72-13,459

BREGMAN, Alvin M., 1945-
THERMAL STRESSES IN BONDED DISSIMILAR
MEDIA CONTAINING PENNY-SHAPED CRACKS.

The City University of New York, Ph.D., 1972
Engineering Mechanics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THERMAL STRESSES IN BONDED DISSIMILAR MEDIA CONTAINING PENNY-SHAPED CRACKS

by

ALVIN M. BREGMAN

A dissertation submitted to the Graduate Faculty in Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1971

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

11/10/1971 — date

Mumtaz K. Kassir — Chairman of Examining Committee

11/10/71 — date

Jacques E. Benveniste — Executive Officer

Prof. Jacques E. Benveniste

Prof. David H. Cheng

Prof. Mumtaz K. Kassir, Chairman

Prof. Raymond Parnes

Supervisory Committee

The City University of New York

ABSTRACT

THERMAL STRESSES IN BONDED DISSIMILAR
MEDIA CONTAINING PENNY-SHAPED CRACKS

by

Alvin M. Bregman

Adviser: Professor Mumtaz K. Kassir

A solution is presented for the problem of a uniform flow of heat disturbing an insulated penny-shaped crack on the bonding surface of two (dissimilar) semi-infinite media. The thermoelastic analysis is separated into independent temperature and mechanical problems which are solved consecutively. By the introduction of appropriate analytic functions, the temperature problem is reduced to a boundary value problem in potential theory with known solution. The axisymmetric mechanical boundary value problem of the infinite space is converted to a potential problem of a half-space and is subsequently reduced to a two-dimensional Hilbert problem by the use of appropriate transformations. Employing the apparatus of the theory of complex variables, a solution to the Hilbert problem is determined from which expressions for displacements, stresses, energy loss and stress-intensity factors are obtained. Stress singularities and oscillations are observed in the

vicinity of the crack tip. The possibility of extending the Griffith-Irwin theory of fracture to cracks in dissimilar media conducting heat under steady-state conditions is also examined.

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my mentor, Professor Mumtaz K. Kassir for his guidance, effort and encouragement during the research and writing of this dissertation. His numerous suggestions were invaluable and are deeply appreciated.

Special thanks to my wife, Jeannette for her continuing patience and understanding and to my parents for their encouragement and assistance throughout my studies.

The financial support of the City University of New York and the National Science Foundation is gratefully acknowledged.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

A	Constant $= \dfrac{A_2 - A_0}{1 - A_0 A_2} A_2$
A_n $(n = 1, 2)$	Constant $= \dfrac{\lambda_n + 2\mu_n}{\mu_n}$
A_0	Constant $= \dfrac{H}{E - 1}$
a	Radius of Crack
$a_n(t)$ $(n = 1, 2)$	Arbitrary Function
B	Constant $= \dfrac{1 - A_0 A_2}{A_2 - A_0} A_2$
B_n $(n = 1, 2)$	Constant $= \dfrac{\mu_n(\lambda_n + 2\mu_n)}{(\lambda_n + 3\mu_n)}$
C_1, C_2	Constants of Integration
D_n $(n = 1, 2)$	Constant $= \dfrac{\mu_n^2}{(\lambda_n + 3\mu_n)}$
E	Constant $= \dfrac{B_1 B_2 + D_1 D_2}{D_1^2 - B_1^2}$
E_1, E_2	Modulus of Elasticity
F_0	Function of x Given by $4K_0 \displaystyle\int_0^\infty \frac{J_{3/2}(as) \sin xs \, ds}{s^{3/2}}$
$F_1(x, y, z)$, $F_2(x, y, z)$	Harmonic Functions
$f(r)$	Integral given by $\displaystyle\int_0^\infty \frac{J_0(rs)\, J_{3/2}(as)\, ds}{s^{3/2}}$

f_1	Function of x given by $-\frac{2(A-B)}{A-\sqrt{AB}}\left[g_1(x)-i\sqrt{AB}\,g_2(x)\right]$
f_2	Function of x given by $-\frac{2(A-B)}{A+\sqrt{AB}}\left[g_1(x)+i\sqrt{AB}\,g_2(x)\right]$
g_1	Function of x given by $4C_2 - 2L_2\gamma k_1 x^2$
g_2	Function of x given by $\frac{F_0(x)}{A - B}$
H	Constant $= \frac{B_1D_2 + B_2D_1}{D_1^2 - B_1^2}$
$J_0, J_{1/2}, J_{3/2}$	Bessel Functions
K_0	Constant $= -\frac{(1-A_2^2)}{(1-A_0A_2)(1-E)}\sqrt{\frac{2}{\pi}}\,\gamma a^{3/2}\left\{(1+\nu_1)\alpha_1 - \frac{(1+\nu_2)\alpha_2 k_1}{2(1-\nu_2)k_2}\left[\frac{\mu_1+\mu_2(1-2\nu_1)}{\mu_1}\right]\right\}$
k_1, k_2	Coefficient of Thermal Conductivity
K_1, K_2	Stress-Intensity Factors
L_n (n=1,2)	Constant $= \frac{(3-4\nu_n)(1+\nu_n)\alpha_n}{2k_n(1-\nu_n)(1-2\nu_n)}$
L' (L'')	Inner (Outer) Segment of Real Axis
m_1 $(= -m_2)$	Constant $= \frac{1}{2\pi i}\ln\frac{A+\sqrt{AB}}{A-\sqrt{AB}}$
$\overline{q}$	Heat Flux Vector
q_x, q_y, q_z	Cartesian Components of Heat Flux Vector
R	Distance from Crack Tip $(= r - a)$
r, θ, z	Cylindrical Coordinates

S^+ (S^-)	Upper (Lower) Half-Space
$T(x,y,z)$	Temperature Distribution
U	Crack Surface Energy
$U_m(x,z)(m=1,2)$	Harmonic Functions
$\bar{u}$	Displacement Vector
u, v, w	Cartesian Components of Displacement
W	Potential Energy
X_{o_n} $(n=1,2)$	Plemelj Function
x, y, z	Rectangular Coordinates
α_1, α_2	Coefficient of Thermal Expansion
γ	Specific Surface Energy
ϵ	Material Constant
$\zeta = x + iz$	Complex Variable
ϑ_{12}, ϑ_k $(k=1,2)$	Material Constant
λ_1, λ_2	Lamé Constant
μ_1, μ_2	Shear Modulus
ν_1, ν_2	Poisson's Ratio
$\bar{\sigma}$	Stress Tensor
σ_{zz}, τ_{xz}, τ_{yz}	Components of Stress on z-plane
τ	Temperature Gradient
$\Phi_1(\zeta)$, $\Phi_2(\zeta)$	Analytic Functions
$\phi_1, \phi_2, \phi_3, \phi_4, \psi$	Harmonic Functions of x,y,z

Ω	Thermoelastic Potential
$\Omega_1(\zeta),\ \Omega_2(\zeta)$	Analytic Functions
ω	$= \epsilon \ln \dfrac{r - at}{r + at}$

CHAPTER 1

INTRODUCTION

When a uniform flow of heat in an elastic solid is disturbed by the presence of cracks or flaws, there is a local intensification of the temperature gradient with associated thermal stress. Disturbances of this type may, in some cases, contribute to the mechanical breakdown of structural members. Hence, a knowledge of the state of thermoelastic stress and deformation in such solids is of significant value in structural analysis and current theories of fracture mechanics.

In most practical applications, the effects of both coupling and inertia may be disregarded and it is thereby possible to separate the original thermal stress problem into two independent problems to be solved consecutively. The first requires a solution for the steady state temperature field and the second is a determination of the induced thermal displacements and stresses in the solid. Such an analysis of a homogeneous medium weakened by a plane of discontinuity, or crack, can be reduced to a mixed boundary value problem for a half-space with the crack plane as a surface of separation of the boundary conditions.

Based on the method of dual integral equations in the Hankel transforms, Florence and Goodier [1]* treated

---

* Numbers in brackets designate References on page 91.

the problem of uniform heat flow disturbed by an insulated penny-shaped crack. The antisymmetry of the temperature and stress fields about the equatorial plane permitted a reduction to a problem of a semi-infinite medium with mixed boundary conditions. Kassir and Sih [2] considered the state of stress induced by a uniform heat flow past an insulated elliptical crack which included the circular crack as a special case. The general solution was constructed in terms of harmonic functions, and expressions for stress and displacement were derived. There is a direct analogy between the problems alluded to above and those of a planar crack under linear shearing stress [3]. Recently, the writer [4] considered the thermoelastic problem of two (or more) parallel coaxial insulated penny-shaped cracks by extending a method developed by Collins [5] for a similar configuration under mechanical loads. The solution was formulated in terms of axially symmetric harmonic functions which were then sought in terms of a complex variable integral with Cauchy type kernels. The problem was then reduced to the solution of Fredholm integral equations of the second kind from which results were obtained by iteration. Other thermoelastic problems involving a linear flow of heat disturbed by cracks or cavities have been treated in [6 - 8] and others.

In all of the preceding investigations, the infinite medium considered was homogeneous. A problem of considerable practical importance is that of determining the induced thermal stresses in bonded semi-infinite bodies of different thermoelastic properties when a uniform heat flow is disturbed by an insulated crack on the surface of bonding. This problem represents an idealization of two dissimilar materials welded together with a flaw developed on the common interface due to faulty joining techniques. The imperfect contact may be due to the existence of an entrapped gas bubble which may have sharp edges and for the purpose of analysis could be regarded as a flat cut or planar crack. It is the purpose of this dissertation to study such a problem. Specifically, a uniform flow of heat is allowed to pass through two bonded semi-infinite media with an insulated penny-shaped crack situated on the bonding surface normal to the direction of the heat flow. It is desired to determine the induced stress-strain distribution in the solid. As the elevated stresses in the vicinity of the crack tip will control the onset of crack propagation and eventual breakdown of the material, special emphasis will be devoted to the finding of the local stresses and the value of the stress-intensity factors.

During the past decade, several two-dimensional problems of bonded dissimilar media containing cracks have been considered in the literature [9 - 12]*. Employing an eigenfunction expansion technique, Williams [9] discovered for the first time that for the problem of two dis-

---

* This list is not intended to be comprehensive. Only representative works are mentioned.

similar materials with a semi-infinite line crack, the stresses possess an oscillatory character of the type $r^{-1/2}$ sin (or cos) of the argument ϵ log r, where r is the radial distance from the crack tip and ϵ is a function of material constants.* The case of a finite crack (or cracks) embedded along the interface of two bonded semi-infinite planes was investigated by Erdogan [10] and Rice and Sih [11] using the complex variable formulation of plane elasticity problems (integrated with the eigenfunction expansion technique in [11]) while Brown [12] calculated the thermally induced stresses when a uniform heat flow is disturbed by an insulated Griffith crack. For the axisymmetric geometry, Mossakovskii and Rybka [13] considered an isothermal problem of a circular crack embedded between bonded dissimilar materials. At large distances from the plane of discontinuity, the two dissimilar half-spaces are acted upon by uniform normal stress. The axisymmetric problem was reduced to a subsidiary problem in the plane which in turn was solved by Muskhelishvili's method in complex function theory. Erdogan [14] presented a solution for the problem of a circular or ring shaped cavity between two dissimilar materials. He utilized Hankel transforms to reduce the problem to a system of simultaneous singular integral equations. Only the dominant part of these equations was considered and an approx-

---

* See equation (5.1-5) for the definition of the bi-elastic constant ϵ .

imate solution was obtained by reduction to a Hilbert problem. In these problems, the oscillatory nature of the stresses and displacements was also encountered. However, as in the two dimensional cases, the zone of oscillation is confined to a very small distance from the crack edge where there would be plastic flow due to yielding of the material and so for all practical purposes the oscillatory behavior of the local stresses, which is physically impossible, may be ignored.

In this dissertation, a solution is presented for the problem of a uniform flow of heat disturbing an insulated penny-shaped crack situated on the bonding surface of two semi-infinite media. The two materials have different thermo-mechanical properties and are assumed to be ideally bonded so that a perfect contact exists, i.e. continuity of temperature, heat flux, displacement and stress. The thermal analysis is separated into independent temperature and mechanical problems governed by the steady state temperature equation and the Navier displacement equation of static equilibrium, respectively (Chapter 2). The temperature field is obtained by first considering the boundary and continuity conditions of temperature and heat flux. By introducing appropriate analytic functions, it is shown that the boundary value problem is analogous to the problem considered by Florence and Goodier [1] for which a solution is known in terms of Bessel functions. The mechanical problem is then solved by finding harmonic functions which satisfy the Cauchy-Navier displacement equations of three-dimensional thermoelasticity. Upon utilizing the boundary conditions on the crack surfaces and the continuity conditions on the bending plane outside the cut, the problem of the infinite space is reduced to that of a half-space with rotational, or polar symmetry. This axisymmetric potential problem is then reduced to a subsidiary potential problem in the plane by using transformations originally introduced by Mossakovskii [15] for this purpose. The two dimensional problem is then shown to reduce to the problem of linear relationship which requires the determination of a sectionally holomorphic function whose boundary values, on the two sides of a line of discontinuity, are linearly related. A solution of this problem, usually known as the Hilbert problem, is then obtained by making use of the Plemelj formulas and Cauchy integrals as developed by Muskhelishvili [16].

Having obtained the solution to the Hilbert problem, it is possible to evaluate the physical quantities of interest of the original axisymmetric problem by employing the appropriate transformations mentioned previously. Results are obtained for the stress and deformation fields, the change in potential energy due to the existence of the crack, and the critical heat flux which initiates crack enlargement. An important practical result is the stress field in the vicinity of the crack edge since this is essential in the investigation of the stability behavior of the crack. The Griffith-Irwin [17] theory of fracture, for example, is based upon the concept that the onset of rapid crack extension occurs when the stress-intensity factor reaches some critical value. The local stresses are found to have the same inverse square root singularity as that found in homogeneous problems but, in addition, the elasticity solution exhibits oscillations (in displacements and stresses) of the form sin or cos of the argument (ϵ log R). Here R is the radial distance from the crack rim and ϵ is a function of the material constants. The oscillation, however, is confined to a very narrow zone near the edge of the cut. This peculiar behavior of the local stresses agrees with results obtained in the two-dimensional crack problems for dissimilar media [9, 10, 11, 12] as well as in the axisymmetric problems [13, 14]. Expressions are obtained for the stress-intensity factors by considering the limiting behavior of the normal and shear stresses as the crack edge is approached, and numerical values are computed for some typical solids.

CHAPTER 2

FUNDAMENTAL EQUATIONS OF THERMOELASTICITY

When the temperature distribution in a solid isotropic body is not uniform, there is a transfer of heat by conduction between any two particles which are at different temperatures. The law of heat conduction may be stated as follows:

$$q_x = -k \frac{\partial T}{\partial x} \quad , \qquad q_y = -k \frac{\partial T}{\partial y} \quad , \qquad q_z = -k \frac{\partial T}{\partial z}$$

where k is the thermal conductivity of the solid, x,y,z are Cartesian coordinates, q_x, q_y, q_z are the Cartesian components of the heat flux vector $\bar{q}$ and T(x,y,z) is the temperature.

For the steady state problem in which there is no internal heat generation, the Fourier heat conduction equation requires the temperature at every point of the solid to satisfy the Laplace equation in three dimensions:

$$\nabla^2 T(x,y,z) = 0 \qquad (2\text{-}1)$$

Once a solution for the temperature field is obtained, it is possible to solve for the resulting displacements and stresses, respectively, from the Navier displacement equation of static equilibrium

$$\frac{1}{1-2\nu} \nabla \nabla \cdot \bar{u} + \nabla^2 \bar{u} = 2\left(\frac{1+\nu}{1-2\nu}\right) \alpha \nabla T \qquad (2\text{-}2)$$

and the Duhamel-Neumann stress displacement relation

$$\bar{\bar{\sigma}} = \mu \left\{ \nabla \bar{u} + \bar{u} \nabla + \frac{2}{1-2\nu} \left[\nu \nabla \cdot \bar{u} - (1+\nu) \alpha T \right] \bar{\bar{I}} \right\} \qquad (2\text{-}3)$$

Here $\bar{u}$ is the displacement vector, $\bar{\bar{\sigma}}$ is the stress tensor and $\bar{\bar{I}}$ is the isotropic tensor. The shear modulus, Poisson's ratio, and the coefficient of linear expansion are designated by μ, ν and α respectively and the material is considered to be homogeneous and isotropic in its thermal and mechanical properties. The gradient and Laplacian operators are represented by ∇ and ∇^2 respectively. It is convenient to seek the temperature field by finding a thermoelastic potential Ω defined by

$$T = \frac{2(1-\nu)}{(1+\nu)\alpha} \frac{\partial \Omega}{\partial z} \qquad (2\text{-}4)$$

The induced displacements and stresses follow upon solving the system of equations (2-2) subject to appropriate boundary conditions across the crack plane and regularity conditions at far away distances.

For the problem involving a uniform flow of heat passing through a solid, a solution of (2-2) may be represented in the form [18]

$$u = \phi_1 + z \frac{\partial \psi}{\partial x} + z \frac{\partial \Omega}{\partial x} \qquad (2\text{-}5a)$$

$$v = \phi_2 + z \frac{\partial \psi}{\partial y} + z \frac{\partial \Omega}{\partial y} \qquad (2\text{-}5b)$$

$$w = \Phi_3 + z\frac{\partial \Psi}{\partial z} + \Omega + z\frac{\partial \Omega}{\partial z} \qquad \text{(2-5c)}$$

where u, v and w are the Cartesian components of the displacement vector $\bar{u}$; $\Phi_j (j = 1,2,3)$ and Ψ are harmonic functions. It is straightforward to confirm that the equations of equilibrium, (2-2), are fulfilled provided that

$$\frac{\partial \Psi}{\partial z} = \frac{1}{(4\nu - 3)}\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} + \frac{\partial \Phi_3}{\partial z}\right) \qquad \text{(2-6)}$$

and some of the components of the stress tensor are

$$\sigma_{zz} = (\lambda + 2\mu)\frac{\partial}{\partial z}(\Phi_3 + \Psi) + \lambda\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right) + 2\mu z\frac{\partial^2}{\partial z^2}(\Psi + \Omega) \qquad \text{(2-7a)}$$

$$\tau_{xz} = \mu\left(\frac{\partial \Phi_1}{\partial z} + \frac{\partial \Phi_3}{\partial x} + \frac{\partial \Psi}{\partial x}\right) + 2\mu\frac{\partial}{\partial x}\left(z\frac{\partial \Psi}{\partial z} + \Omega + z\frac{\partial \Omega}{\partial z}\right) \qquad \text{(2-7b)}$$

$$\tau_{yz} = \mu\left(\frac{\partial \Phi_2}{\partial z} + \frac{\partial \Phi_3}{\partial y} + \frac{\partial \Psi}{\partial y}\right) + 2\mu\frac{\partial}{\partial y}\left(z\frac{\partial \Psi}{\partial z} + \Omega + z\frac{\partial \Omega}{\partial z}\right) \qquad \text{(2-7c)}$$

where σ_{zz} is the normal stress across the z-plane, τ_{xz} and τ_{yz} are the shear stresses on the same plane and λ is the Lamé constant given by

$$\lambda = \frac{2\mu\nu}{1 - 2\nu}$$

The other stress components, σ_{xx} , σ_{yy} and τ_{xy} are not needed for the purpose of satisfying the boundary conditions.

CHAPTER 3

FORMULATION OF TEMPERATURE PROBLEM

Consider two semi-infinite elastic solids bounded together with each material homogeneous and isotropic. Let the medium with thermal conductivity k_1 and coefficient of expansion α_1 occupy the upper semi-space S^+ and the medium with conductivity k_2 and coefficient of expansion α_2 occupy the lower semi-space S^-. Assume the bond to contain a plane circular cavity, or penny-shaped crack, situated in the plane of bonding. Referred to a system of cylindrical coordinates (r,Θ,z) with its origin located at the center of the crack and the z-coordinate along the axis of rotational symmetry, the crack is determined by $z = 0$, $0 \leq r \leq a$ (see Fig.1).

