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A

Topological Model of Simple Siegel Disk Type,

By

Gaofei Zhang

**A dissertation submitted to the Graduate Faculty in
Mathematics in partial fulfillment of the requirements for
the degree of Doctor of Philosophy, The City University of
New York**

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GAOFEI ZHANG

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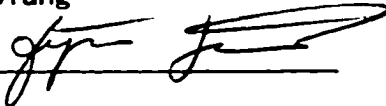
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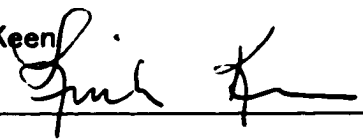
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Abstract

TOPOLOGICAL MODEL OF SIMPLE SIEGEL DISK TYPE

by

Gaofei Zhang

Adviser: Professor Yunping Jiang

Around 1982, Thurston developed a rational realization and rigidity theory for the critically finite branched covering maps. Since then, people have been trying to generalize this theory to critically infinite maps. In this work, we established a Thurston type theory for topological polynomials of simple Siegel disk type. We proved that a topological polynomial of simple Siegel disk type is combinatorially equivalent to a canonical polynomial with a Siegel disk if and only if it has no Thurston obstruction. We then present a way to prove that the Julia set of the canonical polynomial has Lebesgue measure zero, which implies that up to a conformal conjugation, the canonical polynomial is the unique polynomial to realize the topological model such that the boundary of the Siegel disk is a quasi-circle.

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Introduction

Around 1982, Thurston developed a theory to characterize the critically finite rational maps topologically. Since then, people have been trying to generalize this theory to critically infinite rational maps. In this work, we proved that a topological polynomial f of simple Siegel disk type is combinatorially equivalent to a polynomial with a Siegel disk if and only if it has no Thurston obstruction.

A topological polynomial f is a branched covering map such that $f^{-1}(\infty) = \{\infty\}$. Let $\Omega_f = \{x \in S^2 \mid \deg_x f \geq 2\}$ denote the critical set and $P_f = \overline{\cup_{n \geq 1} f^{on}(\Omega_f)}$ denote the post-critical set.

Definition III.1 *We say a topological polynomial f is of simple Siegel disk type if there exists a closed topological disk D and a holomorphic homeomorphism $\phi : D \rightarrow \Delta = \{z \mid |z| \leq 1\}$, such that*

1. $f|_D(z) = \phi^{-1} \circ R_\theta \circ \phi(z)$ for some irrational number θ of bounded type, where $R_\theta : \Delta \rightarrow \Delta$ is defined by $z \rightarrow e^{2\pi i \theta} z$, and
2. $\partial D \cap \Omega_f \neq \emptyset$, and
3. $\text{Orbit}(z) \setminus D$ is a finite set for all $z \in \Omega_f$, where $\text{Orbit}(z) = \{f^{on}(z) \mid n = 1, 2, \dots\}$.

Definition III.2 *Let f and g be two topological polynomials of simple Siegel disk type. We say f and g are combinatorially equivalent to each other if there exists a pair of homeomorphisms $\phi_1, \phi_2 : S^2 \rightarrow S^2$ such that*

- a) $\phi_1 = \phi_2$ are holomorphic when they are restricted on the rotation disk D , and
- b) ϕ_1 is isotopic to ϕ_2 relative to P_f , and
- c) $f = \phi_1^{-1} \circ g \circ \phi_2$.

The following theorem is the main result of this work.

Theorem III.1 *Let f be a topological polynomial of simple Siegel disk type. Then f is combinatorially equivalent to a polynomial if and only if f has no Thurston obstruction. In particular, the boundary of the Siegel disk of the candidate polynomial must be a quasi-circle containing at least one critical point.*

The main idea of the proof is as follows. First we construct a sequence of critical finite and symmetric branched covering maps $\{F_n\}$ such that

- F_n has the same combinatorial structure outside the closed unit disk as that of f , and

- when restricted on the unit circle, F_n is a homeomorphism with rational rotation number θ_n and $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$.

By using the short geodesic analysis, we can prove that F_n has no Thurston obstruction when n is large enough. By using Thurston's theorem and the symmetric property of F_n , we get a sequence of Blaschke product $\{G_n\}$. We then prove that the sequence $\{G_n\}$ obtained in this way is a compact family. This is technically the hardest part of the whole proof. Next, we take a limit map G of a convergent subsequence of $\{G_n\}$. Clearly G is a homeomorphism on the unit circle with rotation number θ and has the same combinatorial structure outside the unit disk as that of f . To get the candidate polynomial, we use Herman-Swiatek's linearization theorem to do a quasi-conformal surgery of G along the unit circle. By this way, we get a topological polynomial G_0 such that

- $G_0(z) = G(z)$ for $z \notin \Delta$, and
- $G_0(z) = H^{-1} \circ R_\theta \circ H(z)$ for $z \in \Delta$ where $H : \Delta \rightarrow \Delta$ is a quasi-conformal map.

Now we can pull back the structure of H by G_0 and get a G_0 -invariant line field $\mu(z)$ on the whole Riemann sphere. Let ϕ be the Beltrami solution for

μ , then $g = \phi \circ G_0 \circ \phi^{-1}$ is the polynomial which is combinatorially equivalent to f . We complete the proof.

One interesting question is if the candidate polynomial constructed in the proof of Theorem 3.1 is the unique polynomial to realize the topological model up to a conformal conjugation. In Chapter 4, we will present an idea to attack this problem. First we will discuss how to prove that the *Julia set* of the candidate polynomial has zero Lebesgue measure. Then by using a theorem of McMullen, we show how to use the zero measure statement to prove that the candidate polynomial is the unique polynomial to realize the topological model such that the boundary of the Siegel disk is a quasi-circle. By Shishikura's recent result, which says that the boundary of the Siegel disk of constant type of any polynomial must be a quasi-circle, we get the rigidity completely. In particular, this implies that the candidate polynomial g does not depend on the construction of the Blaschke model G and the quasi-conformal surgery.

The structure of this thesis is as follows.

In Chapter 1, we will outline the proof of Thurston's theorem following Douady-Hubbard [DH1].

In Chapter 2, we will follow Douady-Hubbard's proof of Thurston's theo-

rem to show a new proof of CJS's theorem on the geometrically finite rational maps.

In Chapter 3, we will prove Theorem 3.1.

In Chapter 4, we will discuss the zero measure problem of the *Julia set* and the rigidity problem.

Chapter 1. Critically Finite Branched Covering Maps and Thurston's Theorem

§1.1. Thurston obstruction

Let $f : S^2 \rightarrow S^2$ be an orientation preserving branched covering map with degree greater than one. We denote by $\deg_x f$ the local degree of f at x . We call

$$\Omega_f = \{x \in S^2 \mid \deg_x f > 1\}$$

the critical set of f , and

$$P_f = \overline{\bigcup_{n>0} f^{on}(\Omega_f)}$$

the post-critical set. We say f is a post-critically finite branched covering map if P_f is a finite set.

Two branched covering mappings f and g are equivalent if and only if there exist homeomorphisms $\theta_1, \theta_2 : (S^2, P_f) \rightarrow (S^2, P_g)$ such that the diagram

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\theta_2} & (S^2, P_g) \\ f \downarrow & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\theta_1} & (S^2, P_g) \end{array}$$

commutes and θ_1 is isotopic to $\theta_2 \text{ rel } P_f$.

If γ is a simple closed curve on $S^2 - P_f$, then the set $f^{-1}(\gamma)$ is a union of disjoint simple closed curves. If γ moves continuously, so does each component of $f^{-1}(\gamma)$. A simple closed curve γ is non-peripheral if each component of $S^2 \setminus \gamma$ contains at least two points of P_f . We will need to consider systems

$$\Gamma = \{\gamma_1, \dots, \gamma_n\}$$

of simple, closed, disjoint, non-homotopic and non-peripheral curves on $S^2 \setminus P_f$. Such a system will be called a *multicurve* on $S^2 \setminus P_f$.

We say a *multicurve* Γ is *f-stable* if for any $\gamma \in \Gamma$, all the non-peripheral components of $f^{-1}(\gamma)$ are homotopic in $S^2 \setminus P_f$ to elements of Γ . To each *f-stable* multicurve Γ we can associate the *Thurston linear transformation*.

$$f_\Gamma : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$$

as follows: Let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $S^2 \setminus P_f$.

Define

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i$$

where

$$d_{i,j,\alpha} = \deg f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \rightarrow \gamma_j.$$

Since f_Γ has a matrix with non-negative entries, there exists a largest eigenvalue $\lambda(\Gamma, f) \in \mathbb{R}_+$. We call a *multicurve* Γ a *Thurston obstruction* if

$$\lambda(\Gamma, f) \geq 1.$$

Lemma I.1 Thurston transformation commutes with the iteration, i.e.,

$$(f^n)_\Gamma = (f_\Gamma)^n.$$

Let $v_f : S^2 \rightarrow Z^+ \cup \{0\}$ be the minimal positive function such that

1. $v_f(x) = 1$ for all $x \notin P_f$.
2. $v_f(x)$ is the integer multiple of $v_f(y) \cdot \deg_y f$ for all $y \in f^{-1}(x)$.

Then we define the orbifold $O_f = (S^2, v_f)$. We say O_f is hyperbolic if

$$\chi(O_f) = 2 - \sum_{x \in S^2} \left(1 - \frac{1}{v_f(x)}\right) < 0.$$

Thurston proved the following remarkable theorem.

Theorem I.2 *A critically finite orientation-preserving branched covering map f with hyperbolic orbifold O_f is combinatorially equivalent to a rational map if and only if f has no Thurston obstruction.*

In the following sections, we will provide an outline of the proof of the theorem. The reader should refer to Douady-Hubbard's wonderful article on this theorem.

§1.2. The Teichmüller space

In the following, we use \mathbb{P}^1 to denote the sphere with the standard complex structure. The Teichmüller space T_f is the Teichmüller space modeled on (S^2, P_f) . It can be obtained in both of the following two ways,

1. The space of smooth complex structures on S^2 , two such structures μ_1 and μ_2 being identified if $\mu_1 = h^* \mu_2$ for some diffeomorphism $h : S^2 \rightarrow S^2$ with $h|_{P_f} = id$ and h isotopic to the identity rel P_f .
2. The space of diffeomorphisms $\phi : (S^2, P_f) \rightarrow \mathbb{P}^1$, with ϕ_1 and ϕ_2 identified if and only if there exists an analytic isomorphism $h : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi_1} & \mathbb{P}^1 \\ id \downarrow & & \downarrow h \\ (S^2, P_f) & \xrightarrow{\phi_2} & \mathbb{P}^1 \end{array}$$

commutes on P_f , and commutes up to isotopy rel P_f .

