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LARGE DEVIATIONS OF LOCAL TIMES OF LÉVY PROCESSES

by

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ABSTRACT

LARGE DEVIATIONS OF LOCAL TIMES OF LÉVY PROCESSES

by

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Consider a real valued symmetric Lévy process, where the exponent ψ of the characteristic function defining the process is regularly varying at infinity with index $1 < \beta \leq 2$. Denote the local time of the Lévy process by L_t^x and define the maximum local time by $L_t^* = \sup_{x \in R} L_t^x$. Results are given, for fixed t, which show the limiting behavior of $\log P(L_t^0 \geq y)$ and $\log P(L_t^* \geq y)$, as y approaches infinity. The estimates are given in terms of ψ.

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1 Introduction

Let $X = \{X(t), t \in R^+\}$ be a real valued symmetric Lévy process, where by symmetry we mean that X and $-X$ have the same distribution. A Lévy process is a stochastic process that has stationary independent increments. The characteristic function of X(t) is

$$E\left(e^{i\lambda X(t)}\right) = \exp(-t\psi(\lambda)). \tag{1}$$

We include Brownian motion in which case $\psi(\lambda) = \lambda^2/2$ and also where ψ has general form

$$\psi(\lambda) = 2\int_0^\infty (1 - \cos\lambda u)\ \nu(du) \tag{2}$$

and ν is a Lévy measure, i.e.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\,\nu(dx) &< \infty \\ \nu(\{0\}) &= 0.\end{aligned}$$

Let $p_t(x)$ be the transition probability density for the Lévy process. By the symmetry

Let $p_t(x)$ be the transition probability density for the Lévy process. By the symmetry

$$p_t(x,y) = p_t(|x-y|)$$

and we write

$$p_t(0,x) = p_t(x).$$

We assume that

$$\int_0^\infty \frac{d\lambda}{\alpha + \psi(\lambda)} < \infty \quad \alpha \geq 0. \tag{3}$$

Define the α-potential density as

$$u^\alpha(x) = \int_0^\infty e^{-\alpha t} p_t(x) dt. \tag{4}$$

For symmetric Lévy processes

$$u^\alpha(x,y) = u^\alpha(|x-y|)$$

and we write

$$u^\alpha(0,x) = u^\alpha(x).$$

By the symmetry

$$u^\alpha(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos \lambda x}{\alpha + \psi(\lambda)} d\lambda. \tag{5}$$

Define

$$\kappa(\alpha) = u^\alpha(0) = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\alpha + \psi(\lambda)}. \tag{6}$$

Note that $\kappa(\alpha) \to 0$ as $\alpha \to \infty$.

A positive measurable function $\psi(x)$ is said to be regularly varying at infinity if

$$\lim_{x\to\infty} \frac{\psi(tx)}{\psi(x)} = t^\beta \tag{7}$$

for every $t > 0$. The number β in (7) is called the index of regular variation of ψ. By the definition, it can be seen that $\psi(x)$ is of the form

$$\psi(x) = x^{\beta} S(x) \tag{8}$$

where $S(x)$ is slowly varying. A positive measurable function varies slowly at infinity if for any $t > 0$

$$\lim_{x \to \infty} \frac{S(tx)}{S(x)} = 1. \tag{9}$$

Throughout the paper ψ is assumed to be regularly varying at infinity with index

$$1 < \beta \leq 2.$$

Note that this means that (3) is satisfied.

The regular variation of ψ at infinity will force κ to be regularly varying at infinity with index $-1/\bar{\beta}$ as will be seen in Lemma 3.4, where

$$1/\beta + 1/\bar{\beta} = 1. \tag{10}$$

Then we have, for all $T > 0$,

$$\int_T^{\infty} \frac{\kappa(y)}{y} dy < \infty. \tag{11}$$

This is because

$$\kappa(x) = x^{-1/\bar{\beta}} S(x)$$

where $S(x)$ is slowly varying at infinity as described in (8) and by [BGT] pp 23 for any fixed $\epsilon > 0$ there exists an N such that for all $x > N$

$$S(x) < x^{\epsilon}. \tag{12}$$

Let $\{L_t^x, (t,x) \in R^+ \times R\}$ denote the local time of $X(t)$ defined as

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t I_{(x-\epsilon, x+\epsilon)}(X(s))ds \tag{13}$$

where $I_A(x)$ is the indicator function of the set A. This definition is valid for all continuous local times. In the next section, it will be seen that the local times we are considering are continuous. By (3) we have that

$$u^\alpha(x) < \infty \qquad\qquad \forall x \in R$$

where the α-potential density is as defined in (4). This insures that the local time of X exists; see Kesten 1969 [K]. L_t^x can be normalized so that it is the occupation density of $X(t)$. i.e. for all measureable sets $B \subset R$

$$\int_0^t I_B(X(s))\, ds = \int_B L_t^x dx.$$

A Lévy process $Y(t)$ is called a symmetric stable process of index $0 < \beta \leq 2$ if it has a characteristic function of the form

$$E\exp(i\lambda Y(t)) = \exp(-t|\lambda|^\beta) \qquad -\infty < \lambda < \infty. \tag{14}$$

Note that this is a special case of (8) with ψ being a pure power; $\psi(\lambda) = |\lambda|^\beta$.

If $1 < \beta \leq 2$ then the local time of $Y(t)$ exists and there is a version of the local time which is jointly continuous in x and t, (see Boylan 1964 [B] or Barlow 1985 [Ba]). It will be shown in the next section that this joint continuity is also the case for our process defined by (1), (2), and (8).

Define the maximum local time as

$$L_t^* = \sup_{x \in R} L_t^x. \tag{15}$$

The main result of this paper is to obtain estimates for $P(L_t^0 \geq y)$ and $P(L_t^* \geq y)$ as y $\to \infty$ for fixed t.

The first result concerns the local time at zero. $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ for $t > 0$ is the usual gamma function.

Theorem 1.1 *Let $X = \{X(t), t \in [0,\infty)\}$ be a symmetric Lévy process with exponent ψ, given in (1). Assume that ψ is regularly varying at infinity with index $1 < \beta \leq 2$. Let L_t^0 be the local time of X at zero at time t. Then for all $t > 0$*

$$\lim_{y\to\infty} \frac{\log P\left(\frac{L_t^0}{t^{1/\beta}} > y\right)}{\psi(y)} = -\frac{\bar{\beta}-1}{(K\bar{\beta})^{\beta}} \tag{16}$$

where

$$K = \Gamma(1/\bar{\beta})\Gamma(1+1/\beta)/\pi. \tag{17}$$

The next result shows that the rates of decay of $P(L_t^0 \geq y)$ and $P(L_t^* \geq y)$ have the same constant in the exponent.

Theorem 1.2 *Let $X = \{X(t), t \in [0,\infty)\}$ be a symmetric Lévy process with exponent ψ, given in (1). Assume that ψ is regularly varying at infinity with* *index* $1 < \beta \le 2$. *Let* L_t^* *be the maximum local time of* X *at time* t. *Then for all* $t > 0$

$$\lim_{y\to\infty} \frac{\log P\left(\frac{L_t^*}{t^{1/\beta}} > y\right)}{\psi(y)} = -\frac{\bar\beta - 1}{(K\bar\beta)^{\beta}} \tag{18}$$

where K *is given in (17).*

For stable processes these results are known. Hawkes [H], proved Theorem 1.1 in 1971 and Lacey [L], proved Theorem 1.2 in 1990.

The following result of Hawkes and Lacey for stable processes is in (1.3) of [L]. Letting L_1^0 be the local time at 0 of the stable process $Y(t)$ of index β then

$$P(L_1^0 \ge y) \sim C_1 y^{-\beta/2} \exp(-C_\beta y^\beta) \qquad y \to \infty \tag{19}$$

where we define $f \sim g$ at infinity to mean $\lim_{y\to\infty} f(y)/g(y) = 1$.

Here $C_1 > 0$ is a constant and

$$\begin{aligned} C_\beta &= \beta^{-1}(\rho/\bar\beta)^{\beta/\bar\beta} \\ \rho^{-1/\bar\beta} &= \Gamma(1+1/\beta)\Gamma(1/\bar\beta)/\pi. \end{aligned}$$

The result in (19) uses the fact that a stable processes $Y(t)$ of index β satisfies a scaling property; i.e. for any constant $a > 0$

$$\{Y(t) : t \ge 0\} \stackrel{d}{=} \{a^{-1/\beta} Y(at) : t \ge 0\}$$

where $\stackrel{d}{=}$ means equal in distribution. This implies that

$$L_t^0 \stackrel{d}{=} t^{1/\bar{\beta}} L_1^0 \tag{20}$$

$$L_t^* \stackrel{d}{=} t^{1/\bar{\beta}} L_1^* \tag{21}$$

for the local time of $Y(t)$. This time rescaling is unavailable for the more general class of Lévy processes considered in this paper. The details of the arguement that leads to (19) are in the following Preliminary Section.

Perkins 1984 [P], using the Ray-Knight theorems, proved for standard Brownian motion, ($\psi(\lambda) = \lambda^2/2$), that

$$\log P(L_1^* \geq y) \sim -\frac{y^2}{2} \qquad\qquad y \to \infty. \tag{22}$$

This proof does not go through for Lévy processes with jumps since there is no version of the Ray-Knight theorems for these processes.

More exact estimates are obtained in Csaki 1989 [C] for the maximum local time of Brownian motion,

$$P(L_1^* \geq y) \sim 4\left(\frac{2}{\pi}\right)^{1/2} y e^{-\frac{y^2}{2}} \qquad\qquad y \to \infty. \tag{23}$$

However Brownian motion has many special properties that can not be used with Lévy processes with jumps.

Lacey [L2], in a 1990 paper proving results for the local times of the Brownian sheet, gives a new proof of (22) for Brownian motion. This proof does not rely on methods unique to Brownian motion. Using similar techniques as in [L], which considers stable processes for $1 < \beta \leq 2$ (neccessary for the local time to exist), he proves that the exponential decay of $P(L_1^* \geq y)$ is the same as $P(L_1^0 \geq y)$. i.e.

$$\log P(L_1^* \geq y) \sim -Ky^\beta \qquad\qquad y \to \infty \tag{24}$$

where K is given in (17).

Throughout the paper, C will be a positive constant which may change from line to line. In addition, when a process starts at a point x we will write P^x for its probability measure. P will generally be written for P^0 and similarly E for E^0, except perhaps when we want to emphasize the start at 0.

