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**ISOPERIMETRY AND LATTICES IN GROUPS
OF HEISENBERG TYPE**

by

Gordon Crandall

**A dissertation submitted to the Graduate Faculty
in Mathematics in partial fulfillment of the require-
ments for the degree of Doctor of Philosophy, The
City University of New York**

2000

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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THE CITY UNIVERSITY OF NEW YORK

Abstract**ISOPERIMETRY AND LATTICES IN GROUPS OF HEISENBERG TYPE****By****Gordon Crandall****Advisor : Professor Jozef Dodziuk**

In the first part of this thesis, an isoperimetric inequality is proved for connected, simply connected nilpotent Lie groups N containing a lattice. The proof is by rough isometry, consequently the inequality is valid only for balls in N of large radius. In the second part, it is proved that every Lie group of H-type contains an embedded lattice. Isoperimetric inequalities on H-type groups then follow from the result obtained in the first part.

Acknowledgements

- I. For Isaac Chavel and Edgar Feldman, the starting pitchers, and most of all, for the closer, Jozef Dodziuk, who set new standards for mentorship.
- II. For dear friends, without which not, Marie Monaco and Ashok Gupta; for dogs Spider, India and Zeb.
- III. For Robert Minkoff (1951-1999):

C'était à Mégara, faubourg de Carthage, dans les jardins d'Hamilcar. Les soldats qu'il avait commandés en Sicile se donnaient un grand festin pour célébrer le jour anniversaire de la bataille d'Éryx, et comme le maître était absent et qu'ils se trouvaient nombreux, ils mangeaient et ils buvaient en pleine liberté.

-Flaubert, Salammbô, Le Festin

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Chapter 1

Introduction and Background

1.1 Introduction

In this thesis, we will prove an isoperimetric inequality in the category of connected, simply connected, nilpotent Lie groups. Our method will be that of rough isometry, as introduced by Masahiko Kanai in [11]. The existence of a rough isometry between a smooth manifold and a discrete net structure embedded in the manifold allows the transferring of a discrete isoperimetric inequality on the net to a “smooth” isoperimetric inequality on the manifold. When the manifold is a Lie group, the role of an embedded net can be played by a uniform lattice in the group. With an isoperimetric inequality on the lattice in hand, the starting point for our theorem — illustrating *de novo* the interplay between the Lie algebra and the group — is the criterion for the existence of a lattice in a connected, simply connected, nilpotent Lie group. The criterion, due to Mal'cev [17], is the existence of a basis for the corresponding Lie algebra of the group, with respect to which the structure coefficients are rational. In the

remainder of this introductory chapter, we will first outline the basic theory of Lie groups and Lie algebras necessary for our work. For a more thorough development, we refer the reader to the first four chapters of [24]. This introductory material is followed by a presentation of the relevant details of smooth isoperimetry on Riemannian manifolds, and by a lengthier discussion of isoperimetry in discrete settings.

In chapter 2, we will prove an isoperimetric inequality on connected, simply connected Lie groups, by transferring a discrete isoperimetric inequality on a finitely generated, discrete, nilpotent group to the smooth ambient group by means of the rough isometry methods of Kanai.

In the third chapter of this thesis, we will prove our main theorem, that it is possible to put a rational structure on the Lie algebras of a more restricted class of connected, simply connected, nilpotent Lie groups, namely those of Heisenberg type or H-type. By Mal'cev's result we know that such groups support co-compact lattices. An isoperimetric inequality on H-type groups can then be proved.

Such groups were first introduced by Aroldo Kaplan ([12], [13], [14]), as having properties nearly identical with the classical Heisenberg groups, namely, the upper triangular matrices with real entries (ones on the main diagonal) of all dimensions ≥ 3 . The H-type groups have played an important role in several areas of modern analysis and differential geometry, particularly as counter-examples in the question of isometry versus isospectrality, and also as counter

examples to the Lichnerowicz conjecture, in the non-compact case. They also serve as important special cases in their connection with non-compact rank one simple groups. Paulo Barbano ([1]) has summarized this material well and provides references to the key papers.

1.2 Lie Groups and Lie Algebras

A (real) Lie group over the real numbers is a group G carrying a differentiable manifold structure such that the product map

$$G \times G \rightarrow G$$

$$(g_1, g_2) \rightarrow g_1 \cdot g_2$$

is also C^∞ . Just as differential geometry is enriched by the interplay of manifold and tangent space, with many key results following from analysis done on the latter, so is the power of Lie theory enhanced through linear algebra and analysis on the “tangent algebra” of a Lie group, namely its Lie algebra.

A (real) Lie algebra \mathfrak{g} is a finite dimensional vector space over the real numbers equipped with an additional bilinear operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called the Lie bracket, such that for all $X, Y \in \mathfrak{g}$:

$$(1) \quad [X, X] = 0 \quad (\Rightarrow [X, Y] = -[Y, X])$$

$$(2) \quad [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \quad (\text{Jacobi Identity}).$$

Starting from the group, the standard construction of the algebra is from the left-invariant vector fields obtained as follows. Let G be a Lie group with

$\tau \in G$. Define ℓ_τ on G as left translation of group elements by the element τ . Let X be a vector field on G , that is a function from G to the tangent bundle of G , $T(G)$, such that

$$\Pi \circ X = \ell_G ,$$

where Π is the natural projection from the tangent bundle to G . X is called left-invariant if for all $\tau \in G$

$$d\ell_\tau \circ X = X \circ \ell_\tau ,$$

where $d\ell_\tau$ is the differential of the left translation by τ on tangent spaces. The set of left invariant vector fields is easily seen to be a vector space, naturally isomorphic to any tangent space of G , and in particular to the tangent space at the identity, $T_e(G)$. The required bracket operation for a Lie algebra is the Lie bracket of vector fields, i.e. for (left-invariant) vector fields X, Y on G

$$[X, Y] = X \circ Y - Y \circ X .$$

This bilinear operation on vector fields satisfies precisely properties (1) and (2) above.

If we continue the analogy of the pair group/algebra to manifold/tangent space, we are led naturally to the question of exponential map. That is, what in the former, plays the role of straight-line or geodesic in the latter? In Lie theory, this is provided by the one-parameter semi-group, i.e., a smooth homeomorphism $f(t)$ from the real numbers as a Lie group to the Lie group G such that $f(t + s) = f(t)f(s)$. These conditions are sufficient to prove that $f(0) = e$,

and that $f'(t)$, the tangent or velocity vector, is constant when moved to the identity by the left translation $f(t)$. Recalling the isomorphism between the tangent space at the identity and the Lie algebra of G , and letting $f_\xi(t)$ represent the one-parameter subgroup with velocity ξ , $\xi \in \mathfrak{g}$, one may define the exponential map, $\exp: \mathfrak{g} \rightarrow G$ by $\exp(\xi) = f_\xi(1)$, precisely as in Riemannian geometry.

It is a standard result that, given any connected Lie group G , the exponential map defined on \mathfrak{g} maps a neighborhood of 0 in \mathfrak{g} onto a neighborhood of e in G diffeomorphically. However, for simply connected nilpotent Lie groups, the exponential map is actually a global diffeomorphism.

We define a lattice in a Lie group G as a discrete, subgroup H , with the condition that the quotient group G/H carry a G -invariant finite measure. Actually, if G is nilpotent all lattices are uniform [3]. Although it can be proved that a necessary condition for a Lie group G to contain a lattice is unimodularity (equality up to scalar multiple of the left invariant and right invariant Haar measures, μ_L and μ_R , so that ΔG , the positive valued modular function on G , defined for any integrable f by

$$\Delta G(x) \cdot \int_G f(g) d\mu_L = \int_G f(g \cdot x) d\mu_L$$

takes on constant value 1), and all nilpotent groups are unimodular, it does not follow that every nilpotent group admits a lattice. Mal'cev has constructed a 2-step nilpotent counterexample in [17]. For connected and simply connected

nilpotent groups, the only known necessary and sufficient condition for the existence of a lattice subgroup is Mal'cev's criterion stated above.

We recall here the definition of a nilpotent group. Suppose that for any vectors x_1, x_2, \dots, x_{p+1} in a Lie algebra \mathfrak{g} of a group G we have

$$[x_1, [x_2, \dots [x_{p+1}, \dots]]] = 0 .$$

Then \mathfrak{g} is termed p -step nilpotent if the above holds for some p , with G nilpotent if and only if \mathfrak{g} is nilpotent. A 1-step nilpotent algebra is called abelian.

1.3 Isoperimetry and Discretizations

We review the fundamentals presented in [5]. The most basic isoperimetric inequality is

$$L^2 \geq 4 \pi A ,$$

for domains in R^2 with area A and boundary length L . This is a direct restatement of the classical inequality stating that of all planar domains with equal area, the disk has shortest boundary length. Analogously, for domains Ω in R^n we have

$$\frac{A(\partial \Omega)}{V(\Omega)^{1-1/n}} \geq \frac{A(S^{n-1})}{V(B^n)^{1-1/n}} = \text{constant}_n ,$$

where A is $(n-1)$ -dimensional measure. Note that the exponent $1-1/n = (n/n-1)^{-1}$ makes the inequality independent of the radius.