A $(-1, 1)$ -differential $\mu \frac{dz}{dz}$ is a collection of functions μ_i defined on a family of open charts U_i such that

$$\mu_i \left(\frac{dz_i}{dz_j} \right)^{-1} \overline{\left(\frac{dz_i}{dz_j} \right)} = \mu_j.$$

where μ_i is defined on U_i in terms of z_i . By the definition, we can speak of $|\mu|$ and therefore $\|\mu\|_\infty$ is well defined. We say a $(-1, 1)$ -differential μ is a

Beltrami differential if $\|\mu\|_\infty < 1$. Let

$$B = \{\mu \mid \|\mu\|_\infty < 1\}.$$

By the Riemann mapping theorem, for each $\mu \in B$, there exists a unique quasiconformal homeomorphism $\phi : S^2 \rightarrow S^2$ which fixes $0, 1, \infty$ such that $\mu(z) = \frac{\phi_z}{\phi_{\bar{z}}}$. A Beltrami path is a piecewisely smooth map $\gamma : [0, 1] \rightarrow B$. We use ϕ_t to denote the correspondent path of quasiconformal homeomorphisms which fixes $0, 1$ and ∞ . Let $\phi : S^2 \rightarrow S^2$ be any quasiconformal homeomorphism, the pull back of ϕ_t by ϕ^{-1} is defined as

$$\phi^*(\phi_t) = \phi_t \circ \phi^{-1}.$$

We use $\phi^*\gamma(t)$ of $\gamma(t)$ to denote the Beltrami coefficient of $\phi^*(\phi_t)$. Let $\xi = \gamma'(t)$ be the tangent vector of $\gamma(t)$ at t , we use $d\phi^*\xi$ to denote the tangent vector of $\phi^*\gamma(t)$ at t . Now we take $\phi = \phi_{t_0}$ and define the Teichmüller norm of ξ (with respect to T_f) is

$$\|\xi\| = \sup \iint q(z) d\phi^*\xi dz d\bar{z}$$

where sup is taken over all the quadratic differentials $q(z)dz^2$ on $(\mathbb{P}^1, \phi(P_f))$ with

$$\iint |q(z)| dz d\bar{z} = 1$$

and $\phi(P_f)$ the set of poles. Since

$$\iint |q(z)| dz d\bar{z} = 1,$$

any pole of $q(z)$ must be simple. For any two elements μ and ν in T_f , the Teichmüller distance between μ and ν is given by

$$d_T(\mu, \nu) = \inf \left\{ \int \|\gamma'(t)\| dt \right\},$$

where \inf is taken over all piecewisely smooth Beltrami paths connecting μ and ν . The reader may refer to [Gardiner, §7.1 for the above].

§1.3. Thurston's pull back algorithm

Let f be a critically finite orientation-preserving branched covering map.

There exists a C^1 branched covering map g such that

1. f is combinatorially equivalent to g ,
2. $\frac{g_z}{g_{\bar{z}}}$ is defined almost everywhere, and $\|\frac{g_z}{g_{\bar{z}}}\| \leq \delta < 1$ for some $\delta > 0$.

Therefore, we can always assume that $h(z) = \frac{f_z}{f_{\bar{z}}}$ is defined almost everywhere, and $\|h(z)\| \leq \delta < 1$ for some $\delta > 0$. Now for each Beltrami differential μ on S^2 , we define the pull back of μ by f as follows,

$$f^*(\mu)(z) = \frac{h(z) + \mu((f(z))\theta(z))}{1 + \overline{h(z)}\mu(f(z))\theta(z)}$$

Where $\theta(z) = \frac{\bar{f}}{f_z}$. Clearly, the pull back of each piecewise C^1 Beltrami path is still a piecewise C^1 Beltrami path. Now let μ be a Beltrami differential on S^2 and $\nu = f^*(\mu)$. Let ϕ_μ, ϕ_ν be the Beltrami solutions for μ and ν , then we have the following diagram,

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi_\nu} & (S^2, \phi_\nu(P_f)) \\ f \downarrow & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\phi_\mu} & (S^2, \phi_\mu(P_f)) \end{array}$$

where g is a rational function. Here we call ϕ_ν the pull back map of ϕ_μ under f . Let $\gamma(t)$ be a smooth Beltrami path passing through μ , then the pull back $f^*\gamma(t)$ is a smooth Beltrami path which passes through ν . Let η be the tangent vector of $\gamma(t)$ at μ , then $\eta' = df^*\eta$ is the tangent vector of $f^*\gamma(t)$ at ν . Let $\xi = d\phi_\mu^*\eta$ and $\xi' = d\phi_\nu^*\eta'$. We have the following diagram,

$$\begin{array}{ccc} \eta' & \xrightarrow{d\phi_\nu^*} & \xi' \\ df^* \uparrow & & \uparrow dg^* \\ \eta & \xrightarrow{d\phi_\mu^*} & \xi \end{array}$$

Let $q'(z)dz^2$ be the quadratic differential on $(\mathbb{P}^1, \phi_\nu(P_f))$. According to [DH1], the push forward of $q'(z)dz^2$ is the quadratic differential on $(\mathbb{P}^1, \phi_\mu(P_f))$ which is given by

$$q(w) = \sum_{g(z)=w} \frac{q'(z)}{[g'(z)]^2}$$

We use $\delta g^*(q')$ to denote the push forward of q' by g . By the above formula, we have $\langle q', \xi' \rangle = \langle \delta g^*(q'), \xi \rangle$, therefore $\|\xi'\| \leq \|\xi\|$ and the pull back will not increase the Teichmüller length of the Beltrami path. Now consider the composition of two consecutive pull back in the following diagram,

$$\begin{array}{ccc}
 \eta'' & \xrightarrow{d\phi_{n+2}^*} & \xi'' \\
 \uparrow df^* & & \uparrow dg_{n+1}^* \\
 \eta' & \xrightarrow{\phi_{n+1}} & \xi' \\
 \uparrow df^* & & \uparrow dg_n^* \\
 \eta & \xrightarrow{\phi_n} & \xi
 \end{array}$$

In [DH1], it is proved that

$$\|\delta g_n^* \delta g_{n+1}^* q''\| < \|q''\|$$

therefore we have

Lemma I.2 $\|\xi''\| < \|\xi\|$.

For the proof of Lemma 1.2, see Proposition 3.3 of [DH].

Lemma I.3 *Let $X \subset \mathbb{P}^1$ be a finite subset such that $0, 1, \infty \in X$. Let m be the cardinality of X and suppose $m \geq 4$. For any $\delta > 0$, if all the simple closed geodesics in $\mathbb{P}^1 \setminus X$ have length greater than δ then the spherical distance between any two distinct points in X has a positive lower bound which depends only on δ and m .*

Proof 1 First we will show that the spherical distance between ∞ and any finite point in X has a positive lower bound which is dependent only on δ and m . Let $X = \{x_1, \dots, x_{m-1}\}$ and $x_m = \infty$. Suppose $|x_1| \leq \dots \leq |x_{m-1}|$. Since $x_2 \leq 1$, Let $M = |x_{m-1}|$, we have

$$\prod_{2 \leq i \leq m-1} \frac{|x_{i+1}|}{|x_i|} = \frac{|x_{m-1}|}{|x_2|} \geq M.$$

So

$$\max\left\{\frac{|x_{i+1}|}{|x_i|}\right\} \geq M^{\frac{1}{m-3}}.$$

Let $A_i = \{z \mid |x_i| < z < |x_{i+1}|\}$, we have an i_0 such that

$$\text{mod}(A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$

Therefore the unique simple closed geodesic γ in A_{i_0} has hyperbolic length

$$\|\gamma\|_{A_{i_0}} = \frac{\pi}{\text{mod}(A_{i_0})} \leq \frac{2\pi^2(m-3)}{\log M}.$$

Since

$$\|\gamma\|_{A_{i_0}} \geq \|\gamma\|_{S^2 \setminus X} \geq \delta,$$

This implies that

$$M \leq e^{\frac{2\pi^2(m-3)}{\delta}},$$

and the spherical distance between ∞ and any finite point in X has a positive lower bound which is dependent only on δ and m .

Now we will show that the spherical distance between any two finite points in X has a positive lower bound dependent only on δ and m . Without loss of generality, we assume

$$d(x_1, x_2) = \min_{x_i \neq x_j} \{d(x_i, x_j)\} \quad \text{and} \quad d(x_1, x_i) \leq d(x_1, x_{i+1})$$

for $i = 1, \dots, m-1$. Since

$$\max\left\{\frac{d(x_1, 0)}{d(x_1, x_2)}, \frac{d(x_1, 1)}{d(x_1, x_2)}\right\} \geq \frac{1}{2d(x_1, x_2)},$$

therefore we have i_0 such that

$$\text{mod}(B_{i_0}) \geq \frac{\log \frac{1}{2d(x_1, x_2)}}{2\pi(m-3)}$$

where

$$B_{i_0} = \{z | d(x_1, x_{i_0}) < d(x_1, z) < d(x_1, x_{i_0+1})\}.$$

Let η be the unique simple closed geodesic in B_{i_0} . Then we have

$$\|\eta\|_{B_{i_0}} = \frac{\pi}{\text{mod}(B_{i_0})} \leq \frac{2\pi^2(m-3)}{\log \frac{1}{2d(x_1, x_2)}}.$$

Since

$$\|\eta\|_{B_{i_0}} \geq \|\eta\|_{S^2 \setminus X} \geq \delta,$$

so we have

$$d(x_1, x_2) \geq \frac{1}{2} e^{-\frac{2\pi^2(m-3)}{\delta}}.$$

so the spherical distance between any two finite points in X has a positive lower bound dependent only on δ and m .