The main results of this paper are Theorems 1.1 and 1.2. The proof of Theorem 1.1, a result on L_t^0, has three steps. First we prove an analogue of a theorem of Marcus and Rosen [MR] that gives good estimates of the moment generating function of L_t^0. We estimate $Ee^{sL_t^0}$ in terms $\kappa^{-1}(\alpha)$ defined in (6). Using the regular variation of $\psi(\lambda)$, the exponent in the characteristic function of $X(t)$, we get an asymptotic relation between $\kappa(\lambda)$ and $\psi^{-1}(\lambda)$ as λ approaches infinity. Finally to get upper bounds on $P^0(L_t^0) > w$ we use Chebyshev's inequality, properties of regular variation of ψ and estimates of the moment generating function of L_t^0 described above. To get lower bounds on $P^0(L_t^0) > w$ we use an inequality of Davies [D] which bounds $P^0(L_t^0) > w$ from below in a way that can be managed with our estimates of $Ee^{sL_t^0}$.

The proof of Theorem 1.2, a result on L_t^*, has three main steps. First it uses the bounds on $P^0(L_t^0 > \lambda)$ established in Theorem 1.1 to get bounds on $P^0(L_t^x > \lambda)$. Secondly it uses an inequality of Barlow [Ba] that provides probability estimates on the difference in the space variable of two local times. Finally it uses an extension of a result in Ledoux and Talagrand [LT] to give estimates on the supremum over a small interval of the difference of two local times.

The organization of this thesis is as follows. In Section 2 we review results about regular variation, Lévy processes and Orlicz spaces that are needed in the proofs of Theorems 1.1 and 1.2. In Section 3 we prove a theorem which estimates the moment generating function of L_t^0 and then use this result to prove Theorem 1.1. Finally in Section 4 Theorem 1.2 is proved.

2 Preliminaries

Since regularly and slowly varying functions, defined in the introduction, are used throughout this paper, we examine some of their properties. All results used about regular variation are contained in Bingham,Goldie,and Teugels [BGT] and Feller Vol II [F].

A simple nontrivial example of a function of slow variation is $\log x$. The case

$$f(x) = \exp\left((\log x)^{1/3}\cos((\log x)^{1/3})\right) \tag{25}$$

shows that a function can be slowly varying and have

$$\begin{aligned} \liminf_{x\to\infty} f(x) &= 0 \\ \limsup_{x\to\infty} f(x) &= \infty. \end{aligned}$$

See [BGT] 1.11.3.

It is a basic property of functions of slow variation at infinity that for any fixed $\delta > 0$ there exists an N such that for all $x > N$

$$x^{-\delta} < S(x) < x^{\delta}. \tag{26}$$

See, e.g. [F] VIII.8 Lemma 2.

We now review some facts about Lévy processes; see e.g. [Br] or Ito 1961 [I]. A process $Y(t)$ is said to be continuous in probability at t_0 if given any $\epsilon > 0$

$$\lim_{t\to t_0} P(|Y_t - Y_{t_0}| > \epsilon) = 0.$$

The Lévy process $X(t)$ defined by (2) and (8) has its characteristic function $E(\exp(i\lambda X(t)))$ continuous at $t = 0$ for every λ. It follows that $X(t)$ is continuous in probability; see Breiman 1968 [Br] pp. 304. Every process with stationary independent increments that is continuous in probability has a version that is in $D([0,\infty))$;i.e. functions defined on $[0,\infty)$ that are right continuous and have left hand limits. See [Br] pp. 306.

Let B a Borel set in R. Define

$$H(t,B) = \#\{s \leq t : X(s) - X(s-) \in B\}.$$

Then H(t,B) is a Poisson process of intensity $\nu(B)$.

By the properties of the Poisson process if $s < t$ then $H(t,B) - H(s,B)$ is independent of $H(s,B)$. Since the holding times of the Poisson process are exponentially distributed, if we define

$$T(B) = \inf\{s : X(s) - X(s-) \in B\}$$

as the hitting time of the set B then $T(B)$ is exponential with parameter $\nu(B)$. In addition, if A is a Borel set disjoint from B then $H(t,A)$ and $H(t,B)$ are independent Poisson processes.

A real valued function $\zeta : R^+ \to R^+$ is called a Young function if it is convex increasing and

$$\begin{aligned} \zeta(0) &= 0 \\ \lim_{x\to\infty} \zeta(x) &= \infty. \end{aligned}$$

The Orlicz space L_ζ is defined as a vector space of all random variables X such that $E\zeta(|X|/c) < \infty$. It is a Banach space with norm

$$\|X\|_\zeta = \inf\{c > 0; E\zeta(|X|/c) < 1\}. \tag{27}$$

When $\zeta(x) = x^p, 1 \leq p < \infty$, then L_ζ is just L_p. See e.g. Ledoux and Talagrand 1991 [LT].

In addition to the power functions, the most commonly used are the exponential functions

$$\zeta_q(x) = \exp(x^q) - 1 \tag{28}$$

for $1 \leq q < \infty$. The example we will use is

$$\zeta_1(x) = \exp(x) - 1. \tag{29}$$

Let (T,d) be a compact pseudo-metric space with pseudo-metric d. A pseudo-metric is a metric that doesn't have to separate points; i.e. $d(s,t) = 0$ doesn't imply $s = t$. The covering numbers

$$N_d(\epsilon) = N(T, d, \epsilon)$$

are defined as the smallest number of open balls in the d-metric of radius $\epsilon > 0$ that cover T. Define the diameter of T as

$$D = D(T) = \sup_{s,t \epsilon T} d(s,t) \tag{30}$$

and the entropy integral as

$$J = J(T, d; \zeta) = \int_0^D \zeta^{-1}(N(T, d; \epsilon))d\epsilon \tag{31}$$

where ζ^{-1} denotes the inverse function of ζ.

We now show that the local times of the Lévy processes we are considering are jointly continuous in x and t. This result goes back to Boylan [B] 1964; we use a different proof.

Lemma 2.1 *Let $X(t)$ be the Lévy process defined by (1) and (2), with $\psi(x) = x^\beta S(x)$ for $\beta > 1$. The local time L_t^x is defined in (13). Then a version of L_t^x exists which is jointly continuous in x and t.*

To prove Lemma 2.1 we use Theorem I of Marcus and Rosen 1993 [MR3] from which results about Lévy processes can be obtained by proving results about related Gaussian processes.

Lemma 2.2 *Let $X(t)$ be the Lévy process defined by (1) and (2), with $\psi(x) = x^\beta S(x)$ for $\beta > 1$. The local time L_t^x is defined in (13). Let $G = \{G(y), y \in S\}$ be the associated Gaussian process, i.e. G is a mean zero Gaussian process with covariance $EG(x)G(y) = u^1(x,y)$ and independent of $X(t)$. Then $\{L_t^x, (t,x) \in R^+ \times R\}$ is continuous almost surely if and only if $\{G(y), y \in R\}$ is continuous almost surely.*

To prove that the associated Gaussian process has continuous paths a.s.

we use Theorem 3.1 of Jain and Marcus 1978 [JM]. If

$$\int_1^\infty \frac{(1-F(x))^{1/2}}{x(\log x)^{1/2}}dx < \infty \tag{32}$$

where F is spectral distribution function of the covariance, then the Gausssian process is continuous. In our case

$$F(x) = \int_{-\infty}^{x} \frac{d\lambda}{1+\psi(\lambda)} \tag{33}$$

and so we consider

$$\int_1^\infty \frac{(\int_x^\infty \frac{d\lambda}{1+\psi(\lambda)})^{1/2}}{x(\log x)^{1/2}}dx. \tag{34}$$

By the regular variation of ψ at ∞, for large enough λ we have that for $\beta-\epsilon>1$

$$\lambda^{\beta-\epsilon} < \psi(\lambda). \tag{35}$$

Finally using this in the integral

$$\int_1^\infty \frac{(1-F(x))^{1/2}}{x(\log x)^{1/2}}dx \quad < \quad \int_1^\infty \frac{x^{\frac{-\beta+1+\epsilon}{2}}}{x(\log x)^{1/2}} \tag{36}$$

$$< \quad \infty \tag{37}$$

since $\frac{1-\beta+\epsilon}{2} < 0$.

The following estimate will be needed in the final section of the paper.

Lemma 2.3 *Define*

$$\delta_0(x) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1-\cos\lambda x}{1+\psi(\lambda)}d\lambda. \tag{38}$$

Then

$$\delta_0(x) \sim \frac{c_\beta}{x\psi(1/x)} \qquad\qquad x \to 0 \tag{39}$$

where $\psi(x) = x^\beta S(x)$ *for* $S(x)$ *slowly varying at infinity and* c_β *is a constant depending only on* $\beta > 1$.

Proof We use a result of Pitman 1968, [Pi]. Let X be a random variable with probability measure P and distribution function $F(x)$, i.e. $F(x) = P(X < x)$. Define for $x \geq 0$

$$H(x) = 1 - F(x) + F(-x). \tag{40}$$

Write

$$U(t) = \int_{-\infty}^{\infty} \cos tx \, dF(x).$$

By Theorem 1 of [Pi] if $H(x)$ is regularly varying of index $-\beta$ when $x \to \infty$, and $0 < \beta < 2$ then

$$1 - U(t) \sim c\,H(1/t) \qquad\qquad t \downarrow 0 \tag{41}$$

where c is a constant depending on the distribution of X.

Define X to be the random variable with density

$$\begin{array}{ll} \dfrac{C}{1+\psi(x)} & \text{for x} > 0 \\ 0 & \text{otherwise.} \end{array}$$

Then we have

$$\begin{aligned} 1 - U(t) &= \int_{-\infty}^{\infty} \frac{C}{1+\psi(x)} dx - \int_{-\infty}^{\infty} \cos tx \frac{C}{1+\psi(x)} dx \\ &= \delta_0(t). \end{aligned} \tag{42}$$

In our case

$$H(x) = \int_x^\infty \frac{C}{1+\psi(t)}\, dt.$$

Let

$$f(x) = \frac{C}{1+\psi(x)}.$$

Since $\psi(x)$ is regularly varying at infinity of index β, $f(x)$ is regularly varying at infinity of index $-\beta$. Thus by Theorem 1.5.11 of [BGT]

$$\frac{x\, f(x)}{\int_x^\infty f(t)dt} \to \beta - 1 \qquad x \to \infty \tag{43}$$

and so $H(x)$ is regularly varying at infinity of index $-\beta+1$. Since $1 < \beta \le 2$ then $0 < \beta - 1 < 2$ and we can apply (41) and (42) to get

$$\begin{aligned} \delta_0(t) &= 1 - U(t) \\ &\sim cH(1/t) \qquad t \downarrow 0 \\ &= \int_{1/t}^\infty \frac{C}{1+\psi(x)} dx. \end{aligned} \tag{44}$$

Rewrite (43) in the form

$$\frac{(1/t)f(1/t)}{\int_{1/t}^\infty f(x)dx} \to \beta - 1 \qquad t \downarrow 0. \tag{45}$$

Using (45) and (44) we see

$$\begin{aligned} \delta_0(t) &\sim \int_{1/t}^\infty \frac{C}{1+\psi(x)} dx \qquad t \downarrow 0 \\ &\sim \frac{C}{t\psi(1/t)} \quad t \downarrow 0 \end{aligned}$$

and the proof of Lemma 2.3 is complete. □

As a final result of this section, we show details of the arguement that leads to (19) in the Introduction. The essential estimate in contained in Hawkes, [H], which quickly leads to (19) as will be seen below.

