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BY

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ABSTRACT

FUNCTIONAL MEASURE IN

QUANTUM FIELD THEORY

BY

ZELJKO ANTUNOVIC

Advisor: Professor Michio Kaku

In this thesis we discuss the local functional measure in non-linear field theories. The local messure is obtained explicitely for quantum gravity, gravity and Yang-Mills theory and supergravity.

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INTRODUCTION

In the quantization of guage invariant field theories in the functional integral method arises the problem of the non-trivial functional measure. Particularly, the problems arise in the theories with singular Lagrangian when the interaction Lagrangian depends non-linearly on the field derivatives.

The most important example is the gravitational field. The functional measure for gravitational field involves the local factor, which is necessary if one is to have covariant and gauge invariant quantization procedure [8,11].

Furthermore, the local factor in the functional measure cancels all the divergencies of the type $\delta^4(0)$ which are present in the theories with derivative couplings.

The functional measure for quantum gravity was first calculated by Fadeev and Popov [17], with the result

$$\prod_x \left[g^{-5/2}(x) \prod_{\mu \leq \nu} d g_{\mu\nu}(x) \right]$$

where $g = \det g_{\mu\nu}$.

Independently, Fradkin and Vilkovisky [8] and Kaku and Senjanovic [15] found:

$$\prod_x \left[g^{-3/2}(x)\, g^{00}(x) \prod_{\mu\leq\nu} d g_{\mu\nu}(x) \right]$$

The measure contains non-covariant looking factor $g^{-3/2}(x)\ g^{00}(x)$, but it is exactly this factor that makes the theory unitary and gauge invariant and free of quartic divergencies [8].

In this thesis we calculate the functional measure for quantum gravity and gravity and Yang-Mills theory using the fact that it is the measure that cancels the quartic divergencies present in these theories. We find the local factor in the functional measure for quantum gravity to be:

$$g^{-3/2}(x)\ g^{00}(x)$$

and for gravity and Yang-Mills:

$$g^{\frac{N^2-7}{4}}(x)\ g^{00\,N^2}(x)$$

where N is the dimension of the internal symmetry group $SU(N)$.

The thesis is organized as follows:
Chapter I is used to review the path integral quantization. In Chapter II the theory of systems with constraints is reviewed, both at a classical and at a quantum level. Chapter III deals with the local factors in the functional

measure. The necessity for the local term in the functional measure is demonstrated on the example of a non-linear - model. Furthermore, the cancellation of the quartic divergences by the functional measure is shown on the simple example and the more general theorem due to Lee and Yang (9) is quoted.

Chapter IV is used to calculate the functional measure for quantum gravity, gravity and Yang-Mills theory and supergravity.

The appendix contains the proof of the theorem refered to in Chapter III.

I. PATH INTEGRAL

Path integral - integral over all trajectories in both configuration and phase space was introduced by Feynman.[(1),(2)]

Let H(p,q) be the Hamiltonian of one dimensional system. Quantization of this system consists in replacing q and p by operators:

(I.1) $$q \to \hat{q} \equiv q \quad ; \quad p \to \hat{p} \equiv -i\hbar\frac{\partial}{\partial q}$$

Footnote 1. These operators are defined in the Hilbert space of the square integrable complex functions $\Psi(q)$.

Time development of the system is given by Schrodinger equation:

(I.2) $$i\frac{\partial \Psi}{\partial t} = H\Psi$$

Where H is the operator obtained from the Hamiltonian of the classical system by substitution (I.1). Formal solution of the Schrodinger equation can be written as:

(I.3) $$\Psi(t) = U(t,t_0)\,\Psi(t_0) \;,$$

Footnote 1. We will use the system of units where ħ=c=1.

Where $U(t,t_0)$ is evolution operator satisfying:

$$U(t,t_0) = \exp\left[i(t-t_0)H\right] . \tag{I.4}$$

Path integral expresses the matrix element of the evolution operator as the average over all trajectories weighted by the factor:

$$\exp\left(i\, S[t_0,t]\right) \tag{I.5}$$

where

$$S[t_0,t] = \int_{t_0}^{t} \left[p(t')\dot{q}(t') - H(p(t'),q(t'))\right] dt' \tag{I.6}$$

is the action for the trajectory $(q(t'), p(t'))$, $(t_0 \leq t' \leq t)$ in the phase space $\dot{q}(t') \equiv \frac{dq(t')}{dt'}$.

The path integral is defined as the limit of the ordinary integral, as follows:

divide the interval (t_0,t) into N equal intervals by points $t'_1, \ldots, t'_{N-1}$

$$[t_0,t'_1),\ (t'_1,t'_2), \ldots, (t'_{N-1},t] . \tag{I.7}$$

We will fix the end points

(I.8) $q(t_0) = q_0$ and $q(t) = q$

and define the trajectory $(q(t'), p(t'))$ by it's values $q(t'_i)$ and $p(t'_i)$ and call them $q_1, \ldots, q_{N-1}$ and $p_1, \ldots, p_N$.

Define the integral

$$W_N\left(q_0, q; t_0, t\right) \equiv (2\pi)^{-N} \int dp_1 dq_1 \ldots dq_{N-1} dp_N \exp\left(iS[t_0, t]\right) \tag{I.9}$$

where $S(t_0, t)$ is the action (I.6) for the trajectory $(q(t'), p(t'))$ defined by $q_1, \ldots, q_N; p_1, \ldots, p_N$.

The basis of the path integral formulation of quantum mechanics is the following theorem:

The limit of the integral (I.9) for $N \to \infty$ is the matrix element of the evolution operator :

$$\lim_{N \to \infty} W_N(q_0, q; t_0, t) = \langle q | \exp[i(t_0 - t)H] | q_0 \rangle \tag{I.10}$$

It is easy to verify this theorem in two special cases when the Hamiltonian depends only on coordinates or only on momenta.

If $H = H(q)$ the action for the trajectory $(q(t'), p(t'))$ is

$$\int_{t_0}^{t} (p\dot{q} - H)\, dt' = p_1(q_1 - q_0) + p_2(q_2 - q_1) + \ldots \tag{I.11}$$

$$\ldots + p_N(q - q_{N-1}) - \int_{t_0}^{t} H[q(t')]\,dt'$$

Integrating over momenta in (I.9) gives the product of δ - functions,

$$\delta(q_1 - q_0)\,\delta(q_2 - q_1)\ldots\delta(q - q_{N-1}) \tag{I.12}$$

which makes integration over $q_1,\ldots,q_{N-1}$ trivial and gives the result

$$\delta(q_0 - q)\exp[i(t_0 - t)H(q_0)] = \langle q_0|\exp[i(t_0-t)H]|q\rangle \tag{I.13}$$

If H = H(p) the action is

$$\int_{t_0}^{t}(p\dot{q} - H)\,dt' = p_1(q_1 - q_0) + p_2(q_2 - q_1) + \ldots \tag{I.14}$$
$$\ldots + p_N(q - q_{N-1}) - \int_{t_0}^{t} H[p(t')]\,dt'$$

Integrating in (I.9) over coordinates and then over momenta gives

$$\frac{1}{2\pi}\int dp\,\exp[ip(q - q_0) + i(t_0 - t)H(p)] \tag{I.15}$$

which is the matrix element of the evolution operator for the Hamiltonian $H = H(p)$.

The proof of (I.10) complicates in general case when the Hamiltonian depends on both coordinates and momenta. The proof of the theorem exists for the cases when $H = H_1(q) + H_2(p)$ and for the Hamiltonians which are parabolic operators. In general case the limit of (I.9) is not equal to the matrix element of the evolution operator.

The path integral is a limit $N \to \infty$ of (I.9) and is denoted as:

$$\int_{q(t_0)}^{q(t)} \exp\left(i S[t_0, t]\right) \prod_{t'} \frac{dp(t')\, dq(t')}{2\pi} \quad . \tag{I.16}$$

It is easy to generalize to a system with n degrees of freedom. The action is

$$S[t_0, t] = \int \sum_{i=1}^{n} \left[p_i \dot{q}_i - H(q, p) \right] dt' \tag{I.17}$$

and to define the path integral we substitute

$$(2\pi)^{-N} \to (2\pi)^{-Nn} \; ; \; dq_k \to \prod_{i=1}^{n} dq_k^i \; ; \; dp_k \to \prod_{i=1}^{n} dp_k^i \tag{I.18}$$

into (I.9) and denote this path integral by:

$$\int_{q(t_0)=q_0}^{q(t)=q} \exp\left(iS[t_0,t]\right) \prod_{t'} \prod_{i=1}^{n} \frac{dq^i(t')\,dp^i(t')}{2\pi}$$

The extension to field theory is made by considering it as a system with infinitely many degrees of freedom. Dividing the space into cubes V_i (i = 1,2,....) of volume v_i , we define the i-th coordinate $q_i(t) = \phi_i(t)$ by

$$\phi_i(t) \equiv \frac{1}{v_i} \int_{V_i} d^3x\ \phi(\vec{x},t) \tag{I.19}$$

and the momenta

$$\pi_i(t) \equiv \frac{1}{v_i} \int_{V_i} d^3x\ \pi(\vec{x},t)\ . \tag{I.20}$$

In the field theory all physical quantities are derivable from the vacuum - to - vacuum transition amplitude in the presence of external sources W(J). The path integral formula for the functional W(J) is

$$W[J] = \int d\phi\, d\pi\ \exp\left[i \int_{-\infty}^{+\infty} d^4x \left\{\pi(x)\dot{\phi}(x) - \right.\right.$$
$$\left.\left. - \mathcal{H}(x) + J(x)\phi(x)\right\}\right]\ . \tag{I.21}$$

The n - point Green's function - the vacuum expectation value of the time - ordered - product of n fields - is given by

$$\text{(I.22)} \qquad \left.\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0} = i^n \langle 0|T[\phi(x_1)\ldots\phi(x_n)]|0\rangle$$

$$= i^n G(x_1,\ldots,x_n) \; .$$

(I.22) gives the complete n - point Green's function. We are more interested in connected Green's functions $G_c\,(x_1,\ldots,x_n)$, which does not include the contribution of disconnected vacuum - to - vacuum graphs. One can show that by defining

$$\text{(I.23)} \qquad W(J) = \exp\,(\,i\,Z(J)\,).$$

One has

$$\text{(I.24)} \qquad G_c(x_1,\ldots,x_n) = (-1)^{n-1} \left.\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}$$

In the case of interacting field theory one writes the Hamiltonian as

$$\text{(I.25)} \qquad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \; ,$$

where $\mathcal{H}_I$ is the interacting part of the Hamiltonian, and the Z(J) becomes

$$Z[J] = \exp\left[-i\int d^4x\ \mathcal{H}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J] \tag{I.26}$$

where
$$Z_0[J] = \int d\phi\, d\pi\ \exp\left[i\int d^4x\left\{\pi(x)\dot{\phi}(x) - \right.\right.$$
$$\left.\left. - \mathcal{H}_0(x) + J(x)\phi(x)\right\}\right]$$

and then uses Wick's theorem to evaluate

$$\exp\left[i\int d^4x\ \mathcal{H}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] = \sum_{n=0}^{\infty}\frac{(i)^n}{n!}\int d^4x\left[\mathcal{H}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]^n \tag{I.27}$$

which is the basis for the perturbation theory.

IIa. SYSTEMS WITH CONSTRAINTS (CLASSICAL THEORY)

Given the Lagrangian of a n - dimensional mechanical system

$$L = L(q,\dot{q}), \tag{II.1}$$

we define the canonical momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (i = 1,\ldots,n) \tag{II.2}$$

We are interested in the case when the Lagrangian is singular, i.e. $\frac{\partial L}{\partial \dot{q}_i}$ are not independent functions of $\dot{q}_i$. Eliminating the $\dot{q}$'s one gets the m constraint equations:

$$T_\alpha (q^i, p_i) = 0 , \qquad (\alpha = 1,\ldots,m) . \tag{II.3}$$

The Hamiltonian is

$$H_0 (q^i, p_i) = p_i \dot{q}^i - L . \tag{II.4}$$

We take the constraints to be the first class, i.e. to satisfy

$$\{T_\alpha , T_\beta\} = T_\gamma U^\gamma_{\alpha\beta} \tag{II.5}$$

$$\{H_o, T_\alpha\} = T_\beta v^\beta_\alpha \tag{II.6}$$

where

$$u = u(p,q) \qquad \text{and} \qquad v = v(p,q).$$

The Poisson's bracket is defined as

$$\{A, B\} = \sum_{i=1}^{n} \left(\frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right)$$

If $u^\gamma_{\alpha\beta}$ = constant, then the constraints $\{T_\alpha\}$ form a Lie algebra.

Our problem is to find the conditional extremum of the canonical action restricted to the constraint surface in phase space, i.e.

$$\mathcal{S} = \int dt \left(p_i \dot{q}^i - H_o \right) \Big|_{T=0} = \int dt \, L \Big|_{T=0} \tag{II.7}$$

and

$$\delta \mathcal{S} = \delta \int dt \left(p_i \dot{q}^i - H_o \right) \Big|_{T=0} = \delta \int dt \, L \Big|_{T=0} . \tag{II.8}$$

For degenerate systems there are many extremals of (II.7) that pass through each point of the constraint surface. Indeed, introducing the Lagrange multipliers

$$S[q,p,\lambda] = \int dt \left(p_i \dot{q}^i - H_0 - \lambda^\mu T_\mu \right) \tag{II.9}$$

one can see that extremals of (II.9) under the choice (II.5) and (II.6' are consistent for any value of λ . We conclude that for initial data

$$(q_0, p_0) \in T(p,q) = 0$$

there are infinitely many classical trajectories enumerated by values of $\lambda^\mu(t)$.