When a uniform flow of heat passes through the solid along the z-direction, i.e. perpendicular to the plane of the crack, the undisturbed temperature field may be taken as τz in S^+ and, in order to ensure perfect thermal contact (continuity of temperature and heat flux along the plane of joining), $\tau k_1 z/k_2$ in S^-. The uniform temperature gradient has been designated by τ. The temperatures in the presence of the plane of discontinuity may be written as (see Fig. 2)

$$\tilde{T}^+(r,z) = \tau z + T^+(r,z) \quad \text{in} \quad S^+ \qquad (3\text{-}1a)$$

$$\tilde{T}^-(r,z) = \frac{k_1}{k_2}\tau z + T^-(r,z) \quad \text{in} \quad S^- \qquad (3\text{-}1b)$$

For the undisturbed temperature, the displacements induced by the uniform flow of heat (obtained by solving (2-2)) are

$$u^+ = \alpha_1 \tau x z \;, \quad v^+ = \alpha_1 \tau y z \;, \quad w^+ = \tfrac{1}{2}\alpha_1 \tau (z^2 - r^2)$$

$$u^- = \alpha_2 \frac{k_1}{k_2} \tau x z \;, \quad v^- = \alpha_2 \frac{k_1}{k_2} \tau y z \;, \quad w^- = \frac{\alpha_2 k_1}{2 k_2} \tau (z^2 - r^2) \tag{3-2}$$

with no accompanying stresses (see, e.g., [19], page 244). In order to ensure continuity of displacement across the plane z = 0, equations (3-2) indicate that one must account for a normal displacement given by

$$\Delta w = -\frac{\tau r^2}{2 k_2}\left[\alpha_1 k_2 - \alpha_2 k_1\right] \;, \quad z = 0 \tag{3-3}$$

This is equivalent to applying the following stresses in the undisturbed solid:

$$\sigma_{xx} = p z \;, \quad \sigma_{yy} = p z \tag{3-4}$$

where

$$p = \frac{\tau E_1 E_2 [\alpha_1 k_2 - \alpha_2 k_1]}{k_2 [E_2(1-\nu_1) - E_1(1-\nu_2)]}$$

The stresses and displacements given in eq. (3-2) and (3-4) are to be superimposed on the values found in the disturbed case in order to obtain the complete solution. Note further that the stresses (3-4) do not affect the boundary conditions for the disturbed case.

For an insulated penny-shaped crack, the axisymmetric temperature $T^+(r,z)$ and $T^-(r,z)$ must satisfy equation (2-1) and the following boundary conditions across the plane $z = 0$:

$$\left.\begin{aligned} \frac{\partial T^+}{\partial z} &= -\gamma \qquad (3\text{-}5a) \\ \frac{\partial T^-}{\partial z} &= -\frac{k_1}{k_2}\gamma \qquad (3\text{-}5b) \end{aligned}\right\} \quad r < a$$

$$\left.\begin{aligned} k_1 \frac{\partial T^+}{\partial z} &= k_2 \frac{\partial T^-}{\partial z} \qquad (3\text{-}5c) \\ T^+ &= T^- \qquad (3\text{-}5d) \end{aligned}\right\} \quad r > a$$

In addition, the temperature field to be determined must vanish at large distances from the crack. Observe that conditions (3-5c) and (3-5d) represent a case of perfect thermal contact between the two bodies. Moreover, it is assumed that there is no generation of heat inside the solid.

The following function is introduced for consideration:

$$T^*(r,z) = T(r,-z) \tag{3-6}$$

It follows that

$$T^{*+}(r,z) = T^-(r,z) \ , \quad T^{*-}(r,z) = T^+(r,z) \tag{3-7}$$

and

$$\frac{\partial T^{*+}(r,z)}{\partial z} = -\frac{\partial T^-(r,z)}{\partial z} \ , \quad \frac{\partial T^{*-}(r,z)}{\partial z} = -\frac{\partial T^+(r,z)}{\partial z} \tag{3-8}$$

Substituting (3-8) into (3-5c) gives

$$k_1 \frac{\partial T^{*-}(r,z)}{\partial z} + k_2 \frac{\partial T^-(r,z)}{\partial z} = 0 \ , \quad r > a, \ z = 0 \tag{3-9}$$

and an identical relation may be established for $r < a$, $z = 0$ by using equations (3-5a), (3-5b) and (3-8). Hence, (3-9) may be extended to the lower semi-space by recalling that if a harmonic function is equal to zero on the boundary of the region for which it is defined, then the function is equal to zero everywhere in the region.* Hence

*This follows from Green's formula (see e.g. [22], p.32)

$$k_1 \frac{\partial T^{*-}(r,z)}{\partial z} + k_2 \frac{\partial T^{-}(r,z)}{\partial z} = 0 , \quad z \leq 0 \tag{3-10}$$

which may be integrated to give

$$k_1 T^{*-}(r,z) + k_2 T^{-}(r,z) = 0, \quad z \leq 0 \tag{3-11}$$

From (3-7), it follows that

$$k_1 T^{+}(r,z) + k_2 T^{-}(r,z) = 0, \quad z \leq 0 \tag{3-12}$$

If the condition of temperature continuity on the bonding surface, (3-5d), is taken into account, (3-12) yields when $z = 0$

$$T^{-}(r,0) = 0, \quad T^{+}(r,0) = 0, \quad r > a \tag{3-13}$$

The boundary conditions for the temperature field now become for S^-

$$\frac{\partial T^{-}(r,z)}{\partial z} = - \frac{\tau k_1}{k_2} , \quad r < a, \ z = 0 \tag{3-14}$$

$$T^{-}(r,0) = 0 , \quad r > a \tag{3-15}$$

and similar conditions for S^+.

Except for a multiplying constant, these conditions are analogous to those encountered by Florence and Goodier [1] for the corresponding problem in a homogeneous solid. Hence, the temperature field in S^- may be written as

$$T^{-}(r,z) = -\sqrt{\frac{2}{\pi}}\,\frac{\tau k_1 a^{1/2}}{k_2}\int_0^\infty e^{\xi z}\,\xi^{-1/2}\,J_{3/2}(\xi a)\,J_0(r\xi)\,d\xi \qquad (3\text{-}16)$$

where J_n is a Bessel function of the first kind and order n. The temperature gradient thus becomes

$$\frac{\partial T^{-}(r,z)}{\partial z} = -\sqrt{\frac{2}{\pi}}\,\frac{\tau k_1 a^{1/2}}{k_2}\int_0^\infty e^{\xi z}\,\xi^{1/2}\,J_{3/2}(\xi a)\,J_0(r\xi)\,d\xi \qquad (3\text{-}17)$$

so that on the plane z = 0, it follows that

$$T^{-}(r,0) = \begin{cases} -\dfrac{2}{\pi}\,\dfrac{\tau k_1}{k_2}\sqrt{a^2-r^2}, & r<a \qquad (3\text{-}18a) \\ 0, & r>a \qquad (3\text{-}18b) \end{cases}$$

and

$$\left[\frac{\partial T^{-}(r,z)}{\partial z}\right]_{z=0} = \begin{cases} -\dfrac{k_1}{k_2}\,\tau, & r<a \qquad (3\text{-}19a) \\ \dfrac{2\tau k_1}{\pi k_2}\left[\dfrac{a}{\sqrt{r^2-a^2}} - \sin^{-1}\dfrac{a}{r}\right], & r>a \qquad (3\text{-}19b) \end{cases}$$

Note that

$$T^{+}(r,0) = \begin{cases} \dfrac{2\tau}{\pi}\sqrt{a^2-r^2}, & r<a \qquad (3\text{-}20a) \\ 0, & r>a \qquad (3\text{-}20b) \end{cases}$$

and

$$\left[\frac{\partial T^{+}(r,z)}{\partial z}\right]_{z=0} = \begin{cases} -\tau & , r < a \quad \text{(3-21a)} \\ \frac{2\tau}{\pi}\left[\frac{a}{\sqrt{r^2-a^2}} - \sin^{-1}\frac{a}{r}\right] & , r > a \quad \text{(3-21b)} \end{cases}$$

At any point of the semi-space, the variation of temperature and temperature gradient could be evaluated from integrals (3-16) and (3-17). Details may be found in [24].

In order to solve for the induced thermal displacements and stresses (see relations (2-5)), it is necessary to evaluate the thermoelastic potential defined in (2-4) and its appropriate derivatives on the plane z = 0. For this purpose, equations (2-4), (3-16) and the regularity of Ω at infinity* give

$$\Omega^{-}(r,z) = -\frac{(1+\nu_2)\alpha_2}{2(1-\nu_2)}\sqrt{\frac{2}{\pi}}\,\frac{\tau k_1 a^{3/2}}{k_2}\int_0^{\infty} e^{\xi z}\,\xi^{-3/2}\,J_{3/2}(\xi a)\,J_0(r\xi)\,d\xi \qquad (3\text{-}22)$$

which on the plane z = 0 becomes

$$\Omega^{-}(r,0) = -\frac{(1+\nu_2)\alpha_2}{2(1-\nu_2)}\sqrt{\frac{2}{\pi}}\,\frac{\tau k_1 a^{3/2}}{k_2}\int_0^{\infty}\frac{J_0(r\xi)\,J_{3/2}(a\xi)}{\xi^{3/2}}\,d\xi \qquad (3\text{-}23)$$

The integral in (3-22) may be evaluated (see [20], p.48; [21], p.54) to yield

---

* Ω represents displacement and therefore must vanish at large distances from the origin (see eq. (2-5e)).

$$\Omega^{-}(r,0) = \begin{cases} -\dfrac{(1+\nu_2)\alpha_2 \gamma k_1}{8k_2(1-\nu_2)}\left[2a^2 - r^2\right] & , r < a \quad \text{(3-24a)} \\ -\dfrac{(1+\nu_2)\alpha_2 \gamma k_1}{4\pi k_2(1-\nu_2)}\left[(2a^2 - r^2)\sin^{-1}\dfrac{a}{r} + a\sqrt{r^2-a^2}\right] & , r > a \quad \text{(3-24b)} \end{cases}$$

It follows that

$$\frac{\partial \Omega^{-}(r,0)}{\partial r} = \begin{cases} \dfrac{(1+\nu_2)\alpha_2 \gamma k_1 r}{4k_2(1-\nu_2)} & , r < a \quad \text{(3-25a)} \\ -\dfrac{(1+\nu_2)\alpha_2 \gamma k_1}{2\pi k_2(1-\nu_2)}\left[\dfrac{a}{r}\sqrt{r^2-a^2} - r\sin^{-1}\dfrac{a}{r}\right] & , r > a \quad \text{(3-25b)} \end{cases}$$

In a similar manner, it is found that for the upper half-space (S^+)

$$\Omega^{+}(r,0) = \begin{cases} -\dfrac{(1+\nu_1)\alpha_1 \gamma}{8(1-\nu_1)}(2a^2 - r^2) & , r < a \quad \text{(3-26a)} \\ -\dfrac{(1+\nu_1)\alpha_1 \gamma}{4\pi(1-\nu_1)}\left[(2a^2 - r^2)\sin^{-1}\dfrac{a}{r} + a\sqrt{r^2-a^2}\right] & , r > a \quad \text{(3-26b)} \end{cases}$$

and

$$\frac{\partial \Omega^{+}(r,0)}{\partial r} = \begin{cases} \dfrac{(1+\nu_1)\alpha_1 \gamma r}{4(1-\nu_1)} & , r < a \quad \text{(3-27a)} \\ -\dfrac{(1+\nu_1)\alpha_1 \gamma}{2\pi(1-\nu_1)}\left[\dfrac{a}{r}\sqrt{r^2-a^2} - r\sin^{-1}\dfrac{a}{r}\right] & , r > a \quad \text{(3-27b)} \end{cases}$$

With the solution to the temperature problem and the values of Ω and $\frac{\partial \Omega}{\partial r}$ on $z = 0$, it is now possible to solve for the induced thermal displacements and stresses.

It should be mentioned that the temperature problem considered in this chapter may be formulated by using Hankel transforms on boundary conditions (3-5a) through (3-5d) without the introduction of the function $T^*(r,z)$. The resulting pair of dual integral equations uncouple immediately and yield the same results as above. This alternate method of solution is presented in Appendix A.

CHAPTER 4

THERMAL STRESS PROBLEM

The disturbed temperature field discussed in the previous chapter gives rise to thermal stresses which become highly intensified at the periphery of the crack. These stresses supply the surface energy required for crack propagation, and a critical value of the temperature gradient may be obtained beyond which crack enlargement occurs. This critical temperature gradient is analogous to the critical fracture stress in the Griffith theory of fracture and therefore is of paramount importance in practical applications.

To obtain the local stresses, the state of thermal stress and deformation in the solid must be determined. For this purpose, the displacement representation (2-5) will be employed and the harmonic functions will be determined by first reducing the problem to an axisymmetric potential problem for a half space depending on two functions which is then transferred into a problem of linear combinations of two analytic functions in a plane whose solution is sought by Muskhelishvili's method in complex variables.

4.1 Derivation of Boundary Conditions

The boundary conditions for the problem are specified on the $z = 0$ plane. A perfect bond between the two bodies is assumed and hence, the components of displacement and stress must be continuous for r > a, while the stress components must vanish on the (free) crack surface. These conditions may be stated as follows:

$$\left.\begin{aligned} \sigma_{zz}^{+} &= 0 \quad , \quad \sigma_{zz}^{-} = 0 \\ \tau_{xz}^{+} &= 0 \quad , \quad \tau_{xz}^{-} = 0 \\ \tau_{yz}^{+} &= 0 \quad , \quad \tau_{yz}^{-} = 0 \end{aligned}\right\} \; r < a , \; z = 0 \qquad \begin{aligned} &(4.1\text{-}1) \\ &(4.1\text{-}2) \\ &(4.1\text{-}3) \end{aligned}$$

$$\left.\begin{aligned} \sigma_{zz}^{+} &= \sigma_{zz}^{-} \\ \tau_{xz}^{+} &= \tau_{xz}^{-} \\ \tau_{yz}^{+} &= \tau_{yz}^{-} \end{aligned}\right\} \; r > a , \; z = 0 \qquad \begin{aligned} &(4.1\text{-}4) \\ &(4.1\text{-}5) \\ &(4.1\text{-}6) \end{aligned}$$

$$\left.\begin{aligned} u^{+} &= u^{-} \\ v^{+} &= v^{-} \\ w^{+} &= w^{-} \end{aligned}\right\} \; r > a , \; z = 0 \qquad \begin{aligned} &(4.1\text{-}7) \\ &(4.1\text{-}8) \\ &(4.1\text{-}9) \end{aligned}$$

where quantities related to the upper half-space are designated by plus, to the lower by minus. In addition, the displacements and stresses must vanish at large distances from the crack.

Making use of equations (2-5) and (2-7), the boundary conditions across the crack surface take the form:

$$(\lambda_1 + 2\mu_1)\frac{\partial}{\partial z}(\Phi_3^+ + \Psi^+) + \lambda_1\left(\frac{\partial \Phi_1^+}{\partial x} + \frac{\partial \Phi_2^+}{\partial y}\right) = 0 \quad (4.1\text{-}10a)$$

$$(\lambda_2 + 2\mu_2)\frac{\partial}{\partial z}(\Phi_3^- + \Psi^-) + \lambda_2\left(\frac{\partial \Phi_1^-}{\partial x} + \frac{\partial \Phi_2^-}{\partial y}\right) = 0 \quad (4.1\text{-}10b)$$

$$\mu_1\left(\frac{\partial \Phi_1^+}{\partial z} + \frac{\partial \Phi_3^+}{\partial x} + \frac{\partial \Psi^+}{\partial x}\right) + 2\mu_1\frac{\partial \Omega^+}{\partial x} = 0 \quad (4.1\text{-}11a)$$

$$\mu_2\left(\frac{\partial \Phi_1^-}{\partial z} + \frac{\partial \Phi_3^-}{\partial x} + \frac{\partial \Psi^-}{\partial x}\right) + 2\mu_2\frac{\partial \Omega^-}{\partial x} = 0 \quad (4.1\text{-}11b)$$

$$\mu_1\left(\frac{\partial \Phi_2^+}{\partial z} + \frac{\partial \Phi_3^+}{\partial y} + \frac{\partial \Psi^+}{\partial y}\right) + 2\mu_1\frac{\partial \Omega^+}{\partial y} = 0 \quad (4.1\text{-}12a)$$

$$\mu_2\left(\frac{\partial \Phi_2^-}{\partial z} + \frac{\partial \Phi_3^-}{\partial y} + \frac{\partial \Psi^-}{\partial y}\right) + 2\mu_2\frac{\partial \Omega^-}{\partial y} = 0 \quad (4.1\text{-}12b)$$

$$\left.\right\} \quad r < a, \; z = 0$$

Here λ_1, μ_1 and λ_2, μ_2, are the Lamé coefficients for the upper and lower half-spaces respectively. The stress continuity conditions exterior to the crack imply

$$(\lambda_1 + 2\mu_1)\frac{\partial}{\partial z}(\Phi_3^+ + \Psi^+) + \lambda_1\left(\frac{\partial \Phi_1^+}{\partial x} + \frac{\partial \Phi_2^+}{\partial y}\right) =$$

$$(\lambda_2 + 2\mu_2)\frac{\partial}{\partial z}(\Phi_3^- + \Psi^-) + \lambda_2\left(\frac{\partial \Phi_1^-}{\partial x} + \frac{\partial \Phi_2^-}{\partial y}\right), \quad r > a, z = 0 \quad (4.1\text{-}13)$$

$$\mu_1\left(\frac{\partial \Phi_1^+}{\partial z} + \frac{\partial \Phi_3^+}{\partial x} + \frac{\partial \Psi^+}{\partial x}\right) + 2\mu_1 \frac{\partial \Omega^+}{\partial x} =$$

$$\mu_2\left(\frac{\partial \Phi_1^-}{\partial z} + \frac{\partial \Phi_3^-}{\partial x} + \frac{\partial \Psi^-}{\partial x}\right) + 2\mu_2 \frac{\partial \Omega^-}{\partial x} \quad , \qquad r > a \, , \; z = 0 \qquad (4.1\text{-}14)$$

$$\mu_1\left(\frac{\partial \Phi_2^+}{\partial z} + \frac{\partial \Phi_3^+}{\partial y} + \frac{\partial \Psi^+}{\partial y}\right) + 2\mu_1 \frac{\partial \Omega^+}{\partial y} =$$

$$\mu_2\left(\frac{\partial \Phi_2^-}{\partial z} + \frac{\partial \Phi_3^-}{\partial y} + \frac{\partial \Psi^-}{\partial y}\right) + 2\mu_2 \frac{\partial \Omega^-}{\partial y} \quad , \quad r > a \, , \; z = 0 \qquad (4.1\text{-}15)$$

while continuity of displacements gives rise to

$$\Phi_1^+ = \Phi_1^- \; , \; \Phi_2^+ = \Phi_2^- \; , \; \Phi_3^+ + \Omega^+ = \Phi_3^- + \Omega^- \; , \quad r > a \, , \; z = 0 \qquad (4.1\text{-}16)$$

In order to simplify the ensuing development, it is convenient to introduce harmonic functions (Φ_4^+ and Φ_4^-) corresponding to the upper and lower half-spaces defined by

$$\Psi^+ = \frac{1}{(4\nu_1 - 3)}\left(\Phi_4^+ + \Phi_3^+\right) \, , \quad \Psi^- = \frac{1}{(4\nu_2 - 3)}\left(\Phi_4^- + \Phi_3^-\right) \qquad (4.1\text{-}17)$$

Making use of (2-6) and (4.1-17), it follows that

$$\frac{\partial \Phi_4^+}{\partial z} = \frac{\partial \Phi_1^+}{\partial x} + \frac{\partial \Phi_2^+}{\partial y} \quad , \qquad \frac{\partial \Phi_4^-}{\partial z} = \frac{\partial \Phi_1^-}{\partial x} + \frac{\partial \Phi_2^-}{\partial y} \qquad (4.1\text{-}18)$$

Hence, boundary conditions (4.1-10a) and (4.1-10b) simplify to

$$(\lambda_1 + 2\mu_1)\frac{\partial}{\partial z}(\Phi_3^+ + \Psi^+) + \lambda_1 \frac{\partial \Phi_4^+}{\partial z} = 0 \quad \Bigg\} \; r < a, \; z = 0 \qquad (4.1\text{-}19a)$$

$$(\lambda_2 + 2\mu_2)\frac{\partial}{\partial z}(\Phi_3^- + \Psi^-) + \lambda_2 \frac{\partial \Phi_4^-}{\partial z} = 0 \qquad (4.1\text{-}19b)$$

Substituting from (2-6) into (4.1-19a) and (4.1-19b), the following equations are obtained:

$$\frac{\partial \Phi_3^+}{\partial z} - \frac{1}{A_1}\frac{\partial \Phi_4^+}{\partial z} = 0 \quad \Bigg\} \; r < a, \; z = 0 \qquad (4.1\text{-}20a)$$

$$\frac{\partial \Phi_3^-}{\partial z} - \frac{1}{A_2}\frac{\partial \Phi_4^-}{\partial z} = 0 \qquad (4.1\text{-}20b)$$

Here, the constants $A_n (n = 1,2)$ are defined by

$$A_n = \frac{\lambda_n + 2\mu_n}{\mu_n}, \quad (n = 1,2) \qquad (4.1\text{-}21)$$

Now, if conditions (4.1-11a) and (4.1-12a) are differentiated with respect to x and y respectively and added, the resulting equation is

$$\mu_1\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\Phi_3^+ + \Psi^+ + 2\Omega^+) + \frac{\partial}{\partial z}\left(\frac{\partial \Phi_1^+}{\partial x} + \frac{\partial \Phi_2^+}{\partial y}\right)\right] = 0, \qquad (4.1\text{-}22)$$

$$r < a, \; z = 0$$

Since Φ_3, Ψ and Ω are harmonic functions, it follows that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\Phi_3^+ + \psi^+ + z\,\Omega^+\right) = -\frac{\partial^2}{\partial z^2}\left(\Phi_3^+ + \psi^+ + z\,\Omega^+\right) \qquad (4.1\text{-}23)$$