For the case of Riemannian manifolds M , the inequality above may not hold with n equal to the dimension of the manifold. We proceed as follows. For $v > 1$, and Ω an open submanifold of M with compact closure and smooth boundary, define the v -isoperimetric quotient as

$$F_v = \frac{A(\partial \Omega)}{V(\Omega)^{1-1/v}},$$

where A and V denote respectively $(n-1)$ - and n -dimensional Riemannian measure. Taking the infimum over all Ω as above, we obtain the v -isoperimetric constant of M :

$$I_v(M) = \inf_{\Omega} \frac{A(\partial \Omega)}{V(\Omega)^{1-1/v}}$$

For $v = \infty$, the isoperimetric constant $I_{\infty}(M) = \inf_{\Omega} \frac{A(\partial \Omega)}{V(\Omega)}$ is called

Cheeger's constant. For a given Riemannian manifold M , isoperimetric analysis proceeds by determining those values of v for which $I_v(M) > 0$. A result of S.T. Yau [26] states that in calculating $I_v(M)$ ($1 < v \leq \infty$), it suffices to let Ω range over connected submanifolds of M .

In what follows, we will only be interested in the case $v \geq n = \dim(M)$, ($M = G$, a Lie group), for which the above is an appropriate formulation. However, a simple argument with $\Omega = B(x, \varepsilon)$, a metric disk, with $\varepsilon \downarrow 0$ shows that $I_v(M) = 0$ for $v < n$. To correct this in the case $1 \leq v < n$ (or for any isoperimetric constant), we may define a modified v -isoperimetric constant, $I_{v,\rho}(M)$, by further

requiring the open sub-manifolds Ω to contain a closed metric disk of radius $\rho > 0$, then taking the infimum over the isoperimetric quotient as before.

A crucial result linking isoperimetry with classical analysis is the theorem of Federer and Fleming, which states simply that for $1 < v \leq \infty$

$$I_v(M) = S_v(M) ,$$

where $S_v(M)$ is the Sobolev constant of M , defined by

$$S_v(M) = \inf_f \frac{\|\text{grad } f\|_1}{\|f\|_{v/v-1}} ,$$

where $\|\cdot\|_p$ is the L^p -norm on M , and f ranges over all smooth functions on M with compact support. A similar result also holds in the "modified" case.

Volume growth of a (non-compact) Riemannian manifold is measured in terms of the volume growth of metric disks contained in the manifold. We say, for example, that a disk centered at x with radius r has polynomial growth if $V(x, r) \leq C \cdot r^D$ ($C, D > 0$). A manifold M has polynomial growth if the above estimate holds independently of $x \in M$. Discretization refers volume growth in the manifold to volume growth of a "disk" in an appropriately discrete structure within the manifold. One may then put the question: is volume growth (i.e. polynomial or exponential) the same in smooth and discrete cases?

In order for the answer to be affirmative, there must be some conditions of local uniformity present at both the smooth and the discrete level. The first example of this type of situation comes from the work of John Milnor [18], and assumes the following set-up: \tilde{M} , a non-compact manifold covers a compact

manifold M , so that the covering map, Π , determines by a finitely generated group Γ of deck-transformations (homeomorphisms γ of \tilde{M} with the property that $\Pi \circ \gamma = \Pi$, for all $\gamma \in \Gamma$) acting on \tilde{M} . In fact, we will assume that Γ acts by isometries and the action is free. The action of Γ , in turn, determines a relatively compact fundamental domain Ω in \tilde{M} , and the images of this domain under elements of Γ tile the covering space, that is

$$\bigcup_{\gamma \in \Gamma} \overline{\gamma(\Omega)} = \tilde{M}, \text{ and}$$

$$\gamma_i(\Omega) \cap \gamma_j(\Omega) = \emptyset, \quad i \neq j.$$

As a result of tiling and the freedom of action of Γ on the fundamental domain, the volume of the metric disk can be estimated in terms of fundamental domains which are in 1-1 correspondence with elements of the discrete group. Thus, volume growth of the disk may be related to volume growth of the group once accurate overestimates and underestimates of the number of fundamental domains containing/contained in a metric disk of given radius (assumed large) are obtained. The details of this "calibration" may be found in Milnor [18] or Chavel [5]. The underestimate is of the form

$$n(\lambda) \leq \frac{V(x; \lambda \mu + \varepsilon)}{V(x; \varepsilon)},$$

where $n(\lambda) = \text{card} \{ \gamma: |\gamma| \leq \lambda \}$ and $|\gamma|$ is the so-called word metric, defined to be the minimum length of γ as a word in a given finite set of generators of Γ . In the case of a non-compact Riemannian manifold covering a compact manifold, the constants μ and ε are independent of x . A similar overestimate in terms of a

ratio of volumes of metric disks can be found for sufficiently large λ :

$$n(\lambda) \geq \frac{V(x; \lambda v - (v + 2\delta))}{V(x; \delta)},$$

for all $\lambda \geq 1 + 3\delta/v$, $x \in M$, where v and δ are positive constants. From the first estimate, it is clear that polynomial growth in the manifold implies polynomial growth in the group. Under the weaker assumption that M has non-negative Ricci curvature, it follows from Bishop's Comparison Theorem (Ricci bounded below case) that

$$\lim_{\lambda \uparrow \infty} \frac{\ln(n(\lambda))}{\lambda} = 0.$$

Hence Γ has what is termed sub-exponential growth. Using the second estimate, it can also be shown that if M is compact with strictly negative curvature (Bishop's estimate in the case of Ricci bounded above is invoked), then the fundamental group of M_0 , $\pi_1(M_0, x_0)$ has exponential growth, that is

$$\limsup_{\lambda \uparrow \infty} \frac{\ln(n_{x_0}(\lambda))}{\lambda} > 0.$$

In summary, a finitely generated group of deck transformations acting by isometries with compact quotient plays the role of discretization of the manifold. We then compare volume of metric balls with the "counted" volume of discrete balls. Of course, the key object here was the fundamental domain which was both a tiling element in the smooth manifold, and, through the 1-1 correspondence between tiling elements and isometries γ of Γ , a counting element in the discrete group.

If we abandon the covering space structure and the machinery that it implies, we must ask ourselves what structure embedded in the manifold will play the role of fundamental group and what "calibrating" assumptions will be required.

The second version of the "discrete group" circle of ideas comes not from the external notion of covering space, but from the "internal" idea of a specified point set within the manifold, chosen in such a way that the graph determined by these points has properties similar to those of the combinatorial structure of a tiling by fundamental domains.

Let M be a complete Riemannian manifold. A countable subset \mathcal{G} , such that for each $\xi \in \mathcal{G}$ there is a non-empty subset $N(\xi) \subseteq \mathcal{G} \setminus \{\xi\}$ with finite cardinality $m(\xi)$ (and with the additional property that $\eta \in N(\xi) \Leftrightarrow \xi \in N(\eta)$) determines a graph structure G , if one postulates the existence of exactly one directed edge from any ξ to its neighbors in $N(\xi)$. A (combinatorial) path in G of length k is defined as a sequence of points $(\xi_0, \xi_1, \dots, \xi_k)$ such that $\xi_j \in N(\xi_{j-1})$ for all $j = 1, \dots, k$. G is connected if any two points in G are connected by a path. The distance $d(\xi, \eta)$ is defined to be the infimum of the lengths of all combinatorial paths linking ξ to η . A natural definition of disks and bounding spheres in G follows. The graph structure of G generalizes the notion of discrete group in the sense that if Γ is a finitely generated group with a generator set A , and if, for $\gamma \in \Gamma$ we define $N(\gamma) = \gamma(A \cup A^{-1})$, then Γ has a (Cayley) graph

structure. Moreover, the word metric on Γ and the combinatorial metric on the induced graph on Γ coincide.

We now discuss the conditions and language in which the Milnor estimates on volume growth of balls, given a finitely generated group of deck transformations (that is, the covering space setup), can be carried over to volume growth of balls, given an appropriate countable point set/graph. A discretization of a complete Riemannian manifold M is a graph G determined by a countable subset of points \mathcal{G} in M with two featured properties:

$$(1) \exists \varepsilon_1 > 0 \text{ such that } \mathcal{G} \text{ is } \varepsilon_1 \text{-separated,}$$

that is, the distance between any two distinct points of \mathcal{G} is greater than or equal to ε_1 , and

$$(2) \exists \varepsilon_2 > 0 \text{ such that } M = \cup_{\xi \in \mathcal{G}} B(\xi, \varepsilon_2).$$

ε_2 is called the covering radius of the discretization. The graph structure G is then determined by defining

$$N(\xi) = \{ \mathcal{G} \cap B(\xi, 2\varepsilon_2) \setminus \xi \}.$$

If M has Ricci curvature bounded below, then it can be shown that:

$$\text{card} \{ G \cap B(x, r) \} \leq \frac{V_k(2r + \varepsilon/2)}{V_k(\varepsilon/2)},$$

where V_k is volume in the constant sectional curvature space M_k . From this estimate it follows immediately that

$$m(\xi) = \text{card} \{ N(\xi) \} \leq \frac{V_k(4\varepsilon_2 + \varepsilon_1/2)}{V_k(\varepsilon_1/2)} = \text{constant}(k, \varepsilon_1, \varepsilon_2).$$

This uniform bound on valence is referred to as “bounded geometry” of G .

To carry over estimates on volume growth of balls in G to similar estimates on metric balls in M , we appeal to the concept of rough isometry. Suppose X and Y are metric spaces with a map (not necessarily continuous) $\varphi : X \rightarrow Y$ and constants $a \geq 1$, $b > 0$ and $\delta > 0$ such that

$$(1/a) \cdot d(x_1, x_2)_X - b \leq d(\varphi(x_1), \varphi(x_2))_Y \leq a \cdot d(x_1, x_2)_X + b$$

for all x_1, x_2 in X , and furthermore, φ is δ -full, that is

$$\cup_{x \in X} B(\varphi(x), \delta) = Y.$$

It is easy to show that if $\varphi_1: X_1 \rightarrow X_2$ and $\varphi_2: X_2 \rightarrow X_3$ are rough isometries, then so is $\varphi_2 \circ \varphi_1$, so that the rough isometry relation is transitive.