We will try to interpret this as dynamics of a system with less degreees of freedom, namely $n - m$ degrees of freedom, corresponding to the phase space of the independent canonical variables (the physical space)

$$q^a , p^a \qquad (a = 1,2,\ldots,n-m)$$

determined by $H_{PHYS.} (q^a, p^a)$.
One has to prove the existence and uniqueness of this new dynamical system.

Let us consider the infinitesimal transformation with parameters F^μ generated in the initial phase space by constraints $\{T^\alpha\}$, so that

$$\delta^F q^i = \{q^i, T^\mu\} F_\mu \tag{II.10}$$

and

$$\delta^F p_i = \{P_i, T^\mu\} F_\mu \quad . \tag{II.10}$$

Now if

$$(q^i, p_i) \in T(q, p) = 0 \ ,$$

then

$$(\bar{q}^i, \bar{p}_i) \in T(q, p) = 0$$

because

$$\bar{q}^i = q^i + \delta^F q^i = q^i + \{q^i, T_\mu\} F^\mu = q^i - \frac{\partial T^\mu}{\partial p^i} F_\mu$$

$$\bar{p}^i = p^i + \delta^F q^i = p^i + \{p^i, T_\mu\} F^\mu = p_i - \frac{\partial T^\mu}{\partial q^i} F_\mu$$

Then from (II.5) we get:

$$T_\alpha(\bar{q}^i, \bar{p}_i) - T_\alpha(q^i, p_i) = T_\alpha(q^i + \delta^F q^i, p_i + \delta^F p_i) - T_\alpha(q^i, p_i) =$$

$$= \frac{\partial T_\alpha}{\partial q^i} \delta^F q^i + \frac{\partial T_\alpha}{\partial p_i} \delta^F p_i = -\frac{\partial T_\alpha}{\partial q^i} \frac{\partial T_\mu}{\partial p_i} F^\mu + \frac{\partial T_\alpha}{\partial p_i} \frac{\partial T_\mu}{\partial p_i} F^\mu =$$

$$= \{T_\alpha, T_\mu\} F^\mu = T_\beta U^\beta_{\alpha\mu} F^\mu = 0$$

on the constraint surface $T(p,q) = 0$.

Therefore, one may apply this transformation to the action (II.7) and get

(II.11) $$\delta^F \mathcal{S} = 0 \quad ,$$

because:

$$\delta^F \mathcal{S} = \int_{t_1}^{t_2} dt \quad \delta^F \left(p_i \dot{q}^i - H_o \right) \Big|_{T=0} =$$

$$= \int_{t_1}^{t_2} dt \left[(\delta^F p_i) \dot{q}^i + p_i \delta^F \dot{q}^i - \frac{\partial H_o}{\partial p_i} \delta^F p_i - \frac{\partial H}{\partial q_i} \delta^F q_i \right]$$

Using the equations of motion

$$\dot{q}^i = \frac{\partial H_o}{\partial p_i} \quad ; \quad \dot{p}_i = - \frac{\partial H_o}{\partial q^i}$$

we get

$$\delta^F \mathcal{S} = \int_{t_1}^{t_2} dt \left(p_i \delta^F \dot{q}^i + \dot{p}_i \delta^F q^i \right) =$$

$$= \int_{t_1}^{t_2} dt \, \frac{d}{dt} \left(p_i \, \delta^F q^i \right) =$$

$$= p_i(t_2) \, \delta^F q^i(t_2) - p_i(t_1) \, \delta^F q^i(t_1) = 0$$

because

$$\delta q^i(t_2) = \delta q^i(t_1) = 0 \, .$$

The transformation (II.10) with constant parameters F^μ is canonical with the generating function $F = T_\mu F^\mu$ because:

$$\bar{H}_o = H_o(\bar{p}, \bar{q}) = H_o\left(q^i + \{q^i, T_\mu\} F^\mu ; p_i + \{p_i, T_\mu\} F^\mu\right) =$$

$$= H_o\left(q^i - \frac{\partial T_\mu}{\partial p_i} F^\mu \, ; \, p_i + \frac{\partial T_\mu}{\partial q^i} F^\mu\right) =$$

$$= H_o(q^i; p_i) - \frac{\partial H_o}{\partial q^i} \frac{\partial T_\mu}{\partial p_i} F^\mu + \frac{\partial H_o}{\partial p_i} \frac{\partial T_\mu}{\partial q^i} F^\mu =$$

$$= H_o(q^i; p_i) + \frac{\partial p_i}{\partial t} \frac{\partial T_\mu}{\partial p_i} F^\mu + \frac{\partial q_i}{\partial t} \frac{\partial T_\mu}{\partial q^i} F^\mu =$$

$$= H_o + \frac{\partial T_\mu}{\partial t} F^\mu = H_o + \frac{d T_\mu}{d t} F^\mu$$

so that:

$$\left(\bar{H}_o - H_o\right) dt = d\left(T_\mu F^\mu\right) = dF.$$

The action (II.9) is also invariant under transformation (II.10), i.e.

$$\delta^F S[q, p, \lambda] = 0 \tag{II.12}$$

provided that one simultaneously transforms the Lagrange multipliers according to the law:

$$\delta^F \lambda^\mu = -\dot{F}^\mu - u^\mu_{\alpha\beta} \lambda^\alpha F^\beta - v^\mu_\alpha F^\alpha. \tag{II.13}$$

Indeed, because of (II.11), if

$$\delta^F \left(\lambda^\mu T_\mu\right) = \frac{dF}{dt} = \frac{d}{dt}\left(T_\alpha F^\alpha\right)$$

and

$$0 = \delta^F\left(\lambda^\mu T_\mu\right) = \frac{d}{dt}\left(T_\alpha F^\alpha\right) = \frac{\partial}{\partial t}\left(T_\alpha F^\alpha\right) + \left\{H, T_\alpha F^\alpha\right\}$$

so that

$$0 = \dot{F}^\alpha T_\alpha + \left\{H_o + \lambda^\mu T_\mu, T_\alpha F^\alpha\right\}$$

and

$$0 = \dot{F}^{\alpha} T_{\alpha} + \{H_0, T_{\alpha}\} F^{\alpha} + \{\lambda_{\mu}, T_{\alpha}\} F^{\alpha} T^{\mu} + \lambda^{\mu} \{T_{\mu}, T_{\alpha}\} F^{\alpha}$$

$$0 = \dot{F}^{\mu} T_{\mu} + v_{\alpha}^{\mu} F^{\alpha} T_{\mu} + (\delta^F \lambda^{\mu}) T_{\mu} + U_{\alpha\beta}^{\mu} \lambda^{\alpha} F^{\beta} T_{\mu}$$

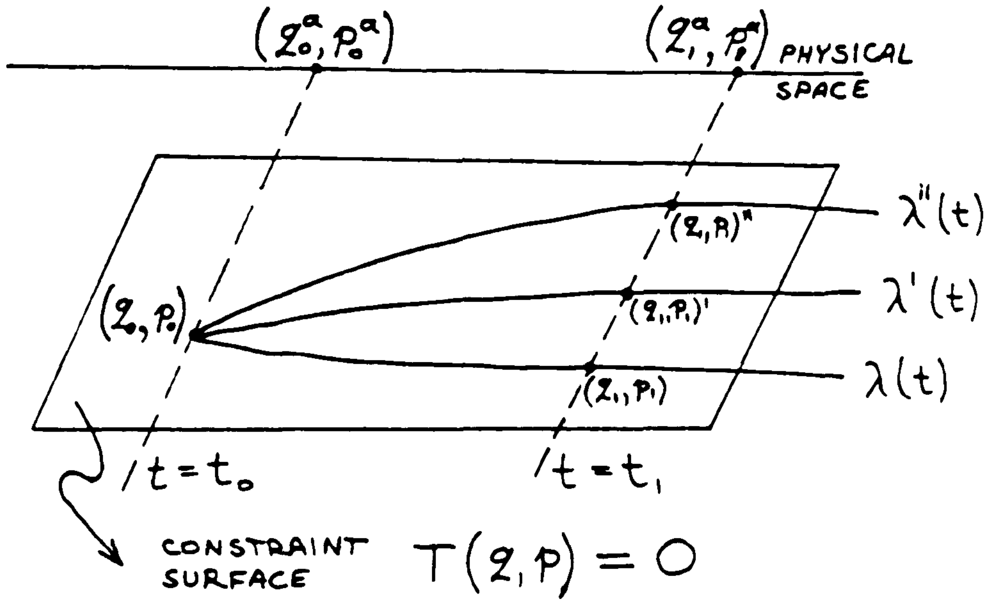
or

$$\delta^F \lambda^{\mu} = -\dot{F}^{\mu} - v_{\alpha}^{\mu} F^{\alpha} - U_{\alpha\beta}^{\mu} \lambda^{\alpha} F^{\beta}.$$

This insures (II.12).

The tranformation (II.10) maps on each other the points (q_1, p_1) , $(q_1, p_1)'$, $(q_1, p_1)''$,.... of the constraint surface. These points are obtained at the instant $t = t_1$ as a result of the development from one and the same initial point (q_0, p_0) at the instant $t = t_0$. We define all such points physically equivalent. The transformation (II.10) divides the constraint surface into equivalence classes, the points belonging to each class being connected to each other by the transformation (II.10).

The physical space is defined as the space of these equivalence classes.



All points in one class (q_1, p_1), $(q_1, p_1)'$, $(q_1, p_1)''$, ... are characterized by one and the same value of the Hamiltonian H_0 and correspond to the same physical state $(q\ , p\ )$.

This definition of the physical space makes possible the separation between invariance and dynamics of the system. The dynamics of the system contained in the generalized Hamilton equations (extremals of (II.9)) deals with the equivalence classes on the whole.

The physical space, being the space of the equivalence classes, one has to choose a representative point for each class. This is done by imposing m additional constraints that break the invariance under the transformation (II.10):

$$\phi^{\alpha}(q^i, p_i) = 0\ , \qquad \alpha = 1, \ldots\ldots, m \tag{II.14}$$

such that

$$\delta^F \phi^{\alpha} = \{\phi^{\alpha}, T_{\beta}\} F^{\beta} \neq 0\ . \tag{II.15}$$

The ϕ constraints are arbitrary apart from the requirement that they break the invariance under (II.10) and are independent. This is ensured by requiring that

$$\det\{\phi^{\alpha}, T_{\beta}\} \neq 0\ . \tag{II.16}$$

Thus the dynamics of the system is described by the action

$$S[q, p, \lambda, \sigma] = \int dt [p_i \dot{q}^i - H_0 - \lambda^{\mu} T_{\mu} - \sigma^{\mu} \phi_{\mu}] \tag{II.17}$$

where σ_{μ} are additional Lagrange multipliers. The dimensionality of the physical space is

$$2n - m - m = 2(n - m).$$

So, to treat the system with degenerate action - which is degenerate because of the existence of the constraints (II.3) subject to conditions (II.5) and (II.6) - one must impose new constraints (II.14), and the total number of constraints doubles.

Let us introduce the compact notation for all constraints:

$$\Theta^{K} = \begin{pmatrix} T_{\alpha} \\ \phi^{\beta} \end{pmatrix} \quad ; \quad \xi_{K} = \left(\lambda^{\alpha}, \sigma_{\beta} \right) , \tag{II.18}$$

where k runs from 1 to 2m, so that the action becomes

$$S[q, p, \lambda, \sigma] = \int dt \left(p_i \dot{q}^i - H_0 - \Theta_K \xi^K \right). \tag{II.19}$$

We also define the new operator: the Dirac bracket

$$\{A, B\}^{*} \equiv \{A, B\} - \{A, \Theta^{K}\} C^{-1}_{K\ell} \{\Theta^{\ell}, B\} \tag{II.20}$$

where

$$C_{K\ell} = \{\Theta_K, \Theta_\ell\} . \tag{II.21}$$

The existence of $C^{-1}_{K\ell}$ is guaranteed because det $C_{K\ell} \neq 0$ by virtue of (II.16).

The Dirac bracket can be shown to have all the usual properties of the Poisson bracket. In addition, it also has two important properties, which are the reason for its introduction. Namely, for any function

A(q,p) we have:

$$\{A,\theta^k\}^* = \{A,\theta^k\} - \{A,\theta^m\} C^{-1}_{mn} \{\theta^n,\theta^k\} = \tag{II.22}$$

$$= \{A,\theta^k\} - \{A,\theta^m\}\,\delta^k_m = 0 ,$$

Footnote 2. Because of (II.20) it is obvious that $\{\theta^k,\theta^l\}^* = 0$ which is a necessary condition for the passage to the quantum theory, i.e. the replacement of classical brackets by commutators.