Taking into account (4.1-18) and (4.1-23), (4.1-22) takes the form

$$\mu_1\left[-\frac{\partial^2}{\partial z^2}\left(\Phi_3^+ + \psi^+ + z\,\Omega^+\right) + \frac{\partial^2 \Phi_4^+}{\partial z^2}\right] = 0 \quad , \quad r < a\,,\ z = 0 \qquad (4.1\text{-}24)$$

A similar expression on the boundary of the lower semi-space (S^-) may be obtained by applying the same procedure to conditions (4.1-11b) and (4.1-12b). Thus

$$\mu_2\left[-\frac{\partial^2}{\partial z^2}\left(\Phi_3^- + \psi^- + z\,\Omega^-\right) + \frac{\partial^2 \Phi_4^-}{\partial z^2}\right] = 0 \quad , \quad r < a\,,\ z = 0 \qquad (4.1\text{-}25)$$

Making use of the relations

$$\frac{\partial^2 \psi^+}{\partial z^2} = \frac{1}{(4\nu_1 - 3)}\left(\frac{\partial^2 \Phi_4^+}{\partial z^2} + \frac{\partial^2 \Phi_3^+}{\partial z^2}\right) \qquad (4.1\text{-}26a)$$

and

$$\frac{\partial^2 \psi^-}{\partial z^2} = \frac{1}{(4\nu_2 - 3)}\left(\frac{\partial^2 \Phi_4^-}{\partial z^2} + \frac{\partial^2 \Phi_3^-}{\partial z^2}\right) \quad , \qquad (4.1\text{-}26b)$$

the conditions that the crack surfaces are free from shear stresses imply the following functional relations:

$$\left.\begin{aligned} \frac{\partial^2 \Phi_3^+}{\partial z^2} - A_1 \frac{\partial^2 \Phi_4^+}{\partial z^2} + \frac{\mu_1}{D_1} \frac{\partial^2 \Omega^+}{\partial z^2} &= 0 \qquad (4.1\text{-}27a) \\ \frac{\partial^2 \Phi_3^-}{\partial z^2} - A_2 \frac{\partial^2 \Phi_4^-}{\partial z^2} + \frac{\mu_2}{D_2} \frac{\partial^2 \Omega^-}{\partial z^2} &= 0 \qquad (4.1\text{-}27b) \end{aligned}\right\} \; r < a,\ z = 0$$

where the constants $D_n (n = 1,2)$ designate

$$D_n = \frac{\mu_n^2}{\lambda_n + 3\mu_n}, \quad (n = 1,2) \qquad (4.1\text{-}28)$$

It should be noted that the value of $\frac{\partial^2 \Omega}{\partial z^2}$ is known, since it is proportional to the temperature gradient across the surfaces of the insulated crack (see (2-4), (3-5a) and (3-5b)). Upon making use of the identities

$$\mu_n = \frac{E_n}{2(1+\nu_n)}, \quad \lambda_n = \frac{E_n \nu_n}{(1+\nu_n)(1-2\nu_n)}, \quad (n = 1,2)$$

equations (4.1-27) take the form

$$\left.\begin{aligned} \frac{\partial^2 \Phi_3^+}{\partial z^2} - A_1 \frac{\partial^2 \Phi_4^+}{\partial z^2} &= L_1 \gamma k_1 \qquad (4.1\text{-}29a) \\ \frac{\partial^2 \Phi_3^-}{\partial z^2} - A_2 \frac{\partial^2 \Phi_4^-}{\partial z^2} &= L_2 \gamma k_1 \qquad (4.1\text{-}29b) \end{aligned}\right\} \; r < a,\ z = 0$$

where the following abbreviations have been introduced for convenience:

$$L_n = \frac{(3-4\nu_n)(1+\nu_n)\alpha_n}{2k_n(1-\nu_n)(1-2\nu_n)} \quad , \quad (n = 1,2) \tag{4.1-30}$$

Since Φ_3 and Φ_4 are axially symmetric harmonic functions, equations (4.1-29) yield

$$\Phi_3^+ - A_1\Phi_4^+ = -\frac{L_1 \gamma k_1 r^2}{4} + C_1 \tag{4.1-31a}$$

$$\Phi_3^- - A_2\Phi_4^- = -\frac{L_2 \gamma k_1 r^2}{4} + C_2 \tag{4.1-31b}$$

$$r < a, \; z = 0$$

where C_1 and C_2 are arbitrary constants subject to determination.

Thus, the boundary conditions across the crack surfaces have been reduced to two conditions which determine Φ_3 and Φ_4. These conditions may be taken for either crack surface and for that in S^-, are given by (4.1-20b) and (4.1-31b).

The remaining boundary conditions governing the solution are (4.1-4) - (4.1-9) which represent continuity of stress and displacement across the bonding surface exterior to the crack. Before proceeding to reduce these conditions, it should be noted that substitution of (2-6) and (4.1-18) into (4.1-13) gives

$$B_1\frac{\partial\Phi_3^+}{\partial z} - D_1\frac{\partial\Phi_4^+}{\partial z} = B_2\frac{\partial\Phi_3^-}{\partial z} - D_2\frac{\partial\Phi_4^-}{\partial z} \;, \; r > a, z = 0 \tag{4.1-32}$$

where

$$B_n = \frac{\mu_n(\lambda_n + 2\mu_n)}{\lambda_n + 3\mu_n}, \qquad (n = 1,2) \tag{4.1-33}$$

Moreover, differentiating (4.1-14) and (4.1-15) with respect to x and y, adding and taking into account (4.1-18) and (4.1-23) results in

$$\mu_1\left[-\frac{\partial^2}{\partial z^2}(\Phi_3^+ + \Psi^+ + 2\Omega^+) + \frac{\partial^2 \Phi_4^+}{\partial z^2}\right] =$$

$$\mu_2\left[-\frac{\partial^2}{\partial z^2}(\Phi_3^- + \Psi^- + 2\Omega^-) + \frac{\partial^2 \Phi_4^-}{\partial z^2}\right], \qquad r > a,\ z = 0 \tag{4.1-34}$$

while the substitution of (4.1-26) into (4.1-34) yields

$$D_1\frac{\partial^2\Phi_3^+}{\partial z^2} - B_1\frac{\partial^2\Phi_4^+}{\partial z^2} + \mu_1\frac{\partial^2\Omega^+}{\partial z^2} = D_2\frac{\partial^2\Phi_3^-}{\partial z^2} - B_2\frac{\partial^2\Phi_4^-}{\partial z^2} + \mu_2\frac{\partial^2\Omega^-}{\partial z^2},$$

$$r > a,\ z = 0 \tag{4.1-35}$$

Equations (4.1-32) and (4.1-35) represent continuity of normal and shear stress respectively across the bonding surface. These conditions must necessarily be true not only on the bonding surface, but across the crack surfaces as well (since the crack surfaces are subjected to zero stresses). This may be confirmed by multiplying (4.1-20a) and (4.1-20b) by B_1 and B_2 respectively and subtracting. The result is

$$B_1 \frac{\partial \Phi_3^+}{\partial z} - \frac{B_1}{A_1} \frac{\partial \Phi_4^+}{\partial z} = B_2 \frac{\partial \Phi_3^-}{\partial z} - \frac{B_2}{A_2} \frac{\partial \Phi_4^-}{\partial z} \quad , \; r < a, \; z = 0 \qquad (4.1\text{-}36)$$

Noting that

$$\frac{B_n}{A_n} = D_n \quad , \quad (n = 1, 2) \quad ,$$

equation (4.1-36) takes the form

$$B_1 \frac{\partial \Phi_3^+}{\partial z} - D_1 \frac{\partial \Phi_4^+}{\partial z} = B_2 \frac{\partial \Phi_3^-}{\partial z} - D_2 \frac{\partial \Phi_4^-}{\partial z} \quad , \; r < a, \; z = 0 \qquad (4.1\text{-}37)$$

which represents continuity of normal stress across the crack opening. A similar procedure may be applied to (4.1-27) to obtain an expression for shear stress continuity across the cut analogous to (4.1-35). (In this case, (4.1-27a) and (4.1-27b) are multiplied by D_1 and D_2 respectively and subtracted.) The result is

$$D_1 \frac{\partial^2 \Phi_3^+}{\partial z^2} - B_1 \frac{\partial^2 \Phi_4^+}{\partial z^2} + \mu_1 \frac{\partial^2 \Omega^+}{\partial z^2} =$$

$$D_2 \frac{\partial^2 \Phi_3^-}{\partial z^2} - B_2 \frac{\partial^2 \Phi_4^-}{\partial z^2} + \mu_2 \frac{\partial^2 \Omega^-}{\partial z^2} \quad , \; r < a, \; z = 0 \qquad (4.1\text{-}38)$$

Thus, the equations of stress continuity, which are valid on the (whole) z = 0 plane, may be expressed as

$$B_1 \frac{\partial \Phi_3^+}{\partial z} - B_2 \frac{\partial \Phi_3^-}{\partial z} - D_1 \frac{\partial \Phi_4^+}{\partial z} + D_2 \frac{\partial \Phi_4^-}{\partial z} = 0 \; , \; z = 0 \qquad (4.1\text{-}39)$$

$$D_1 \frac{\partial^2 \Phi_3^+}{\partial z^2} - D_2 \frac{\partial^2 \Phi_3^-}{\partial z^2} - B_1 \frac{\partial^2 \Phi_4^+}{\partial z^2} + B_2 \frac{\partial^2 \Phi_4^-}{\partial z^2} = \mu_2 \frac{\partial^2 \Omega^-}{\partial z^2} - \mu_1 \frac{\partial^2 \Omega^+}{\partial z^2}, \quad z = 0 \qquad (4.1\text{-}40)$$

where the thermoelastic potentials Ω^+ and Ω^- have been determined previously in Chapter 3.

The conditions of displacement continuity (equations (4.1-16)) across the bonded surface provide the remaining equations needed for the determination of Φ_3 and Φ_4. Differentiating the first two equations of (4.1-16) by x and y respectively and adding gives

$$\frac{\partial \Phi_1^+}{\partial x} + \frac{\partial \Phi_2^+}{\partial y} = \frac{\partial \Phi_1^-}{\partial x} + \frac{\partial \Phi_2^-}{\partial y}, \quad r > a, \ z = 0 \qquad (4.1\text{-}41)$$

Upon making use of (4.1-18), equation (4.1-41) gives

$$\frac{\partial \Phi_4^+}{\partial z} = \frac{\partial \Phi_4^-}{\partial z} \qquad (4.1\text{-}42)$$

At this point, it is convenient to introduce the following functions for consideration:

$$\Phi_3^*(x,y,z) = \Phi_3(x,y,-z), \quad \Phi_4^*(x,y,z) = \Phi_4(x,y,-z) \qquad (4.1\text{-}43)$$

The purpose of these functions is to reduce conditions (4.1-39) and (4.1-40) (which apply to both half-spaces) to conditions valid for a half-space only (the lower one, say) thereby facilitating the solution of the problem under consideration. Now, (4.1-43) imply

$$\left.\begin{aligned}
&\Phi_i^{*+}(x,y,z) = \Phi_i^{-}(x,y,z) \quad , \quad \Phi_i^{*-}(x,y,z) = \Phi_i^{+}(x,y,z) && (4.1\text{-}44a)\\
&\frac{\partial \Phi_i^{*+}(x,y,z)}{\partial z} = -\frac{\partial \Phi_i^{-}(x,y,z)}{\partial z} \; , \; \frac{\partial \Phi_i^{*-}(x,y,z)}{\partial z} = -\frac{\partial \Phi_i^{+}(x,y,z)}{\partial z} && (4.1\text{-}44b)\\
&\frac{\partial^2 \Phi_i^{*+}(x,y,z)}{\partial z^2} = \frac{\partial^2 \Phi_i^{-}(x,y,z)}{\partial z^2} \; , \; \frac{\partial^2 \Phi_i^{*-}(x,y,z)}{\partial z^2} = \frac{\partial^2 \Phi_i^{+}(x,y,z)}{\partial z^2} && (4.1\text{-}44c)
\end{aligned}\right\} (i = 3,4)$$

and hence conditions (4.1-39) and (4.1-40) take the form

$$\left.\begin{aligned}
&-B_1 \frac{\partial \Phi_3^{*-}}{\partial z} - B_2 \frac{\partial \Phi_3^{-}}{\partial z} + D_1 \frac{\partial \Phi_4^{*-}}{\partial z} + D_2 \frac{\partial \Phi_4^{-}}{\partial z} = 0 && (4.1\text{-}45)\\
&D_1 \frac{\partial^2 \Phi_3^{*-}}{\partial z^2} - D_2 \frac{\partial^2 \Phi_3^{-}}{\partial z^2} - B_1 \frac{\partial^2 \Phi_4^{*-}}{\partial z^2} + B_2 \frac{\partial^2 \Phi_4^{-}}{\partial z^2} = \\
&\qquad \mu_2 \frac{\partial^2 \Omega^{-}}{\partial z^2} - \mu_1 \frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1} \frac{\partial^2 \Omega^{-}}{\partial z^2} && (4.1\text{-}46)
\end{aligned}\right\} z = 0$$

where equations (2-4) and (3-5e) have been employed. Since a harmonic function that is zero everywhere on the boundary of a region is zero everywhere within the region*, equations (4.1-45) and (4.1-46) may be extended to include $z \leq 0$. Integrating these two expressions in S^-, and making use of the regularity conditions at infinity of the potential functions involved, it follows that

---

*** See footnote on page 14.**

$$\left.\begin{array}{l} -B_1\Phi_3^{*-} - B_2\Phi_3^- + D_1\Phi_4^{*-} + D_2\Phi_4^- = 0 \\ \\ D_1\Phi_3^{*-} - D_2\Phi_3^- - B_1\Phi_4^{*-} + B_2\Phi_4^- \\ \qquad - \mu_2\Omega^- + \mu_1\frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1}\Omega^- = 0 \end{array}\right\} z \le 0 \quad \begin{array}{r}(4.1\text{-}47)\\ \\ (4.1\text{-}48)\end{array}$$

It is a simple matter to deduce from these equations that

$$\Phi_3^{*-} = E\Phi_3^- - H\Phi_4^- - \frac{D_1}{D_1^2 - B_1^2}\left[\mu_1\frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1}\Omega^- - \mu_2\Omega^-\right] \quad (4.1\text{-}49)$$

and

$$\Phi_4^{*-} = H\Phi_3^- - E\Phi_4^- - \frac{B_1}{D_1^2 - B_1^2}\left[\mu_1\frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1}\Omega^- - \mu_2\Omega^-\right] \quad (4.1\text{-}50)$$

where the following abbreviations have been adopted:

$$E = \frac{B_1 B_2 + D_1 D_2}{D_1^2 - B_1^2}, \quad H = \frac{B_1 D_2 + B_2 D_1}{D_1^2 - B_1^2} \quad (4.1\text{-}51)$$

Differentiating (4.1-50) with respect to z gives

$$\frac{\partial\Phi_4^{*-}}{\partial z} = H\frac{\partial\Phi_3^-}{\partial z} - E\frac{\partial\Phi_4^-}{\partial z} - \frac{B_1}{D_1^2 - B_1^2}\left[\mu_1\frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1}\frac{\partial\Omega^-}{\partial z} - \mu_2\frac{\partial\Omega^-}{\partial z}\right] \quad (4.1\text{-}52)$$

From the definitions of Φ_3^* and Φ_4^*, it is possible to eliminate the star quantities from (4.1-49) and (4.1-52). Substituting from (4.1-44) into (4.1-49) and (4.1-52) results in

$$\Phi_3^+ = E\,\Phi_3^- - H\,\Phi_4^- - \frac{D_1}{D_1^2 - B_1^2}\left[\mu_1 \frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1} - \mu_2\right]\Omega^- \qquad (4.1\text{-}53)$$

$$-\frac{\partial \Phi_4^+}{\partial z} = H\frac{\partial \Phi_3^-}{\partial z} - E\frac{\partial \Phi_4^-}{\partial z} - \frac{B_1}{D_1^2 - B_1^2}\left[\mu_1 \frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1} - \mu_2\right]\frac{\partial \Omega^-}{\partial z} \qquad (4.1\text{-}54)$$

If attention is now focused on the bending surface ($r > a$, $z = 0$), it is possible to use the physical conditions of the problem to simplify (4.1-53) and (4.1-54). Noting that $\frac{\partial \Omega^-}{\partial z} = 0$ for $r > a$, $z = 0$ and making use of the conditions of continuity of displacements across the bond, conditions (4.1-53) and (4.1-54) reduce to

$$\Phi_3^- - A_0\,\Phi_4^- = \frac{1}{(1-E)}\left[\Omega^+ - \Omega^- - \frac{D_1}{D_1^2 - B_1^2}\left(\mu_1 \frac{(1+\nu_1)(1-\nu_2)\alpha_1 k_2}{(1+\nu_2)(1-\nu_1)\alpha_2 k_1} - \mu_2\right)\Omega^-\right],$$

$$r > a,\quad z = 0 \qquad (4.1\text{-}55)$$

$$\frac{\partial \Phi_3^-}{\partial z} - \frac{1}{A_0}\frac{\partial \Phi_4^-}{\partial z} = 0, \quad r > a,\ z = 0 \qquad (4.1\text{-}56)$$

where A_0 stands for

$$A_0 = \frac{H}{E - 1} \qquad (4.1\text{-}57)$$

Equations (4.1-20b), (4.1-31b), (4.1-55) and (4.1-56) represent the boundary conditions required for a so-

lution to the thermal stress problem.

For the purpose of reducing the axisymmetric potential problem into a subsidiary problem in the plane, it is convenient to recast boundary conditions (4.1-20b), (4.1-31b), (4.1-55) and (4.1-56) in terms of functions $F_1(x,y,z)$ and $F_2(x,y,z)$ defined by

$$F_1(x\,y\,z) = \Phi_3^-(x\,y\,z) - A_2\,\Phi_4^-(x\,y\,z) \tag{4.1-58a}$$

$$F_2(x\,y\,z) = \Phi_3^-(x\,y\,z) - \frac{1}{A_2}\,\Phi_4^-(x\,y\,z) \tag{4.1-58b}$$

which, when solved simultaneously, imply

$$\Phi_3^- = \frac{\frac{F_1}{A_2} - A_2 F_2}{\frac{1}{A_2} - A_2} \quad , \quad \Phi_4^- = \frac{F_1 - F_2}{\frac{1}{A_2} - A_2} \tag{4.1-59}$$

Substituting (4.1-59) into (4.1-55) and (4.1-56) and evaluating the right side of (4.1-55) from Chapter 3 (see (3-24) and (3-26)) leads to

$$\left.\begin{aligned} F_1(x\,y\,0) - AF_2(x\,y\,0) &= K_0 f(r) \\ \left[\frac{\partial F_1(x\,y\,z)}{\partial z} - B\,\frac{\partial F_2(x\,y\,z)}{\partial z}\right]_{z=0} &= 0 \end{aligned}\right\}\; r > a \qquad \begin{aligned}&(4.1\text{-}60)\\ &(4.1\text{-}61)\end{aligned}$$

where the constants A and B represent

$$A = \frac{A_2 - A_0}{1 - A_0 A_2}\,A_2 \quad , \quad B = \frac{1 - A_0 A_2}{A_2 - A_0}\,A_2 \tag{4.1-62}$$

and

$$K_0 = -\frac{(1-A_2^2)}{(1-A_0A_1)(1-E)}\sqrt{\frac{2}{\pi}}\,\tau a^{3/2}\Big\{(1+\nu_1)\alpha_1 - \frac{(1+\nu_2)\alpha_2 k_1}{2(1-\nu_2)k_2}\left[\frac{\mu_1+\mu_2(1-2\nu_1)}{\mu_1}\right]\Big\} \qquad (4.1\text{-}63)$$

$$f(r) = \int_0^\infty \frac{J_0(rs)\,J_{3/2}(as)\,ds}{s^{1/2}} = \sqrt{\frac{2}{\pi}}\,\frac{1}{4a^{3/2}}\left[(2a^2-r^2)\sin^{-1}\frac{a}{r} + a\sqrt{r^2-a^2}\right] \qquad (4.1\text{-}64)$$

while from (4.1-20b), (4.1-31b) and (4.1-58), it follows that

$$F_1(x,y,0) = -\frac{L_2\tau k_1 r^2}{4} + C_2 \qquad (4.1\text{-}65)$$

$$\left[\frac{\partial F_2(x,y,z)}{\partial z}\right]_{z=0} = 0 \qquad (4.1\text{-}66)$$

$$r < a$$

Equations (4.1-60), (4.1-61), (4.1-65) and (4.1-66) are boundary conditions of an axially symmetric problem in potential theory in three dimensions. Since the functions F_1 and F_2 were defined for the lower semi-space, S^-, it is clear that the original boundary value problem for an infinite solid has been reduced to that of a semi-infinite space.

