma 2.1, and $r \geq \rho$, we obtain

$$|K^t(p,q)| = O(r^{-1}) = O(\rho^{-1}) .$$

Let c denote a positive constant. For the radius ϵ of E sufficiently small (independent of p), we obtain in local polar coordinates

$$\int_E |\phi(q)||K(p,q)|\, d\sigma(q) \leq \int_0^{2\pi}\int_0^{\epsilon} |\phi(q)|\, c\rho^{-1}\rho\, d\rho d\theta$$

$$\leq 2\pi c\epsilon \operatorname*{ess\,sup}_{q \in E} |\phi(q)|$$

$$\leq a\epsilon \operatorname*{ess\,sup}_{q \in E} |\phi(q)|$$

where $a = 2\pi c$. The same holds for K replaced by K^t.

From here on we will only consider surface elements small enough for Lemma 2.4 to hold. Let E be the set of edge points of Σ and S_m the set of points of Σ having distance from E at least $1/2^m$. On $C^o \cap L^1(\Sigma)$ we can define a family of norms for $m = 1,2,3,\ldots$ by $\|f\|_m = \sup_{p \in S_m} |f(p)|$. These appear in the next lemma and play a major role in Chapter 3.

<u>Lemma 2.5</u> For $\phi(q) \in C^o \cap L^1(\Sigma)$, $u(p) = K\phi(p)$ exists at all points $p \in S_m \subset \Sigma$ and is bounded as follows. For all $p \in S_m$,

$$|K\phi(p)| \le a\|\phi\|_{m+1}\epsilon + \frac{b}{\epsilon^2}\|\phi\|$$

where a, b, and ϵ are positive constants. a and b are independent of ϕ and m and ϵ is sufficiently small. The Lemma also holds with $K\phi(p)$ replaced by $K^t\phi(p)$.

<u>Proof</u>. Let F be the face of Σ that contains p. Let B be a ball of radius ϵ about p. Choose $\epsilon < 1/2^{m+1}$ and small enough that $(B \cap \Sigma) \subset F$. Let E be an element of surface at p of radius ϵ. $E \subset S_{m+1}$, so $\sup_{q \in E} |\phi(q)| = \|\phi\|_{m+1}$, hence by Lemmas 2.2 and 2.4,

$$|K\phi(p)| \le \int_{\Sigma \cap B} |\phi(q)|\,|K(p,q)|\,d\sigma(q) + \int_{\Sigma - B} |\phi(q)|\,|K(p,q)|\,d\sigma(q)$$

$$\le a\,\|\phi\|_{m+1}\,\epsilon + \frac{b}{\epsilon^2}\|\phi\|$$

as required.

Similar reasoning produces the following corollary.

Corollary 2.5 If $\phi \in$ L.B. , E is a surface element of sufficiently small radius ϵ at $p_o \in \Sigma$, and

$\operatorname{ess\,sup}_{p \in E} |\phi(p)| = c$, then

$$|K\phi(p)| \leq ac\epsilon + \frac{b}{\epsilon^2} \|\phi\| .$$

2.2 Continuity of $u(p)$ and $v(p)$.

Lemma 2.6 For density functions $\phi(q)$, $g(q)$ in $L^1(\Sigma)$, $u(p)$ and $v(p)$ exist and are continuous at all points not on Σ .

Proof. Let $N(p_o)$ be a neighborhood of $p_o \notin \Sigma$ chosen so that it has positive distance from the surface Σ . $Q(p,q)$ and $K(p,q)$ are continuous and bounded in $N(p_o)$. ($r = |p-q|$ is bounded away from zero for $p \in N(p_o)$ and $q \in \Sigma$.) Call the bounds on the respective absolute values A and B . Then

$$|v(p)| \leq \int_\Sigma |g(q)|\, |Q(p,q)|\, d\sigma(q)$$

and

$$|u(p)| \leq \int_\Sigma |\phi(q)|\, |K(p,q)|\, d\sigma(q)$$

have integrands bounded by integrable functions $|g(q)|A$ and $|\phi(q)|B$ respectively. Thus $v(p)$ and $u(p)$ exist for $p \in N(p_o)$ and by Lebesgue's dominated convergence theorem we can pass to the limit under the integral sign. Thus

$$\lim_{p \to p_0} v(p) = \int_{\Sigma} \lim_{p \to p_0} g(q)Q(p,q)\, d\sigma(q) \quad ,$$

which shows that $v(p)$ is continuous at p_0. Similarly, $u(p)$ is continuous at p_0.

Similar reasoning produces the following corollary.

Corollary 2.6 $u(p), v(p) \in C^{\infty}(R^3 - \Sigma)$.

Lemma 2.7 Let p_0 be any point in $\dot{\Sigma}$. If the density function g is in L.B., then $v(p_0)$ exists, and $v(p)$ is a continuous function at p_0.

Proof. Let B_1 and B_2 be balls centered at p_0 with radii δ and $\delta/2$ respectively. Consider a tangent - normal coordinate system S_1 at p_0 (Def.1). Denote the tangent plane at p_0 by π. Let $E = \Sigma \cap B_1$, confine p to B_2, and split the integral for $v(p)$.

$$v(p) = \int_E g(q)Q(p,q)\, d\sigma(q) + \int_{\Sigma - E} g(q)Q(p,q)\, d\sigma(q) \quad .$$

Call the first integral $v_1(p)$ and the second $v_2(p)$. By the dominated convergence theorem $v_2(p)$ is a continuous function of p at p_0.

Let ρ, θ denote a local polar coordinate system at p_0. Let $q = (x,y,z)$ and let q' be the orthogonal projection of q onto π. Let $C = \operatorname{ess\,sup}_{q \in E} |g(q)|$ and note that $|Q(p_o,q)| = |e^{iwr}/r| \leq 1/r$. Here, $r = |p_o-q| = \sqrt{x^2+y^2+z^2}$, and $\rho = |p_o-q'| = \sqrt{x^2+y^2}$, so $\rho \leq r$. Therefore,

$$|v_1(p_o)| \leq \int_E |g(q)|\ |Q(p_o,q)|\ d\sigma(q)$$
$$\leq c \int_0^{2\pi}\int_0^{\delta} \frac{1}{\rho}\rho\, d\rho d\theta$$

shows the existence of $v_1(p_o)$.

To show the continuity of $v(p)$ at p_o examine $v_1(p)$ for p in B_2 . Let $r = |p-q|$.

(1) $$v_1(p) = \int_E |g(q)|\ |Q(p,q)|\ d\sigma(q)$$
$$\leq c\int_E \frac{1}{r}\, d\sigma(q) \quad .$$

Let p' be the orthogonal projection of p onto π . Let p' be the origin of another polar coordinate system, S_2 , in plane π with coordinates θ and $\hat{\rho}$, where

(2) $$\hat{\rho} = |p'-q'| \leq r = |p-q| \quad .$$

We want to express (1) in terms of coordinate system S_2 . For $p \in B_2$ and $q \in E$

(3) $$r \leq |q-p_o| + |p-p_o|$$
$$\leq \delta + \delta/2$$
$$\leq (3/2)\,\delta \quad .$$

From (1) , (2) and (3) , for $p \in B_2$,

$$(4) \qquad |v_1(p)| \leq ac\int_0^{2\pi}\int_0^{(3/2)\delta} \frac{1}{\rho}\rho \, d\rho \, d\theta$$

$$\leq 3\pi ac\,\delta$$

where a is a positive constant. Then

$$(5) \qquad |v(p)-v(p_o)| \leq |v_1(p)-v_1(p_o)| + |v_2(p)-v_2(p_o)|$$

$$\leq |v_1(p)| + |v_1(p_o)| + |v_2(p)-v_2(p_o)|$$

$$\leq 6\pi ac\,\delta + |v_2(p)-v_2(p_o)| \quad .$$

Given an ϵ, a δ can be found to make the first term less than $\epsilon/2$. Then, with δ fixed, p can be chosen close enough to p_o to make the second term less than $\epsilon/2$. Therefore, $v(p)$ is continuous at p_o.

2.3 Jump relations for $u(p)$ and $(\partial v/\partial N)(p)$, as p approaches the surface.

The "jump relations" in Lemma 2.9, will provide the integral equations that are used to solve the three classical boundary value problems. We will prove this lemma for summable density functions, that is, of class $L^1(\Sigma)$. The jump relations will hold at Lebesgue points.

Definition 2.2 $p_o \in \dot{\Sigma}$ will be called a Lebesgue point of $f \in L^1(\dot{\Sigma})$, if there is an element of surface E of radius δ at p_o with the property that

$$\lim_{\delta \to 0} \frac{1}{\pi \delta^2} \int_E |f(p)-f(p_o)| \, d\sigma(p) = 0$$

The set of non Lebesgue points has measure zero [4] . Further, if $f \in C^o \cap L^1(\dot{\Sigma})$ then all points in $\dot{\Sigma}$ are Lebesgue points of f .

Let l be the normal line to the surface at $p_o \in \dot{\Sigma}$ and let N be the outward normal vector at p_o . The derivative $\frac{\partial v}{\partial N}(p)$ exists at points p on l $(p \neq p_o)$. Let $\frac{\partial v}{\partial N}(p_o)_-$, $\frac{\partial v}{\partial N}(p_o)_+$ denote the limits of $\frac{\partial v}{\partial N}(p)$ as $p \to p_o$ along l from inside, outside Σ . Let $u(p_o)_-$, $u(p_o)_+$ denote the limit of $u(p)$ as $p \to p_o$ from inside, outside Σ . Let

$$u(p_o) = \int_\Sigma \phi(q) \, K(p_o,q) \, d\sigma(q)$$

$$\frac{\partial v}{\partial N}(p_o) = \int_\Sigma g(q) \, \frac{\partial}{\partial N_p} \, Q(p_o,q) \, d\sigma(q)$$

These are called the direct values of $u(p)$ and $\frac{\partial v}{\partial N}$ at p_o . Their existence is implied by the following lemma [3] p.112 .

<u>Lemma 2.8</u> Consider a tangent - normal coordinate system at p_o and a surface element E of radius δ at p_o . Let $q = (x,y,z)$ be an arbitrary point in E , and $\rho = \sqrt{x^2+y^2}$. If p_o is a Lebesgue point of $f \in L^1(\dot{\Sigma})$ then the integral

$$\int_E \frac{f(q) - f(p_o)}{\rho} \, d\sigma(q)$$

is convergent and has limit zero as $\delta \to 0$. Moreover, for $p \in l$, that is, $p = (0,0,z)$, the integral

$$|z| \int_E \frac{|f(q) - f(p_o)|}{(\rho^2+z^2)^{3/2}} \, d\sigma(q)$$

tends uniformly to zero as $\delta \to 0$, for all nonzero values of z .

The technical lemma just stated will also be used in the proof of the jump relations.

<u>Lemma 2.9</u> Let $u(p)$ be a double layer potential with density ϕ and let $v(p)$ be a single layer potential with density g . If p_o is a Lebesgue point of the functions $\phi, g \in L^1(\Sigma)$ then

(1)
$$u(p_o)_+ = +2\pi\phi(p_o) + u(p_o)$$
$$u(p_o)_- = -2\pi\phi(p_o) + u(p_o)$$

(2)
$$\frac{\partial v}{\partial N}(p_o)_+ = -2\pi g(p_o) + \frac{\partial v}{\partial N}(p_o)$$
$$\frac{\partial v}{\partial N}(p_o)_- = +2\pi g(p_o) - \frac{\partial v}{\partial N}(p_o)$$

<u>Proof</u>. The proof will be given in two parts. In Part A we will prove, for unit density, (1) then (2) . In Part B we will prove for summable density, based on Part A, (2) then (1) .

<u>Part A(1)</u>. Let Ω be the region interior to Σ and let p be a point in the exterior of Σ. From Green's identity for the functions 1 and $Q(p,q)$ we write

$$\int_{\Sigma} K(p,q)\,d\sigma(q) = \iint_{\Omega} (\Delta_q Q(p,q)+w^2Q(p,q))\,d\Omega(q) - \iint_{\Omega} w^2Q(p,q)\,d\Omega(q)$$

The first integral is zero because Q satisfies the reduced wave equation in Ω. The second integral is a continuous function of p, call it $H(p)$. Thus

$$u(p) = -H(p) \tag{3}$$

Now consider p in Ω. Let B be a ball of radius δ about p. Green's identity for $\Omega - B$ gives

$$\int_{\Sigma} K(p,q)\,d\sigma(q) - \int_{\partial B} K(p,q)\,d\sigma(q) = -\iint_{\Omega - B} w^2Q(p,q)\,d\Omega(q) \quad .$$

On ∂B, $\frac{\partial Q}{\partial N} = \frac{\partial Q}{\partial r} = e^{iw\delta}\left(\frac{iw}{\delta} - \frac{1}{\delta^2}\right)$ and

$$\int_{\partial B} K(p,q)\,d\sigma(q) = -4\pi e^{iw\delta} + O(\delta) \quad ,$$

so in the limit as $\delta \to 0$ we obtain

$$u(p) = -4\pi - H(p) \quad . \tag{4}$$

Lastly, consider $p = p_0$. Let $\Omega_1 = \Omega \cap B$, $S_1 = \partial B \cap \Omega$, and $S_2 = \Sigma \cap B$. Green's identity for $\Omega - \Omega_1$ gives

$$\int_{\Sigma - S_2} K(p_0,q)\,d\sigma(q) - \int_{S_1} K(p_0,q)\,d\sigma(q) = -\iint_{\Omega - \Omega_1} w^2Q(p_0,q)\,d\Omega(q)$$

and

$$\int_{S_1} K(p,q)\, d\sigma(q) = -2\pi e^{iw\delta} + O(\delta) \quad ,$$

so in the limit as $\delta \to 0$ we get

$$(5) \qquad u(p_o) = -2\pi - H(p_o)$$

Comparing equations (3), (4), and (5) in the limit as $p \to p_o$, we obtain

$$u(p_o)_+ = 2\pi + u(p_o)$$

$$u(p_o)_- = -2\pi + u(p_o) \quad .$$

<u>Part A(2)</u>. For a point p on l , $(p \neq p_o)$

$$(6) \quad \frac{\partial v}{\partial N}(p) = \frac{\partial}{\partial N} \int_\Sigma Q(p,q)d\sigma(q) = \int_\Sigma \frac{\partial}{\partial N} Q(p,q)d\sigma(q) \quad .$$

Set up a tangent - normal coordinate system at p_o with the z axis in the direction of the exterior normal N . Let $z = q(x,y)$ describe an element of surface E of radius δ at p_o . An arbitrary point $q \in E$ is denoted by (x,y,z) , whereas $p \in l$ is given by $(0,0,z)$. The derivatives to be taken are with respect to the p variable. $r = |p-q| = \sqrt{x^2+y^2+(Z-z)^2}$ and $\rho = \sqrt{x^2+y^2}$. Let $A(r) = e^{iwr}(iwr-1)$ and let A be a bound for $|A(r)|$ for q in a neighborhood of p_o .

$$(7) \qquad \frac{\partial Q}{\partial N} = \frac{\partial Q}{\partial r}\frac{\partial r}{\partial Z} = \frac{A(r)}{r^3}(Z-z)$$

Now consider the potential of a double layer with unit density, namely $u(p) = \int_\Sigma K(p,q)d\sigma(q)$. Using $K(p,q)$

from (4) of Lemma 2.2 gives us

$$(8) \qquad u(p) = \int_{\Sigma} \frac{A(r)}{r^3} \left[xN_q^x + yN_q^y + (z-Z)N_q^z \right] d\sigma(q) \quad .$$

Substitute (7) into (6) , then add (6) and (8) . The result is

$$\frac{\partial v}{\partial N}(p) + u(p) = \int_{\Sigma} \frac{A(r)}{r^3} \left[xN_q^x + yN_q^y + (Z-z)(1-N_q^z) \right] d\sigma(q) \quad .$$

Call the left side $F(p)$ and split the right side into an integral over E , call it $F_1(p)$, and one over $\Sigma - E$, call it $F_2(p)$. We want to show that $F(p)$ is a continuous function on 1 at point p_o . $Z-z = O(1)$, and from Lemma 2.3 $N_q^x = O(\rho)$, $N_q^y = O(\rho)$ and $1-N_q^z = O(\rho^2)$, hence

$$xN_q^x + yN_q^y + (Z-z)(1-N_q^z) = O(\rho^2) \quad .$$

The last estimate and local polar coordinates at p_o give, independent of z ,

$$F_1(p) = \int_0^{\delta} O(r^{-3})\, O(\rho^3)\, d\rho = O(\delta)$$

Consider

$$|F(p)-F(p_o)| \leq |F_1(p)| + |F_1(p_o)| + |F_2(p)-F_2(p_o)| \quad .$$

Given an ϵ , a δ can be found so that each of the first two terms on the right is less than $\epsilon/3$, independent of z. For δ now fixed, by the continuity of $F_2(p)$, for sufficiently small z ,the third term is less than $\epsilon/3$.

Therefore,

$$\frac{\partial v}{\partial N}(p) = -u(p) + F(p)$$

where $F(p)$ is continuous on l . Hence,

$$\frac{\partial v}{\partial N}(p_o)_+ = -u(p_o)_+ + F(p_o)$$

$$\frac{\partial v}{\partial N}(p_o)_- = -u(p_o)_- + F(p_o)$$

$$\frac{\partial v}{\partial N}(p_o) = -u(p_o) + F(p_o) \quad .$$

By comparison of the above equations, and use of the results of Part A(1), we obtain

$$\frac{\partial v}{\partial N}(p_o)_+ = -2\pi + \frac{\partial v}{\partial N}(p_o)$$

$$\frac{\partial v}{\partial N}(p_o)_- = +2\pi + \frac{\partial v}{\partial N}(p_o) \quad .$$

Part B(2). Let (r_{pq},N) denote the angle between the vector $p-q$ and the vector N . For p on l $(p \neq p_o)$ we write, by (5) of Lemma 2.2,

$$(9) \quad \frac{\partial v}{\partial N}(p) = \int_{\Sigma} g(q)A(r_{p_oq})\frac{\cos(r_{p_oq},N)}{r^2_{p_oq}}\,d\sigma(q)$$

$$+ g(p_o)\left[\int_{\Sigma} A(r_{pq})\frac{\cos(r_{pq},N)}{r^2_{pq}}\,d\sigma - \int_{\Sigma} A(r_{p_oq})\frac{\cos(r_{p_oq},N)}{r^2_{p_oq}}\,d\sigma\right]$$

$$+ \int_{\Sigma} (g(q)-g(p_o))A(r_{pq})\,\frac{\cos(r_{pq},N)}{r^2_{pq}}\,d\sigma$$

(continued on next page)

$$-\int_{\Sigma}(g(q)-g(p_o))A(r_{p_oq})\,\frac{\cos(r_{p_oq},N)}{r^2_{p_oq}}\,d\sigma\ .$$

The first integral on the right in equation (9) is the direct value $(\partial v/\partial N)(p_o)$. The function in square brackets is, for unit density, the normal derivative of the single layer minus its direct value. By Part A(2), this has value $-2\pi, (+2\pi)$ as $p \to p_o$ from the exterior (interior) of Σ. To prove Part B(2) it remains only to show that the difference of the last two integrals, call it $I(p)-I(p_o)$, tends to zero as $p \to p_o$. Split Σ into $E, \Sigma - E$. For fixed δ, $[I(p)-I(p_o)]_{\Sigma - E}$ is continuous, and tends to 0 as $p \to p_o$, so we need only consider

$$|I(p)-I(p_o)|_E \le A\int_E |g(q)-g(p_o)|\,\frac{|\cos(r_{pq},N)|}{r^2_{pq}}\,d\sigma(q)$$

$$+ A\int_E |g(q)-g(p_o)|\,\frac{|\cos(r_{p_oq},N)|}{r^2_{p_oq}}\,d\sigma(q)\ .$$

Recall $p = (0,0,Z)$, $q = (x,y,z)$, and let $q_o = (x,y,0)$. By considering the triangle p,q,q_o and using Lemma 2.1 we get

$$\left|r_{pq} - \sqrt{\rho^2+Z^2}\right| \le |z| \le b\rho^2$$

where b is a positive constant. Since $\rho \le r_{pq}$,

$$\left|1 - \frac{\sqrt{\rho^2+Z^2}}{r_{pq}}\right| \le b\rho\ .$$

Hence, for $q \in E$

$$\frac{1}{r_{pq}} \leq \frac{1+b\rho}{\sqrt{\rho^2+Z^2}} \leq \frac{1+b\delta}{\sqrt{\rho^2+Z^2}} \quad .$$

Then,

$$|\cos(r_{pq},N)| = \frac{|Z-z|}{r_{pq}} \leq \frac{|Z|}{r_{pq}} + \frac{|z|}{r_{pq}}$$
$$\leq \frac{|Z|\,(1+b\delta)}{\sqrt{\rho^2+Z^2}} + \frac{b\rho^2}{\rho} \quad .$$

From the above inequalities it follows that

$$(10) \quad \frac{|\cos(r_{pq},N)|}{r_{pq}^2} \leq \frac{|Z|\,(1+b\delta)^3}{(\rho^2+Z^2)^{3/2}} + \frac{b}{\rho}\,(1+b\delta)^2 \quad .$$

Also, from Lemma 2.4

$$\frac{|\cos(r_{p_oq},N)|}{r_{p_oq}^2} < \frac{c}{\rho} \quad ,$$

where c is a positive constant. Lemma 2.8 together with (10) and (11) show that $|I(p)-I(p_o)|_E \to 0$ as $\delta \to 0$.

This finishes Part B(2).

Part B(1). The proof is similar to that of Part B(2). Consider p on l $(p \neq p_o)$. In place of (9) we have now

$$u(p) = u(p_o) \pm 2\pi\phi(p_o)$$
$$+ \int_{\Sigma}(\phi(q)-\phi(p_o))A(r_{pq})\,\frac{\cos(r_{pq},N_q)}{r_{pq}^2}\,d\sigma(q)$$
$$- \int_{\Sigma}(\phi(q)-\phi(p_o))A(r_{p_oq})\,\frac{\cos(r_{p_oq},N_q)}{r_{p_oq}^2}\,d\sigma(q) \quad .$$

To finish the proof it suffices to show that the difference of the last two integrals, call it $I(p)-I(p_o)$, tends to zero as $p \to p_o$. We need only consider $|I(p)-I(p_o)|_E$.
Let u_{pq} be a unit vector in the direction p to q .
Then, by Lemma 2.3 we get

$$(12) \qquad |\cos(r_{pq},N_q) - \cos(r_{pq},N)| \leq |u_{pq}|\,|N_q-N| \leq c r_{p_o q} \quad ,$$

where c is a positive constant. Consider

$$(13) \quad |I(p)| = \left| \int_E (\phi(q)-\phi(p_o))A(r_{pq}) \frac{\cos(r_{pq},N_q)}{r_{pq}^2} d\sigma(q) \right|$$

$$\leq A\int_E |\phi(q)-\phi(p_o)| \frac{|\cos(r_{pq},N)|}{r_{pq}^2} d\sigma(q)$$

$$+ A\int_E |\phi(q)-\phi(p_o)| \frac{|\cos(r_{pq},N_q)-\cos(r_{pq},N)|}{r_{pq}^2} d\sigma(q).$$

The first integral was shown to tend to zero as $\delta \to 0$ in Part B(2). By (12) and $r_{p_o q} < 2\rho$, the second integral has the bound

$$2Ac \int_E \frac{|\phi(q)-\phi(p_o)|}{\rho} d\sigma(q) \quad .$$

This goes to zero as $\delta \to 0$ by Lemma 2.8 . (12), (13), and the consequences stated above hold with p replaced by p_o , hence $|I(p)-I(p_o)|_E \to 0$ as $\delta \to 0$. This finishes Part B(2) and completes the proof of Lemma 2.9 .

2.4 Geometric estimates and properties of auxiliary functions.

The next major lemma (Lemma 2.18) requires extended preperations consisting of Lemmas 2.10 to 2.17 .

Notation. Let $r_{pq} = |p-q|$. Denote the angle between the vectors $p-q$ and N_q by (r_{pq},N_q) .

Lemma 2.10 If p and q are on the same face of $\dot{\Sigma}$, then

$$|\cos(r_{pq},N_q)| = O(r_{pq}) \quad .$$

Proof. In a tangent - normal coordinate system at p , by Lemmas 2.1 and 2.3 ,

$$\cos(r_{pq},N_q) = \frac{(x,y,z)}{r}\cdot N_q = O(r_{pq}) \quad .$$

Lemma 2.11 If v denotes an arbitrary direction, and m, m_1,m_2 are arbitrary distinct points in space, then

$$|\cos(r_{mm_1},v)-\cos(r_{mm_2},v)| \le \frac{2r_{m_1m_2}}{r_{mm_1}} \quad .$$

Proof. Let x,y,z be a rectilinear coordinate system with x axis coinciding with the v direction. Denote the x coordinates of m,m_1,m_2 by x,x_1,x_2 , then

$$\cos(r_{mm_1},v)-\cos(r_{mm_2},v) = \frac{x-x_1}{r_{mm_1}} - \frac{x-x_2}{r_{mm_2}}$$

$$= \frac{x_2-x_1}{r_{mm_1}} + \frac{r_{mm_2}-r_{mm_1}}{r_{mm_1}}\cdot\frac{x-x_2}{r_{mm_2}} \quad .$$

Observe that $|x_2-x_1| \leq r_{m_1m_2}$, and by the triangle inequality $|r_{mm_2}-r_{mm_1}| \leq r_{m_1m_2}$, so we then obtain

$$|\cos(r_{mm_1},v)-\cos(r_{mm_2},v)| \leq \frac{2r_{m_1m_2}}{r_{mm_1}} \quad .$$

Lemma 2.12 Let p_1 and p_2 be two points in R^3 . Let p_o be the midpoint of the straight line segment $\overline{p_1p_2}$, and B_{12} a ball with center p_o and radius $r_{12} = |p_1-p_2|$. For any point q in R^3-B_{12} ,

$$\frac{1}{4} < \frac{r_{p_1q}}{r_{p_2q}} < 4 \quad .$$

Proof. By the triangle inequality

$$|p_2-q| \leq |p_1-q| + |p_1-p_2| \quad .$$

Divide by $|p_1-q|$ to get

$$\frac{|p_2-q|}{|p_1-q|} \leq 1 + \frac{|p_1-p_2|}{|p_1-q|} \quad .$$

Since $|p_1-q| \geq \frac{|p_1-p_2|}{2}$ we have $\frac{|p_2-q|}{|p_1-q|} \leq 3$. By reversal of the roles of p_1 and p_2 , also $\frac{|p_1-q|}{|p_2-q|} \leq 3$.

Going to the next integer to obtain strict inequalities, we obtain the assertion of the lemma.