Let R_d be the set of all the rational functions with degree $d > 1$. For a sequence $\{f_n\} \subset R_d$ and $f \in R_d$, we say $f_n \rightarrow f$ if f_n is convergent to f uniformly in the spherical metric. Therefore we have a topological space R_d . We define an equivalent relation in R_d as follows: for any f and g in R_d , we say f is equivalent to g if $f = h_1 \circ g \circ h_2$ for some linear transformation $h_i, i = 1, 2$. We denote the quotient space by S_d . For $\delta > 0$, let

$$\mathcal{F}_{d,\delta} = \{[f] \in S_d \mid d(x, y) \geq \delta, \text{ for all distinct critical values } x, y \text{ of } f\}$$

Lemma I.4 $\mathcal{F}_{d,\delta}$ is compact.

Proof 2 For $f \in \mathcal{F}_{d,\delta}$, we use X to denote the set of its critical values. If $|X| = 1$, the number of the preimages of the critical value would be $\sum d_i \geq \sum (d_i - 1) + 1 = 2d - 1$. So we get $d \geq 2d - 1$ and therefore $d \leq 1$ which is a contradiction with $d > 1$. If $|X| = 2$, f is equivalent to z^d . So $\mathcal{F}_{d,\delta}$ consists of one element and must be compact. So we need only consider the case when $|X| \geq 3$. By a composition with a linear transformation on the left and right, we can assume both X and $f^{-1}(X)$ contain $0, 1, \infty$.

Since $f : \mathbb{P}^1 \setminus f^{-1}(X) \rightarrow S^2 \setminus X$ is a covering map, any simple short geodesic in $\mathbb{P}^1 \setminus f^{-1}(X)$ will be mapped to a short closed geodesic in $\mathbb{P}^1 \setminus X$. So the length of the closed geodesic in $\mathbb{P}^1 \setminus f^{-1}(X)$ has a positive lower bound. Since $f^{-1}(X)$ contains $0, 1, \infty$, so by Lemma 1.3, we have that $d(w, v) \geq \epsilon$ for any $w \neq v \in f^{-1}(X)$. Now take any sequence f_n in $\mathcal{F}_{d,\delta}$, by taking a subsequence we can assume $X_n \rightarrow X$ and $f_n^{-1}(X_n) \rightarrow Y$. Since X_n contains $0, 1, \infty$, so $\{f_n\}$ is normal in $\mathbb{P}^1 \setminus Y$. Let g be the limit point of any convergent subsequence f_{n_k} of $\{f_n\}$. Then g is holomorphic in $\mathbb{P}^1 \setminus Y$. Clearly any point in Y can not be an essential pole of g , so g must be a rational function. Now we'll prove that $f_{n_k} \rightarrow g$ uniformly on the sphere under the spherical metric. For each $x \in X$, there is a small open disk centered at x with boundary circle γ such that the closure of the disk doesn't contain any other points in X . We denote this disk by $D(\gamma)$. We can take D_γ small enough such that any component of $g^{-1}(D(\gamma))$ must be a open topological disk contains only one point y in Y with a smooth boundary curve η . We denote such a component by $D(\eta)$. We have $g : \eta \rightarrow \gamma$ is a covering map with degree $\deg g_y$. Since $f_{n_k} \rightarrow g$ uniformly in any compact subset of $S^2 \setminus Y$, therefore, as k is large enough, we have η_{n_k} near η such that $f_{n_k} : \eta_{n_k} \rightarrow \gamma$ is a covering map with the same degree as $g : \eta \rightarrow \gamma$. Since when k is large enough, $D(\eta_{n_k})$ contains

only one point in $f_{n_k}^{-1}(X_{n_k})$, so $f_{n_k} : D(\eta_{n_k}) \rightarrow D_\gamma$ is a branched covering map with the same degree as $g : D(\eta) \rightarrow D(\gamma)$. Therefore $f_{n_k} \rightarrow g$ uniformly on the whole sphere. So g is rational map of degree d and $\text{dist}(x, y) \geq \delta$ for any two distinct critical values of g . Therefore $g \in F_{d, \delta}$. We have proved that, $F_{d, \delta}$ is compact.

Now we can have

Lemma I.5 *Let $\{\phi_n\}$ be the sequence of the pull back of the identity map. If there exists a $\delta > 0$ independent of n such that the hyperbolic length of any simple closed geodesic in $\mathbb{P}^1 \setminus \phi_n(P_f)$ is greater than δ , then the pull back uniformly contracts the Teichmüller metric, and therefore, $\{\phi_n\}$ is exponentially convergent in the Teichmüller metric.*

Proof 3

$$\begin{array}{ccc}
 (S^2, P_f) & \xrightarrow{\phi_{n+2}} & (\mathbb{P}^1, \phi_{n+2}(P_f)) \\
 f \downarrow & & \downarrow g_{n+1} \\
 (S^2, P_f) & \xrightarrow{\phi_{n+1}} & (\mathbb{P}^1, \phi_{n+1}(P_f)) \\
 f \downarrow & & \downarrow g_n \\
 (S^2, P_f) & \xrightarrow{\phi_n} & (\mathbb{P}^1, \phi_n(P_f))
 \end{array}$$

By lemma 1.2, we have

$$\|\delta g_n^* \delta g_{n+1}^* q_{n+2}\| < \|q_{n+2}\|$$

for any quadratic differential q_{n+2} on $\mathbb{P}^1 \setminus \phi_{n+2}(P_f)$. Since the space $\{\|q\| = 1\}$ is compact, so we have a ϵ_n , such that $1 > \epsilon_n > 0$ and

$$\|\delta g_n^* \delta g_{n+1}^* q_{n+2}\| \leq (1 - \epsilon_n) \|q_{n+2}\|.$$

Let

$$\epsilon_0 = \inf\{\epsilon_n\}$$

By the assumption and Lemma 1.3, we know all the quadratic differentials are contained in a compact set. By lemma 1.4, we know the family of $\{g_n\}$ is a compact family. Therefore we can assume $q_{n+2} \rightarrow q$, $g_n \rightarrow h_1$ and $g_{n+1} \rightarrow h_2$ such that

$$\epsilon_0 = 1 - \frac{\|\delta h_1^* \delta h_2^* q\|}{\|q\|}$$

By Lemma 1.2 we get $\epsilon_0 > 0$.

§1.4 two lemmas in hyperbolic geometry on Riemann surface

Lemma I.6 *Let S be a hyperbolic Riemann surface. Then any closed geodesic γ with length $l(\gamma) < \log(\sqrt{2} + 1)$ must be a simple closed curve and any two such short geodesics are disjoint.*

For the proof, see Corollary 6.6 and Proposition 6.7 in [DH]. From now on, we say a closed geodesic γ is short if the length $l(\gamma) < \log(\sqrt{2} + 1)$.

Lemma I.7 *Let S be a hyperbolic Riemann surface. Let $X \subset S$ be a finite set with $|X| = p$. Let γ be a short closed geodesic in S . Let $S' = S \setminus X$. Let $\gamma_i, i = 1, \dots, n$ be all short closed geodesics on S' which are homotopic to γ in S . Then $|\sum_{1 \leq i \leq n} \frac{1}{l(\gamma_i)} - \frac{1}{l(\gamma)}| < C(p)$, where $C(p)$ is a constant only dependent on p .*

For the proof, see Theorem 7.1 in [DH].

§1.5 the hyperbolic length of simple closed curve

Let f be a critically finite orientation-preserving branched covering map. Let $\gamma \subset S^2 \setminus P_f$ be a simple closed curve. Let $\phi : S^2 \rightarrow \mathbb{P}^1$ be a quasi-conformal homeomorphism. Consider the hyperbolic Riemann surface $\mathbb{P}^1 \setminus \phi(P_f)$. Let ξ be the unique simple closed geodesic in the homotopy class of $\phi(\gamma)$. We define the ϕ -norm $\|\gamma\|_\phi$ of γ to be the hyperbolic length of ξ in $\mathbb{P}^1 \setminus \phi(P_f)$. Furthermore, for a subset $X \in S^2$ and a simple closed curve $\gamma \in S^2 \setminus X$, we use $\|\gamma\|_{\phi, X}$ to denote the hyperbolic length of $\phi(\gamma)$ in $\mathbb{P}^1 \setminus \phi(X)$.

Lemma I.8 *For any simple closed curve $\gamma \subset S^2 \setminus P_f$, the map $\tau : T(S^2, P_f) \rightarrow \mathbb{R}$ by $[\phi] \rightarrow \log \|\gamma\|_{\phi}^{-1}$ is a Lipschitz map with Lipschitz constant 2.*

Now let $\phi_0 = Id$. Let $\{\phi_n\}$ be the sequence of the pull back of ϕ_0 by f .

We need only to prove

Lemma I.9 *There exists a positive number δ , such that for all simple closed curve $\gamma \subset S^2 \setminus P_f$, $\|\gamma\|_{\phi_n} > \delta$ for all $n = 1, 2, \dots$.*

We will give the proof of Lemma 1.9 in §1.7.

§1.6 the Gap Lemma

Lemma I.10 *Let $p = |P_f| < \infty$. There is a sequence $\delta_k \rightarrow 0$ and a sequence $J_k \rightarrow \infty$ such that for any $\phi : S^2 \rightarrow S^2$ there is a sequence of intervals $[a_k^\phi, b_k^\phi]$ with*

- $\delta < a_k^\phi < b_k^\phi$, and
- $b_k^\phi / a_k^\phi = J_k$, and
- $\|\gamma\|_{\phi} < a_k^\phi$ or $\|\gamma\|_{\phi} > b_k^\phi$ for any short simple closed ϕ -short geodesic.

For the proof see Proposition 8.1 of [DH].

§1.7 the proof of Thurston's theorem

In this section, we will prove the sufficiency of the Thurston's theorem. For the necessary part, see section 4 of [DH].

Proof 4 *From Lemma 1.4 and Lemma 1.5, we need only to prove Lemma 1.9. Since there can only be finite distinct Thurston linear transformation, so by Lemma 1.1, we have a universal m , such that*

$$(f_\Gamma)^{om} = A^m = (b_{ij})$$

with $\|A\| < \frac{1}{2}$.