Indeed, if

$$\theta^k|\alpha\rangle = 0 \quad \text{AND} \quad \theta^l|\alpha\rangle = 0$$

then

$$[\theta^k,\theta^l]|\alpha\rangle = 0$$

which corresponds to the classical equation

$$\{\theta^k,\theta^l\} = 0$$

and because of the consistency conditions

(II.23) $$\dot{\theta}^{k} = \{H, \theta^{k}\} = 0$$

where

(II.24) $$H = H_{0} + \xi_{k} \theta^{k}$$

we have

(II.25) $$\{H, A\}^{*} = \{H, A\} - \{H, \theta^{k}\} C^{-1}_{kl} \{\theta^{l}, A\} = \{H, A\}$$

and so:

(II.26) $$\dot{A} = \{H, A\}^{*} \quad .$$

From (II.21) we also have

$$0 = \dot{\theta}^{k} = \{H, \theta^{k}\} = \{H_{0}, \theta^{k}\} + \xi_{l}\{\theta^{l}, \theta^{k}\} + \{\xi_{l}, \theta^{k}\}\theta^{l}$$

so, that in the physical space:

(II.27) $$\theta^{k}(q^{i}, p_{i}) = 0 \quad , \quad k = 1, \ldots, 2m$$

we have

$$\xi_\ell \{\theta^\ell, \theta^\kappa\} = -\{H_0, \theta^\kappa\}$$

and finally

$$\xi_\kappa = - C^{-1}_{\kappa\ell}\{\theta^\ell, H_0\} \quad . \tag{II.28}$$

Equations of motion are

$$\dot{q}^i = \{H, q^i\}^* = \{H_0, q^i\}^* + \{\xi_\kappa \theta^\kappa, q^i\}^* =$$

$$= \{H_0, q^i\}^* + \{\xi_\kappa, q^i\}^* \theta^\kappa = \{H_0, q^i\}^*$$

because of (II.20) and (II.27).

Thus, under the restriction to the physical space (II.27) the Cauchy problem acquires the unique solution determined by

$$\dot{q}^i = \{H_0, q^i\}^* \; ; \; \dot{p}_i = \{H_0, p_i\}^* \tag{II.29}$$

while the Lagrange multipliers are determined by (II.28).

Now we can introduce the canonical coordinates in the physical space: (q^a, p^a). To do this one has to solve the system of equations:

$$q^i = q^i(q^a, p^a) \;\; ; \;\; p_i = p_i(q^a, p^a)$$

(II.30)

$$\theta^K = \theta^K\left[q^i(q^a, p^a), p_i(q^a, p^a)\right] = 0$$

where $a = 1, \ldots, n - m$,

in such a way that $2(n - m)$ independent variables (q^a, p^a) are canonical variables, i.e.

(II.31) $$\int p_i \dot{q}^i \, dt = \int p_a \dot{q}^a \, dt \; .$$

Then under the substitution of functions (II.30) into (II.27), (II.28) and (II.29) the equations of motion take the canonical form:

(II.32) $$\dot{q}^a = \frac{\partial H_{PHYS.}(q^b, p^b)}{\partial p^a} \;\; ; \;\; -\dot{p}^a = \frac{\partial H_{PHYS.}(q^b, p^b)}{\partial q^a}$$

with the Hamiltonian:

$$H_{PHYS.}(q^a, p^a) = H_0(q^i, p_i)\Big|_{\theta = 0} \; .$$

This follows from the property of Dirac brackets which we cite without proof:[(3)]

$$\{A,B\}^*\Big|_{\theta=0} = \sum_{a=1}^{n-m}\left(\frac{\partial A}{\partial p^a}\frac{\partial B}{\partial q^a} - \frac{\partial A}{\partial q^a}\frac{\partial B}{\partial p^a}\right) \tag{II.33}$$

where

$$A = A(q^i, p_i)\Big|_{\theta=0} = A\left[q^i(p^a, q^a), p_i(p^a, q^a)\right]$$

$$B = B(q^i, p_i)\Big|_{\theta=0} = B\left[q^i(p^a, q^a), p_i(p^a, q^a)\right] .$$

The initial data for equations (II.32) are 2(n - m) independent initial data

$$q_0^a = q^a(t)\Big|_{t=0} \; ; \; p_0^a = p^a(t)\Big|_{t=0} . \tag{II.34}$$

The canonical equations of motion (II.32) can be derived from the action

$$\int dt\left(p^a \dot{q}_a - H_{PHYS.}(q^a, p^a)\right) = \tag{II.35}$$

$$= \int dt\left(p_i \dot{q}^i - H_0(q^i, p_i)\right)\Big|_{T=\Phi=0} .$$

The uniqueness of the physical dynamics is guaranteed by the fact that the replacement of the additional constraints $\emptyset$ in (II.35) by some other additional constraints is,equivalent to a canonical transformation of the variables (q^{α}, p^{α}). Thus, an equivalence class on the surface $T(q,p) = 0$ projects as a whole on one point of the physical space, and the arbitrariness in the choice of a member of a class - $\emptyset(q,p) = 0$ - reduces to a standard canonical transformation in the physical space.

The classical theory of the systems with constraints has been developed by many authors.[3-7]

IIb. QUANTIZATION OF SYSTEMS WITH CONSTRAINTS

In section IIa. we have seen that mechanical system (II.1), with 2n degrees of freedom and 2m constraints (II.18), is equivalent to a system with 2(n - m) independent degrees of freedom (II.30) with canonical variables (q^a, p^a) where $a = 1,2,\ldots,n-m$. To quantize this system one imposes the canonical commutation relations upon the independent initial data (II.34):

$$i\left[q_0^a, p_0^b\right] = \left\{q_0^a, p_0^b\right\} \equiv \delta^{ab}$$

$$i\left[q_0^a, q_0^b\right] = \left\{q_0^a, q_0^b\right\} \equiv 0 \qquad \text{(II.36)}$$

$$i\left[p_0^a, p_0^b\right] = \left\{p_0^a, p_0^b\right\} \equiv 0$$

and solves the Hamilton equations (II.32) for the Heisenberg operators $q^a(t)$ and $p^a(t)$. The needed quantities are vacuum expectation values of time ordered products of these operators:

$$\langle 0 | T\left[q^a(t)\ldots q^b(t')p^c(t'')\ldots p^d(t''')\right] | 0 \rangle . \qquad \text{(II.37)}$$

Path integral expression for this Green's function can be obtained from the functional:

$$Z_{PHYS.} = \int dq^a dp_a \exp\left[i\int dt\left\{p^\ell \dot{q}_\ell - H_{PHYS.}(q^\ell, p_\ell)\right\}\right] \tag{II.38}$$

which determines the S - matrix in the Hilbert space of physical states.

The problem of quantization of systems with constraints consists in the equivalent reformulation of the commutation relations (II.36), dynamical equations (II.32) and the solution (II.38) for the S - matrix in terms of the initial phase space (q^i, p_i) and the explicitly given quantities: the Hamiltonian H_0, constraints T_α and the arbitrary additional conditions ϕ_β.

The commutation relations for the original variables are:

$$\begin{aligned} i[q^i, p_j] &= \{q^i, p_j\}^* \equiv \delta^i_j - \frac{\partial\theta^k}{\partial p_i} C^{-1}_{k\ell} \frac{\partial\theta^\ell}{\partial q^j} \\ i[q^i, q^j] &= \{q^i, q^j\}^* \equiv \frac{\partial\theta^k}{\partial p_i} C^{-1}_{k\ell} \frac{\partial\theta^\ell}{\partial p_j} \\ i[p_i, p_j] &= \{p_i, p_j\}^* \equiv \frac{\partial\theta^k}{\partial q^i} C^{-1}_{k\ell} \frac{\partial\theta^\ell}{\partial q^j} \end{aligned} \tag{II.39}$$

and due to the property (II.33) of Dirac brackets the above relations reduce to (II.36) under the restriction to the constraint surface: $\theta^{\kappa} = 0$.

We already know the form of the dynamical equations for the original variables. They are equations (II.29) and (II.28) for the constrained variables (q^i, p_i).

The main result of this section is the expression for the S - matrix first obtained in reference (3):

$$\text{(II.40)} \quad Z_{\text{PHYS.}} = \int \left(\prod_t dq^i dp_i\right)\left(\prod_t d\lambda^\mu db_\mu\right)\left(\prod_t \det\{T_\alpha, \phi^\beta\}\right) \exp\left[i\int dt\left(p_i \dot{q}^i - H_0 - \lambda^\mu T_\mu - b^\mu \phi_\mu\right)\right].$$

It contains the classical action $S(q,p,\lambda,b)$ (II.17) and the non-trivial measure of integration over the trajectories in the initial phase space

$$\text{(II.41)} \quad d\mu(q^i, p_i) = \prod_t (dq^i dp_i) \det\{T_\alpha, \phi^\beta\}.$$

The proof of the expression (II.40) consists in showing that it is identical to the functional integral (II.38) over independent canonical variables (q^α, p^α).

First we rewrite the measure as

$$\text{(II.42)} \quad \int \prod_t \left(d\lambda^\mu d b_\mu\right) \exp\left[i\int dt\left(-\lambda^\mu T_\mu - b^\mu \phi_\mu\right)\right] d\mu(q^i, p_i) =$$

$$= \int \left(\prod_t dq^i dp_i\right) \left[\prod_\mu \delta(T_\mu)\,\delta(\phi^\mu)\right] \det\{T_\alpha, \phi^\beta\}.$$

The main idea of the proof is in showing that is possible to perform the canonical transformation such that the guage conditions are the new canonical momenta:

$$\text{(II.43)} \quad \phi^\alpha(q^i, p_i) = p^\alpha \quad , \qquad \alpha = 1,\ldots,m.$$

The consistency condition (II.16) now becomes

$$\text{(II.44)} \quad \det\left\{\frac{\partial T_\alpha}{\partial q^\beta}\right\} \neq 0$$

where q^α are the coordinates conjugate to p^α. The condition (II.44) ensures that the constraints (II.3) can be solved for q^α. The constrained surface in the phase space (q^i, p_i) is then defined by

$$\text{(II.45)} \quad p^\alpha = 0 \quad ; \qquad q^\alpha = q^\alpha(q^a, p^a)$$

where $\alpha = 1,\ldots,m$ and $a = 1,\ldots,n\text{-}m$, and where

$$T^{\beta}(q^{\alpha}, 0, q^{a}, p^{a}) = 0 \tag{II.46}$$

and (q^{a}, p^{a}) are the remaining $2(n - m)$ canonical variables which act as independent variables in the physical space.

In the new canonical representation the measure (II.42) becomes

$$\prod_{t} dq^{i} dp_{i} \prod_{\alpha} \delta(T^{\alpha})\, \delta(\Phi_{\alpha})\, \det\{T_{\alpha}, \Phi^{\beta}\} =$$

$$\prod_{t} dq^{i} dp_{i} \prod_{\alpha} \delta(T^{\alpha})\, \delta(p_{\alpha})\, \det\left\{\frac{\partial T^{\alpha}}{\partial q^{\beta}}\right\} = \tag{II.47}$$

$$\prod_{t} dq^{a} dp_{a} \prod \delta(p_{\alpha})\, \delta\left[q^{\alpha} - q^{\alpha}(q^{a}, p^{a})\right] dq^{\alpha} dp_{\alpha} .$$

After a trivial intergration over p_{α} and q^{α} the measure becomes

$$\prod_{t} dq^{a} dp_{a} \tag{II.48}$$

which is indeed the integral measure for the integration over the independent canonical variables (q^{a}, p^{a}) in the physical space.

For application to the quantum field theory one still has to show that the above canonical transformation is equivalent to a unitary transformation of the field operators. This can be done in the case of the Yang - Mills field theory, but it meets serious obstacles in the gravity theory.[8] The problem is that, in contrast to the Yang - Mills gauge transformations, the general coordinate transformation affect not only the form of the field operators, but also their space-time argument. For this reason a coordinate transformation in the gravitational theory does not reduce to a unitary transformation of the field operators, but it also involves a change of time-ordering in the S - matrix.

IIIa. LOCAL FACTORS IN FUNCTIONAL MEASURE

The local factors in the functional measure arise in theories with singular Lagrangians when the interaction Lagrangian depends nonlinearly on the field derivatives. To illustrate this let us consider the particle on the n - dimensional sphere (non-linear σ - model).

The Lagrangian is:

$$L = \frac{1}{2}\dot{\vec{x}}^2 , \qquad \vec{x} = (x_1, \ldots, x_n) \tag{III.1}$$

with the constraint,

$$\vec{x}^2 = x_i x^i = 1 , \qquad i = 1, \ldots, n. \tag{III.2}$$

Let us introduce the independent coordinates

$$q_i = x_i , \qquad i = 1, \ldots, n-1 \tag{III.3}$$

with

$$x_n = \left(1 - \vec{q}^2\right)^{1/2} . \tag{III.4}$$

In terms of the new variables the Lagrangian is

$$\text{(III.5)} \qquad L = \frac{1}{2}\dot{\vec{q}}^{\,2} + \frac{1}{2}\frac{\vec{q}\cdot\dot{\vec{q}}\;\;\vec{q}\cdot\dot{\vec{q}}}{1-\vec{q}^{\,2}} =$$

$$= \frac{1}{2}\sum_{i,j=1}^{n-1} \dot{q}_i\, G_{ij}\, \dot{q}_j$$

where

$$\text{(III.6)} \qquad G_{ij}(q) = \delta_{ij} + \frac{q_i\, q_j}{1-\vec{q}^{\,2}} \quad .$$

Due to the constraint (III.2) the functional integral for the theory is obviously not given by the simple expression $\displaystyle\int \prod_{i=1}^{n-1} dq_i \, \exp\left(i\int L\, dt\right)$.