4.2 Reduction of Axially Symmetric Problem in a Half-Space to Subsidiary Problem in a Plane

The axially symmetric potential problem formulated in the previous section requires a determination of the harmonic functions $F_1(x,y,z)$ and $F_2(x,y,z)$ subject to boundary conditions (4.1-60), (4.1-61), (4.1-65) and (4.1-66). Since $F_1(x,y,z)$ and $F_2(x,y,z)$ are independent of θ and vanish for large z ($z \leq 0$), they may be expressed in the form

$$F_n(r,z) = \int_0^\infty a_n(t) J_0(rt) e^{tz} dt \quad , \quad (n=1,2) \qquad (4.2\text{-}1)$$

where $J_0(rt)$ is a Bessel function of the first kind and $a_n(t)$, $(n = 1,2)$ are arbitrary functions. Note that as $z \to -\infty$ (in S^-), F_1 and F_2 vanish. Differentiating (4.2-1) with respect to z gives

$$\frac{\partial F_n(r,z)}{\partial z} = \int_0^\infty a_n(t)\, t\, J_0(rt) e^{tz} dt \quad , \quad (n=1,2) \qquad (4.2\text{-}2)$$

The application of boundary and continuity conditions (4.1-60), (4.1-61), (4.1-65) and (4.1-66) to these expressions for F_n and $\frac{\partial F_n}{\partial z}$, $(n = 1,2)$ results in two sets of coupled dual integral equations for the unknown functions $a_n(t)$, $(n = 1,2)$ which are difficult to solve. However, a simpler procedure is to obtain the solution of the problem under consideration by transferring the axisymmetric problem to a two dimensional problem and applying the theory of complex variables to plane elasticity problems as developed by Muskhelishvili [16]. In order to reduce the axially symmetric formulation to a plane problem, harmonic functions $U_1(x,z)$ and $U_2(x,z)$ are introduced as

$$U_n(x,z) = \int_0^\infty \frac{4}{t}\, a_n(t)\sin(tx)\, e^{tz}\, dt \,, \quad (n=1,2) \tag{4.2-3}$$

where $U_n(x,z)$, $(n = 1,2)$ are defined in the lower half-space S^- and are antisymmetric with respect to x. The derivative of (4.2-3) with respect to z yields

$$\frac{\partial U_n(x,z)}{\partial z} = \int_0^\infty 4\, a_n(t)\sin(tx)\, e^{tz}\, dt \,, \quad (n=1,2) \tag{4.2-4}$$

Following the procedure of Mossakovskii and Rybka [13], the following transformations, which illustrate the correspondence between the axisymmetric functions $F_n(r,z)$ and the plane functions $U_n(x,z)$, may be established:

$$\frac{\partial}{\partial x}\int_0^x \frac{F_n(r,z)}{\sqrt{x^2-r^2}}\, r\, dr = \frac{1}{4}\,\frac{\partial U_n(x,z)}{\partial x} \tag{4.2-5a}$$

$$\int_0^x \frac{\partial F_n(r,z)}{\partial z}\,\frac{r\, dr}{\sqrt{x^2-r^2}} = \frac{1}{4}\,\frac{\partial U_n(x,z)}{\partial z} \tag{4.2-5b}$$

$$-\frac{\partial}{\partial x}\int_x^\infty \frac{F_n(r,z)}{\sqrt{r^2-x^2}}\, r\,dr = \frac{1}{4}\frac{\partial U_n(x,z)}{\partial z} \qquad (4.2\text{-}5c)$$

$$\frac{1}{2\pi}\int_0^r \frac{\partial U_n(x,z)}{\partial x}\frac{dx}{\sqrt{r^2-x^2}} = F_n(r,z) \qquad (4.2\text{-}5d)$$

$$\frac{1}{2\pi r}\frac{\partial}{\partial r}\int_0^r \frac{\partial U_n(x,z)}{\partial z}\frac{x\,dx}{\sqrt{r^2-x^2}} = \frac{\partial F_n(r,z)}{\partial z} \qquad (4.2\text{-}5e)$$

$$-\frac{1}{2\pi r}\frac{\partial}{\partial r}\int_r^\infty \frac{\partial U_n(x,z)}{\partial x}\frac{x\,dx}{\sqrt{x^2-r^2}} = \frac{\partial F_n(r,z)}{\partial z} \qquad (4.2\text{-}5f)$$

$$\frac{1}{2\pi}\int_r^\infty \frac{\partial U_n(x,z)}{\partial z}\frac{dx}{\sqrt{x^2-r^2}} = F_n(r,z) \qquad (4.2\text{-}5g)$$

$$-\int_x^\infty \frac{\partial F_n(r,z)}{\partial z}\frac{r\,dr}{\sqrt{r^2-x^2}} = \frac{1}{4}\frac{\partial U_n(x,z)}{\partial x} \qquad (4.2\text{-}5h)$$

$$(n = 1,2)$$

Employing these equations, it is now possible to transform the boundary conditions on $F_n(r,z)$ $(n = 1,2)$ to corresponding boundary conditions on $U_n(x,z)$ $(n = 1,2)$ in the xz-plane. Substituting (4.1-65) into (4.2-5a) gives

$$\frac{\partial U_1(x,0)}{\partial x} = g_1(x), \quad |x| < a \qquad (4.2\text{-}6)$$

where $g_1(x) = 4C_2 - 2L_2\gamma k_1 x^2$.

From (4.1-66) and (4.2-5b), a second two-dimensional boundary condition is obtained as

$$\left[\frac{\partial U_2(x,z)}{\partial z}\right]_{z=0} = 0 \quad , \quad |x| < a \tag{4.2-7}$$

Making use of (4.2-5h), boundary condition (4.1-61) is reduced to

$$\frac{\partial U_1(x,0)}{\partial x} - B\,\frac{\partial U_2(x,0)}{\partial x} = 0 \quad ; \quad |x| > a \tag{4.2-8}$$

The remaining external boundary condition, (4.1-60), may be reduced by employing (4.2-5c). Before achieving this some manipulation is necessary. Thus transformation (4.2-5c) implies

$$\left[\frac{\partial U_1(x,z)}{\partial z} - A\,\frac{\partial U_2(x,z)}{\partial z}\right]_{\substack{z=0\\|x|>a}} = -4\,\frac{\partial}{\partial x}\int_x^{\infty}\frac{r\,dr}{\sqrt{r^2-x^2}}\left[F_1(r,0) - AF_2(r,0)\right] \tag{4.2-9}$$

Substituting for $F_1 - AF_2$ from (4.1-60) and (4.1-64) and making the following permissible change of order of integration

$$\int_x^{\infty}\frac{r\,dr}{\sqrt{r^2-x^2}}\int_0^{\infty}\frac{J_0(rs)\,J_{V_2}(as)\,ds}{s^{3/2}} = \int_0^{\infty}\frac{J_{V_2}(as)\,ds}{s^{3/2}}\int_x^{\infty}\frac{J_0(rs)\,r\,dr}{\sqrt{r^2-x^2}} \tag{4.2-10}$$

and replacing the inner integral by the equivalent result

$$\int_x^{\infty}\frac{r\,J_0(rs)\,dr}{\sqrt{r^2-x^2}} = \frac{\cos xs}{s} \quad , \tag{4.2-11}$$

it follows that

$$\frac{\partial U_1(x,z)}{\partial z} - A\,\frac{\partial U_2(x,z)}{\partial z} = F_0(x)\;,\quad z=0,\quad |x|>a \qquad (4.2\text{-}12)$$

where $F_0(x)$ designates the discontinuous Weber-Sonine-Schafheitlin integral given by

$$F_0(x) = 4K_0\int_0^{\infty}\frac{J_{3/2}(as)\sin xs\,ds}{s^{3/2}} \qquad (4.2\text{-}13)$$

which may be shown to be given by

$$F_0(x) = \begin{cases} 2K_0\sqrt{\frac{a}{2\pi}}\left\{\left[1-\frac{x^2}{a^2}\right]\ln\left(\frac{|x|+a}{|x|-a}\right)+\frac{2|x|}{a}\right\}, & |x|>a \qquad (4.2\text{-}14a) \\ 2K_0|x|\sqrt{\frac{2}{\pi a}}\left\{1-\frac{(a^2-x^2)}{2a|x|}\ln\left(\frac{a-|x|}{a+|x|}\right)\right\}, & |x|<a \qquad (4.2\text{-}14b) \end{cases}$$

Thus, the desired objective of reducing the original axisymmetric boundary value problem into an equivalent problem in the plane has been achieved. The harmonic functions $U_1(x,z)$ and $U_2(x,z)$, which are defined in S^-, must be determined subject to boundary conditions (4.2-6), (4.2-7), (4.2-8) and (4.2-12). This will be done in the next section using the theory of complex variables.

4.3 Reduction of Two-Dimensional Problem to Problem of Linear Relationship

The solution to the subsidiary problem in the plane may be obtained by formulating a problem of linear relationship, or Hilbert problem, between the functions U_1 and U_2. This procedure is initiated by regarding the functions $U_1(x,z)$ and $U_2(x,z)$, harmonic in S^-, as the real parts of analytic functions $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ of the complex variable $\zeta = x + iz$. Thus

$$U_n(x,z) = \frac{1}{2}\Phi_n(\zeta) + \frac{1}{2}\overline{\Phi_n(\zeta)} \tag{4.3-1}$$

Differentiation of (4.3-1) gives

$$\frac{\partial U_n}{\partial x} = \frac{1}{2}\Phi_n'(\zeta) + \frac{1}{2}\overline{\Phi_n'(\zeta)} \tag{4.3-2a}$$

$$\frac{\partial U_n}{\partial z} = \frac{i}{2}\Phi_n'(\zeta) - \frac{i}{2}\overline{\Phi_n'(\zeta)} \tag{4.3-2b}$$

Since $U_1(x,z)$ and $U_2(x,z)$ were defined for the lower half-space S^- where $z \leq 0$, the same must be true for the functions $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$. In the Hilbert problem, however, there must be a linear relationship of the boundary values of a certain sectionally holomorphic function on both sides of a line of discontinuity. In the present case, this means that a relation between plus and minus quantities at $z = 0$ is required. This may be accomplished by defining (see reference [16], page 294 for more details on the extension of analytic functions) $\overline{F}(\zeta)$ as the function having the conjugate complex value of $F(\zeta)$ at the point $\bar{\zeta}$, where $\bar{\zeta}$ is the conjugate complex value of ζ. Thus,

$$\overline{F}(\zeta) = \overline{F(\bar{\zeta})} \quad , \quad \overline{F(\zeta)} = \overline{F}(\bar{\zeta}) \tag{4.3-3}$$

Therefore, in the present case

$$\overline{\Phi}_n(\bar{\zeta}) = \overline{\Phi_n(\zeta)} \quad , \quad (n = 1, 2) \tag{4.3-4}$$

Since $\Phi_n(\zeta)$ is defined in the lower semi-space S^-, it follows that

$$\overline{\Phi}_n^{+}(x) = \overline{\Phi_n^{-}(x)} \quad \text{on} \quad z = 0 \quad (n = 1, 2) \tag{4.3-5}$$

Thus, if (4.3-2) is considered on z = 0, and substituted into (4.2-6), (4.2-7), (4.2-8) and (4.2-12), the following boundary conditions on $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ are obtained:

$$\Phi_1'^{-} + \overline{\Phi}_1'^{+} = 2 g_1(x) \quad \text{on} \quad L' \tag{4.3-6a}$$

$$\Phi_2'^{-} - \overline{\Phi}_2'^{+} = 0 \quad \text{on} \quad L' \tag{4.3-6b}$$

$$\Phi_1'^{-} - \overline{\Phi}_1'^{+} - A\Phi_2'^{-} + A\overline{\Phi}_2'^{+} = -2i F_2(x) \quad \text{on} \quad L'' \tag{4.3-7a}$$

$$\Phi_1'^{-} + \overline{\Phi}_1'^{+} - B\Phi_2'^{-} - B\overline{\Phi}_2'^{+} = 0 \quad \text{on} \quad L'' \tag{4.3-7b}$$

where L' denotes the region of the real axis for which $-a < x < a$ and L'' refers to the remaining part of the axis. (see Fig. 4)

The boundary value problem posed in equations (4.3-6) and (4.3-7) may be simplified by introducing functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$, analytic over the entire plane, with the exception of a slit coinciding with L', as follows:

$$\Phi_1'(\zeta) - B\Phi_2'(\zeta) = \Omega_1(\zeta) \qquad \text{in } S^- \tag{4.3-8a}$$

$$\Phi_1'(\zeta) - A\Phi_2'(\zeta) + iF_0(\zeta) = \Omega_2(\zeta) \qquad \text{in } S^- \tag{4.3-8b}$$

$$\overline{\Phi}_1'(\zeta) - B\overline{\Phi}_2'(\zeta) = -\Omega_1(\zeta) \qquad \text{in } S^+ \tag{4.3-8c}$$

$$\overline{\Phi}_1'(\zeta) - A\overline{\Phi}_2'(\zeta) - iF_0(\zeta) = \Omega_2(\zeta) \qquad \text{in } S^+ \tag{4.3-8d}$$

From these relations, it can be shown that

$$\Phi_1'(\zeta) = \frac{A\Omega_1(\zeta) - B\Omega_2(\zeta) + iBF_0(\zeta)}{A - B} \qquad \text{in } S^- \tag{4.3-9a}$$

$$\overline{\Phi}_1'(\zeta) = -\frac{\left[A\Omega_1(\zeta) + B\Omega_2(\zeta) + iBF_0(\zeta)\right]}{A - B} \qquad \text{in } S^+ \tag{4.3-9b}$$

$$\Phi_2'(\zeta) = \frac{\Omega_1(\zeta) - \Omega_2(\zeta) + iF_0(\zeta)}{A - B} \qquad \text{in } S^- \tag{4.3-9c}$$

$$\overline{\Phi}_2'(\zeta) = -\frac{\left[\Omega_1(\zeta) + \Omega_2(\zeta) + i F_0(\zeta)\right]}{A - B} \quad \text{in } S^+ \qquad (4.3\text{-}9d)$$

Using these relations, it follows that (4.3-7a) and (4.3-7b) are satisfied identically. This condition motivated the choice of $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ in (4.3-8) and permits a determination of the unknown functions from the conditions on L' only. Substituting (4.3-9) into (4.3-6) leads to

$$\left.\begin{aligned} &\frac{A}{A-B}\,\Omega_1^- - \frac{B}{A-B}\,\Omega_2^- - \frac{A}{A-B}\,\Omega_1^+ - \frac{B}{A-B}\,\Omega_2^+ = 2\,g_1(x) \qquad &(4.3\text{-}10a) \\ &\frac{1}{A-B}\,\Omega_1^- - \frac{1}{A-B}\,\Omega_2^- + \frac{1}{A-B}\,\Omega_1^+ + \frac{1}{A-B}\,\Omega_2^+ = -2i\,g_2(x) \qquad &(4.3\text{-}10b) \end{aligned}\right|_{\text{on } L'}$$

where $g_2(x) = \frac{F_0(x)}{A-B}$ and $g_1(x)$ is defined in (4.2-6).

Before concluding the general development of the present section, eq. (4.3-10a) is multiplied by $\frac{(A-B)}{A+\sqrt{AB}}$ and (4.3-10b) by $\frac{\sqrt{AB}\,(A-B)}{A+\sqrt{AB}}$. Addition and subtraction of the two resulting equations results in the standard form of the problem of linear relationship, or Hilbert problem, as follows:

$$\left[\Omega_1(x) - \frac{B+\sqrt{AB}}{A+\sqrt{AB}}\,\Omega_2(x)\right]^- - \frac{A-\sqrt{AB}}{A+\sqrt{AB}}\left[\Omega_1(x) - \frac{B+\sqrt{AB}}{A+\sqrt{AB}}\,\Omega_2(x)\right]^+ =$$

$$\frac{2(A-B)}{A+\sqrt{AB}}\left[g_1(x) - i\sqrt{AB}\,g_2(x)\right] \qquad \text{(4.3-11a)}$$

$$\left[\Omega_1(x) - \frac{B-\sqrt{AB}}{A-\sqrt{AB}}\,\Omega_2(x)\right]^- - \frac{A+\sqrt{AB}}{A-\sqrt{AB}}\left[\Omega_1(x) - \frac{B-\sqrt{AB}}{A-\sqrt{AB}}\,\Omega_2(x)\right]^+ =$$

$$\frac{2(A-B)}{A-\sqrt{AB}}\left[g_1(x) + i\sqrt{AB}\,g_2(x)\right] \qquad \text{(4.3-11b)}$$

4.4 Solution of Hilbert Problem

Briefly, the Hilbert problem requires a solution of the equation

$$F^+(t) - G(t)F^-(t) = f(t) \quad \text{on } L \qquad \text{(4.4-1)}$$

which relates the boundary values ($F^+(t)$ and $F^-(t)$) of a sectionally holomorphic function, $F(\zeta)$, as a line of discontinuity (L) is approached from above and below. $G(t)$ and $f(t)$ are functions given on L and $G(t) \neq 0$ everywhere on L. Both $G(t)$ and $f(t)$ must satisfy the Holder condition on L, i.e. for every pair of points t_1 and t_2 of L, the following inequalities hold:

$$|G(t_2) - G(t_1)| \leq K|t_2 - t_1|^{\delta} \quad ,$$

$$|f(t_2) - f(t_1)| \leq K|t_2 - t_1|^{\delta}$$

where K and δ are positive constants. Equations (4.3-11) are of the above type where G(t) is a constant, say G_0. In such a case, the solution of the Hilbert problem depends on the Plemelj function $X_0(\zeta)$ which has the important property that

$$X_0^+(t) = G_0 X_0^-(t) \quad \text{on } L \tag{4.4-2}$$

For the problem under discussion, the Plemelj functions are

$$X_0(\zeta) = \left(\frac{\zeta - a}{\zeta + a}\right)^{m_n}, \quad (n = 1, 2) \tag{4.4-3}$$

where the two values of n refer to the first and second equations of (4.3-11) and m_n are complex constants. It can be shown that $X_0(\zeta)$ will satisfy (4.4-2) provided

$$G_0 = e^{2\pi i m} \tag{4.4-4}$$

Taking (4.3-11), (4.4-1) and (4.4-4) into account yields the following values for $m_n (n = 1,2)$:

$$m_1 = \frac{1}{2\pi i} \ln \frac{A + \sqrt{AB}}{A - \sqrt{AB}} = -m_2 \tag{4.4-5}$$

The reader is referred to [16], page 427, for further discussion of the Hilbert problem.

It should be noted that the form of the Plemelj function gives rise to the oscillation of the stress field at the periphery of the crack.

The solution of the Hilbert problem as posed in (4.3-11), is given by [16]

$$\Omega_1(\zeta) - \frac{B+\sqrt{AB}}{A+\sqrt{AB}}\,\Omega_2(\zeta) = \frac{X_{0_1}(\zeta)}{2\pi i}\int_{L'}\frac{f_1(x)\,dx}{X_{0_1}^{+}(x)(x-\zeta)} \qquad \textbf{(4.4-6a)}$$

and

$$\Omega_1(\zeta) - \frac{B-\sqrt{AB}}{A-\sqrt{AB}}\,\Omega_2(\zeta) = \frac{X_{0_2}(\zeta)}{2\pi i}\int_{L}\frac{f_2(x)\,dx}{X_{0_2}^{+}(x)(x-\zeta)} \qquad \textbf{(4.4-6b)}$$

where

$$f_1(x) = -\frac{2(A-B)}{A-\sqrt{AB}}\left[g_1(x) - i\sqrt{AB}\,g_2(x)\right] \qquad \textbf{(4.4-7a)}$$

$$f_2(x) = -\frac{2(A-B)}{A+\sqrt{AB}}\left[g_1(x) + i\sqrt{AB}\,g_2(x)\right] \qquad \textbf{(4.4-7b)}$$

The integrals in (4.4-6) may be evaluated from complex function theory. Specifically, using the residue theorem and Cauchy integral formula, it follows that

$$\int_{L'}\frac{f_n(x)\,dx}{X_{0n}^{+}(x)(x-\zeta)} = \frac{2\pi i}{1-G_{0n}}\left\{\frac{f_n(\zeta)}{X_{0n}(\zeta)} - \gamma_q\zeta^{q} - \gamma_{q-1}\zeta^{q-1} - \dots - \gamma_0\right\}, \quad (n=1,2) \qquad \textbf{(4.4-8)}$$

where $\gamma_q, \gamma_{q-1}, \ldots, \gamma_0$ represent the coefficients of $\zeta^q, \zeta^{q-1}, \ldots, \zeta^0$ in the expansion of $\frac{F_n(\zeta)}{X_1(\zeta)}$ for large ζ.