Observe that $|x_2-x_1| \leq r_{m_1m_2}$, and by the triangle inequality $|r_{mm_2}-r_{mm_1}| \leq r_{m_1m_2}$, so we then obtain

$$|\cos(r_{mm_1},v)-\cos(r_{mm_2},v)| \leq \frac{2r_{m_1m_2}}{r_{mm_1}} \quad .$$

<u>Lemma 2.12</u> Let p_1 and p_2 be two points in R^3 . Let p_o be the midpoint of the straight line segment $\overline{p_1p_2}$, and B_{12} a ball with center p_o and radius $r_{12} = |p_1-p_2|$. For any point q in R^3-B_{12} ,

$$\frac{1}{4} < \frac{r_{p_1q}}{r_{p_2q}} < 4 \quad .$$

<u>Proof</u>. By the triangle inequality

$$|p_2-q| \leq |p_1-q| + |p_1-p_2| \quad .$$

Divide by $|p_1-q|$ to get

$$\frac{|p_2-q|}{|p_1-q|} \leq 1 + \frac{|p_1-p_2|}{|p_1-q|} \quad .$$

Since $|p_1-q| \geq \frac{|p_1-p_2|}{2}$ we have $\frac{|p_2-q|}{|p_1-q|} \leq 3$. By reversal of the roles of p_1 and p_2 , also $\frac{|p_1-q|}{|p_2-q|} \leq 3$.

Going to the next integer to obtain strict inequalities, we obtain the assertion of the lemma.

<u>Lemma 2.13</u> For p_1, p_2 on the same face F of $\dot{\Sigma}$ and $q \in \Sigma - B_{12}$,

1) $$\left|\cos(r_{p_1q}, N_q) - \cos(r_{p_2q}, N_q)\right| = O(r_{12})$$

2) $$\left|\cos(r_{p_1q}, N_{p_1}) - \cos(r_{p_2q}, N_{p_2})\right| = O(r_{12}) \quad ,$$

independent of the location of p_1, p_2 in F.

<u>Proof</u>. In a tangent - normal coordinate system at p_1, $p_1 = (0,0,0)$. Let $q = (x,y,z)$ and $p_2 = (x_2, y_2, z_2)$. For brevity, let $r_1 = r_{p_1q}$ and $r_2 = r_{p_2q}$. Let e_x, e_y, e_z be unit vectors in the x,y,z directions.

(1) $$\cos(r_1, N_q) - \cos(r_2, N_q) = \left[\cos(r_1, e_x) - \cos(r_2, e_x)\right] \cos(N_q, e_x)$$
$$+ \text{ analogous terms in } y \text{ and } z \quad .$$

Now, $\cos(N_q, e_x) = N_q^x = O(r_{p_1q})$ from Lemma 2.3, together with Lemma 2.12, gives

(2) $$\left|\cos(r_{p_1q}, e_x) - \cos(r_{p_2q}, e_x)\right| \left|\cos(N_q, e_x)\right| \le \frac{2r_{12}}{r_{p_1q}} O(r_{p_1q})$$
$$= O(r_{12}) \quad .$$

The same estimate holds for y in place of x in (2).

Now consider

(3) $$\left|\cos(r_{p_1q}, e_z) - \cos(r_{p_2q}, e_z)\right| \left|\cos(N_q, e_z)\right| \le \left|\frac{z}{r_{p_1q}} - \frac{z - z_2}{r_{p_2q}}\right| \cdot 1$$
$$\le |z| \left|\frac{1}{r_{p_1q}} - \frac{1}{r_{p_2q}}\right| + \frac{|z_2|}{r_{p_2q}} \quad .$$

Observe that

$$(4) \quad \left|\frac{1}{r_{p_1q}} - \frac{1}{r_{p_2q}}\right| = \left|\frac{r_{p_2q} - r_{p_1q}}{r_{p_1q}\, r_{p_2q}}\right| \leq \frac{r_{12}}{r_{p_1q} r_{p_2q}}$$

follows from the triangle inequality. Also, $z = O(r^2_{p_1q})$ and $z_2 = O(r^2_{12})$ from Lemma 2.3 . Using these, we obtain the following estimate for the right side of (3):

$$O(r^2_{p_1q}) \frac{r_{12}}{r_{p_1q}\, r_{p_2q}} + \frac{O(r^2_{12})}{r_{p_2q}} \quad .$$

Using Lemma 2.12 in the first term, and $r_{12} \leq 2r_{p_2q}$ for $q \in \Sigma - B_{12}$ in the second term, we see that they both have order r_{12} . Therefore, each of the three terms on the right side of (1) have order r_{12} .

Now to prove 2), start with

$$\cos(r_{p_1q}, N_{p_1}) - \cos(r_{p_2q}, N_{p_2}) =$$

$$\left[\cos(r_{p_1q}, N_{p_1}) - \cos(r_{p_2q}, N_{p_1})\right] + \left[\cos(r_{p_2q}, N_{p_1}) - \cos(r_{p_2q}, N_{p_2})\right].$$

Examine the first term. Because the coordinate system was set up at p_1 ,

$$\cos(r_{p_1q}, N_{p_1}) - \cos(r_{p_2q}, N_{p_1}) = \cos(r_{p_1q}, e_z) - \cos(r_{p_2q}, e_z)$$
$$= O(r_{12}) \quad .$$

The order of magnitude was shown in 1) starting at (3).

For the second term, let u be a unit vector in the direction of p_2q. Then, by Lemma 2.3

$$\begin{aligned} |\cos(r_{p_2q}, N_{p_1}) - \cos(r_{p_2q}, N_{p_2})| &= |u \cdot N_{p_1} - u \cdot N_{p_2}| \\ &\le |u|\,|N_{p_1} - N_{p_2}| \\ &\le O(r_{12}) \end{aligned}$$

This completes the proof of Lemma 2.13.

The next lemma is the starting point of a development that leads to existence of tangential derivatives of the double layer. The lemma itself follows directly from equation (5) of Lemma 2.9.

<u>Lemma 2.14</u> For p in $\dot{\Sigma}$,

$$\int_{\Sigma} K(p,q)\, d\sigma(q) = -2\pi - w^2 \int_{\Omega} Q(p,q)\, d\Omega(q)$$

where Ω is the region of space bounded by Σ.

<u>Lemma 2.15</u> Define the volume integral $G(p)$ by:

$$G(p) = \int_{\Omega} Q(p,q)\, d\Omega(q) \quad .$$

$G(p)$ is a differentiable function of p for p in $\dot{\Sigma}$.

<u>Proof</u>. Let p_1 and p_2 be points in a surface element $E \subset \dot{\Sigma}$. Let p_o be the midpoint of the straight line segment $\overline{p_1p_2}$, and let B be a ball with center at p_o and radius $\epsilon > 2|p_1 - p_2|$. Consider,

$$G(p_2) - G(p_1) = \int_{\Omega} \frac{e^{iwr_{p_2q}}}{r_{p_2q}} - \frac{e^{iwr_{p_1q}}}{r_{p_1q}} \, d\Omega(q) \quad .$$

Split this integral into one over $B \cap \Omega$, call it I_1, and one over $\Omega - B$, call it I_2. Consider I_1 written thus

$$(1) \quad I_1 = \int_{B\cap\Omega} \frac{e^{iwr_{p_2q}} - e^{iwr_{p_1q}}}{r_{p_2q}} + e^{iwr_{p_1q}}\left[\frac{1}{r_{p_2q}} - \frac{1}{r_{p_1q}}\right] d\Omega(q).$$

Now consider

$$(2) \quad \left| e^{iw|p_2-q|} - e^{iw|p_1-q|} \right| \leq \int_{p_1}^{p_2} \left|\nabla_p \, e^{iw|p-q|}\right| ds(p)$$

where the integration is performed over the shortest path on Σ from p_1 to p_2. $|\nabla_p \, e^{iwr}| = O(1)$, hence

$$\left| e^{iw|p_2-q|} - e^{iw|p_1-q|} \right| = O(|p_1 - p_2|) = O(r_{12}) \quad .$$

Using this estimate, and (4) of Lemma 2.13 in (1) yields

$$|I_1| \leq \int_{B\cap\Omega} \frac{O(r_{12})}{r_{p_2q}} + \frac{r_{12}}{r_{p_1q}\, r_{p_2q}} \, d\Omega(q) \quad .$$

Consider ball B_{12} of radius r_{12} about the midpoint of line segment $\overline{p_1p_2}$. Split the integral for I_1 into one over $B_{12} \cap \Omega$, call it I_1', and one over $(B - B_{12}) \cap \Omega$, call it I_1''. Let π be the plane perpendicular to $\overline{p_1p_2}$ and passing through its midpoint. π splits $B_{12} \cap \Omega$ into a set H_1 containing p_1 and a set H_2 containing p_2.

For $q \in H_2$, $r_{p_1q} \geq \frac{1}{2} r_{12}$ so $\frac{1}{r_{p_1q}} \leq \frac{2}{r_{12}}$. Then

$$\int_{H_2} \frac{O(r_{12})}{r_{p_2q}} + \frac{r_{12}}{r_{p_1q}\, r_{p_2q}}\, d\Omega(q)$$

$$\leq \int_{H_2} \frac{O(r_{12})}{r_{p_2q}} + \frac{2}{r_{p_2q}}\, d\Omega(q)$$

$$\leq c_1 \int_{H_2} r_{p_2q}^{-1}\, d\Omega(q) \leq c_2 \int_0^{2r_{12}} r\, dr = O(r_{12}^2)$$

where we used srherical coordinates about p_2 with $r=r_{p_2q}$. c_1, c_2, c_3, c_4 denote positive constants. A result similar to the above holds for the integral over H_1 , and since $r_{12} < \epsilon$, we can write $|I_1'| \leq r_{12}\, O(\epsilon)$. Now examine I_1'' . By Lemma 2.12 we can write

$$|I_1''| \leq \int_{(B-B_{12})\cap\Omega} \frac{O(r_{12})}{r_{p_2q}} + \frac{r_{12}}{r_{p_2q}^2}\, d\Omega(q)$$

$$\leq c_3 r_{12} \int_{(B-B_{12})\cap\Omega} r_{p_2q}^{-2}\, d\Omega(q) \quad .$$

Again use spherical coordinates about p_2 to obtain

$$|I_1''| \leq c_4 r_{12} \int_{\frac{1}{2}r_{12}}^{\epsilon} dr = c_4 r_{12} \left(\epsilon - \tfrac{1}{2} r_{12} \right)$$

$$\leq r_{12}\, O(\epsilon) \quad .$$

Hence, $|I_1| = r_{12}\, O(\epsilon)$.

Now consider the difference quotient

$$(3) \qquad \frac{G(p_2) - G(p_1)}{|p_2 - p_1|} = \frac{I_1}{|p_2-p_1|} + \frac{I_2}{|p_2-p_1|} \quad .$$

Call the first term on the right $T_1(\epsilon)$ and the second term $T_2(\epsilon)$. Let p_2 approach p_1 along the tangential direction specified by the unit vector t. By the dominated convergence theorem, for fixed ϵ

$$\lim_{p_2 \to p_1} T_2(\epsilon) = \int_{\Omega - B} \nabla_p Q(p_1,q) \cdot t \, d\Omega(q) \ ,$$

and furthermore the limit of the above integral as $\epsilon \to 0$ exists because $\nabla_p Q(p,q) = O(r^{-2})$. The estimate for $|I_1|$ shows that $T_1(\epsilon)$ has order ϵ independent of $|p_2-p_1|$. This implies the differentiability of $G(p)$ at p_1 and completes the proof of the lemma.

Lemma 2.16 The derivative $G'(p)$ of $G(p) = \int_\Omega Q(p,q)d\Omega(q)$ is a Hölder continuous function of p for $p \in \Sigma$. Specifically, $|G'(p_1)-G'(p_2)| = O(|p_1-p_2|^{1/4})$ for p_1, p_2 in a neighborhood of p.

Proof. It is sufficient to consider $\frac{\partial Q}{\partial t}$ where t is a local parameter of the surface at point $p = (x(t),y(t),z(t))$.

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial r} \nabla_p r \cdot \frac{\partial r}{\partial t}$$

$$= \frac{1}{r^2}\left[e^{iwr}(iwr-1)\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right)\cdot\left(\frac{\partial x}{\partial t},\frac{\partial y}{\partial t},\frac{\partial z}{\partial t}\right)\right] \quad .$$

Denote the function in the square brackets above by $f(p,q)$ so that $\frac{\partial Q}{\partial t} = \frac{1}{r^2} f(p,q)$. Also denote $\int_\Omega \frac{\partial Q}{\partial t}\, d\Omega(q)$ by $G'(p)$.

$$(1)\ G'(p_1)-G'(p_2) = \int_\Omega \frac{f(p_1,q)}{r^2_{p_1q}} - \frac{f(p_2,q)}{r^2_{p_2q}}\, d\Omega(q)$$

$$= \int_\Omega \frac{f(p_1,q)-f(p_2,q)}{r^2_{p_1q}} + f(p_2,q)\left[\frac{1}{r^2_{p_1q}} - \frac{1}{r^2_{p_2q}}\right] d\Omega(q).$$

Let p_o be the midpoint of line segment $\overline{p_1p_2}$ and let B be the ball with center p_o and radius $\lambda = r^u_{12}$ where $0 < u \leq 1$ and $r_{12} < 1$. Split the integral in (1) into one over $B \cap \Omega$, call it I_1 , and one over $\Omega - B$, call it I_2 .

$$|I_1| \leq \int_{B\cap\Omega} \frac{|f(p_1,q)| + |f(p_2,q)|}{r^2_{p_1q}} + |f(p_2,q)|\left[\frac{1}{r^2_{p_1q}} + \frac{1}{r^2_{p_2q}}\right] d\Omega(q).$$

Integrating each term serarately, and using spherical coordinates about p_1 or p_2 as appropriate, we obtain with the help of $|f(p,q)| < c$ where c is a positive constant, $|I_1| = O(\lambda)$.

Now consider I_2 . We will use the following inequality for $q \in \Omega - B$

(2) $$\left|\frac{1}{r^2_{p_1q}} - \frac{1}{r^2_{p_2q}}\right| = \left|\frac{1}{r_{p_1q}} - \frac{1}{r_{p_2q}}\right|\left|\frac{1}{r_{p_1q}} + \frac{1}{r_{p_2q}}\right|$$

$$\leq \frac{r_{12}}{r_{p_1q}\, r_{p_2q}} \frac{r_{p_1q} + r_{p_2q}}{r_{p_1q}\, r_{p_2q}}$$

$$\doteq \frac{O(r_{12})}{r^3_{p_1q}} \quad .$$

This follows from Lemma 2.12 and (4) of lemma 2.13. By direct calculation it can be seen that $\nabla_p\, f(p,q) = O(r^{-1})$. Now notice that for p on the arc (shortest path) p_1p_2, and $q \in R^3-B$, $r_{pq} \geq \lambda - \frac{r_{12}}{2}$ and $r_{12} \leq \lambda$, so

(3) $$r_{pq} \geq \lambda/2 \quad .$$

Therefore, $\nabla_p\, f(p,q) = O(\lambda^{-1})$. Then

(4) $$|f(p_1,q)-f(p_2,q)| \leq \int_{p_2}^{p_1} |\nabla_p\, f(p,q)|\, ds(p) = |p_1-p_2|\, O(\lambda^{-1}).$$

Consider

$$|I_2| \leq \int_{\Omega-B} \frac{|f(p_1,q)-f(p_2,q)|}{r^2_{p_1q}} + |f(p_2,q)| \left|\frac{1}{r^2_{p_1q}} - \frac{1}{r^2_{p_2q}}\right| d\Omega(q) \quad .$$

Estimate (3) holds for $p = p_1$, so $r^{-2}_{p_1q} = O(\lambda^{-2})$. This, (2), (4), and $|f(p_2,q)| < c$, yield

$$|I_2| \leq \int_{\Omega-B} r_{12}\, O(\lambda^{-3})\, d\Omega(q)$$

$$\leq r_{12}\, O(\lambda^{-3}) \text{ volume of } \Omega \quad .$$

Therefore,

$$|G(p_1) - G(p_2)| \le |I_1| + |I_2|$$
$$= O(\lambda) + r_{12}\, O(\lambda^{-3})$$
$$= O(r_{12}^{u}) + O(r_{12}^{1-3u}) \quad .$$

Now set $u = 1-3u$, from which we choose $u = 1/4$, then conclude that

$$|G(p_1) - G(p_2)| = O(r_{12}^{1/4}) \quad .$$

This completes the proof of Lemma 2.16.

The following is a technical lemma that we will repeatedly make reference to. It allows us to make more extended use of estimates of differences.

Lemma 2.17 Let F , h be functions defined on $\dot{\Sigma}$. Suppose that $|F(p)|$ is bounded by a positive constant c_1 in a neighborhood of p_1 that includes p_2 . Suppose that $h(p)$ satisfies a Lipschitz condition, $|h(p_1) - h(p_2)| \le c|p_1-p_2|$, and $|h(p)| < H$ where c and H are positive constants. Then,

$$|h(p_1)F(p_1) - h(p_2)F(p_2)| \le H|F(p_1)-F(p_2)| + O(|p_1-p_2|) \ .$$

Proof. By use of the general identity

(1) $$A_1B_1 - A_2B_2 = (A_1-A_2)B_1 + A_2(B_1-B_2) \quad ,$$

we obtain the inequality

$$|h(p_1)F(p_1) - h(p_2)F(p_2)|$$
$$\leq |h(p_1)|\,|F(p_1)-F(p_2)| + |h(p_1)-h(p_2)|\,|F(p_2)|$$
$$\leq H\ |F(p_1)-F(p_2)| + c|p_1 - p_2|\, c_1 \quad .$$

2.5 Hölder continuity and differentiability of $u(p)$ on the surface.

Definition 2.3: A function ϕ in $L^1(\Sigma)$ is locally Hölder continuous at $p_o \in \Sigma$ if there is a surface element E at p_o and positive constants c, u $(0 < u \leq 1)$ dependent on E, p_o such that for any p_1, p_2 in E

$$|\phi(p_1) - \phi(p_2)| \leq c\,|p_1-p_2|^u \quad .$$

Lemma 2.18 If the density function ϕ is in $L^1(\Sigma)$ and satisfies a local Hölder condition at $p_o \in \Sigma$, namely $|\phi(p_1) - \phi(p_2)| = O(r_{12}^u)$ where $0 < u \leq 1$, then the double layer potential $u(p) = \int_\Sigma \phi(q)K(p,q)\, d\sigma(q)$ is locally Hölder continuous at p_o with exponent 1.

Proof. From Lemmas 2.15 and 2.16, $G(p)$ is Hölder continuous with exponent 1. Define $T(p_1,p_2)$ as follows:

$$(1)\quad T(p_1,p_2) = \int_\Sigma \left[\phi(q)-\phi(p_1)\right]\left[K(p_1,q)-K(p_2,q)\right] d\sigma(q)$$
$$= \int_\Sigma \phi(q)K(p_1,q)d\sigma(q) - \int_\Sigma \phi(q)K(p_2,q)d\sigma(q)$$

(continued on next page)

$$- \phi(p_1)\left[\int_{\Sigma} K(p_1,q)d\sigma(q) \quad - \quad \int_{\Sigma} K(p_2,q)d\sigma(q)\right] \quad .$$

From (1), by Lemma 2.14 we have

(2) $\quad T(p_1,p_2) = u(p_1) - u(p_2) + w^2\phi(p_1)\left[G(p_1)-G(p_2)\right] \quad ,$

which implies

(3) $|u(p_1)-u(p_2)| \leq |T(p_1,p_2)| + w^2|\phi(p_1)||G(p_1)-G(p_2)| \quad .$

The conclusion will follow if we can show $T(p_1,p_2) = O(r_{12})$.

Let $A(r) = e^{iwr}(iwr-1)$, $A_1 = A(r_{p_1q})$ and $A_2 = A(r_{p_2q})$. Let A be a bound for $|A(r)|$. By (3) of Lemma 2.2,

(4) $\quad T(p_1,p_2) = \displaystyle\int_{\Sigma}\left[\phi(q)-\phi(p_1)\right]\left[A_1\frac{\cos(r_{p_1q},N_q)}{r^2_{p_1q}} - A_2\frac{\cos(r_{p_2q},N_q)}{r^2_{p_2q}}\right]d\sigma(q).$

Let $E_1 = E(\delta,p_o)$ denote an element of surface of radius δ at p_o in which ϕ is locally Hölder continuous. Let p_1, p_2 be any two points in $E_2 = E(\delta/2,p_o)$. Let B_{12} be a ball of radius r_{12} centered at the midpoint of line segment $\overline{p_1p_2}$. Split Σ into, $S_1 = \Sigma \cap B_{12}$, $S_2 = \Sigma \cap (E_1-B_{12})$, and $S_3 = \Sigma - E_1$. Denote the corresponding integrals I_1, I_2, I_3 . $I_3(p)$ is $C^{\infty}(E_2)$, hence Hölder continuous with exponent 1 in E_2 . Next, consider

$$|I_1| \leq A\int_{S_1} |\phi(q)-\phi(p_1)| \left[\frac{|\cos(r_{p_1q},N_q)|}{r^2_{p_1q}} + \frac{|\cos(r_{p_2q},N_q)|}{r^2_{p_2q}}\right] d\sigma(q)$$

$$\leq \int_{S_1} O(r^u_{p_1q}) \left[\frac{O(r_{p_1q})}{r^2_{p_1q}} + \frac{O(r_{p_2q})}{r^2_{p_2q}}\right] d\sigma(q) \quad .$$

This follows from the hypothesis and Lemma 2.10. Because, $r_{p_1q} < 2r_{12}$ for $q \in B_{12}$, we can write

$$|I_1| \leq r^u_{12}\int_{S_1} O(r^{-1}_{p_1q}) + O(r^{-1}_{p_2q}) \, d\sigma(q) \quad .$$

Now, for instance, in local polar coordinates about p_1

$$\int_{S_1} O(r^{-1}_{p_1q}) \, d\sigma(q) \leq \int_0^{2r_{12}} \frac{c}{\rho} \rho \, d\rho = O(r_{12}) \quad ,$$

where c is a positive constant. Therefore,

$$(5) \qquad |I_1| = r^u_{12}\, O(r_{12}) = O(r^{1+u}_{12}) \quad .$$

To treat I_2, first consider the inequality

$$(6) \qquad \left|\frac{A_1\cos(r_{p_1q},N_q)}{r^2_{p_1q}} - \frac{A_2\cos(r_{p_2q},N_q)}{r^2_{p_2q}}\right|$$

$$\leq \frac{|A_1\cos(r_{p_1q},N_q) - A_2\cos(r_{p_2q},N_q)|}{r^2_{p_1q}}$$

$$+ |A_2\cos(r_{p_2q},N_q)| \left|\frac{1}{r^2_{p_1q}} - \frac{1}{r^2_{p_2q}}\right|$$

$$\leq \frac{O(r_{12})}{r^2_{p_1q}} + O(r_{p_2q}) \frac{O(r_{12})}{r^3_{p_1q}} \quad .$$

The last line follows from Lemmas 2.13 and 2.17, Lemma 2.10, and (2) of Lemma 2.16. Further, by Lemma 2.12, $O(r_{p_1q}) = O(r_{p_2q})$, so we obtain from (6), for $q \in \Sigma - B_{12}$

$$\left| \frac{A_1 \cos(r_{p_1q}, N_q)}{r^2_{p_1q}} - \frac{A_2 \cos(r_{p_2q}, N_q)}{r^2_{p_2q}} \right| = \frac{O(r_{12})}{r^2_{p_1q}} \quad .$$

Therefore,

$$|I_2| \leq \int_{S_2} |\phi(q) - \phi(p_1)| \, \frac{O(r_{12})}{r^2_{p_1q}} \, d\sigma(q)$$

$$\leq \int_{S_2} O(r^u_{p_1q}) \, \frac{O(r_{12})}{r^2_{p_1q}} \, d\sigma(q)$$

$$= r_{12} \int_{S_2} O(r^{-2+u}_{p_1q}) \, d\sigma(q) \quad .$$

Now use local polar coordinates about p_1 and let c_1, c_2, c_3 denote positive constants.

$$|I_2| \leq r_{12} \int_{(1/2)r_{12}}^{2\delta} c_1 \rho^{-2+u} \rho \, d\rho \quad .$$

After integrating, we obtain

$$|I_2| \leq r_{12} \, c_2 \left[(2\delta)^u - (\tfrac{1}{2} r_{12})^u \right]$$

$$\leq c_2 \, (2\delta)^u \, r_{12} = O(r_{12}) \quad .$$

In conclusion, because I_1, I_2, I_3 are each of order r_{12}, so also is $T(p_1,p_2)$, hence by (3), $|u(p_1)-u(p_2)| = O(r_{12})$ for p_1, p_2 in $E(\delta/2,p_o)$. This completes the proof of the lemma.

Lemma 2.19 Consider a density function $\phi \in L^1(\Sigma)$. Let $\|\phi\| = \int_\Sigma |\phi(q)| d\sigma(q)$. Suppose either of the following:

1) ϕ is locally essentially bounded on $\dot{\Sigma}$ (Def. 2.1). p_o is a point in $\dot{\Sigma}$ and F_1 is the face containing p_o . Let B_2 be a ball of radius r_o about p_o , in which almost everywhere $|\phi(q)| \leq b$. b is a positive constant. Furthermore, assume that r_o is small enough that $\Sigma \cap B_2$ contains only points of F_1 .

2) Σ is smooth (C^2) and $\phi \in W(\Sigma)$ (cf. Chapter 3). Let $\|\phi\|_w^2 = \sum_{m=1}^{\infty} 2^{-m}\|\phi\|_m^2$ (cf. paragraph preceeding Lemma 2.5).

Then the double layer potential is locally Hölder continuous (Def 2.3) as stated below. Let ϵ , c_j, $j = 1$ to 5, denote positive constants independent of ϕ and m . ϵ is arbitrarily small.

From hypothesis 1).

$$|u(p_1)-u(p_2)| \leq \left[c_1 b + c_2\frac{\|\phi\|}{r_o^2}\right]|p_1-p_2|^{1-\epsilon} .$$

This holds for p_1, p_2 in a surface element E of radius $r_o/2$ about p_o .

<u>From hypothesis 2)</u>.

$$|u(p_1)-u(p_2)| \leq (c_3 \|\phi\|_{m+1} + c_4 \|\phi\|_W + c_5 \|\phi\|) |p_1-p_2|^{1/4} \quad .$$

This holds for p_1, p_2 in S_m and $|p_1-p_2| \leq 1/2^{m+1}$.

<u>Proof</u>. <u>Part 1)</u>. Let $A(r) = e^{iwr}(iwr-1)$, $A_1 = A(r_{p_1q})$, $A_2 = A(r_{p_2q})$ and let A be a bound for $|A(r)|$. Also let a_1, a_2 be positive constants. Let p_1, p_2 be any two points in E and let B_{12} be a ball with center at the midpoint p_{oo} of line segment $\overline{p_1p_2}$ and radius r_{12} .

$$(1) \quad u(p_1)-u(p_2) = \int_{\Sigma} \phi(q) \left[\frac{A_1 \cos(r_{p_1q}, N_q)}{r^2_{p_1q}} - \frac{A_2 \cos(r_{p_2q}, N_q)}{r^2_{p_2q}} \right] d\sigma(q)$$

Let $S_1 = \Sigma \cap B_{12}$, $S_2 = \Sigma \cap (B_2 - B_{12})$ and $S_3 = \Sigma - B_2$. Split the integral over Σ into integrals over the above regions and denote their absolute values by I_1, I_2, I_3 .