The correct theory is given by

$$\text{(III.7)} \qquad Z = \int \prod_{i=1}^{n-1} dq_i \, dx_n \, \delta(\vec{x}^{\,2}-1) \exp\left(\frac{i}{2}\int \dot{\vec{x}}^{\,2} dt\right)$$

$$= \int \prod_{i=1}^{n-1} dq_i \left\{ dx_n \, \delta\left[x_n^2 - (1-\vec{q}^{\,2})\right]\right\} \exp\left(i\int L(q)\, dt\right).$$

The additional term present is:

$$\text{(III.8)} \qquad \int dx_n \, \delta\left[x_n^2 - (1-\vec{q}^{\,2})\right] =$$

$$= \int dx_n \frac{1}{2(1-\vec{q}^{\,2})^{1/2}} \left[\delta\left(x_n - (1-\vec{q}^{\,2})^{1/2}\right) + \delta\left(x_n + (1-\vec{q}^{\,2})^{1/2}\right) \right]$$

$$= \frac{1}{2(1-\vec{q}^{\,2})^{1/2}} + \frac{1}{2(1-\vec{q}^{\,2})^{1/2}} = \frac{1}{(1-\vec{q}^{\,2})^{1/2}}$$

and the correct functional integral is

$$Z = \int d\vec{q}\, \frac{1}{(1-\vec{q}^{\,2})^{1/2}} \exp\left[i \int L(q)\, dt \right]. \qquad \text{(III.9)}$$

We see that the correct functional measure requires a local factor $(1 - \vec{q}^{\,2})^{-1/2}$ due to the constraint (III.2).

But how would one quantize the theory given by the Lagrangian (III.5):

$$L = \frac{1}{2} \sum_{i,j=1}^{m} \dot{q}_i\, G_{ij}(q)\, \dot{q}_j$$

without any constraints?

The correct way is the canonical quantization. The momenta are given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{m} G_{ij}\, \dot{q}_j \qquad \text{(III.10)}$$

so, that $\displaystyle \dot{q}_i = \sum_{j=1}^{m} (G^{-1})_{ij}\, p_j$

and the Hamiltonian is

(III.11) $$H = \frac{1}{2} \sum_{i,j=1}^{m} p_i \left(G^{-1}\right)_{ij} p_j \quad .$$

The functional integral in the phase space is then

(III.12) $$Z = \int \prod_{i=1}^{m} dq_i \, dp_i \exp\left[i \int dt \left(\vec{p} \cdot \dot{\vec{q}} - \frac{1}{2} \vec{p} \cdot G^{-1} \vec{p} \right) \right] .$$

To go to the configuration space one has to integrate over momenta

$$\int d\vec{p} \exp\left[i \int dt \left(-\frac{1}{2} \vec{p} \cdot G^{-1}(q) \cdot \vec{p} + \vec{p} \cdot \dot{\vec{q}} \right) \right] =$$

$$= \left(\det G^{-1} \right)^{-1/2} \exp\left[i \int dt \, \frac{1}{2} \dot{\vec{q}} \cdot G \cdot \dot{\vec{q}} \right]$$

so, that the correct functional integral is

(III.13) $$Z = \int d\vec{q} \left(\det G \right)^{1/2} \exp\left[i \int dt \, L(q) \right]$$

and we again recover the local factor in the functional measure (det G $)^{1/2}$. It is easy to calculate it:

$$\left(\det G \right)^{1/2} = \left[\det \left(\delta_{ij} + \frac{q_i q_j}{1 - \vec{q}^2} \right) \right]^{1/2} =$$

$$= \exp\left[\frac{1}{2} \log \det \left(\delta_{ij} + \frac{q_i q_j}{1 - \vec{q}^2} \right) \right] = \exp\left[\frac{1}{2} \operatorname{Tr} \log \left(\delta_{ij} + \frac{q_i q_j}{1 - \vec{q}^2} \right) \right]$$

$$= \exp\left[\frac{1}{2}\,\mathrm{Tr}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\left(\frac{q_i q_j}{1-\vec{q}^{\,2}}\right)^n\right] = \exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\vec{q}^{\,2}}{1-\vec{q}^{\,2}}\right)^n\right] =$$

$$= \exp\left[\frac{1}{2}\log\left(1+\frac{\vec{q}^{\,2}}{1-\vec{q}^{\,2}}\right)\right] = \exp\left[\frac{1}{2}\log\frac{1}{1-\vec{q}^{\,2}}\right]$$

so, that

$$(\det G)^{1/2} = (1-\vec{q}^{\,2})^{-1/2} \tag{III.14}$$

which is identical to (III.8).

In gauge invariant quantum field theories the local factor in functional measure is important for the proof of gauge invariance and unitarity of the theory.(8) Furthermore, the local factor in the functional measure cancels all quartic divergencies which are present in the non-linear theories.(9-11)

Let us demonstrate this on a simple example.

IIIb. CANCELLATION OF $\delta^4(0)$ DIVERGENCIES

We start with a scalar field

(III.15) $$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi \; .$$

It's S - matrix is given by a functional integral

(III.16) $$Z = \int d\phi \; \exp\left[i \int d^4x \, \mathcal{L}(x)\right] \; .$$

Let us perform a non-linear point transformation on a field $\phi(x)$

(III.17) $$\phi(x) \longrightarrow \Psi(x) + \varkappa \Psi^2(x) \; .$$

The transformation of the functional measure is

(III.18) $$d\phi \rightarrow d\Psi \left(1 + 2\varkappa\Psi\right)$$

and includes a local factor $\prod_x \left[1 + 2\varkappa\Psi(x)\right]$.

Under the transormation (III.17) the Lagrangian changes to

(III.19) $$\mathcal{L} = \frac{1}{2}\partial_\mu \Psi \left(1 + 4\varkappa\Psi + 4\varkappa^2\Psi^2\right)\partial^\mu \Psi =$$

$$= \frac{1}{2}\partial_\mu \Psi (1+2\kappa\Psi)^2 \partial^\mu \Psi$$

and the functional integral (III.16) becomes:

$$Z = \int d\Psi \, (1+2\kappa\Psi) \exp\left[i\int d^4x \, \mathcal{L}(\Psi)\right]. \tag{III.20}$$

We can exponentiate the local factor in the functional measure as:

$$\prod_x (1+2\kappa\Psi) = \exp\left[\sum_x \log\left(1+2\kappa\Psi(x)\right)\right] =$$

$$= \exp\left[\frac{1}{\Delta x}\sum_x \Delta x \log\left(1+2\kappa\Psi(x)\right)\right] \xrightarrow[\Delta x \to 0]{}$$

$$\longrightarrow \exp\left[\delta^4(0)\int d^4x \log\left[1+2\kappa\Psi(x)\right]\right]$$

so, that (III.20) becomes

$$Z = \int d\Psi \exp\left(i\int d^4x \left\{\mathcal{L}(\Psi) - i\,\delta^4(0)\log\left[1+2\kappa\Psi\right]\right\}\right). \tag{III.21}$$

The effective action is

$$S_{\text{EFF.}} = \int d^4x \left\{\mathcal{L}(\Psi) - i\,\delta^4(0)\log\left[1+2\kappa\Psi\right]\right\}. \tag{III.22}$$

Let us carefully investigate the contents of this theory. The momentum conjugate to $\Psi(x)$ is

(III.23) $$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \Psi)} = (1+2\varkappa\Psi)^2 \partial_0 \Psi$$

and $$\partial_0 \Psi = (1+2\varkappa\Psi)^{-2}\pi .$$

The Hamiltonian is

$$\mathcal{H} = \pi\partial_0\Psi - \mathcal{L} = \pi(1+2\varkappa\Psi)^{-2}\pi - \tfrac{1}{2}\partial_\mu\Psi(1+2\varkappa\Psi)^2\partial^\mu\Psi$$

and

(III.24) $$\mathcal{H} = \frac{1}{2}\pi(1+2\varkappa\Psi)^{-2}\pi - \frac{1}{2}\partial_i\Psi(1+2\varkappa\Psi)^2\partial^i\Psi .$$

Let us define:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

(III.25)

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$$

where:

$$\mathcal{L}_0 = \frac{1}{2}\partial_0\Psi\partial^0\Psi$$

(III.26) $$\mathcal{L}_I = \frac{1}{2}\partial_0\Psi(4\varkappa\Psi + 4\varkappa^2\Psi^2)\partial^0\Psi + \frac{1}{2}\partial_i\Psi(1+2\varkappa\Psi)^2\partial^i\Psi$$

$$\mathcal{H}_0 = \frac{1}{2}\pi\pi$$

$$\mathcal{H}_I = -\frac{1}{2}\pi\frac{4\kappa\psi+4\kappa^2\psi^2}{(1+2\kappa\psi)^2}\pi - \frac{1}{2}\partial_i\psi(1+2\kappa\psi)^2\partial^i\psi.$$

Because

$$\pi\frac{4\kappa\psi+4\kappa^2\psi^2}{(1+2\kappa\psi)^2}\pi = \partial_0\psi(1+2\kappa\psi)^2(4\kappa\psi+4\kappa^2\psi^2)\partial^0\psi$$

$$= \partial_0\psi(4\kappa\psi+4\kappa^2\psi^2)\partial^0\psi + \partial_0\psi(4\kappa\psi+4\kappa^2\psi^2)^2\partial^0\psi$$

we see that interaction Hamiltonian is not the negative of the interaction Lagrangian, i.e.

$$\mathcal{H}_I = -\mathcal{L}_I - \frac{1}{2}\partial_0\psi(4\kappa\psi+4\kappa^2\psi^2)^2\partial^0\psi. \tag{III.27}$$

Furthermore, because the interaction Lagrangian is quadratic in the field derivatives, non-covariant propagators appear in the theory. Indeed, defining the contraction

$$\psi(x)^{\bullet}\psi(y)^{\bullet} \equiv \langle 0|T\{\psi(x)\psi(y)\}|0\rangle \tag{III.28}$$

we have

$$\psi(x)^{\bullet}\partial_\mu\psi(y)^{\bullet} = \partial_\mu\langle 0|T\{\psi(x)\psi(y)\}|0\rangle \tag{III.29}$$

but

(III.30) $$\partial_\mu \Psi(x)^{\bullet}\, \partial_\nu \Psi(x)^{\bullet} =$$

$$= \partial_\mu \partial_\nu \langle 0|T\{\Psi(x)\Psi(y)\}|0\rangle - i\,\delta_{\mu 0}\delta_{\nu 0}\,\delta^4(x-y)$$

because: $$\langle 0|T\{\partial_\mu \Psi(x)\,\partial_\nu \Psi(x)\}|0\rangle =$$

$$= \theta(x_0-y_0)\,\partial_\mu \Psi(x)\,\partial_\nu \Psi(y) + \theta(y_0-x_0)\,\partial_\nu \Psi(y)\,\partial_\mu \Psi(x) =$$

$$= \partial_\mu \langle 0|T\{\Psi(x)\partial_\nu \Psi(y)\}|0\rangle - \delta_{\mu 0}\,\delta(x_0-y_0)\left[\Psi(x), \partial_\nu \Psi(y)\right] =$$

$$= \partial_\mu \partial_\nu \langle 0|T\{\Psi(x)\Psi(y)\}|0\rangle - i\,\delta_{\mu 0}\delta_{\nu 0}\,\delta^4(x-y) \; .$$

The non-covariant piece of the propagator (III.30) will contribute divergent terms of the $\delta^4(o)$ - type to the S- matrix. This contribution can be calculated using the following theorem, which is proved in the appendix (9),(10)

A system

(III.31) $$\mathcal{L} = \frac{1}{2}\,\partial_\mu \phi^a G_{ab}(\phi)\,\partial^\mu \phi^b$$

with the contraction

$$\partial_\mu \phi^a(x)^\bullet \partial_\nu \phi^b(y)^\bullet = -\partial_\mu \partial_\nu \left\{ \phi^a(x)^\bullet \phi^b(y)^\bullet \right\} - - i\, \delta_{\mu 0} \delta_{\nu 0} \delta^{ab} \delta^4(x-y) \tag{III.32}$$

and the non-covariant interaction Hamiltonian

$$\mathcal{H}_I = -\mathcal{L}_I - \frac{1}{2} \partial_0 \phi^a (G-1)^2_{ab} \partial^0 \phi^b, \tag{III.33}$$

has the same S - matrix,

$$S = T \exp\left[-i \int d^4x\, \mathcal{H}_I(x) \right], \tag{III.34}$$

as the covariant system with

$$\mathcal{H}_I = -\mathcal{L}_I + \frac{i}{2} \delta^4(0)\, \mathrm{Tr} \log G_{ab}(\phi) \tag{III.35}$$

and the contraction

$$\partial_\mu \phi^a(x)^\bullet \partial_\nu \phi^b(x)^\bullet = -\partial_\mu \partial_\nu \left\{ \phi^a(x)^\bullet \phi^b(y)^\bullet \right\}. \tag{III.36}$$

In other words, we can treat this system as given by:

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L} = \mathcal{L} + \frac{i}{2}\delta^4(0)\,\mathrm{Tr}\log G_{ab}(\phi), \tag{III.37}$$

where in evaluating the S - matrix in the perturbation theory one has to take only the covariant term in the propagator (III.32). The additional term in the action

$$\delta S = \int d^4x\, \delta\mathcal{L}_I = i\int d^4x\, \delta^4(0) \log(1+2\kappa\Psi) \tag{III.38}$$

is the contribution from the non-covariant term in the propagator (III.30).

In the functional formalism this term is cancelled by the local functional measure

$$\prod_x (1+2\kappa\Psi) = \exp\left[-i\int d^4x\, \delta^4(0) \log(1+2\kappa\Psi)\right]$$

so:

The local factor in the functional measure cancels the $\delta^4(0)$ -type divergences coming from the non-covariant term in the propagator of the theory with the derivative couplings.

The cancellation of quartic divergencies by the functional measure occurs in the similiar way in the gravity theory.[11] For cur purpose of calculating the functional measure in quantum gravity we need to know only that the cancellation occurs on the one loop level. We will calculate the contribution of the non-covariant propagators in the theory in the one loop approximation which must be cancelled by the first term in the expansion of the local measure.