Solving for $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$, the following results are obtained:

$$\Omega_1(\zeta) = \frac{(A-B)}{\sqrt{AB}}\left\{-\frac{i\sqrt{AB}}{(A-B)}F_0(\zeta) + 2L_2\gamma k_1 m_1 a\zeta\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{-m_1}+\left(\frac{\zeta-a}{\zeta+a}\right)^{m_1}\right]\right.$$

$$\left.-\left[L_2\gamma k_1\zeta^2 + 2L_2\gamma k_1 m_1^2 a^2 - 2C_2\right]\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{-m_1}-\left(\frac{\zeta-a}{\zeta+a}\right)^{m_1}\right]\right\} \qquad (4.4\text{-}9a)$$

$$\Omega_2(\zeta) = \frac{(A-B)}{B}\left\{-4C_2 + 2L_2\gamma k_1\zeta^2 + 2L_2\gamma k_1 m_1 a\zeta\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{-m_1}-\left(\frac{\zeta-a}{\zeta+a}\right)^{m_1}\right]\right.$$

$$\left.-\left[L_2\gamma k_1\zeta^2 + 2L_2\gamma k_1 m_1^2 a^2 - 2C_2\right]\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{-m_1}+\left(\frac{\zeta-a}{\zeta+a}\right)^{m_1}\right]\right\} \qquad (4.4\text{-}9b)$$

The constant of integration, C_2, may be evaluated by considering the solution at infinity. The functions $\Omega_n(\zeta)$, $(n = 1,2)$ were defined in terms of derivatives of analytic functions $\Phi_n(\zeta)$, $(n = 1,2)$ (see (4.3-8)) and must therefore vanish at infinity as $\frac{1}{\zeta^2}$. This condition gives

$$C_2 = \frac{iK_1A_2}{3m_1(A-B)}\sqrt{\frac{a}{2\pi}} + \frac{L_2\gamma k_1 a^2}{6}(1+2m_1^2) \qquad (4.4\text{-}10)$$

Once the analytic functions $\Omega_n(\zeta)$, n = 1,2, governing the formulation of the thermal stress problem are known, the physical quantities of interest in the original solid may be obtained by substituting the values of $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ into the appropriate transformations which relate the plane problem solved in this section to the original axisymmetric problem under discussion. This will be done in the next chapter.

CHAPTER 5

EVALUATION OF PHYSICAL QUANTITIES

The Hilbert solution of the previous chapter may be used to evaluate all the physical quantities of the thermal stress problem. The quantities of interest include displacements and stresses on the bending plane, potential energy loss due to the existence of the crack, the critical value of the heat flux and the stress intensity factors. These quantities are essential in the application of the Griffith-Irwin theory of fracture. The details involved in calculating these quantities are quite lengthy. They involve expressing the stresses and displacements in terms of the harmonic functions Φ_3 and Φ_4 and then in terms of F_1 and F_2 as defined in (4.1-58). Making use of the transformations derived in eqs. (4.2-5), the stresses and displacements are expressed in terms of the functions $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial z}$ which are related to the analytic functions Ω_1 and Ω_2 obtained from the Hilbert solution in (4.4-9) (see (4.3-2) and (4.3-8)). Only the pertinent results and essential steps in the procedure will be indicated in the sequel. The details of the calculations will be left out.

5.1 Displacements Across Plane $z = 0$

For the lower half-space, the radial displacement across the plane $z = 0$ may be expressed by

$$u_r^- = \frac{1}{2\pi r\left(\frac{1}{A_1} - A_1\right)} \int_0^r \left[\frac{\partial U_1(x,z)}{\partial z} - \frac{\partial U_2(x,z)}{\partial z} \right] \frac{x\,dx}{\sqrt{r^2 - x^2}} \quad , \; z = 0 \qquad (5.1\text{-}1)$$

where use has been made of equations (2-5), (4.1-18), (4.1-59) and (4.2-5e). Taking account of (4.2-7), the radial displacement on the lower crack surface may now be written as

$$u_r^- = \frac{1}{2\pi r\left(\frac{1}{A_1} - A_1\right)} \int_0^r \frac{\partial U_1(x,z)}{\partial z} \frac{x\,dx}{\sqrt{r^2 - x^2}} \quad , \; r < a, \; z = 0 \qquad (5.1\text{-}2)$$

where $\frac{\partial U_1}{\partial z}$ is obtained from equations (4.3-2b) and (4.3-9) as

$$\frac{\partial U_1}{\partial z} = \frac{-i}{2(A-B)} \left\{ A(\Omega_1^+ + \Omega_1^-) - B(\Omega_2^+ + \Omega_2^-) + 2iBF_0(x) \right\} ,$$

$$|x| < a , \quad z = 0 \qquad (5.1\text{-}3)$$

and $F_0(x)$ is defined in (4.2-14). Evaluating the limiting values as $z \to 0$ from above and below the crack (see Appendix B for further details), expression (5.1-3) becomes

$$\frac{\partial U_1}{\partial z} = F_0(x) + 2\sqrt{A}/(A-B) \left\{ \frac{m_1 a \alpha_1 \pi k_1 x}{\sqrt{AB}} \left[\left(\frac{a-x}{a+x}\right)^{m_1} + \left(\frac{a-x}{a+x}\right)^{-m_1} \right] + \right.$$

$$\left. \left[\frac{\alpha_1 \pi k_1 x^2}{2\sqrt{AB}} - \frac{a^2 \alpha_1 \pi k_1 (1 - 4m_1^2)}{6\sqrt{AB}} + \frac{K_1}{3m_1(A-B)} \sqrt{\frac{a}{2\pi}} \right] \left[\left(\frac{a-x}{a+x}\right)^{m_1} - \left(\frac{a-x}{a+x}\right)^{-m_1} \right] \right\} ,$$

$$|x| < a , \; z = 0 \qquad (5.1\text{-}4)$$

where $m_1 = i\epsilon$ and ϵ is the bielastic constant given by

$$\epsilon = -\frac{1}{2\pi} \ln \frac{E_1(1+\nu_2)(3-4\nu_1) + E_2(1+\nu_1)}{E_1(1+\nu_2) + E_2(1+\nu_1)(3-4\nu_2)} \qquad (5.1\text{-}5)$$

Substituting (5.1-4) inte (5.1-2) yields

$$u_r^- = \frac{1}{2\pi r(\bar{A}_2 - \bar{A})} \int_0^r \Bigg[F_1(x) + 2\sqrt{A(A-B)} \Big\{ -\frac{2\epsilon L_2 \gamma k x}{\sqrt{AB}} \Big[\left(\frac{a-x}{a+x}\right)^{i\epsilon} + \left(\frac{a-x}{a+x}\right)^{-i\epsilon} \Big] + \Big[\frac{L_2 \gamma k x}{2\sqrt{AB}} - \frac{a^2 L_2 \gamma}{\sqrt{AB}} (1+4\epsilon^2) - \frac{K_1}{3\epsilon(A-B)} \sqrt{\frac{a}{2\pi}} \Big] \cdot \Big[\left(\frac{a-x}{a+x}\right)^{i\epsilon} - \left(\frac{a-x}{a+x}\right)^{-i\epsilon} \Big] \Big\} \Bigg] \frac{x\,dx}{\sqrt{r^2 - x^2}}, \quad r < a,\ z = 0 \qquad (5.1\text{-}6)$$

A similar precedure gives fer the upper surface of the crack

$$u_r^+ = \left(E - \frac{H}{A_2}\right) u_r^- - \frac{B_1 \gamma k_1}{2\pi k_2 (D^2 - B^2)} \left[\frac{E_1 \alpha_1 k_2}{(1-\nu_1) k_1} - \frac{E_2 \alpha_2}{(1-\nu_2)} \right] \left[\frac{a^3}{3r} - \frac{(a^2 - r^2)^{3/2}}{3r} \right],$$

$$r < a, \quad z = 0 \qquad (5.1\text{-}7)$$

where u_r^- is given by (5.1-6).

The radial displacement en the bending plane eut-side the cut may be evaluated frem

$$u_r^- = \frac{1}{2\pi r(\bar{A}_2 - A_2)}\int_r^\infty \left[\frac{\partial U_2(x,0)}{\partial x} - \frac{\partial U_1(x,0)}{\partial x} \right] \frac{x\,dx}{\sqrt{x^2 - r^2}} \qquad (5.1\text{-}8)$$

Consideration of (4.3-2a) and (4.3-9) leads to

$$\frac{\partial U_1}{\partial x} = -\frac{B}{(A-B)}\, u_1^{+}, \quad \frac{\partial U_2}{\partial x} = -\frac{u_2^{+}}{(A-B)}, \quad z = 0,\ |x| > a \qquad (5.1\text{-}9)$$

since $u_1^+ = u_1^-$ and $u_2^+ = u_2^-$ for z = 0, r > a. Substituting (5.1-9) into (5.1-8) and evaluating $U_i(x)$ at z = 0, |x| > a (Appendix B) results in

$$u_r^- = \frac{L_1 K_1 (B-1)}{\pi B r(\bar{A}_2 - A_2)}\int_r^\infty \Big\{ \epsilon a x \left[\left(\frac{x-a}{x+a}\right)^{i\epsilon} - \left(\frac{x-a}{x+a}\right)^{-i\epsilon} \right]$$

$$+ \left(\frac{a^2}{6} + \frac{2}{3}\epsilon^2 a^2 - \frac{x^2}{2} \right)\left[\left(\frac{x-a}{x+a}\right)^{i\epsilon} + \left(\frac{x-a}{x+a}\right)^{-i\epsilon} \right] - \frac{a^2}{3}(1 - 2\epsilon^2) + x^2 \Big\} \frac{x\,dx}{\sqrt{x^2 - r^2}}$$

$$+ \frac{K_2 A_1 (B-1)}{2\pi B \epsilon (A-B)(\bar{A}_2 - A_2)}\sqrt{\frac{a}{2\pi}}\ \frac{1}{r}\int_r^\infty \left[\left(\frac{x-a}{x+a}\right)^{-i\epsilon} + \left(\frac{x-a}{x+a}\right)^{i\epsilon} - 2 \right] \frac{x\,dx}{\sqrt{x^2 - r^2}},$$

$$r > a, \quad z = 0 \qquad (5.1\text{-}10)$$

and an identical expression for u_r^+ on z = 0, r > a. This checks the continuity condition $u_r^+ = u_r^-$ outside the crack.

For the homogeneous case, i.e. when the two bonded materials have the same thermomechanical properties ($k_1 = k_2 = k$, $\nu_1 = \nu_2 = \nu$, $\alpha_1 = \alpha_2 = \alpha$), (5.1-6) reduces to

$$u_r^{\mp} = \mp \frac{2(1+\nu)\alpha \tau r}{3\pi} \sqrt{a^2 - r^2} \;, \quad r < a, \; z = 0 \qquad (5.1\text{-}11)$$

Performing a similar reduction on (5.1-10) yields

$$u_r^{\mp} = 0 \;, \quad r > a, \; z = 0 \qquad (5.1\text{-}12)$$

These results agree with those found by Florence and Goodier [1].

The normal displacement across the plane z = 0 is given by

$$w = \Phi_3 + \omega \;, \quad z = 0 \qquad (5.1\text{-}13)$$

Since ω is known from the temperature solution (see (3-24) and (3-26)), w may be obtained if Φ_3 is evaluated. Taking into account (4.1-59) and (4.2-5d), it follows that

$$\Phi_3^{-} = \frac{1}{\left(\frac{1}{A_3} - A_1\right)} \int_x^a \left[\frac{1}{A_3} \frac{\partial U_1}{\partial x} - A_1 \frac{\partial U_2}{\partial x} \right] \frac{dx}{\sqrt{x^2 - a^2}} \;, \quad z = 0 \qquad (5.1\text{-}14)$$

When $|x| < a$, $\frac{\partial U_1(x,0)}{\partial x}$ is given by (4.2-6) and $\frac{\partial U_2(x,0)}{\partial x}$ may be found by considering (4.3-2a) and (4.3-9). Thus,

$$\frac{\partial U_2}{\partial x} = \frac{1}{2(A-B)} \left[(\Omega_1^{-} - \Omega_1^{+}) - (\Omega_2^{-} + \Omega_2^{+}) \right] \;, \quad |x| < a, \; z = 0 \qquad (5.1\text{-}15)$$

which may be evaluated by considering limiting values ($z \to 0$) of $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ from (4.4-9) (Appendix B). Substituting (5.1-15) and (4.2-6) into (5.1-14) and taking into account (3-24a), (4.4-9) and (5.1-13), it can be shown that the normal displacement on the lower crack surface takes the form

$$w^- = \frac{L_2 \gamma k_1}{2\pi(\frac{1}{A_1} - A_2)} \Bigg[\frac{\pi a^2}{3A_2}(1-2\epsilon^2) - \frac{\pi r^2}{2A_2} - \frac{A_1}{8}\int_0^r \Big\{ \frac{2\sqrt{A}\,B}{\sqrt{A}}\Big[-\frac{a^2}{6} - \frac{2}{3}\epsilon^2 a^2 + \frac{x^2}{2}\Big] \cdot$$

$$\cdot \Big[\Big(\frac{a-x}{a+x}\Big)^{i\epsilon} + \Big(\frac{a-x}{a+x}\Big)^{-i\epsilon}\Big] + \frac{2i\epsilon a x\sqrt{A}\,B}{\sqrt{A}}\Big[\Big(\frac{a-x}{a+x}\Big)^{i\epsilon} - \Big(\frac{a-x}{a+x}\Big)^{-i\epsilon}\Big] +$$

$$2\Big[\frac{a^2}{3}(1-2\epsilon^2) - x^2\Big]\Big\}\frac{dx}{\sqrt{r^2-x^2}}\Bigg] + \frac{K_1\sqrt{A}}{\pi\epsilon(\frac{1}{A_1} - A_2)\sqrt{A}\cdot B}\sqrt{\frac{a}{2\pi}}\int_0^r \Big[\Big(\frac{a-x}{a+x}\Big)^{-i\epsilon} +$$

$$\Big(\frac{a-x}{a+x}\Big)^{i\epsilon}\Big]\frac{dx}{\sqrt{r^2-x^2}} + \frac{(1-A)K_2}{\epsilon(\frac{1}{A_1} - A)A\cdot B}\sqrt{\frac{a}{2\pi}} - \frac{(1+\nu_2)\alpha_2\gamma k_1}{8(1-\nu_2)k_2}(2a^2 - r^2)$$

$$r < a, \; z = 0 \qquad (5.1\text{-}16)$$

The normal displacement on the upper crack surface may be determined from a knowledge of Φ_3^+ and Ω^+. It can be seen from (4.1-53) that Φ_3^+ may be expressed in terms of Φ_3^-, Φ_A^- and Ω^-. Since Ω^+ and Ω^- are known from the temperature solution and Φ_3^- was found in computing w^-, it follows that w^+ may be obtained if the value of Φ_A^- is determined. Substituting (4.2-5d) into (4.1-59) leads to

$$\Phi_4^- = \frac{1}{2\pi(\frac{1}{A_2} - A_2)} \int_0^r \left[\frac{\partial U_1}{\partial x} - \frac{\partial U_2}{\partial x} \right] \frac{dx}{\sqrt{r^2 - x^2}} \; , \quad r < a, \; z = 0 \qquad (5.1\text{-}17)$$

which may be evaluated by considering (4.2-6) and (5.1-15). If (4.1-53) and (3-26a) are taken into account, it can be shown from (5.1-13) that

$$w^+ = \frac{(1+\nu_1)\,\tau(2a^2 - r^2)}{8} \Bigg\{ - \frac{\alpha_1}{(1-\nu_1)} - \frac{(1-2\nu)\alpha_1 k}{k} \left[\frac{\alpha_1 k}{(1-\nu)\alpha_1 k} - \frac{E_1}{E_1(1-\nu_2)} \right] \Bigg\}$$

$$+ \frac{L_2 \tau k}{E(1-A_2^2)} \left[\frac{a^2}{r}(1-2\epsilon^2) - \frac{r^2}{4} \right] \left[E(B - A_2^2) - H A_2 (B-1) \right]$$

$$+ \frac{K_1}{3\epsilon(1-A_2^2)(A-B)} \sqrt{\frac{a}{2\pi}} \left[E A_2 (1-A) - H (A_2^2 - A) \right] +$$

$$\frac{(H - E A_2)}{\pi(\frac{1}{A_2} - A_2)} \int_0^r \Bigg\{ \frac{2L_2 \tau k \sqrt{A-B}}{3\sqrt{A}} \left[-\frac{a^2}{6} - \frac{2}{3}\epsilon^2 a^2 + \frac{x^2}{2} \right] \left[\left(\frac{a-x}{a+x}\right)^{i\epsilon} + \left(\frac{a-x}{a+x}\right)^{-i\epsilon} \right]$$

$$- \frac{2K_0\sqrt{A}}{3\epsilon A_2\sqrt{A-B}} \sqrt{\frac{a}{2\pi}} \left[\left(\frac{a-x}{a+x}\right)^{-i\epsilon} + \left(\frac{a-x}{a+x}\right)^{i\epsilon} \right]$$

$$+ \frac{2L_2 \tau k_1 i\epsilon a x \sqrt{A-B}}{3\sqrt{A}} \left[\left(\frac{a-x}{a+x}\right)^{i\epsilon} - \left(\frac{a-x}{a+x}\right)^{-i\epsilon} \right] \Bigg\} \frac{dx}{\sqrt{r^2 - x^2}} \; ,$$

$$r < a, \; z = 0 \qquad (5.1\text{-}18)$$

A similar procedure may be used to evaluate the normal displacement outside the crack. Without going into the details of the procedure, the results are

$$w^{+} = w^{-} = -\frac{(1+\nu_2)\alpha_1\tau k_1}{4\pi k_2(1-\nu_2)}\left[(2a^2-r^2)\sin^{-1}\frac{a}{r} + a\sqrt{r^2-a^2}\right]$$

$$+ \frac{1}{2\pi(1-A_2^2)}\int_{r}^{\infty}\left\{F_1(x) - \frac{2iV_0(A_2^2-A)}{3\epsilon(A-B)}\sqrt{\frac{a}{2\pi}}\left[\left(\frac{x-a}{x+a}\right)^{i\epsilon} - \left(\frac{x-a}{x+a}\right)^{-i\epsilon}\right]\right\}\frac{dx}{\sqrt{x^2-r^2}}$$

$$+ \frac{L_2\tau k}{\pi A_2}\,\frac{(A_2^2-A)}{(A_2^2-1)}\int_{r}^{\infty}\left\{\left(-\frac{a^2}{[illegible]} - \frac{2}{3}\epsilon^2 a^2 + \frac{x^2}{2}\right)\left[\left(\frac{x-a}{x+a}\right)^{i\epsilon} - \left(\frac{x-a}{x+a}\right)^{-i\epsilon}\right]\right.$$

$$\left. + i\epsilon a x\left[\left(\frac{x-a}{x+a}\right)^{i\epsilon} + \left(\frac{x-a}{x+a}\right)^{-i\epsilon}\right]\right\}\frac{dx}{\sqrt{x^2-r^2}},$$

$$r > a, \quad z = 0 \qquad (5.1\text{-}19)$$

For the case of homogeneous materials, (5.1-16) reduces to

$$w^{-} = \frac{(1+\nu)\alpha\tau}{[illegible]}\left[4a^2(2-\nu) - [illegible]\,r^2(1-\nu)\right], \quad r < a, \quad z = 0 \qquad (5.1\text{-}20)$$

while outside the cut, the normal displacement on z = 0, given by (5.1-19), takes the following form:

$$w^{+} = w^{-} = \frac{(1+\nu)\alpha\tau}{6\pi k(1-\nu)}\left\{\sin^{-1}\frac{a}{r}\left[-3r^2(1-\nu) + 2a^2(2-\nu)\right] + \right.$$

$$\left. 3a(1-\nu)\sqrt{r^2-a^2}\right\}, \quad r > a, \quad z = 0 \qquad (5.1\text{-}21)$$

These expressions agree with the corrected results of Florence and Goodier [1] as noted by Kassir and Sih ([2], p.134-135).