By lemma 1.10, there exists a δ_k such that for any $\phi_n, n = 0, 1, \dots$, if

$$\{\gamma \subset S^2 \setminus P_f \mid \|\gamma\|_{\phi_n} < \delta_k\} \neq \emptyset,$$

then

$$\mathcal{F}_{\phi_n} = \{\gamma \subset S^2 \setminus P_f \mid \|\gamma\|_{\phi_n} < a_k^{\phi_n}\}$$

is a f^{om} - stable family. Let

$$\|\mathcal{F}_{\phi_n}\| = \sum_{\gamma \in \mathcal{F}_{\phi_n}} \frac{1}{\|\gamma\|_{\phi_n}}$$

Let \mathcal{G} be the family which consists of the non-peripheral components of the

preimage of \mathcal{F}_{ϕ_n} by $f^{\circ m}$. Let $P_m = f^{-m}(P_f)$. Then we have

$$\begin{array}{ccc} (S^2, P_m) & \xrightarrow{\phi_{n+m}} & (\mathbb{P}^1, \phi_{n+m}(P_m)) \\ f^{\circ m} \downarrow & & \downarrow g_m \\ (S^2, P_f) & \xrightarrow{\phi_n} & (\mathbb{P}^1, \phi_n(P_f)) \end{array}$$

Since $g_m : (\mathbb{P}^1, \phi_{n+m}(P_m)) \rightarrow (\mathbb{P}^1, \phi_n(P_f))$ is a holomorphic covering map, so

$$\sum_{\xi \in \mathcal{G}} \|\xi\|_{\phi_{n+m}, P_m}^{-1} = \sum_{i,j} b_{ij} \|\gamma_i\|_{\phi_n}^{-1} \leq \frac{1}{2} \|\mathcal{F}_{\phi_n}\|$$

By Lemma 1.7, we have

$$\|\mathcal{F}_{\phi_{n+m}}\| \leq \sum_{\xi \in \mathcal{G}} \|\xi\|_{\phi_{n+m}, P_m}^{-1} + M$$

where the constant M is dependent only on f and m . Let $M' = \max\{M, (p-2)/\delta_k\}$, so we get

$$\|\mathcal{F}_{\phi_{n+m}}\| \leq \frac{1}{2} \|\mathcal{F}_{\phi_n}\| + M'$$

for any n . Let $N = \max\{\|\mathcal{F}_{\phi_0}\|, \dots, \|\mathcal{F}_{\phi_0}\|, 2M'\}$, we have

$$\|\mathcal{F}_{\phi_k}\| \leq 3N$$

for all k . So \mathcal{F}_{ϕ_n} has a uniform upper bound for all n . Therefore there is a positive lower bound of $\|\gamma\|_{\phi_n}$ for all simple closed curve $\gamma \subset S^2 \setminus P_f$ and for

all $n = 1, 2, \dots$. We complete the proof by Lemma 1.9.

Chapter 2. Geometrically Finite Branched Covering Maps and A New Proof of Cui-Jiang-Sullivan's Theorem

§2.1 Statements and Definitions

In this Chapter, we will follow the idea of Chapter 1 to give a new proof of Cui-Jiang-Sullivan's theorem.

Definition II.3 *Let f be an orientation preserving branched covering map.*

We say f is of geometrically finite type if the post-critical set P_f is an infinite set and P_f has only finite accumulation points.

Lemma II.11 *If f is a branched covering map of geometrically finite type, then every accumulation point of P_f must be periodic.*

Proof 5 *Let $\Lambda_f = \{a_1, \dots, a_m\}$ be the set of the accumulation points of P_f .*

First we prove Λ_f is closed under f . Take any $x \in \Lambda_f$. We know there is a critical point c and a subsequence $\{f^{n_k}(c)\}$ such that $f^{n_k}(c) \rightarrow x$. Therefore we have $f^{n_k+1}(c) \rightarrow f(x)$. So $f(x) \in \Lambda_f$ and we have proved Λ_f is closed

under f . Now let

$$Q = \{x \in \Lambda_f \mid x \text{ is periodic point of } f\}$$

we need only prove $\Lambda_f \setminus Q$ is empty. Suppose $\Lambda_f \setminus Q$ is not empty. For any $x \in Q$, since $f(x) \in Q$, therefore there is an $\epsilon > 0$ such that

$$f(B_\epsilon(x)) \cap B_\epsilon(y) = \emptyset$$

for all $x \in Q$ and all $y \in \Lambda_f \setminus Q$. By making ϵ smaller, we assume $d(x_1, x_2) > \epsilon$ for any two points $x_1, x_2 \in \Lambda_f$.

Now take any $y \in \Lambda_f \setminus Q$ and let l be a number such that $f^l(y) = x \in Q$.

Let c be a critical point of f such that $f^{n_k}(c) \rightarrow y$. Therefore $f^{n_k+l}(c) \rightarrow x$.

Let M be the number such that

$$f^k(c) \in \bigcup_{x \in \Lambda_f} B_\epsilon(x)$$

for all $k > M$. Since $f^{n_k+l}(c) \rightarrow x$ for some $x \in Q$, so we have $n > M$ such that $f^n(c) \in B_\epsilon(x)$. Since $f^{n+1}(c) \in \bigcup_{x \in \Lambda_f} B_\epsilon(x)$ and $f(B_\epsilon(x)) \cap B_\epsilon(y) = \emptyset$ for all $y \in \Lambda_f \setminus Q$, so $f^{n+1}(c) \in B_\epsilon(z)$ for some $z \in Q$. By induction, we can prove $f^k(c) \in \bigcup_{x \in Q} B_\epsilon(x)$ for all $k \geq n$. This is a contradiction with $f^{n_k}(c) \rightarrow y$.

In [McM1], McMullen proved that a critically infinite rational map having no Siegel disks and Herman rings has no Thurston Obstruction. In particular,

any geometrically finite rational map has no *Thurston*

Obstruction. This raised the following question: If f is a geometrically finite map without *Thurston Obstruction*, is f always combinatorially equivalent to a rational map? The answer is negative. In the work of Cui, Jiang, and Sullivan, they show that there exists a geometrically finite branched covering map with no *Thurston Obstruction* but not combinatorially equivalent to any rational map. (See [CJS] or [CJS1], [CJS2]) The way they studied that is to construct an invariant shrinking family of curves and introduce a linearizable condition at the accumulation points. The linearizable condition is

Definition II.4 *Let f be a branched covering map of geometrically finite type. For any accumulation point a of P_f , let p be the period of f at a . We say f is linearizable at a if there exist homeomorphisms $\phi, \psi : S^2 \rightarrow \overline{\mathbb{C}}$ such that*

1. $\phi(a) = 0$,
2. ϕ is isotopic to ψ rel P_f ,
3. $(\phi \circ f \circ \psi^{-1})^{\circ p}(z) = \lambda z$ for some $|\lambda| < 1$, if $\deg_a f^{\circ p} = 1$ or $(\phi \circ f \circ \psi^{-1})^{\circ p}(z) = z^d$ if $\deg_a f^{\circ p} = d > 1$.

They then proved

Theorem II.3 (Cui-Jiang-Sullivan) *A geometrically finite branched covering map f is combinatorially equivalent to a rational map if and only if f has no Thurston Obstruction and f is locally linearizable at all the accumulation points of P_f .*

In this chapter, we will use Thurston's argument as in Chapter 1 to give a new proof of this theorem. For the original proof of Theorem 2.1, please refer to [CJS], [CJS1] and [CJS2].

Since f is linearizable at all the accumulation points of P_f , so for each accumulation point $a_i \in \Lambda_f$, $1 \leq i \leq m$, we can assume there is an open disk D_i with center at a_i such that $f^{\circ p}|_{D_i}(z) = a_i + \lambda(z - a_i)$ for some λ with $0 < |\lambda| < 1$ or $f^{\circ p}|_{D_i}(z) = a_i + (z - a_i)^n$ for some $n \geq 2$. By making each D_i small enough, we can assume D_i and D_j are disjoint for all $1 \leq i \neq j \leq m$,

Lemma II.12 *We can take D_i such that for each D_i there is an annulus A_i which is mapped into some D_j by f and does not any post-critical point.*

Proof 6 *Suppose D is one of the linearization disk such that $a \in D \cap \Lambda_f$ has period p . So we have $f(D) \subset D$. Take an open topological disk B such that $f(D) \subset B \subset D$ and $B \setminus f(D)$ does not contain any point in P_f . Next we divide $B \setminus f(D)$ into p subannulus E_1, \dots, E_p .*

If we denote $E_0 = \emptyset$, then the new linearization disks are taken to be

$$D_j = f(f^{\circ p}(D) \cup (\cup_{0 \leq i \leq j} E_i))$$

for $j = 0, 1, \dots, p-1$. The annulus A_j around D_j can be taken to be

$$A_j = f^{\circ j}(B_j)$$

Here is the idea of our proof. Let

$$\Omega = \cup_{i=1}^m D_i$$

$$P_1 = \{z \in P_f | z \notin \Omega\}$$

$$X = \partial\Omega \cup P_1$$

We construct the Teichmüller space T_f modeled on the Riemann surface

$$R = (S^2 \setminus \Omega, X)$$

and consider the Thurston's pull back

$$\sigma_f : T_f \rightarrow T_f.$$

We will prove that σ_f contracts the Teichmüller metric. We consider the push forward δ_f which is the dual of $d\sigma_f$. Let $q(z)dz^2$ be any holomorphic quadratic differential on R with $\|q\| = 1$ and all the poles contained in $P_f \setminus \Omega$.

Since $\|\delta_f q\|$ will not count the integral of $|q|$ over A_i for all $i = 1, \dots, m$, so $\|\delta_f q\| < \|q\|$. Next by a compact argument, we can prove that σ_f is a strictly contracting map. To prove σ_f uniformly contracts the Teichmüller metric, we need only to prove that the geometry of $\phi_n(R)$ is uniformly bounded. This is proved by showing that the length of the short closed geodesic in $\phi_n(R)$ has a uniform positive lower bound. Therefore, the Thurston pull back uniformly contracts the Teichmüller metric. Then we use the fixed point argument to obtain the candidate rational map.

§2.2. A New Proof of Theorem 2.1

Let $\Lambda_f = \{a_1, \dots, a_m\}$ be the set of the accumulation points of P_f . For each $a_i \in \Lambda_f$, we can assume that f is already linearized in a open neighborhood D_i around a_i such that

1. D_i and D_j are disjoint for all $1 \leq i \neq j \leq m$,
2. for each D_i , there is an annulus A_i attached the boundary D_i from outside which is mapped into some D_j by f .