IVa. THE FUNCTIONAL MEASURE FOR QUANTUM GRAVITY

The Lagrangian for the theory of gravity[12] is:

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{GF.} + \mathcal{L}_{GH.} \tag{IV.1}$$

where

$$\mathcal{L}_2 = -\frac{2}{\mathcal{K}^2}\sqrt{-g}\,R = -\frac{2}{\mathcal{K}^2}\sqrt{-g}\;g^{\mu\nu}R_{\mu\nu} \tag{IV.2}$$

with $\mathcal{K}^2 = 32\pi G$,

where G is Newton's constant.

The gauge fixing and ghost Lagrangians will be specified later. The Lagrangian (IV.2) contains the total derivative term[13] which can be eliminated to give

$$\mathcal{L}_2 = \frac{2}{\mathcal{K}^2}\sqrt{-g}\;g^{\mu\nu}\left(\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\right) \tag{IV.3}$$

where the Christoffel's symbols are:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\,g^{\alpha\beta}\left(g_{\mu\beta|\nu} + g_{\nu\beta|\mu} - g_{\mu\nu|\beta}\right) \tag{IV.4}$$

and (IV.3) becomes

$$\mathcal{L}_2 = \frac{1}{2\mathcal{K}^2}\sqrt{-g}\;g_{\alpha\beta|\gamma}\,g_{\mu\nu|\delta}\left[g^{\alpha\beta}g^{\mu\nu}g^{\gamma\delta} - \right. \tag{IV.5}$$

$$-2g^{\alpha\beta}g^{\mu\rho}g^{\nu\sigma}-g^{\alpha\mu}g^{\beta\nu}g^{\rho\sigma}+2g^{\alpha\mu}g^{\beta\sigma}g^{\nu\rho}\Big] .$$

To quantize the theory one introduces the quantum gravitational fields $h_{\mu\nu}$ by

$$g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu} \tag{IV.6}$$

where $\eta_{\mu\nu}$ is the metric tensor of Minkowski space with the signature (+---). The quadtratic part of the Lagrangian (IV.5) is the Pauli-Fierz[(14)] free field Lagrangian for a massless spin 2 particle:

$$\bar{\mathcal{L}}_2^{(0)} = \frac{1}{2} h_{\alpha\beta|\rho} h_{\mu\nu|\sigma}\Big[\eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\rho\sigma} - \tag{IV.7}$$

$$-2\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\sigma}+2\eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\nu\rho}\Big] .$$

This Lagrangian is singular due to the invariance under general coordinate transformations:

$$\delta g_{\mu\nu} = \xi^{\alpha}_{\ |\mu} g_{\alpha\nu} + \xi^{\alpha}_{\ |\nu} g_{\mu\alpha} + \xi^{\alpha} g_{\mu\nu|\alpha} \tag{IV.8}$$

and requres the gauge breaking term. We take the harmonic gauge:

$$\text{(IV.9)} \quad F^{\mu} = h^{\mu\alpha}{}_{|\alpha} - \frac{1}{2} h_{\alpha}^{\alpha\,|\mu} =$$

$$= \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\beta\rho} + \eta^{\mu\beta}\eta^{\alpha\rho} - \eta^{\mu\rho}\eta^{\alpha\beta}\right) h_{\alpha\beta|\rho}$$

and the gauge fixing Lagrangian:

$$\text{(IV.10)} \quad \mathcal{L}_{G.F.} = -\eta_{\mu\nu} F^{\mu} F^{\nu} =$$

$$= \frac{1}{2} h_{\alpha\beta|\rho} h_{\mu\nu|\sigma}\left[\frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma} + 2\eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\nu\sigma}\right].$$

Adding (IV.7) and (IV.10) we get the quadratic part of the gravitational Lagrangian:

$$\text{(IV.11)} \quad \mathcal{L}_2^{(o)} = -\frac{1}{2} h_{\alpha\beta|\rho} h_{\mu\nu|\sigma} P^{\alpha\beta,\mu\nu} \eta^{\rho\sigma}$$

where

$$\text{(IV.12)} \quad P^{\alpha\beta,\mu\nu} = \frac{1}{2}\left(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}\right),$$

with the property

$$\text{(IV.13)} \quad P^{\alpha\beta,\rho\sigma} P_{\rho\sigma,\mu\nu} = \frac{1}{2}\left(\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta} + \delta_{\nu}^{\alpha}\delta_{\mu}^{\beta}\right) = \mathbb{1}^{\alpha\beta}_{\mu\nu}.$$

The canonical momenta are

$$\pi^{\alpha\beta} = \frac{\delta \mathcal{L}_2^{(0)}}{\delta h_{\alpha\beta|0}} = -P^{\alpha\beta,\mu\nu} h_{\mu\nu|0} \tag{IV.14}$$

so, that

$$h_{\mu\nu|0} = -P_{\mu\nu,\rho\sigma}\,\pi^{\rho\sigma} \;. \tag{IV.15}$$

The canonical commutation relations are then

$$\left[h_{\alpha\beta}(x), h_{\mu\nu|0}(y)\right]_{x_0=y_0} = i P_{\alpha\beta,\mu\nu}\,\delta^3(\vec{x}-\vec{y}) \;. \tag{IV.16}$$

The non-covariant propagator appear in

$$T\{h_{\alpha\beta|\rho} h_{\mu\nu|\sigma}\} = \partial_\rho \partial_\sigma T\{h_{\alpha\beta} h_{\mu\nu}\} - i\,\delta_{\rho 0}\delta_{\sigma 0} P_{\alpha\beta,\mu\nu}\,\delta^4(0).$$

We are interested only in the quartic divergent terms, so we will take only the non-covariant term in the propagator, i.e.

$$T\{h_{ab} h_{\alpha\beta|\rho} h_{\mu\nu|\sigma}\} = -i\,\delta_{\rho 0}\delta_{\sigma 0} P_{\alpha\beta,\mu\nu}\,\delta^4(0)\, h_{ab} \;. \tag{IV.17}$$

To find the one loop contibution of the quartic divergent terms to S-matrix we start from (IV.5) and use:

$$\sqrt{-g} = 1 + \tfrac{1}{2}\eta^{ab} h_{ab} + \dots\dots$$

$$g_{\alpha\beta|\rho} = \varkappa\, h_{\alpha\beta|\rho}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \varkappa\, h^{\alpha\beta} = \eta^{\alpha\beta} - \varkappa\, \eta^{a\alpha}\eta^{b\beta} h_{ab}\,,$$

to obtain the three graviton vertices:

(IV.18) $$\mathcal{L}_2^{(1)} = \frac{1}{2}\varkappa\, h_{ab}\, h_{\alpha\beta|\rho}\, h_{\mu\nu|\sigma}\, V^{ab,\alpha\beta,\mu\nu,\rho\sigma}$$

where:

(IV.19) $$V^{ab,\alpha\beta,\mu\nu,\rho\sigma} = \frac{1}{2}\eta^{ab}\eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\rho\sigma} -$$

$$- \eta^{ab}\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{2}\eta^{ab}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\sigma} + \eta^{ab}\eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\nu\rho} -$$

$$-2\eta^{a\alpha}\eta^{b\beta}\eta^{\mu\nu}\eta^{\rho\sigma} + 2\eta^{a\alpha}\eta^{b\beta}\eta^{\mu\rho}\eta^{\nu\sigma} + 2\eta^{a\alpha}\eta^{b\mu}\eta^{\beta\nu}\eta^{\rho\sigma} -$$

$$-2\eta^{a\alpha}\eta^{b\mu}\eta^{\beta\sigma}\eta^{\nu\rho} + 2\eta^{a\alpha}\eta^{b\rho}\eta^{\beta\sigma}\eta^{\mu\nu} + 2\eta^{a\alpha}\eta^{b\sigma}\eta^{\beta\rho}\eta^{\mu\nu} -$$

$$-4\eta^{a\alpha}\eta^{b\sigma}\eta^{\beta\mu}\eta^{\nu\rho} - \eta^{a\rho}\eta^{b\sigma}\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{a\rho}\eta^{b\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}.$$

Using (IV.17) we find the quartic divergent terms to be:

$$T\left\{\mathcal{L}_2^{(1)}\right\} = -i\varkappa\,\delta^4(0)\left[\tfrac{1}{2}\eta^{ab} + \eta^a_0\eta^b_0\right] h_{ab}. \tag{IV.20}$$

The only other contribution comes from the ghost Lagrangian:

$$\mathcal{L}_{GH.} = \varkappa\,\bar{\Lambda}_\mu \frac{\delta F^\mu}{\delta \xi^\nu}\Lambda_\nu \tag{IV.21}$$

where Λ_μ is anticommuting ghost field and

$$\delta h_{\alpha\beta} = \frac{1}{\varkappa}\left(\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha\right) + \left(h_{\alpha\lambda}\partial_\beta + h_{\beta\lambda}\partial_\alpha + h_{\alpha\beta|\lambda}\right)\xi^\lambda, \tag{IV.22}$$

where we used (IV.8).

For the quadratic part of the ghost Lagrangian we find:

$$\delta\bar{F}^\mu = P^{\mu\rho,\alpha\beta}\partial_\rho\left(\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha\right) = \partial_\rho\partial^\rho\xi^\mu$$

and

$$\mathcal{L}_{GH.}^{(0)} = -\bar{\Lambda}_{\mu|\rho}\Lambda^{\mu|\rho}. \tag{IV.23}$$

The canonical momenta are:

$$\pi^\mu = \frac{\delta\mathcal{L}_{GH.}^{(0)}}{\delta\Lambda_{\mu|0}} = -\bar{\Lambda}^{\mu|0}$$

and the canonical anticommutation relations

$$\left\{\bar{\Lambda}_{\mu|0}(x),\, \Lambda_\nu(y)\right\}_{x_0=y_0} = i\,\delta_{\mu\nu}\,\delta^3(\vec{x}-\vec{y})\,. \tag{IV.24}$$

The non-covariant term in the propagator is

$$T\{\bar{\Lambda}_{\mu|\rho}\Lambda_{\nu|\sigma}\} = i\,\delta_{\rho 0}\,\delta_{\sigma 0}\,\delta_{\mu\nu}\,\delta^4(0)$$

so, that the quartic divergent term is:

$$T\{h_{ab}\bar{\Lambda}_{\mu|\rho}\Lambda_{\nu|\sigma}\} = i\,\delta_{\mu\nu}\delta_{\rho 0}\delta_{\sigma 0}\,\delta^4(0)\,h_{ab}\,. \tag{IV.25}$$

The one-loop contribution is found from the ghost-graviton vertices. From

$$\delta\bar{\bar{F}} = P^{\alpha\beta,\mu\rho}\partial_\rho\left(h_{\alpha\nu}\partial_\beta + h_{\beta\nu}\partial_\alpha + h_{\alpha\beta|\nu}\right)\xi^\nu$$

we see that the last term doesn't contribute to quartic divergencies, and the first two terms give:

$$\mathcal{L}^{(1)}_{GH.} = -\varkappa\,\bar{\Lambda}_{\mu|\rho}\,P^{\alpha\beta,\mu\rho}\left(h_{\alpha\nu}\eta_{\beta\sigma} + h_{\beta\nu}\eta_{\alpha\sigma}\right)\Lambda^{\nu|\sigma}. \tag{IV.26}$$

Using (IV.25) we find the one-loop quartic divergent contribution to S-matrix:

$$T\{\mathcal{L}^{(1)}_{GH.}\} = -i\varkappa\,\delta^4(0)\,\eta^{ab}h_{ab}\,. \tag{IV.27}$$

Combining (IV.20) and (IV.27) we get the quartic divergent part in the one-loop approximation for the S-matrix:

(IV.28) $$T\{\mathcal{L}^{(1)}\} = T\{\mathcal{L}_2^{(1)} + \mathcal{L}_{GH.}^{(1)}\} = = -i\varkappa\,\delta^4(0)\left[\frac{3}{2}\eta^{ab} + \eta^{ao}\eta^{bo}\right]h_{ab}.$$

This must be cancelled by the contribution coming from the functional measure, which means, that the term of the first order in $\varkappa$ in the functional measure must be:

(IV.29) $$\mathcal{L}_M^{(1)} = i\varkappa\,\delta^4(0)\left[\frac{3}{2}\eta^{ab} + \eta^{ao}\eta^{bo}\right]h_{ab}.$$

This is the first term in the expansion of

(IV.30) $$\mathcal{L}_M = -i\,\delta^4(0)\,\log\left[g^{-3/2}g^{oo}\right]$$

because:

$$g^{oo} = 1-\varkappa h^{oo} = 1-\varkappa\,\eta^{ao}\eta^{bo}h_{ab}\,.$$

We conclude that the local factor in the functional measure for quantum gravity must be:

(IV.31) $$\prod_{x} g^{-3/2}(x)\, g^{00}(x)$$

and the complete functional measure is [8,15]:

(IV.32) $$\prod_{x} \left[g^{-3/2}(x)\, g^{00}(x) \prod_{\mu \leq \nu} dg_{\mu\nu}(x) \right] .$$

IVb. THE FUNCTIONAL MEASURE FOR THE GRAVITATIONAL AND YANG-MILLS FIELDS

The Lagrangian is given by:

(IV.33) $$\mathcal{L} = \mathcal{L}_{G.} + \mathcal{L}_{Y.M.} = -\frac{2}{\kappa^2}\sqrt{-g}\,R - \frac{1}{4}\sqrt{-g}\,g^{\alpha\mu}g^{\beta\nu}F^a_{\alpha\beta}F^a_{\mu\nu},$$

where

(IV.39) $$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + \kappa f_{abc} A^b_\alpha A^c_\beta \quad .$$

The fields A^a_α belong to adjoint representation of internal symmetry group SU(N), so that $a = 1,\ldots,N^2 - 1$.