For $0 < \alpha < 1$, define $\tau(t)$ to have characteristic function

$$E\exp(i\lambda\tau(t)) = \exp(-t\cos(\pi\alpha/2)|\lambda|^{\alpha}(1 - i\,\mathrm{sgn}(\lambda)\tan(\pi\alpha/2). \tag{46}$$

It can be seen that $\tau(t)$ has increasing sample paths. $\tau(t)$ is called the stable subordinator of index α.

Lemma 2.4 *(Hawkes) Let $\tau(t)$ be a stable subordinator of index α. Then as $x \to 0^+$*

$$P(\tau(t) \le t^{1/\alpha}x) \sim c_1 x^{\alpha/(2(1-\alpha))}\exp\left(-c_2 x^{-(\alpha/(1-\alpha))}\right) \tag{47}$$

where

$$c_1 = c_1(\alpha) = \left(2\pi(1-\alpha)\alpha^{\alpha(2(1-\alpha))}\right)^{-1/2}$$

$$c_2 = c_2(\alpha) = (1-\alpha)\alpha^{\alpha(1-\alpha)}.$$

Let L_t^0 be the local time at 0 of $Y(t)$, the symmetric stable process of index $0 < \beta \le 2$, as defined in (14). Stone 1963 [S] showed that if $U_t(\omega)$ is defined by

$$U_t(\omega) = \inf\{v : L_v^0(\omega) > t\} \tag{48}$$

then

$$U_t(\omega) = \rho\tau_t(\omega) \tag{49}$$

where $\tau_t(\omega) = \tau(t,\omega)$ is the stable subordinator of index α and

$$\alpha = 1 - 1/\beta = 1/\bar{\beta} \tag{50}$$

$$\rho^{-1/\bar{\beta}} = \Gamma(1 + 1/\beta)\Gamma(1/\bar{\beta})/\pi. \tag{51}$$

Since by (50) we have a relation between a stable process of index β and the stable subordinator of index α, we can apply Lemma 2.4 and (47) to see that

$$\begin{aligned} P(\tau(t) \leq t^{1/\alpha}x) &= P(U_t \leq \rho t^{1/\alpha}x) \\ &= P(L^0(U_t) \leq L^0(\rho t^{1/\alpha}x)) \\ &= P(L^0(\rho t^{1/\alpha}x) \geq t). \end{aligned}$$

U_t is the right continuous inverse of $L^0(t) = L^0_t$.

Then, as in Lacey [L], using the scaling in (20) and Lemma 2.4 we obtain

$$\begin{aligned} P(L^0(\rho t^{1/\alpha}x) \geq t) &= P((\rho t^{1/\alpha}x)^\alpha L^0_1 \geq t) \\ &= P(L^0_1 \geq \rho^{-\alpha}x^{-\alpha}). \end{aligned}$$

Using (47) we get

$$P(L^0_1 \geq \rho^{-\alpha}x^{-\alpha}) \sim c_1 x^{\alpha/(2(1-\alpha))} \exp\left(-c_2 x^{-(\alpha/(1-\alpha))}\right). \tag{52}$$

Letting

$$x = (y/\rho)^{-1/\alpha}$$

we have

$$P(L^0_1 \geq y) \sim C_1 y^{-\beta/2} \exp(-C_\beta y^\beta) \quad y \to \infty.$$

For Brownian motion the above result can be arrived at more quickly since more is known about the distribution of the local time of Brownian motion. By a well known result of P. Lévy for a Brownian motion $B(t)$ and its local time L_1^0

$$P(L_1^0 > y) = P(\sup_{0<t<1} B(t) > y).$$

By the reflection principle

$$\begin{aligned} P(\sup_{0<t<1} B(t) > y) &= 2\,P(B(1) > y) \\ &= \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-x^2/2} dx. \end{aligned}$$

Finally using the Gaussian tail estimates

$$\frac{y}{1+y^2} e^{-y^2/2} \leq \int_y^\infty e^{-x^2/2} dx \leq \frac{1}{y} e^{-y^2/2}$$

we get the result

$$P(L_1^0 \geq y) \sim \frac{1}{\sqrt{2\pi}} y^{-1} \exp\left(-\frac{y^2}{2}\right) \qquad y \to \infty.$$

3 Local Time at zero

We consider

$$P(L_t^0 > y)$$

for a fixed t and large y. Good estimates will be needed for this in order to prove Theorem 1.1. By Chebyshev's inequality

$$P\left(L_t^0 > y\right) = P(e^{sL_t^0} \geq e^{sy}) \leq \frac{E(e^{sL_t^0})}{e^{sy}}.$$

To estimate $E(e^{sL_t^0})$ we will use the following theorem.

Theorem 3.1 *Let $X = \{X(t), t \in [0,\infty)\}$ be a symmetric Lévy process with local time L_t^x. Let $\kappa(u)$ be as defined in (6) and let $\kappa(u)$ be regularly varying at infinity with index $-1/\bar{\beta}$. Then there exists an s_0 such that for all $s \in [s_0, \infty)$ we have*

$$|E^0 e^{sL_t^0} - f(s)\, e^{\kappa^{-1}(1/s)t}| \leq C\, e^{(1/2)\kappa^{-1}(1/s)t} \quad \forall t > 0. \tag{53}$$

Here κ^{-1} is the inverse of κ and $1/\beta + 1/\bar{\beta} = 1$. The function $f(s)$ is defined so that $\lim_{s\to\infty} f(s) = \bar{\beta}$ and the constant C depends only on s_0 and $\bar{\beta}$.

Proof of Theorem 3.1:

The proof takes the same approach as Marcus and Rosen 1992 [MR] Theorem 2.2. they consider s small and the same methods work for s large.

First we need some properties of the analytic extension of $\kappa(\alpha)$ to the right half plane. For $z = x + iy,\ x > 0$ set

$$\kappa(z) = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{z + \psi(\lambda)}. \tag{54}$$

Let $N = N(\bar{\beta})$ be a fixed positive number such that

$$\sqrt{2}\left(\frac{2}{N+1}\right)^{1/\bar{\beta}} < \frac{1}{2}. \tag{55}$$

Since κ is regularly varying at infinity with index $-1/\bar{\beta}$, given $\epsilon > 0$, let x_0' be large enough so that for all $x_0 \in [x_0', \infty)$

$$1 - \epsilon < v^{1/\bar{\beta}}\,\frac{\kappa(vx_0)}{\kappa(x_0)} < 1 + \epsilon \quad 1 \le v \le 2N. \tag{56}$$

Two lemmas will be needed. The first is

Lemma 3.2 *Let κ be as defined in (6) then*

$$-\lambda\kappa'(\lambda) \sim \frac{1}{\bar{\beta}}\kappa(\lambda) \qquad\qquad \lambda \to \infty. \tag{57}$$

Proof:

This proof uses a variant of the Monotone Density Theorem. See Bingham,Goldie, and Teugels [BGT] Theorem 1.7.2. As in [MR] which had $\lambda \to 0$ set

$$\kappa(\alpha) = -\int_\alpha^\infty \kappa'(u)du$$

and note that $-\kappa(u)$ is monotone decreasing as $u \to \infty$. Then proceed as in (1.7.5) and below in [BGT] to get (57). □

Lemma 3.3 *Let κ be regularly varying at infinity with index $-1/\bar{\beta}$. Let $z = x_0 + iy, x_0 > 0$ where x_0 and N are such that (55) and (56) are satisfied. Then*

$$
\begin{aligned}
|\kappa(z)| &\leq \sqrt{2}\kappa(x_0 + y) \qquad \forall y \geq 0 && (58)\\
\int_{Nx_0}^{\infty} \frac{\kappa(y)}{y} dy &\leq C\kappa(Nx_0) && (59)\\
|\kappa(x_0 + iy)| &\geq \kappa\left(\frac{5x_0}{4}\right) \qquad 0 \leq y \leq x_0/2 && (60)\\
|\text{Im }\kappa(x_0 + iy)| &\geq C\kappa((N+1)x_0) \qquad \frac{x_0}{2} \leq y \leq Nx_0 && (61)
\end{aligned}
$$

Proof

For (58) we have

$$
\begin{aligned}
|\kappa(z)| &\leq \frac{1}{\pi}\int_0^{\infty} \left|\frac{d\lambda}{z + \psi(\lambda)}\right| \\
&= \frac{1}{\pi}\int_0^{\infty} \frac{d\lambda}{((\psi(\lambda) + x_0)^2 + y^2)^{1/2}} \\
&\leq \frac{\sqrt{2}}{\pi}\int_0^{\infty} \frac{d\lambda}{\psi(\lambda) + x_0 + y}.
\end{aligned}
$$

For the proof of (59) use (11) and the regular variation of κ at infinity. We use Feller Vol II [F] pp 281 with $p = -1$ which implies that

$$\lim_{x_0\to\infty} \frac{\kappa(Nx_0)}{\int_{Nx_0}^{\infty} y^{-1}\kappa(y)dy} = C.$$

To obtain (60) write that

$$\begin{aligned}
|\kappa(x_0+iy)| &> \operatorname{Re}\kappa(x_0+iy) \\
&= \frac{1}{\pi}\int_0^\infty \frac{\psi(\lambda)+x_0}{(\psi(\lambda)+x_0+iy)(\psi(\lambda)+x_0-iy)}d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\psi(\lambda)+x_0+\frac{y^2}{\psi(\lambda)+x_0}} \\
&\geq \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\psi(\lambda)+x_0+\frac{x_0^2}{4(\psi(\lambda)+x_0)}}
\end{aligned}$$

where the last step uses the fact that $y < x_0/2$. Then since $\psi > 0$

$$\begin{aligned}
\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\psi(\lambda)+x_0+\frac{x_0^2}{4(\psi(\lambda)+x_0)}} &\geq \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\psi(\lambda)+x_0+\frac{x_0}{4}} \\
&= \kappa\left(\frac{5x_0}{4}\right).
\end{aligned}$$

For (61) we have

$$\begin{aligned}
|\operatorname{Im}\kappa(z)| &= \frac{1}{\pi}\int_0^\infty \frac{y}{(\psi(\lambda)+x_0)^2+y^2}d\lambda \\
&\geq \frac{1}{\pi}\int_0^\infty \frac{x_0/2}{(\psi(\lambda)+x_0)^2+(Nx_0)^2}d\lambda
\end{aligned}$$

since $\frac{x_0}{2} \leq y \leq Nx_0$. Then

$$\begin{aligned}
\frac{1}{\pi}\int_0^\infty \frac{x_0/2}{(\psi(\lambda)+x_0)^2+(Nx_0)^2}d\lambda &\geq \frac{x_0}{2\pi}\int_0^\infty \frac{d\lambda}{(\psi(\lambda)+(N+1)x_0)^2} \\
&= \frac{x_0}{2}(-\kappa'((N+1)x_0) \\
&\geq C\kappa((N+1)x_0)
\end{aligned}$$

where for the last inequality we use Lemma 3.2 and large enough x_0. □

Continuing with the proof of Theorem 3.1, define

$$f(s,t) = E^0 e^{sL_t^0}. \tag{62}$$

Denote the Laplace transform of $f(s,t)$, for fixed s and a function of t, by

$$\mathcal{L}f(s,t) = \int_0^\infty e^{-\alpha t} f(s,t)\, dt.$$

We estimate $f(s,t)$ by estimating the inverse Laplace transform of $(\alpha(1-\kappa(\alpha)s)^{-1}$.