The key result linking discretization to rough isometry is that if M is a complete manifold with Ricci curvature bounded uniformly from below and G is a discretization of M , then M is roughly isometric to any of its discretizations and the rough isometry is realized by the inclusion map $i: G \rightarrow M$. Transitivity implies further that any two discretizations are roughly isometric.

In order to extend the concepts of isoperimetry to discretizations, we need the discrete analogue of $F_\nu(\Omega) =$

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/\nu}},$$

where Ω is an open submanifold of a smooth manifold M . By subgraph of G , we understand a finite subgraph K of G , arising from a finite subset of points \mathcal{Z} of \mathcal{G} ,

\mathcal{V} being the set of vertices of the graph G . By the boundary of K , ∂K we mean the subset of the set of edges of G (denoted \mathcal{E}) which connect points of \mathcal{K} to points in $\mathcal{V} \setminus \mathcal{K}$. The discrete area measure on \mathcal{E} is the counting measure, that is, the cardinality of a set of oriented edges. Area of the boundary is then the count of oriented edges on the boundary.

There is another definition of the boundary of a finite subgraph of a graph G , namely:

$$\partial K = \{ \xi \in \mathcal{E} : d(\xi, K) = 1 \}$$

d being the discrete path metric. Then ∂K has a natural area measure as the cardinality of a subset of \mathcal{E} . This definition is usually easier in making proofs, and if G has bounded geometry then the area measures defined on the two different subcollections of boundaries are equivalent in the sense that their quotients are uniformly bounded away from 0 and ∞ . Now for $\nu \geq 1$ and any finite subgraph K in G , we define the isoperimetric quotient of K as

$$F_\nu(K) = \frac{A(\partial K)}{V(K)^{1-1/\nu}},$$

and the isoperimetric constant $I_\nu(G)$ as the infimum of $F_\nu(K)$ where K varies over all finite subgraphs of G .

After this preparation, we are able to state the key theorem which will allow us to transfer discrete isoperimetric inequalities on discretizations (e.g., lattices in connected, simply-connected nilpotent Lie groups) to smooth isoperimetric inequalities on manifolds with bounded geometry. We note that in

the context of manifolds, as opposed to graphs, bounded geometry requires not only that (1) Ricci curvature is bounded below, but, also that (2) the injectivity radius of M is uniformly bounded away from 0. The following theorem is a result of M. Kanai [11].

Theorem (1.3.1) Let M have bounded geometry. Then for any $\nu \geq n$ ($= \dim M$), we have $I_\nu(M) > 0$, if and only if $I_\nu(G) > 0$, for any discretization G of M .

Chapter 2

An Isoperimetric Inequality on Nilpotent Lie Groups

2.1 The discrete case

We are now in a position to use the methods described in Chapter 1 to transfer an isoperimetric inequality on a net, i.e. a discretization, to a similar isoperimetric inequality on a Riemannian manifold which contains the discrete structure. We first state a discrete isoperimetric inequality which is the work of T. Coulhon and L. Saloff-Coste [6].

Theorem (2.1.1). Let Γ be a finitely generated infinite discrete group with generating set $\{g_1, \dots, g_k\}$, and for $n \geq 1$, let

$$B(n) = \{x \in \Gamma \mid x = g_{i_1}^{\gamma_1} \dots g_{i_n}^{\gamma_n}, i_1, \dots, i_n \in \{1, \dots, k\}, \gamma_j = 0, \pm 1\}.$$

Put $V(n) = \text{card } B(n)$. Then if $V(n) \geq Cn^D$, we have

$$\inf_{\Omega \subset \Gamma} \{ \text{card } (\partial\Omega) / (\text{card } \Omega)^{(D-1)/D} \} = I_D(\Gamma) > 0,$$

where Ω is any subset of G , and $\partial\Omega = \{x \in \Omega \mid \exists i \in \{1, \dots, k\}, \gamma = \pm 1 \text{ such that } xg_i^\gamma \notin \Omega\}$.

Then we need to determine whether there is a constant $D > 1$, for which $\text{card}(B(n)) \geq Cn^D$. This question is answered in the affirmative by results of J. Wolf [25] and H. Bass [2], following directly the work of J. Milnor [18].

Theorem (2.1.2). Let Γ be a finitely generated p -step nilpotent group, and S a finite generating set for Γ . Then Γ satisfies the following growth condition

$$g_S(m) \geq cm^{E(\Gamma)},$$

with $E(\Gamma) = \sum_{k=0}^{p-1} (k+1) n_k$, where $g_S(m)$ (equivalent to $V(n)$ above), is the growth function of Γ counting the number of distinct elements of Γ expressible as words in S of length equal to or less than m . n_k is the rank of the finitely generated abelian group Γ_k / Γ_{k+1} and the Γ_k belong to the lower central series decomposition for Γ , with $\Gamma_{j+1} = [\Gamma, \Gamma_j]$.

Thus, we conclude that

Theorem (2.1.3) $I_D(\Gamma) > 0$ for Γ discrete and finitely generated, with $D = E(\Gamma)$ as above.

N. Varopoulos [23] refers to D as the Dirichlet dimension of N , D. Robinson [22] calls it the dimension at infinity of N , and P. Pansu [20] refers to it as the Carnot-Caratheodory dimension.

Theorem (2.2.1): Suppose N is a connected, simply connected nilpotent Lie group containing a lattice. For any left invariant metric on the Lie algebra \mathfrak{g} of N , $I_D(N) > 0$, with $D = E(\Gamma)$.

Proof: N is a homogeneous Riemannian space, and any such space is always geodesically complete. The exponential map $\mathfrak{g} \rightarrow N$ is a global diffeomorphism, and consequently, the injectivity radius of N , is trivially bounded away from 0. Thus using Lemma 2.2.2 below we conclude that N has bounded geometry.

Lemma (2.2.2) $\text{Ric}(X, X)$ is bounded below by $(n-1)k$, $k \leq 0$, for all $X \in \mathfrak{g}$. Therefore N has bounded geometry.

Lemma (2.2.3) A co-compact lattice in N is equivalent to a discretization of N .

We conclude that $I_D(N) > 0$ by Theorem 1.3.1.

Proof of Lemma (2.2.2).

If a Lie group N contains a co-compact lattice Γ , then

$$\inf_{g \in N} \text{Ric } g = \inf_{g \in N\Gamma} \text{Ric } g ,$$

and the right hand side is bounded from below by the compactness of N/Γ .

Proof of (2.2.3)

First, we need to prove that our discretization Γ (here Γ is a discrete, finitely generated subgroup of N , which is a connected, simply connected Lie group) is ε_1 separated, that is, for any α, β in Γ , we have $d(\alpha, \beta) \geq \varepsilon_1$, where $d(\cdot, \cdot)$ is distance measured in the ambient manifold N . The inspiration here is [18], and the discussion in [5].

Choose $e \in N$, and define the function $d(e, \gamma \cdot e) = d(e, \gamma) = \|\gamma\|_\bullet$, the so-called displacement norm on Γ . The Milnor theorems then give a constant $c \geq 1$, such that

$$c^{-1}|\gamma|_\Gamma \leq \|\gamma\|_\bullet \leq c|\gamma|_\Gamma,$$

where $|\gamma|_\Gamma$, is the word metric on the finitely generated group Γ . If we then insert Γ into N by $\varphi(\gamma) = \gamma \cdot e = \gamma$, it follows that φ is a quasi-isometry, that is, for all distinct α and β in Γ we have

$$c^{-1}|\beta^{-1}\gamma|_\Gamma \leq d(\varphi(\beta), \varphi(\gamma)) \leq c|\beta^{-1}\gamma|.$$

Now we have $d(\varphi(\beta), \varphi(\gamma)) = d(\beta, \gamma) \geq c^{-1}$, since $|\beta^{-1}\gamma| \geq 1$. So we take $\varepsilon_1 = c^{-1}$.

Secondly we need to find ε_2 , such that $N = \cup_{\gamma \in \Gamma} B(\gamma, \varepsilon_2)$. Since N/Γ is compact, we may cover it with finitely many balls $B(\varepsilon_2', n_i\Gamma)$, $i=1,2, \dots,p$, belonging to N/Γ . Then $N = \cup_{\gamma \in \Gamma} \cup_{i=1, \dots, p} B(\varepsilon_2', n_i\gamma)$. We can cover the inner union by a single ball $B(\varepsilon_2, n_i\gamma)$, and then we have easily that $N = \cup_{\gamma \in \Gamma} B(\varepsilon_2, \gamma)$.

Chapter 3

H - Type Algebras and Lattices

In the second part of this thesis, we will prove that a significant class of Lie groups, namely the connected, simply connected, 2-step nilpotent Lie groups of H-type, contain (co-compact) lattices. To do this we will use Mal'cev's criterion for the existence of such lattices, namely the rationality of the structure constants for the Lie algebras corresponding to the Lie groups under consideration.