5.2 Contact Stresses

On the plane $z = 0$, it follows from (2-6), (2-7), (4.1-18) and (4.1-58a) that

$$\tau_{rz}^{-} = 2D_2 \frac{\partial F_1}{\partial r} + 2\mu_2 \frac{\partial \Omega^{-}}{\partial r}, \qquad z = 0 \tag{5.2-1}$$

Since $\frac{\partial \Omega}{\partial r}$ is known from the temperature solution (see (3-25)), the shear stress may be obtained if $\frac{\partial F_1}{\partial r}$ is determined. It is evident from (4.2-5g) that

$$\frac{\partial F_1}{\partial r} = \frac{1}{2\pi} \frac{\partial}{\partial r} \int_r^{\infty} \frac{\partial U_1}{\partial z} \frac{dx}{\sqrt{x^2 - r^2}}, \qquad z = 0 \tag{5.2-2}$$

Noting from (4.3-2b) and (4.3-9) that

$$\frac{\partial U_1}{\partial z} = \frac{A}{(A-B)} \Omega_1^{+} - \frac{B}{(A-B)} F_1(x) \tag{5.2-3}$$

it follows that for the region outside the cut,

$$\frac{\partial F_1}{\partial r} = \frac{1}{2\pi (A-B)} \frac{\partial}{\partial r} \int_r^{\infty} \left[A\Omega_1^{+} - BF_1 \right] \frac{dx}{\sqrt{x^2 - r^2}}, \quad r > 1, \; z = 0 \tag{5.2-4}$$

Substituting from (B-9) into (5.2-4) results in

$$\frac{\partial F_1}{\partial r} = \frac{iAL_2\gamma k_1}{\pi\sqrt{AB}}\frac{\partial}{\partial r}\int_r^{\infty}\left\{\left[-\frac{a^2}{6}(1-2\epsilon^2)-\epsilon^2a^2+\frac{x^2}{2}\right]\left[\left(\frac{x-a}{x+a}\right)^{i\epsilon}-\left(\frac{x-a}{x+a}\right)^{-i\epsilon}\right]\right.$$

$$\left.+\,i\epsilon a x\left[\left(\frac{x-a}{x+a}\right)^{i\epsilon}+\left(\frac{x-a}{x+a}\right)^{-i\epsilon}\right]\right\}\frac{dx}{\sqrt{r^2-x^2}}$$

$$+\frac{1}{2\pi}\frac{\partial}{\partial r}\int_r^{\infty}\left\{F_2+\frac{2iAK_1}{3\epsilon(A-B)}\sqrt{\frac{a}{2\pi}}\left[\left(\frac{x-a}{x+a}\right)^{-i\epsilon}-\left(\frac{x-a}{x+a}\right)^{i\epsilon}\right]\right\}\frac{dx}{\sqrt{x^2-r^2}}\,,$$

$$r \to a,\ z = 0 \qquad (5.2\text{-}5)$$

With a change of variables to $t = \frac{r}{x}$, and noting that $\left(\frac{r-at}{r+at}\right)^{\pm i\epsilon} = \cos\omega \pm i\sin\omega$, where $\omega = \epsilon\ln\frac{r-at}{r+at}$, it is found that

$$\tau_{rz}^{-} = \frac{E_2\alpha_2\gamma k_1}{\pi k_2(1-\nu_2)}\frac{\left[E_1(1-\nu_2^2)+E_2(1-\nu_1^2)\right]}{\left[E_1(1+\nu_2)(1-2\nu_2)-E_2(1+\nu_1)(1-2\nu_1)\right]}\int_1^0\left\{\left[\frac{2r}{t^2}-\frac{4a^2r\epsilon^2}{r^2-a^2t^2}\right]\sin\omega\right.$$

$$\left.+\left[\frac{2a\epsilon}{t}+\frac{2at}{(r^2-a^2t^2)}\left(-\frac{4}{3}a^2\epsilon^3-\frac{a^2\epsilon}{2}+\frac{r^2\epsilon}{t^2}\right)\right]\cos\omega\right\}\frac{dt}{t\sqrt{1-t^2}}$$

$$+\frac{2K_0E_2(1-2\nu_2)}{\pi(1+\nu_2)(3-4\nu_2)}\sqrt{\frac{a}{2\pi}}\int_0^1\left\{\frac{2}{3\epsilon}+\frac{r\omega}{[\text{illegible}]}+\frac{2Aat}{3(A-B)(r^2-a^2t^2)}\right\}\cos\omega\,\frac{dt}{t\sqrt{1-t^2}}$$

$$-\frac{E_2\alpha_2\gamma k}{2\pi(1-\nu_2)k_2}\left[\frac{a}{r}\sqrt{r^2-a^2}-r\,[\text{illegible}]\,\frac{[\text{illegible}]}{r}\right],\quad r \to a,\ z = 0 \qquad (5.2\text{-}6)$$

In the special case when the bonded materials have the same thermoelastic properties, the above expression reduces to

$$\tau_{rz}^{-} = -\frac{E\alpha\gamma a^3}{3\pi k(1-\nu)r\sqrt{r^2-a^2}} \qquad (5.2\text{-}7)$$

where $k_1 = k_2 = k$, $\alpha_1 = \alpha_2 = \alpha$, $\nu_1 = \nu_2 = \nu$, $E_1 = E_2 = E$. This result is identical to the one obtained by Florence and Goodier [1] for the homogeneous problem. It should be noted that $\tau_{rz}^{+} = \tau_{rz}^{-}$ when $r > a$, $z = 0$ since the stresses are continuous across the contact surface.

On the $z = 0$ plane, the normal stress is obtained from (2-7a) in the form

$$\sigma_{zz} = (\lambda + 2\mu)\frac{\partial}{\partial z}(\Phi_3 + \Psi) + \lambda\left(\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y}\right), \quad z = 0 \qquad (5.2\text{-}8)$$

Taking into account (2-6), (4.1-18), (4.1-58b), (4.2-5f) and (5.1-9), the above expression reduces to

$$\sigma_{zz}^{-} = \frac{\mu_2(\lambda_1 + 2\mu_2)}{(\lambda_1 + 3\mu_1)\pi r}\frac{\partial}{\partial r}\int_r^{\infty}\frac{\Omega_2^{+}}{(A-B)}\frac{\kappa\, d\kappa}{\sqrt{\kappa^2 - r^2}}, \quad r > a,\ z = 0 \qquad (5.2\text{-}9)$$

Substituting (B-10) into (5.2-9) and changing variables as in (5.2-6) leads to

$$\sigma_{zz}^{-} = -\frac{E_2 \alpha_2 \tau k_1}{\pi k_2 (1-\nu_2)} \frac{[E_1(1-\nu_2^2) + E_2(1-\nu_1^2)]}{[E_1(1-\nu_2)(1-2\nu_2) - E_2(1+\nu_1)(1-2\nu_1)]} \cdot$$

$$\int_0^1 \left\{ \frac{a^2}{3r}(1-2t^2) - \frac{3r}{t^2} + \left[\frac{3r}{t^2} - \frac{4a^2t^2}{3r} - \frac{a^2}{3r} - \frac{4a^2t^2 r}{(r^2-a^2t^2)} \right] \cos\omega \right.$$

$$\left. + \left[\frac{2at}{(r^2-a^2t^2)} \left(\frac{a^2}{3} - \frac{r^2}{t^2} + \frac{4a^2t^2}{3} \right) - \frac{4at}{t} \right] \sin\omega \right\} \frac{dt}{t^2\sqrt{1-t^2}}$$

$$+ \frac{4K_0 A_2}{2\epsilon(A-B)\pi r} \sqrt{\frac{a}{2\pi}} \frac{E_2(1-2\nu_2)}{(1+\nu_2)(3-4\nu_2)} \frac{[E_1(1-\nu_2^2) + E_2(1-\nu_1^2)]}{[E_1(1-2\nu_2)(1+\nu_2) - E_2(1-2\nu_1)(1+\nu_1)]} \cdot$$

$$\int_0^1 \left[-1 + \cos\omega - \frac{2rat}{(r^2-a^2t^2)} \sin\omega \right] \frac{dt}{t^2\sqrt{1-t^2}}, \quad r > a,\ z = 0$$

(5.2-10)

Integrals in (5.2-6) and (5.2-10) may be obtained by numerical evaluation. From the condition of stress continuity, $\sigma_{zz}^{+} = \sigma_{zz}^{-}$ on $r > a$, $z = 0$. For the case in which the bonded materials have the same thermoelastic properties, (5.2-10) reduces to

$$\sigma_{zz}^{-} = 0, \quad r > a, \ z = 0$$

which is expected for the homogeneous problem.

5.3 Loss in Potential Energy and Critical Heat Flux

The existence in the body of a circular crack of radius a reduces its potential energy by an amount

$$W = \int_0^{2\pi}\int_0^a \left[\tau_{rz}^{+} u_r^{+} - \tau_{rz}^{-} u_r^{-} \right] r\, dr\, d\theta, \quad r < a,\ z = 0 \qquad (5.3\text{-}1)$$

The shear stress which contributes to the loss in potential energy due to the heat flux is given by

$$\tau_{zr} = -2\mu\frac{\partial\Omega}{\partial r}, \quad r < a,\ z = 0 \tag{5.3-2}$$

which may be evaluated from eqs. (3-25a) and (3-27a). Thus,

$$\tau_{rz}^{-} = -\frac{E_2\alpha_2\tau k_1 r}{4k_2(1-\nu_2)} \tag{5.3-3a}$$

$$\tau_{rz}^{+} = -\frac{E_1\alpha_1\tau r}{4(1-\nu_1)} \tag{5.3-3b}$$

$$r < a,\ z = 0$$

Taking into account (5.3-1) and (5.3-3), the energy may be expressed as

$$W = W^{+} + W^{-} \tag{5.3-4}$$

where

$$W^{+} = -\frac{\pi E_1\alpha_1\tau}{2(1-\nu_1)}\int_0^a u_r^{+}\, r^2\, dr \tag{5.3-5a}$$

$$W^{-} = -\frac{\pi E_2\alpha_2\tau k_1}{2k_2(1-\nu_2)}\int_0^a u_r^{-}\, r^2\, dr \tag{5.3-5b}$$

$$r < a,\ z = 0$$

Thus, the energy may be evaluated using the expressions previously obtained for u_r^{+} and u_r^{-} (see (5.1-7) and

and (5.1-6)). Substituting (5.1-6) into (5.3-5b) and changing the order of integration leads to

$$W^- = -\frac{E_2\alpha_2 L_2\gamma^2 k_1^2\sqrt{B(A-B)}}{2k_2(1-\nu_2)B(\frac{1}{A_2}-A_2)}\int_0^a\left\{i\epsilon a x\left[\left(\frac{a-x}{a+x}\right)^{i\epsilon}+\left(\frac{a-x}{a+x}\right)^{-i\epsilon}\right]\right.$$

$$\left.+\left[\frac{x^2}{2}-\frac{a^2}{6}(1+4\epsilon^2)\right]\left[\left(\frac{a-x}{a+x}\right)^{i\epsilon}-\left(\frac{a-x}{a+x}\right)^{-i\epsilon}\right]\right\}\sqrt{a^2-x^2}\;x\,dx$$

$$-\frac{E_2\alpha_2\gamma k}{4k_2(1-\nu_2)(\frac{1}{A_2}-A_2)}\int_0^a F(x)\,x\sqrt{a^2-x^2}\,dx$$

$$+\frac{E_2\alpha_2\gamma_1 k K\sqrt{A}}{6k_2\epsilon(1-\nu_2)(\frac{1}{A_2}-A_2)\sqrt{A-B}}\sqrt{\frac{a}{2\pi}}\int_0^a\left[\left(\frac{a-x}{a+x}\right)^{i\epsilon}-\left(\frac{a-x}{a+x}\right)^{-i\epsilon}\right]x\sqrt{a^2-x^2}\,dx\,,$$

(5.3-6)

Evaluating the integrals in (5.3-6) (see Appendix C) results in

$$W^- = \frac{\pi E_2\alpha_2 L_2\gamma^2 k_1^2\sqrt{B(A-B)}}{4k_2(1-\nu_2)B(\frac{1}{A_2}-A_2)}\,a^5\left(\epsilon^2+\frac{1}{4}\right)\left(\epsilon^3+\epsilon\right)\frac{\pi}{\cosh\pi\epsilon}$$

$$+\frac{2E_2\alpha_2\gamma k_1 K_1\sqrt{A}}{9k_2(1-\nu_2)(\frac{1}{A_2}-A_2)\sqrt{A-B}}\sqrt{\frac{a}{2\pi}}\,a^3\left(\epsilon^2+\frac{1}{4}\right)\frac{\pi}{\cosh\pi\epsilon}$$

$$-\frac{E_2\alpha_2\gamma k_1 K_1 a^4}{20k_2(1-\nu_2)(\frac{1}{A_2}-A_2)}\sqrt{\frac{2\pi}{a}} \qquad (5.3\text{-}7)$$

The value of W^+ may be determined by substituting (5.1-7) into (5.3-5a) and performing a similar procedure. The result in this case is given by

$$W^{+} = \left(E - \frac{H}{A_2}\right)\frac{E_1\alpha_1\tau k_2(1-\nu_2)}{E_2\alpha_2\tau k_1(1-\nu_1)}W^{-}$$

$$- \frac{\alpha_1\tau^2 k_1(1+\nu_1)a^5}{10k_2}\left[\frac{E_1\alpha_1 k_2}{(1-\nu_1)k_1} - \frac{E_2\alpha_2}{(1-\nu_2)}\right] \qquad (5.3\text{-}8)$$

Equations (5.3-7) and (5.3-8) may now be combined according to (5.3-4) to yield an expression for the total energy loss. Taking into account the fact that

$$\cos \pi i \epsilon = \frac{A}{\sqrt{A^2 - A_2^2}}$$

it can be shown that

$$W = \frac{\pi a^5\alpha_2\tau^2 k_1 E_2}{45k_2^2(1-\nu_2)}\left[(1+\nu_1)\alpha_1 k_2 + (1+\nu_2)\alpha_2 k_1\right]\cdot$$

$$\cdot\frac{\left[\mathcal{V}_2 E_1^2 + \mathcal{V}_1 E_2^2 + 2E_1E_2\mathcal{V}_{12}(1+\nu_1)(1+\nu_2)\right]\left[\epsilon(4\epsilon^2+1)(\epsilon^2+1)\right]}{\left[E_1(1+\nu_2)(1-2\nu_2) - E_2(1+\nu_1)(1-2\nu_1)\right]\left[E_1(1+\nu_2)(1-\nu_2) + E_2(1+\nu_1)(1-\nu_1)\right]}$$

$$+ \frac{K_0a^3(1-2\nu_2)E_2\tau\sqrt{2a\pi}}{90k_2(1+\nu_2)(3-4\nu_2)}\left[(1+\nu_1)\alpha_1 k_2 + (1+\nu_2)\alpha_2 k_1\right]\left[20(\epsilon^2+\tfrac{1}{4}) - 9\right]$$

$$- \frac{(1+\nu_1)\alpha_1\tau^2 a^5}{10k_2(1-\nu_1)(1-\nu_2)}\left[E_1\alpha_1(1-\nu_2)k_2 - E_2\alpha_2 k_1(1-\nu_1)\right] \qquad (5.3\text{-}9)$$

where the following representations have been employed:

$$\mathcal{V}_{12} = (1-2\nu_1)(1-2\nu_2) + 4(1-\nu_1)(1-\nu_2) \qquad (5.3\text{-}10a)$$

$$\mathcal{V}_k = (1+\nu_k)^2(3-4\nu_k), \quad (k = 1,2) \qquad (5.3\text{-}10b)$$

In the homogeneous case, when the two bonded materials have the same thermomechanical properties, equation (5.3-9) reduces to

$$W = -\frac{4a^5\alpha^2\gamma^2E(1+\nu)}{45(1-\nu)}$$

which agrees with that reported by Kassir [3].

According to the Griffith theory of fracture, the condition necessary for the enlargement of the crack may be expressed in the form

$$\frac{\partial}{\partial a}(W - U) = 0 \qquad (5.3\text{-}11)$$

where U represents the crack surface energy given by

$$U = -2\pi a^2\gamma \qquad (5.3\text{-}12)$$

with γ representing the specific surface energy. The crack in the dissimilar case will not extend along the bond surface. Its extension is not known a priori. However, an approximate Griffith type failure criterion may be obtained if it is assumed that the crack will extend along the z = 0 plane. Making use of (5.3-9), the (approximate) criterion of failure given by (5.3-11) takes the somewhat complicated expression

$$\frac{\pi a^4 \alpha_2 \gamma_c^2 k_1 E_2}{9 k_2^2 (1-\nu_2)} \left[(1+\nu_1)\alpha_1 k_2 + (1+\nu_2)\alpha_2 k \right] \cdot$$

$$\cdot \frac{\left[2\vartheta_2 E_1^2 + 2\vartheta_1 E_2^2 + 2E_1 E_2 2\vartheta_2 (1+\nu_1)(1+\nu_2) \right] \left[\ldots (4\epsilon^2 + \ldots)(\epsilon^2+1) \right]}{\left[E_1 (1+\nu_2)(1-2\nu_2) - E_2 (1+\nu_1)(1-2\nu_1) \right] \left[E_1 (1-\nu_2^2) + E_2 (1-\nu_1^2) \right]}$$

$$+ \frac{\ldots (1+\nu_2) E_2 \gamma_c^2}{18 k_2 (1-\nu_2)(3-4\nu_2)^2} \left[(1+\nu_1)\alpha_1 k_2 + (1+\nu_2)\alpha_2 k \right] \left[20(\epsilon^2 + \tfrac{1}{4}) - 9 \right] \sqrt{2\pi}\, a^4 \cdot$$

$$\cdot \frac{(1-A_2^2)}{(1-A_1 A_2)^2 - \ldots} \sqrt{\frac{2}{\pi}} \left\{ \frac{(1+\nu_2)\alpha_2 k}{2(1-\nu_2) k_2 \ldots} \left[1 - \frac{E_2 (1+\nu_1)(1-2\nu_1)}{E_1 (1+\nu_2)} \right] - \ldots \right\}$$

$$- \frac{(1+\nu_2) \ldots \gamma_c^2 \ldots}{\ldots k_2 (1-\nu_1)(1-\nu_2) \ldots} \left[E_1 \ldots (1+\nu_2) k_2 - E_2 \ldots \right] \ldots - 4 \ldots \tag{5.3-13}$$

where γ_c represents the approximate critical value of the heat flux.

For the homogeneous case, (5.3-13) reduces to

$$\gamma_{c\,(\ldots)} = 3\sqrt{\frac{\pi\, \ldots k\, (1-\nu)}{\ldots\, \alpha\, \ldots^2}}$$

5.4 Stress Intensity Factors

A knowledge of the stress field in the vicinity of the crack edge is essential in the investigation of the stability behavior of the crack. In this region, the stresses are infinitely large in magnitude and change their sign an infinite number of times with the thermal loading contributing to both stress intensity factors, K_1 and K_2. According to the Griffith-Irwin theory of fracture, the onset of rapid crack extension occurs when the magnitude of the crack border stress field reaches some critical value. (For the dissimilar problem, this value is given in terms of an assumed function of K_1 and K_2.) For the purpose of obtaining closed form expressions for the stress intensity factors, it is sufficient to study the singular parts of the external bend stresses when $(r - a) \rightarrow 0^+$.