By Lemma 2.10,

$$I_1 \leq A \int_{S_1} |\phi(q)| \frac{|\cos(r_{p_1q}, N_q)|}{r^2_{p_1q}} d\sigma(q) + A \int_{S_1} |\phi(q)| \frac{|\cos(r_{p_2q}, N_q)|}{r^2_{p_2q}} d\sigma(q)$$

$$\leq A \int_{S_1} |\phi(q)| \frac{O(r_{p_1q})}{r^2_{p_1q}} d\sigma(q) + A \int_{S_1} |\phi(q)| \frac{O(r_{p_2q})}{r^2_{p_2q}} d\sigma(q) \quad .$$

Therefore,

$$I_1 \leq a_1 b \, r_{12}$$

Now consider the following bound for I_2:

$$(2) \quad I_2 \leq \int_{S_2} |\phi(q)| \, \frac{|A_1 \cos(r_{p_1 q}, N_q) - A_2 \cos(r_{p_2 q}, N_q)|}{r_{p_1 q}^2} \, d\sigma(q)$$

$$+ \int_{S_2} |\phi(q)| \, |\cos(r_{p_2 q}, N_q)| \, \left| \frac{A_1}{r_{p_1 q}^2} - \frac{A_2}{r_{p_2 q}^2} \right| d\sigma(q) \quad .$$

Use Lemmas 2.13 and 2.17 in the first integral and Lemma 2.10, (2) of Lemma 2.16, and Lemma 2.17 in the second integral. Then use Lemma 2.12 to combine into one integral as follows:

$$I_2 \leq O(r_{12}) \int_{S_2} |\phi(q)| \, \frac{1}{r_{p_1 q}^2} \, d\sigma(q) \quad .$$

Because $\frac{1}{2} r_{12} < r_{p_1 q}$, for q outside B_{12} we can write

$$I_2 \leq O(r_{12}^{1-\epsilon}) \, b \int_{S_2} r^{-2+\epsilon} \, d\sigma(q) \quad ,$$

and in local polar coordinates

$$I_2 \leq O(r_{12}^{1-\epsilon}) \, b \int_0^{2\pi} \int_{(1/2) r_{12}}^{2 r_0} \rho^{-1+\epsilon} \, d\rho \, d\theta$$

$$\leq a_2 \, r_{12}^{1-\epsilon} \, b \quad .$$

For I_3 we follow the same initial steps as for I_2 , then

$$I_3 \leq O(r_{12}) \int_{S_3} |\phi(q)| \frac{1}{r^2_{p_1 q}} d\sigma(q)$$

$$\leq O(r_{12}) \frac{1}{(r_o/2)^2} \|\phi\| \quad .$$

Therefore,

$$|u(p_1)-u(p_2)| \leq \left[c_1 b + c_2 \frac{\|\phi\|}{r_o^2} \right] |p_1-p_2|^{1-\epsilon} \quad .$$

This finishes Part 1).

Part 2). At first we use the notation and results of Part 1). Let p_1 and p_2 be in S_m with $r_{12} = |p_1-p_2| \leq 1/2^{m+1}$. Then $b = \|\phi\|_{m+1}$, and from 1) $I_1 \leq a_1 \|\phi\|_{m+1} r_{12}$. Let B_2 be a ball of radius $r_o = 1/2^{m+1}$ about p_{oo} . We obtain from 1), $I_2 \leq a_2 \|\phi\|_{m+1} r_{12}^{1-\epsilon}$.

Now, let B_4 be a ball of radius r_4 (small but independent of m) . Let $S_4 = B_4 - B_2$ and $S_7 = \Sigma - B_4$. Denote the corresponding integrals I_4, I_7 . Σ is assumed smooth (C^2) but still with faces (sections), and $\phi \in W(\Sigma)$. The initial estimate for I_2 holds also for I_4 .

$$I_4 \leq O(r_{12}) \int_{S_4} |\phi(q)| \frac{1}{r^2_{pq}} d\sigma(q) \quad .$$

Split S_4 into $S_4 \cap F_1$ and $S_4 \cap OF$, where F_1 is the face containing p_1 and p_2 , and OF stands for the other faces (sections). Call the corresponding integrals

I_5 and I_6. Examine I_5. We will integrate over strips $R_{m+k} = (S_4 \cap F_1) \cap (S_{m+k} - S_{m+k-1})$, then sum from $k=1$ to ∞. To do this, start by letting p_3 denote the edge point closest to p. Let p_3' be its orthogonal projection onto the tangent plane at p. Set up a tangent - normal coordinate system at p with positive x axis in the direction p to p_3'. See Figure 3.2. Let $\rho = \sqrt{x^2+y^2}$. By Lemma 2.1, $r_{pq} = \rho + O(\rho^2) = O(\rho)$. Let $I_{R_{m+k}}$ denote the following integral.

$$\int_{R_{m+k}} \frac{|\phi(q)|}{r_{pq}^2} \, d\sigma(q) \leq \|\phi\|_{m+k} \int_{x_1}^{x_2} 2 \int_0^{r_o} \frac{c}{x^2+y^2} \, dy dx$$

Here, c is a positive constant and x_1, x_2 are, respectively, the smallest and largest x coordinates in the strip R_{m+k}. The y integration corresponds to integrating along the length of R_{m+k} and the x integration corresponds to integrating across its width.

$$2 \int_0^{r_o} \frac{c}{x^2+y^2} \, dy = \frac{2c}{x} \tan^{-1} \frac{y}{x}\Big]_{y=0}^{r_o} \leq \frac{c\pi}{x}$$

and

$$\int_{x_1}^{x_2} \frac{\pi}{x} \, dx \leq \frac{\pi}{x_1} (x_2 - x_1) \leq \frac{\pi}{2^{k-1}}$$

because $x_1 \geq 1/2^{m+1}$ and $x_2 - x_1 = 1/2^{m+k}$.

Therefore, $I_{R_{m+k}} \leq \|\phi\|_{m+k} \frac{c\pi}{2^{k-1}}$. Because $r_{12} < 1/2^{m+1}$,

$r_{12}^{3/4}\, 2^{-k+1} \le 2^{-(3/4)(m+1)-k+1} \le 2^{-(3/4)(m+k)}$, so we get

$$I_5 = O(r_{12}) \sum_{k=2}^{\infty} I_{R_{m+k}}$$

$$\le O(r_{12}^{1/4}) \sum_{k=2}^{\infty} 2^{-(3/4)(m+k)} \,|\!|\!|\phi|\!|\!|_{m+k}$$

$$\le O(r_{12}^{1/4}) \,|\!|\!|\phi|\!|\!|_W \quad .$$

The last line follows from setting $j=m+k$, then using the Cauchy inequality, $\sum a_j b_j \le \sqrt{\sum a_j^2 \sum b_j^2}$, with $a_j = 2^{-j/4}$ and $b_j = 2^{-j/2}\, |\!|\!|\phi|\!|\!|_j$.

Because the distance from p_o to the strips R_{m+k} in other faces is greater than that in F_1 , an estimate similar to that for I_5 holds for I_6, hence

$$I_4 = I_5 + I_6 \le c_4 \,|\!|\!|\phi|\!|\!|_W\, O(r_{12}^{1/4}) \quad .$$

Lastly, as for I_3 in 1),

$$I_7 \le O(r_{12}) \int_{\Sigma - B_4} |\phi(q)| \, \frac{1}{r_{p_1 q}^2}\, d\sigma(q)$$

$$\le O(r_{12}) \frac{1}{r_4^2} \,\|\phi\| \quad .$$

Using the estimates for I_1, I_2, I_4, and I_7, we obtain

$$|u(p_1)-u(p_2)| \le (c_3 |\!|\!|\phi|\!|\!|_{m+1} + c_4 |\!|\!|\phi|\!|\!|_W + c_5 \|\phi\|)\ |p_1 - p_2|^{1/4} \ .$$

This completes the proof of Lemma 2.19.

Corollary 2.19 For $g \in$ L.B., the single layer potential, $v(p) = Qg(p) = \int_{\Sigma} g(q)Q(p,q)\, d\sigma(q)$ where $Q(p,q) = e^{iwr}/r$, is locally Holder continuous at points of $\dot{\Sigma}$.

Proof. The proof of Lemma 2.19 can be traced through with minor modifications for the simpler kernal Q .

Lemma 2.20 If the density function ϕ is in $L^1(\Sigma)$ and satisfies a local Holder condition at p_o (Def.2.3), $|\phi(p_1)-\phi(p_2)| = O(r_{12}^u)$ where $0 < u \leq 1$, then the double layer potential $u(p)$ is differentiable on $\dot{\Sigma}$ at p_o .

Proof. Take p_1 equal to p_o . From (2) of Lemma 2.18 we write the difference quotient

$$(1) \qquad \frac{u(p_1)-u(p_2)}{|p_1-p_2|} = \frac{T(p_1,p_2)}{|p_1-p_2|} - w^2\phi(p_1)\,\frac{G(p_1)-G(p_2)}{|p_1-p_2|}$$

Let the unit vector t specify a tangential direction on $\dot{\Sigma}$ at p_1 . By Lemma 2.15 $G(p)$ is differentiable on $\dot{\Sigma}$ so the second term on the right has a limit as $p_2 \to p_1$ along t . Therefore, to show the differentiability of $u(p)$ we need only consider $T(p_1,p_2)/|p_1-p_2|$. In Lemma 2.18 we split $T(p_1,p_2)$ into three parts I_1, I_2, I_3 . First consider $I_1 + I_2$. In Lemma 2.18 ball B had a fixed radius r_o . Now regard the radius as no longer fixed. Denote it by ϵ . For $r_{12} < \epsilon$, (5) and (7) of Lemma 2.18 can be written $|I_1| = \epsilon^u O(r_{12})$ and $|I_2| \leq c_2 \epsilon^u r_{12}$. Now consider,

$$(2) \qquad \frac{T(p_1,p_2)}{|p_1-p_2|} = \frac{I_1 + I_2}{|p_1-p_2|} + \frac{I_3}{|p_1-p_2|} \quad .$$

By the above estimates for $|I_1|$ and $|I_2|$ we have

$$(3) \qquad \left|\frac{I_1 + I_2}{p_1-p_2}\right| = O(\epsilon^u)$$

independent of $|p_1-p_2|$. As $p_2 \to p_1$ along t , for any fixed ϵ ,

$$\lim_{p_2 \to p_1} \frac{I_3(\epsilon)}{|p_1-p_2|} = \int_{\Sigma - E} (\phi(q)-\phi(p_1))\, \nabla_p K(p_1,q)\cdot t \, d\sigma(q)$$

by the dominated convergence theorem. This integral exists and possesses a limit as $\epsilon \to 0$ because $\nabla_p K(p,q)\cdot t$ has $O(r^{-2+u})$. This follows by an explicit formula for $\nabla_p K(p,q)\cdot t$, (2) of Lemma 2.23, together with estimates (5) of Lemma 2.23. In fact, by (2) and (3), in the limit as $\epsilon \to 0$,

$$(4) \qquad T'(p_1) = \int_{\Sigma} (\phi(q)-\phi(p_1))\, \nabla_p K(p_1,q)\cdot t \, d\sigma(q)$$

Then by (1), $u'(p_1)$ exists and

$$(5) \qquad u'(p_1) = T'(p_1) - w^2 \phi(p_1)\, G'(p_1) \quad .$$

This completes the proof of Lemma 2.20.

To show that the tangential derivative of the double layer is Hölder continuous (L. 2.23), we need certain preparations, (L. 2.21 and L. 2.22).

2.6 Geometric estimates

Lemma 2.21 Referred to a fixed set of local coordinates, let t_1 and t_2 denote unit vectors specifying corresponding tangential directions at points p_1, p_2 respectively. p_1 and p_2 are on the same face of Σ. Then, for a positive constant a

$$\left.\begin{array}{l} |\cos(e_x,t_1) - \cos(e_x,t_2)| \\ |\cos(e_y,t_1) - \cos(e_y,t_2)| \\ |\cos(e_z,t_1) - \cos(e_z,t_2)| \end{array}\right\} \leq a\, r_{12}$$

Proof. Suppose the surface is described by the vector $X = (x_1(u,v), x_2(u,v), x_3(u,v))$, where u, v are local coordinates. Let $t = X_u / X_u$. By hypothesis $X \in C^2(\Sigma)$. This implies that $t(u,v)$ satisfies a Lipschitz condition, which is equivalent to the assertion of the lemma.

Lemma 2.22 under the same hypotheses as in Lemma 2.21, if p_1, p_2 and q $(q \neq p_1, p_2)$ are all on the same face of Σ, then

1) $|\cos(N_q,t_1) - \cos(N_q,t_2)| \leq 3a\, r_{12}$

$$2)\qquad |\cos(r_{p_1q},t_1)-\cos(r_{p_2q},t_2)| \leq \frac{6r_{12}}{r_{p_1q}} + 3\,a\,r_{12}$$

where a is the positive constant in the statement of Lemma 2.21.

Proof. Consider,

$$\cos(N_q,t_1) - \cos(N_q,t_2) = N_q\cdot(t_1-t_2) \quad .$$

In terms of direction cosines this is:

$$\cos(N_q,e_x)(\cos(e_x,t_1)-\cos(e_x,t_2)) + \text{analogous terms in } e_y \text{ and } e_z$$

$$\leq 3\,a\,r_{12} \quad .$$

This follows from Lemma 2.21. That proves 1).

For 2) start by letting u_{p_1q} and u_{p_2q} be unit vectors in the directions of p_1q and p_2q respectively. Then,

$$\cos(r_{p_1q},t_1) - \cos(r_{p_2q},t_2) = u_{p_1q}\cdot t_1 - u_{p_2q}\cdot t_2 \quad .$$

The right side can be written in terms of direction cosines. From that comes the inequality:

$$(1)\quad |\cos(r_{p_1q},t_1) - \cos(r_{p_2q},t_2)|$$

$$\leq |\cos(r_{p_1q},e_x)\cos(e_x,t_1) - \cos(r_{p_2q},e_x)\cos(e_x,t_2)|$$

$$+ \text{ analogous terms in } e_y \text{ and } e_z$$

$$\leq |(\cos(r_{p_1q},e_x)-\cos(r_{p_2q},e_x))\cos(e_x,t_1)| +$$

$$|(\cos(e_x,t_1)-\cos(e_x,t_2))\cos(r_{p_2q},e_x)| + \text{analogous terms} \quad .$$

By Lemma 2.11

$$|\cos(r_{p_1q}, e_x) - \cos(r_{p_2q}, e_x)| \leq \frac{2r_{12}}{r_{p_1q}} \quad ,$$

and by Lemma 2.21

$$|\cos(e_x, t_1) - \cos(e_x, t_2)| \leq a\, r_{12} \quad ,$$

with similar estimates for the analogous terms. Using these estimates in the right side of (1) yields assertion 2).

2.7 Hölder continuity of $u'(p)$, $v'(p)$ on the surface. Results for $(\partial v/\partial n)(p)$.

Lemma 2.23 If the density function ϕ is in $L^1(\Sigma)$ and satisfies a local Hölder condition at $p_o \in \Sigma$,

(1) $$|\phi(p_1)-\phi(p_2)| = O(|p_1-p_2|^u) \quad \text{where} \quad 0 < u \leq 1 \quad :$$

then the double layer potential $u(p)$ has a tangential derivative that is also locally Hölder continuous (with exponent $u/2$) at p_o .

Proof. Let t be a unit vector tangent to Σ at p_o . Let a prime denote differentiation at p_o in the direction of t . From (5) of Lemma 2.20,

$$u'(p) = T'(p) - w^2\, \phi(p)\, G'(p) \quad .$$

Since $G(p)$ is Hölder continuous by Lemma 2.16, we need only consider $T'(p)$. By (4) of Lemma 2.20,

$$T'(p) = \int_\Sigma (\phi(q)-\phi(p))\, \nabla_p K(p,q)\cdot t\, d\sigma(q) \quad .$$

We will calculate to obtain a workable expression for $T'(p)$. Using the notation and formulas of Lemma 2.2,

$$K(p,q) = \left[e^{iwr}(iwr^{-2}-r^{-3})\right]\left[(x-X)N_q^x+(y-Y)N_q^y+(z-Z)N_q^z\right]$$

Call the first factor $f(p,q)$ and the second $g(p,q)$.

$$K'(p) = \nabla_p K\cdot t = -fN_q\cdot t + rf_r\frac{(q-p)}{r}\cdot N_q\,\frac{(p-q)}{r}\cdot t$$

In terms of direction cosines

$$(2)\quad K'(p) = \frac{1}{r^3}\left[A(r)\cos(N_q,t) - B(r)\cos(r_{pq},t)\cos(r_{pq},N_q)\right]$$

where $r = r_{pq}$ and

$$A(r) = e^{iwr}(iwr-1)$$

$$B(r) = e^{iwr}(-w^2r^2 - 3iwr + 3)\quad .$$

Let A and B be bounds for $|A(r)|$ and $|B(r)|$ respectively.

Now we must show local Hölder continuity of

$$(3)\qquad T'(p) =$$

$$\int_\Sigma\left[\phi(q)-\phi(p)\right]\frac{1}{r^3}\left[A(r)\cos(N_q,t)-B(r)\cos(r_{pq},t)\cos(r_{pq},N_q)\right]d\sigma(q).$$

Let $E_1 = E(\delta,p_o)$ denote the element of surface of radius δ at p_o in which ϕ is locally Hölder continuous. Let p_1, p_2 be any two points in $E_2 = E(\delta/2,p_o)$. Let B_{12} be a ball of radius r_{12} centered at the midpoint of line segment $\overline{p_1p_2}$. For brevity let $A_1 = A(r_{p_1q})$, $A_2 = A(r_{p_2q})$ and similarly for B_1, B_2. t_1 and t_2 are defined in Lemma 2.21.

(4) $$T'(p_1) - T'(p_2) =$$

$$\iint_{\Sigma}[\phi(q)-\phi(p_1)]\frac{1}{r^3_{p_1q}}[A_1\cos(N_q,t_1)-B_1\cos(r_{p_1q},t_1)\cos(r_{p_1q},N_q)]\,d\sigma(q)$$

$$-\iint_{\Sigma}[\phi(q)-\phi(p_2)]\frac{1}{r^3_{p_2q}}[A_2\cos(N_q,t_2)-B_2\cos(r_{p_2q},t_2)\cos(r_{p_2q},N_q)]\,d\sigma(q)$$

Split Σ into $\Sigma \cap B_{12}$, $\Sigma \cap (E_1-B_{12})$ and $\Sigma - E_1$. Denote these regions by S_1, S_2, S_3 and the corresponding integrals I_1, I_2, I_3. I_3 is $C^{\infty}(E_2)$, hence Hölder continuous in E_2. To treat I_1 and I_2 we will need these estimates

(5)
$$|\cos(N_q,t_1)| = O(r_{p_1q}) \qquad |\cos(N_q,t_2)| = O(r_{p_2q})$$
$$|\cos(r_{p_1q},N_q)| = O(r_{p_1q}) \qquad |\cos(r_{p_2q},N_q)| = O(r_{p_2q})$$

To show the first estimate, consider a tangent - normal coordinate system at p_1. The unit vector t_1 can be represented by $(a,b,0)$. Then by Lemma 2.3

$$\cos(N_q,t_1) = N_q\cdot t_1 = (O(r),O(r),O(1))\cdot(a,b,0)$$
$$= O(r_{p_1q}) .$$

Similarly, $\cos(N_q,t_2) = O(r_{p_2q})$. The second two estimates in (5) follow directly from Lemma 2.10.

Now we use estimates (5), the boundedness of $A(r)$, $B(r)$ and (1) to obtain, based on extending the integrals on the right side of (4) over S_1 instead of Σ, replacing the brackets by absolute value signs, and adding the second integral instead of subtracting it:

$$|I_1| \leq \int_{S_1} r^u_{p_1q} \frac{O(r_{p_1q})}{r^3_{p_1q}} d\sigma(q) + \int_{S_1} r^u_{p_2q} \frac{O(r_{p_2q})}{r^3_{p_2q}} d\sigma(q) .$$

Use of local polar coordinates at p_1 and then again at p_2 yields

$$|I_1| \leq c \int_0^{2r_{12}} \rho^{u-1}\, d\rho = O(r^u_{12})$$

where c is a positive constant.

Now consider I_2 written as follows (compare with (4)):

(6) $$I_2 =$$

$$\left[\phi(p_2)-\phi(p_1)\right]\int_{S_2} \frac{1}{r^3_{p_1q}}\left[A_1\cos(N_q,t_1)-B_1\cos(r_{p_1q},t_1)\cos(r_{p_1q},N_q)\right] d\sigma(q)$$

$$+\int_{S_2}\left[\phi(q)-\phi(p_2)\right]\left\{\frac{1}{r^3_{p_1q}}\left[A_1\cos(N_q,t_1)-B_1\cos(r_{p_1q},t_1)\cos(r_{p_1q},N_q)\right]\right.$$

$$\left.-\frac{1}{r^3_{p_2q}}\left[A_2\cos(N_q,t_2)-B_2\cos(r_{p_2q},t_2)\cos(r_{p_2q},N_q)\right]\right\}d\sigma(q).$$

Call the first term above I_2' and the second I_2''. By (1), estimates (5), and the boundedness of $A(r)$, $B(r)$, we obtain

(7) $$|I_2'| \leq r_{12}^{u} \int_{S_2} \frac{O(r_{p_1q})}{r_{p_1q}^3}\, d\sigma(q)$$

$$\leq r_{12}^{u/2} \int_{S_2} O(r_{p_1q}^{-2+u/2})\, d\sigma(q) \quad .$$

The last line follows from $\frac{1}{2} r_{12} < r_{p_1q}$. Using local polar coordinates at p_1 yields

(8) $$|I_2'| \leq O(r_{12}^{u/2}) \int_{(1/2)r_{12}}^{2\delta} \rho^{-1+u/2}\, d\rho$$

$$\leq O(r_{12}^{u/2}) \quad .$$

Now consider $|I_2''|$ written as follows:

(9) $$|I_2''| \leq \int_{S_2} |\phi(q)-\phi(p_2)| \left| \frac{A_1\cos(N_q,t_1)}{r_{p_1q}^3} - \frac{A_2\cos(N_q,t_2)}{r_{p_2q}^3} \right| d\sigma(q)$$

$$+\int_{S_2} |\phi(q)-\phi(p_2)| \times$$

$$\left| \frac{B_2\cos(r_{p_2q},t_2)\cos(r_{p_2q},N_q)}{r_{p_2q}^3} - \frac{B_1\cos(r_{p_1q},t_1)\cos(r_{p_1q},N_q)}{r_{p_1q}^3} \right| d\sigma(q).$$

We will need the estimate

(10) $$\left| \frac{1}{r_{p_1q}^3} - \frac{1}{r_{p_2q}^3} \right| \leq 4\, \frac{r_{12}}{r_{p_1q}^4} \quad ,$$

valid for $q \in R^3 - B_{12}$. This follows by factorization, then use of Lemma 2.12. To estimate (9) we use identity (1) of Lemma 2.17 repeatedly, followed by Lemmas 2.12, 2.13, 2.17, 2.22, estimates (5) and (10), and the hypothesis (1), to obtain

$$|I_2''| \leq \int_{S_2} r_{p_1q}^u \frac{O(r_{12})}{r_{p_1q}^3} \, d\sigma(q) \quad .$$

Since for q outside B_{12}, $(1/2)r_{12} < r_{p_1q}$, we can write

$$|I_2''| \leq r_{12}^u \int_{S_2} \frac{O(r_{p_1q})}{r_{p_1q}^3} \, d\sigma(q) \quad .$$

This is the same expression as for $|I_2'|$ in (7), which leads to estimate (8). Therefore, we conclude likewise here

$$|I_2''| \leq O(r_{12}^{u/2}) \quad .$$

Then, for p_1, p_2 in $E(\delta/2, p_o)$,

$$|T'(p_1) - T'(p_2)| = O(r_{12}^{u/2}) \quad .$$

This completes the proof of Lemma 2.23.

<u>Corollary 2.23</u> If the density function ϕ satisfies the hypotheses of Lemma 2.23, then the single layer potential $v(p) = Q\phi(p) = \int_{\Sigma} \phi(q)Q(p,q) \, d\sigma(q)$ has a Hölder continuous derivative on $\dot{\Sigma}$.

<u>Proof</u>. The proof of Lemma 2.23 and its ancillary lemmas can be traced through with appropriate modifications for the simpler kernal Q.

Lemmas 2.19, 2.20, and 2.23 are true if we replace $u(p)$ by $(\partial v/\partial N)(p)$, where v is the potential of a single layer and N is the outward normal to Σ at $p \in \Sigma$. See the statements below. Sources of proofs in the potential theory case (w=0) for a smooth closed surface are indicated.

Lemma 2.19′ If $p \in \Sigma$ and the density function $\phi \in$ L.B. then the normal derivative of the single layer potential $(\partial v/\partial N)(p)$ is Hölder continuous on Σ .

Proof. [3] page 61, [18] page 633.

Lemma 2.23′ If the density function ϕ satisfies a Hölder condition on Σ then the normal derivative of the single layer potential $(\partial v/\partial N)(p)$ possesses a tangential derivative on Σ that is also Hölder continuous.

Proof. [3] page 312.

2.8 Continuity of $(\partial u/\partial N)(p)$ as p approaches the surface.

Lemma 2.24 Consider the double layer potential $u(p) = \int_\Sigma \phi(q)K(p,q)\, d\sigma(q)$. Let p_o be an arbitrary point in Σ . Suppose $\phi(q) \in L^1(\Sigma)$ has Hölder continuous derivatives in a neighborhood of p_o . Let l be the normal line to the surface at p_o and N_o the outward normal at p_o . Then, as p approaches p_o along l, $(\partial u/\partial N_o)(p)$ has a limit $(\partial u/\partial N_o)(p_o)_-$ from inside Σ, and a limit $(\partial u/\partial N_o)(p_o)_+$ from outside Σ, and these limits are equal.