Following the work of Aroldo Kaplan ([12], [13], [14]), we take as given finite-dimensional inner product spaces U and V of dimension m and n respectively, along with a bilinear map $J : U \rightarrow \text{End}(V)$ satisfying the properties

$$|J_z(v)|_V = |z|_U |v|_V \quad (\text{orthogonal multiplication}) \quad (3.1), \text{ and}$$

$$J_z^2(v) = J_z(J_z(v)) = -|z|^2 \cdot v, \text{ for all } z \in U \text{ and } v \in V. \quad (3.2)$$

A Lie algebra structure is obtained on $U \oplus V$ by polarizing conditions (3.1)

and (3.2) above:

$$\langle J_{z_1}(v), J_{z_2}(v) \rangle = \langle z_1, z_2 \rangle |v|^2, \text{ and}$$

$$\langle J_z(v_1), J_z(v_2) \rangle = |z|^2 \langle v_1, v_2 \rangle.$$

These together imply immediately that

$$\langle J_z(v_1), v_2 \rangle + \langle v_1, J_z(v_2) \rangle = 0,$$

and consequently, we have a "pre-bracket" $[\cdot, \cdot]: V \times V \rightarrow U$ defined by

$$\langle z, [v_1, v_2] \rangle = \langle J_z(v_1), v_2 \rangle.$$

Now we set $N = U \oplus V$, the orthogonal direct sum of the inner product spaces U and V , and extend the pre-bracket to all of N , by

$$[z_1 + v_1, z_2 + v_2]_{N \times N} = [v_1, v_2]_{V \times V}.$$

It follows immediately that N is a two-step nilpotent Lie algebra with center U .

The algebra N thus constructed is called a Lie algebra of H-type.

If vectors v_1, \dots, v_n and z_1, \dots, z_m constitute orthonormal bases for V and U , then the only non-zero structure constants for the Lie algebra $N = U \oplus V$ are real numbers A_{ij}^k appearing in the equations

$$[v_i, v_j] = \sum_{k=1, \dots, n} A_{ij}^k z_k. \quad (3.3)$$

This follows from the fact that the bracket on $N \times N$ takes values in U , and the fact that the brackets $[U, V] = [U, U] = \{0\}$.

Clifford coefficients on V are defined by

$$J_{z_\alpha}(v_i) = \sum_{s=1, \dots, n} \tilde{A}_{is}^\alpha v_s. \quad (3.4)$$

It is apparent that

$$\begin{aligned}\tilde{A}_{ij}^\alpha &= \langle \sum_{s=1 \dots m} \tilde{A}_{is}^\alpha v_s, v_j \rangle = \langle J_{z_\alpha}(v_i), v_j \rangle \\ &= \langle z_\alpha, [v_i, v_j] \rangle = \langle z_\alpha, \sum_{k=1 \dots n} A_{ij}^k z_k \rangle = A_{ij}^\alpha,\end{aligned}$$

so that Clifford coefficients and H-type algebra structure constants are equal, and we will prove the rationality of the former.

It will be convenient for us to recast the given data on the spaces U and V , as well as the properties of the J -map linking them, in the language of Clifford algebras and Clifford modules. Starting from a fixed space U_k , with associated inner product and orthonormal basis e_1, \dots, e_k , a standard construction (see [10] and [18] for details) gives rise to a Clifford algebra, $C(U_k) = C_k$, with generators e_1, \dots, e_k , and satisfying the relations

$$e_i^2 = -1, \quad 1 \leq i \leq k, \quad \text{and} \quad (3.5)$$

$$e_i e_j = -e_j e_i, \quad 1 \leq i, j \leq k, \quad i \neq j. \quad (3.6)$$

U_k is naturally embedded in C_k . The properties (3.5) and (3.6) of the J -map allow it to be factored as $J = \rho \circ i$, where i is the embedding of U_k into C_k and ρ is the inclusion of the algebra generated by $\{J_z \mid z \in U\}$ in $\text{End}(V)$. The image $\rho(\alpha) = \rho(\alpha)(v) = \alpha \cdot v$ is termed Clifford multiplication. Under the action of ρ , $V = V_k$ becomes a Clifford module over the Clifford algebra C_k .

In what follows, it will be natural for us to use the language and formalism of Clifford algebras and Clifford modules. In this formulation, the J -map and the map ρ are equivalent, and verifying property (3.1) is the same as showing that Clifford multiplication by elements of U is an orthogonal multiplication. Also, verifying property (3.2) is unnecessary, given the relations (3.5) and (3.6) in the

algebra C_k . However, the main reason for preferring the second model of interaction between the spaces U and V is that it provides us with the classification theory of Clifford modules, which has been worked out completely (see [10] and [16] for details). In particular, classification tells us that for each C_k with $k \geq 1$ there is a unique irreducible Clifford module V_k over C_k , except when $k \equiv 3 \pmod{4}$, when there are two irreducible modules that are non-isomorphic. Classification also gives us the exact dimension of the irreducible Clifford module V_k over the Clifford algebra C_k . This is the classical Hurwitz formula. If we know the dimension of the irreducible module V_k , irreducibility will never be an issue in what follows. We will never need to prove irreducibility for any Clifford module, as long as it has the dimension given by the Hurwitz formula. If such a module were reducible, then it could be factored, producing (eventually) an irreducible module of dimension necessarily lower than that determined by the Hurwitz formula. This is a contradiction. A table of the dimensions of irreducible Clifford modules taken from [15] is given below for the benefit of the reader.

$\dim U_k$	$8r$	$8r+1$	$8r+2$	$8r+3$	$8r+4$	$8r+5$	$8r+6$	$8r+7$
$\dim V_k$	2^{4r}	2^{4r+1}	2^{4r+2}	2^{4r+2}	2^{4r+3}	2^{4r+3}	2^{4r+3}	2^{4r+3}

Given a Clifford algebra C_k , we will show for every $k \geq 1$ how to construct an irreducible Clifford module V_k with an induced inner product and a distinguished orthonormal basis v_1, \dots, v_m such that

$J_z v = z \cdot v$, as Clifford multiplication, is an orthogonal multiplication (3.7),

and

$J_{e_i} v_j = \pm v_l$ for some l , $1 \leq l \leq m$, that is J_{e_i} permutes the basis elements v_1, \dots, v_m of V_k up to sign. (3.8)

Properties (3.7) and (3.8) taken together are sufficient to show that $U_k \oplus V_k$ is an irreducible Lie algebra of H-type with integral structure constants.

In order to construct V_k inductively from modules V_l , with $l < k$, it is convenient to work in the category of Z_2 -graded Clifford modules. In fact, the first step in constructing V_k , is to construct a Z_2 -graded Clifford module over C_k , which we will denote W_k . W_k , too, will have an inner product and a distinguished orthonormal basis w_1, \dots, w_m of W_k^0 , the 0-component of W_k , such that

- (i) $W_k^0 \perp W_k^1$,
- (ii) $J_z w = z \cdot w$, as Clifford multiplication, is an orthogonal multiplication, and
- (iii) For $i \neq j$, $J_{e_i} \cdot J_{e_j}$ permutes the basis w_1, \dots, w_m up to sign.

If we treat W_k as an ungraded module with orthonormal basis w_1, \dots, w_m , $e_1 w_0, \dots, e_1 w_m$, then properties (ii) and (iii) imply (3.7) and (3.8) for W_k . The implication property (ii) \Rightarrow (3.7) is obvious. To show that property (iii) \Rightarrow (3.8),

we check that

$$e_i(e_1 w_j) = e_i e_1 w_j = \pm w_a, \text{ for some } a, \text{ and}$$

$$e_k w_l = -e_1 2e_k w_l$$

$$\begin{aligned}
 &= -e_1 e_1 e_k w_1 \\
 &= \pm e_1 w_2 .
 \end{aligned}$$

It turns out that in certain dimensions ($k \equiv 3 \pmod{4}$), the dimension of W_k as a real vector space is twice the dimension of V_k . We will show how to split W_k in these cases into an orthogonal direct sum $W_k = V_k \oplus V_k'$ of non-isomorphic Clifford modules that will, however, yield isomorphic H-type algebras. We will also show how to construct orthonormal bases (with respect to an inner product inherited from W_k) in V_k and V_k' by using the oriented volume element in the underlying inner product space of C_k . These bases will satisfy properties (3.7) and (3.8).

If $k \equiv 5$ or $6 \pmod{8}$, we can obtain an irreducible Clifford module by restriction of the split module V_j , $j \equiv 7 \pmod{8}$. If k is not $\equiv 3 \pmod{4}$ and not $\equiv 5$ or $6 \pmod{8}$, then classification tells us that W_k is irreducible as an ungraded module, and we may simply take $W_k = V_k$ as the unique irreducible Clifford module over C_k . By either of these three methods, we obtain for all $k \geq 1$ an irreducible Clifford module satisfying (3.7) and (3.8). Consequently, for every $k \geq 1$ there is a unique (up to isomorphism) irreducible Lie algebra of H-type, $U_k \oplus V_k$, with integral structure constants. Since an arbitrary H-type Lie algebra may be written as

$$N = U \oplus V_1 \oplus V_2 \oplus \dots \oplus V_r ,$$

with each $U \oplus V_i$ irreducible, it follows from the decomposition $J = J_1 \oplus J_2 \oplus \dots \oplus J_r$, where $J_i = J|V_i$ that every H-type algebra possesses a

basis with integral Lie structure constants. Using Mal'cev's criterion, we reach our main result which we state as follows:

Theorem 3.1 Every Lie group of H-type contains an embedded lattice.

Our construction of the Clifford modules V_k will be inductive, and to start the induction, we will refer to the work of P. Barbano [1], who has worked out a presentation of U_k and irreducible V_k for $0 \leq k \leq 7$. Not all of the V_k obtained by Barbano are Z_2 -graded, and our first concern will be to construct corresponding W_k satisfying properties (i), (ii), (iii) above. Secondly, we will need to construct W_8 as a Z_2 -graded irreducible module over C_8 . These two constructions are instances of the same general procedure and will constitute Lemma 3.2 below. With W_k , $k \leq 8$, constructed, we proceed to the induction step. Lemma 3.3 below will verify that if W_k is irreducible and has an induced inner product and basis satisfying (i), (ii), and (iii), then $W_k \widehat{\otimes} W_8$, considered as a Z_2 -graded module over $C_{k+8} \cong C_k \widehat{\otimes} C_8$, also possesses an induced inner product and basis with the same three properties, and is irreducible. This will complete the inductive construction of the Z_2 -graded Clifford modules W_k for all $k \geq 1$. Lemma 3.4, which follows, treats the splitting of W_k , when $k \equiv 3 \pmod{4}$. Finally, we will show that the non-isomorphic Clifford modules V_k and $V_{k'}$ obtained in Lemma 3.4 give rise to isomorphic H-type Lie algebras.