(see Lemma 2.2)

For each $a_i \in \Lambda_f$, if a_i is not a critical value of f , we can assume that D_i does not contain any critical value by taking a smaller D_i . We divide our proof into the following subsections.

The Teichmüller space T_f and the Thurston pull back

As before, we use the following notation

$$\Omega = \cup_{i=1}^m D_i$$

$$P_1 = \{z \in P_f | z \notin \Omega\}$$

$$X = \partial\Omega \cup P_1$$

By assumption, P_1 is a finite set. Let M be the set of all the Beltrami coefficients on the sphere. Consider the subset $M^* \subset M$

$$M^* = \{\mu \in M \mid \mu(z) = 0 \text{ for } z \in \Omega\}$$

There is a unique quasi-conformal homeomorphism $\phi_\mu : S^2 \rightarrow S^2$ which has complex dilation $\mu(z)$ and fixes $0, 1$ and ∞ . Without loss of the generality, we can assume that $0, 1$, and ∞ belong to P_f .

We define an equivalence relation in M^* : for $\mu, \nu \in M^*$, $\mu \sim \nu$ if and only if ϕ_μ is isotopic to ϕ_ν rel X , therefore the set of the equivalence class M^* / \sim

is exactly the Teichmüller space modeled on $(S^2 \setminus \Omega, X)$. In the following, we use T_f to denote this Teichmüller space. Let μ be a Beltrami coefficient on the sphere S^2 , then the pull back of μ by f is defined to be

$$f^*(\mu)(z) = \frac{h(z) + \mu(f(z))\theta(z)}{1 + \overline{h(z)}\mu(f(z))\theta(z)}$$

where $h(z) = \frac{f_x}{f_z}$ and $\theta(z) = \frac{\overline{f_x}}{f_z}$.

Lemma II.13 *The pull back $f^* : M^* \rightarrow M^*$ induces a map $\sigma_f : T_f \rightarrow T_f$. Furthermore, the pull back of each C^1 Beltrami path is still a C^1 Beltrami path.*

Proof 7 *Since f is analytic contracting inside Ω , so $f^*(\mu) \in M^*$ for any $\mu \in M^*$. Let $\nu = f^*(\mu)$, we have*

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi_\nu} & (S^2, P_g) \\ f \downarrow & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\phi_\mu} & (S^2, P_g) \end{array}$$

From this diagram, we see the isotopy rel X can be lifted, so f^ preserves the equivalence class. Therefore f^* induces a map $\sigma_f : T_f \rightarrow T_f$. Clearly the pull back by f of any C^1 Beltrami path is still a C^1 Beltrami path.*

The push forward of quadratic differential

Let $\gamma : [0, 1] \rightarrow M^*$ be any C^1 Beltrami path. Take any $t \in [0, 1]$. Let $\mu = \gamma(t)$ and $\eta = \gamma'(t)$. We use T_μ to denote the Teichmüller space modeled on $(S^2 \setminus \phi_\mu(\Omega), \phi_\mu(\partial\Omega \cup P_1))$. Since ϕ_μ induces an isometric mapping between T_f and T_μ , so $\|\eta\| = \|\xi\|$, where $\xi = d\phi_\mu^*(\eta)$. Write $\xi = \xi(z) \frac{dz}{d\bar{z}}$, $q = q(z) dz^2$, we know

$$\|\xi\| = \sup \left| \int \int_{S^2 \setminus \phi_\mu(\Omega)} q(z) \xi(z) dz \wedge d\bar{z} \right|$$

where the *sup* is taken over all the quadratic differentials q on $S^2 \setminus \phi_\mu(\Omega)$ such that

1. All the poles of q must be simple and be contained in $\phi_\mu(P_1)$,
2. $\int \int_{S^2 \setminus \phi_\mu(\Omega)} |q(z)| dz \wedge d\bar{z} = 1$.

Let $\eta' = df^*(\eta)$, and $\nu = f^*(\mu)$. Let $\xi' = d\phi_\nu^*(\eta')$. We have $\|\xi'\| = \|\eta'\|$.

Since the following diagram is commutative,

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi_\nu} & (S^2, P_g) \\ f \downarrow & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\phi_\mu} & (S^2, P_g) \end{array}$$

We have $\xi' = dg^*(\xi)$. We can use the following diagram to show the relation among η , η' , ξ , and ξ' .

$$\begin{array}{ccc} \eta' & \xrightarrow{d\phi_\nu^*} & \xi' \\ df^* \uparrow & & \uparrow dg^* \\ \eta & \xrightarrow{d\phi_\mu^*} & \xi \end{array}$$

Lemma II.14 *For each quadratic differential $q(z)dz^2$ on $S^2 \setminus \phi_\nu(\Omega)$, there is a unique quadratic differential $q'(w)dw^2$ on $S^2 \setminus \phi_\mu(\Omega)$ such that*

$$\int \int_{S^2 \setminus \phi_\mu(\Omega)} \xi'(z)q(z)dz \wedge d\bar{z} = \int \int_{S^2 \setminus \phi_\nu(\Omega)} \xi(w)q'(w)dw \wedge d\bar{w}$$

Proof 8 q' is given by

$$q' = \sum_{g(z)=w} \frac{q(z)}{[f'(z)]^2}$$

The quadratic differential q' in Lemma 2.3 is called the push forward of q .

The Thurston back contracts the Teichmüller metric

We need only to show there is a number δ , $0 < \delta < 1$, such that $\|\xi'\| < \delta\|\xi\|$

for some uniform $\delta < 1$. From Lemma 2.3, we have

$$\begin{aligned}
\|q'\| &= \int \int_{S^2 \setminus \phi_\nu(\Omega)} |q'(w)| dw \wedge d\bar{w} \\
&= \int \int_{S^2 \setminus \phi_\nu(\Omega)} \left| \sum_{g(z)=w} \frac{q(z)}{[g'(z)]^2} \right| dw \wedge d\bar{w} \\
&\leq \int \int_{S^2 \setminus \phi_\nu(\Omega) \cup \phi_\nu(A_i)} |q(z)| dz \wedge d\bar{z} \tag{1} \\
&= \int \int_{S^2 \setminus \phi_\nu(\Omega)} |q(z)| dz \wedge d\bar{z} - \int \int_{\cup \phi_\nu(A_i)} |q(z)| dz \wedge d\bar{z} \\
&= \|q\| - \int \int_{\cup \phi_\nu(A_i)} |q(z)| dz \wedge d\bar{z}
\end{aligned}$$

Since

$$\|q\| = 1,$$

We need only to prove that

$$\int \int_{\phi_\nu(\cup A_i)} |q(z)| dz \wedge d\bar{z} > \epsilon$$

for some uniform $\epsilon > 0$.

The bounded ϕ_n -geometry of (S^2, P_f)

Let m_0 be the number of the points in P_1 .

Definition II.5 Let $\Lambda = \{D_i, A_i, P_1 \mid 1 \leq i \leq m\}$. We say Λ has a bounded

ϕ_n - geometry if there exists an $\epsilon > 0$ such that

1. $d(\phi_n(z_i), \phi_n(A_j)), d(\phi_n(A_i), \phi_n(A_j)), d(\phi_n(z_i), \phi_n(z_j)) > \epsilon$ for $z_i \in P_1, A_i \in \Lambda$,
2. $\phi_n(D_i)$ contains a disk with radius $> \epsilon$ and $\text{mod}(\phi_n(A_i)) > \epsilon$ for all $i = 1, \dots, m$.

Lemma II.15 *Suppose Λ has a ϵ -bounded geometry. Then there exists a $\delta > 0$, dependent only on m, n, ϵ , such that for any quadratic differential $q(z)dz^2$ on $S^2 \setminus \Omega$, with possible simple poles at z_i for some $z_i \in P_1$, satisfying*

$$\int \int_{S^2 \setminus \Omega} |q(z)| dz \wedge d\bar{z} = 1$$

we have

$$\int \int_{\cup A_i} |q(z)| dz \wedge d\bar{z} > \delta$$

where $A_i = E_i \setminus D_i$ for $\int \int_{\cup A_i} |q(z)| dz \wedge d\bar{z} > \delta$

Proof 9 *We assume $\infty \in D_1$. Suppose the lemma is not true. Then we can get a sequence of quadratic differentials $q_k(z)dz^2$ where*

$$q_k(z) = \sum_{1 \leq i \leq n} \frac{a_{i,k}}{z - z_i} + g_k(z), k = 1, 2, \dots$$

such that

1. $\int \int |q_k(z)| dz \wedge d\bar{z} = 1$ for all $k = 1, 2, \dots$,

2. $\int \int_{\phi_k(\Omega)} |q_k(z)| dz \wedge d\bar{z} \rightarrow 0$ as $k \rightarrow \infty$,

3. $g_k(z)$ is holomorphic in $\mathbb{P}^1 \setminus \phi_k(\Omega)$.

Since

$$\int \int_{\phi_k(\Omega)} |q_k(z)| dz \wedge d\bar{z} \rightarrow 0$$

as $k \rightarrow \infty$, so we can take $\Gamma_k = \{\gamma_1, \dots, \gamma_m\}$ where $\gamma_i \subset \phi_k(A_i)$ is a simple closed curve in the homotopy class of $\phi_k(A_i)$ such that

$$\int_{\Gamma_k} |q_k(z)| dz \rightarrow 0$$

as $k \rightarrow \infty$. For each z_i in $\phi_k(P_1)$, let

$$\eta_i = \{z | d(z, z_i) = \frac{\epsilon}{2}\}$$

For $z \in \eta_i$. we have

$$\begin{aligned} g_k(z) &= \frac{1}{2\pi i} \int_{\Gamma_k} \frac{g_k(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma_k} \frac{q_k(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \sum_{1 \leq i \leq m_0} \int_{\Gamma} \frac{a_{i,k}}{(\xi - z_i)(\xi - z)} d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma_k} \frac{q_k(\xi)}{\xi - z} d\xi + \sum_{1 \leq i \leq m_0} \frac{a_{i,k}}{z - z_i} \end{aligned} \quad (2)$$

Since $d(\eta_i, \Gamma_k) > \frac{\epsilon}{2}$, so

$$\int_{\Gamma_k} |q_k(z)| dz \rightarrow 0$$

uniformly on η_i as $k \rightarrow \infty$, so

$$\frac{1}{2\pi i} \int_{\Gamma_k} \frac{q_k(\xi)}{\xi - z} d\xi \rightarrow 0$$

Therefore,

$$g_k(z) - \sum_{1 \leq i \leq m_0} \frac{a_{i,k}}{z - z_i} \rightarrow 0$$

uniformly for all $z \in \eta_i$. Thus we have

$$a_{i,k} = \frac{1}{2\pi} \int_{\eta_i} (z - z_i) \left(g_k(z) - \sum_{1 \leq j \leq m_0} \frac{a_{j,k}}{z - z_j} \right) dz \rightarrow 0$$

This implies

$$g_k(z) \rightarrow 0$$

and

$$\sum_{1 \leq i \leq m_0} \frac{a_{i,k}}{z - z_i} \rightarrow 0$$

Furthermore,

$$\int \int_{S^2 \setminus \Omega} |q(z)| dz \wedge d\bar{z} \rightarrow 0$$

as $k \rightarrow \infty$, which is a contradiction.