The quadtratic part of the Yang-Mills Lagrangian is

(IV.35) $$\mathcal{L}^{(0)}_{Y.M.} = -\frac{1}{2}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)A^a_{\alpha|\beta}A^a_{\mu|\nu}$$

and in the gauge: $A^a_\alpha = 0$, we find the canonical momenta:

$$\pi^a_\alpha = -A^a_{\alpha|0}$$

The commutation relations are

(IV.36) $$\left[A^a_\alpha(x), A^b_{\beta|0}(y)\right]_{x_0=y_0} = -i\,\delta^{ab}\left(\delta_{\alpha\beta} - \delta_{\alpha 0}\delta_{\beta 0}\right)\delta^3(\vec{x}-\vec{y}).$$

The non-covariant part of the propagator is:

$$T\left\{A^{a}_{\alpha|\beta} A^{a}_{\mu|\nu}\right\} = i(N^2-1)\left(\delta_{\alpha\mu} - \delta_{\alpha 0}\delta_{\mu 0}\right)\delta_{\beta 0}\delta_{\nu 0}\,\delta^4(0) \tag{IV.37}$$

where the factor (N^2-1) results from the summation over SU(N) indices.

The first order Yang-Mills Langrangian is

$$\mathcal{L}^{(1)}_{Y.M.} = \frac{1}{2}\kappa\, h_{ak}\, A^{i}_{\alpha|\beta} A^{i}_{\mu|\nu}\, D^{ak,\alpha\beta,\mu\nu} \tag{IV.38}$$

with

$$D^{ak,\alpha\beta,\mu\nu} = \frac{1}{2}\eta^{ak}\eta^{\alpha\mu}\eta^{\beta\nu} - \frac{1}{2}\eta^{ak}\eta^{\alpha\nu}\eta^{\beta\mu} - \tag{IV.39}$$

$$- \eta^{a\alpha}\eta^{k\mu}\eta^{\beta\nu} + 2\eta^{a\alpha}\eta^{k\nu}\eta^{\beta\mu} - \eta^{a\beta}\eta^{k\nu}\eta^{\alpha\mu} .$$

The quartic divergent term is then:

$$T\left\{\mathcal{L}^{(1)}_{Y.M.}\right\} = i\kappa(N^2-1)\,\delta^4(0)\left[\frac{1}{4}\eta^{ak} - \eta^{a}_{0}\eta^{k}_{0}\right] h_{ak} \tag{IV.40}$$

where we used (IV.37).

This term must be cancelled by the first order contribution from the fuctional measure, i.e.

$$\mathcal{L}^{(1)}_{M_{Y.M.}} = -i\kappa\,\delta^4(0)(N^2-1)\left[\tfrac{1}{4}\eta^{ab} - \eta^{a0}\eta^{b0}\right] h_{ab}. \tag{IV.41}$$

which is the first term in the expansion of

$$\mathcal{L}_{M_{Y.M.}} = -i\,\delta^4(0)\,\log M_{Y.M.} \tag{IV.42}$$

Finally we get the local factor in the functional measure for the Yang-Mills fields:

$$M_{Y.M.} = g^{N^2-1/4}\,(g^{00})^{N^2-1} \tag{IV.43}$$

where we used:

$$g = 1 + \kappa\,\eta^{ab} h_{ab} + \cdots$$

$$g^{00} = 1 - \kappa h^{00} = 1 - \kappa\,\eta^{a0}\eta^{b0} h_{ab}$$

The total functional measure for gravitational and Yang-Mills fields is then:

$$\prod_x \left[g^{N-1/4}(x)\, g^{00\,N^2-1}(x) \prod_{\mu\leq\nu} dg_{\mu\nu}(x) \prod_{a,\alpha} dA^a_\alpha \right. . \tag{IV.44}$$

This result was first obtained using different method by Aragone[16].

IVc. THE FUNCTIONAL MEASURE FOR SUPERGRAVITY

In this section we will calculate the functional measure for supergravity. The result we obtain is the same as the one obtained by authors in ref. (18), although we work in different gauges.

The Lagrangian of the theory is

(IV.45) $$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{3/2} + \mathcal{L}_4$$

with

$$\mathcal{L}_2 = -\frac{2}{\kappa^2}\sqrt{-g}\,R$$

$$\mathcal{L}_{3/2} = -\frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\overline{\psi}_\mu \gamma_5 \gamma_\rho D_\sigma \psi_\nu$$

$\mathcal{L}_4$ is a term quartic in spin 3/2 Majorana fields $\psi_\mu(x)$ and is not important for our considerations. Our conventions are the same as those in ref. (19). The spin 2 Lagrangian is given in terms of the vierbein fields $e_{\mu a}$ instead of the metric $g_{\mu\nu}$. Greek indices refer to the Riemman space while atin indices refer to locally flat space. The basic formulae are:

$$g_{\mu\nu} = e_{\mu a} e^a_\nu \qquad \eta_{ab} = e_{a\mu} e^\mu_b$$

$$\gamma^\mu = e^{\mu a}\gamma_a \qquad D_\rho = \partial_\rho + \frac{1}{2}\,\omega_{\rho,ab}\,\sigma^{ab}$$

$$R = e^{\mu a} R_{\mu a} \qquad R_{\mu a} = e^{\nu b} R_{\mu\nu ab}$$

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu,ab} - \partial_\nu \omega_{\mu,ab} + \omega_{\mu,a}{}^c \omega_{\nu,bc} - \omega_{\nu,a}{}^c \omega_{\mu,bc}$$

$$\omega_{\nu,ab} = \frac{1}{2}\left[e^\mu_a\left(e_{\mu b|\nu} - e_{\nu b|\mu}\right) + e^\rho_a e^\sigma_b e^c_\nu\, e_{\rho c|\sigma}\right] - (a \leftrightarrow b)$$

Using the above and discarding the surface term the spin 2 Lagrangian can be written as:

(IV.46) $$\mathcal{L}_2 = \frac{\sqrt{-g}}{4\kappa^2}\left(e^{a\mu}e^{b\nu} - e^{a\nu}e^{b\mu}\right)\omega_{\mu,a}{}^{c}\omega_{\nu,bc} \, .$$

To quantize the theory we introduce the quantum vierbein field $C_{\mu a}$ by:

(IV.47) $$\begin{aligned} e_{\mu a} &= \eta_{\mu a} + \kappa C_{\mu a} \\ e^{\mu a} &= \eta^{\mu a} - \kappa C^{\mu a} + \kappa^2 C^{\mu b}\eta_{b\nu}C^{\nu a} - \ldots \end{aligned}$$

in terms of which we have:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\eta_{\mu a}C^{a}_{\nu} + \kappa C_{\mu a}\eta^{a}_{\nu} + \kappa^2 C_{\mu a}C^{a}_{\nu}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa\eta^{\mu a}C^{\nu}_{a} - \kappa C^{\mu a}\eta^{\nu}_{a} + 3\kappa^2 C^{\mu a}C_{a}^{\nu} - \ldots$$

$$\sqrt{-g} = \left(1 + 2\kappa\eta^{\mu a}C_{\mu a} + \ldots\right)^{1/2} = 1 + \kappa\eta^{\mu a}C_{\mu a} + \ldots$$

$$\omega^{(1)}_{\nu,ab} = \kappa\left(C_{\nu a|b} - C_{\nu b|a}\right)$$

$$\omega^{(2)}_{\nu,ab} = \frac{\kappa^2}{2}\eta^{\rho c}\Big[-C_{\nu c}C_{\rho a|b} + 2C_{\rho a}C_{\nu b|c} - C_{\rho a}C_{\nu c|b} - C_{\rho a}C_{\mu c|\nu}\eta^{\mu}_{b}\Big] - (a \leftrightarrow b)$$

Collecting the terms ofzeroth and first order in $\mathcal{H}$ from (IV.46) we find:

(IV.48)
$$\mathcal{L}_2^{(0)} = \frac{1}{4} C_{\mu a|\rho} C_{\nu h|b} \Big[\eta^{\mu a}\eta^{\nu h}\eta^{\rho b} - \eta^{\mu a}\eta^{\nu\rho}\eta^{hb} - \eta^{\mu\rho}\eta^{ab}\eta^{\nu h} - \eta^{\mu\nu}\eta^{ah}\eta^{\rho b} + \eta^{\mu\nu}\eta^{a\rho}\eta^{hb} + \eta^{\mu\nu}\eta^{ab}\eta^{h\rho} \Big]$$

and

(IV.49)
$$\begin{aligned} \mathcal{L}_2^{(1)} = \frac{\mathcal{H}}{4} C_{\alpha a} C_{\mu i|\rho} C_{\nu j|b} \Big[& \eta^{\alpha a}\eta^{\mu i}\eta^{\nu j}\eta^{\rho b} - \\ & -2\eta^{\alpha a}\eta^{\mu i}\eta^{\nu\rho}\eta^{jb} - \eta^{\alpha a}\eta^{\mu\nu}\eta^{ij}\eta^{\rho b} + \eta^{\alpha a}\eta^{\mu\nu}\eta^{i\rho}\eta^{jb} + \\ & + \eta^{\alpha a}\eta^{\mu\nu}\eta^{ib}\eta^{j\rho} - 4\eta^{\alpha\mu}\eta^{ai}\eta^{\nu j}\eta^{\rho b} + 4\eta^{\alpha\mu}\eta^{ai}\eta^{\nu\rho}\eta^{jb} + \\ & + 4\eta^{\alpha\mu}\eta^{a\nu}\eta^{ij}\eta^{\rho b} - 2\eta^{\alpha\mu}\eta^{a\nu}\eta^{i\rho}\eta^{jb} - 2\eta^{\alpha\mu}\eta^{a\nu}\eta^{ib}\eta^{j\rho} + \\ & + 4\eta^{\alpha\mu}\eta^{a\rho}\eta^{ib}\eta^{\nu j} - 4\eta^{\alpha\mu}\eta^{a\rho}\eta^{ij}\eta^{\nu b} + 4\eta^{\alpha\mu}\eta^{ab}\eta^{i\rho}\eta^{\nu j} - \\ & - 4\eta^{\alpha\mu}\eta^{ab}\eta^{ij}\eta^{\nu\rho} - 2\eta^{\alpha\rho}\eta^{ab}\eta^{\mu i}\eta^{\nu j} + 2\eta^{\alpha\rho}\eta^{ab}\eta^{\mu\nu}\eta^{ij} \Big] \end{aligned}$$

To define the propgator we need to specify the gauge. Taking the gauge condition as in ref. (19)

(IV.50)
$$F_\mu = C_{\mu a}{}^{|a} - \frac{1}{2}\eta^{\nu\alpha} C_{\nu\alpha|\mu}$$

we get a gauge fixing Lagrangian

$$\mathcal{L}_{G.F.} = -\frac{1}{2} F_\mu \eta^{\mu\nu} F_\nu =$$

$$= -\frac{1}{2} G_{\mu a|\rho} C_{\nu h|b} \Big[\frac{1}{4} \eta^{\mu a}\eta^{\nu h}\eta^{\rho b} - \frac{1}{2}\eta^{\mu a}\eta^{\nu\rho}\eta^{hb} - \frac{1}{2}\eta^{\mu\rho}\eta^{ab}\eta^{\nu h} + \eta^{\mu\nu}\eta^{a\rho}\eta^{hb} \Big] \,. \tag{IV.51}$$

From (IV.48) and (IV.51) we have:

$$\mathcal{L}_0 = \frac{1}{4} G_{\mu a|\rho} C_{\nu h|b} \Big[\frac{1}{2}\eta^{\mu a}\eta^{\nu h}\eta^{\rho b} - \frac{1}{2}\eta^{\mu\nu}\eta^{ah}\eta^{\rho b} - \frac{1}{2}\eta^{\mu h}\eta^{\nu a}\eta^{\rho b} \Big] = \tag{IV.52}$$

$$= \frac{1}{4} G_{\mu a|\rho} C_{\nu h|b} P^{\mu a,\nu h} \eta^{\rho b}$$

with

$$P^{\mu a,\nu h} = \frac{1}{2}\left(\eta^{\mu a}\eta^{\nu h} - \eta^{\mu\nu}\eta^{ah} - \eta^{\mu h}\eta^{\nu a} \right) \tag{IV.53}$$

and it's inverse

$$P^{-1}_{\nu h,\rho c} = \frac{1}{2}\left(\eta_{\nu h}\eta_{\rho c} - \eta_{\nu\rho}\eta_{hc} - \eta_{\nu c}\eta_{h\rho} \right). \tag{IV.54}$$

The canonical momenta are

$$\pi^{\mu a} = \frac{\delta \mathcal{L}^{(0)}}{\delta G_{\mu a|0}} = \frac{1}{2} P^{\mu a,\nu h} C_{\nu h|0}$$

so, that

$$C_{\nu h|0} = 2 P^{-1}_{\nu h,\rho c} \pi^{\rho c} \,.$$

Taking the commutation relations:

$$\left[C_{\mu a}(x), \pi^{\rho b}(y)\right]_{x_0=y_0} = -i\, \mathbb{1}_{\mu a}{}^{\rho c}\, \delta^3(\vec{x}-\vec{y})$$

and

$$\left[C_{\mu a}(x), C_{\nu e|b}(y)\right]_{x_0=y_0} = -2i\, P^{-1}_{\mu a,\nu e}\, \delta_{b0}\, \delta^3(\vec{x}-\vec{y}),$$

the quartic divergent part becomes

$$T\left\{C_{\mu a|\rho}\, C_{\nu e|b}\right\} = 2i\, P^{-1}_{\mu a,\nu e}\, \delta_{\rho 0}\, \delta_{b0}\, \delta^4(0)\,. \tag{IV.55}$$

The one-loop contribution to quartic divergences is found from (IV.49) to be:

$$T\left\{\mathcal{L}_2^{(1)}\right\} = i\varkappa\, \delta^4(0)\, C_{\mu a}\left[-\eta^{\mu a} - 2\,\eta^{\mu}_{0}\,\eta^{a}_{0}\right]. \tag{IV.56}$$

The only other contribution comes from the ghost Lagrangian

$$\mathcal{L}_{GH.} = \varkappa\, \bar{\Lambda}^{\mu}\, \frac{\delta F_\mu}{\delta \xi^\nu}\, \Lambda^\nu$$

where the gauge parameter $\xi^\nu(x)$ is a vector field.