Before proving Lemma 3.2, we motivate it by giving some additional details concerning the results of P. Barbano [1] on irreducible Lie algebras of H-type for $0 \leq \dim U \leq 7$. These are summarized in the table below.

k	0	1	2	3	4	5	6	7
U_k	{0}	iR	H_2^*	H_3^*	O_4^*	O_5^*	O_6^*	O_7^*
V_k	R	C	H	H	O	O	O	O

H and O are the standard algebras of quaternions and octonions. H_j^* and O_k^* respectively denote purely imaginary subspaces of dimension j and k. By using canonical bases for C, H, and O and the standard inner product

$$\langle x, y \rangle = \text{Re } x \bar{y}$$

for x and y in C, H or O, it is very easy to check that properties (3.7) and (3.8) hold for $J_z(v) = zv$. Since the Clifford algebra relations are not invoked, it must be verified that $J_z^2 v = -|z|^2 \cdot v$. This is straightforward for C and H, but non-trivial for O, which is not associative. It is possible to make a proof of this fact by using the multiplication formula

$$(h_1, w_1)(h_2, w_2) = (h_2 h_1 - \bar{w}_1 w_2, w_1 h_2 + w_2 \bar{h}_1),$$

valid in $O = H \oplus H$ given by L.E. Dickson in [9].

For $z = (h_1, w_1) \in U_k = O_k^*$, $4 \leq k \leq 7$, we have $h_1^2 = -|h_1|^2$ and $w_1^2 = -|w_1|^2$.

Taking $v = (h_2, w_2)$ and using associativity in H, we obtain

$$\begin{aligned} z(zv) &= (h_1, w_1)(h_2 h_1 - \bar{w}_1 w_2, w_1 h_2 + w_2 \bar{h}_1) \\ &= \{(h_2 h_1 - \bar{w}_1 w_2) h_1 - \bar{w}_1 (w_1 h_2 + w_2 \bar{h}_1), w_1 (h_2 h_1 - \bar{w}_1 w_2) + (w_1 h_2 + w_2 \bar{h}_1) \bar{h}_1\} \\ &= \{h_2 h_1^2 - \bar{w}_1 w_1 h_2 - \bar{w}_1 w_2 h_1 - \bar{w}_1 w_2 h_1, w_1 h_2 h_1 - w_1 \bar{w}_1 w_2 + w_1 h_2 \bar{h}_1 + w_2 (\bar{h}_1)^2\} \\ &= \{h_2 (h_1^2 - |w_1|^2), w_2 (\bar{h}_1^2 - |w_1|^2)\} \\ &= -(|h_1|^2 + |w_1|^2)(h_2, w_2) \end{aligned}$$

$$= -|z|^2 v.$$

Verification that the V_k are irreducible proceeds by showing that C , H , and O contain no non-trivial submodules with respect to action by iR , H_j^* , and O_k^* , as shown by Barbano in [1].

In order to build inductively on these irreducible modules to create Clifford modules over Clifford algebras C_k for arbitrary $k \geq 1$, we will work with Z_2 -graded modules over Z_2 -graded Clifford algebras. We will see that Clifford algebras have a natural Z_2 -grading. However, not all of the modules above are Z_2 -graded with respect to their corresponding Clifford algebras. To see this clearly, we give a brief discussion concerning Z_2 -grading of algebras and modules.

We will say that an algebra A is Z_2 -graded if there is a direct sum decomposition

$$A = A^0 \oplus A^1, \text{ and}$$

$$A^i A^j \subseteq A^{i+j}, \text{ with indices taken modulo 2.}$$

For tensor algebras $A \otimes B$ with A and B Z_2 -graded, we can put a Z_2 -graded structure on $A \otimes B$ by means of the multiplication

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg(b)\deg(a')} (aa' \otimes bb'),$$

where elements b and a' are assumed to be of pure degree 0 or 1. The resulting graded tensor algebra is denoted $A \widehat{\otimes} B$, and has the direct-sum decomposition

$$(A \widehat{\otimes} B)^0 = A^0 \otimes B^0 + A^1 \otimes B^1, \text{ and}$$

$$(A \widehat{\otimes} B)^1 = A^0 \otimes B^1 + A^1 \otimes B^0,$$

where the tensor product on the right hand side is the usual tensor product

determined by

$$(a \otimes b) \cdot (a' \otimes b') = (aa' \otimes bb').$$

Similarly, we define a Z_2 -graded module over a Z_2 -graded algebra A as an A -module possessing a decomposition

$$M = M^0 \oplus M^1, \text{ and}$$

$$A^i M^j \subseteq M^{i+j}, \text{ for } i, j \in Z_2.$$

To see that the Clifford algebra C_k is Z_2 -graded, we consider the automorphism

$$\tilde{\alpha} : C_k \rightarrow C_k,$$

which extends the linear map

$$\alpha : U_k \rightarrow C_k,$$

$$u \rightarrow -i(u),$$

where i is the embedding of U_k into $C(U_k)$.

See [10] for details. Under the action of α on the Clifford algebra, we obtain a decomposition

$$C_k = C_k^0 \oplus C_k^1,$$

where C_k^0 and C_k^1 are the $+1$ and -1 eigenspaces of $\tilde{\alpha}$. It is easy to check that

$$(C_k^i)(C_k^j) \subseteq C_k^{i+j},$$

so that C_k is Z_2 -graded. Because of the defining relations of the Clifford algebra, C_k^0 is the linear span of products of even numbers of generators of C_k , and C_k^1 is the span of products of odd numbers of the generators. It follows directly from this that $\dim C_k^0 = \dim C_k^1 = 2^{k-1}$, and that C_k^0 is a subalgebra of C_k , while C_k^1 is a

submodule over the algebra C_k^0 . We will say that a Clifford module is a module V over a Clifford algebra C_k , and that W_k is a Z_2 -graded Clifford module if it is Z_2 -graded as module over C_k considered as a Z_2 -graded algebra.

Examining our modules V_k above for $k \leq 7$, we see that if $k = 1$ and $U_1 = i\mathbb{R}$, with a single generator i , then $C_1^0 = \mathbb{R}$, $C_1^1 = i\mathbb{R}$. If we decompose $C = V_1 = V_1^0 \oplus V_1^1 = \mathbb{R} \oplus i\mathbb{R}$, then it is obvious that $C = V_1$ is a Z_2 -graded Clifford module over C_1 . For $k = 2$, suppose $U_2 = H_2^* = \text{span}_{\mathbb{R}} \{j, k\}$. Checking the multiplication table for H , it follows that linear combinations of products of even numbers of j and k are spanned by 1 and i , while linear combinations of products of odd numbers of j and k are spanned by j and k . So $C_2^0 = \text{span}_{\mathbb{R}} \{1, i\}$ and $C_2^1 = \text{span}_{\mathbb{R}} \{j, k\}$. It is clear that $(C_2^i)(C_2^j) = C_2^{i+j}$. If we decompose $H = V_2$ in the same way, then it is easy to check that H is Z_2 -graded over the graded Clifford algebra C_2 . Thus $V_1 = W_1$ and $V_2 = W_2$.

Finally, for $k = 4$, we consider $U_4 = O_4^* = \text{span}_{\mathbb{R}} \{o_4, o_5, o_6, o_7\}$. In the Clifford algebra C_4 , these elements become generators e_1, e_2, e_3, e_4 , which act on the Clifford module O by octonion multiplication. O itself is Z_2 -graded with decomposition $O = V_4^0 \oplus V_4^1 = \text{span}_{\mathbb{R}} \{1, o_1, o_2, o_3\} \oplus \text{span}_{\mathbb{R}} \{o_4, o_5, o_6, o_7\}$. To verify that $O = V_4$ is a Z_2 -graded Clifford module over C_4 , we need to check the inclusion relation

$$(C_4^i)(C_4^j) \subseteq C_4^{i+j}$$

Since C_4^i , $i = 0, 1$, are determined by products of an even number of generators and by products of an odd number of generators respectively, we need only

check that single products e_i acting on the basis $\{1, o_1, o_2, o_3\}$ take it to $\pm \{o_4, o_5, o_6, o_7\}$, and that double products $e_i e_k$ take $\{1, o_1, o_2, o_3\}$ to $\pm \{1, o_1, o_2, o_3\}$. But this is obvious from a multiplication table for octonions.

Remark: It can be shown that C_4 itself is isomorphic to the 16-dimensional space of 2×2 matrices with entries in the quaternion algebra H . See [10] for details.

Lemma 3.2 If V is an ungraded module over the algebra C_n^0 , then $W = C_n \otimes_{C_n^0} V$ is a Z_2 -graded module over C_n of twice the dimension of V . The Z_2 -grading is given by $W^0 = C_n^0 \otimes_{C_n^0} V = \{1 \otimes v \mid v \in V\}$ and $W^1 = C_n^1 \otimes_{C_n^0} V = \{e_i \otimes v \mid v \in V\}$.