By (2.5) of [MR]

$$\mathcal{L}f(s,t) = \frac{1}{\alpha(1-\kappa(\alpha)s)}$$

for $\kappa(\alpha)s < 1$. Since

$$\frac{1}{\alpha(1-\kappa(\alpha)s)} = \frac{1}{\alpha} + \frac{\kappa(\alpha)s}{\alpha(1-\kappa(\alpha)s)}$$

and

$$\mathcal{L}(1) = \frac{1}{\alpha}$$

we have

$$\mathcal{L}(f(s,t)-1) = \frac{\kappa(\alpha)s}{\alpha(1-\kappa(\alpha)s)}. \tag{63}$$

Then by the inversion formula

$$f(s,t)-1 = \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}\kappa(z)s}{z(1-\kappa(z)s)}dy \tag{64}$$

where $\Gamma = x' + iy$ for some fixed x' for which $\kappa(x')s < 1$.

For each s choose x_0 such that

$$\kappa(2x_0) = \frac{1}{s}.$$

However only consider those $s \in [s_0, \infty)$ where s_0 is large enough so that (56) holds for x_0. Note $1/s$ is small so x_0 is large since $\lim_{x\to\infty}\kappa(x) = 0$. Take $x' > 2x_0$.

The right side of (64) equals

$$\bar{f}(s,t) - 1 = e^{x't}\mathcal{F}^{-1}\mathcal{F}(e^{-x't}(f(s,t)-1))$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transforms. We shall prove the integral

$$\frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}\kappa(z)s}{z(1-\kappa(z)s)}dy$$

is in L^1 and so $\bar{f}(s,t) = f(s,t)$ for all t.

To evaluate this integral we shall consider a contour consisting of the straight line segments $\{x' + iy, -M \le y \le M\}$ $\{x_0 + iy, -M \le y \le M\}$ and the horizontal lines that join them at y equal M and $-M$.

Now pass to the limit as M $\to \infty$. The value of the integral parallel to the x-axis,

$$\int \frac{e^{xt}e^{iyt}\kappa(z)s}{z(1-\kappa(z)s)}dx$$

goes to 0 as M gets large. This is because x is bounded, $|e^{iyt}| = 1$, $\kappa(z)$ gets small for $|z|$ large and z is away from the pole $\kappa(z) = 1/s$.

Then since $z = \kappa^{-1}(1/s)$ is a simple pole and

$$\int_C f(z)dz = 2\pi i(\text{Res}(f(z_0))$$

we get

$$\begin{aligned} f(s,t) - 1 &= \text{Res}\left\{\frac{e^{zt}\kappa(z)s}{z(1-\kappa(z)s)} : z = \kappa^{-1}(1/s)\right\} + \\ &\quad e^{x_0 t}\text{Re}\,\frac{1}{\pi i}\int_0^\infty \frac{e^{iyt}}{x_0 + iy}\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}dy \end{aligned}$$

where

$$\text{Res}\{\frac{e^{zt}\kappa(z)s}{z(1-\kappa(z)s)} : z = \kappa^{-1}(1/s)\} = e^{2x_0 t}\lim_{z\to\kappa^{-1}(1/s)} \frac{z - \kappa^{-1}(1/s)}{z(1-\kappa(z)s)}.$$

To evaluate the residue term, we use L'Hospitals rule to get

$$\lim_{z\to\kappa^{-1}(1/s)} \frac{e^{2x_0 t}}{(1-\kappa(z)s) + z(-\kappa'(z)s)} = \frac{e^{2x_0 t}}{\kappa^{-1}(1/s)(-\kappa'(\kappa^{-1}(1/s))s}.$$

Let

$$\begin{aligned} w(s) &= \frac{-1}{\kappa^{-1}(1/s)(-\kappa'(\kappa^{-1}(1/s))s} \\ s &= \frac{1}{\kappa(\alpha)}. \end{aligned}$$

Then by Lemma 3.2

$$\begin{aligned}\lim_{s\to\infty} w(s) &= \lim_{s\to\infty} -\frac{\kappa(\alpha)}{\alpha\kappa'(\alpha)} \\ &= \lim_{\alpha\to\infty} -\frac{\kappa(\alpha)}{\alpha\kappa'(\alpha)} \\ &= \bar{\beta}.\end{aligned}$$

Thus

$$\begin{aligned}Ee^{sL_t^0}-1 &= f(s,t)-1 \\ &= e^{2x_0t}w(s)+e^{x_0t}\mathrm{Re}\frac{1}{\pi i}\int_0^\infty \frac{e^{iyt}}{x_0+iy}\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}dy \\ &= e^{\kappa^{-1}(1/s)t}w(s)+e^{\frac{1}{2}\kappa^{-1}(1/s)t}\mathrm{Re}\frac{1}{\pi i}\int_0^\infty \frac{e^{iyt}}{x_0+iy}\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}dy.\end{aligned}$$

We will show that

$$\int_0^\infty \frac{|e^{iyt}|}{(x_0^2+y^2)^{1/2}}\left|\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}\right|dy < C \tag{65}$$

which completes the proof since the 1 can be absorbed into $C\ \exp(\frac{1}{2}\kappa^{-1}(1/s)t)$ to get (53).

To obtain (65) we first show that

$$\left|\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}\right| < C \qquad 0\le y\le Nx_0 \tag{66}$$

which gives

$$\int_0^{Nx_0} \frac{1}{(x_0^2+y^2)^{1/2}}\left|\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}\right|dy < NC. \tag{67}$$

We show (66) for $0\le y\le x_0/2$. By the proof of (60)

$$1-\frac{\kappa(2x_0)}{\mathrm{Re}\kappa(z)} \ge 1-\frac{\kappa(2x_0)}{\kappa(5x_0/4)}.$$

By (56)

$$1 - \epsilon \leq (8/5)^{1/\bar{\beta}} \frac{\kappa(2x_0)}{\kappa(5x_0/4)} \leq 1 + \epsilon$$

since $x_0 \in [x_0', \infty)$. Thus

$$1 - \frac{\kappa(2x_0)}{\kappa(5x_0/4)} \geq (1 - \epsilon)(1 - (5/8)^{1/\bar{\beta}})$$

and so

$$\mathrm{Re}\,\kappa(z) - \frac{1}{s} \geq C\,\mathrm{Re}\,\kappa(z) \qquad 0 \leq y \leq x_0/2. \tag{68}$$

Using (68) for $0 \leq y \leq x_0/2$

$$\begin{aligned} \left|\frac{s\kappa(z)}{1 - s\kappa(z)}\right|^2 &= \frac{(\mathrm{Re}\,\kappa(z))^2 + (\mathrm{Im}\,\kappa(z))^2}{(\mathrm{Re}\,\kappa(z) - 1/s)^2 + (\mathrm{Im}\,\kappa(z))^2} \\ &\leq \frac{(\mathrm{Re}\,\kappa(z))^2 + (\mathrm{Im}\,\kappa(z))^2}{2(\mathrm{Re}\,\kappa(z))^2 + (\mathrm{Im}\,\kappa(z))^2} \\ &\leq C. \end{aligned}$$

Now consider when $x_0/2 \leq y \leq Nx_0$. By (58) and (61)

$$\begin{aligned} \left|\frac{s\kappa(z)}{1 - s\kappa(z)}\right|^2 &\leq 1 + \frac{(\mathrm{Re}\,\kappa(z))^2}{(\mathrm{Im}\,\kappa(z))^2} \\ &\leq 1 + \frac{2\kappa^2(x_0 + y)}{C^2\kappa^2((N+1)x_0)} \\ &\leq 1 + \frac{2\kappa^2(3x_0/2)}{C^2\kappa^2((N+1)x_0)} \\ &\leq C. \end{aligned}$$

Thus (66) and (67) have now been proven for $0 \leq y \leq Nx_0$.

Now we examine the second part of the integral in (65) and want to show that

$$\int_{Nx_0}^{\infty} \frac{1}{(x_0^2+y^2)^{1/2}} \left| \frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s} \right| dy \le C.$$

For $y \ge Nx_0$

$$\begin{aligned} |s\kappa(z)| &\le \frac{\sqrt{2}\kappa((N+1)x_0)}{\kappa(2x_0)} \\ &\le \sqrt{2}\left(\frac{2}{N+1}\right)^{1/\bar{\beta}} \\ &\le \frac{1}{2} \end{aligned}$$

by (58), the fact that κ is decreasing and (56),(55). So

$$\left| \frac{s\kappa(z)}{1-s\kappa(z)} \right| \le C\, s|\kappa(z)|$$

since the denominator is bounded. Then

$$\begin{aligned} \int_{Nx_0}^{\infty} \frac{1}{(x_0^2+y^2)^{1/2}} \left| \frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s} \right| dy &\le Cs \int_{Nx_0}^{\infty} \frac{\kappa(x_0+y)}{y} dy \\ &\le Cs \int_{Nx_0}^{\infty} \frac{\kappa(y)}{y} dy \end{aligned}$$

by the above inequality and (58). By (59)

$$\begin{aligned} Cs \int_{Nx_0}^{\infty} \frac{\kappa(y)}{y} dy &\le Cs\kappa(Nx_0) \\ &\le C. \end{aligned}$$

This finally yields that

$$\int_{0}^{\infty} \frac{1}{(x_0^2+y^2)^{1/2}} \left| \frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s} \right| dy \le C$$

and the proof of Theorem 3.1 is complete. □

The following lemma will be needed for the proof of Theorem 1.1. We present it here to minimize the interruptions in the proof.

Lemma 3.4 *Let κ and ψ be as defined in (6),(2) then*

$$\lambda\kappa(\lambda) \sim K\psi^{-1}(\lambda) \qquad \lambda \to \infty \tag{69}$$

where $K = \Gamma(1/\bar{\beta})\Gamma(1+1/\beta)/\pi$.