Proof. Let E be an element of surface of radius ϵ at p_o. Split the integral for $u(p)$ as follows:

$$u(p) = \int_E \phi(q)K(p,q)d\sigma(q) + \int_{\Sigma-E} \phi(q)K(p,q)d\sigma(q) \quad .$$

Call the first integral $u_1(p)$ and the second $u_2(p)$. The dominated convergence theorem justifies the passage to to the limit under the integral sign in the following two assertions. Because $K(p,q)$ and $(\partial K/\partial N_o)(p,q)$ are bounded (r is bounded away from zero), and $\phi(q)$ is integrable, we have

$$(\partial u_2/\partial N_o)(p) = \int_{\Sigma-E} \phi(q)\ (\partial K/\partial N_o)(p,q)\ d\sigma(q) \quad ,$$

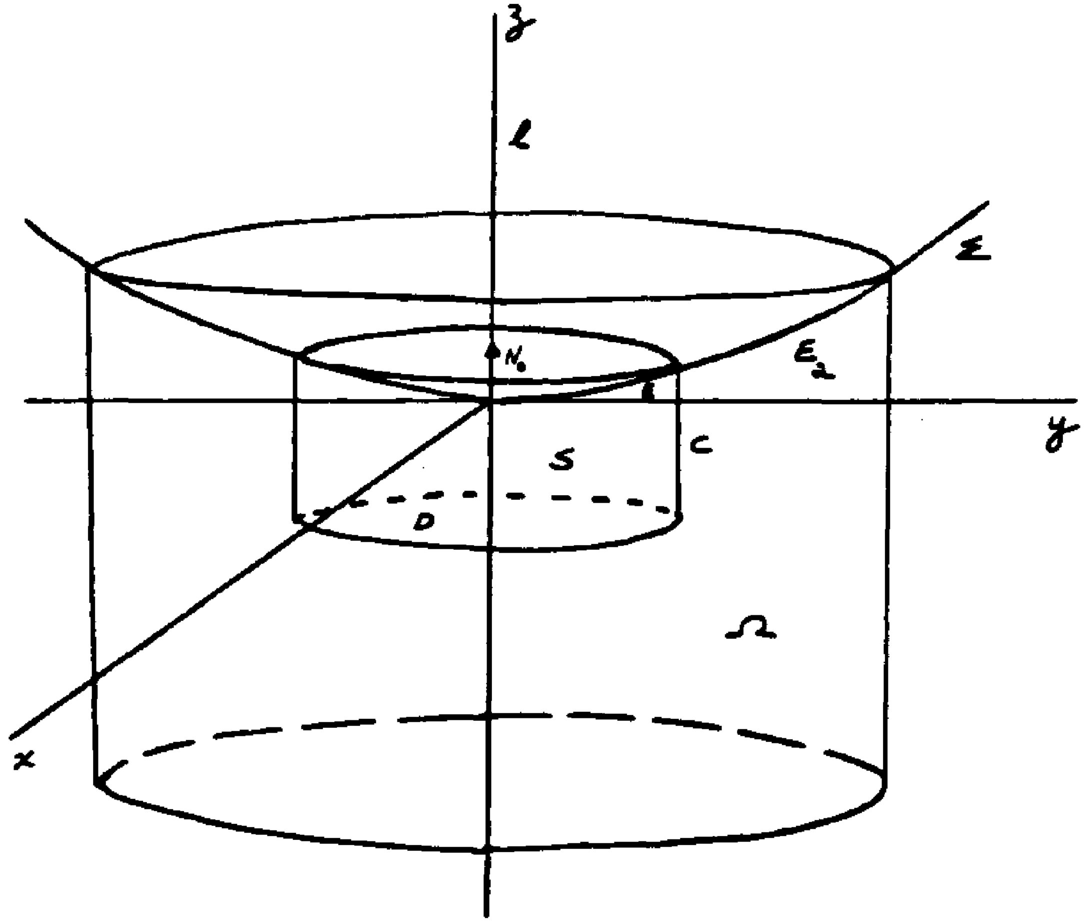
$$\lim_{p\to p_o} (\partial u_2/\partial N_o)(p) = \int_{\Sigma-E} \phi(q) \lim_{p\to p_o} (\partial K/\partial N_o)(p,q)\ d\sigma(q) \ .$$

The limit here is independent of the direction of approach. In view of the continuity of $(\partial u_2/\partial N_o)(p)$ at p_o, just shown, the lemma will be proved if we can show

$$(\partial u_1/\partial N_o)(p_o)_- = (\partial u_1/\partial N_o)(p_o)_+ \quad .$$

Start by letting Ω be a local cylinderical region of radius 2ϵ at p_o. Let $\partial\Omega$ be its boundary and E_2 the corresponding surface element. See Figure 2.1 on page 64.

Figure 2.1



Let $F(q)$ be the extension of $\phi(q)$ to $\partial\Omega$ described in Def. 1.4.

By well known solutions of the Dirichlet problem of potential theory there is a function $h(q)$ which is harmonic in the interior of Ω and agrees with $F(q)$ on $\partial\Omega$. From the Hölder continuity of $F'(q)$ (from $\phi'(q)$) a theorem of Kellogg [5] implies the axistence and continuity of $(\partial h/\partial N)(q)$ on the boundary at interior points of the faces (not including the edges) of the local cylindrical surface $\partial\Omega$. For later use we will need integrability of $(\partial h/\partial N)(q)$ on a local cylindrical surface. To ensure this we will consider the nested local cylindrical region S of radius ϵ at p_o corresponding to the surface element E (See Fig. 2.1 and Def. 1.3.). We will show that $(\partial h/\partial N)(q)$ is continuous on each closed face of ∂S. Denote the faces of ∂S as follows: element of surface E, truncated cylinder C, disc D. Because $h(q)$ is harmonic in the larger region Ω, it possesses a continuous normal derivative on D and on C. By Kellogg's theorem this derivative possesses a limit as $q \in C$ approaches an edge point on E. Also from Kellogg's theorem $h(q)$ has a continuous normal derivative on all of E.

Green's second identity for a domain R with boundary ∂R is:

$$\iint_R u\Delta v - v\Delta u \, d\Omega = \int_{\partial R} u\, v_N - v\, u_N \, d\sigma .$$

We will use this identity with $v = Q(p,q)$ and $u = h(q)$. To do this, the domain R must exclude the singular point p of $Q(p,q)$. Therefore, define R as follows:

<u>Case 1</u>. For $p \in 1$ and $p \notin S$ let $R = S$.

<u>Case 2</u>. For $p \in 1$ and $p \in S$ let $R = S-B$ where B is a ball of radius δ about p .

$$(1) \iint_R h(q)\Delta_q Q(p,q)d\alpha(q) = \int_{\partial R} h(q)(\partial Q/\partial N_q)(p,q)d\sigma(q) - \int_{\partial R} Q(p,q)(\partial h/\partial N)(q)d\sigma(q) \quad .$$

The first integral on the right side of (1) can be written: $\int_{\partial R} h(q)K(p,q)d\sigma(q)$. Split it into

$$(2) \qquad \int_E h(q)K(p,q)d\sigma(q) + \int_{\partial R-E} h(q)K(p,q)d\sigma(q) \quad .$$

The first integral in (2) is just $u_1(p)$. Besides these changes, add $w^2\iint_R Q(p,q)d\alpha(q)$ to both sides of (1).

The new left side is:

$$\iint_R h(q)\left[\Delta_q Q(p,q) + w^2 Q(p,q)\right] d\alpha(q) \quad .$$

The factor in the brackets is zero because Q is a solution of the Helmholtz equation in R . Hence, from (1) we obtain an equation that can be organized thus

$$(3) \quad 0 = u_1(p) - \int_{\partial R} Q(p,q)(\partial h/\partial N)(q)d\sigma(q) + \int_{\partial R-E} h(q)K(p,q)d\sigma(q) + w^2\iint_R Q(p,q)d\alpha(q) \quad .$$

In Case 1 $p \notin S$, so we simply replace R by S in (3). Label the resulting equation (4). In Case 2 $p \in S$, so $R = S-B$. Equation (3) then yields a new equation composed of (4) plus three additional terms on the right side, namely

$$(5)\quad \int_{\partial B} Q(p,q)(\partial h/\partial N)(q)d\sigma(q) - \int_{\partial B} h(q)K(p,q)d\sigma(q) - w^2\iint_B Q(p,q)d\alpha(q) \quad .$$

By the argument leading to (4) of Lemma 2.9, as the radius δ of B goes to zero, the second term in (5) approaches $4\pi h(p)$. A similar argument shows that the first integral in (5) approaches zero as $\delta \to 0$. The third integral when written in spherical coordinates can easily be seen to approach zero as $\delta \to 0$. From these observations, (3), in Case 2 where $p \in S$, becomes in the limit as $\delta \to 0$

$$(6)\quad 0 = u_1(p) + 4\pi h(p) - \int_{\partial S} Q(p,q)(\partial h/\partial N)(q)d\sigma(q) + \int_{\partial S-E} h(q)K(p,q)d\sigma(q) + w^2\iint_S Q(p,q)d\alpha(q) \quad .$$

Now apply the operator $(\partial/\partial N_o)$ to (4) and (6). The resulting equations are listed below as $(4)'$ and $(6)'$. The result for the last two terms on the right in both equations is:

$$\int_{\partial S-E} h(q)(\partial K/\partial N_o)(p,q)d\sigma(q) + w^2\iint_S (\partial Q/\partial N_o)(p,q)d\alpha(q) \quad .$$

The derivatives here are with respect to p . Let A(p) denote the sum of these two terms. The derivative under the integral sign as well as the continuity of $A(p)$ follows from the dominated convergence theorem. Thus we have

$$(4)' \quad 0 = (\partial u_1/\partial N_o)(p) - (\partial\ /\partial N_o)\int_{\partial S} Q(p,q)(\partial h/\partial N)(q)d\sigma(q) + A(p)$$

$$(6)' \quad 0 = (\partial u_1/\partial N_o)(p) + 4\pi(\partial h/\partial N_o)(p) - (\partial\ /\partial N_o)\int_{\partial S} Q(p,q)(\partial h/\partial N)(q)d\sigma(q) + A(p) \quad .$$

Now let $p \to p_o$ from outside S in $(4)'$ and from inside S in $(6)'$. The integral

$$I(p) = \int_S Q(p,q)(\partial h/\partial N)d\sigma(q)$$

can be thought of as a single layer potential with density $(\partial h/\partial N)(q)$. As such, by Lemma 2.9

$$(\partial I/\partial N_o)(p) \longrightarrow -2\pi(\partial h/\partial N_o)(p_o) + C(p_o)$$

as $p \to p_o$ from outside S in $(4)'$ and

$$(\partial I/\partial N_o)(p) \longrightarrow +2\pi(\partial h/\partial N_o)(p_o) + C(p_o)$$

as $p \to p_o$ from inside S in $(6)'$, where

$$C(p_o) = \int_{\partial S} (\partial Q/\partial N_o)(p_o,q)\ (\partial h/\partial N)(q)\ d\sigma(q) \quad .$$

By the continuity of $A(p)$: $A(p) \longrightarrow A(p_o)$. By Kellogg's theorem the term $4\pi(\partial h/\partial N_o)(p)$ in $(6)'$ approaches a limit $4\pi(\partial h/\partial N_o)(p_o)$ as $p \to p_o$ from inside S . The resulting equations in the limit are:

(4)'' $0 = (\partial u_1/\partial N_o)(p_o)_+ + 2\pi(\partial h/\partial N_o)(p_o) - C(p_o) + A(p_o)$

(6)'' $0 = (\partial u_1/\partial N_o)(p_o)_- + 4\pi(\partial h/\partial N_o)(p_o)$

$- 2\pi(\partial h/\partial N_o)(p_o) - C(p_o) + A(p_o)$.

The equations (4)'' and (6)'' show that the separate limits $(\partial u_1/\partial N_o)(p_o)_+$ and $(\partial u_1/\partial N_o)(p_o)_-$ exist. Comparison of the equations shows that the limits are equal. This completes the proof of Lemma 2.24.

<u>Corollary 2.24</u> Denote the value of $\lim_{p \to p_o} (\partial u/\partial N_o)(p)$ by $P\phi(p_o)$. As a function of p_o , $P\phi(p_o)$ is in $C^o \cap L^1(\Sigma)$.

<u>Proof</u>. From (4)'' of Lemma 2.24, for $p_o \in \Sigma$

$$(\partial u_1/\partial N_o)(p_o) = -2\pi(\partial h/\partial N_o)(p_o) + C(p_o) - A(p_o) \quad .$$

By Kellogg's theorem $(\partial h/\partial N_o)(p_o)$ is continuous. By Lemma 2.4 and the dominated convergence theorem, $C(p_o) - A(p_o)$ is continuous. $(\partial u_2/\partial N_o)(p_o)$ is also continuous so $P\phi(p_o)$ is continuous for $p_o \in \Sigma$. For the integrability see [18] page 630.

Chapter Three

K and K^t on the space $W(\Sigma)$

3.1 The function space $W(\Sigma)$.

Σ is a C^2 manifold with corners (cf. Section 1.2). Let E be the set of edge points of Σ and S_m the set of points of Σ having distance from E at least $1/2^m$. Further, let $\mathring{\Sigma} = \Sigma - E$, and denote by D the class of functions $C^0 \cap L^1(\mathring{\Sigma})$. For $f \in D$ we can define a family of norms for $m = 1, 2, 3, \ldots$ as follows:

$$\|f\|_m = \sup_{p \in S_m} |f(p)| \quad .$$

A norm defined in terms of these is

$$\|f\|_W^2 = \sum_{m=1}^{\infty} 2^{-m} \|f\|_m^2 \quad .$$

Call this the W^2 norm. D equipped with the W^2 norm is a function space $(D, \|\ \|_W)$ which for brevity we denote by $W(\Sigma)$.

In this chapter we investigate properties of $W(\Sigma)$ and the operators K, K^t defined on $W(\Sigma)$.

Lemma 3.1 Let $\|f\|_0$ denote the L^2 norm of f on Σ. If $f \in W(\Sigma)$ then f is square integrable on Σ and

$$\|f\|_0^2 \leq (2A + L)\ \|f\|_W^2 \tag{1}$$

where A is the area of Σ and L is a positive constant independent of f.

Proof. Define a sequence of functions as follows:

$$g_m(p) = f^2(p) \quad \text{for} \quad p \in S_m$$
$$= 0 \quad \text{for} \quad p \in \Sigma - S_m \ .$$

Clearly, $\{g_m\}$ is a monotonic sequence of functions. Observe that

$$\int_\Sigma g_1(p)\, d\sigma = \int_{S_1} f^2(p)\, d\sigma \leq A\ \|f\|_1^2$$
$$\leq 2\,A\ \|f\|_W^2$$

where A is the area of Σ. Let $R_{m+1} = S_{m+1} - S_m$. The width of $R_{m+1} \leq c/2^{m+1}$ where c is a positive constant independent of m. Let L_1 denote the length of the edges of Σ, then Area of $R_{m+1} \leq cL_1/2^{m+1}$.

Now consider,

$$\int_\Sigma g_{j+1}(p)d\sigma = \int_{S_{j+1}} f^2(p)d\sigma$$
$$= \int_{S_j} f^2(p)d\sigma + \int_{R_{j+1}} f^2(p)d\sigma$$
$$\leq \int_\Sigma g_j(p)d\sigma + \frac{cL_1}{2^{j+1}}\ \|f\|_{j+1}^2 \ .$$

Let $L = cL_1$ and use induction to obtain

$$\int_\Sigma g_m(p)d\sigma \leq 2A\ \|f\|_W^2 + L \sum_{j=2}^{m} \frac{\|f\|_j^2}{2^j}$$
$$\leq (2A + L)\ \|f\|_W^2 \ .$$

This shows the uniform boundedness of $\|g_m\|$.

By the monotone convergence theorem we conclude that there is an integrable function h(p) such that almost everywhere on Σ: $h(p) = \lim_{m\to\infty} g_m(p)$. This limit also equals $f^2(p)$ and

$$\int_\Sigma f^2(p)d\sigma = \int_\Sigma h(p)d\sigma = \lim_{m\to\infty}\int_\Sigma g_m^2(p)d\sigma$$
$$\leq (2A + L)\ \|f\|_W^2 \ .$$

This completes the proof of the lemma.

<u>Remark</u>: As is well known, $f \in L^2$ implies $f \in L^1$ on a space of finite measure. For later use we need specific bounds. By Hölder's inequality,

$$\int_\Sigma |f(p)|\ d\sigma \leq \sqrt{\int_\Sigma 1\ d\sigma}\sqrt{\int_\Sigma f^2(p)\ d\sigma} \ .$$

Let b_1 be the square root of the area of Σ so that

$$\|f\| \leq b_1\ \|f\|_o \ .$$

Now use (1) and let $k = b_1\sqrt{2A + L}$ to obtain

(3) $$\|f\| \leq k\ \|f\|_W \ .$$

<u>Lemma 3.2</u> $W(\Sigma)$ is a Banach space.

<u>Proof</u>. It is easily seen that $\|\ \|_W$ possesses the properties of a norm.

To show completeness, let $\{f_k\}$ be a Cauchy sequence of functions of $W(\Sigma)$. Given an ϵ there is an N such that for all $j, k \geq N$

$$(1) \qquad \epsilon \geq \|f_j - f_k\|_W^2 = \sum_{m=1}^{\infty} 2^{-m} \|f_j - f_k\|_m^2$$

We start by showing Cauchy convergence of $\{f_k\}$ on each S_m in the sup. norm. From (1) we obtain the estimate

$$\|f_j - f_k\|_m^2 \leq 2^m \|f_j - f_k\|_W^2$$

Now, given any ϵ, by the hypothesis an N can be found, so that $\|f_j - f_k\|_W^2 < \epsilon/2^m$ for all $j, k \geq N$ and hence by the above estimate $\|f_j - f_k\|_m^2 \leq 2^m \epsilon/2^m = \epsilon$ as required. Since the convergence is uniform, f_k converges on S_m to a continuous function f. This holds for any m so $f \in C^0(\Sigma)$.

Now, split the sum in (1) as follows:

$$\sum_{m=1}^{r} 2^{-m} \|f_j - f_k\|_m^2 + \sum_{m=r+1}^{\infty} 2^{-m} \|f_j - f_k\|_m^2 \leq \epsilon$$

where r is an arbitrary positive integer. Consider the first sum. If we fix k and pass to the limit as $j \to \infty$ we obtain

$$\sum_{m=1}^{r} 2^{-m} \|f - f_k\|_m^2 \leq \epsilon$$

Since the inequality is valid for arbitrary r we can pass to the limit as $r \to \infty$ and obtain

$$\|f - f_k\|_W^2 = \sum_{m=1}^{\infty} 2^{-m} \|f - f_k\|_m^2 \leq \epsilon \quad .$$

Since ϵ was arbitrary, we conclude that f_k approaches f in the W^2 norm. This completes the proof of Lemma 3.2.

3.2 Boundedness of K and K^t on $W(\Sigma)$.

Theorem 3.1 Suppose Σ is an arbitrary C^2 manifold with corners (cf. Section 1.2). Then, K and K^t are bounded operators on the space $W(\Sigma)$.

Proof. We will only distinguish between K and K^t as it becomes necessary. When a relation holds for both K and K^t we will generally write it only for K .

We must show, for a positive constant c ,

$$\|K\phi\|_W^2 \leq c \ \|\phi\|_W^2$$

or equivalently,

$$\sum_{m=1}^{\infty} 2^{-m} \ \|K\phi\|_m^2 \leq c \ \sum_{m=1}^{\infty} 2^{-m} \ \|\phi\|_m^2 \quad .$$

This depends on estimates for

$$\|K\phi\|_m = \sup_{p \in S_m} |K\phi(p)| = \sup_{p \in S_m} \left| \int_\Sigma \phi(q) K(p,q) d\sigma(q) \right| \quad .$$

The proof is long and is organized as follows. First we divide the finding of estimates (and showing that their contribution to $\|K\phi\|_W^2$ is bounded) into two parts according to the location of p . Let u be a fixed positive integer large enough that Lemma 2.4 holds for any element of surface of radius $2D$ where $D = 1/2^u$.

Part 1. $p \in S_m$ for $m \leq u$

Part 2. $p \in S_m - S_u$ for $m > u$.

We will divide Part 2 after introducing some notation. Denote by p_o , the nearest edge point to p . Let B be a ball with center at p_o and radius $2D$. Let F_1 denote the face of Σ that contains p . In Part 2 we split the integral bounding $|K\phi(p)|$ into integrals over complimentary subsets of Σ , namely

2a). $(F_1 \cap B) \cup (\Sigma - B)$

2b). $(\Sigma - F_1) \cap B$.

Most of the work of proving the theorem belongs to Part 2b), in which p and q are on opposite faces and near to each other. The proof of Part 2b) is built up by means of successively more general cases.

Case 1. No more than two faces of Σ meet at any edge and Σ is planar at all points within distance $2D$ of any edge.

Case 2. No more than two faces of Σ meet at any edge.

Case 3. Σ is an arbitrary C^2 manifold. (n faces may meet at a corner.)

Now we begin to prove the parts and cases.

Part 1. From Lemma 2.5 there are constants a, b, and ϵ such that for any p in $S_m \leq S_u$

(1) $$|K\phi(p)| \leq a\|\phi\|_{m+1}\epsilon + \frac{b}{\epsilon^2}\|\phi\| .$$

We state an inequality here for future use:

(2) $$\left[\sum_{j=1}^{n} b_j\right]^2 \leq n \sum_{j=1}^{n} b_j^2 .$$

This follows from the Cauchy inequality (page 51, for $a_j=1$). Relations (1) and (2) give, taking $n=2$,

(3) $$\|K\phi\|_m^2 \leq 2\left[a^2\|\phi\|_{m+1}^2\epsilon^2 + \frac{b^2}{\epsilon^4}\|\phi\|^2\right]$$

Now consider the sum for $\|K\phi\|_W^2$ and split it as follows:

(4) $$\|K\phi\|_W^2 = \sum_{m=1}^{u} 2^{-m}\|K\phi\|_m^2 + \sum_{m=u+1}^{\infty} 2^{-m}\|K\phi\|_m^2 .$$

Call the first sum S_1 and the second sum S_2 . By (3), the definition of $\|\phi\|_W^2$, and (3) of Lemma 3.1 we obtain

$$S_1 \leq 2\sum_{m=1}^{u} 2^{-m}\left[a^2\epsilon^2\|\phi\|_{m+1}^2 + \frac{b^2}{\epsilon^4}\|\phi\|^2\right]$$

$$\leq 2^2a^2\epsilon^2\sum_{m=1}^{\infty}\frac{\|\phi\|_{m+1}^2}{2^{m+1}} + 2\frac{b^2}{\epsilon^4}\|\phi\|^2\sum_{m=1}^{\infty}2^{-m}$$

$$\leq 2^2a^2\epsilon^2\|\phi\|_W^2 + 2\frac{b^2}{\epsilon^4}k^2\|\phi\|_W^2 .$$

Let $c_1 = 2^2a^2\epsilon^2 + 2b^2k^2/\epsilon^4$, so that

(5) $$S_1 \leq c_1\|\phi\|_W^2 .$$

Part 2. To treat S_2 it is sufficient to consider an arbitrary point $p \in S_m - S_u$ for $m > u$, for if $\sup_{p \in S_m} |K\phi(p)|$ were achieved for $p \in S_u$, by (3) it would lead to the bound (5), already shown. We want to obtain an estimate of $|K\phi(p)|$:

(6) $$|K\phi(p)| \leq \int_{(F_1 \cap B) \cup (\Sigma - B)} |\phi(q)||K(p,q)|\, d\sigma(q) \quad + \quad \int_{(\Sigma - F_1) \cap B} |\phi(q)||K(p,q)|\, d\sigma(q).$$

We partition $F_1 \cap B$ further into $D_1 = F_1 \cap B \cap S_{m+1}$ and $(F_1 \cap B) - D_1$, as shown in the table below. The corresponding integral is denoted on the right.

Region	Integral
Part 2a).	
$D_1 = F_1 \cap B \cap S_{m+1}$	$I_1(m)$
$D_2 = (F_1 \cap B) - D_1$	$I_2(m)$
$D_3 = \Sigma - B$	$I_3(m)$
Part 2b).	
$D_4 = (\Sigma - F_1) \cap B$	$I_4(m)$

Figure 3.2 on page 78 illustrates F_1 near an edge.

With the notation from the table we write, based on (6), for $p \in S_m - S_u$ and $m > u$,

(7) $$|K\phi(p)|^2 \leq \left[I_1(m) + I_2(m) + I_3(m) + I_4(m)\right]^2 .$$

Figure 3.1

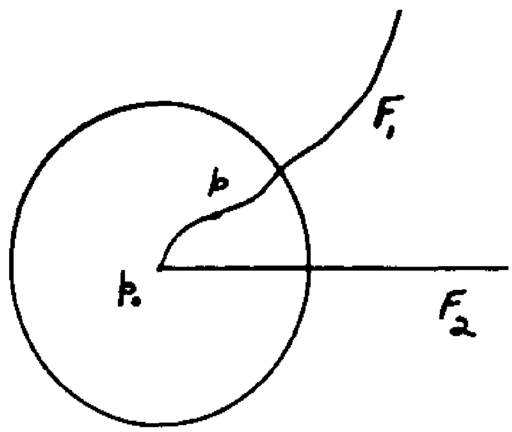
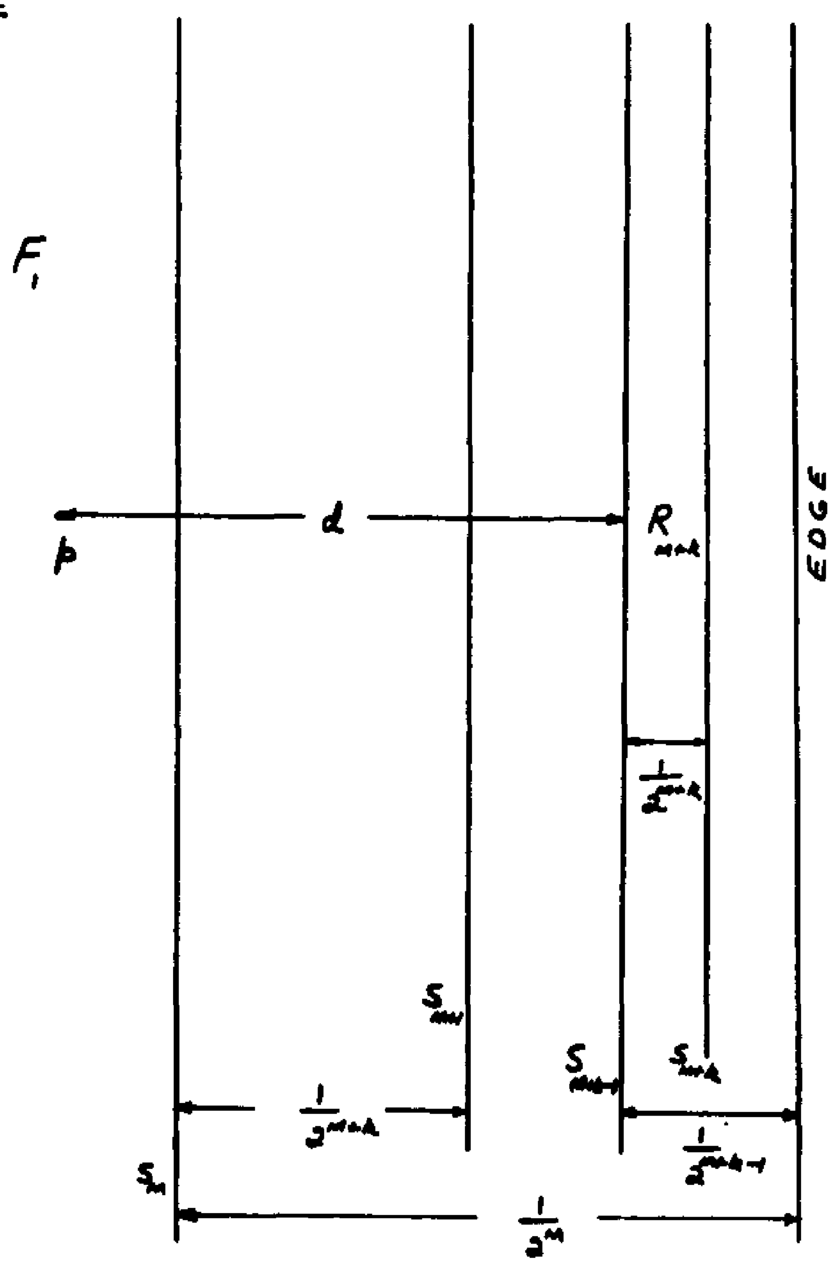


Figure 3.2



Part 2a). In this part we will find estimates for I_1, I_2, I_3 and their contribution to $\|K\phi\|_W^2$. By Lemma 2.4 there are constants a and ϵ independent of ϕ and m such that for an element of surface E of radius ϵ at p

$$\int_E |\phi(q)||K(p,q)|\, d\sigma(q) \leq a\, \epsilon \sup_{q \in E} |\phi(q)| \quad .$$

This holds also for the truncated element of surface $D_1 = B \cap F_1 \cap S_{m+1}$, so

$$I_1(m) \leq a\ 2\ D\ \|\phi\|_{m+1} \quad .$$

Next, we want to integrate over $D_2 = (F_1 \cap B) - D_1$. We will do this by integrating over each strip

$$R_{m+k} = B \cap (S_{m+k} - S_{m+k-1}) \quad \text{for} \quad k = 2,3,4,\ldots \quad .$$

By Lemma 2.4 there is a positive constant c_1 independent of m, such that $|K(p,q)| \leq c_1/r$ for p, q in D_2. Let d be the distance from p to R_{m+k}. For $p \in S_m$ and $k \geq 2$, $d \geq 1/2^{m+1}$. See Figure 3.2. Let

$$J(k) = \int_{R_{m+k}} |\phi(q)||K(p,q)|\, d\sigma(q) \quad .$$

This can be estimated as follows:

$$J(k) \leq \sup_{q \in R_{m+k}} |\phi(q)| \sup_{q \in R_{m+k}} |K(p,q)| \text{ Area of } R_{m+k}$$

$$\leq \|\phi\|_{m+k}\, (c_1/d) \quad c_2(2/2^u)\ c_3(1/2^{m+k})$$

where $c_2(2/2^u)$ is a bound on the length of R_{m+k},

$c_3(1/2^{m+k})$ is a bound on the width of R_{m+k} and c_2, c_3 are positive constants. These bounds follow from the smoothness of the face F_1 and the fact that $R_{m+k} \subset B$. Now, let $c = c_1 c_2 c_3$, substitute for d , and simplify to obtain

$$J(k) \leq \|\phi\|_{m+k} \quad c/2^{u+k}$$

Therefore,

$$I_2(m) \leq \sum_{k=2}^{\infty} J(k) \leq \sum_{k=2}^{\infty} \|\phi\|_{m+k} \quad c/2^{u+k}$$

Now consider $q \in D_3 = \Sigma - B$ in which $r = |q-p| \geq D = 1/2^u$.