Furthermore:

(i) If V is equipped with an inner product, then W^0 and W^1 have a natural inner product, and we define W^0 and W^1 to be orthogonal.;

(ii) If $\langle ze_n v, ze_n v \rangle = |z|^2 \|v\|^2$ for all $z \in U$, $v \in V$, then the Clifford multiplication $J_z w = z \cdot w$ is orthogonal; and

(iii) If V is equipped with an orthonormal basis v_1, \dots, v_m such that multiplication by products $e_i e_j$ of generators of C_n permute the v_i up to sign, then similar multiplication by $e_i e_j$ permute the basis elements $1 \otimes v_1, \dots, 1 \otimes v_m$, for W^0 up to sign.

Proof: It is obvious that $C_n \otimes_{C_n^0} V$ is a module over C_n . Now we decompose $C_n \otimes_{C_n^0} V$ as $C_n^0 \otimes_{C_n^0} V \oplus C_n^1 \otimes_{C_n^0} V$. By transposing even products of generators from left to right and using the fact that V is a module over C_n^0 , we can express any element of $C_n^0 \otimes_{C_n^0} V$ as $1 \otimes v$, for some $v \in V$. Carrying out

the same procedure on $C_n^1 \otimes_{C_n^0} V$, we obtain elements of the form $e_k \otimes v$ ($k < n$).

If in addition, we multiply by $1 = -e_n^2$, use $e_n e_k = -e_k e_n$, and transpose, we see that any element of $C_n^1 \otimes_{C_n^0} V$ can be expressed as $e_n \otimes v$ for some $v \in V$. If $k = n$

above, then this conclusion is obvious. Generators of C_n act on the module by

left multiplication. We check that for $1 \otimes v_1 \in C_n^0 \otimes_{C_n^0} V$

$$\begin{aligned} e_k(1 \otimes v_1) &= e_k \otimes v_1, \quad k \neq n \\ &= -e_n^2 e_k \otimes v_1 \\ &= -e_n \otimes e_n e_k v_1 \\ &= e_n \otimes v_2 \in C_n^1 \otimes_{C_n^0} V, \end{aligned}$$

and $e_n(1 \otimes v_1) = e_n \otimes v_1 \in C_n^1 \otimes_{C_n^0} V$.

Similarly, for $e_n \otimes v_1 \in C_n^1 \otimes_{C_n^0} V$

$$\begin{aligned} e_k(e_n \otimes v_2) &= e_k e_n \otimes v_2, \quad k \neq n \\ &= -e_n e_k \otimes v_2 \\ &= -1 \otimes e_n e_k v_2 \\ &= 1 \otimes v_2 \in C_n^0 \otimes_{C_n^0} V, \end{aligned}$$

and $e_n(e_n \otimes v_1) = -1 \otimes v_1 \in C_n^0 \otimes_{C_n^0} V$. We conclude that $C_n \otimes_{C_n^0} V$ is a Z_2 -graded module over C_n .

(i) It is clear that $W^0 = \{1 \otimes v \mid v \in V\} \cong V$ and $W^1 = \{e_n \otimes v \mid v \in V\} \cong V$.

We have $W^0 + W^1 = \{1 \otimes v + e_n \otimes w \mid (v, w) \in V^2\}$. Without loss of generality, we may consider $W^0 + W^1$ as $W^0 \oplus W^1$, a direct sum, and define $W^0 \perp W^1$.

(ii) Before computing $\|J_z w\|^2$, we observe that by polarization, and by the hypothesis in (ii)

$$\langle z_1 e_n v, z_2 e_n v \rangle = \langle z_1, z_2 \rangle \|v\|^2.$$

In particular, we have

$$\langle e_i e_n v, e_j e_n v \rangle = \delta_{ij} \|v\|^2,$$

so that by putting $j = n$,

$$\langle e_i e_n v, v \rangle = 0, \quad i < n.$$

We now proceed to calculate

$$\begin{aligned} \|J_z w\|^2 &= \|J_z (1 \otimes v_1) + J_z (e_n \otimes v_2)\|^2 \\ &= \|z \otimes v_1 + z e_n \otimes v_2\|^2 \\ &= \|z \otimes v_1\|^2 + \|z e_n \otimes v_2\|^2, \text{ since} \end{aligned}$$

$z \otimes v_1 \in W^1$ and $z e_n \otimes v_2 \in W^0$ (recall that z is a linear combination of the e_i , and transpose double products to the right, using $e_n^2 = -1$ as necessary), and $W^1 \perp W^0$.

We then decompose z as $z = z' + \alpha e_n$, where $z' \in \text{span}\{e_1, \dots, e_{n-1}\}$. We now have

$$\|z' \otimes v_1 + \alpha e_n \otimes v_1\|^2 + \|z' e_n \otimes v_2 + \alpha e_n \otimes v_2\|^2.$$

By writing $z' = -e_n^2 z' = e_n z' e_n$, so that $z' \otimes v_1 = e_n z' e_n \otimes v_1 = e_n \otimes z' e_n v_1$ (by transposition of products $e_i e_n$, for $i < n$), and using the relation $\langle e_i e_n v, v \rangle = 0$ for $i < n$, we conclude that $z' \otimes v_1 \perp e_n \otimes v_1$, so that

$$\begin{aligned} \|z' \otimes v_1 + \alpha e_n \otimes v_1\|^2 &= \|z' e_n v_1\|^2 + \alpha^2 \|v_1\|^2 \\ &= |z'|^2 \|v_1\|^2 + \alpha^2 \|v_1\|^2, \text{ by hypothesis,} \\ &= (|z'|^2 + \alpha^2) \|v_1\|^2 \end{aligned}$$

$$= |z|^2 ||v_1||^2$$

For $||z'e_n \otimes v_2 + \alpha e_n \otimes v_2||^2$, it is easy to see, by transposing terms $e_i e_n$ for $i < n$, that $z'e_n \otimes v_2 \in W^0$ and $\alpha e_n \otimes v_2 \in W^1$. Consequently

$$\begin{aligned} ||z'e_n \otimes v_2 + \alpha e_n \otimes v_2||^2 &= ||z' e_n v_2||^2 + \alpha^2 ||v_2||^2 \\ &= |z'|^2 ||v_2||^2 + \alpha^2 ||v_2||^2, \text{ by hypothesis,} \\ &= (|z'|^2 + \alpha^2) ||v_2||^2 \\ &= |z|^2 ||v_2||^2. \end{aligned}$$

We conclude that $||J_2 w|| = |z| ||w||$, and we are done.

(iii) To verify that double products permute basis elements of the form $1 \otimes v_i$ up to sign, we check first that for $i, j < n$

$$\begin{aligned} e_i e_j (1 \otimes v_i) &= (1 \otimes e_i e_j v_i), \text{ since } e_i e_j \text{ is even,} \\ &= \pm (1 \otimes v_j), \end{aligned}$$

using the assumption that the double products $e_i e_j$, $i \neq j$ act on v_k by permuting them up to sign. Similarly for $i = n$, for $j < n$,

$$\begin{aligned} e_n e_j (1 \otimes v_i) &= (1 \otimes e_n e_j v_i), \\ &= \pm (1 \otimes v_j). \end{aligned}$$

We will apply Lemma 3.2 for two different purposes. Our first aim is to complete the construction of Z_2 -graded modules W_k , $k \leq 7$. For $k = 3$, $V = V_3 = H$, and for $k = 7$, $V = V_7 = O$. Since H is a Clifford module over C_3 , it is also a module over C_3^0 , so that by Lemma 3.2, there is a distinguished basis with respect to which $C_3 \otimes_{C_3^0} H$ is a Z_2 -graded module over C_3 . Similarly, O is a Clifford module

over C_7 , hence a module over C_7^0 , and it follows, again by Lemma 3.2, that there is a distinguished basis with respect to which $C_7 \otimes_{C_7^0} H$ is a Z_2 -graded module over C_7 .

The second use of Lemma 3.2 is to construct an irreducible Z_2 -graded Clifford module over C_8 . To do this, we identify C_7 with C_8^0 by means of the algebra isomorphism:

$$\begin{aligned} \Psi : C_7 &\cong C_8^0 \\ x^0 + x^1 &\rightarrow x^0 + x_1 e_8 . \end{aligned}$$

We conclude that once again, $W_8 = C_8 \otimes_{C_8^0} O$ possesses a basis with respect to which it is Z_2 -graded. It is then irreducible because it has the dimension of the unique (up to isomorphism) irreducible Clifford module. Thus $V_8 = W_8$. The Clifford modules constructed in this manner also satisfy properties (i), (ii), and (iii). Remark: In our three applications of Lemma 3.2, it is easy to verify the assumption in (i) that $\langle ze_n v, ze_n v \rangle = |z|^2 \|v\|^2$ for all $z \in U$, $v \in V$. One has only to examine the standard inner products on H and O .

We have now constructed irreducible Z_2 -graded modules for $k = 1$, $k = 2$, $k = 4$, and $k = 8$. For $k = 3$ and $k = 7$ we have constructed Z_2 -graded modules W_k of dimension (as a real vector space) twice that of V_k . In Lemma 3.4, we will show how these W_k may be split to yield ungraded irreducible modules. For $k = 5$ and $k = 6$, we begin with the graded module over C_7 of real dimension 16, and split it, using Lemma 3.4, to obtain V_7 , an ungraded irreducible module of dimension 8. C_7 acts on V_7 with generators $\{e_1, \dots, e_7\}$. To obtain V_6 and V_5 , we

have only to restrict the action to the subalgebras $C_5, C_6 \subset C_7$. It is obvious that Clifford coefficients for V_5 and V_6 are then subsets of those for V_7 and integrality is assured once we have proved it for V_7 . The initial stage of our inductive argument ($k \leq 8$) is complete.