Proof: By the form of the characteristic function

$$\exp(-t\psi(\lambda)) = E(e^{i\lambda X(t)}) = \int_{-\infty}^{\infty} e^{i\lambda y} p_t(y)dy. \tag{70}$$

Taking the inverse Fourier transform in (70) we have

$$p_t(0) = \frac{1}{\pi}\int_0^{\infty} e^{-t\psi(\lambda)}d\lambda.$$

Define the two monotone functions as follows

$$\hat{\psi}(\lambda) = \sup_{0\le x\le\lambda} \psi(x)$$

$$\check{\psi}(\lambda) = \inf_{x\ge\lambda} \psi(x).$$

Since ψ is regularly varying at infinity with index $\beta > 0$ it is known that

$$\hat{\psi}(\lambda) \sim \psi(\lambda) \qquad \lambda \to \infty$$

$$\check{\psi}(\lambda) \sim \psi(\lambda) \qquad \lambda \to \infty.$$

See e.g. [BGT] Theorem 1.5.3,Theorem 1.5.12. Since $\hat{\psi}$ and $\check{\psi}$ are monotone they are invertable and so

$$\psi(\hat{\psi}^{-1}(\lambda)) \sim \psi(\check{\psi}^{-1}(\lambda)) \sim \lambda \qquad \lambda \to \infty.$$

Clearly

$$\frac{1}{\pi}\int_0^\infty e^{-t\lambda} d\hat{\psi}^{-1}(\lambda) \le p_t(0) \le \frac{1}{\pi}\int_0^\infty e^{-t\lambda} d\check{\psi}^{-1}(\lambda).$$

Now we use [BGT] Theorem 1.7.1. Since ψ^{-1} is regularly varying with index $1/\beta$

$$p_t(0) \sim \frac{\Gamma(1+1/\beta)}{\pi}\psi^{-1}(1/t) \qquad t \to 0+.$$

Since

$$\lambda\kappa(\lambda) = \lambda\int_0^\infty e^{-\lambda t} p_t(0)dt$$

we can use [BGT] Theorem 1.7.6 to get

$$\lambda\kappa(\lambda) \sim \frac{\Gamma(1+1/\beta)\Gamma(1-1/\beta)}{\pi}\,\psi^{-1} \text{ as } \lambda \to \infty.$$

This completes the proof of Lemma 3.4. □

Proof of Theorem 1.1: We first obtain an upper bound for $P(L_t^0 > w)$. Using Theorem 3.1 and Chebyshev's inequality, there exists an s_0 such that for all $s \ge s_0$

$$P(L_t^0 > w) \le C\exp\left(\kappa^{-1}(1/s)t - sw\right). \tag{71}$$

Let

$$s = 1/\kappa(a/t).$$

For fixed t, $\kappa(a/t)$ goes to 0 as a goes to infinity and s gets large as required. So

$$P(L_t^0 > w) \le C\exp\left(a - \frac{w}{\kappa(a/t)}\right). \tag{72}$$

Let

$$w = t^{1/\bar{\beta}}\kappa(a)y$$

in (72) which yields

$$P(L_t^0 > t^{1/\bar{\beta}}\kappa(a)y) \leq C\exp\left(a - \frac{t^{1/\bar{\beta}}\kappa(a)y}{\kappa(a/t)}\right). \tag{73}$$

Fix $\epsilon > 0$. Since $\kappa(a)$ is regularly varying at infinity with index $-1/\bar{\beta}$ we have

$$\lim_{a\to\infty} \frac{\kappa(a)}{\kappa(a/t)} t^{1/\bar{\beta}} = 1. \tag{74}$$

Therefore given any $\epsilon > 0$

$$P(L_t^0 > t^{1/\bar{\beta}}\kappa(a)y) \leq C\exp\left(a - (1-\epsilon)y\right). \tag{75}$$

Note that we have required a to be sufficiently large twice; once above and once in the application of Theorem 3.1. Let

$$y = \frac{u}{\kappa(a)}.$$

Substituting this value in (75) we see that

$$P(L_t^0 > t^{1/\bar{\beta}}u) \leq C\exp\left(a - (1-\epsilon)\frac{u}{\kappa(a)}\right) \tag{76}$$

To get a good upper bound we want to minimize the exponent in (76). We show below that the y that minimizes is a simple multiple of a. Differentiating the exponent in (6) with respect to a and setting it to zero yields

$$1 + (1-\epsilon)u\frac{\kappa'(a)}{\kappa^2(a)} = 0.$$

Use Lemma 3.2 to approximate $\kappa'(a)$.

$$1 - \frac{(1-\epsilon)u}{\bar{\beta}a\kappa(a)} \approx 0$$

which implies

$$y = \bar{\beta}a$$

since $u = y\kappa(a)$. Substituting for y in (75) yields

$$P(L_t^0 > t^{1/\bar{\beta}}\kappa(a)a\bar{\beta}) \leq C\exp(a - a\bar{\beta}). \tag{77}$$

By Lemma 3.4

$$a\kappa(a) \sim K\psi^{-1}(a)$$

and so

$$P(L_t^0 > t^{1/\bar{\beta}}K\bar{\beta}\psi^{-1}(a)/(1-\epsilon)) \leq C\exp(-a(\bar{\beta}-1)). \tag{78}$$

Letting $a = \psi(y)$ we get

$$P\left(\frac{L_t^0}{t^{1/\bar{\beta}}K\bar{\beta}} > y\right) \leq C\exp(-\psi(y)(\bar{\beta} - 1 - \eta)) \tag{79}$$

where $\eta > 0$ can be made small by taking y large. This completes the proof of the upper bound of (16) in Theorem 1.1.

Continuing with the proof of Theorem 1.1, we need a lower bound on $P^0(L_t^0 > w)$. For this, an inequality from Davies [D] is very useful. (See e.g. [MR] in the proof of Lemma 3.2.).

Lemma 3.5 *For any u,t > 0, $0 < \delta < 1$ and $0 < w < y$*

$$\begin{aligned} P^0(L_t^0 > w) \geq & \; e^{-(1-\delta)uy} E^0(e^{(1-\delta)uL_t^0}) \\ & - e^{-uy} E^0(e^{uL_t^0}) \\ & - e^{-(1-\delta)uy} \int_0^w e^{(1-\delta)uz} dP^0(L_t^0 \leq z) \end{aligned} \tag{80}$$

Label the parts in the inequality above as follows

$$P^0(L_t^0 > w) \geq J_1 - J_2 - J_3. \tag{81}$$

First consider

$$J_2 = e^{-uy} E^0(e^{uL_t^0}).$$

Apply Theorem 3.1 to get

$$J_2 \leq e^{-uy} C \; \exp\left(\kappa^{-1}(1/u)t\right).$$

Let

$$u = \frac{1}{\kappa(a/t)} \tag{82}$$

for t fixed and large a. Then

$$J_2 \leq C \; \exp\left(a - \frac{y}{\kappa(a/t)}\right).$$

Let

$$y = a\kappa(a)t^{1/\bar{\beta}}\bar{\beta} \tag{83}$$

then

$$\exp\left(a - \frac{y}{\kappa(a/t)}\right) = \exp\left(a - \frac{a\kappa(a)t^{1/\bar{\beta}}\bar{\beta}}{\kappa(a/t)}\right).$$

Using the regular variation of κ and (74) we see that, with $\epsilon = \delta^4$,

$$J_2 \leq C \ \exp\left(a - a\bar{\beta}(1 - \epsilon)\right) = C \ \exp\left(-a\left(\bar{\beta}(1-\epsilon) - 1\right)\right).$$

Now by Lemma 3.4

$$K\psi^{-1}(a)(1-\epsilon) \leq a\kappa(a) \leq K\psi^{-1}(a)(1+\epsilon)$$

where K is defined in Theorem 1.1.

Even though ψ is not monotone, for large a, ψ can be treated as though it is invertable in asymptotic relations. This is well known, see e.g. [BGT] Theorem 1.5.12.

$$a(1-\epsilon) \leq \psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right) \leq a(1+\epsilon) \tag{84}$$

Substituting for a we get

$$J_2 \leq C\exp\left(-\psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right)(1-\epsilon')(\bar{\beta}(1-\epsilon)-1)\right)$$

Letting ϵ and $\epsilon' = \delta^4$ we have

$$J_2 \leq C\exp\left(-\psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right)(\bar{\beta} - 1 + O(\delta^3))\right) \tag{85}$$

Next consider

$$J_1 = e^{-(1-\delta)uy} E^0(e^{(1-\delta)uL_t^0})$$

By Theorem 3.1 for large u and fixed t

$$J_1 \geq C \, \exp\left(-(1-\delta)uy \; + \; \kappa^{-1}\left(\frac{1}{u(1-\delta)}\right)t\right). \tag{86}$$

Now consider the exponent. Define

$$A_1 = \left((1-\delta)uy \; - \; \kappa^{-1}\left(\frac{1}{u(1-\delta)}\right)t\right). \tag{87}$$

We will bound A_1 above. Substituting for u defined in (82)

$$A_1 = \frac{(1-\delta)y}{\kappa(\frac{a}{t})} \; - \; \kappa^{-1}\left(\frac{\kappa(\frac{a}{t})}{(1-\delta)}\right)t. \tag{88}$$

By the regular variation of κ at infinity, the $1-\delta$ comes out as a power for large a. So

$$A_1 \leq \frac{(1-\delta)y}{\kappa(\frac{a}{t})} \; - \; a(1-2\epsilon)(1-\delta)^{\bar{\beta}}. \tag{89}$$

Since $\epsilon = \delta^4$

$$A_1 \leq (1+\delta^3)\left(\frac{(1-\delta)y}{\kappa(\frac{a}{t})} \; - \; a(1-\delta)^{\bar{\beta}}\right). \tag{90}$$

From (83) we have $y = a\kappa(a)t^{\frac{1}{\beta}}\bar{\beta}$ and using the regular variation of κ

$$\begin{aligned} A_1 \;&\leq\; (1+\delta^3)\left((1-\delta)a\bar{\beta}(1+\epsilon) \; - \; a(1-\delta)^{\bar{\beta}}\right) \\ &=\; (1+\delta^3)\left(a((1-\delta)\bar{\beta}(1+\epsilon) \; - \; (1-\delta)^{\bar{\beta}})\right). \end{aligned}$$

Expanding $(1-\delta)^{\bar{\beta}}$

$$A_1 \leq (1+\delta^3)\left(a(\epsilon(1-\delta)\bar{\beta} \; + \; \bar{\beta}(1-\delta) - 1 \; + \bar{\beta}\delta - \frac{\bar{\beta}(\bar{\beta}-1)}{2}\delta^2 + O(\delta^3)\right).$$

Thus

$$A_1 \le a\left(\bar{\beta} - 1 - \frac{\bar{\beta}(\bar{\beta}-1)}{2}\delta^2 + O(\delta^3)\right)$$

after absorbing $\epsilon = \delta^4$ and δ^3 terms into $O(\delta^3)$ and so

$$\exp(-A_1) \ge \exp\left(-a\left(\bar{\beta} - 1 - \frac{\bar{\beta}(\bar{\beta}-1)}{2}\delta^2 + O(\delta^3)\right)\right).$$

By (84)

$$a = \psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right)(1 + O(\delta^3)) \tag{91}$$

so

$$J_1 \ge \exp\left(-\psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right)\left(\bar{\beta} - 1 - \frac{\bar{\beta}(\bar{\beta}-1)}{2}\delta^2 + O(\delta^3)\right)\right). \tag{92}$$

Having bounded J_1 from below and J_2 from above, we need an upper bound for J_3. This will complete the lower bound for $P^0(L_t^0 > w)$.