Let $\phi_0 = Id$, and ϕ_n be the sequence of the pull back of ϕ_0 by f . We see ϕ_n are all univalent in the linearization disks D_1, \dots, D_m .

Remember $\Omega = \cup_{i=1}^m D_i$, $P_1 = \{z \in P_f | z \notin D_i \text{ for all } i = 1, \dots, m\}$, $X = \partial\Omega \cup P_1$ and $0, 1, \infty \in P_f$.

Let $R = S^2 \setminus \Omega$ and $P_1 = P_f \setminus \Omega$. From Lemma 2.4, we know we need only to prove that $\phi_n(R \setminus P_1)$ has bounded geometry for all $n = 1, 2, \dots$. This is equivalent to prove that the length of the simple closed geodesics in $\phi_n(R \setminus P_1)$ has a uniform positive low bound for all n .

Remark II.1 *Since f is holomorphic in all the linearization disks D_i , by Koebe distortion theorem and Theorem 6.3 in [DH], any short simple closed geodesic in $\phi_n(S^2 \setminus P_f)$ is either lies in $\phi_n(R \setminus P_1)$ or lies deeply in the super attracting disks.*

For each D_i , let a_i be the unique periodic point in D_i and b_i be a point in $D_i \cap P_f$ such that $d(a_i, b_i) > \alpha \text{diam}(D_i)$ for some $\alpha > 0$. Let $P_2 = P_1 \cup (\cup_{1 \leq i \leq m} \{a_i, b_i\})$.

For each simple closed curve $\gamma \in R \setminus P_1$, we use $\|\gamma\|_{\phi, P_2}$ to denote the hyperbolic length of the geodesic in $\mathbb{P}^1 \setminus \phi(P_2)$ which is in the same homotopy class as $\phi(\gamma)$. We will prove

Lemma II.16 *There exists a uniform $\delta > 0$ such that $\|\gamma\|_{\phi_n, P_2} > \delta$ for all simple closed curves in $S^2 \setminus P_2$ and all $n = 1, 2, \dots$.*

Let

$$\mathcal{F}_\delta = \{[\gamma_i] \mid \gamma_i \subset R \setminus P_2 \text{ and } \|\gamma\|_{\phi_n, P_2} < \delta\}$$

Before we prove Lemma 2.6, let's prove the following lemma first,

Lemma II.17 *For δ small enough, if the family \mathcal{F}_δ is not empty, then \mathcal{F}_δ must be a f – stable family.*

Proof 10 *Let $\gamma \in \mathcal{F}$ and ξ be a non-peripheral component of $f^{-1}(\gamma)$. Then ξ must be in $R \setminus P_2$. Let $P_3 = P_2 \cup f(P_2)$. Since ϕ_n is univalent in all $D_i, i = 1, \dots, m$, there exists a number $K > 1$ independent of n and γ , such that*

$$\|\gamma\|_{\phi_n, P_3} < K \|\gamma\|_{\phi_n, P_2}$$

On the other hand, we have

$$\|\xi\|_{\phi_{n+1}, P_2} < d \|\gamma\|_{\phi_n, P_3}$$

so we get

$$\|\xi\|_{\phi_{n+1}, P_2} < dK \|\gamma\|_{\phi_n, P_2}$$

If δ is small enough, by the Gap Lemma(lemma 1.10), we know $\xi \in \mathcal{F}_\delta$.

Therefore \mathcal{F} is a f – stable family.

Now let's prove Lemma 2.6. By lemma 1.6, we know any two simple closed geodesics with length less than $\log(\sqrt{2} + 1)$ are disjoint. Therefore there exists a k dependent only on f such that for all $n = 1, 2, \dots$, there are at most k simple closed geodesics in $\phi_n(\mathbb{P}^1 \setminus P_1)$ with length less than $\log(\sqrt{2} + 1)$.

When δ is small enough, By Lemma 2.6, we know \mathcal{F}_δ is a stable family. Similar to Chapter 1, for each of such stable family \mathcal{F}_δ , we can define the Thurston linear transformation and it's associated linear transformation matrix A . Since the size of A is bounded by k , and each entry of A can only be one of the finite values, so there are only finite many such linear transformation matrixes.

Since f has no *Thurston Obstruction*, so we have a uniform number l such that

$$\|A^l\|_{max} \leq \frac{1}{2}$$

for the Thurston transformation A associated to the stable family \mathcal{F}_δ when δ is small enough.

Now let

$$\mathcal{F}_{\phi_n} = \sum_{\gamma \in \mathcal{F}_\delta} \|\gamma\|_{\phi_n, P_2}^{-1}$$

We will prove that \mathcal{F}_{ϕ_n} has a positive lower bound for all $n = 1, 2, \dots$. This will imply Lemma 2.6 immediately. Let \mathcal{G} be the family consisting of all the non-peripheral components of the preimage of \mathcal{F}_δ by f^{ol} in $S^2 \setminus P_2$.

Let

$$Y = P_2 \cup f^{ol}(P_2)$$

and

$$Y_l = f^{-l}(Y)$$

Therefore $P_2 \subset Y_m$. By Lemma 1.7, we have

$$\mathcal{F}_{\phi_{n+l}} \leq \sum_{\xi \in \mathcal{G}} \|\xi\|_{\phi_{n+l}, Y_l}^{-1} + M$$

where the constant M is dependent only on f and l .

On the other hand, from the following diagram,

$$\begin{array}{ccc} (S^2, Y_l) & \xrightarrow{\phi_{n+m}} & (S^2, \phi_{n+m}(Y_l)) \\ f^{ol} \downarrow & & \downarrow g_n \\ (S^2, Y) & \xrightarrow{\phi_n} & (S^2, \phi_n(Y)) \end{array}$$

we see $g_n : (S^2, \phi_{n+l}(Y_l)) \rightarrow (S^2, \phi_n(Y))$ is a holomorphic covering map. As before we denote

$$\mathcal{F}_\delta = \{[\gamma_i] \mid \gamma_i \subset R \setminus P_2 \text{ and } \|\gamma_i\|_{\phi_n, P_2} < \delta\}$$

Therefore we have

$$\sum_{\xi \in \mathcal{G}} \|\xi\|_{\phi_{n+m}, Y_m}^{-1} = \sum_{i,j} b_{ij} \|\gamma_i\|_{\phi_n, Y}^{-1} \leq \sum_j b_{ij} \|\gamma_i\|_{\phi_n, P_2}^{-1}$$

where $\sum_j b_{ij}$ is exactly the (i, j) entry in A^l and therefore we have

$$\sum_j b_{ij} \leq \frac{1}{2}$$

so we have

$$\sum_{\xi \in \mathcal{G}} \|\xi\|_{\phi_{n+l}, Y_m}^{-1} \leq \frac{1}{2} \mathcal{F}_{\phi_n}$$

and therefore

$$\mathcal{F}_{\phi_{n+l}} \leq \frac{1}{2} \mathcal{F}_{\phi_n} + M$$

For any $j = 0, 1, \dots, l-1$ and $k = 1, 2, \dots$, We get that

$$\mathcal{F}_{\phi_{kl+j}} \leq \frac{1}{2^k} (\mathcal{F}_{\phi_j} - 2M) + 2M$$

So \mathcal{F}_{ϕ_n} has a uniform upper bound for all n . Therefore there is a positive lower bound of $\|\gamma\|_{\phi_n, P_2}$ for all $\gamma \in \mathcal{F}_\delta$ and $n = 1, 2, \dots$. We complete the proof of Lemma 2.6. This implies the pull back sequence by f has a bounded

ϕ_n – geometry. By Lemma 2.5, we know the Thurston pull back is a strict contraction so it has a unique fixed point, which is the candidate polynomial.

We have completed the proof of Theorem 2.1.

The following result is a direct consequence of the proof given above,

Theorem II.4 *Let f and g be two rational maps such that*

$$(1) f(z) = \psi^{-1} \circ g \circ \phi(z)$$

(2) $\phi, \psi : S^2 \rightarrow S^2$ are two homeomorphisms such that ϕ is isotopic to ψ relative to P_f ,

(3) for each accumulation point z_0 of P_f , there is an open neighborhood U of z_0 such that $\phi(z) = \psi(z)$ for all $z \in U$.

Then f is conformally conjugate to g .

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Chapter 3. Rational Realization of Topological Polynomial of Simple Siegel Disk Type

§3.1 topological polynomial of simple Siegel disk type

We say a branched covering map f is a topological polynomial if $f^{-1}(\infty) = \{\infty\}$. Let θ be an irrational number. It's said to be of bounded type if $\sup\{a_n\} < \infty$ where $[a_1, a_2, \dots]$ is the continuous fraction of θ . Let $\Delta = \{z \mid |z| < 1\}$ be the unit disk. We use

$$R_\theta(z) = e^{2\pi i \theta} z : \Delta \rightarrow \Delta$$

to denote the rotation by angle θ . For a topological polynomial f , we still use

$$\Omega_f = \{z \mid \deg_z f > 1\} \quad \text{and} \quad P_f = \overline{\bigcup_{k \geq 1} f^k(\Omega_f)}$$

to denote the critical set and the post-critical set. We also use

$$\text{Orbit}(f, z) = \{z_k = f^k(z) \mid k = 0, 1, \dots\}$$

Definition III.6 *A topological polynomial is said to be of simple Siegel disk type if there exists a closed topological disk D (called it the rotation disk) and a holomorphic homeomorphism $\phi : D \rightarrow \overline{\Delta}$ such that*

1. $f|_D(z) = \phi^{-1} \circ R_\theta \circ \phi(z)$ for some θ of bounded type,
2. $\partial D \cap \Omega_f \neq \emptyset$, and
3. $\text{Orbit}(f, z) \setminus D$ is a finite set for every $z \in \Omega_f$.