Lemma 2.2 gives

$$\begin{aligned} I_3(m) &\leq \int_{\Sigma - B} |\phi(q)| |K(p,q)| \, d\sigma(q) \\ &\leq c_1/D^2 \quad \|\phi\| \\ &\leq c_1 \, 2^{2u} \quad \|\phi\| \quad . \end{aligned}$$

Going back to S_2 in (4), and using (7), then (2), we have

$$S_2 \leq 4 \sum_{m=u+1}^{\infty} 2^{-m} \left[I_1^2(m) + I_2^2(m) + I_3^2(m) + I_4^2(m) \right] \quad .$$

Introduce the notation, for $j = 1,2,3,4$:

$$T_j = \sum_{m=u+1}^{\infty} 2^{-m} \, I_j^2(m)$$

so that

(8) $$S_2 \leq 4 \, (T_1 + T_2 + T_3 + T_4) \quad .$$

T_4 belongs to Part 2b) and will be treated later. Let

$$S_3 = 4\ (T_1 + T_2 + T_3)$$

To estimate S_3, use the previous bounds obtained for $I_j(m)$ for $j = 1,2,3$. The results are:

$$T_1 \leq a^2\ (2/2^{2u})^2\ 2 \sum_{m=u+1}^{\infty} 2^{-(m+1)}\ \|\phi\|^2_{m+1} \quad ,$$

$$T_2 \leq c^2 \sum_{m=u+1}^{\infty} \left[\sum_{k=2}^{\infty} \frac{2^{-m/2}}{2^{u+k}}\ \|\phi\|_{m+k}\right]^2 \quad ,$$

$$T_3 \leq c_1^2\ \|\phi\|^2\ 2^{4u} \sum_{m=u+1}^{\infty} 1/2^m \quad .$$

Thus,

$$T_1 \leq a^2\ 2^{3-4u}\ \|\phi\|^2_W \quad ,$$

and, using estimate (3) of Lemma 3.1 for $\|\phi\|^2$, we obtain

$$T_3 \leq c_1^2\ 2^{4u}\ k^2\ \|\phi\|^2_W \quad .$$

For T_2 we use a generalized Minkowski's inequality [1]:

$$\left[\sum_{j=1}^{n} \left[\sum_{k=1}^{m} a_{jk}\right]^2\right]^{1/2} \leq \sum_{k=1}^{m} \left[\sum_{j=1}^{n} a_{jk}^2\right]^{1/2} \quad .$$

Thus,

$$\text{(10)} \qquad \sqrt{T_2} \leq c \sum_{k=2}^{\infty} \left[\sum_{m=u+1}^{\infty} \left[\frac{2^{-m/2}}{2^{u+k}}\ \|\phi\|_{m+k}\right]^2\right]^{1/2} \quad .$$

Let the sum over m in (10) be denoted by T_5.

$$T_5 = \sum_{m=u+1}^{\infty} \frac{1}{2^m 2^{2u+2k}}\ \|\phi\|^2_{m+k}$$

$$\leq 2^{-k} \sum_{m=u+1}^{\infty} 2^{-(m+k)}\ \|\phi\|^2_{m+k} \quad .$$

Therefore, $T_5 \leq 2^{-k} \|\phi\|_W^2$. Substitute this back into (10) to get

(11) $$\sqrt{T_2} \leq c \ \|\phi\|_W \sum_{k=2}^{\infty} 2^{-k/2} \quad .$$

Denote the sum by R and square (11) to obtain

$$T_2 \leq c^2 R^2 \|\phi\|_W^2 \quad .$$

To finish our treatment of S_3 ,use the estimates for T_1, T_2, and T_3 in (9), and obtain

$$S_3 \leq 4 \ (a^2 \ 2^{3-4u} \|\phi\|_W^2 + c^2 \ R^2 \ \|\phi\|_W^2 + c_1^2 \ 2^{4u} \ k^2 \ \|\phi\|_W^2) \quad .$$

Let $c_3 = 4 \ (a^2 \ 2^{3-4u} + c^2 \ R^2 + c_1^2 \ k^2 \ 2^{4u})$, then

(12) $$S_3 \leq c_3 \ \|\phi\|_W^2 \quad .$$

This, together with (4), (5), (8), and (9), completes the proof of Part 2a).

At this point we would like to show how we can finish up the proof of the theorem if we grant the result to be shown later in Part 2b) that

$$T_4 \leq c_4 \ \|\phi\|_W^2$$

where c_4 is a positive constant. Then, (8), with the help of (12) can be written

$$\begin{aligned} S_2 &\leq S_3 + 4 \ T_4 \\ &\leq c_3 \ \|\phi\|_W^2 + 4 \ c_4 \ \|\phi\|_W^2 \quad . \end{aligned}$$

Let $c_2 = c_3 + 4\ c_4$ so that

(13) $$S_2 \leq c_2\ \|\phi\|_W^2 \quad .$$

Estimates (5) and (13), together with (4), produces

(14) $$\|K\phi\|_W^2 \leq c_1\ \|\phi\|_W^2 + c_2\ \|\phi\|_W^2 = (c_1+c_2)\ \|\phi\|_W^2 \quad .$$

This shows the boundedness of the operator K on the space $W(\Sigma)$.

<u>Part 2b)</u>. In terms of our previous notation (cf. (4)) we will now investigate

$$T_4 = \sum_{m=u+1}^{\infty} 2^{-m}\ I_4^2(m)$$

where

$$I_4(m) = \int_{(\Sigma-F_1)\cap B} |\phi(q)||K(p,q)|\ d\sigma(q) \quad .$$

$p \in S_m - S_u$ with $m > u$ and $p \in$ face F_1 . We intend to show that $T_4 \leq c_4\ \|\phi\|_W^2$ where c_4 is a positive constant.

<u>Case 1</u>. No more than two faces of Σ meet at an edge and Σ is planar at all points within distance $2\ D$ of any edge. Let F_2 be the face adjacent to face F_1 . F_1 and F_2 are planar inside B . See Figure 3.3 on page 84. Consider an $x\ y\ z$ rectilinear coordinate system at p_0 , with x axis along the edge, and y axis in F_2 . p is in F_1 at distance t from p_0 and $q = (x,y,0)$ is in F_2 . The pertinent vectors are:

Figure 3.3 Σ inside B . F_1 is seen on edge.

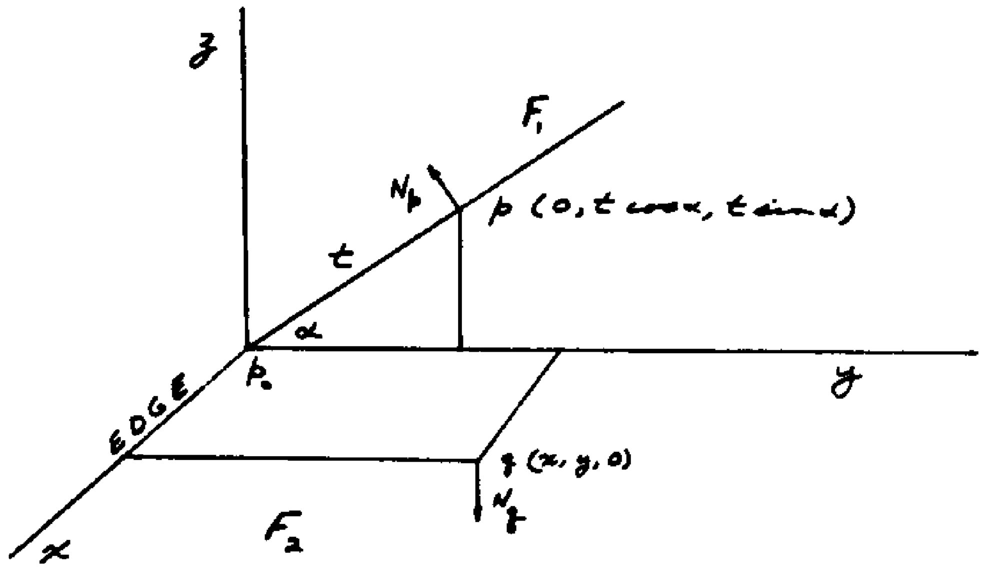
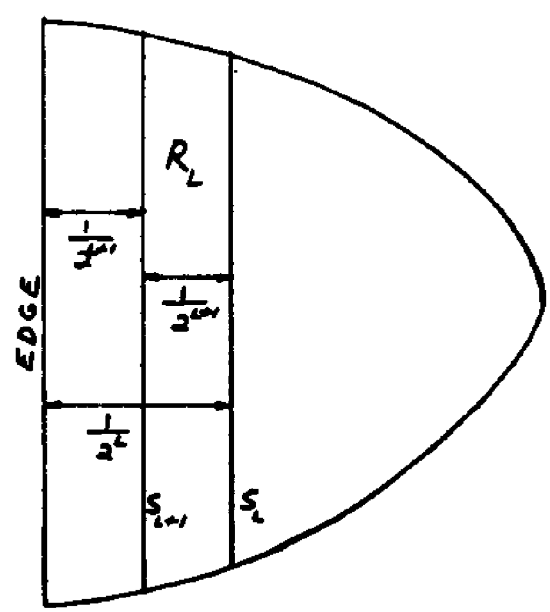


Figure 3.4 $B \cap F_2$.



$$r = p-q = (-x, t\cos\alpha - y, t\sin\alpha)$$

$$N_p = (0, -\sin\alpha, \cos\alpha) \quad , \quad N_q = (0,0,-1) \quad .$$

Then,

$$K(p,q) = \frac{e^{iwr}(iwr-1)}{r^2} \frac{q-p}{r} \cdot N_q$$

where, here $r = |p-q|$. Let A be a bound for $|e^{iwr}(iwr-1)|$. Then,

(15) $\quad |K(p,q)| \leq (A/r^2)\ |\cos(r_{pq}, N_q)| \leq A/r^2 \quad ,$

(16) $\quad |K(p,q)| \leq (A/r^3)\ \ t \sin\alpha \quad .$

Similarly,

$$K^t(p,q) = K(q,p) = \frac{e^{iwr}(iwr-1)}{r^2} \frac{p-q}{r} \cdot N_p \quad .$$

Hence,

(17) $\quad |K^t(p,q)| \leq (A/r^2)\ |\cos(r_{pq}, N_p)| \leq A/r^2 \quad ,$

$\quad |K^t(p,q)| \leq (A/r^3)\ \ y \sin\alpha \quad .$

Let $a^2 = (t\cos\alpha - y)^2 + (t\sin\alpha)^2$: then $r^2 = x^2 + a^2$. We have the same bound A/r^2 for both K and K^t so the following calculations hold for either operator.

(18) $$I_4(m) \leq A \int_{B\cap F_2} \frac{|\phi(q)|}{r^2}\, d\sigma(q)$$

$$\leq A \iint_{B\cap F_2} \frac{|\phi(q)|}{x^2 + a^2}\, dxdy \quad .$$

Let $R_L = B \cap (S_{L+1} - S_L)$ as shown in Figure 3.4. For p in $S_m \cap F_1$, let $I(m,L)$ be the integral over the strip R_L .

We estimate it as follows:

$$I(m,L) \leq A \cdot \text{width of } R_L \cdot \|\phi\|_{L+1} \cdot J$$

where

$$J = 2 \int_0^{2D} \frac{1}{x^2+a^2}\, dx \quad ,$$

$a^2 = (t \cos\alpha - y^*)^2 + (t \sin\alpha)^2$, y^* is that value of y between $2^{-(L+1)}$ and 2^{-L} for which the integral is a maximum, and $2D$ is the radius of B .

$$J = \frac{2}{a} \arc\tan \frac{x}{a}\Big]_0^{2D}$$

$$< \frac{\pi}{a} \quad ,$$

and width of $R_L \leq c\ (1/2^{L+1})$ where c is a positive constant; therefore

$$I(m,L) < A\ c\ \pi\ \frac{\|\phi\|_{L+1}}{2^{L+1}}\ \frac{1}{a} \quad .$$

Rather than sum over L corresponding to $q \in B \cap F_2$, we use the larger bound

$$I_4(m) \leq \sum_{L=1}^{\infty} I(m,L) \quad .$$

Now examine T_4 . Change index, letting $s = L+1$.

$$\text{(19)} \qquad T_4 = \sum_{m=u+1}^{\infty} 2^{-m}\ I_4^2(m) \leq \sum_{m=1}^{\infty} 2^{-m}\left[\sum_{L=1}^{\infty} I(m,L)\right]^2$$

$$\leq (Ac\pi)^2 \sum_{m=1}^{\infty} 2^{-m}\left[\sum_{s=2}^{\infty} \frac{\|\phi\|_s}{2^s}\ \frac{1}{a}\right]^2 \quad .$$

To proceed further we need to make a case study depending on estimates for a^2. Recall,

$$a^2 = (t\cos\alpha - y^*)^2 + (t\sin\alpha)^2 .$$

Note that a is the distance from p in F_1 to q^* in F_2 where $p = (0, t\cos\alpha, t\sin\alpha)$ and $q^* = (0, y^*, 0)$. $p \in S_m$ so $2^{-m} < t$ and $q^* \in R_{L+1} = S_{L+1} - S_L$ so $2^{-(L+1)} < y^* < 2^{-L}$.

To distinguish the forthcoming cases, and to establish the validity of the estimates which come from the geometry, please see Figures 3.5 and 3.6 on page 88.

Case I. $90^\circ \leq \alpha < 180^\circ$

$$a \geq \sqrt{(2^{-m}\cos\alpha - 2^{-(L+1)})^2 + (2^{-m}\sin\alpha)^2} .$$

Case II. $0^\circ < \alpha < 90^\circ$

We split this case into two subcases:

IIa). For $(1/2^L) \leq (1/2^m)\cos\alpha$

$$a \geq \sqrt{(2^{-m}\cos\alpha - 2^{-L})^2 + (2^{-m}\sin\alpha)^2} .$$

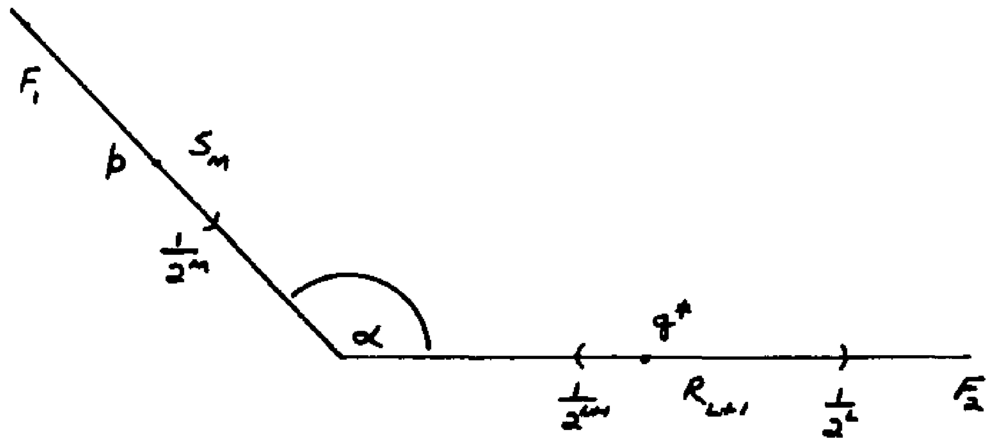
IIb). For $(1/2^L) > (1/2^m)\cos\alpha$

$$a \geq (1/2^{L+1})\sin\alpha .$$

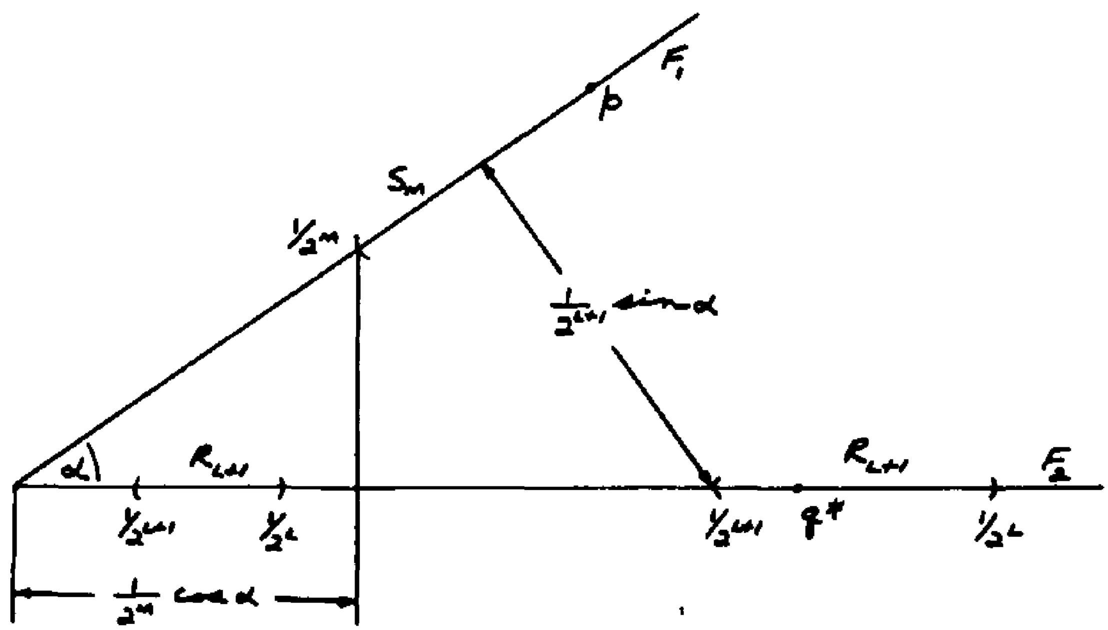
Case I. Rewrite the square of the sum over s appearing in (19) as a double sum,

$$(20) \quad T_4 \leq (Ac\pi)^2 \sum_{m=1}^{\infty} 2^{-m} \sum_{s=2}^{\infty} \sum_{n=2}^{\infty} \frac{\|\phi\|_s}{2^s a(m,s)} \cdot \frac{\|\phi\|_n}{2^n a(m,n)} .$$

<u>Figure 3.5</u> Case I $90^\circ \leq \alpha < 180^\circ$



<u>Figure 3.6</u> Case II $0^\circ < \alpha < 90^\circ$



Let G be the general term of the double sum and rewrite it as follows:

$$G = \frac{\|\phi\|_s}{2^{3s/4+n/4}\sqrt{a(m,s)a(m,n)}} \cdot \frac{\|\phi\|_n}{2^{3n/4+s/4}\sqrt{a(m,s)a(m,n)}}$$

$$\leq \frac{1}{2}\left[\frac{\|\phi\|_s^2}{2^s 2^{(s+n)/2}\, a(m,s)a(m,n)} + \frac{\|\phi\|_n^2}{2^n 2^{(s+n)/2}\, a(m,s)a(m,n)}\right] .$$

The last line follows from the inequality: $ab \leq \frac{1}{2}(a^2+b^2)$.

When we substitute this bound for G, we obtain two sums which differ only in notation for indices. These can be combined. With some rearrangement and change of order we have:

$$(21) \quad T_4 \leq (Ac\pi)^2 \sum_{s=2}^{\infty} \frac{\|\phi\|_s^2}{2^s} \sum_{m=1}^{\infty} \frac{1}{2^{(m+s)/2} a(m,s)} \sum_{n=1}^{\infty} \frac{1}{2^{(m+n)/2} a(m,n)} .$$

Call the sum over s ; S_1 and the sum over n ; S_2 .

Using the estimate of a for Case I, we have

$$S_1 \leq \sum_{m=1}^{\infty} \frac{1}{2^{(m+s)/2}\sqrt{(2^{-m}\cos\alpha - 2^{-s})^2 + (2^{-m}\sin\alpha)^2}}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{\sqrt{(2^{(s-m)/2}\cos\alpha - 2^{(m-s)/2})^2 + (2^{(s-m)/2}\sin\alpha)^2}} .$$

Change index, letting $j = m-s$.

$$(22) \quad S_1 \leq \sum_{j=1-s}^{\infty} \frac{1}{\sqrt{(2^{-j/2}\cos\alpha - 2^{j/2})^2 + (2^{-j/2}\sin\alpha)^2}} .$$

Extend the lower limit to $-\infty$, which makes the bound independent of s . Let D_j be the expression under the root sign above. Upon squaring and adding, we find

(23) $$D_j = 2^{-j} - 2\cos\alpha + 2^{j} \ .$$

For $90^\circ \leq \alpha \leq 180^\circ$, $-2\cos\alpha \geq 0$. $D_o = 2(1-\cos\alpha)$, and for all non zero j , $D_j > 2^{|j|}$. Hence, combining the sums over positive and negative j , we have, independent of s ,

$$S_1 < \frac{1}{\sqrt{2(1-\cos\alpha)}} + 2\sum_{j=1}^{\infty} \frac{1}{2^{j/2}}$$
$$< \frac{1}{\sqrt{2(1-\cos\alpha)}} + \frac{2}{\sqrt{2}-1} \ .$$

The same bound holds for S_2, independent of m , so from (21) we get

$$T_4 \leq (Ac\pi)^2 \left[\frac{1}{\sqrt{2(1-\cos\alpha)}} + \frac{2}{\sqrt{2}-1}\right]^2 \|\phi\|_w^2 \ .$$

This completes Case I.

<u>Case II</u>. The range for Case IIa), $2^{-L} \leq 2^{-m}\cos\alpha$, can be expressed as $2^m \leq 2^L\, 2^{\log_2\cos\alpha}$. Since $\log_2\cos\alpha$ is negative, denote it $-b$. Then $m \leq L - [b]$ or equivalently $m + [b] \leq L$. We can split the sum (19) that bounds T_4 as follows:

$$T_4 \leq (Ac\pi)^2 \sum_{m=1}^{\infty} 2^{-m}\left[\sum_{L=1}^{m+[b]-1} \frac{\|\phi\|_{L+1}}{2^{L+1}}\frac{1}{a} + \sum_{L=m+[b]}^{\infty} \frac{\|\phi\|_{L+1}}{2^{L+1}}\frac{1}{a}\right]^2 \ .$$

Call the first sum in the parentheses S_1 (which corresponds to Case IIb) and the second S_2 (which corresponds to Case IIa). By (2), for $n = 2$, we obtain

$$(24) \qquad T_4 \leq 2(Ac\pi)^2 \left[\sum_{m=1}^{\infty} 2^{-m} S_1^2 + \sum_{m=1}^{\infty} 2^{-m} S_2^2 \right] .$$

Call the first sum S_1' and the second S_2'.

$$S_2' = \sum_{m=1}^{\infty} 2^{-m} \sum_{L=m+[b]}^{\infty} \frac{\|\phi\|_{L+1}}{2^{L+1}} \frac{1}{a} .$$

Now use the estimate of a for Case IIa) and extend the lower index to $L = 1$.

$$S_2' \leq \sum_{m=1}^{\infty} 2^{-m} \sum_{L=1}^{\infty} \frac{\|\phi\|_{L+1}}{2^{L+1}} \frac{1}{\sqrt{(2^{-m}\cos\alpha - 2^{-L})^2 + (2^{-m}\sin\alpha)^2}} .$$

Multiply the denominator by $2^1 \cdot 2^{-1}$ to get

$$S_2' \leq \sum_{m=1}^{\infty} 2^{-(m+1)} \sum_{L=1}^{\infty} \frac{\|\phi\|_{L+1}}{2^{L+1}} \frac{1}{\sqrt{(2^{-m-1}\cos\alpha - 2^{-L-1})^2 + (2^{-m-1}\sin\alpha)^2}} .$$

Change index, letting $s = L+1$ and replacing $m+1$ by m; then

$$S_2' \leq \sum_{m=2}^{\infty} 2^{-m} \sum_{s=2}^{\infty} \frac{\|\phi\|_s}{2^s} \frac{1}{\sqrt{(2^{-m}\cos\alpha - 2^{-s})^2 + (2^{-m}\sin\alpha)^2}} .$$

The sum here is bounded by that in Case I, (19), (20), which leads to (21), hence

$$(25) \quad S_2' \leq \sum_{s=2}^{\infty} \frac{\|\phi\|_s^2}{2^s} \sum_{m=1}^{\infty} \frac{1}{2^{(m+s)/2} a(m,s)} \sum_{n=1}^{\infty} \frac{1}{2^{(m+n)/2} a(m,n)} .$$

Denote the sum over m as S. Just as before, (22), (23), we obtain

$$S \leq \sum_{j=-\infty}^{\infty} \frac{1}{\sqrt{2^{-j} - 2\cos\alpha + 2^{j}}} ,$$

only now $\cos\alpha \geq 0$. Let D_j denote the expression under the root sign. $D_o = 2(1-\cos\alpha)$, $D_1 = D_{-1} = (5/2)-2\cos\alpha \geq 1/2$, and for $j \geq 2$,

$$D_j = 2^{j-1}(2 + 2^{-2j+1} - 2^{2-j}\cos\alpha) > 2^{j-1} .$$

Therefore, splitting off the terms for D_o, D_1, D_{-1}, and combining the sum for $|j| \geq 2$, we have, independent of s ,

$$S \leq \frac{1}{\sqrt{2(1-\cos\alpha)}} + 2^{3/2} + 2 \sum_{j=2}^{\infty} \frac{1}{2^{(j-1)/2}}$$

$$\leq \frac{1}{\sqrt{2(1-\cos\alpha)}} + 2^{3/2} + \frac{2}{\sqrt{2} - 1} .$$

The same bound holds for the sum over n in (25), independent of m , so from (25) we get

$$(26) \qquad S_2' \leq \left[\frac{1}{\sqrt{2(1-\cos\alpha)}} + \frac{6 - 2\sqrt{2}}{\sqrt{2} - 1}\right]^2 \cdot \|\phi\|_W^2 .$$

Now consider S_1' , (24). Using the estimate

$a \geq (1/2^{L+1}) \sin\alpha$, for Case IIb), and letting $s = L+1$, we obtain

$$S_1' \leq \sum_{m=1}^{\infty} 2^{-m} \left[\sum_{s=2}^{m+[b]} \frac{\|\phi\|_s}{2^s} \frac{2}{\sin\alpha}\right]^2$$

$$\leq \frac{1}{\sin^2\alpha} \sum_{m=1}^{\infty} 2^{-m} \sum_{s=2}^{m+[b]} \sum_{n=2}^{m+[b]} \|\phi\|_s \|\phi\|_n .$$

Rewrite the general term of the double sum thus:

$$\|\phi\|_s \, \|\phi\|_n = \frac{\|\phi\|_s}{2^{(s-n)/4}} \, \frac{\|\phi\|_n}{2^{(n-s)/4}} \leq \frac{1}{2}\left[\frac{\|\phi\|_s^2}{2^{(s-n)/2}} + \frac{\|\phi\|_n^2}{2^{(n-s)/2}}\right].$$

When we substitute this into the bound for S_1', we get two sums which differ only in the letters used for the indices. Combining these, and changing the order, we get

$$S_1' \leq \frac{1}{\sin^2\alpha} \sum_{s=2}^{\infty} \frac{\|\phi\|_s^2}{2^s} \sum_{m=s-[b]}^{\infty} \frac{1}{2^{(m-s)/2}} \sum_{n=2}^{m+[b]} \frac{1}{2^{(m-n)/2}} \quad .$$

Change index, letting $j = m-s$ and $k = n-m$.

$$S_1' \leq \frac{1}{\sin^2\alpha} \sum_{s=2}^{\infty} \frac{\|\phi\|_s^2}{2^s} \sum_{j=-[b]}^{\infty} \frac{1}{2^{j/2}} \sum_{k=2-m}^{[b]} \frac{1}{2^{k/2}} \quad .$$

Each of the last two sums is bounded by

$$\sum_{j=-\infty}^{\infty} \frac{1}{2^{j/2}} \leq \frac{\sqrt{2}+1}{\sqrt{2}-1} \quad ,$$

hence

(27) $$S_1' \leq \frac{1}{\sin^2\alpha}\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]^2 \|\phi\|_W^2 \quad .$$

Substituting (26) and (27) into (24) yields

$$T_4 \leq 2(Ac\pi)^2\left[\frac{1}{\sin^2\alpha}\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]^2 + \left[\frac{1}{\sqrt{2(1-\cos\alpha)}} + \frac{6-2\sqrt{2}}{\sqrt{2}-1}\right]^2\right]\|\phi\|_W^2 \quad .$$

This completes Case II and Part 2b) Case 1.