For

$$0 < \gamma < w \tag{93}$$

$$\begin{aligned} J_3 &= e^{-(1-\delta)uy}\int_0^w e^{(1-\delta)uz} dP(L_t^0 \le z) \\ &\le e^{-(1-\delta)uy}\left(e^{(1-\delta)u\gamma} + \int_\gamma^w e^{(1-\delta)uz} dP(L_t^0 \le z)\right) \\ &\le 2e^{-(1-\delta)u(y-\gamma)} + u(1-\delta)e^{-(1-\delta)uy}\int_\gamma^w e^{(1-\delta)uz} P(L_t^0 \ge z)dz \end{aligned} \tag{94}$$

using integration by parts for the last inequality.

Substituting for y and u which are given in (83), (82) and recalling $\epsilon = \delta^4$

$$\begin{aligned} \exp(-(1-\delta)uy) &= \exp(-(1-\delta)\frac{a\kappa(a)}{\kappa(a/t)}t^{1/\bar{\beta}}\bar{\beta}) \\ &\le \exp(-(1-\delta)a\bar{\beta}(1-\epsilon)) \\ &\le \exp(-a(\bar{\beta}(1-\delta) + O(\delta^3))) \end{aligned} \tag{95}$$

for a sufficiently large.

Let

$$\gamma = \frac{a\kappa(a)t^{1/\bar{\beta}}}{2} = \frac{y}{2\bar{\beta}} \tag{96}$$

then by (95)

$$\begin{aligned} \exp(-(1-\delta)u(y-\gamma)) &= \exp(-(1-\delta)uy(1-\frac{1}{2\bar{\beta}}) \\ &\leq \exp(-a(\bar{\beta}(1-\delta)-(1-\delta)/2+O(\delta^3))). \end{aligned} \tag{97}$$

Using (92) and (91) for δ sufficiently small

$$\bar{\beta}(1-\delta)-(1-\delta)/2 > \bar{\beta}-1-\frac{\bar{\beta}(\bar{\beta}-1)}{2}\delta^2$$

as long as

$$\delta < \frac{1}{2\bar{\beta}-1}.$$

Now we consider the integrand in (94). By (79)

$$\begin{aligned} e^{(1-\delta)uz}P(L_t^0 \geq z) &\leq \exp((1-\delta)uz - \psi\left(\frac{z}{t^{1/\bar{\beta}}\bar{\beta}K}\right)(\bar{\beta}-1-\delta^4)) \\ &\stackrel{\text{def}}{=} \exp(h(z)) \end{aligned} \tag{98}$$

for $z > z_0$. By (96) and (93), since y approaches infinity, so does γ and therefore w. Let

$$w = (1-c\delta)y \tag{99}$$

where c is determined by β and $c\delta < 1$.

The first term of $h(z)$

$$\frac{(1-\delta)z}{\kappa(a/t)} \leq \frac{(1+\epsilon)(1-\delta)az}{tK\psi^{-1}(a/t)} \qquad \text{for } a \geq a_0 \tag{100}$$

by Lemma 3.4. Note that by (84), for large a, $a(1-\epsilon) \leq \psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right) \leq a(1+\epsilon)$ and so a and y both approach infinity.

Let

$$z = s\,y$$

where

$$\frac{1}{2\bar{\beta}} \leq s \leq 1 - c\delta$$

so that

$$\gamma \leq z \leq w.$$

Then by Lemma 3.4, (74), (83) and (84) for sufficiently large y

$$\begin{aligned}\frac{(1+\epsilon)(1-\delta)a(sy)}{tK\psi^{-1}(a/t)} &\leq \frac{(1+C\epsilon)(1-\delta)a^2\kappa(a)\bar{\beta}s}{K\psi^{-1}(a)} \\ &\leq (1+C\epsilon)(1-\delta)a\bar{\beta}s \\ &\leq \psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right)(1+C\epsilon)(1-\delta)\bar{\beta}s \end{aligned} \tag{101}$$

where $C > 1$ is a constant not necessarily the same at each occurrence.

The second term of the exponent $h(z)$ is

$$\psi\left(\frac{sy}{t^{1/\bar{\beta}}\bar{\beta}K}\right)(\bar{\beta}-1-\delta^4) \geq \psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right)(1-\epsilon)(\bar{\beta}-1-\delta^4)s^{\bar{\beta}} \quad \text{as } y \to \infty \tag{102}$$

using the regular variation of ψ of index β at infinity. Define

$$g(z) = \psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right)((1+C\delta^4)(1-\delta)\bar{\beta}s - (1-\delta^4)(\bar{\beta}-1)s^{\beta}). \tag{103}$$

It has been established that

$$h(z) \le g(z) \quad \text{for } z > z_0. \tag{104}$$

We now show that g is increasing for $\gamma \le z \le w$. For this calculation g will be considered as a function of s with y fixed. The derivative of

$$(1+C\delta^4)(1-\delta)\bar{\beta}s - (1-\delta^4)(\bar{\beta}-1)s^{\beta} \tag{105}$$

must be greater than or equal to zero to have $g(z)$ increasing. This is true if

$$(1-\delta)\bar{\beta} - \frac{(1-\delta^4)}{(1+C\delta^4)}\beta(\bar{\beta}-1)s^{\beta-1} \ge 0.$$

Since $\frac{(1-\delta^4)}{(1+C\delta^4)} < 1$, using $\bar{\beta} = \beta(\bar{\beta}-1)$ from (10) means that it sufficient that

$$(1-\delta) - (1-c\delta)^{\beta-1} \ge 0$$

since $s < 1 - c\delta$. Since $\beta > 1$ is fixed, we can choose

$$c > c_1 \stackrel{\text{def}}{=} \frac{1}{\beta-1} \tag{106}$$

so that

$$(1-\delta) > (1-c\delta)^{\beta-1}$$

for δ sufficiently small. Thus $g(z)$ is increasing for $\gamma \le z \le w$.

By (98), (103) and (104)

$$\begin{aligned}
&\exp\left((1-\delta)uz - \psi\left(\frac{z}{t^{1/\bar\beta}\bar\beta K}\right)(\bar\beta - 1 - \delta^4))\right) \\
&\leq \exp\left(\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)((1+C\delta^4)(1-\delta)\bar\beta s - (1-\delta^4)(\bar\beta-1)s^{\beta})\right) \\
&\leq \exp\left(\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)(1+C\delta^4)(1-\delta)\bar\beta(1-c\delta)\right) \\
&\qquad \exp\left(-\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)(1-\delta^4)(\bar\beta-1)(1-c\delta)^{\beta}\right) \qquad (107)
\end{aligned}$$

where since $g(z)$ is increasing, s can be replaced by $1-c\delta$. Using (94), (98), and (107)

$$\begin{aligned}
&u(1-\delta)e^{-(1-\delta)uy}\int_{\gamma}^{w} e^{(1-\delta)uz}P(L_t^0 > z)dz \\
&\leq ue^{-(1-\delta)uy}\int_{\gamma}^{w}\exp\left((1-\delta)uz - \psi\left(\frac{z}{t^{1/\bar\beta}\bar\beta K}\right)(\bar\beta-1-\delta^4))\right)dz \\
&\leq u(w-\gamma)\exp(-(1-\delta)uy) \\
&\qquad \exp\left(\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)((1+C\delta^4)(1-\delta)\bar\beta(1-c\delta) - (1-\delta^4)(\bar\beta-1)(1-c\delta)^{\beta})\right) \\
&\leq uy\exp\left(-\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)(1-\delta)(1-C\delta^4)\bar\beta\right) \\
&\qquad \exp\left(-\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)((1-\delta^4)(\bar\beta-1)(1-c\delta)^{\beta} - (1+C\delta^4)(1-\delta)\bar\beta(1-c\delta))\right) \\
&\leq uy\exp\left(-\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)(1-\delta)\bar\beta\right) \\
&\qquad \exp\left(-\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)((\bar\beta-1)(1-c\delta)^{\beta} - (1-\delta)\bar\beta(1-c\delta) - C\delta^3\right) \qquad (108)
\end{aligned}$$

using (95) to recall estimates on uy.

Since

$$uy \leq \exp\left(\psi\left(\frac{y}{t^{1/\bar\beta}\bar\beta K}\right)\delta^3\right)$$

for large y, using the series expansion for $(1-c\delta)^{\beta}$ the expression in (108) is

$$\begin{aligned}
&\leq \exp\left(-\psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right)((1-\delta)\bar{\beta}c\delta+(\bar{\beta}-1)(1-\beta c\delta+\frac{\beta(\beta-1)}{2}(c\delta)^2)-C\delta^3)\right)\\
&\leq \exp\left(-\psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right)((1-\delta)\bar{\beta}c\delta+\bar{\beta}-1-\bar{\beta}(c\delta)+\frac{\bar{\beta}(\beta-1)}{2}(c\delta)^2)-C\delta^3)\right)\\
&\leq \exp\left(-\psi\left(\frac{y}{t^{1/\bar{\beta}}\bar{\beta}K}\right)(\bar{\beta}-1-(c\delta)(\bar{\beta}\delta)+\frac{\bar{\beta}(\beta-1)}{2}(c\delta)^2-C\delta^3)\right).
\end{aligned}$$

Using (92) we will need

$$\bar{\beta}-1-(c\delta)(\bar{\beta}\delta)+\frac{\bar{\beta}(\beta-1)}{2}(c\delta)^2>\bar{\beta}-1-\frac{\bar{\beta}(\bar{\beta}-1)}{2}\delta^2$$

which means that

$$c\left(\frac{c\,\bar{\beta}(\beta-1)}{2}-\bar{\beta}\right)>-\frac{\bar{\beta}(\bar{\beta}-1)}{2}.$$