Lemma 3.3 If W_k and W_l are Z_2 -graded Clifford modules over the algebras C_k and C_l with $\{v_1, \dots, v_n\}$ and $\{e_1 v_1, \dots, e_l v_n\}$ orthonormal bases for W_k^0 and W_k^1 , and $\{w_1, \dots, w_m\}$ and $\{f_1 w_1, \dots, f_l w_m\}$ orthonormal bases for W_l^0 and W_l^1 , then $W_k \widehat{\otimes} W_l$ is a Z_2 -graded module over $C_k \widehat{\otimes} C_l \cong C_l$ with $\{v_i \otimes w_j\} \cup \{e_i v_k \otimes f_l w_j\}$ a basis for $(W_k \widehat{\otimes} W_l)^0$ and $\{v_r \otimes f_l w_s\} \cup \{e_1 v_1 \otimes w_d\}$ a basis for $(W_k \widehat{\otimes} W_l)^1$. Furthermore:

- (i) $W_k^0 \perp W_k^1$ and $W_l^0 \perp W_l^1 \Rightarrow (W_k \widehat{\otimes} W_l)^0 \perp (W_k \widehat{\otimes} W_l)^1$;
- (ii) If C_k acts on W_k by orthogonal multiplication and C_l acts on W_l by orthogonal multiplication, then $C_k \widehat{\otimes} C_l$ acts by orthogonal multiplication on $W_k \widehat{\otimes} W_l$; and
- (iii) If double products of generators $e_m e_n$ for C_k and $f_p f_q$ for C_l permute the base of W_k^0 and W_l^0 up to sign, then double products of generators of $C_k \widehat{\otimes} C_l$, namely $\{e_i \otimes 1, \dots, e_k \otimes 1, 1 \otimes f_1, \dots, 1 \otimes f_l\}$ permute basis elements of $(W_k \widehat{\otimes} W_l)^0$ up to sign.

Proof. (i) Follows easily from the decomposition $(W_k \widehat{\otimes} W_l)^0 = (W_k^0 \otimes W_l^0) + (W_k^1 \otimes W_l^1)$ and $(W_k \widehat{\otimes} W_l)^1 = (W_k^0 \otimes W_l^1) + (W_k^1 \otimes W_l^0)$.

(ii) To verify orthogonal multiplication, it will be more convenient to use a property equivalent to it, given that we are working in the category of Clifford algebras where the property that $J_u^2 v = -|u|^2 v$ is assumed. We show below

that $\|J_u v\| = |u| \|v\|$ is equivalent to $J_u^* v = -J_u v$, where J_u^* is the adjoint operator to J_u . Indeed, if the latter property is true, then

$$\begin{aligned} \|J_u v\|^2 &= \langle J_u v, J_u v \rangle \\ J_u v &= \langle v, J_u^* (J_u v) \rangle \\ &= \langle v, -J_u^2 v \rangle \\ &= \langle v, |u|^2 v \rangle = \|v\|^2 |u|^2 . \end{aligned}$$

On the other hand, if $\|J_u v\|^2 = |u|^2 \|v\|^2$, then

$$\begin{aligned} \langle v, J_u^* J_u v \rangle &= \langle J_u v, J_u v \rangle = |u|^2 \langle v, v \rangle \\ &= \langle v, |u|^2 v \rangle \\ &= \langle v, -J_u^2 v \rangle . \end{aligned}$$

Since this is true for all v , we have $J_u^* J_u = -J_u^2$. But J_u is a non-singular, invertible operator, and we may conclude that $J_u^* = -J_u$.

Now, if W_k and W_l are inner product spaces, there is an induced inner product defined on $W_k \widehat{\otimes} W_l$ defined by

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle_{W_k \otimes W_l} = \langle v_1, w_1 \rangle_{W_k} \cdot \langle v_2, w_2 \rangle_{W_l} .$$

We define J acting on elements $v_1 \otimes v_2$ over span $\{e_i \otimes 1, 1 \otimes f_j\}$, e_i and f_j being the generators of C_k and C_l . In particular

$$\begin{aligned} J_{e_i \otimes 1} (v_1 \otimes v_2) &= e_i v_1 \otimes v_2 = J_{e_i} v_1 \otimes v_2, \text{ and} \\ J_{1 \otimes f_j} (v_1 \otimes v_2) &= (-1)^{\deg v_1} v_1 \otimes f_j v_2 \\ &= (-1)^{\deg v_1} v_1 \otimes J_{f_j} v_2 , \end{aligned}$$

using the Z_2 -graded tensor product.

Recalling that the Clifford algebra $C_{k+1} \cong C_k \hat{\otimes} C_1$ is generated by $\{e_i \otimes 1, 1 \otimes f_j\}$, we can express an arbitrary element in $U_{k+1} = \text{span}_R\{e_i \otimes 1, 1 \otimes f_j\}$ as

$$z = z_1 \otimes 1 + 1 \otimes z_2 ,$$

where $z_1 = \sum_i \alpha_i e_i \in C_k$, and $z_2 = \sum_j \beta_j f_j \in C_1$.

We need to check that $J_z^* = -J_z$, and since $J_z = J_{z_1 \otimes 1} + J_{1 \otimes z_2}$ and the operation of taking the adjoint is linear, it is enough to check that $J_{z_1 \otimes 1}^* = -J_{z_1 \otimes 1}$ and that $J_{1 \otimes z_2}^* = -J_{1 \otimes z_2}$. The first equality follows from the simple calculation

$$\begin{aligned} \langle J_{z_1 \otimes 1}^* (v_1 \otimes v_2), (w_1 \otimes w_2) \rangle &= \langle v_1 \otimes v_2, J_{z_1 \otimes 1} (w_1 \otimes w_2) \rangle \\ &= \langle v_1 \otimes v_2, z_1 w_1 \otimes w_2 \rangle \\ &= \langle v_1, z_1 w_1 \rangle \langle v_2, w_2 \rangle \\ &= \langle J_{z_1}^* v, w_1 \rangle \langle v_2, w_2 \rangle \\ &= -\langle J_{z_1} v_1 \otimes v_2, w_1 \otimes w_2 \rangle, \text{ using the induction hypothesis,} \\ &= \langle -J_{z_1 \otimes 1} (v_1 \otimes v_2), w_1 \otimes w_2 \rangle, \end{aligned}$$

which implies that $J_{z_1 \otimes 1}^* = -J_{z_1 \otimes 1}$, since $w_1 \otimes w_2$ was arbitrary.

To prove the second equality, we assume without loss of generality that elements v_1, v_2, w_1 , and w_2 are of pure degree. We then follow the previous calculation:

$$\begin{aligned} \langle J_{1 \otimes z_2}^* (v_1 \otimes v_2), w_1 \otimes w_2 \rangle &= \langle v_1 \otimes v_2, J_{1 \otimes z_2} (w_1 \otimes w_2) \rangle \\ &= (-1)^{(\text{deg } z_2)(\text{deg } w_1)} \langle v_1 \otimes v_2, w_1 \otimes z_2 w_2 \rangle \\ &= (-1)^{(\text{deg } w_1)} \langle v_1, w_1 \rangle \langle v_2, z_2 w_2 \rangle, \text{ since } \text{deg } z_2 = 1, \end{aligned}$$

$$\begin{aligned}
&= (-1)^{(\deg w_1)} \langle v_1, w_1 \rangle \langle v_2, J_{z_2} w_2 \rangle \\
&= (-1)^{(\deg w_1)} \langle v_1, w_1 \rangle \langle J_{z_2}^* v_2, w_2 \rangle \\
&= (-1)^{(\deg w_1)} \langle v_1, w_1 \rangle \langle -J_{z_2} v_2, w_2 \rangle, \text{ by assumption,} \\
&= (-1)^{(\deg w_1)} \langle -v_1 \otimes J_{z_2} v_2, w_1 \otimes w_2 \rangle \\
&= (-1)^{(\deg w_1 + \deg v_1)} \langle -J_{1 \otimes z_2} (v_1 \otimes v_2), w_1 \otimes w_2 \rangle,
\end{aligned}$$

from which we conclude that $J_{1 \otimes z_2}^* = -J_{1 \otimes z_2}$. We note that as long as v_1 and w_1 have the same degree, $(-1)^{\deg w_1 + \deg v_1} = 1$. If v_1 and w_1 have unequal degrees, then v_1 and w_1 are orthogonal, and both sides of the equality above vanish.

(iii) To verify that double products of generators of $C_k \widehat{\otimes} C_l$ permute basis elements of $(W_k \widehat{\otimes} W_l)^0$, we need to check Clifford multiplication of $(v_i \otimes w_j)$ and $(e_i v_k \otimes f_l w_l)$ by double products of 3 types, namely $e_m e_n \otimes 1$, $1 \otimes f_p f_q$, and $e_n \otimes f_q$.

$$(1) (e_m e_n \otimes 1) (v_i \otimes w_j) = e_m e_n v_i \otimes w_j = \pm v_n \otimes w_j, \text{ by hypothesis.}$$

$$(2) (1 \otimes f_p f_q) (v_i \otimes w_j) = v_i \otimes f_p f_q w_j = \pm v_i \otimes w_a, \text{ for some } a.$$

$$(3) (e_n \otimes f_q) (v_i \otimes w_j) = e_n v_i \otimes f_q w_j$$

$$= -e_i^2 e_n v_i \otimes -f_q^2 f_q w_j$$

$$= e_i (e_i e_n v_i) \otimes f_q (f_q f_q w_j)$$

$$= \pm e_i v_n \otimes f_q w_a.$$

$$(4) (e_m e_n \otimes 1) (e_i v_k \otimes f_l w_l) = e_m e_n e_i v_k \otimes f_l w_l$$

$$= e_i e_m e_n v_k \otimes f_l w_l, \text{ by two transpositions of } e_i,$$

$$= \pm e_i v_n \otimes f_l w_l.$$

$$\begin{aligned}
(5) \quad (1 \otimes f_p f_q) (e_1 v_k \otimes f_1 w_l) &= e_1 v_k \otimes f_p f_q f_1 w_l \\
&= e_1 v_k \otimes f_1 f_p f_q w_l \\
&= \pm e_1 v_k \otimes f_1 w_s, \text{ as in (iv)}.
\end{aligned}$$

$$\begin{aligned}
(6) \quad (e_n \otimes f_q) (e_1 v_k \otimes f_1 w_l) &= e_n e_1 v_k \otimes f_q f_1 w_l \\
&= \pm v_n \otimes w_s \text{ by assumption,}
\end{aligned}$$

and the permutation of bases property is proved. The proof of Lemma 3.3 is finished.