The transformation law for the veirbein field is from(IV.8):

$$\delta C_{\mu a} = \left[\frac{1}{\varkappa}\eta_{a\nu}\partial_\mu + C_{\nu a}\partial_\mu + C_{\mu a|\nu}\right]\xi^\nu \tag{IV.57}$$

so, that the ghost Lagrangian becomes:

(IV.58) $$\mathcal{L}^{(o)}_{GH.} = -\bar{\Lambda}^{\mu}_{\ 1\nu}\Lambda^{\nu}_{\ 1\mu} + \frac{1}{2}\bar{\Lambda}^{\mu}_{\ 1\mu}\Lambda^{\nu}_{\ 1\nu}$$

(IV.59) $$\mathcal{L}^{(1)}_{GH.} = -\varkappa\,\bar{\Lambda}^{\mu 1a} C_{a\nu}\Lambda^{\nu}_{\ 1\mu} + \frac{\varkappa}{2}\bar{\Lambda}^{\mu}_{\ 1\mu} C_{\nu a}\Lambda^{\nu 1a} - \\ -\varkappa\,\bar{\Lambda}^{\mu 1a} C_{\mu a 1\nu}\Lambda^{\nu} + \frac{\varkappa}{2}\bar{\Lambda}^{\mu}_{\ 1\mu}\eta^{a\beta} C_{a\beta 1\nu}\Lambda^{\nu}$$

The last two terms in $\mathcal{L}^{(1)}_{GH.}$ do not contribute to $\delta^4(0)$ terms and we need concern ourselves with the first two terms only. From (IV.58) we find:

$$\pi_0 = -\frac{1}{2}\bar{\Lambda}^{0}_{\ 10} + \frac{1}{2}\bar{\Lambda}^{i}_{\ i} \qquad (i = 1,2,3)$$

$$\pi_i = -\bar{\Lambda}^{0}_{\ 1i}$$

and

$$\bar{\Lambda}^{0}_{\ 1\mu} = -(\delta_{\mu\nu} + \delta_{\mu 0}\delta_{\nu 0})\pi^{\nu} + \bar{\Lambda}^{i}_{\ 1i}\,\delta_{\mu 0}\,.$$

The commutation relations are

$$\{\pi_\mu(x), \Lambda_\nu(y)\}_{x_0 = y_0} = -i\,\delta_{\mu\nu}\,\delta^3(\vec{x}-\vec{y})$$

$$\{\bar{\Lambda}^{\alpha}_{\ 1\mu}(x)\Lambda_\nu(y)\}_{x_0 = y_0} = i\,\delta^{\alpha 0}(\delta_{\mu\nu} + \delta_{\mu 0}\delta_{\nu 0})\,\delta^3(\vec{x}-\vec{y})$$

and the non-covariant part is:

(IV.60) $$T\{\bar{\Lambda}_{\alpha 1\mu}\Lambda_{\nu 1\rho}\} = i\,\delta_{\alpha 0}(\delta_{\mu\nu} + \delta_{\mu 0}\delta_{\nu 0})\,\delta_{\rho 0}\,\delta^4(0)\,.$$

From (IV.60) and (IV.59) the ghost contribution to quartic divergences is:

$$T\{\mathcal{L}^{(1)}_{GH.}\} = i\kappa\,\delta^4(0)\,G_{\mu a}\left[-\eta^{\mu a}\right]. \tag{IV.61}$$

Now we need to find the quartic divergences associated with spin 3/2 fields. Using the formula:

$$\begin{aligned}\epsilon^{\mu\nu\rho b}\gamma_5\gamma_\rho &= i\sqrt{-g}\left[g^{\mu\nu}\gamma^b - g^{\mu b}\gamma^\nu - \gamma^\mu g^{\nu b} + \gamma^\mu\gamma^b\gamma^\nu\right]\\ &= \frac{i}{2}\sqrt{-g}\left[\gamma^\mu\gamma^b\gamma^\nu - \gamma^\nu\gamma^b\gamma^\mu\right]\end{aligned}$$

we can write the Lagrangian as:

$$\mathcal{L}_{3/2} = \frac{i}{4}\sqrt{-g}\,\overline{\psi}_\mu\left(\gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\nu\right)D_\rho\psi_\nu\,. \tag{IV.62}$$

The quadratic part of this Lagrangian is

$$\mathcal{L}^{(0)}_{3/2} = \frac{i}{4}\,\overline{\psi}_a\left(\gamma^b\gamma^c\gamma^a - \gamma^a\gamma^c\gamma^b\right)\partial_c\psi_b\,.$$

Taking the gauge:

$$F_{(i)} = \eta^{a\mu}\,\gamma_{a_{(ij)}}\psi_{\mu_{(j)}}\;;\quad \overline{F}_{(j)} = \eta^{\mu a}\,\overline{\psi}_{\mu_{(i)}}\gamma_{a_{(ij)}} \tag{IV.63}$$

(indices in parenthesis are spinoreal),

and the gauge fixing Lagrangian:

$$\mathcal{L}_{GF.} = \frac{i}{4}\,\overline{F}_{(\alpha)}\,\partial\!\!\!/_{(\alpha\beta)}\,F_{(\beta)} = \frac{i}{4}\,\overline{\psi}_a\,\gamma^a\gamma^c\gamma^b\partial_c\psi_b$$

we get:

Canonical momenta and commutation relations are:

$$\Pi^{b} = \frac{\delta \mathcal{L}^{(0)}}{\delta(\partial_0 \psi_b)} = \frac{i}{4} \overline{\psi}_a \gamma^b \gamma^0 \gamma^a$$

and

$$\overline{\psi}_a = \Pi^{b}\left(-i \gamma_a \gamma_0 \gamma_b\right)$$

so, that:

$$\left\{ \Pi_{c_{(i)}}(x),\ \psi_{b_{(j)}}(y) \right\}_{x_0 = y_0} = -i\, \delta_{bc}\, \delta_{ij}\, \delta^3(\vec{x}-\vec{y})$$

and

$$\left\{ \overline{\psi}_{a_{(i)}}(x),\ \psi_{b_{(j)}}(y) \right\}_{x_0 = y_0} = -\left(\gamma_a \gamma_0 \gamma_b - 4\delta_{a0}\delta_{b0}\gamma_0\right)_{(ji)} \delta^3(\vec{x}-\vec{y})$$

which ensures consistency with the gauge condition (IV.63).

The quartic divergent part is then:

$$T\left\{ \overline{\psi}_{a_{(i)}}\, M_{(ij)}\, \partial_\rho\, \psi_{b_{(j)}} \right\} = \tag{IV.64}$$

$$= -\frac{1}{2}\delta^4(0)\, \mathrm{Tr}\left[M\left(\gamma_a\gamma_0\gamma_b - 4\delta_{a0}\delta_{b0}\gamma_0\right)\right]\delta_{\rho 0}\,,$$

where M_{ij} is any combination of γ-matrices.

½ is present on the right side of (IV.64) because a Majorana field has half as many degrees of freedom as complex Rarita-Schwinger field.

Neglecting the terms in which the derivatives of the veirbein appear, (their contribution to quartic divergences being zero), the first order Lagrangian is:

$$\mathcal{L}^{(1)}_{3/2} = \frac{i\kappa}{4} G_{\mu a}\, \eta^{\mu a}\, \overline{\Psi}_{b} \left(\gamma^{c}\gamma^{i}\gamma^{b} - \gamma^{b}\gamma^{i}\gamma^{c}\right)\partial_i \Psi_c +$$

$$+ \frac{i\kappa}{4} G_{\mu a}\, \overline{\Psi}_{b} \Big[-\eta^{\mu c}\gamma^{a}\gamma^{i}\gamma^{b} - \eta^{\mu i}\gamma^{c}\gamma^{a}\gamma^{i} -$$

$$- \eta^{\mu b}\gamma^{c}\gamma^{i}\gamma^{a} + \eta^{\mu b}\gamma^{a}\gamma^{i}\gamma^{c} + \eta^{\mu i}\gamma^{b}\gamma^{a}\gamma^{c} + \eta^{\mu c}\gamma^{b}\gamma^{i}\gamma^{a} \Big]\partial_i \Psi_c$$

Using (IV.64) we have:

$$\text{(IV.65)} \qquad T\left\{\mathcal{L}^{(1)}_{3/2}\right\} = i\kappa\, \delta^4(0)\, G_{\mu a}\left[-2\eta^{\mu a} + 2\eta^{\mu}_{\ 0}\eta^{a}_{\ 0}\right] .$$

Finally, combining (IV.56), (IV.61) and (IV.65) one gets;

$$\text{(IV.66)} \qquad T\left\{\mathcal{L}^{(1)}_{2} + \mathcal{L}^{(1)}_{GH.} + \mathcal{L}^{(1)}_{3/2}\right\} = i\kappa\, \delta^4(0)\, G_{\mu a}\left[-4\eta^{\mu a}\right] .$$

This must be cancelled by the first order term comming from the measure, which means that:

$$\mathcal{L}^{(1)}_{M.} = i\kappa\, \delta^4(0)\, G_{\mu a}\left[4\eta^{\mu a}\right]$$

and the measure term in the Lagrangian is

$$\mathcal{L}_{M} = -i\,\delta^4(0) \log e^{-4} = -i\,\delta^4(0) \log g^{-2}$$

giving the local measure for the supergravity:

$$\prod_x g^{-2}(x) \ .$$

APPENDIX

We prove the theorem cited in the section IIIb. The result was obtained by Lee and Yang[9] by explicitly summing the relevant graphs. This proof is due to Gerstein and all.[10]

Let us consider a system with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a G_{ab}(\phi) \partial^\mu \phi^b \tag{A1}$$

where

$$G_{ab}(\phi) = \delta_{ab} + \bar{G}_{ab}(\phi) \; . \tag{A2}$$

We divide the Lagrangian into free and interacting part

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

given by

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi_a$$

$$\mathcal{L}_I = \frac{1}{2} \partial_\mu \phi^a \bar{G}_{ab} \partial^\mu \phi^b \; . \tag{A3}$$

The momemta are defined as

$$\pi_a = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a} = G_{ab} \partial^0 \phi^b$$

$$\partial_0 \phi_a = G^{-1}_{ab} \pi^b$$

and the Hamiltonian of the theory is

(A4) $$\mathcal{H} = \mathcal{H}_o + \mathcal{H}_I = \frac{1}{2}\pi^a G^{-1}_{ab}\pi^b - \frac{1}{2}\partial_i\phi^a G_{ab}\partial^i\phi^b$$

with

(A5) $$\mathcal{H}_o = \frac{1}{2}\pi_a\pi^a - \frac{1}{2}\partial_i\pi^a\partial^i\pi_a$$

and

(A6) $$\mathcal{H}_I = -\frac{1}{2}\partial_\mu\phi^a\bar{G}_{ab}\partial^\mu\phi^b - \frac{1}{2}\partial_o\phi^a\bar{G}^2_{ab}\partial^a\phi^b =$$

$$= -\mathcal{L}_I - \frac{1}{2}\partial_o\phi^a\bar{G}^2_{ab}\partial^o\phi^b .$$

The S - matrix for the theory is

(A7) $$S = T\ \exp\left[-i\int d^4x\ \mathcal{H}_I(x)\right]$$

where the operator $\mathcal{H}_I$ should be written in terms of the interaction picture operators. The transformation to the interaction picture is easy once the interaction Hamiltonian is written in terms of ϕ^a, $\partial_i\phi^a$ and π^a. Then we can make the substitution:

$$\phi_a \longrightarrow \mathcal{S}_a$$

$$\partial_i \phi_a \longrightarrow \partial_i \varphi_a$$

$$\pi_a \longrightarrow \partial_0 \varphi_a$$

and

$$\partial_0 \phi_a \longrightarrow G^{-1}_{ab} \partial^0 \varphi^b$$

so, that the Hamiltonian in the interaction picture becomes:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$$

where

(A8) $$\mathcal{H}_0 = \frac{1}{2} \partial_0 \varphi^a \partial^0 \varphi_a - \frac{1}{2} \partial_i \varphi^a \partial^i \varphi_a$$

and

(A9) $$\mathcal{H}_I = -\frac{1}{2} \partial_\mu \varphi^a \bar{G}_{ab} \partial^\mu \varphi^b + \frac{1}{2} \partial_0 \varphi^a \left[\bar{G}^2 (1+\bar{G})^{-1} \right]_{ab} \partial_0 \varphi^b.$$

It is easy to verify that there are non-covariant propagators in this theory:

$$\delta_{ab} \Delta(k^2) = \int d^4x \, e^{ikx} \langle 0 | T\{\varphi_a(x) \varphi_b(0)\} | 0 \rangle$$

(A10) $$\delta_{ab} \Delta_\mu(k) = \int d^4x \, e^{ikx} \langle 0 | T\{\partial_\mu \varphi_a(x) \varphi_b(0)\} | 0 \rangle$$

$$\delta_{ab}\Delta_{\mu\nu}(k) = \int d^4x\, e^{ikx} \langle 0|T\{\partial_\mu \varphi_a(x)\partial_\nu \varphi_b(0)\}|0\rangle$$

the explicit form of which is

(A11) $$\Delta(k^2) = \frac{i}{k^2 + i\epsilon}$$

(A12) $$\Delta_\mu(k) = \frac{k_\mu}{k^2 + i\epsilon}$$

(A13) $$\Delta_{\mu\nu}(k) = \frac{i k_\mu k_\nu}{k^2 + i\epsilon} - g_{\mu 0} g_{\nu 0} .$$

One should prove that the S - matrix for this theory is identical to the S - matrix for the theory given by:

$$\mathcal{H}_I = -\frac{1}{2}\partial_\mu \varphi^a \overline{G}_{ab} \partial^\mu \varphi^b + \delta H$$

with

$$\Delta_{\mu\nu}(k) = \frac{i\, k_\mu k_\nu}{k^2 + i\epsilon} .$$

Let us rewrite the interaction Hamiltonian (A9) as:

(A14) $$\mathcal{H}_I = -\frac{1}{2}\partial^\mu \varphi^a H_{ab,\mu\nu}(\varphi)\partial^\nu \varphi^b$$

where

(A15) $$H_{ab,\mu\nu}(\varphi) = \overline{G}_{ab} g_{\mu\nu} - \left[\overline{G}^2(1+\overline{G})^{-1}\right]_{ab} n_\mu n_\nu .$$

We introduced a unit timelike vector n_μ .