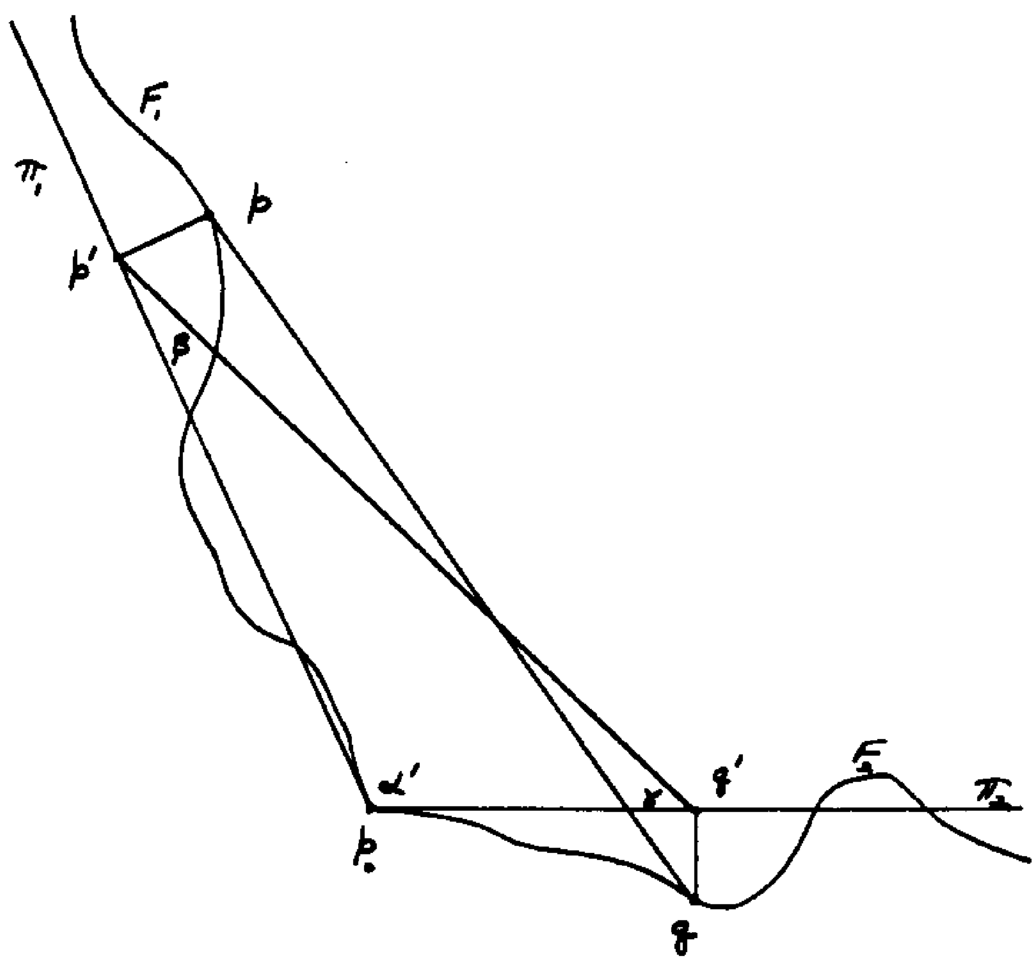
The idea of the next part of the proof is to show that $|K\phi(p)|$, for a surface with general faces, is bounded by a constant times the integral bounding $|K\phi(p)|$ in the case just treated, in which the surface is planar near the edges.

Part 2b). Case 2. No more than two faces meet at any edge of Σ . Everything is as in Case 1 except that the adjacent faces F_1 and F_2 need not be planar anywhere. We need the following preparations.

Let p_o be any edge point, not a corner, of an arbitrary face F . By hypothesis the normal vector at any interior point p of face F approaches a limiting value N as p approaches p_o from within F . Let $\pi(p_o,F)$ denote the plane perpendicular to N at p_o . This plane possesses the usual properties of a tangent plane for points in face F . Now, denote by p_o the nearest edge point to p . The adjacent faces at p_o , F_1 and F_2 , have tangent planes $\pi(p_o,F_1)$ and $\pi(p_o,F_2)$. Call these simply π_1 and π_2 .

Let p' be the orthogonal projection of p onto π_1 and q' the orthogonal projection of q onto π_2 . See Figure 3.7 on page 95. Let α be the dihedral angle between π_1 and π_2 . Let α' be the angle $p' p_o q'$, β the angle $p_o p' q'$, and γ the angle $p_o q' p'$. Denote $|p'-p_o|$ by t and $|p_o-q'|$ by ρ . From the law of sines in triangle $p' p_o q'$, and $0^\circ < \alpha', \beta, \gamma < 180^\circ$, we obtain:

<u>Figure 3.7</u>



$$\frac{\rho}{r_{p'q'}} = \frac{\sin\beta}{\sin\alpha'} \le \frac{1}{\sin\alpha'} \quad ; \qquad \rho \le \frac{1}{\sin\alpha'}\, r_{p'q'}$$

$$\frac{t}{r_{p'q'}} = \frac{\sin\gamma}{\sin\alpha'} \le \frac{1}{\sin\alpha'} \quad ; \qquad t \le \frac{1}{\sin\alpha'}\, r_{p'q'} \; .$$

From Lemma 2.1, and the above relations, we obtain

$$r_{pp'} \le ct^2 \le \frac{c}{\sin^2\alpha'}\, r^2_{p'q'} \quad ; \qquad r_{qq'} \le c\,\rho^2 \le \frac{c}{\sin^2\alpha'}\, r^2_{p'q'} \; ,$$

where c is a positive constant. Now, observe that

$$\begin{aligned} |r_{pq} - r_{p'q'}| &\le r_{pp'} + r_{qq'} \\ &\le \frac{2c}{\sin^2\alpha'}\, r^2_{p'q'} \\ &\le \frac{1}{2}\, r_{p'q'} \; . \end{aligned}$$

The last line follows for ball B, hence $r_{p'q'}$, sufficiently small. Therefore,

$$\frac{1}{2}\, r_{p'q'} \le r_{pq} \le \frac{3}{2}\, r_{p'q'} \; .$$

This, together with (15), gives

$$|K(p,q)| \le \frac{A}{r^2_{pq}} \le \frac{4A}{r^2_{p'q'}} \; .$$

Suppose, $q = f(q')$ is the local representation of F_2 with respect to π_2. Then, for an appropriate positive constant c :

$$\int_{F_2} |\phi(q)|\, |K(p,q)|\, d\sigma(q) \le c \int_{\pi_2} |\phi(f(q))|\, \frac{4A}{r^2_{pq}}\, d\sigma(q') \; .$$

The type of integral on the right was treated in Part 2b) Case 1 (18). It leads to a contribution to $\|K\phi\|_W^2$ that is bounded.

Part 2b). Case 3. Σ is an arbitrary C^2 manifold with corners. Let p_o be a corner point of Σ and B a ball of radius $2D$ about p_o. For $m > u$, let p be an arbitrary point in $(S_m - S_u) \cap B$. Denote the face containing p by F_1 and the adjacent faces at p_o by $F_2, F_3, \ldots, F_n$. Split the integral over $(\Sigma - F_1) \cap B$ and write it symbolically thus:

$$\int_{(\Sigma - F_1)\cap B} |\phi(q)||K(p,q)|\, d\sigma(q) = \int_{F_2 \cap B} + \int_{F_3 \cap B} + \ldots + \int_{F_n \cap B} \quad .$$

Let the supremums over $p \in (S_m - S_u) \cap B$ of each of the integrals on the right be denoted by $H_2(m), \ldots, H_n(m)$. T_4 denotes their contribution to $\|K\phi\|_W^2$ (cf. (8)).

$$T_4^2 = \sum_{m=u+1}^{\infty} 2^{-m} \left[H_2(m) + H_3(m) + \ldots + H_n(m)\right]^2$$

$$\leq n\left[\sum_{m=u+1}^{\infty} 2^{-m} H_2^2(m) + \ldots + \sum_{m=u+1}^{\infty} 2^{-m} H_n^2(m)\right]$$

by inequality (2). Each of the sums in the parentheses was shown to be bounded, by the arguments of Part 2b) Case 2. Letting $c_j \|\phi\|_W^2$ denote the bounds, we have

$$T_4^2 \leq n \sum_{j=1}^{n} c_j \|\phi\|_W^2 \leq c_4 \|\phi\|_W^2 \quad ,$$

where $c_4 = n \sum_{j=1}^{n} c_j$. This is the last estimate we needed to show. The boundedness of T_4^2 was used in the argument leading to the boundedness of $\|K\phi\|_W^2$, (14). This finishes the proof of Theorem 3.1.

3.3 Compactness of K and K^t for Σ smooth.

Theorem 3.2 As a specialization of the C^2 manifold Σ (cf. Section 1.2), suppose all of Σ is smooth (C^2). The faces of Σ are now sections. Then, K and K^t are compact operators on the space $W(\Sigma)$.

Proof. We will work with K throughout the proof. The demonstration holds word for word with K^t in place of K .

Let $S = \{f_j\}$ be a bounded sequence of functions, specifically,

(1) $$\|f_j\|_W \leq c \quad .$$

We must show that the sequence of functions $\{Kf_j\}$ is compact, that is, there is a convergent subsequence $\{Kf_n\}$ in the W^2 norm. We already know that $\{Kf_j\} \subset W(\Sigma)$ by Theorem 3.1.

From (1), $c^2 \geq \|f_j\|_W^2 = \sum_{m=1}^{\infty} 2^{-m} \|f_j\|_m^2$, we obtain the bound

(2) $$\|f_j\|_m \leq 2^{m/2} c \quad .$$

From (3) of Lemma 3.1, we obtain, letting r denote the constant kc, a bound on the L^1 norm of f_j:

$$\|f_j\| \leq k\ \|f_j\|_W \leq r \quad . \tag{3}$$

By Lemma 2.5, for $f \in C^0 \cap L^1(\Sigma)$, $\|Kf\|_m$ is bounded as follows:

$$\|Kf\|_m \leq a\|f\|_{m+1}\epsilon + \frac{b}{\epsilon^2}\|f\| \quad ,$$

where a, b, and $\epsilon(m)$ are positive constants. Replacing f by f_j and using the bounds (2) and (3), we have

$$\|Kf_j\|_m \leq a\ 2^{(m+1)/2}c\epsilon + \frac{b}{\epsilon^2} r \quad . \tag{4}$$

Therefore, $\|Kf_j\|_m$ is uniformly bounded for all $f_j \in S$.

By Lemma 2.19, for Σ smooth (C^2) and $f \in W(\Sigma)$, the function Kf is Hölder continuous on S_m, as follows:

for p_1, p_2 in S_m and $|p_1-p_2| \leq 1/2^{m+1}$,

$$|Kf(p_1)-Kf(p_2)| \leq (c_3\|f\|_{m+1} + c_4\|f\|_W + c_5\|f\|)\,|p_1-p_2|^{1/4}, \tag{5}$$

where the constants c_3, c_4, c_5 are independent of f and m. Replacing f by f_j then using (1), (2), and (3) we have

$$\begin{aligned}|Kf_j(p_1)-Kf_j(p_2)| &\leq (c_3 2^{(m+1)/2}c + c_4 c + c_5 r)\,|p_1-p_2|^{1/4} \\ &\leq a_1\ 2^{(m+1)/2}\,|p_1-p_2|^{1/4} \quad ,\end{aligned} \tag{6}$$

where a_1 is a positive constant. This shows that the sequence of functions $\{Kf_j\}$ is equicontinuous on S_m.

This, together with the uniform boundedness of $\{Kf_j\}$ on S_m, (4), yields by Ascoli's theorem the existence of a subsequence of $\{Kf_j\}$ uniformly convergent on S_m. Denote the subsequence by $\{Kf\}_m$. This is true for each $m = 1,2,3,\ldots$.

Construct a diagonal sequence as follows. Let Kf_1 be the first function of $\{Kf\}_1$, Kf_2 the second function of $\{Kf\}_2$ and so on. Denote the resulting sequence by $\{Kf_n\}$. It is convergent on each S_m for $m = 1,2,3,\ldots$ in the corresponding norm. Let g be the function to which $\{Kf_n\}$ converges. By the sup. norm convergence on S_m, $g \in C^o(S_m)$. This is true for each S_m, hence $g \in C^o(\dot{\Sigma})$. The convergence of Kf_n to g in each m norm implies Cauchy convergence on each set S_m. That is, given an ϵ, there is an $N(m)$ depending on m, such that, for n, m greater than $N(m)$, $\|Kf_n - Kf_k\|_m < \epsilon$, where Kf_n and Kf_k are functions from $\{Kf_n\}$. Introduce the notation $h = f_n - f_k$. From (2) we obtain the bound

$$\|h\|_m \leq \|f_n\|_m + \|f_k\|_m \leq 2\ 2^{(m+1)/2}c$$

Similarly, $\|h\| \leq 2r$ and $\|h\|_W \leq 2c$. Using these bounds in (5) produces, where $a_2 = 2a_1$,

(7) $$|Kh(p_1) - Kh(p_2)| \leq a_2 \, 2^{(m+1)/2} |p_1 - p_2|^{1/4} .$$

Now we start a development that will show Cauchy convergence of $\{Kf_n\}$ in the W^2 norm. First we need to increase the allowable distance between p_1, p_2 in (5), (6), (7) which hold for p_1, p_2 in S_m and $|p_1 - p_2| \leq 1/2^{m+1}$.

Let s_1, s_4 be points in S_m and $|s_1 - s_4| \leq 1/2^m$. Let s_2, s_3 be intermediate points on the arc (shortest path) $s_1 s_4$ so that $|s_j - s_{j+1}| \leq (1/2)|s_1 - s_4|$ for $j=1,2,3$ (For m sufficiently large these points exist.) Then,

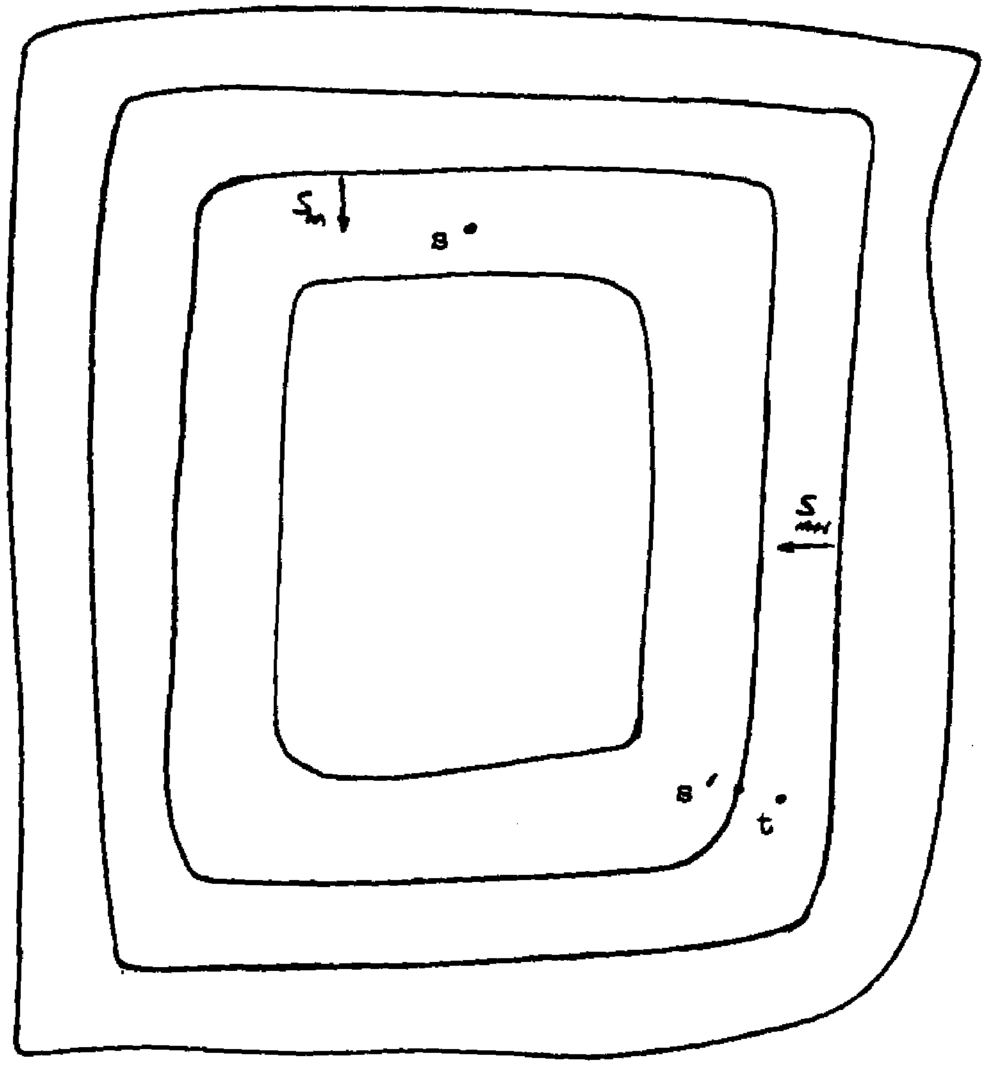
$$\begin{aligned} |Kh(s_1) - Kh(s_2)| &\leq \sum_{j=1}^{3} |Kh(s_j) - Kh(s_{j+1})| \\ &\leq \sum_{j=1}^{3} a_2 \, 2^{(m+1)/2} |s_j - s_{j+1}|^{1/4} \\ &\leq a_2 \, 2^{(m+1)/2} \, 3(1/2)^{1/4} |s_1 - s_4|^{1/4} . \end{aligned}$$

Let $a_3 = a_2 \, 3 \, 2^{1/4}$. Thus, we obtain for p_1, p_2 in S_m and $|p_1 - p_2| \leq 1/2^m$,

(8) $$|Kh(p_1) - Kh(p_2)| \leq a_3 \, 2^{m/2} \, |p_1 - p_2|^{1/4} .$$

To find an estimate for the growth of $\|Kh\|_m$ as m increases, we will investigate the geometry of the system of sets $\{S_m\}$. Figure 3.8 on page 102 illustrates a face of Σ. Because S_m is closed, $\|Kh\|_m = \sup_{p \in S_m} |Kh(p)|$ is

Figure 3.8 A face of Σ .



achieved at some point $s \in S_m$. Similarly, $\|Kh\|_{m+1}$ is achieved at some point $t \in S_{m+1}$. Note that s and t may be on different faces. If $t \in S_m$ then $\|Kh\|_m$ equals $\|Kh\|_{m+1}$, hence estimate (11), that we are working towards, is certainly satisfied. Otherwise, and more importantly, assume $t \in S_{m+1} - S_m$. Let s' be the point in S_m closest to t . Because s' is also in S_{m+1} , we can use the Hölder continuity (8) to obtain

(9) $$|Kh(t)-Kh(s')| \leq a_3 \, 2^{(m+1)/2} |t-s'|^{1/4} .$$

Since, $|Kh(s')| \leq |Kh(s)| \leq |Kh(t)|$, it follows that

(10) $$|Kh(t)| - |Kh(s)| \leq |Kh(t)| - |Kh(s')| .$$

Using (9), (10), the definitions of s and t , and $|t-s'| \leq$ width of $S_{m+1} = 1/2^{m+1}$, we have

(11) $$\begin{aligned} \|Kh\|_{m+1} - \|Kh\|_m &\leq a_3 \, 2^{(m+1)/2} \, (1/2^{m+1})^{1/4} \\ &\leq a_3 \, 2^{(m+1)/4} . \end{aligned}$$

Equivalently,

$$\|Kh\|_{m+1} \leq \|Kh\|_m + a_3 \, 2^{(m+1)/4} .$$

Then, by induction, we obtain a growth relation:

(12) $$\begin{aligned} \|Kh\|_{m+j} &\leq \|Kh\|_m + a_3 \, 2^{m/4} \left[2^{1/4} + \dots + 2^{j/4} \right] \\ &\leq \|Kh\|_m + a \, 2^{(m+j)/4} , \end{aligned}$$

where $a = (a_3 \, 2^{1/2})/(2^{1/4}-1)$.

Now consider the W^2 norm of Kh :

$$\|Kh\|_W^2 = \sum_{m=1}^{\infty} 2^{-m} \|Kh\|_m^2 \quad .$$

Let r be a positive integer that we will choose later, and split the sum.

$$\|Kh\|_W^2 = \sum_{m=1}^{r} 2^{-m}\|Kh\|_m^2 + \sum_{j=1}^{\infty} 2^{-(r+j)}\|Kh\|_{r+j}^2 \quad .$$

Call the first sum T_1 and the second sum T_2 .

$$T_2 = \sum_{j=1}^{\infty}\left[2^{-(r+j)/2}\|Kh\|_{r+j}\right]^2 \leq \left[\sum_{j=1}^{\infty} 2^{-(r+j)/2}\|Kh\|_{r+j}\right]^2 \quad .$$

Call the sum inside the parentheses T_3 , and use the growth relation (12) to estimate it.

$$T_3 \leq \sum_{j=1}^{\infty} 2^{-(r+j)/2}\left[\|Kh\|_r + a\ 2^{(r+j)/4}\right]$$

$$\leq 2^{-r/2}\|Kh\|_r \sum_{j=1}^{\infty} 2^{-j/2} + a\ 2^{-r/4} \sum_{j=1}^{\infty} 2^{-j/4}$$

$$\leq c_1\ 2^{-r/2}\|Kh\|_r + a\ c_2\ 2^{-r/4} \quad ,$$

where c_1, c_2 denote the sums of the geometric series.

Now we will show that $\|Kh\|_W^2$, which is the same as $\|Kf_n - Kf_k\|_W^2$, is arbitrarily small, for n, k large enough. Given an ϵ , choose r large enough that $ac_2\ 2^{-r/4} < \epsilon$. With r now fixed, choose n, k sufficiently large that $c_1\ 2^{-r/2}\|Kh\|_r < \epsilon$, and $\|Kh\|_r < \epsilon$. Then $T_2 \leq T_3^2 \leq 4\,\epsilon^2$, and by the nature of the sup. norm $\|Kh\|_m < \epsilon$ for every $m \leq r$, so $T_1 < \epsilon^2 \sum_{m=1}^{r} 2^{-m} < \epsilon^2$. Gathering the estimates, we obtain $\|Kh\|_W^2 < 5\epsilon^2$. This completes the proof that $\{Kf_n\}$ is a Cauchy sequence in the W^2 norm. By completeness (Lemma 3.2) there is a function $g \in W(\Sigma)$ such that $\|Kf_n - g\|_W \longrightarrow 0$. This must be the same function g to which the diagonal sequence converges, because convergence in the W^2 norm implies convergence in each m norm. This completes the proof of Theorem 3.2.

3.4 Normalization of Q, K, K^t. Fredholm Alternative.

We wish to normalize Q (hence K and K^t). This normalization will be in effect here and in Chapter 5. Let $Q = (e^{iwr})/2\pi r$. Hence, the new operators K, K^t equal the old operators K, K^t divided by 2π.

By Theorem 3.2, for Σ smooth (C^2), K and K^t are compact operators on $W(\Sigma)$, so the Fredholm Alternative (F.A.) [17] holds for the pair of equations:

A. $\quad v = (\pm I + K)u$ and $X = (\pm I + K^*)Y$

where $v, u \in W(\Sigma)$ and $X, Y \in W(\Sigma)^*$.

The F.A. also holds for the pair of equations:

B. $\quad v = (\pm I + K^t)u$ and $X = (\pm I + K^{t*})Y$.

The next lemma will provide us with a practical criterion for solvability of the pair of eguations A. For an operator T let $N(T)$ denote its nullspace.

<u>Lemma 3.3</u> $N(I+K^t) = N(I+K^*)$.

<u>Proof</u>. Let $a(q)$, $b(q)$ denote functions in $W(\Sigma)$. By Lemma 3.1 they are in $L^2(\Sigma)$, so $\int_\Sigma a(q)b(q)d\sigma(q)$ exists and defines a pairing $\langle a,b\rangle$. $b\in W(\Sigma)$ can be thought of as a linear functional $a \mapsto \langle a,b\rangle$ acting on the function $a\in W(\Sigma)$.