Since the right part of the above inequality is less than zero it is sufficient that

$$\frac{c(\beta-1)}{2}-1>0.$$

Let

$$c_2=\frac{2}{\beta-1}. \tag{109}$$

Choosing

$$c>c_2>c_1$$

where c_1 was defined in (106), we have that

$$J_3\leq \exp\left(-\psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right)(\bar{\beta}-1-\eta\delta^2)\right) \tag{110}$$

where

$$\eta < \frac{\bar{\beta}(\bar{\beta}-1)}{2}.$$

From (81), we have

$$P^0(L_t^0 > w) \geq J_1 - J_2 - J_3$$

where J_1, J_2, J_3 are in terms of y. However $w = (1 - c\delta)y$ and δ approaches 0. Then using (110), (92), and (85) and the regular variation of ψ at infinity with index β, we have that for all $y > y_0$

$$P^0(L_t^0 > y) \geq K_1 - K_2 - K_3$$

with

$$\begin{aligned}
K_1 &\geq \exp\left(-\psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right)(\bar{\beta} - 1 - \mu\delta^3 + O(\delta^4))\right) \\
K_2 &\leq C\exp\left(-\psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right)(\bar{\beta} - 1 + O(\delta^4))\right) \\
K_3 &\leq \exp\left(-\psi\left(\frac{y}{t^{\frac{1}{\beta}}\bar{\beta}K}\right)(\bar{\beta} - 1 - \nu\delta^3 + O(\delta^4))\right)
\end{aligned}$$

and

$$\nu < \mu.$$

This completes the proof of the lower bound of (16) and so Theorem 1.1 is proven. □

Remark 3.6 *Our result agrees with Lacey's result (see the Introduction) on the constant in the exponent in the case X(t) is a symmetric stable Lévy process where $\psi(\lambda) = \lambda^\beta$. From [L] we have that*

$$\lim_{y\to\infty} \frac{\log P\left(\frac{L_t^0}{t^{1/\beta}} > y\right)}{y^{\beta}} = -C_\beta$$

where $C_\beta = \beta^{-1}(\rho/\bar{\beta})^{\beta/\bar{\beta}}$ *and* $\rho^{-1/\bar{\beta}} = K$. *Our result for the stable process is*

$$\lim_{y\to\infty} \frac{\log P\left(\frac{L_t^0}{t^{1/\beta}} > y\right)}{y^{\beta}} = -\frac{\bar{\beta}-1}{(K\bar{\beta})^{\beta}}$$

and

$$C_\beta = \frac{\bar{\beta}-1}{(K\bar{\beta})^{\beta}}.$$

4 Maximum Local Time

Before proving Theorem 1.2, some estimates are needed on the size of $|L_t^x - L_t^y|$ when x and y are near each other, for fixed t. Barlow [Ba] has such a result for 1-dimensional Lévy processes.

Denote E^a as expectation with respect to P^a, where P^a is the law of X starting at a. Define T_b to be the first hitting time of b. i.e.

$$T_b = \inf\{t \geq 0 : X_t = b\}. \tag{111}$$

Following Barlow define

$$h(a,b) = h(b-a) = E^a L^a_{T_b}. \tag{112}$$

The next lemma is a weaker version of Lemma 2.8 in [Ba] in the sense that it does not consider the supremum over t.

Lemma 4.1 *For each* $\lambda > 0, \quad p > 0$

$$P(|p \wedge L_t^a - p \wedge L_t^b| > \lambda) \leq 2 \exp\left(-\frac{\lambda^2}{4p\, h(a,b)}\right). \tag{113}$$

It is shown in [Ba] Lemma 2.4 that for sufficiently small x

$$C_1\, \delta_0(x) \leq h(x) \leq C_2\, \delta_0(x) \tag{114}$$

for constants $0 < C_1 \leq C_2$ where

$$\delta_0(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos \lambda x}{1 + \psi(\lambda)} d\lambda$$

as defined in (38).

Using Lemma 2.3 which estimates $\delta_0(x)$ as $x \to 0$ and (113) and (114), there exists an $\nu > 0$ such that for $|a-b| < \nu$

$$P(|p \wedge L_t^a - p \wedge L_t^b| > \lambda) \leq 2\exp\left(-\frac{C\lambda^2}{p\,\frac{1}{|a-b|\psi(1/|a-b|)}}\right)$$
$$= 2\exp\left(-\frac{C\lambda^2(\frac{1}{|a-b|})^{\beta-1}S(\frac{1}{|a-b|})}{p}\right). \quad (115)$$

We used the representation of the regular variation of ψ at infinity, as in (8) and recalled that S is a slowly varying function.

As discussed in the preliminary section, we can control a function of slow variation at infinity in the following way. For any fixed $\delta > 0$ there exists an ν' such that for all $|a-b| < \nu'$

$$\left(\frac{1}{|a-b|}\right)^{-\delta} < S\left(\frac{1}{|a-b|}\right) < \left(\frac{1}{|a-b|}\right)^{\delta}. \quad (116)$$

Since $\beta > 1$, δ can be chosen small enough so that

$$\eta \stackrel{\text{def}}{=} \beta - 1 - \delta > 0. \quad (117)$$

Then using (116) and (115) we have that

$$P(|p \wedge L_t^a - p \wedge L_t^b| > \lambda) < 2\exp\left(-\frac{C\lambda^2}{p|a-b|^{\eta}}\right). \quad (118)$$

Let (T,d) be a compact pseudo-metric space with pseudo-metric d and covering numbers $N_d(\epsilon)$ as described in the Preliminary Section. In all that follows we assume that the metric d is monotone. In the application of the lemma, and in the proof of Theorem 1.2, the metric we will define is monotone.

Lemma 4.2 *Let $X = (X_t)_{t \in T}$ be a random process such that*

$$P(|X_s - X_t| > \lambda) \le C \exp\left(-\frac{\lambda}{d(s,t)}\right) \tag{119}$$

and for all $0 < \epsilon < D$

$$l(\epsilon) = \int_0^\epsilon \log N_d(u) du < \infty. \tag{120}$$

Then for all $\lambda, \epsilon > 0$

$$P\left(\sup_{\substack{d(s,t)<\epsilon \\ s,t \in T}} |X_s - X_t| > \lambda + l(\epsilon)\right) \le \frac{1}{d^{-1}(\epsilon)} \exp\left(-\left(\frac{\lambda}{C\epsilon}\right)\right) \tag{121}$$

where C is a constant and d^{-1} is the inverse of d.

Proof We use the following result in [LT] pp 300 to prove Lemma 4.2. Here as described in the Preliminary Section, $D = D(T) = \sup_{s,t \epsilon T} d(s,t)$ is the diameter of T and the entropy integral $J = J(T, d; \zeta) = \int_0^D \zeta^{-1}(N(T, d; \epsilon)) d\epsilon$.

Lemma 4.3 *Let ζ be a Young function and let $X = (X_t)_{t \in T}$ be a random process in $L_1(\Omega, A, P)$ such that for all measureable sets A in Ω and all s, t in (T, d)*

$$\int_A |X_s - X_t| dP \le d(s,t) P(A) \zeta^{-1}\left(\frac{1}{P(A)}\right). \tag{122}$$

Then for every A

$$\int_A \sup_{s,t\in T} |X_s - X_t| dP \leq 8P(A) \int_0^D \zeta^{-1}\left(\frac{N(T,d;\epsilon)}{P(A)}\right) d\epsilon. \qquad (123)$$

Further it is shown in [LT] 11.4 pp 302, under the assumption of (122) and using Lemma 4.3, that for Young functions of the exponential type, for every $u > 0$

$$P\left(\sup_{s,t\in T} |X_s - X_t| > 8C(J+u)\right) \leq \left(\zeta\left(\frac{u}{D}\right)\right)^{-1}. \qquad (124)$$

Here C depends only on the Young function ζ.

We now assume the condition of Lemma 4.2; that a process satisfies an exponential inequality in the form

$$P(|X_s - X_t| > \lambda) \leq C \exp\left(-\frac{\lambda}{d(s,t)}\right) \qquad (125)$$

for some fixed $C > 0$. By the definition of the Orlicz space norm with Young function ζ_1, defined in (29),

$$\|X_s - X_t\|_{\zeta_1} = \inf\{c > 0; E\zeta_1(|X_s - X_t|/c) < 1\}. \qquad (126)$$

Since ζ_1 is increasing we can use integration by parts to get that

$$\begin{aligned} E\,\zeta_1(|X_s - X_t|/c) &= \int_0^\infty P\left(\frac{|X_s - X_t|}{c} > t\right) e^t dt \\ &< C\int_0^\infty \exp\left(-\frac{tc}{d(s,t)}\right) e^t\, dt \end{aligned}$$

using (125). Taking the infimum over c means that

$$\|X_s - X_t\|_{\zeta_1} \leq d(s,t). \tag{127}$$

It is shown in [LT] pp 300 that (127) implies that (122) is satisfied. We therefore have that (125) implies (124), with C depending on ζ_1 only;

$$P\left(\sup_{s,t\in T} |X_s - X_t| > 8C(J+u)\right) \leq \exp\left(-\frac{u}{D}\right). \tag{128}$$

This gives a modulus of continuity result over T. Cover T with balls B_k of radius 2ϵ and centers c_k such that the c_k are within ϵ of each other, all in the d−metric. Then for $s,t \in T$ and $d(s,t) < \epsilon$ there is some B_k such that $s,t \in B_k$. This is because for any $s \in T$ for some c_k we have $d(s,c_k) < \epsilon$ and so

$$d(t,c_k) \leq d(s,t) + d(s,c_k) = 2\epsilon.$$

Therefore

$$\begin{aligned} & P\left(\sup_{\substack{d(s,t)<\epsilon \\ s,t\in T}} |X_s - X_t| > C(l(2\epsilon)+u)\right) \\ \leq\ & P\left(\sup_{B_k} \sup_{\substack{d(s,t)<\epsilon \\ s,t\in B_k}} |X_s - X_t| > C(l(2\epsilon)+u)\right) \\ \leq\ & \sum_{B_k} P\left(\sup_{\substack{d(s,t)<\epsilon \\ s,t\in B_k}} |X_s - X_t| > C(l(2\epsilon)+u)\right) \end{aligned} \tag{129}$$

by the properties of probability measures.

Then using (128) and that the diameter of all the balls B_k is 2ϵ we have

$$P\left(\sup_{\substack{d(s,t)<\epsilon \\ s,t\in B_k}} |X_s - X_t| > C(l(2\epsilon)+u)\right) \leq \exp\left(-\frac{u}{2\epsilon}\right). \tag{130}$$

Since d is monotone, the number of balls with centers that are ϵ apart covering T is less than or equal to

$$\frac{1}{d^{-1}(\epsilon)} \tag{131}$$

where d^{-1} is the inverse of d. Therefore we have that

$$P\left(\sup_{\substack{d(s,t)<\epsilon \\ s,t\in T}} |X_s - X_t| > C(l(2\epsilon) + u)\right) \leq \frac{1}{d^{-1}(\epsilon)} \exp\left(-\frac{u}{2\epsilon}\right)$$

which is (121) after changing the constant in the exponent and so we have proven Lemma 4.2. □

Proof of Theorem 1.2:

By Theorem 1.1 since

$$P(L_1^0 > \lambda) < P(L_1^* > \lambda)$$

only the upper bound in (18) needs to be proven.