The S - matrix is

$$S = T\exp\left[\frac{i}{2}\int d^4x\, \partial^\mu \varphi^a H_{ab,\mu\nu} \partial^\nu \varphi^b\right] . \tag{A16}$$

The inverse of the interaction Hamiltonian (A14) is:

$$H^{-1}_{ab,\mu\nu} = \bar{G}^{-1}_{ab} g_{\mu\nu} + \delta_{ab} n_\mu n_\nu \tag{A17}$$

because

$$H_{ab,\mu\nu} H^{-1}_{bc,\rho\sigma} g^{\nu\rho} = \delta_{ac} g_{\mu\sigma} . \tag{A18}$$

The propagators of the theory are

$$\langle 0|T\{\varphi_a(x)\varphi_b(y)\}|0\rangle = i\,\Delta_{ab}(x-y) \tag{A19a}$$

$$\langle 0|T\{\partial_\mu\varphi_a(x)\varphi_b(y)\}|0\rangle = i\,\partial_\mu \Delta_{ab}(x-y) \tag{A19b}$$

$$\begin{aligned}\langle 0|T\{\partial_\mu\varphi_a(x)\partial_\nu\varphi_b(y)\}|0\rangle &= -i\,\partial_\mu\partial_\nu \Delta_{ab}(x-y) - \\ &- i\, n_\mu n_\nu \delta_{ab} \delta^4(x-y) \equiv i\,\Delta_{ab,\mu\nu}(x-y)\end{aligned} \tag{A19c}$$

We want to collect all terms in S that contain $\Delta_{ab,\mu\nu}(x-y)$, i.e. we want to perform all the contractions between $\partial_\mu\varphi_a$ and $\partial_\nu\varphi_b$. To achieve that we replace $\partial_\mu \varphi_a(x)$ by:

$$\partial^\mu \varphi_a(x) \rightarrow \partial^\mu \varphi_a(x) + i\int d^4x' \Delta^{\mu\mu'}_{aa'}(x-x') \frac{\delta}{\delta\, \partial^{\mu'} \varphi_{a'}(x')} \quad .$$

From (A16) we get:

$$\text{(A20)} \quad S = :\exp\left\{\frac{i}{2}\int d^4x \left[\partial^\mu \varphi_a(x) + i\int d^4x' \Delta^{\mu\mu'}_{aa'}(x-x') \frac{\delta}{\delta \partial^{\mu'} \varphi_{a'}(x')}\right]\right.$$

$$\left. H_{ab,\mu\nu}(x)\left[\partial^\nu \varphi_b(x) + i\int d^4x'' \Delta^{\nu\nu'}_{bb'}(x-x') \frac{\delta}{\delta\, \partial^{\nu'} \varphi_{b'}(x'')}\right]\right\}:$$

because $H_{ab,\mu\nu}(x)$ doesn't depend on $\partial^\mu \varphi_a(x)$, although it does depend on $\varphi_a(x)$. To simplify the equations we introduce matrix notation:

$$\partial^\mu \varphi_a(x) \equiv Q$$

Q being a vector with three indices,

a - isospin index

μ - space-time index

x - "coordinate" index

and matrices:

$$\Delta \equiv \Delta^{\mu\nu}{}_{ab}(x-y)$$

$$H \equiv H_{ab,\mu\nu}(x)\, \delta^4(x-y)$$

$$H^{-1} \equiv H^{-1}_{ab,\mu\nu}(x)\, \delta^4(x-y)$$

The S - matrix is then:

(A21) $$\mathcal{S} = :\exp\left\{\frac{i}{2}\left[Q + i\left(\Delta\frac{\delta}{\delta Q}\right)\right] H \left[Q + i\left(\Delta\frac{\delta}{\delta Q}\right)\right]\right\}: .$$

Differentiating with respect to Q we have

(A22) $$\frac{\delta \mathcal{S}}{\delta Q} = \left(iHQ - H\Delta\frac{\delta}{\delta Q}\right)\mathcal{S}$$

with the solution:

$$\frac{\delta \mathcal{S}}{\delta Q} + H\Delta\frac{\delta \mathcal{S}}{\delta Q} = iHQ\mathcal{S}$$

$$(1 + H\Delta)\frac{\delta \mathcal{S}}{\delta Q} = iHQ\mathcal{S}$$

$$(H^{-1} + \Delta)\frac{\delta \mathcal{S}}{\delta Q} = iQ\mathcal{S}$$

$$\frac{\delta \mathcal{S}}{\mathcal{S}} = i(H^{-1} + \Delta)^{-1} Q\,\delta Q$$

$$\log\frac{\mathcal{S}}{C} = \frac{1}{2} i\, Q(H^{-1} + \Delta)^{-1} Q$$

(A23) $$\mathcal{S} = C:\exp\left[\frac{i}{2} Q(H^{-1} + \Delta)^{-1} Q\right]: .$$

To determine the constraint C we can use

$$\frac{\delta \mathcal{S}}{\delta Q} = i(H^{-1} + \Delta)^{-1} Q\mathcal{S}$$

and differentiate (A21) and (A23) with respect to H to get:

$$\frac{\delta S}{\delta H} = \left\{ \frac{i}{2} \left[Q + i\Delta \frac{\delta}{\delta Q} \right] \left[Q + i\Delta \frac{\delta}{\delta Q} \right] \right\} S =$$

$$= \left\{ \frac{i}{2} Q^2 - \frac{1}{2}\left(\Delta \frac{\delta}{\delta Q}\right) Q - \frac{1}{2} Q \Delta \frac{\delta}{\delta Q} - \frac{i}{2}\left(\Delta \frac{\delta}{\delta Q}\right)\left(\Delta \frac{\delta}{\delta Q}\right) \right\} S =$$

$$= \left\{ \frac{i}{2} Q^2 - \frac{1}{2}\Delta - \frac{1}{2} Q \Delta \frac{\delta}{\delta Q} + \frac{1}{2}\left(\Delta \frac{\delta}{\delta Q}\right) \Delta \left(H^{-1} + \Delta\right)^{-1} Q \right\} S =$$

$$= \left\{ \frac{i}{2} Q^2 - \frac{1}{2}\Delta - \frac{1}{2} Q \Delta \frac{\delta}{\delta Q} + \frac{1}{2} \Delta \left(H^{-1} + \Delta\right)^{-1} \Delta \right\} S$$

and because

$$S\Big|_{Q=0} = C$$

we have

$$\frac{\delta S}{\delta H}\Big|_{Q=0} = -\frac{1}{2}\Delta C + \frac{1}{2}\Delta\left(H^{-1} + \Delta\right)^{-1} \Delta C =$$

$$= -\frac{1}{2}\Delta\left[1 - \left(H^{-1} + \Delta\right)^{-1}\Delta\right] C =$$

$$= -\frac{1}{2}\Delta\left(H^{-1} + \Delta\right)^{-1} H^{-1} C$$

and

$$\frac{\delta S}{\delta H}\Big|_{Q=0} = -\frac{1}{2}\left(\Delta^{-1} + H\right)^{-1} C \tag{A24}$$

and from (A23):

(A25)
$$\left.\frac{\delta S}{\delta H}\right|_{Q=0} = \frac{\delta C}{\delta H}$$

so, that:

(A26)
$$\frac{\delta C}{\delta H} = -\frac{1}{2}\left(\Delta^{-1}+H\right)^{-1} C$$

with the solution:

$$\frac{\delta C}{C} = -\frac{1}{2}\,\Delta\left(1+\Delta H\right)^{-1}\delta H$$

and

(A27)
$$C = A\ \det\left(1+\Delta H\right)^{-1/2}.$$

To fix the constant A we require

$$S\big|_{H=0} = C\big|_{H=0} = 1 \implies A = 1$$

so, that

(A28)
$$C = \det\left(1+\Delta H\right)^{-1/2}.$$

The S - matrix is

(A29)
$$S = \exp\left[-\tfrac{1}{2}\log\det H\right]\exp\left[-\tfrac{1}{2}\log\det\left(H^{-1}+\Delta\right)\right]$$
$$:\exp\left[\tfrac{1}{2}\,iQ\left(H^{-1}+\Delta\right)^{-1}Q\right]:$$

To separate the non-covariant terms in S we introduce from (A17) and (A19c):

$$\bar{\Delta} = \Delta + n \qquad\qquad \Delta = \bar{\Delta} - n$$

$$\bar{H}^{-1} = H^{-1} - n \qquad\qquad H^{-1} = \bar{H}^{-1} + n$$

where the non-covariant matrix n is:

$$n = n^{\mu} n^{\nu} \delta_{ae}\, \delta^4(x-y)$$

From (A29) is then :

$$S = \exp\left[\tfrac{1}{2}\log\det(\bar{H}^{-1}+n)\right]\exp\left[-\tfrac{1}{2}\log\det(\bar{\Delta}+\bar{H}^{-1})\right]:\exp\left[\tfrac{i}{2}Q(\bar{\Delta}+\bar{H}^{-1})^{-1}Q\right]:$$

Now:

$$\tfrac{1}{2}\log\det(\bar{H}^{-1}+n) = \tfrac{1}{2}\log\det\left[\bar{H}^{-1}(1+n\bar{H})\right] =$$

$$= \tfrac{1}{2}\log\bar{H}^{-1} + \tfrac{1}{2}\log\det(1+n\bar{H}) =$$

$$= -\tfrac{1}{2}\log\det\bar{H} + \tfrac{1}{2}\log\det(1+n\bar{H})$$

and we have

$$S = \exp\left[\tfrac{1}{2}\log\det(1+n\bar{H})\right]\exp\left[-\tfrac{1}{2}\log\det\bar{H}\right] \tag{A30}$$

$$\exp\left[-\tfrac{1}{2}\log\det(\bar{H}^{-1}+\bar{\Delta})\right]:\exp\left[\tfrac{i}{2}Q(\bar{\Delta}+\bar{H}^{-1})^{-1}Q\right]:$$

The only non-covariant term is:

(A31)
$$\delta S = \exp\left[\frac{1}{2}\log\det(1+n\bar{H})\right]$$

because:

$$\bar{H} = \bar{G}_{ab}(x)\, g_{\mu\nu}\, \delta^4(x-y)$$

$$\bar{\Delta} = -\partial^\mu \partial^\nu \Delta_{ab}(x-y)$$

and

(A32)
$$\bar{H} n = \bar{G}_{ab}(x)\, n_\mu n_\nu\, \delta^4(x-y)\,.$$

Now,

$$\log\det(1+\bar{H}n) = \mathrm{Tr}\log(1+\bar{H}n) =$$
$$= \mathrm{Tr}\log\left[1+\bar{G}_{ab}(x)\, n_\mu n_\nu\, \delta^4(x-y)\right] =$$
$$= \mathrm{Tr}\int d^4x\, \delta^4(0)\, \log\left[1+\bar{G}_{ab}(x)\, n_\mu n_\nu\right] =$$
$$= \int d^4x\, \delta^4(0)\, \mathrm{Tr}\log\left[1+\bar{G}_{ab}(x)\right]$$

and finally:

(A33)
$$\log\det(1+\bar{H}n) = \int d^4x\, \delta^4(0)\, \mathrm{Tr}\log G(x)$$

so, that:

$$\delta S = \exp\left[\frac{1}{2}\int d^4x\, \delta^4(0)\, \mathrm{TR}\log G(x)\right] . \tag{A34}$$

The covariant part of the S - matrix is computed from the interaction Hamiltonian

$$-\frac{1}{2}\partial^\mu \varphi^a \bar{G}_{ab} \partial_\mu \varphi^b$$

with the covariant propagator

$$\bar{\Delta}^{\mu\nu}{}_{ab}(x-y) = -\partial^\mu \partial^\nu \Delta_{ab}(x-y) .$$

Therefore, S - matrix may be evaluated with the naive Feynam rules, provided the interaction Hamiltonian is chosen to be:

$$\mathcal{H}_I = -\frac{1}{2}\partial^\mu \varphi^a \bar{G}_{ab}\partial_\mu \varphi^b + \frac{1}{2} i \delta^4(0)\, \mathrm{TR}\log G(x) . \tag{A35}$$

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