<u>Claim 1</u>. K^t can be considered as an operator on some of the functionals of $W(\Sigma)^*$, the conjugate of $W(\Sigma)$. $\langle a,K^t b\rangle$ agrees with $K^* b(a)$, and

$$\dim N(I+K^t) \leq \dim N(I+K^*) \tag{1}$$

<u>Proof of the claim</u>. The definition of K^*b is $K^*b(a) = b(Ka)$. Now observe that:

$$\begin{aligned} \langle Ka,b\rangle &= \int_\Sigma \left[(Ka)(p)\right] b(p)\, d\sigma(p) \\ &= \int_\Sigma \left[\int_\Sigma a(q)K(p,q)\, d\sigma(q)\right] b(p)\, d\sigma(p) \\ &= \int_\Sigma \int_\Sigma a(q)b(p)K(p,q)\, d\sigma(q)\, d\sigma(p) \\ &= \int_\Sigma \left[\int_\Sigma b(p)K(p,q)\, d\sigma(p)\right] a(q)\, d\sigma(q) \\ &= \int_\Sigma \left[(K^t b)(q)\right] a(q)\, d\sigma(q) \\ &= \langle a,K^t b\rangle . \end{aligned} \tag{2}$$

We justify the change of order of integration as follows. By Theorem 3.1 for $a\in W(\Sigma)$, $\int_\Sigma |a(q)||K(p,q)|\, d\sigma(q) \in W(\Sigma)$, hence the function defined by the integral is a square integrable function of p, by Lemma 3.1. Therefore,

$$\int_{\Sigma} \int_{\Sigma} |a(q)||K(p,q)| \, d\sigma(q) \, |b(p)| \, d\sigma(p)$$

exists. This is sufficient to justify the interchange of order of integration in (2) (consequence of Fubini's Theorem [6], p.362)

We don't know that all the linear functionals in $W(\Sigma)^*$ can be realized by the pairing $\langle a,b \rangle$, but we can conclude that $N(I+K^t) \subseteq N(I+K^*)$, and hence

(3) $$\dim N(I+K^t) \leq \dim N(I+K^*) \quad .$$

<u>Claim 2</u>. K can be considered as an operator on some of the functionals of $W(\Sigma)^*$, $\langle a,Kb \rangle$ agrees with $K^{t*}b(a)$, and

(4) $$\dim N(I+K) \leq \dim N(I+K^{t*}) \quad .$$

The proof is similar to that given for Claim 1.

The F.A. [17] applied to the pair of equations A yields

(5) $$\dim N(I+K^*) = \dim N(I+K) \quad ,$$

and F.A. applied to the pair of equations B yields

(6) $$\dim N(I+K^{t*}) = \dim N(I+K^t) \quad .$$

Therefore,

$$\dim N(I+K^*) = \dim N(I+K) \leq \dim N(I+K^{t*}) = \dim N(I+K^t)$$

by (5), (4), and (6), hence

$$\dim N(I+K^*) \leq \dim N(I+K^t) \quad .$$

This, together with (3), implies

$$\dim N(I+K^t) = \dim N(I+K^*) \quad .$$

Since the domain of $I+K^t$ is a subset of the domain of $I+K^*$, and $\dim N(I+K^t) = \dim N(I+K^*)$, we conclude that $N(I+K^t) = N(I+K^*)$.

By replacement of I by $-I$ throughout the proof of Lemma 3.5 we obtain the following result.

<u>Corollary 3.3</u> $N(-I+K^t) = N(-I+K^*)$.

The F.A. applied to the pair of equations A , together with Corollary 3.3, gives the last result of the chapter.

<u>Lemma 3.4</u> For Σ smooth (C^2) and the operators K, K^t defined on $W(\Sigma)$, $v = (-I+K)u$ has a solution u , if $\langle v,Y \rangle = 0$, for all solutions Y of $0 = (-I+K^t)Y$.

Chapter Four

K, K^t on the spaces $L^\infty(\Sigma)$, $L^1(\Sigma)$

4.1 Boundedness of operator K on $L^\infty(\Sigma)$.

Lemma 4.1 There is a positive constant c such that

$$\int_\Sigma |K(p,q)|\, d\sigma(q) \le c$$

for all $p \in \dot{\Sigma}$.

Proof. We will consider three cases.

Case 1. p is near an edge but not a corner.

Case 2. p is not near an edge nor a corner.

Case 3. p is near a corner.

Case 1. Let p_o be an edge point of Σ. By hypothesis the normal vector at $p \in \dot{\Sigma}$ approaches a limiting value N as $p \to p_o$ from within a face F. Let $\pi(p_o,F)$ denote the plane perpendicular to N at p_o. This plane possesses the usual properties of a tangent plane for points in face F. Let F_1 and F_2 be the adjacent faces at p_o and $\pi(p_o,F_1)$, $\pi(p_o,F_2)$ the corresponding planes. From here on, call them simply π_1 and π_2.

Let B be a ball of radius h about p_o and let p be a point in $F_1 \cap B$. By Lemma 2.1, the distance by which a point p in F_1 departs from its projection onto plane π_1 is $O(t^2)$ where t is the distance from p to p_o.

Similarly, for a point $q \in F_2 \cap B$ at distance s from p_o, $d(q, \pi_2) = O(s^2)$. Figure 4.1 on page 111 illustrates the situation. Let α denote the angle between π_1 and π_2. Note that $0 < \alpha < \pi$ and region R in Fig. 4.1 may be in either the interior or exterior of Σ.

Figure 4.2 on page 111 illustrates the special case, which we will handle first, where the faces are planar, that is, $F_1 = \pi_1$ and $F_2 = \pi_2$. Let $T = \pi_1 \cap \pi_2$. Consider a rectilinear coordinate system with xy plane equal to π_2, the x axis parallel to T and the z axis through p. In this coordinate system, the outward normal vector to π_2 is $N_q = (0,0,-1)$, and the vector $r = q-p$ is given by $(x,y,-t\sin\alpha)$.

$$K(p,q) = \frac{e^{iwr}(iwr-1)}{r^3}\, r \cdot N_q \quad .$$

Let

$$F(p) = \int_{B\cap\pi_2} |K(p,q)|\, d\sigma(q) \quad ,$$

which we prefer to write in polar coordinates, where $\rho^2 = x^2+y^2$. Let $A(r) = e^{iwr}(iwr-1)$ and let A be a bound on $|A(r)|$.

$$F(p) = \iint_{B\cap\pi_2} \frac{A(r)\; t\,\sin\alpha}{(\rho^2+t^2\sin^2\alpha)^{3/2}}\, \rho\, d\rho\, d\theta$$

$$\leq A\, |t\sin\alpha| \int_0^{2\pi}\!\!\int_0^{2h} \frac{\rho\, d\rho\, d\theta}{(\rho^2+t^2\sin^2\alpha)^{3/2}}$$

Figure 4.1

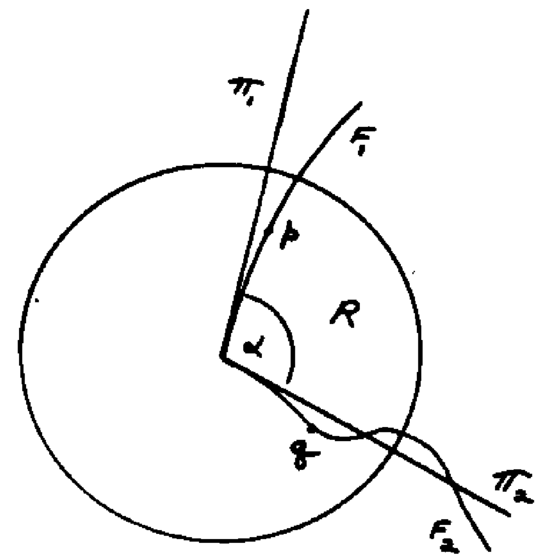
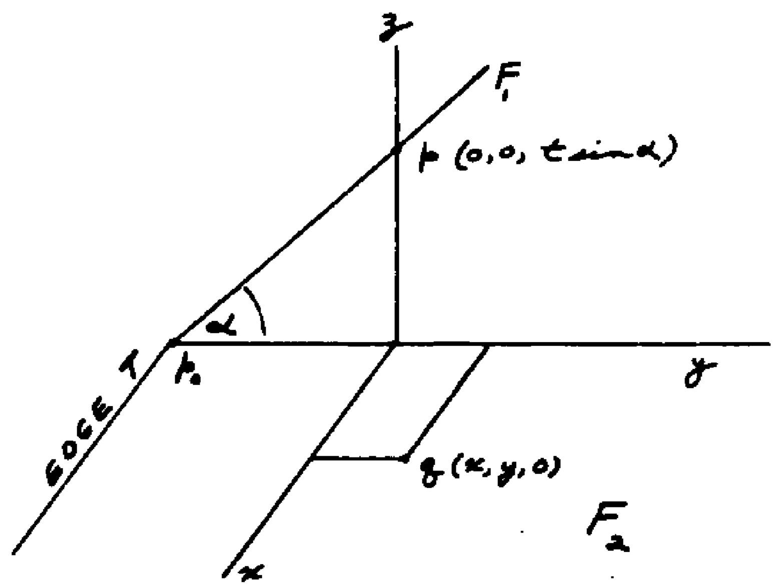


Figure 4.2 F_1 is seen on edge.



$$F(p) \leq 2\pi A\,|t\sin\alpha|\left[\frac{1}{(t^2\sin^2\alpha)^{1/2}} - \frac{1}{(4h^2+t^2\sin^2\alpha)^{1/2}}\right]$$

$$\leq 2\pi A\left[1 - \frac{1}{\left[\frac{4h^2}{t^2\sin^2\alpha} + 1\right]^{1/2}}\right]$$

$$\leq 2\pi A \quad .$$

Note that Fig. 4.2 shows α between 0 and $\pi/2$ but the estimates hold for $0 < \alpha < \pi$.

Now, we consider F_1 and F_2 not necessarily planar. We are still treating $F(p) = \int_{B\cap F_2} |K(p,q)|\,d\sigma(q)$ for $p \in F_1 \cap B$. As stated earlier in the proof $d(p,\pi_1) = O(t^2)$ and $d(q,\pi_2)$ has order equal to the square of $d(p_0,q)$, that is, $d(q,\pi_2) = O(x^2+(y+t\cos\alpha)^2)$. Based on these estimates we write

(1) $$r = (x,y,t\sin\alpha + O(t^2) + O(x^2+(y+t\cos\alpha)^2)) \quad .$$

From Lemma 2.3, the unit normal vector at q can be expressed as

$$N_q = (O(d(p_0,q)),\ O(d(p_0,q)),\ O(1))$$

where

$$d(p_0,q) = \sqrt{x^2 + (y+t\cos\alpha)^2}$$

$$\leq |x| + |y| + |t\cos\alpha| \ .$$

Also,

$$x^2 + (y+t\cos\alpha)^2 = \rho^2 + 2yt\cos\alpha + t^2\cos^2\alpha \quad .$$

Now calculate $r \cdot N_q$ in terms of orders of magnitude. We use the fact that x and y are of order ρ, combine terms of the same order, and absorb terms of higher order into those of lower order to obtain

$$r \cdot N_q = O(\rho^2) + O(\rho)O(t) + O(t) \quad .$$

Now examine r^2 . From (1) we write

$$r^2 = \rho^2 + \left[t\sin\alpha + O(\rho^2) + O(t^2) + O(\rho)O(t)\right]^2$$
$$= (\rho^2 + t^2\sin^2\alpha)(1 + O(t) + O(\rho) + O(\rho)O(t)) \quad .$$

Because t and ρ are bounded and independent, we absorb the $O(\rho)O(t)$ term into $O(t) + O(\rho)$. Next, use the definition of the order notation from which we can then find a single positive constant c , such that

$$r^2 \geq (\rho^2 + t^2\sin^2\alpha)(1 - ct - c\rho) \quad .$$

Remark: Because h is chosen small at the outset, t and ρ are small, hence the factor $1 - ct - c\rho$ will be positive.

Now consider $F(p)$, for t restricted to lie between 0 and $1/3c$. Also restrict the domain of integration to $0 < \rho < 1/3c$. Call the resulting integral $F_1(p)$, and observe that $1 - ct - c\rho \geq 1/3$, so that

$$F_1(p) \leq \int_0^{2\pi}\int_0^{2h} \frac{O(t)+O(t)O(\rho)+O(\rho^2)}{\left[\frac{1}{3}(\rho^2+t^2\sin^2\alpha)\right]^{3/2}}\rho\, d\rho\, d\theta \quad .$$

Split off the integral corresponding to the first two terms of the numerator. These contain $O(t)$ and contribute a bounded integral by essentially the same argument as in the planar case. The integral corresponding to the third term will be bounded because it has a numerator of order ρ^3

which is not exceeded by the order of the denominator.

To proceed further we recall two earlier results. We state them in a form suitable to the present context.

I. Let p be in Σ. Let E be an element of surface of radius ϵ about p (Def. 1.2) in which the surface is smooth (C^2) then by Lemma 2.4, there is a constant a such that $\int_E |K(p,q)|\, d\sigma(q) \leq a\epsilon$. Because p may be arbitrarily close to an edge, E may be truncated at the edge. This does not affect the estimate.

II. Let D be a domain of integration in which the distance $r = |q-p|$ is bounded away from zero. By Lemma 2.2, $\int_D |K(p,q)|\, d\sigma(q)$ is bounded.

Now we will finish up $F(p)$. From (1) we see that for $\rho \geq 1/3c$ and for any t, r^2 is bounded away from zero, so by II the corresponding part of $F(p)$ is bounded.

To complete Case 1 ($p \in B \cap F_1$) observe that I takes care of the integration for $q \in E \supset (B \cap F_1)$ and II applies to q outside $B \cup E$, hence $F(p)$ is bounded.

<u>Case 2</u>. Here, p is in a face and the distance of p from the nearest edge point is greater than h. I and II imply the boundedness of $F(p)$.

<u>Case 3</u>. Suppose n faces $F_1, F_2, \ldots, F_n$ meet at a vertex p_0. Let B be a ball of radius h about p_0 and let p be an arbitrary point in $B \cap F_1$. The arguments in Case 1

apply to q in $B \cap F_j$ for $j = 2,\ldots,n$ as well as to $q \in B \cap F_1$ or $q \in \Sigma - B$. Hence, we conclude that $F(p)$ is bounded.

<u>Lemma 4.2</u> K is a bounded operator on the space $L^{\infty}(\Sigma)$.

<u>Proof</u>.

$$\|K\phi(p)\|_{\infty} = \operatorname*{ess\,sup}_{p \in \Sigma} |K\phi(p)|$$
$$\leq \operatorname*{ess\,sup}_{p \in \Sigma} \int_{\Sigma} |\phi(q)||K(p,q)|\, d\sigma(q)$$
$$\leq \|\phi\|_{\infty} \operatorname*{ess\,sup}_{p \in \Sigma} \int_{\Sigma} |K(p,q)|\, d\sigma(q)$$
$$\leq \|\phi\|_{\infty}\, c.$$

The last line follows from Lemma 4.1.

4.2 <u>P.C. Surfaces</u>.

<u>Definition 4.1</u> P.C. Surfaces.

Let Σ now be polyhedral or cylindrical in the following sense. In the neighborhood of any corner, Σ is part of a polyhedron for which the angle between adjacent faces is at least 90°. In the neighborhood of any edge point that is not a corner, Σ is part of a right circular cylinder or one of the above polyhedrons. Call these P.C. surfaces. Examples: a box, cylinder, or pipe.

The P.C. surfaces are more limited than we would like. It seems clear that the next lemma would hold for somewhat more general surfaces. The definitive class of such surfaces is a topic for futher investigation.

Let S_h be the set of points of Σ that have distance at least h from the set of edge points of Σ.

Lemma 4.3 For any P.C. surface, there is an $h_1 > 0$, so that, for all $p \in \dot{\Sigma}$ and all $h < h_1$,

$$I(p) = \int_{\Sigma - S_h} |K(p,q)| \, d\sigma(q) < 2\pi \quad .$$

Proof. At first, consider only two faces, F_1 containing p, and an adjacent face F_2. See Figure 4.3 on page 117. Let T be the edge in common to F_1 and F_2. F_1 can be non planar. F_2 is planar in the neighborhood of T. Let p_o be an edge point closest to p. Suppose, for now, that p is in $\Sigma - S_h$, that is, $d(p_o,p) < h$. Let B_1, B_2 be balls about p_o with radii $2h$ and D respectively, where $2h < D = 1/20\,|w|$. To avoid excessive detail, the estimates below will tend to be generous. By Lemma 2.4, there are constants A_1, A_2 so that

$$I_1 = \int_{B_1 \cap F_1} |K(p,q)| \, d\sigma(q) \leq A_1 \; 4h \quad ,$$

and, letting $S = F_1 \cap (B_2 - B_1) \cap (\Sigma - S_h)$

$$I_2 = \int_S |K(p,q)| \, d\sigma(q) \leq \int_S \frac{A_2}{r_{pq}} \, d\sigma(q) \quad .$$

Figure 4.3

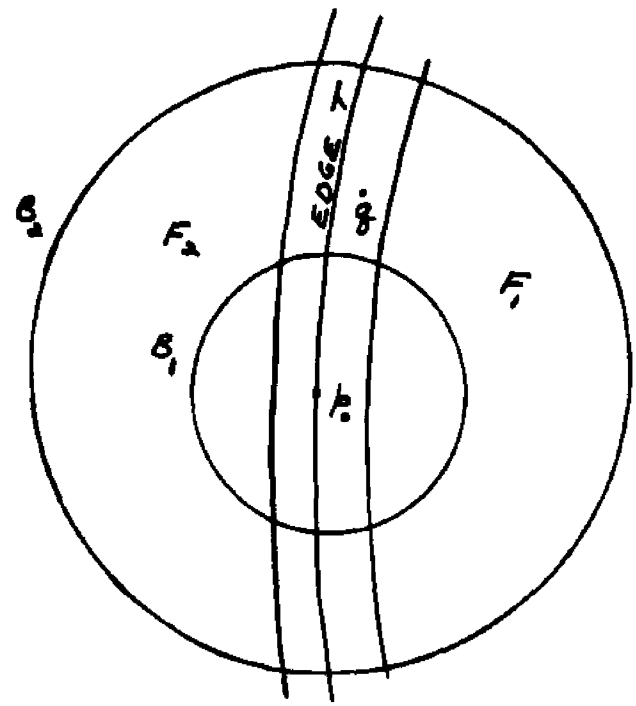
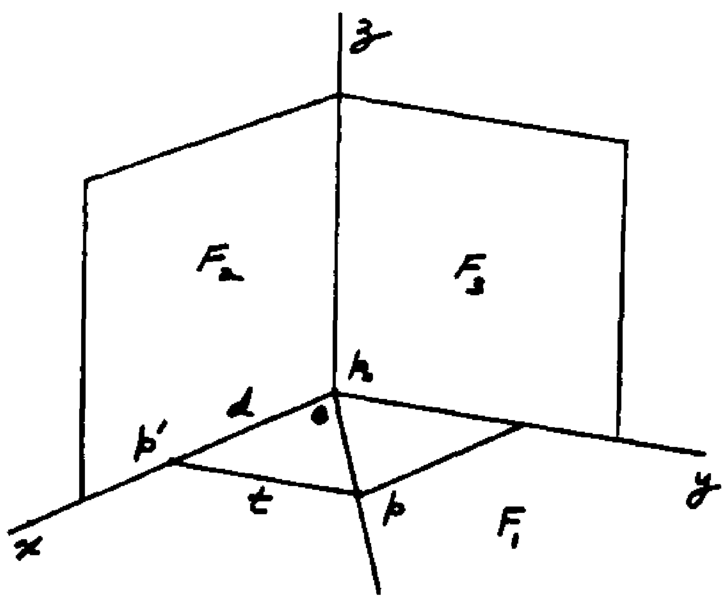


Figure 4.4



Let s represent arc length along the edge T, measured from p_0. For $q \in B_2 - B_1$, there are positive constants c_1, c_2 so that $c_1 s < r_{pq} < c_2 s$, and a constant c_3 so that width of $\Sigma - S_h < c_3 h$. Therefore, there is a constant c such that

$$I_2 \leq \left[2\int_h^{2D} \frac{c}{s}\, ds\right]\left[\text{width of } \Sigma - S_h\right]$$

$$\leq 2c(\log 2D - \log h)c_3 h \quad .$$

Now, with q no longer restricted to F_1,

$$I_3 = \int_{\Sigma - B_2} |K(p,q)|\, d\sigma(q) \leq \int_{\Sigma - B_2} \frac{A}{r^2}\, d\sigma(q)$$

$$\leq \frac{A}{D^2}\,(\text{area of } \Sigma - B_2) \leq \frac{A}{D^2}\, c_4 h$$

where $c_4 = (\text{length of the edges of } \Sigma)\, c_3$.

Next, consider the integral

$$I_4 = \int_{B_2 \cap F_2} |K(p,q)|\, d\sigma(q) \leq A \int_{B_2 \cap F_2} \frac{t\sin\alpha}{r_{pq}^3}\, d\sigma(q) \quad .$$

The estimate follows from (16) of Theorem 3.1. See Fig. 3.3 and the displayed calculations on page 85.

$$I_4 \leq A\, t\sin\alpha \int_0^D \int_{-D}^D \frac{1}{(x^2+a^2)^{3/2}}\, dx\, dy$$

where $a^2 = (y - t\cos\alpha)^2 + (t\sin\alpha)^2$.

$$\int_0^D \frac{1}{(x^2+a^2)^{3/2}}\, dx = \left.\frac{x}{a^2(x^2+a^2)^{1/2}}\right]_0^D$$

$$= \frac{1}{a^2}\, \frac{1}{\left[1 + \frac{a^2}{D^2}\right]^{1/2}} < \frac{1}{a^2} \quad .$$

Therefore,

$$I_4 \leq A\ t\sin\alpha\ 2\int_0^D \frac{1}{(y-t\cos\alpha)^2 + (t\sin\alpha)^2}\,dy$$

$$\leq 2A\ t\sin\alpha \frac{1}{t\sin\alpha} \text{arc tan}\left(\frac{y-t\cos\alpha}{t\sin\alpha}\right)\Big]_0^D$$

$$\leq 2A\left[\frac{\pi}{2} - \text{arc tan}(-\cot\alpha)\right]$$

$$\leq 2A\left[\frac{\pi}{2} - \text{arc tan}(-\tan(\frac{\pi}{2} - \alpha))\right] \quad .$$

In B_2 , $|e^{iwr}(iwr-1)| \leq 1 + |w|2D$, which serves in place of A , hence $I_4 \leq 2(1 + |w|2D)(\pi - \alpha)$.

Gathering the estimates for I_1 through I_4 we have

$$I(p) \leq 4A_1h + 2c\left[\log 2D - \log h\right]c_3h + Ah\frac{c_4}{D^2} + I_4 \quad .$$

By hypothesis, $\alpha \geq \pi/2$. Since $D = 1/20|w|$, we obtain $I_4 \leq 2(11/10)(\pi/2) = 1.1\pi$. After that, h can be chosen small enough to make the first three terms less than $\pi/10$. Then

(1) $$I(p) \leq .1\pi + I_4$$
$$< 2\pi \quad .$$

If $h \leq d(p_o,p) < 1/20|w|$, by Lemma 2.4,

$$I_1 = \int_{B_1}|K(p,q)|\,d\sigma(q) \leq \int_{E_2-E_1}|K(p,q)|\,d\sigma(q) \leq 4A_1h$$

where E_1 and E_2 are elements of surface with center at p and radii $d(p,B_1)$, $d(p_o,p) + 2h$ respectively. The rest of the estimates go through as before yielding (1).

If $1/20|w| \leq d(p_o,p)$, then choose $D < 1/40|w|$, so that $d = d(p, \Sigma - S_h) \geq 1/40\,|w|$. Then by Lemma 2.2,

$$I(p) = \int_{\Sigma - S_h} |K(p,q)|\, d\sigma(q) \leq \int_{\Sigma - S_h} \frac{A}{r^2}\, d\sigma(q)$$

$$\leq \frac{A}{d^2} \text{ area of } (\Sigma - S_h)$$

$$\leq (40\,|w|)^2 A\, c_4\, h \quad .$$

h can be chosen small enough that (1) again holds.

For p near a corner, let p_o be the corner point. B_1, B_2 about p_o now include parts of Σ on two adjacent faces F_2, F_3. The integral I_4 (for $p \in F_1$) can be written

$$I_4 = \int_{B_2 \cap F_2} |K(p,q)|\, d\sigma(q) + \int_{B_2 \cap F_3} |K(p,q)|\, d\sigma(q)$$

$$= \int_0^D \int_0^D |K(p,q)|\, dy\, dz + \int_0^D \int_0^D |K(p,q)|\, dx\, dz \quad .$$

Call the integrals on the right $I_{4,2}$ and $I_{4,3}$ respectively. We will carry out the proof for $B_2 \cap \Sigma$ being part of a cube with corner p_o. See Figure 4.4 on page 117. The proof is similar when the angle between adjacent pairs of faces at the corner is greater than $90°$. The axes are set up so that $B \cap F_1 \subset xy$ plane, $B \cap F_2 \subset xz$ plane, and $B \cap F_3 \subset yz$ plane. Consider p on a ray making angle θ with the x axis and q in $B \cap F_2$. The argument will be similar to that already given for just two faces.

Let $p = (d,t,0)$, $q = (x,0,z)$ and

$$a = \sqrt{(z - t\cos\alpha)^2 + (t\sin\alpha)^2} ,$$

where α denotes the angle between F_1 and F_2 . (Later we will set $\alpha = \pi/2$.) Observe that

$$a/d \geq t/d = \tan\theta .$$

Now set up new axes X,Y,Z parallel to axes x,y,z respectively, but with origin $p' = (d,0,0)$. Then,

$$I_{4,2} \leq A\, t\sin\alpha \int_0^D \int_{-d}^{D-d} \frac{1}{(X^2+a^2)^{3/2}}\, dX\, dZ .$$

$$\int_{-d}^{D-d} \frac{1}{(X^2+a^2)^{3/2}}\, dX = \left. \frac{X}{a^2(X^2+a^2)^{1/2}} \right]_{-d}^{D-d}$$

$$= \frac{1}{a^2}\left[\left[1 + \frac{a^2}{(D-d)^2}\right]^{-1/2} + \left[1 + \frac{a^2}{d^2}\right]^{-1/2}\right]$$

$$\leq \frac{1}{a^2}\left[1 + (1+\tan^2\theta)^{-1/2}\right] .$$

The rest of the integration is like that of I_4 for just two faces. Thus with simplification,

$$I_{4,2} \leq (1 + 2|w|D)(\pi - \alpha)(1 + \cos\theta) .$$

The same procedure for $I_{4,3}$ gives

$$I_{4,3} \leq (1 + 2|w|D)(\pi - \alpha)(1 + \sin\theta) .$$

Therefore, for $\alpha = \pi/2$, $0 \leq \theta \leq \pi/2$, and $D = 1/20|w|$,

$$I_4 \leq (1 + 2|w|D)(\pi/2)(2 + \cos\theta + \sin\theta)$$
$$\leq 1.1\pi(1 + \sqrt{2}/2) < 1.88\pi .$$

This bound again yields the validity of inequality (1) for $I(p)$.