Definition III.7 Let f and g be two topological polynomials of simple Siegel disk type. We say f and g are combinatorially equivalent to each other if there exists a pair of homeomorphisms $\phi_1, \phi_2 : S^2 \rightarrow S^2$ such that

- a) $\phi_1 = \phi_2$ are holomorphic when they are restricted on the rotation disk D ,
- b) ϕ_1 is isotopic to ϕ_2 relative to P_f , and
- c) $f = \phi_1^{-1} \circ g \circ \phi_2$.

Definition III.8 Let f be a topological polynomial of simple Siegel disk type.

A Thurston obstruction of f is a f -stable multicurve such that

- i) $\Gamma \subset S^2 \setminus (P_f \cup D)$, where D is the rotation disk, and
- ii) $f_\Gamma : R^\Gamma \rightarrow R^\Gamma$ has an eigenvalue ≥ 1 .

The main result in this chapter is

Theorem III.5 *Let f be a topological polynomial of simple Siegel disk type. Then f is combinatorially equivalent to a polynomial with one Siegel disk if and only if f has no Thurston Obstruction.*

We will provide the proof of Theorem 3.1 in the following sections. The main idea of the proof is to construct an appropriate Blaschke product and then use the quasiconformal surgery to get the candidate polynomial.

§3.2. Construction of a sequence of symmetric critically finite branched covering maps

For each $w \in \mathbb{C}$, let w^* be the symmetric image of w about the unit circle. Without loss of generality, we assume that the rotation disk $D = \overline{\Delta}$ and $f|_D(z) = e^{2\pi i\theta} z$. Define

$$X = \{z \in \Omega_f \mid \text{Orbit}(f, z) \text{ enters into } \Delta \text{ and does not contain } 0\}.$$

For each $z \in X$, use $i_0 \geq 0$ be the first moment for $\text{Orbit}(f, z)$ such that $z_{i_0} \in \Delta$. Furthermore, we assume $z_{i_0} = w^*$ for some $w \in X$. We will consider the general case in §3.6.

First we define $F : S^2 \rightarrow S^2$ by $F(z) = f(z)$ if $|z| \geq 1$ and $F(z) = f^*(z^*)$ if $|z| < 1$. Then F is a symmetric branched covering map of S^2 . Next, we

take a sequence of rational numbers $\theta_n = p_n/q_n$ such that $(p_n, q_n) = 1$ and $\theta_n \rightarrow \theta$ as n goes to ∞ . Let $A(a, b) = \{z | a < |z| < b\}$ the annulus of inner radius a and outer radius b . Since $Orbit(F, z) \setminus \partial D$ is a finite set for every $z \in \Omega_F$, we can find $0 < r < 1 < R$ such that

$$\Omega_F \cap A(r, 1), \quad \Omega_F \cap A(1, R), \quad P_F \cap A(r, 1), \quad \text{and} \quad P_F \cap A(1, R)$$

are all empty. Now we define a map h_n as follows. Set $h_n(z) = z$ for $z \notin A(r, R)$ and $h_n(z) = e^{2\pi i(\theta_n - \theta)} z$ for $|z| = 1$. Then extend h_n homeomorphically to $A(r, 1) \cup A(1, R)$ such that $h_n(z) = h_n^*(z^*)$. Furthermore, we let $h_n(z) \rightarrow z$ uniformly on the whole sphere S^2 . Define

$$F_n = h_n \circ F.$$

Thus we defined a sequence of critically finite symmetric branched covering map F_n . Note that from our construction, $P_{F_n} \setminus D = P_f \setminus D$.

§3.3 No Thurston obstruction for large n

In this section, we define Levy cycles for F_n and discuss their relation with Thurston obstruction.

Definition III.9 *Let $g : S^2 \rightarrow S^2$ be an orientation-preserving branched covering map. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a multi-curve. We assume $\gamma_{n+1} = \gamma_1$.*

We say Γ is a Levy cycle if for each $i = 1, \dots, n$, there is a component of η_i of $f^{-1}(\gamma_{i+1})$ such that η_i is homotopic to γ_i rel P_g and $g : \eta_i \rightarrow \gamma_{i+1}$ is a homeomorphism.

Take $\phi_0 = Id$. For each fixed $n \geq 1$, pull it back by F_n inductively like in §1.3 as follows:

$$\begin{array}{ccc} (S^2, P_{F_n}) & \xrightarrow{\phi_{m+1}} & (\mathbb{P}^1, \phi_{m+1}(P_{F_n})) \\ F_n \downarrow & & \downarrow G_m \\ (S^2, P_{F_n}) & \xrightarrow{\phi_m} & (\mathbb{P}^1, \phi_m(P_{F_n})) \end{array}$$

Since $F_n(z^*) = F_n(z)^*$, $\phi_m(z^*) = \phi_m(z)^*$ for $m = 0, 1, \dots$. So we get $G_m(z^*) = G_m(z)^*$ for $m = 0, 1, \dots$. Therefore, G_m is a Blaschke product on \mathbb{P}^1 . We write

$$G_m(z) = \lambda z^d \prod_{1 \leq k \leq d-1} \frac{z - p_k}{1 - \bar{p}_k z}$$

where d is the degree of f , $\lambda = e^{2\pi i \tau}$ for some $0 \leq \tau \leq 1$, and $|p_k| > 1$ for $k = 1, \dots, d-1$. Let $Y_n = F_n^{-1}(P_{F_n})$.

We first discuss a short geodesic which separates D and ∞ .

Lemma III.18 *There exists a $\delta > 0$ independent of m such that $\|\gamma\|_{\phi_m, P_{F_n}} > \delta$ for all simple closed curve $\gamma \subset S^2 \setminus P_{F_n}$ which separates D and ∞ .*

Proof 11 Let $\Gamma_\delta = \{\gamma_1, \dots, \gamma_n\} \subset S^2 \setminus D$ such that $\|\gamma_i\|_{\phi_m, P_{F_n}} < \delta$. By

Lemma 1.10, when δ is small enough, Γ_δ is a f -stable family. Since f has no Thurston Obstruction, $f_{\Gamma_\delta}^{\circ n} \rightarrow 0$ (the zero transformation) as n goes to ∞ .

Now consider the subset G_δ consisting of all $\gamma \in \Gamma_\delta$ such that γ separates ∞ and D . Suppose $\Gamma_\delta = G_\delta \cup H_\delta$. Then the matrix of f_{Γ_δ} with respect to (G_δ, H_δ) is

$$[f_{\Gamma_\delta}] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Let $f_{G_\delta} : R^{G_\delta} \rightarrow R^{G_\delta}$ be the restriction with the matrix A . Clearly, $f_{G_\delta}^{\circ n} \rightarrow 0$ as $n \rightarrow \infty$. We can assume that $\|A\| \leq \frac{1}{2}$.

Let ξ be a short simple closed geodesic in $\mathbb{P}^1 \setminus \phi_{m+1}(Y_n)$. Let Ω_1 and Ω_2 be the two components of $\mathbb{P}^1 \setminus \xi$ such that $\infty \in \Omega_1$ and $0 \in \Omega_2$. As $\delta \rightarrow 0$, we have

1. $|\frac{z}{p_i}| \rightarrow \infty$ for $z \in \xi$ and $p_i \in \Omega_2$ and

2. $|\frac{p_i}{z}| \rightarrow \infty$ for $z \in \xi$ and $p_i \in \Omega_1$.

From (1) and (2), we know $G_m|_{z \in \xi} = az^k + o(z^k)$ for some constant $a \neq 0$.

Therefore when δ is small enough, $G_m(\xi)$ still separates ∞ and D . Note that it is a simple closed geodesic in $\mathbb{P}^1 \setminus \phi_m(P_{F_n})$. Let

$$Z_{\delta,m} = \sum_{\gamma \in G_\delta} \frac{1}{\|\gamma\|_{\phi_m, P_{F_n}}}.$$

We consider the family of all the non-peripheral components of preimages of G_δ by F_n which separates ∞ and D . We use \mathcal{G} to denote this family. Since

$$\begin{array}{ccc} (S^2, Y_n) & \xrightarrow{\phi_{m+1}} & (S^2, \phi_{m+1}(Y_n)) \\ F_n \downarrow & & \downarrow G_m \\ (S^2, P_{F_n}) & \xrightarrow{\phi_m} & (S^2, \phi_m(P_{F_n})) \end{array}$$

and since $G_m : (\mathbb{P}^1, \phi_{m+1}(Y)) \rightarrow (\mathbb{P}^1, \phi_m(P_{F_n}))$ is a holomorphic covering map,

$$\sum_{\xi \in \mathcal{G}} \frac{1}{\|\xi\|_{\phi_{m+1}, Y}} = \sum_j \sum_{\gamma_i \in G_\delta} b_{ij} \frac{1}{\|\gamma_i\|_{\phi_m, P_{F_n}}} \leq \frac{1}{2} Z_{\delta, m}$$

where $A = (b_{ij})$.

By Lemma 1.8, we have

$$Z_{\delta, m+1} \leq \sum_{\xi \in \mathcal{G}} \frac{1}{\|\xi\|_{\phi_{m+1}, Y}} + M$$

where the constant M is only dependent on F_n (but independent of m).

Therefore we get

$$Z_{\delta, m+1} \leq \frac{1}{2} Z_{\delta, m} + M.$$

By the same argument as that we used in the proof of Lemma 1.9 (or Lemma 2.5), we can prove $Z_{\delta, m}$ has a uniform upper bound. This implies Lemma 3.1.