The proof uses several ideas from Lacey 1990 [L] and Griffin 1985 [G]. As in [L] we first consider the supremum of L_t^x over an interval I of length one. Define for a fixed $\lambda > 1$

$$Z_x = 2\lambda \wedge L_t^x.$$

Let the metric d be as follows

$$d(x,y) = C|x-y|^\eta \tag{132}$$

where $\eta > 0$ is as defined in (117). Using (118) we have that

$$P(|Z_s - Z_t| > \lambda) \leq C \exp\left(-\frac{\lambda}{d(s,t)}\right)$$

which is the first requirement in Lemma 4.2.

Using (30), the diameter D of (I, d) is just a constant. Setting

$$d^{-1}(u) = \left(\frac{u}{C}\right)^{1/\eta} \tag{133}$$

we get a bound on the entropy numbers

$$N(u) \leq 1 + (Cu^{-1})^{1/\eta}. \tag{134}$$

Let $\epsilon = \lambda^{-\beta}$ and note that

$$d^{-1}(\epsilon) = \left(\frac{\lambda}{C}\right)^{-\beta/\eta} \tag{135}$$

and that

$$\begin{aligned} l(\epsilon) &= \int_0^\epsilon (\log N(u))du \\ &< \int_0^\epsilon C \log(1/u) du \\ &< C\epsilon \log(1/\epsilon). \end{aligned} \tag{136}$$

Therefore

$$\lim_{\epsilon \to 0} l(\epsilon) = 0. \tag{137}$$

Define I_ϵ to be a set of points in I such that

$$\sup_{x \in I} \min_{y \in I_\epsilon} d(x, y) \leq \epsilon$$

and the cardinality of I_ϵ as small as possible. I_ϵ is called an ϵ-net on I with the metric d. Since $\epsilon = \lambda^{-\beta}$ using (135) we have that

$$\operatorname{card}(I_\epsilon) \le C\lambda^{\beta/\eta}. \tag{138}$$

Now consider

$$\begin{aligned} P\left(\sup_{x\in I} L_t^x > \lambda + \lambda^{3/4}\right) &= P\left(\sup_{x\in I} Z_x > \lambda + \lambda^{3/4}\right) \\ &< P\left(\sup_{x\in I_\epsilon} Z_x > \lambda\right) + \\ &\quad P\left(\sup_{\substack{d(x,y)<\epsilon \\ x,y\in I}} |Z_x - Z_y| > \lambda^{3/4}\right) \end{aligned} \tag{139}$$

by a version of the triangle inequality.

For the first term in the inequality we have that

$$P\left(\sup_{x\in I_\epsilon} Z_x > \lambda\right) \le \operatorname{card}(I_\epsilon) \sup_{x\in I} P(L_t^x > \lambda) \tag{140}$$

since the ϵ-net is finite. Then

$$\begin{aligned} \sup_{x\in I} P(L_t^x > \lambda) &\le P(L_t^0 > \lambda) \\ &< \exp\left(-\psi(\lambda)\frac{(\bar{\beta}-1)}{(K\bar{\beta})^\beta}\right) \end{aligned} \tag{141}$$

by the strong Markov property and our estimate on L_t^0 in (16). Thus

$$P\left(\sup_{x\in I_\epsilon} Z_x > \lambda\right) \le C\lambda^{\beta/\eta} \exp\left(-\psi(\lambda)\frac{(\bar{\beta}-1)}{(K\bar{\beta})^\beta}\right) \tag{142}$$

Considering the second expression in (139) we have

$$P\left(\sup_{\substack{d(x,y)<\epsilon \\ x,y\in I}} |Z_x - Z_y| > \lambda^{3/4}\right) < \frac{1}{d^{-1}(\epsilon)} \exp\left(-\left(\frac{\lambda^{3/4}}{C\epsilon}\right)\right)$$

$$
\begin{aligned}
&< \lambda^{\beta/\eta} \exp\left(-\left(\frac{\lambda^{3/4}}{C\epsilon}\right)\right) \\
&< \lambda^{\beta/\eta} \exp\left(-\frac{\lambda^{\beta+3/4}}{C}\right). \qquad (143)
\end{aligned}
$$

Here we used (121) in Lemma 4.2 to control the supremum increment process with $\epsilon = \lambda^{-\beta}$ and we recall that $d^{-1}(\epsilon) = \left(\frac{\lambda}{C}\right)^{-\beta/\eta}$.

Since $\psi(\lambda) = \lambda^{\beta} S(\lambda)$ we have that

$$\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\lambda^{\beta+3/4}} = 0.$$

Therefore the exponent in (143) becomes much smaller than the one in (141) as λ approaches infinity and so

$$P\left(\sup_{x\in I} L_t^x > \lambda + \lambda^{3/4}\right) < C\lambda^{\beta/\eta} \exp\left(-\psi(\lambda)\frac{(\bar{\beta}-1)}{(K\bar{\beta})^{\beta}}\right). \qquad (144)$$

We then have the result of Theorem 1.2 but with L_t^* replaced by $\sup_{x\in I} L_t^x$, i.e.

$$\lim_{\lambda\to\infty} \frac{\log P\left(\frac{\sup_{x\in I} L_t^x}{t^{1/\beta}} > \lambda\right)}{\psi(\lambda)} = \frac{1-\bar{\beta}}{(K\bar{\beta})^{\beta}}. \qquad (145)$$

We now can make an extension of the result in (144) by taking the supremum over a larger, but still finite, interval. Namely that

$$
\begin{aligned}
P\left(\sup_{|x|\le\lambda^{2\beta}} L_t^x > \lambda + \lambda^{3/4}\right) &\le \sum_{-\lambda^{2\beta}\le j\le\lambda^{2\beta}} P\left(\sup_{j\le x\le j+1} L_t^x > \lambda + \lambda^{3/4}\right) \\
&\le 2\lambda^{2\beta} C\lambda^{\beta/\eta} \exp\left(-\psi(\lambda)\frac{(\bar{\beta}-1)}{(K\bar{\beta})^{\beta}}\right) \\
&< \lambda^{C} \exp\left(-\psi(\lambda)\frac{(\bar{\beta}-1)}{(K\bar{\beta})^{\beta}}\right) \qquad (146)
\end{aligned}
$$

where we used (144) when the supremum is taken over the unit interval. This result will be needed in the last steps of the proof of Theorem 1.2.

As described in Griffin [G], $X(t)$ can be decomposed as the sum of two independent Lévy processes

$$X(t) = X_1(t) + X_2(t)$$

where

$$\begin{aligned} X_2(t) &= \sum_{s \le t} (X(s) - X(s-)) I\{|X(s) - X(s-)| > 1\} \\ X_1(t) &= X(t) - X_2(t) \end{aligned}$$

because the jumps are in disjoint sets.

Since $X_1(t)$ is symmetric and has bounded jumps it has a moment generating function

$$E \exp(aX_1(t)) = \exp(t\phi(a)) \tag{147}$$

where

$$\phi(a) = \int_{-1}^{1} (e^{ax} - 1)\nu(dx).$$

Note that $\phi(a) \to 0$ as $a \to 0$.

Define stopping times by

$$\begin{aligned} T_0 &= 0 \\ T_1 &= \inf\{s : |X(s) - X(s-)| > 1\} && (148) \\ T_{k+1} &= \inf\{s > T_k : |X(s) - X(s-)| > 1\}. && (149) \end{aligned}$$

Because X is a Lévy process $T_1, T_2 - T_1, \ldots$ are independent exponential random variables with parameter

$$\nu(|x| > 1) = b$$

as discussed in the Preliminary Section. Define

$$J = \max\{j : T_j \leq 1\}.$$

J then has a Poisson distribution with parameter b. Finally define

$$M_j = \sup_{T_{j-1} \leq s < T_j} |X(s) - X(T_{j-1})| \tag{150}$$
$$M^* = \max\{M_j : j \leq J\}. \tag{151}$$

We now are ready to prove the following lemma due to Lacey in [L].

Lemma 4.4 *There exists an $\alpha > 0$ and a $C > 0$ such that for all $\lambda \geq C$*

$$P(J > \lambda) \leq Ce^{-\alpha\lambda} \tag{152}$$
$$P(M_1 > \lambda) \leq Ce^{-\alpha\lambda} \tag{153}$$
$$P(M^* > \lambda) \leq Ce^{-\alpha\lambda} \tag{154}$$

<u>Proof</u> Let

$$X_1^*(t) = \sup_{0 \leq s \leq t} |X_1(s)|.$$

Then by (147) there is an $a > 0$ so that

$$E\ \exp(aX_1(t)) \leq \exp(t) \tag{155}$$

since $\phi(a) \to 0$ as $a \to 0$. Then for $t < \lambda a/2$

$$\begin{aligned} P(X_1^*(t) > \lambda) &\leq C\,P(X_1(t) > \lambda) \\ &\leq C\exp(-a\lambda + t) \\ &\leq C\exp(-a\lambda/2) \end{aligned}$$

by Levý's and Chebyshev's inequalities using (155). Then for $\lambda > 4/a$ we have

$$\begin{aligned} P(M_1 > \lambda) &\leq P(T_1 > a\lambda/4) + P(X_1^*(a\lambda/4) > \lambda) \\ &\leq C\exp(-\alpha\lambda) \end{aligned}$$

which proves (153). (154) follows from (153) and (152).

Now the proof of Theorem 1.2 can be finished. If

$$J \leq \lambda^{2\beta}$$

and

$$M^* \leq \lambda^{2\beta}$$

for large enough λ then

$$\sup_x L_t^x \subset \bigcup_{j<\lambda^{2\beta}} (X(T_j) - \lambda^{2\beta}, X(T_j) + \lambda^{2\beta}).$$

This is because the above union is the maximum range of the process $X(t)$ and the local time is zero at a point x if the process doesn't hit x. Therefore

$$\begin{aligned} P\left(L_t^* > \lambda + \lambda^{3/4}\right) &\leq P(J > \lambda^{2\beta}) + P(M^* > \lambda^{2\beta}) \\ &+ CP\left(\sup_{|x|\leq\lambda^{2\beta}} L_t^x > \lambda + \lambda^{3/4}\right) \qquad (156) \\ &\leq 2C\exp(-C\lambda^{2\beta}) + C\exp\left(\frac{1-\bar{\beta}}{(K\bar{\beta})^{\beta}}\psi(\lambda)\right) \\ &\leq C\exp\left(\frac{1-\bar{\beta}}{(K\bar{\beta})^{\beta}}\psi(\lambda)\right). \qquad (157) \end{aligned}$$

Here Lemma 4.4 was used to bound $P(J > \lambda^{2\beta})$ and $P(M^* > \lambda^{2\beta})$. To handle the supremum over $|x| \leq \lambda^{2\beta}$ we used (146). The proof of Theorem 1.2 is now complete.

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