Now consider p near the edge of a right circular cylinder. p in the curved portion F_2 has been treated already. Consider p in the planar part F_1 . We need only consider, for $p \in B_2 \cap F_1$,

$$I_4 = \int_{B_2 \cap F_2} |K(p,q)| \, d\sigma(q) \quad .$$

Set up a rectilinear coordinate system with origin at the center of the circle of radius R which is the base of the cylinder. The x axis goes through p and the z axis is the axis of the cylinder. Let p_o be the point where the x axis intersects the edge of the cylinder. The intersection of the ball B_2 of radius D about p_o with the curved face F_2 corresponds to the ranges $-k \le \theta \le k$ and $0 \le z \le D$, where $k = \tan^{-1}(D/R)$. Suppose $p \in F_1$ is given by $p = (t,0,0)$. $q \in F_2$ is given by $q = (x,y,z)$, vector $r_{pq} = (x-t,y,z)$ and vector $N_q = (1/R)(x,y,0)$. By (4) of Lemma 2.2, letting $A(r) = e^{iwr}(iwr-1)$, we have

$$K(p,q) = \frac{A(r)}{r_{pq}^3} r_{pq} \cdot N_q = \frac{A(r)}{r^3} \; \frac{R^2-tx}{R} \quad .$$

Use $x^2+y^2 = R^2$, $x = R\cos\theta$ and let A be a bound on $|A(r)|$.

Then,

$$I_4 \leq \frac{A}{R} \int_0^D \int_{-k}^{k} \frac{R^2 - tR\cos\theta}{(R^2 - 2tR\cos\theta + t^2 + z^2)^{3/2}} R \, d\theta \, dz$$

Change variables, letting $s = t/R$, $u = z/R$, and $U = D/R$.

$$I_4 \leq 2A \int_0^k \int_0^U \frac{1 - s\cos\theta}{(u^2 + s^2 + 1 - 2s\cos\theta)^{3/2}} \, du \, d\theta$$

Let $a^2 = s^2 + 1 - 2s\cos\theta$. Upon integrating, we see that

$$\int_0^U (u^2 + a^2)^{-3/2} \, du \leq 1/a^2 \quad .$$

Therefore,

$$I_4 \leq 2A \int_0^k \frac{1 - s\cos\theta}{s^2 + 1 - 2s\cos\theta} \, d\theta \quad .$$

Call the integral above, I .

$$I = \int_0^k \frac{d\theta}{(s^2+1) - 2s\cos\theta} - s \int_0^k \frac{\cos\theta}{(s^2+1) - 2s\cos\theta} \, d\theta$$

$$= \frac{\theta}{2} + \frac{(1-s^2)}{2} \int_0^k \frac{d\theta}{(s^2+1) - 2s\cos\theta}$$

$$= \frac{\theta}{2} + \frac{(1-s^2)}{2} \left[\frac{2}{s^2-1} \tan^{-1}\left(\frac{(s^2-1)\tan(\theta/2)}{(s-1)^2} \right) \right]_0^k$$

$$= \frac{\theta}{2} - \tan^{-1}\left(\frac{s+1}{s-1} \tan\frac{\theta}{2} \right) \Big]_0^k$$

$$\leq \frac{k}{2} + \frac{\pi}{2} \quad .$$

This holds for $0 \leq s \leq 1$, or equivalently $0 \leq t \leq R$.

Hence,

$$I_4 \leq 2AI = A(k + \pi) \quad .$$

Choose $D \leq \mathrm{Min}(1/20|w|, R\tan(\pi/10))$. Then, for p in B_2, $|A(r)| \leq 1+|w|2D \leq 1.1$, and $k = \tan^{-1}(D/R) \leq \pi/10$, therefore

$$I_4 \leq 1.1\,(.1\pi + \pi) = 1.21\pi \quad .$$

This, together with the estimates for q outside $B_2 \cap F_2$, again yields, as in (1), $I(p) < 2\pi$. The proof of Lemma 4.3 is now complete.

4.3 <u>Projection operator</u> π_h, <u>compactness of</u> $\pi_h \bullet K$ <u>on</u> <u>$L^\infty(\Sigma)$</u>.

Recall that S_h is the set of points of Σ at distance at least h from the set of edge points of Σ .

<u>Definition 4.2</u>

$$\begin{aligned} \pi_h\,\phi(p) &= \phi(p) \quad \text{for} \quad p \in S_h \\ &= 0 \quad \text{for} \quad p \in \Sigma - S_h \quad . \end{aligned}$$

<u>Lemma 4.4</u> If Σ is a P.C. surface then, for all sufficiently small h ,

$$\|K - \pi_h \bullet K\|_\infty < 2\pi \quad .$$

<u>Proof</u>. Because, $K\phi = \pi_h \bullet K\phi$ on S_h, and $\pi_h \bullet K\phi = 0$ on $\Sigma - S_h$, we have

$$\begin{aligned} \|(K - \pi_h \bullet K)\phi\|_\infty &= \operatorname*{ess\,sup}_{p \in \Sigma} |(K - \pi_h \bullet K)\phi(p)| \\ &\leq \operatorname*{ess\,sup}_{p \in \Sigma} \int_{\Sigma - S_h} |\phi(q)||K(p,q)|\,d\sigma(q) \end{aligned}$$

$$\|(K - \pi_h \circ K)\phi\|_\infty \leq \|\phi\|_\infty \operatorname*{ess\,sup}_{p \in \Sigma} \int_{\Sigma - S_h} |K(p,q)|\, d\sigma(q)$$
$$< \|\phi\|_\infty \; 2\pi \; .$$

The last line follows from Lemma 4.3.

<u>Lemma 4.5</u> Given any $h > 0$, $\pi_h \circ K$ is a compact operator on the space $L^\infty(\Sigma)$.

<u>Proof</u>. h is fixed. Let $\{f_n\}$ denote a bounded sequence of functions of $L^\infty(\Sigma)$, $\|f_n\|_\infty \leq b$, where b is a positive constant. We prepare to use Lemma 2.19. Let d be the minimum distance between the components of S_h . A ball B_2 of radius $d/2$ about any point p of S_h will contain only points of S_h on the same face as p . Let $\|f_n\|$ denote the L^1 norm of f_n . Let a denote the area of Σ . Then $\|f_n\| \leq ba$. Lemma 2.19 yields, for $|p_1 - p_2| < d/4$,

$$|\pi_h \circ Kf_n(p_1) - \pi_h \circ Kf_n(p_2)| \leq \left(c_1 b + c_2 \frac{ba}{(d/2)^2}\right) |p_1 - p_2|^{1-\epsilon} ,$$

where c_1 and c_2 are positive constants independent of p and f_n , and ϵ is an arbitrarily small positive constant. This shows the equicontinuity of $\{\pi_h \circ Kf_n(p)\}$ on $\dot{\Sigma}$. From Lemma 4,2, $\|\pi_h \circ Kf_n\|_\infty \leq \|Kf_n\|_\infty \leq bc$. By Ascoli's theorem, there is a uniformly convergent subsequence of $\{\pi_h \cdot Kf_n\}$. $L^{\infty}(\Sigma)$ is known to be complete, so the lemma is proved.

4.4 A condition leading to the Fredholm Alternative.

The following lemma is due to R. Leis [8] .

Lemma 4.6 Let B be a Banach space and B^* its conjugate space. Let T be a linear transformation on B and T^* its conjugate transformation on B^* . If there exists a compact linear transformation $\bar{T}$ in B with $\|T-\bar{T}\| < 1$, then the Fredholm Alternative (F.A.) is valid for the pair of equations:

$$v = (I + T)u \quad \text{and} \quad X = (I + T^*)Y \quad ,$$

where I is the identity transformation. Specifically,

1). If $(v,Y) = 0$ for every solution Y of $0 = (I + T^*)Y$, then $v = (I + T)u$ has a solution u .

2). If $(X,u) = 0$ for every solution u of $0 = (I + T)u$, then $X = (I + T^*)Y$ has a solution Y .

Proof. Let $H = T - \bar{T}$. Let $G = (I + H)^{-1}$. This inverse exists and is given by the series $G = (I + H)^{-1} = I-H+H^2-H^3+\ldots$ which converges because $\|H\| < 1$.

Claim. The following are equivalent equations:

(1) $\quad v = (I + T)u$

(2) $\quad Gv = v_1 = (I + G\bar{T})u$.

<u>Proof of the claim</u>. Given

$$v = (I + T)u = (I + H + \bar{T})u$$
$$= (I + H)(I + G\bar{T})u \quad ,$$

apply G . Let $Gv = v_1$.

$$Gv = v_1 = G(I + H)(I + G\bar{T})u$$
$$= (I + G\bar{T})u \quad ,$$

which proves the claim.

Now observe, from the series expression for G , that G is bounded, which together with $\bar{T}$ compact, implies that $G\bar{T}$ is also compact. Therefore, the F.A. [17] holds for the pair of equations:

(2) $$Gv = v_1 = (I + G\bar{T})u$$

(3) $$L = (I + (G\bar{T})^*)M \quad .$$

<u>Claim</u>. The following are equivalent equations:

(3) $$L = (I + (G\bar{T})^*)M$$

(4) $$L = (I + T^*)G^*M \quad .$$

<u>Proof of the claim</u>. First note that $G^* = (I + H)^{-1*} = (I + H^*)^{-1}$, and $T^* = (H + \bar{T})^* = H^* + \bar{T}^*$, then

$$L = (I + (G\bar{T})^*)M = (I + \bar{T}^*G^*)M$$
$$= (I + \bar{T}^*G^*)(I + H^*)G^*M$$
$$= (I + H^* + \bar{T}^*)G^*M$$
$$= (I + T^*)G^*M \quad ,$$

which proves the claim.

Now we will prove the two assertions of the lemma.

1). Suppose $(v,Y) = 0$ for every solution Y of $0 = (I + T^*)Y$. Let $y = G^{*-1}Y$ so $Y = G^*y$ then $(Gv,y) = (v,G^*y) = (v,Y) = 0$ for every solution $G^*M = G^*y$ of homogeneous (4), $0 = (I + T^*)G^*M$, which is equivalent to $(Gv,y) = 0$ for every solution y of homogeneous (3), $0 = (I + (G\mathbb{T})^*)y$, which by the F.A. for compact operators [17], applied to (2), (3), implies there is a solution u of (2), $Gv = v_1 = (I + G\mathbb{T})u$, which is equivalent to u being a solution of (1), $v = (I + T)u$.

2). The proof is similar to that of 1), based on statement 2) in the F.A. for compact operators ([17], p.191).

<u>Corollary 4.6</u> Under the hypotheses of Lemma 4.6, the F.A. holds for the pair of equations:

$$v = (-I + K)u \quad \text{and} \quad X = (-I + K^*)Y \quad .$$

<u>Proof</u>. Follow the same steps as in Lemma 4.6, using

$$G = (-I + H)^{-1} = -(I - H)^{-1} = -(I + H + H^2 + \dots) \quad .$$

4.5 <u>Boundedness of K^t</u>. <u>Compactness of $K^t \cdot \pi_h$ on $L^1(\Sigma)$</u>.

<u>Lemma 4.7</u> $\pi_h \cdot K = (K^t \cdot \pi_h)^*$.

<u>Proof</u>. All linear functionals on $L^1(\Sigma)$ can be represented by the pairing $\langle f,g \rangle = \int_\Sigma f(p)g(p)\, d\sigma(p)$, where

$f \in L^1(\Sigma)$ and $g \in L^\infty(\Sigma)$ [14]. We will think of $\langle f,g \rangle$ as the linear functional g acting on f. The definition of $(K^t \circ \pi_h)^*$ is $\langle (K^t \circ \pi_h)f,g \rangle = \langle f,(K^t \circ \pi_h)^* g \rangle$, but

$$\begin{aligned}
\langle (K^t \circ \pi_h)f,g \rangle &= \int_\Sigma (K^t \circ \pi_h f(p))g(p)\, d\sigma(p) \\
&= \int_\Sigma \int_{S_h} f(q)K(q,p)\, d\sigma(q)\, g(p)\, d\sigma(p) \\
&= \int_{S_h} f(q) \int_\Sigma g(p)K(q,p)\, d\sigma(p)\, d\sigma(q) \\
&= \langle f,(\pi_h \circ K)g \rangle .
\end{aligned}$$

Therefore, $\pi_h \circ K = (K^t \circ \pi_h)^*$.

<u>Corollary 4.7</u> $K = (K^t)^*$.

<u>Lemma 4.8</u> On $L^1(\Sigma)$, K^t is a bounded operator and for arbitrary $h > 0$, $K^t \circ \pi_h$ is a compact operator.

<u>Proof</u>. The boundedness of K^t comes from Lemma 4.2, Corollary 4.7, and the general fact that T^* bounded, implies that T is bounded, [6] p. 233. The compactness of $K^t \circ \pi_h$ comes from Lemmas 4.5, 4.7, and the general fact that T^* compact, implies that T is compact, [17] p. 187.

<u>Lemma 4.9</u> If Σ is a P.C. surface, then for all sufficiently small h , the L^1 norm of $K^t - K^t \circ \pi_h$ is bounded as follows: $\|K^t - K^t \circ \pi_h\| < 2\pi$.

<u>Proof</u>.

$$\|K^t - K^t \circ \pi_h\| = \|(K^t - K^t \circ \pi_h)^*\|_\infty$$
$$= \|K - \pi_h \circ K\|_\infty$$
$$< 2\pi \; .$$

This follows, because the norm of an operator and its adjoint are equal, [6] p. 233, $(K^t - K^t \circ \pi_h)^* = K - \pi_h \circ K$ from Lemma 4.7 and its corollary, and Lemma 4.4.

4.6 <u>Normalization of Q, K, K^t</u>. <u>Fredholm Alternative</u>.

In order to use Lemma 4.6, we will normalize Q (hence K and K^t) as follows. Let $Q(p,q) = (e^{iwr})/2\pi r$. We will continue with this normalization from here on. In terms of the new operator K^t (equal to the original operator K^t divided by 2π), from Lemma 4.9 we obtain, for P.C. surfaces and h sufficiently small, $\|K^t - K^t \circ \pi_h\| < 1$. From this, by Lemmas 4.6 and 4.8, we obtain the concluding result of this chapter.

<u>Lemma 4.10</u> If Σ is an arbitrary P.C. surface, then the Fredholm Alternative is valid for the equations:

$$v = (\pm I + K^t)u \quad \text{and} \quad X = (\pm I + K)Y \quad ,$$

where v and u are in $L^1(\Sigma)$ and X and Y are in $L^\infty(\Sigma)$.

Chapter Five

Existence Proof

5.1 Introduction and set up for existence proof.

In this chapter we will find a solution u(p) , of the Neumann problem in R^3 , in the exterior Σ_+ , of a C^2 manifold with corners Σ , (cf. Chapter 1.). Let E be the set of edges of Σ and $\dot{\Sigma} = \Sigma - E$. A statement of the problem is:

1). A function g defined on $\dot{\Sigma}$ is given.

2). Find a function u(p) , defined in Σ_+ , that satisfies:

a). the reduced wave equation,

$$\Delta u + w^2 u = 0 \quad \text{in } \Sigma_+ \quad (w \text{ is a real constant.})$$

b). $u_N = g$ on $\dot{\Sigma}$,

c). the Sommerfeld radiation condition:

$$(\partial u/\partial r) - iwu = o(r^{-1}) \quad ,$$

uniformly along all rays from the origin.

We will attempt to find a solution u(p) that is the difference of a single layer potential v(p) , with density the same as the given Neumann data function g(q) , and a double layer potential w(p) , with a density function h(q) , yet to be determined. Specifically, let

$$v(p) = \int_\Sigma g(q)Q(p,q)\, d\sigma(q) \quad ,$$

$$w(p) = \int_\Sigma h(q)K(p,q)\, d\sigma(q) \quad ,$$

where $Q = (e^{iwr})/2\pi r$, and let

$$u(p) = \frac{1}{2}\left[w(p) - v(p)\right] \quad .$$

Both $v(p)$ and $w(p)$ satisfy 2a) and 2c), so the problem is to find h so that

$$u_{N_+}(p) = \frac{1}{2}\left[w_{N_+}(p) - v_{N_+}(p)\right] = g(p) \quad .$$

Here, $u_{N_+}(p)$, for example, means $\lim(\partial u/\partial N)(p_+)$ as $p_+ \rightarrow p \in \dot{\Sigma}$ from Σ_+ .

Introduce the following notation. Let L.B. denote the set of integrable functions on Σ that are locally essentially bounded on $\dot{\Sigma}$, (Def.2.1, p.15). The functions of $W(\Sigma)$ (cf. Chapter 3.) or $L^{\infty}(\Sigma)$ are contained in L.B. . Within the set of integrable functions on Σ , let H denote the set of locally Hölder continuous functions on $\dot{\Sigma}$ (Def.2.3, p.42), and H' the set of functions with locally Hölder continuous first partial derivatives on $\dot{\Sigma}$. For p in $\dot{\Sigma}$ we regard Q and K as operators and write $v(p) = Qg(p)$ and $w(p) = Kh(p)$.

We will find a solution to the exterior Neumann problem under either of the following hypotheses:

1). All of Σ is smooth (C^2) and $g \in W(\Sigma)$, (cf. Chapter 3.).

2). Σ is a P.C. surface and $g \in C^0 \cap L^{\infty}(\Sigma)$, (cf. Chapter 4.).

The demonstration from here on is valid for either 1) or 2). The supporting lemmas to cover both 1) and 2) will be cited as needed.

5.2 Summary of Chapter 2.

The following two theorems serve to summarize the results of Chapter 2 as well as lay out the facts that we will draw on. The statements of the jump lemmas have been changed to reflect the normalization based on $Q = (e^{iwr})/2\pi r$. The supporting lemma or corollary is given in parentheses.

Theorem 5.1 If $g \in$ L.B. (or H) then Qg (C.2.19, C.2.23) and $K^t g \in H$ (or H') (L.2.19', L.2.20'). If $g \in$ L.B. then $v = Qg \in C^o(R^3 - E)$ (L.2.6, L.2.7). If $p \in \dot{\Sigma}$, then $v_{N_\pm}(p) = (K^t \mp I)g(p)$, for $g \in C^o \cap L^1(\Sigma)$ (L.2.9).

Theorem 5.2 If $h \in$ L.B. (or H) then $Kh \in H$ (or H') (L.2.19, L.2.23). As $p_\pm \to p \in \dot{\Sigma}$, where $p_\pm \in \Sigma_\pm$, $w(p_\pm) \longrightarrow (K \pm I)h(p) = w_\pm(p)$, for $h \in C^o \cap L^1(\dot{\Sigma})$ (L.2.9). If $h \in H'$, then the limits of $(\partial w/\partial N)(p_\pm)$ exist as $p_\pm \to p \in \dot{\Sigma}$ and are equal (L.2.24). The common value will be denoted by $Ph(p)$ and $Ph(p) \in C^o \cap L^1(\dot{\Sigma})$ (C.2.24).

The next two sections employ the proof ideas of Leis [7], and Sacksteder [16].

5.3 Relations among Q, K, K^t, P.

Theorem 5.3 $KQ = QK^t$ maps $C^o \cap L^1(\dot{\Sigma})$ to H' and $PQ = (K^t)^2 - I$ maps H to H, $Qg = 0$ if and only if $g = K^t g$, and Q maps any invariant subspace belonging to an eigenvalue $\lambda \neq 1$ of K^t isomorphically onto a corresponding subspace for K.

Proof. If $p_+ \in \Sigma_+$, then $Q(p_+,q)$ is, as a function of q, a solution of the reduced wave equation in Ω, the region of R^3 bounded by Σ. So also is v. Consider the equation

$$\iint_{\Omega} Q(p_+,q)(\Delta v(q)+w^2 v) - v(q)(\Delta_q Q(p_+,q)+w^2 Q)\, d\Omega(q)$$
$$= \int_{\Sigma}\left[Q(p_+,q)v_{N_-}(q) - v(q)Q_{N_q}(p_+,q)\right] d\sigma(q).$$

Without the terms $w^2 v$ and $w^2 Q$ in parentheses on the left side, the above equation is Green's identity. Inserting those terms does not change the value of the expression, but does show that the left side is zero. Therefore,

$$(1) \qquad \int_{\Sigma} v(q)K(p_+,q)\, d\sigma(q) = \int_{\Sigma} v_{N_-}(q)Q(p_+,q)\, d\sigma(q) \quad .$$

Let p_+ approach $p \in \Sigma$. On the left side, by Theorem 5.2, we obtain $(K + I)v(p) = (K + I)Qg(p) \in H$, and on the right we obtain, by Theorem 5.1 (continuity of $Qv_{N_-}(p)$, then $v_{N_-}(p) = (K^t + I)g(p)$)

$$Qv_{N_-}(p) = Q(K^t + I)g(p) \in H .$$

Comparing the last two equations gives $KQ = QK^t$.

Apply the operator $(\partial / \partial N_p)$ to both sides of (1), then let p_+ approach $p \in \Sigma$. Assuming $g \in H$, by Theorem 5.1, $v = Qg \in H'$ then on the left, by theorem 5.2, we obtain

$Pv(p) = PQg$. On the right, two applications of Theorem 5.1 give, for $g \in C^0 \cap L^1(\dot{\Sigma})$: $(K^t - I)v_{N_-} = (K^t - I)(K^t + I)g$. Comparing the last two equations gives: $PQ = (K^t)^2 - I$, and shows that PQg is defined by means of the above formula, for $g \in C^0 \cap L^1(\dot{\Sigma})$. Further, this shows that P makes sense on the range of Q as well as on H' (cf.Th.5.2). Therefore, if E denotes the smallest subspace of $C^0 \cap L^1(\dot{\Sigma})$ containing H' and the range of Q , then E is a natural domain for P .

By Theorem 5.1, $v_{N_+} = 0$ if and only if $g = K^t g$, and by the uniqueness theorem of Levine [11] , $v_{N_+} = 0$ is equivalent to $v \equiv 0$ in Σ_+ , so $v = 0$ on Σ , or equivalently $Qg = 0$. The remaining assertions follow from what has already been proved.

<u>Theorem 5.4</u> $PK = K^t P$ and $QP = K^2 - I$, it being understood that in both cases the domain is taken to be H'. $Ph = 0$ if and only if $h = -Kh$, and P maps any invariant subspace belonging to an eigenvalue $\lambda \neq -1$ of K isomorphically onto a corresponding subspace for K^t .

<u>Proof</u>. The proof is similar to that of Theorem 5.3. Start by replacing v by w in (1):

$$(2) \quad \int_\Sigma w_-(q)K(p_+,q)\, d\sigma(q) = \int_\Sigma w_{N_-}(q)Q(p_+,q)\, d\sigma(q) \quad .$$

Let $p_+ \to p \in \dot{\Sigma}$. On the left, by Theorem 5.2 twice, we get

$$(K+I)w_-(p) = (K+I)(K-I)h(p) \quad .$$

On the right, by Theorem 5.1 then Theorem 5.2, we get $Qw_{N_-}(p) = QPh(p)$. Therefore, $K^2-I = QP$.

Apply $(\partial/\partial N_p)$ to (2), then let $p_+ \to p$. On the left, by Theorem 5.2 twice, we get $Pw_-(p) = P(K-I)h(p)$. On the right, by Theorem 5.1 then Theorem 5.2, we get $(K^t-I)w_{N_-}(p) = (K^t-I)Ph$. Therefore, $PK = K^tP$.

By the uniqueness theorem of Levine [11], we obtain: $w_+(p) = (K+I)h(p) = 0 \iff w(p) \equiv 0$ in $\Sigma_+ \iff w_N(p) = 0$ on Σ, or equivalently, $Ph = 0$. The remaining assertions follow from what has been proved.

5.4 Completion of existence proof.

Claim. To complete the existence proof it suffices to find an h in E such that the following two integral equations are satisfied.

(3) $\qquad Qg = (K-I)h$,

(4) $\qquad (K^t+I)g = Ph_+$.

Proof of the claim. The boundary condition that must be met is:

$$u_{N_+} = \tfrac{1}{2}(w_{N_+} - v_{N_+}) = g$$

$$= \tfrac{1}{2}(Ph_+ - (K^t-I)g) = g \ .$$

The last equation is equivalent to (4). The role of (3) will be seen shortly.

To find a function h in E that satisfies both (3) and (4), start with (3). By Lemmas 3.4 and 4.10, the Fredholm Alternative holds for the pair of equations (3) and $X = (K^t - I)Y$. Hence, (3) can be solved for h in $L^\infty(\Sigma)$ or $W(\Sigma)$, if $\langle Qg, s\rangle = 0$ for all solutions s of $(K^t - I)s = 0$. Here, $\langle a,b\rangle$ denotes the pairing $\int_\Sigma a(q)b(q)\, d\sigma(q)$ for a, b in $W(\Sigma)$, or for one of a, b in $L^\infty(\Sigma)$ and the other in $L^1(\Sigma)$. It should be noted that h , the solution of (3), will automatically be in H , since $g, h \in$ L.B. implies that Qg and Kh are in H .

Now, observe that: $\langle Qg, s\rangle = \langle Qg, K^t s\rangle = \langle g, QK^t s\rangle = \langle g, KQs\rangle = \langle g, K0\rangle = 0$, by $s = K^t s$, the symmetry of the operator Q , $QK^t = KQ$, and $s = K^t s$ implies $Qs = 0$, from Theorem 5.3. This argument provides a solution $h_1 \in H$ of (3), but h_1 is not uniquely determined if $\lambda = 1$ is in the spectrum of K . (That is, if $(K-I)h = 0$ has a solution h_2 , then $h = h_1 + h_2$ is also a solution of (3). Note that $h_2 \in$ L.B. and $h_2 = Kh_2$, implies $h_2 \in H$, then $h_2 \in H'$, by Theorem 5.2 twice, so Ph_2 exists.)

It still needs to be shown that for some h_2 satisfying $(K-I)h_2 = 0$, there is an $h = h_1 + h_2$ which satisfies (4), that is, $(K^t + I)g = P(h_1 + h_2)$, or equivalently,

$$Ph_2 = (K^t + I)g - Ph_1 \quad .$$

(Note that P applied to (3) gives $PQg = P(K-I)h_1$, hence Ph_1 exists, because PQg and PKh_1 exist.)

Theorem 5.4 shows that P maps the space of all h_2 in H satisfying $h_2 = Kh_2$ (Note $\lambda \neq -1$.) isomorphically onto the space of solutions of $\Theta = K^t\Theta$. Let $(K^t+I)g-Ph_1$ be denoted by Θ_1. If we can show that $(K^t-I)\Theta_1 = 0$, then by Theorem 5.4, there exists a definite function h_2, satisfying $(K-I)h_2 = 0$, with the property that $Ph_2 = \Theta_1$, as required. Now to show $(K^t-I)\Theta_1 = 0$:

$$\begin{aligned}(K^t-I)\left[(K^t+I)g - Ph_1\right] &= \left[(K^t)^2-I\right]g - (K^t-I)Ph_1 \\ &= PQg - P(K-I)h_1 \\ &= 0 \quad ,\end{aligned}$$

by Theorem 5.3, Theorem 5.4, and P applied to (3) with $h = h_1$. This completes the existence proof.

The methods of this chapter are also applicable to the exterior Dirichlet and mixed problems as well as to interior problems [16].

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