In view of (5.2-1), (4.2-5d) and (3-25b), the contact shear stress outside the crack may be determined by the relation

[illegible] (5.4-1)

and from (5.2-8), (2-6), (4.1-18), (4.1-58b) and (4.2-5e), the normal contact stress is given by

[illegible] (5.4-2)

These expressions may be evaluated by noting from (4.2-6), (4.2-7), (5.1-9), (4.3-2b), (4.3-9) and Appendix B that

[illegible] (5.4-3)

$$\left.\frac{\partial u_2}{\partial z}\right|_{z=0} = \frac{2}{\Lambda_2} \operatorname{Im}\Big[\Big(L_2 \gamma k\, x^2 - 2 L_2 \gamma k \text{ [illegible]} a^2 - 2 \text{ [illegible]}$$

$$- 2 L_2 \gamma k \text{ [illegible]} x\Big) \text{ [illegible]} \frac{x - a}{x + a}\Big)^{\text{[illegible]}} \Big] H(x - a) \qquad (5.4\text{-}4)$$

where H(x - a) denotes the Heaviside step function and the real and imaginary parts of a function of a complex variable have been denoted by Re and Im, respectively. Substituting expressions (5.4-3) and (5.4-4) in relations (5.4-1) and (5.4-2), and making use of the change of variables

$$x = a \coth \alpha \quad , \quad r = a \coth \beta \qquad (5.4\text{-}5)$$

results in the following expressions for the contact stresses outside the crack:

$$\sigma_{zz}(r, 0) = \frac{2 \Omega_2}{\pi} \operatorname{Re} \frac{\partial}{\partial r} \int_{\beta}^{\infty} \Big(\text{[illegible]} + 2 L_2 \gamma k \text{ [illegible]} a^2 \coth \alpha$$

$$- 2 L_2 \gamma k \text{ [illegible]} - 2 C_2 \Big) \text{ [illegible]} \frac{\text{[illegible]}\, d\alpha}{\sqrt{\coth^2 \beta - \coth^2 \alpha}} \qquad (5.4\text{-}6)$$

$$\sigma_{rz}(r, 0) = \frac{\text{[illegible]}}{\pi \coth \beta} \operatorname{Im} \frac{\partial}{\partial r} \int_{\beta}^{\infty} \Big(L_2 \gamma k\, a^2 \coth^2 \alpha - 2 L_2 \text{ [illegible]} a^2 \coth \alpha$$

$$- 2 L_2 \gamma k \text{ [illegible]} - 2 C_2 \Big) \text{ [illegible]} \frac{\coth \alpha \operatorname{cosech}^2 \alpha \, d\alpha}{\sqrt{\coth^2 \beta - \coth^2 \alpha}} \qquad (5.4\text{-}7)$$

Upon taking the limit as r → a⁺, and noting that for large α and β the following relations hold (see Appendix D for more details)

$$\coth\alpha = 1 + 2e^{-2\alpha} + \cdots$$

$$(\coth^2\beta - \coth^2\alpha)^{-1/2} = \frac{e^{\beta}}{2\sqrt{\pi}}\Big[1 - 2e^{-2\alpha} - 2e^{-2\beta} + \sum_{n=1}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}e^{-2(n\ldots)} \text{[illegible]}\Big]$$

it is not difficult to show that the limiting values of the stresses in (5.4-6) and (5.4-7) become

$$\tau_{rz}(r,0) = \text{[illegible]} - 2c_2\Big]e^{(1-2i\epsilon)\beta}\left(i\epsilon+\frac{1}{2}\right)\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\text{[illegible]}\,(1+i\epsilon+n)}\Big\}$$

$$+ \text{[illegible] terms} \qquad (5.4\text{-}8)$$

$$\sigma_{zz}(r,0) = -\frac{\text{[illegible]}}{\text{[illegible]}}\,\mathrm{Im}\Big\{\Big[\text{[illegible]}\; - 2c_2\Big]e^{(1-2i\epsilon)\beta}\left(i\epsilon+\frac{1}{2}\right)\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)(1+i\epsilon+n)}\Big\}$$

$$+ \text{[illegible] terms} \qquad (5.4\text{-}9)$$

where $e^{\beta(1-2i\epsilon)}$ **may be written in the equivalent form**

$$e^{\beta(1-2i\epsilon)} = \frac{\sqrt{2r}}{\sqrt{r-a}}\left[\cos\left(\epsilon\ln\frac{r-a}{r+a}\right) + i\sin\left(\epsilon\ln\frac{r-a}{r+a}\right)\right] \qquad (5.4\text{-}10)$$

Making use of Dougall's formula (see, for example [21]) for the summation of the gamma function

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)(i\epsilon+1+n)} = \frac{\pi\,\Gamma(1+i\epsilon)}{\Gamma(\frac{3}{2}+i\epsilon)}$$

expressions (5.4-8) and (5.4-9) may be written in the standard form

$$\tau_{rz}(r,0) = -\frac{1}{\sqrt{2R}}\left[K_1 \sin\left(\epsilon \ln \frac{R}{R+2a}\right) - K_2 \cos\left(\epsilon \ln \frac{R}{R+2a}\right)\right] + \text{finite quantity} \qquad (5.4\text{-}11)$$

$$\sigma_{zz}(r,0) = -\frac{1}{\sqrt{2R}}\left[K_1 \cos\left(\epsilon \ln \frac{R}{R+2a}\right) + K_2 \sin\left(\epsilon \ln \frac{R}{R+2a}\right)\right] + \text{finite quantity} \qquad (5.4\text{-}12)$$

where R = r - a and the stress-intensity factors K_1 and K_2 are given by

$$K_1 = \frac{2D_2}{\sqrt{\pi a}}\, \mathrm{Im}\left\{\left[L_2\gamma k_1 a^2 + 2L_2\gamma k_1 i\epsilon a^2 - 2L_1\gamma k\,\epsilon^2 a^2 - 2C_2\right]\frac{\Gamma(2+i\epsilon)}{(1+i\epsilon)\Gamma(\frac{1}{2}+i\epsilon)}\right\} \qquad (5.4\text{-}13)$$

$$K_2 = \frac{2D_2}{\sqrt{\pi a}}\, \mathrm{Re}\left\{\left[L_2\gamma k\, a^2 + 2L_2\gamma k\, i\epsilon a^2 - 2L_1\gamma k\,\epsilon^2 a^2 - 2C_2\right]\frac{\Gamma(2+i\epsilon)}{(1+i\epsilon)\Gamma(\frac{1}{2}+i\epsilon)}\right\} \qquad (5.4\text{-}14)$$

with the constants D_2, L_2 and C_2 being determined by

(4.1-28), (4.1-30) and (4.4-10) respectively. Note that the gamma function of complex arguments has been tabulated extensively [26] so that there is no difficulty in computing the numerical values of K_1 and K_2 from eqns. (5.4-13) and (5.4-14).

In the homogeneous problem, $\epsilon = 0$ (see (5.1-5)) so that (5.4-13) and (5.4-14) reduce to

$$K_1 = 0$$

$$K_2 = \frac{4 D_2 L_2 \tau k_1 a^2}{3\pi\sqrt{a}}$$

For the general case, the stress-intensity factors may be evaluated by substituting the values of the material constants of the bonded materials into (5.4-13) and (5.4-14). Values for steel-glass and steel-copper are as follows:

	K_1	K_2
Steel-Glass	$0.11583\,\tau\, a^{3/2}$	$-1.1866\,\tau\, a^{3/2}$
Steel-Copper	$61.358\,\tau\, a^{3/2}$	$1906.8\,\tau\, a^{3/2}$

where lengths are measured in inches and temperatures in °F. In each case, steel is the material occupying the upper semi-space. Expressions (5.4-13) and (5.4-14) control the onset of crack propagation due to the influence of the linear heat flow in the solid. It is interesting to note that both K_1 and K_2 modes (the crack opening and edge sliding modes) are operating in the general dissimilar solid.

CHAPTER 6

DISCUSSION, CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In the preceding study, a method is presented for solving the thermoelastic problem of two bonded semi-infinite media with different properties containing a penny-shaped crack on the interface and subjected to a uniform flow of heat. This represents an important extension to the works previously considered in the literature which have been restricted to two-dimensional studies of dissimilar media or axisymmetric homogeneous problems.

The uncoupling of the temperature and stress distributions permits an independent determination of the temperature field followed by a solution to the elastic part of the problem. The transformation of the original formulation for an infinite space to that of a semi-infinite space is the first significant step in the solution and once this is accomplished, appropriate integral transformations are employed in order to reduce the axisymmetric formulation to a subsidiary problem in the plane. This forms the crucial step in the solution and permits a further reduction to a Hilbert problem from which closed form expressions for displacement, stress, energy loss, critical heat flux and stress-intensity factors are obtained.

It is shown that the stresses at the crack tip have singularities of the form $1/\sqrt{R}$ (where R represents the distance from the crack rim) and exhibit oscillations of the form sin (or cos) ϵ ln R (where ϵ is a function of material constants). The stresses depend upon the material constants and the radius of the crack and are linearly proportional to the heat flux. Thus, the direction of applied heat flux determines whether tensile or compressive stresses are present. The stresses are shown to involve two stress-intensity factors representing different modes of fracture, as opposed to the single factor in homogeneous problems. Thus, for the dissimilar problem, some function of K_1 and K_2, say $f(K_1,K_2)$, must be assumed, which will cause the crack to grow upon reaching some critical value.

The physical quantities obtained in Chapter 5 are evaluated for the special case in which the two bonded materials are identical and are shown to agree with previous studies involving homogeneous materials. It should be noted that in the homogeneous case, the right side of (4.1-60) vanishes so that all terms involving K_o vanish from the resulting solution.

The success of the approach taken in the present study suggests the possibility of further application to several preb ms not considered in the literature. Examples include the case of an external penny-shaped crack under conditions similar to those presently considered, and problems involving dissimilar media containing cracks of different shapes. The same method can also be considered in dealing with solids containing more than one plane of discontinuity (either cracks or rigid inclusions). In each of the above mentioned configurations, the type of loading may be thermal, mechanical, or both; however, certain thermoelastic crack problems may be treated from an isothermal point of view. It is shown in the preceding work that the uniform heat flow problem is related to an isothermal problem with applied shear stress.

The results of this study contribute to the understanding of failure of dissimilar materials when used in conjunction with the current theories of fracture mechanics.

APPENDIX A

ALTERNATE FORMULATION OF TEMPERATURE PROBLEM

The temperature field in the upper and lower semi-spaces may be represented respectively, in the form

$$T^{+}(r,z) = \int_0^\infty A_1(\xi)\, J_0(r\xi)\, e^{-\xi z}\, d\xi \qquad \text{(A-1)}$$

$$T^{-}(r,z) = \int_0^\infty A_2(\xi)\, J_0(r\xi)\, e^{\xi z}\, d\xi \qquad \text{(A-2)}$$

where $A_1(\xi)$ and $A_2(\xi)$ are arbitrary functions to be determined from the thermal boundary conditions.

In view of eq. (3-16), the Hankel transform approach taken here will yield a solution identical to that of Chapter 3 if it can be shown that

$$A_2(\xi) = -\sqrt{\frac{2}{\pi}}\; \frac{\gamma k_1}{k_2}\, a^{3/2}\, \xi^{-1/2}\, J_{3/2}(\xi a) \qquad \text{(A-3)}$$

Differentiation of (A-1) and (A-2) yields

$$\frac{\partial T^{+}(r,z)}{\partial z} = -\int_0^\infty \xi A_1(\xi)\, J_0(r\xi)\, e^{-\xi z}\, d\xi \qquad \text{(A-4)}$$

$$\frac{\partial T^{-}(r,z)}{\partial z} = \int_0^\infty \xi A_2(\xi)\, J_0(r\xi)\, e^{\xi z}\, d\xi \qquad \text{(A-5)}$$

so that boundary conditions (3-5a) and (3-5b) give

$$\left.\begin{aligned} \int_0^\infty \xi A_1(\xi)\, J_0(r\xi)\, d\xi &= \gamma \qquad \text{(A-6)} \\ \int_0^\infty \xi A_2(\xi)\, J_0(r\xi)\, d\xi &= -\frac{k}{k_2}\gamma \qquad \text{(A-7)} \end{aligned}\right\} r < a$$

while conditions (3-5c) and (3-5d) yield

$$\left.\begin{aligned} -k_1\int_0^\infty \xi A_1(\xi)\, J_0(r\xi)\, d\xi &= k_2\int_0^\infty \xi A_2(\xi)\, J_0(r\xi)\, d\xi \qquad \text{(A-8)} \\ \int_0^\infty A_1(\xi)\, J_0(r\xi)\, d\xi &= \int_0^\infty A_2(\xi)\, J_0(r\xi)\, d\xi \qquad \text{(A-9)} \end{aligned}\right\} r > a$$

Multiplying (A-6) and (A-7) by k_1 and k_2 respectively and adding results in

$$\int_0^\infty \left[k_1 A_1(\xi) + k_2 A_2(\xi)\right] \xi J_0(r\xi)\, d\xi = 0 \;, \quad r < a \qquad \text{(A-10)}$$

while for $r > a$, (A-8) may be written as

$$\int_0^\infty \left[k_1 A_1(\xi) + k_2 A_2(\xi)\right] \xi J_0(r\xi)\, d\xi = 0 \;, \quad r > a \qquad \text{(A-11)}$$

Therefore, by the Hankel inversion theorem

$$k_1 A_1(\xi) = -k_2 A_2(\xi) \qquad \text{(A-12)}$$

so that (A-7) and (A-9) yield

$$\int_0^\infty \xi A_2(\xi) J_0(r\xi)\, d\xi = -\frac{k}{k_2}\tau \quad , \quad r < a \tag{A-13}$$

$$\int_0^\infty A_2(\xi) J_0(r\xi)\, d\xi = 0 \quad , \quad r > a \tag{A-14}$$

The solution to these dual integral equations (see [24], p.65-70) is

$$A_2 = -\sqrt{\frac{2}{\pi}}\, \frac{\tau k}{k_2}\, a^{3/2} \xi^{-1/2} J_{3/2}(\xi a) \tag{A-15}$$

which is in agreement with eq. (3-16) and thus leads to the same results as those obtained in Chapter 3.

APPENDIX B

LIMITING FORMS OF $\Omega_1(\zeta)$ AND $\Omega_2(\zeta)$ AS $z \to 0$

Expressions for $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ on the $z = 0$ plane are obtained by evaluating eqs. (4.4-9) as $z \to 0$. The limiting values of the terms

$$\left(\frac{\zeta - a}{\zeta + a}\right)^{m_1}, \qquad \left(\frac{\zeta - a}{\zeta + a}\right)^{-m_1}$$

may be determined by first recalling that

$$(\zeta - c)^{\pm m_1} = |\zeta - c|^{\pm m_1} e^{\pm i m_1 \theta}, \quad (c = a, -a) \tag{B-1}$$

where $\theta = \arg(\zeta - c)$. Now, if ζ describes a closed path beginning at a point x of the arc L' (see Fig. B-1) and leading, without intersecting L', from the top of L' around the end a to the bottom of the arc as shown, it follows that

$$\left[(\zeta - a)^{\pm m_1}\right]^{+} = |x - a|^{\pm m_1} e^{\pm i m_1 \pi} \tag{B-2a}$$

$$\left[(\zeta - a)^{\pm m_1}\right]^{-} = |x - a|^{\pm m_1} e^{\mp i m_1 \pi} \tag{B-2b}$$

$$\left[(\zeta + a)^{\pm m_1}\right]^{+} = |x + a|^{\pm m_1} \tag{B-2c}$$

$$\left[(\zeta + a)^{\pm m_1}\right]^{-} = |x + a|^{\pm m_1} \tag{B-2d}$$

$$(z = 0, |x| < a)$$

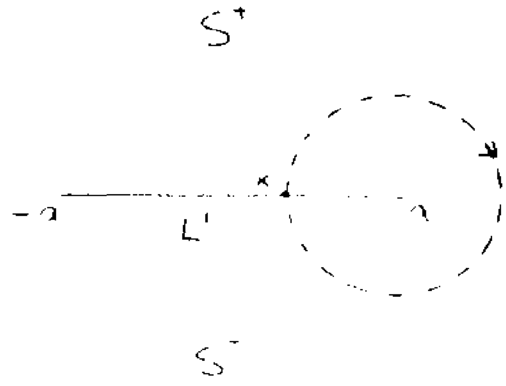


Figure B-1

Hence,

$$\left[\left(\frac{\zeta - a}{\zeta + a}\right)^{m}\right]^{+} = \left(\frac{a - x}{a + x}\right)^{m} e^{i\pi m} \qquad \text{(B-3a)}$$

$$\left[\left(\frac{\zeta - a}{\zeta + a}\right)^{m}\right]^{-} = \left(\frac{a - x}{a + x}\right)^{m} e^{-i\pi m} \qquad \text{(B-3b)}$$

$z = 0,\ |x| < a$

$$\left[\left(\frac{\zeta - a}{\zeta + a}\right)^{-m}\right]^{+} = \left(\frac{a - x}{a + x}\right)^{-m} e^{-i\pi m} \qquad \text{(B-3c)}$$

$$\left[\left(\frac{\zeta - a}{\zeta + a}\right)^{-m}\right]^{-} = \left(\frac{a - x}{a + x}\right)^{-m} e^{i\pi m} \qquad \text{(B-3d)}$$

For the case when $z = 0$ and $|x| > a$, it follows from (B-1) that

$$\left[\left(\frac{\zeta - a}{\zeta + a}\right)^{m_1}\right]^{+} = \left[\left(\frac{\zeta - a}{\zeta + a}\right)^{m_1}\right]^{-} = \left(\frac{x - a}{x + a}\right)^{m_1} \qquad \text{(B-4a)}$$

$$\left[\left(\frac{\zeta - a}{\zeta + a}\right)^{-m_1}\right]^{+} = \left[\left(\frac{\zeta - a}{\zeta + a}\right)^{-m_1}\right]^{-} = \left(\frac{x - a}{x + a}\right)^{-m_1} \qquad \text{(B-4b)}$$

since the argument of ($\zeta - a$) ($\zeta = -a\,a$) does not change in going from plus to minus values. Making use of eqs. (B-3), (B-4) and (4.2-14) and noting (see (4.4-5)) that

$$e^{m_1 \pi} + e^{-m_1 \pi} = \frac{2\sqrt{A}}{\sqrt{A - B}}$$

$$e^{m_1 \pi} - e^{-m_1 \pi} = \frac{2\sqrt{B}}{\sqrt{A - B}}$$

the following expressions are obtained from eqs. (4.4-9):

$$\left(\Omega_1^{+} + \Omega_1^{-}\right)_{\substack{|x|<a \\ z=0}} = -2 F_1(x)\Big|_{|x|<a} +$$

$$4\sqrt{\frac{A-B}{B}}\, L_2 \gamma k\, m_1 a x \left[\left(\frac{a-x}{a+x}\right)^{-m_1} + \left(\frac{a-x}{a+x}\right)^{m_1}\right]$$

$$-\sqrt{\frac{A-B}{B}}\left[2 L_2 \gamma k\, x^2 + 4 L_2 \gamma k\, m_1^2 a^2 - 4 C_1\right] \cdot$$

$$\left[\left(\frac{a-x}{a+x}\right)^{-m_1} - \left(\frac{a-x}{a+x}\right)^{m_1}\right] \qquad \text{(B-5)}$$

$$\left(\Omega_1^+ - \Omega_1^-\right)_{\substack{|x|<a \\ z=0}} = -4\sqrt{\frac{A-B}{A}}\, L_2 \gamma k\, m_1 a x \left[\left(\frac{a-x}{a+x}\right)^{-m_1} - \left(\frac{a-x}{a+x}\right)^{m_1}\right]$$

$$+\, 2\sqrt{\frac{A-B}{A}}\left[L_2\gamma k\, x^2 + 2L_2\gamma k\, m_1^2 a^2 - 2C_2\right]\left[\left(\frac{a-x}{a+x}\right)^{-m_1} + \left(\frac{a-x}{a+x}\right)^{m_1}\right]$$

(B-6)

$$\left(\Omega_2^+ + \Omega_2^-\right)_{\substack{|x|<a \\ z=0}} = \frac{(A-B)}{B}\Bigg\{ -8C_2 + 4L_2\gamma k\, x^2 +$$

$$4\sqrt{\frac{A}{A-B}}\, L_2\gamma k\, m_1 a x \left[\left(\frac{a-x}{a+x}\right)^{-m_1} - \left(\frac{a-x}{a+x}\right)^{m_1}\right]$$

$$-\, 2\sqrt{\frac{A}{A-B}}\left[L_2\gamma k\, x^2 + 2L_2\gamma k\, m_1^2 a^2 - 2C_2\right]\left[\left(\frac{a-x}{a+x}\right)^{-m_1} + \left(\frac{a-x}{a+x}\right)^{m_1}\right]$$

(B-7)

$$\left(\Omega_2^+ - \Omega_2^-\right)_{\substack{|x|<a \\ z=0}} = -4\sqrt{\frac{A-B}{B}}\, L_2\gamma k\, m_1 a x \left[\left(\frac{a-x}{a+x}\right)^{-m_1} + \left(\frac{a-x}{a+x}\right)^{m_1}\right]$$

$$+\, 2\sqrt{\frac{A-B}{B}}\left[L_2\gamma k\, x^2 + 2L_2\gamma k\, m_1^2 a^2 - 2C_2\right]\left[\left(\frac{a-x}{a+x}\right)^{-m_1} - \left(\frac{a-x}{a+x}\right)^{m_1}\right]$$

(B-8)

$$\Omega_1^+\Big|_{\substack{|x|>a \\ z=0}} = \Omega_1^-\Big|_{\substack{|x|>a \\ z=0}} = \frac{(A-B)}{\sqrt{AB}}\Bigg\{ -\frac{4\sqrt{AB}}{(A-B)} F_0(x)\Big|_{x>a} + 2L_2\gamma k\, m_1 a x \cdot$$

$$\cdot\left[\left(\frac{x-a}{x+a}\right)^{-m_1} + \left(\frac{x-a}{x+a}\right)^{m_1}\right] - \left[L_2\gamma k\, x^2 + 2L_2\gamma k\, m_1^2 a^2 - 2C_2\right]\cdot$$

$$\cdot\left[\left(\frac{x-a}{x+a}\right)^{-m_1} - \left(\frac{x-a}{x+a}\right)^{m_1}\right]\Bigg\}$$

(B-9)

$$\Omega_2^+\Big|_{\substack{|x|>a\\ z=0}} = \Omega_2^-\Big|_{\substack{|x|>a\\ z=0}} = \frac{(A-B)}{B}\Bigg\{-4C_2 + 2L_2\gamma k\, x^2 + 2L_2\gamma k\, m_1 a x\left[\left(\frac{x-a}{x+a}\right)^{-m_1} - \left(\frac{x-a}{x+a}\right)^{m_1}\right] - \Big[L_2\gamma k\, x^2 + 2L_2\gamma k\, m_1^2 a^2 - 2C_2\Big]\left[\left(\frac{x-a}{x+a}\right)^{-m_1} + \left(\frac{x-a}{x+a}\right)^{m_1}\right]\Bigg\} \qquad \text{(B-10)}$$