Corollary: $W_{k \leftrightarrow s} = W_k \widehat{\otimes} W_s$ is an irreducible Z_2 -graded Clifford module.

Proof: Classification, assuming that W_k was irreducible.

Suppose W_k is a Z_2 -graded module over C_k , where $k \equiv 3 \pmod{4}$, that is, $k \equiv 3$ or $7 \pmod{8}$. Let $\omega = e_1 e_2 \dots e_k$ be the oriented volume element in the underlying inner product space U . Recalling the relations $e_i e_j = -e_j e_i$ ($i \neq j$) and $e_i^2 = -1$ for generators of the Clifford algebra, it follows easily that

$$(i) \quad \omega^2 = 1,$$

$$(ii) \quad v\omega = \omega v, \text{ for all } v \in C_k \text{ (} \omega \text{ is central in } C_k \text{), and}$$

(iii) multiplication by ω , as an automorphism of W_k , is self-adjoint.

We then have the following

Lemma 3.4 There is a direct sum decomposition

$$W_k = W_k^+ \oplus W_k^- = V_k \oplus V_k',$$

where W_k^+ and W_k^- are the +1 and -1 eigenspaces for the automorphism multiplication by ω . W_k^+ and W_k^- are non-isomorphic ungraded submodules of W_k .

Proof We define two maps on W_k :

$$\phi_1(w) = ((1 + \omega)/\sqrt{2}) w, \text{ and}$$

$$\phi_2(w) = ((1 - \omega)/\sqrt{2}) w .$$

We check first that $\text{Im}\phi_1 = W_k^+$, and $\text{Im}\phi_2 = W_k^-$.

First suppose that $w \in \text{Im}\phi_1$. Then there is a $z \in W_k$ such that

$$((1 + \omega)/\sqrt{2}) z = w$$

$$\omega ((1 + \omega)/\sqrt{2}) z = \omega w$$

$$((\omega + \omega^2)/\sqrt{2}) z = \omega w$$

$$((\omega + 1)/\sqrt{2}) z = \omega w$$

$$w = \omega w ,$$

so $w \in W_k^+$. Next suppose that $w \in W_k^-$, so that

$$\omega w = w$$

$$(\omega + 1) w = 2w$$

$$((\omega + 1)/\sqrt{2}) (w/\sqrt{2}) = w ,$$

so there is a $z \in W_k$ (namely $w/\sqrt{2}$) such that $\phi_1(z) = w$. Hence $w \in \text{Im}\phi_1$. The proof for ϕ_2 is precisely the same.

To show that ϕ_1 and ϕ_2 are module homomorphisms, we must verify that for ϕ_1 , say

$$\phi_1(v w) = v \phi_1(w) , \text{ for all } v \in C_k , w \in W_k .$$

The left hand side is given by

$$\begin{aligned}
 & ((1 + \omega)/\sqrt{2}) (v w) \\
 &= (v w + \omega v w) / \sqrt{2} \\
 &= (v w + v \omega w) / \sqrt{2}, \text{ since } \omega \text{ is central,} \\
 &= v (w + \omega w) / \sqrt{2} \\
 &= v ((1 + \omega)/\sqrt{2})(w) = v \phi_1(w), \text{ the right hand side .}
 \end{aligned}$$

The proof for ϕ_2 is the same. From these facts it follows that W_k^+ and W_k^- are submodules of W_k . Since

$$((1 + \omega)/\sqrt{2}) ((1 - \omega)/\sqrt{2}) = ((1 - \omega)/\sqrt{2}) ((1 + \omega)/\sqrt{2}) = 0,$$

we have $\ker \phi_1 = W_k^-$ and $\ker \phi_2 = W_k^+$. Since

$$((1 + \omega)/\sqrt{2})^2 = 1 + \omega, \text{ and } ((1 - \omega)/\sqrt{2})^2 = 1 - \omega,$$

we conclude that $\phi_1^2 (W_k^+) = W_k^+$ and $\phi_2^2 (W_k^-) = W_k^-$. W_k^+ and W_k^- are then proper submodules and the decomposition is completed with the observation that

$$((1 + \omega)/2) + ((1 - \omega)/2) = 1.$$

That W_k^+ and W_k^- are non-isomorphic Clifford modules follows from the fact that any isomorphism ϕ from the first to the second must satisfy

$$\phi(\omega w) = \omega \phi(w) = -\omega \phi(w),$$

which is impossible. W_k^+ and W_k^- are, however, isomorphic as vector spaces, and this implies that each as a submodule has half the dimension of W_k .

Corollary: If the vectors w_1, \dots, w_n are an orthonormal basis for the \mathbb{Z}_2 -graded module W_k such that Clifford coefficients are integral then the vectors

$((1 + \omega)/\sqrt{2}) w_1, \dots, ((1 + \omega)/\sqrt{2}) w_n$ are again orthonormal and constitute a basis for the (ungraded) submodule W_k^+ , such that Clifford coefficients calculated with respect to this new basis are again integral.

Proof: That the given vectors are orthonormal and span W_k^+ follows directly from the decomposition given above. We check the integrality of the new Clifford coefficients

$$\begin{aligned} & \langle J_{e_i} ((1+\omega)/\sqrt{2}) w_k, ((1+\omega)/\sqrt{2}) w_l \rangle, e_i \text{ a generator of } C_k \\ &= \frac{1}{2} \langle e_i w_k + e_i \omega w_k, w_l + \omega w_l \rangle. \end{aligned}$$

Observe that $e_i \omega w_k$ and w_l are elements of W_k^0 , while $e_i w_k$ and ωw_l belong to W_k^1 .

The above inner product simplifies then to

$$\begin{aligned} & \frac{1}{2} \{ \langle e_i w_k, \omega w_l \rangle + \langle e_i \omega w_k, w_l \rangle \} \\ &= \frac{1}{2} \cdot 2 \langle e_i w_k, \omega w_l \rangle, \text{ since multiplication by } \omega \text{ is self-adjoint.} \end{aligned}$$

Now ωw_l can be expressed as e_i times pairs $e_j e_k$, which permute basis elements of w_k . So $\langle e_i w_k, e_i w_m \rangle = \langle w_k, w_m \rangle = \delta_{k,m}$ by orthogonal multiplication, and the proof is complete.

We now show how to continue the construction inductively, using Lemma 3.3, to obtain irreducible Clifford modules V_k , for all $k \geq 1$. Irreducibility is automatic because the dimensions of the W_k agree with those given by classification. First of all

$$\begin{aligned} W_{8k} &= W_{8(k-1)} \widehat{\otimes} W_8 = V_{8k}, \\ W_{8k+1} &= W_{8(k-1)+1} \widehat{\otimes} W_8 = V_{8k+1}, \end{aligned}$$

$$W_{8k+2} = W_{8(k-1)+2} \widehat{\otimes} W_8 = V_{8k+2}, \text{ and}$$

$$W_{8k+4} = W_{8(k-1)+4} \widehat{\otimes} W_8 = V_{8k+4},$$

and bases for these modules have the appropriate properties stipulated in Lemma 3.3. Similarly,

$$W_{8k+3} = W_{8(k-1)+3} \widehat{\otimes} W_8, \text{ and}$$

$$W_{8k+7} = W_{8(k-1)+7} \widehat{\otimes} W_8,$$

and again bases have appropriate properties. Then using Lemma 3.4, we split these Z_2 -graded modules to obtain W_{8k+3}^* and W_{8k+3} , and similarly W_{8k+7}^* and W_{8k+7} . We choose $W_{8k+3}^* = V_{8k+3}$ and $W_{8k+7}^* = V_{8k+7}$. Of course, we could equally well have chosen the other submodule.

Finally, V_{8k+5} and V_{8k+6} are equal to V_{8k+7} as vector spaces, with Clifford multiplication obtained from the embeddings $C_{8k+5} \subset C_{8k+7}$ and $C_{8k+6} \subset C_{8k+7}$.

As a last observation, we note the following. Had we calculated Clifford coefficients for $\phi_2 = (1 - \omega)/\sqrt{2}$, instead of ϕ_1 , we would have found that

$$\langle J_{e_i} ((1+\omega)/\sqrt{2})w_k, ((1+\omega)/\sqrt{2})w_l \rangle = - \langle J_{e_i} ((1-\omega)/\sqrt{2})w_k, ((1-\omega)/\sqrt{2})w_l \rangle.$$

Recalling that two H type Lie algebras are isomorphic if and only if there exist isometries Ψ and Φ (Φ not necessarily the identity map), such that

$$\Psi (J_v v) = J_{\Phi(v)} \Psi (v),$$

we see that if we take $\Phi = -1$, the Clifford coefficients, which are equal to Lie structure constants are identical for W_k^* and W_k . Since we have shown before

that W_k^+ and W_k^- are not isomorphic as modules, we have an example where non-isomorphic Clifford modules yield the same H type Lie algebra.

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