APPENDIX C

EVALUATION OF INTEGRALS OF SECTION 5.3

Employing the well known formula

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \text{(C-1)}$$

expressions for the integrals in (5.3-6) are obtained as follows:

$$\int_0^a \left[\left(\frac{a-x}{a+x}\right)^{m_1} - \left(\frac{a-x}{a+x}\right)^{-m_1}\right] x\sqrt{a^2-x^2}\,dx =$$

$$\frac{4a^3}{\Gamma(4)}\left[\Gamma(\tfrac{5}{2}-m_1)\Gamma(\tfrac{3}{2}+m_1) - \Gamma(\tfrac{5}{2}+m_1)\Gamma(\tfrac{3}{2}-m_1)\right] \qquad \text{(C-2)}$$

$$\int_0^a \left[\left(\frac{a-x}{a+x}\right)^{m} + \left(\frac{a-x}{a+x}\right)^{-m_1}\right] x^2\sqrt{a^2-x^2}\,dx =$$

$$2a^4\left\{\frac{4\Gamma(\tfrac{7}{2}-m_1)\Gamma(\tfrac{3}{2}+m_1)}{\Gamma(5)} - \frac{4\Gamma(\tfrac{5}{2}-m_1)\Gamma(\tfrac{3}{2}+m_1)}{\Gamma(4)}\right.$$

$$+ \frac{\Gamma(\tfrac{3}{2}-m_1)\Gamma(\tfrac{3}{2}+m_1)}{\Gamma(3)} + \frac{4\Gamma(\tfrac{7}{2}+m_1)\Gamma(\tfrac{3}{2}-m_1)}{\Gamma(5)}$$

$$\left. - \frac{4\Gamma(\tfrac{5}{2}+m_1)\Gamma(\tfrac{3}{2}-m_1)}{\Gamma(4)} + \frac{\Gamma(\tfrac{3}{2}+m_1)\Gamma(\tfrac{3}{2}-m_1)}{\Gamma(3)}\right\} \qquad \text{(C-3)}$$

$$\int_0^a \left[\left(\frac{a-x}{a+x}\right)^m - \left(\frac{a-x}{a+x}\right)^{-m}\right] x^3 \sqrt{a^2-x^2}\, dx =$$

$$2a^5 \left\{ \frac{8\Gamma(\frac{9}{2}-m)\Gamma(\frac{3}{2}+m)}{\Gamma(6)} - \frac{12\Gamma(\frac{7}{2}-m)\Gamma(\frac{3}{2}+m)}{\Gamma(5)} \right.$$

$$+ \frac{6\Gamma(\frac{5}{2}-m)\Gamma(\frac{3}{2}+m)}{\Gamma(4)} - \frac{\Gamma(\frac{3}{2}-m)\Gamma(\frac{3}{2}+m)}{\Gamma(3)}$$

$$- \frac{8\Gamma(\frac{9}{2}+m)\Gamma(\frac{3}{2}-m)}{\Gamma(6)} + \frac{12\Gamma(\frac{7}{2}+m)\Gamma(\frac{3}{2}-m)}{\Gamma(5)}$$

$$\left. - \frac{6\Gamma(\frac{5}{2}+m)\Gamma(\frac{3}{2}-m)}{\Gamma(4)} + \frac{\Gamma(\frac{3}{2}+m)\Gamma(\frac{3}{2}-m)}{\Gamma(3)} \right\} \qquad \text{(C-4)}$$

The gamma functions in eqs. (C-2) - (C-4) are evaluated by means of the relations (see [21], p.2)

$$\left.\begin{aligned} \Gamma(1+z) &= z\Gamma(z), \\ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, &\quad \Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) = \frac{\pi}{\cos \pi z} \end{aligned}\right\} \qquad \text{(C-5)}$$

so that the above integrals take the form

$$\int_0^a \left[\left(\frac{a-x}{a+x}\right)^{m_1} - \left(\frac{a-x}{a+x}\right)^{-m_1}\right] x\sqrt{a^2-x^2} =$$

$$- \frac{4a^3 m_1}{3}\left(\frac{1}{4} - m_1^2\right)\frac{\pi}{\cos \pi m_1} \qquad \text{(C-6)}$$

$$\int_0^a \left[\left(\frac{a-x}{a+x}\right)^{m_1} + \left(\frac{a-x}{a+x}\right)^{-m_1}\right] x^2\sqrt{a^2-x^2}\, dx =$$

$$4a^4\left(\frac{m_1^2}{6} + \frac{1}{8}\right)\left(\frac{1}{4} - m_1^2\right)\frac{\pi}{\cos \pi m_1} \qquad \text{(C-7)}$$

$$\int_0^a \left[\left(\frac{a-x}{a+x}\right)^{m_1} - \left(\frac{a-x}{a+x}\right)^{-m_1}\right] x^3\sqrt{a^2-x^2}\, dx =$$

$$- \frac{a^5}{15}\left(4m_1^3 + 11m_1\right)\left(\frac{1}{4} - m_1^2\right)\frac{\pi}{\cos \pi m_1} \qquad \text{(C-8)}$$

The remaining integral in (5.3-6) may be integrated by parts and yields

$$\int_0^a F_0(x)\, x\sqrt{a^2-x^2}\, dx = \frac{K_0 a^4}{5}\sqrt{\frac{2\pi}{a}} \qquad \text{(C-9)}$$

where $F_0(x)$ is given by (4.2-14b).

APPENDIX D

SERIES EXPANSION OF SECTION 5.4

For large values of α and β,

$$\coth \alpha = 1 + 2e^{-2\alpha} + \ldots \tag{D-1}$$

$$\coth \beta = 1 + 2e^{-2\beta} + \ldots \tag{D-2}$$

from which it follows that

$$\left(\coth^2\beta - \coth^2\alpha\right)^{-1/2} = \frac{e^{\beta}}{2}\left[1 - e^{-2(\alpha-\beta)}\right]^{-1/2} \tag{D-3}$$

Expansion of (D-3) leads to

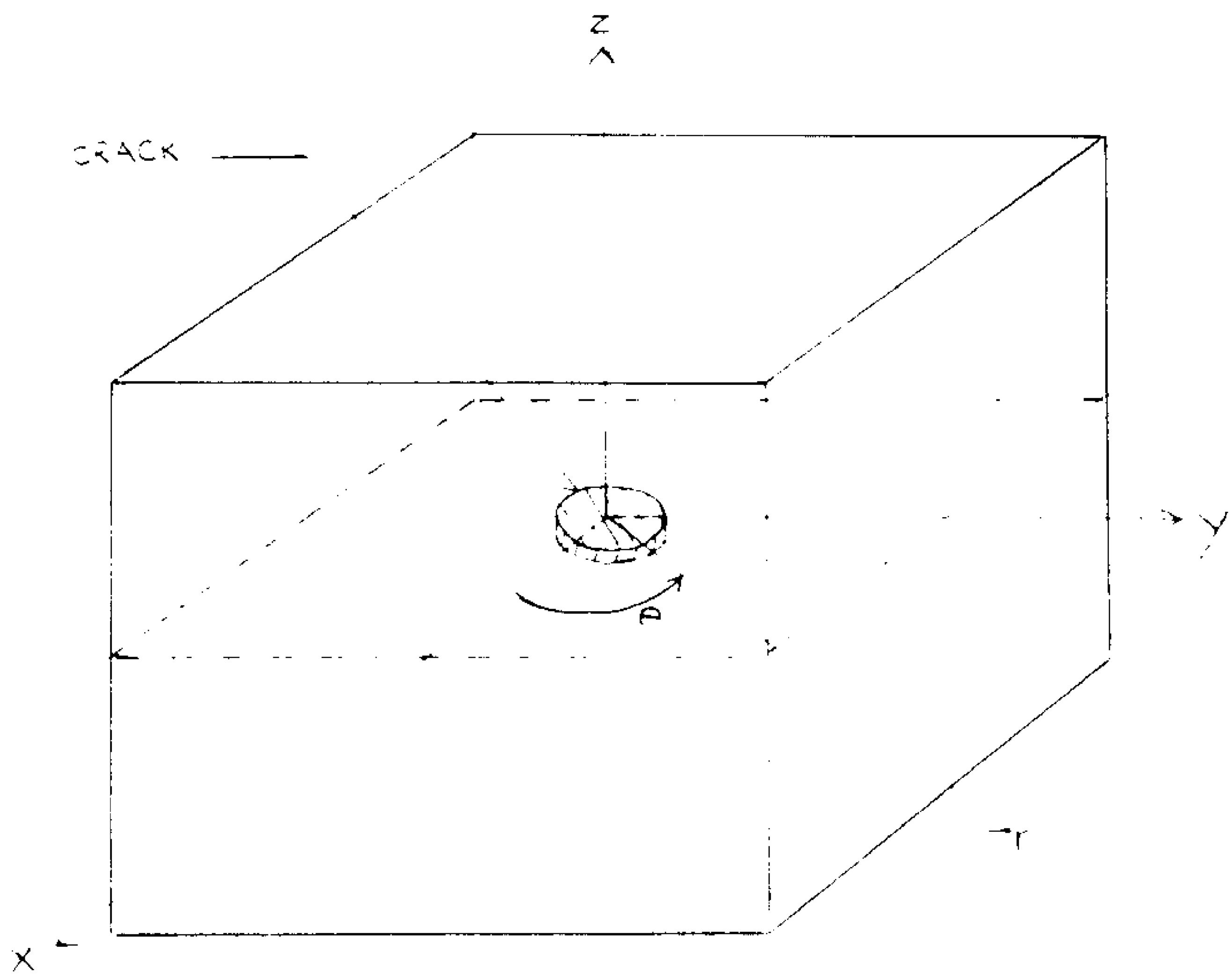
$$\left(\coth^2\beta - \coth^2\alpha\right)^{-1/2} = \frac{e^{\beta}}{2}\left[1 + \frac{1}{2}e^{-2(\alpha-\beta)} + \frac{1\cdot 3}{2\cdot 4}e^{-4(\alpha-\beta)} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^{-6(\alpha-\beta)} + \ldots\right] \tag{D-4}$$

which may be written in the form

$$\left(\coth^2\beta - \coth^2\alpha\right)^{-1/2} = \frac{e^{\beta}}{2}\sum_{n=0}^{\infty} b_n e^{-2n(\alpha-\beta)} \tag{D-5}$$

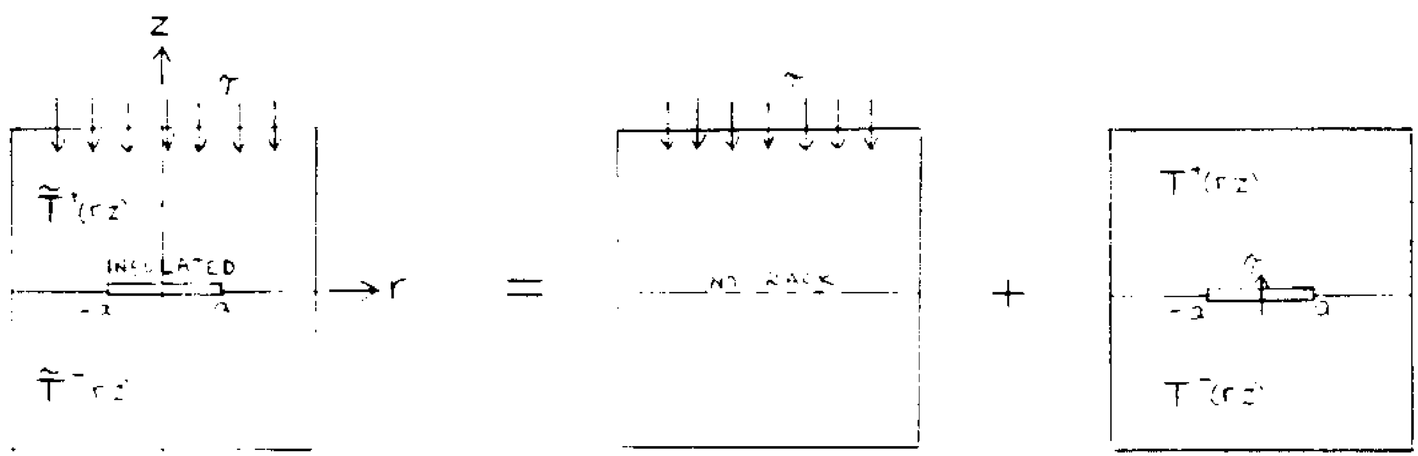
where

$$b_n = \frac{(n-\frac{1}{2})!}{n!\sqrt{\pi}} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\sqrt{\pi}}$$



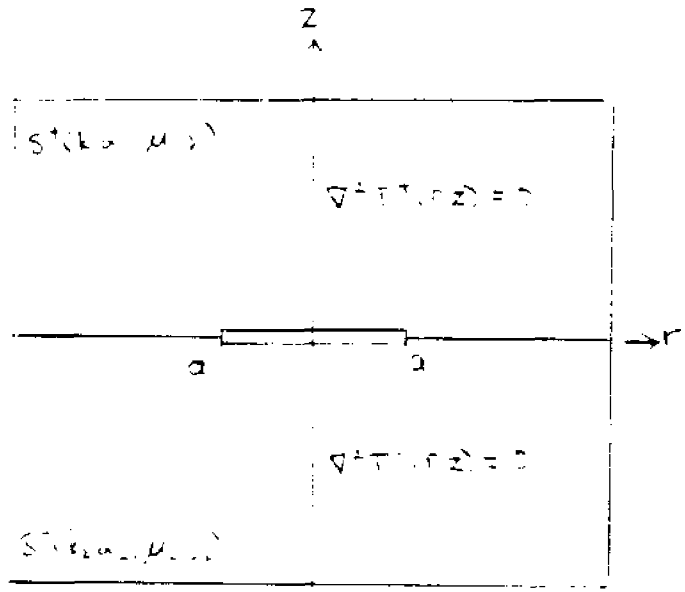
PENNY-SHAPED CRACK

FIGURE 1



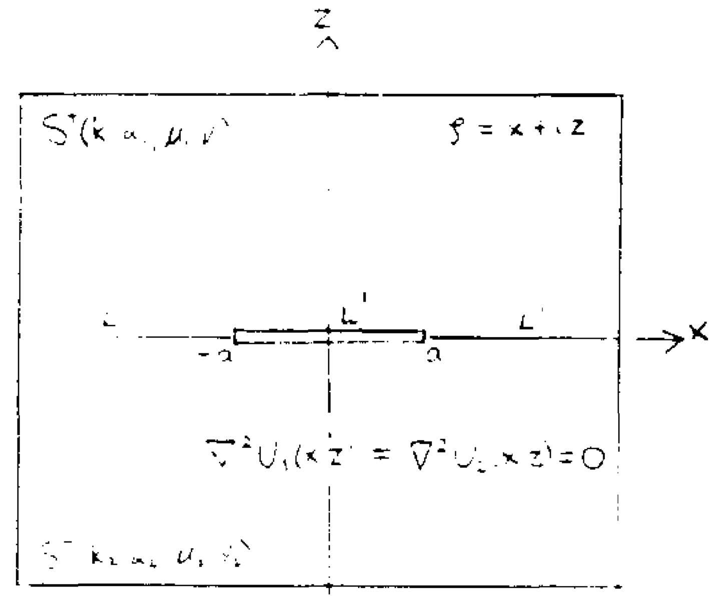
UNIFORM HEAT FLOW AT INFINITY

FIGURE 2



ORIGINAL PLANE

FIGURE 3



ζ - PLANE

FIGURE 4

REFERENCES

1. A. L. Florence and J. N. Goodier, "The Linear Thermoelastic Problem of Uniform Heat Flow Disturbed by a Penny-Shaped Insulated Crack", Int. J. Engng. Sci., Vol. 1, 1963, pp. 533-540.

2. M. K. Kassir and G. C. Sih, "Three-Dimensional Thermoelastic Problems of Planes of Discontinuities or Cracks in Solids", Developments in Theoretical and Applied Mechanics, Vol. 3, 1967, pp. 117-146.

3. M. K. Kassir, "Thermal Crack Propagation", Journal of Basic Engineering, Trans. ASME, Paper No. 71-Met-N, 1971.

4. M. K. Kassir and A. M. Bregman, "Thermal Stresses in a Solid Containing Parallel Circular Cracks", presented at the Fourth National Symposium on Fracture Mechanics, Carnegie-Mellon University, Pittsburgh, Pa., August 24-26, 1970.

5. W. D. Collins, "Some Axially Symmetric Stress Distributions in Elastic Solids Containing Penny-Shaped Cracks", Proc. Roy. Soc., London, England, Ser. A, Vol. 266, 1962, pp. 359-386.

6. A. L. Florence and J. N. Goodier, "Thermal Stresses Due to Disturbance of Uniform Heat Flow by an Insulated Ovaloid Hole", J. Appl. Mech., Vol. 27, Trans. ASME, 1960, pp. 635-639.

7. A. L. Florence and J. N. Goodier, "Thermal Stress Due to Disturbance of Uniform Heat Flow by an Insulated Spheroidal Cavity, Proc. 4th U.S. National Congress of Applied Mechanics, 1962, pp. 595-602.

8. M. A. Hussain and S. L. Pu, "Thermal Stresses Near a Prolate Spheroidal Inclusion", J. Appl. Mech., Trans. ASME, June, 1970, pp. 403-408.

9. M. L. Williams, "The Stresses Around a Fault or Crack in Dissimilar Media", Bulletin of the Seismological Society of America, Vol. 49, 1959, pp. 199-204.

10. F. Erdogan, "Stress Distribution in Bonded Dissimilar Materials with Cracks", J. Appl. Mech., Vol. 32, Trans. ASME, 1965, pp. 403-410.

11. J. R. Rice and G. C. Sih, "Plane Problems of Cracks in Dissimilar Media", J. Appl. Mech., Vol. 32, Trans. ASME, 1965, pp. 418-423.

12. E. J. Brown, "Thermal Stresses and Elastic-Plastic Stress Distribution in Bonded Dissimilar Media", Ph.D. Dissertation, Lehigh University, Bethlehem, Pa., 1966. See also E. J. Brown and F. Erdogan, "Thermal Stresses in Bonded Materials Containing Cuts on the Interface", Int. J. Engng. Sci., Vol. 6, 1968, pp. 517-529.

13. V. I. Mossakovskii and M. T. Rybka, "Generalization of the Griffith-Sneddon Criterion for the Case of a Non-Homogeneous Body, PMM, Journal of Applied Mech. and Math., Vol. 4, 1964, pp. 1277-1286.

14. F. Erdogan, "Stress Distribution in Bonded Dissimilar Materials Containing Circular or Ring-Shaped Cavities", J. Appl. Mech., Trans. ASME, Dec. 1965, pp. 829-836.

15. V. I. Mossakovskii, "Fundamental Mixed Problem in the Theory of Elasticity for a Half-Space with a Circular Line Separating the Boundary Conditions", PMM, Vol. 18, No. 2, 1954.

16. N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, P. Noordhoff Ltd., Groningen, Holland, 1953.

17. G. R. Irwin, "Fracture Mechanics", Structural Mechanics, Edited by J. N. Goodier and N. J. Hoff, Pergamon Press, New York, 1960, p. 557.

18. E. Trefftz, "Mathematische Elastizatstheorie", Handbuch der Physik, edited by H. Geiger and K. Scheel, Vol. 6, p. 92, Springer (1928).

19. B. A. Boley and J. H. Weiner, Theory of Thermal Stresses, John Wiley and Sons, Inc., New York, 1960.

20. A. Erdelyi (Ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, Bateman Manuscript Project, California Institute of Technology, Tables of Integral Transforms, Vol. 2, McGraw Hill Book Company, 1954.

21. W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag New York Inc., 1966.

22. C. E. Weatherburn, Advanced Vector Analysis, G. Bell and Sons, Ltd., London, 1928.

23. H. B. Dwight, Tables of Integrals and Other Mathematical Data, 4th ed., The MacMillan Co., 1961

24. I. N. Sneddon, <u>Fourier Transforms</u>, McGraw-Hill Book Company, Inc., 1951.

25. G. Petit Bois, <u>Tables of Indefinite Integrals</u>, Dover Publications, Inc., New York, 1961.

26. <u>Table of the Gamma Function for Complex Arguments</u>, National Bureau of Standards, Applied Mathematics Series, 34, 1954.

VITA

Alvin M. Bregman was born in New York City on February 7, 1945.

He was graduated from the Bronx High School of Science in June 1961. Attending the City College of New York, Mr. Bregman was elected to Tau Beta Pi and Chi Epsilon and was a member of A. S. C. E. He graduated cum laude in August 1965 with the Bachelor of Engineering degree.

Continuing at the City College in the City University of New York doctoral program, he was awarded a National Science Foundation Graduate Traineeship (1965-69) and received the Master of Engineering degree in February 1967. During the course of his graduate studies, Mr. Bregman co-authored a paper on "Thermal Stresses in a Solid Containing Parallel Circular Cracks" which was presented at the Fourth National Symposium on Fracture Mechanics. In September 1970, he accepted his current position as Instructor of Mathematics at Brooklyn College of the City University of New York. Working towards his Ph.D. degree, he completed his doctoral dissertation in 1971.

Mr. Bregman is married to the former Jeannette Lang and they have a daughter, Shari.