Lemma III.19 *There are constants $C_0 = (d+1)/(d-1) > 1$ and $C_1 > C_0$*

independent of m such that

$$C_0 \leq p_k \leq C_1$$

for all G_m and $1 \leq k \leq d-1$.

Proof 12 By lemma 3.1, we know that the length of the geodesic in $\mathbb{P}^1 \setminus \phi_m(Y)$ has a positive lower bound. This implies p_k can not approach to ∞ too much (see the proof of Lemma 1.3). To prove the lower bound, we observed that G_m is an orientation-preserving diffeomorphism when restricted on the unit circle. Define

$$T(\theta) = \frac{1}{2\pi i} \log(G_m(e^{2\pi i \theta}))$$

We have

$$T'(\theta) = d + \sum_{1 \leq l \leq d-1} \frac{1 - |p_l|^2}{|1 - \bar{p}_l e^{2\pi i \theta}|^2} \geq 0.$$

Therefore, we have

$$d \geq \frac{|p_l|^2 - 1}{|1 - \bar{p}_l e^{2\pi i \theta}|^2}$$

for all $l = 1, \dots, d-1$. By taking θ such that $\bar{p}_l e^{2\pi i \theta} = |p_l|$ for each l , we

then get

$$d \geq \frac{|p_l| + 1}{|p_l| - 1}.$$

So

$$|p_l| \geq \frac{d+1}{d-1}$$

Now we discuss a general short geodesics. Let $\gamma \subset S^2 \setminus P_{F_n}$ such that $\|\gamma\|_{\phi_m} < \delta$. Let $D(\gamma)$ be the component of $S^2 \setminus \gamma$ which does not contain ∞ .

Lemma III.20 *If δ is small enough, then $\phi_m(D(\gamma))$ contains no p_i or $1/\bar{p}_i$ for all $i = 1, \dots, d - 1$.*

Proof 13 *By Lemma 1.7 and Lemma 1.8, there is a closed curve ξ in $\mathbb{P}^1 \setminus \phi_m(Y_n)$ which is a short simple closed ϕ_{m+1} -geodesic and which is homotopic to $\phi_m(\gamma)$ in $\mathbb{P}^1 \setminus \phi_m(P_{F_n})$. By Lemma 3.2, we know that $\phi_m(D(\gamma))$ can not contain a zero and a pole of G_m at same time. Therefore, by the symmetry, we need only to prove that $\phi_m(D(\gamma))$ contains no zero of G_m . If this is not true, $G_m(D(\phi_{m+1}(\xi)))$ would contain 0 and its boundary $G_m(\phi_{m+1}(\xi))$ is also a short geodesic. This implies that $G_m(\phi_{m+1}(\xi))$ is inside D . (From Lemma 3.1, it can not be outside of D). But the symmetric closed curve of $G_m(\phi_{m+1}(\xi))$ is also a short geodesic which separates ∞ and D . This contradicts to Lemma 3.1.*

Lemma III.21 *If F_n has a Thurston Obstruction, then there exists an innermost Levy cycle of F_n .*

Proof 14 *We assume without loss of generality that Γ is minimal in the sense that every sub-multicurve $\Gamma' \subset \Gamma$ that is still a Thurston obstruction*

is in fact Γ . From the proof presented in Chapter 1, we can assume that Γ is a Thurston Obstruction consisting of short ϕ_m -geodesics. Let A_Γ be the matrix of the Thurston linear transformation associated to Γ .

We say $\gamma \in \Gamma$ is an essential curve if $A_\Gamma^m \gamma$ does not go to 0. Otherwise, we say γ is a negligible curve. We say an essential curve $\gamma \in \Gamma$ is innermost if $D(\gamma)$ does not contain any other essential curves in Γ .

Let $\mathcal{I} \subset \Gamma$ be the subset consisting of all the innermost curves in Γ . Clearly \mathcal{I} is non-empty. We will prove \mathcal{I} is a Levy cycle of F_n . We call such a Levy cycle an innermost Levy cycle.

For each $\gamma \in \mathcal{I}$, $F_n^{-1}(\gamma)$ has an essential component. Let ξ be one of the essential components of $F_n^{-1}(\gamma)$. We claim that ξ is innermost. The proof is the following. By Lemma 3.3, $\phi_{m+1}(D(\xi))$ can not contain poles $\frac{1}{\beta_i}$, $i = 1, \dots, d-1$, of G_m . Therefore $G_m : \phi_{m+1}(D(\xi)) \rightarrow \phi_m(D(\gamma))$ is a proper covering map. If ξ is not innermost, then there is another essential short geodesic $\phi_{m+1}(\eta) \subset \phi_{m+1}(D(\xi))$. So $F_n(\eta) \subset D(\gamma)$ must be an essential curve in Γ . (Otherwise, $\Gamma \setminus \{\eta\}$ is a Thurston obstruction. This contradicts that Γ is minimal.) This is a contradiction with that γ is innermost. So we proved that ξ is innermost.

Let $\Phi(\gamma)$ be the set of all innermost components of $F_n^{-1}(\gamma)$. Since all

short ϕ_m -geodesic are disjoint, $\Phi(\gamma) \cap \Phi(\gamma') = \emptyset$ for all $\gamma \neq \gamma' \in \mathcal{I}$. But

$$\mathcal{I} = \cup_{\gamma \in \mathcal{I}} \Phi(\gamma).$$

So $\Phi(\gamma)$ contains exactly one element in \mathcal{I} . Therefore, Φ induces a permutation of \mathcal{I} . We still denote this permutation as Φ .

Let $\mathcal{I} = \{\gamma_0, \dots, \gamma_{k-1}\}$ where

$$\gamma_0 \rightarrow \gamma_1 \cdots \rightarrow \gamma_k = \gamma_0$$

forms a cycle. We claim \mathcal{I} is an innermost Levy cycle.

Here is the proof of the claim. By the minimal property of Γ , we have

$$\Gamma = \{\gamma_0, \dots, \gamma_{k-1}, \text{negligible curves}\}$$

and

$$F_{n,\Gamma}(\gamma_{i+1}) = a_i \gamma_i + \beta$$

where $\beta \in R^\Gamma$ has only negligible components. By Lemma 3.3, there is only one component γ' in $F_n^{-1}(\gamma_{i+1})$ such that γ' is homotopic to γ_i , so $a_i = \frac{1}{d_i}$ where d_i is the covering degree of $F_n : \gamma' \rightarrow \gamma_{i+1}$. So the matrix of the Thurston linear transformation becomes

$$A_\Gamma = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$$

$B^l \rightarrow 0$ as $l \rightarrow 0$. The matrix A has the form

$$\begin{pmatrix} 0 & a_1 & * & \cdots & * \\ 0 & 0 & a_2 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_{k-1} \\ a_0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The maximal eigenvalue of A_Γ is equal to the maximal eigenvalue of A . But the maximal eigenvalue λ of A satisfies $\lambda^k - a_0 \cdots a_{k-1} = 0$. Thus $a_0 \cdots a_{k-1} \geq 1$. Since $a_i \leq 1$, so $a_i = 1$ for all $i = 0, \dots, k-1$. This implies that \mathcal{I} is a Levy cycle.

One of the conclusions from the above lemma is that $D(\gamma)$ for every γ in \mathcal{I} does not contain any critical point of F_n .

Lemma III.22 F_n has no innermost Levy cycle for n large enough.

Proof 15 We prove it by contradiction. Assume F_n has an innermost Levy cycle \mathcal{I} . For each $\gamma \in \mathcal{I}$, let $D(\gamma)$ be the bounded domain not containing ∞ . Then $D(\gamma)$ does not contain any critical point of F_n . We claim that the closed curves of \mathcal{I} can not be all contained in $S^2 \setminus D$ (or D). The reason of the claim is that, otherwise, this Levy cycle \mathcal{I} is also the Levy cycle of f (or

all symmetric closed curves of those curves in Levy cycle form a Levy cycle because F_n is symmetric about the unit circle).

Since every γ is a short ϕ_m -geodesic, if it crosses the unit circle, it must be symmetric about the unit circle. The reason is that the symmetric curve, of γ is also a short ϕ_m -geodesic and short ϕ_m geodesics are disjoint. Now we claim that every γ in \mathcal{I} must cross the unit circle. If not, then we met two cases: one is that we can find a γ_{i+1} disjoint with the unit circle but γ_i crosses the unit circle. The other is that we can find γ_{i+1} contained in the open unit disk and γ_i contained in the outsider of the closed unit disk.

We first prove that the first case is impossible. Otherwise, one of components of $F_n^{-1}(\gamma_i)$ is isotopic to γ_{i+1} rel P_{F_n} . Use η to denote this component. Since γ_{i+1} crosses the unit circle, it must contain one point $w \in P_{F_n} \cap D$ and $w^ \in P_{F_n} \cap \Delta^c$. So η must cross the unit circle. This implies $\gamma_i = F_n(\eta)$ must cross the unit circle. It contradicts the assumption.*

Now we prove the other case is also impossible. If not, then we can find a γ_{i+1} contained in the open unit disk and γ_i contained in the outside of closed unit disk. Since $D(\gamma_i)$ contains a point $z \in P_f$ and $w = f(z)$ first falls into the open unit disk, by our assumption in §3.2, w is a symmetric reflection about the unit circle of a critical point of f . Thus w is a critical point of F_n .

It contradicts to that F_n on $D(\gamma_{i+1})$ is homeomorphic.

Thus we proved that the Levy cycle \mathcal{I} consists of all symmetric closed curves about the unit circle. In the following we will prove that we can not have such a Levy cycle if n is a large positive integer. Again we prove this, by contradiction.

Assume \mathcal{I} is a Levy cycle consisting of all symmetric closed curves about the unit circle. Each γ_i in \mathcal{I} contains at least two points from P_{F_n} . Denote $S_i = D(\gamma_i) \cap P_{F_n}$. Then $F_n : S_i \rightarrow S_{i+1}$ is bijective. It is clear that $F_n : S_i \cap S^1 \rightarrow S_{i+1} \cap S^1$ is bijective. Moreover, $F_n : S_i \cap \Delta \rightarrow S_{i+1} \cap \Delta$ and $F_n : S_i \cap (S^2 \setminus D) \rightarrow S_{i+1} \cap (S^2 \setminus D)$ are also bijective. The reason is that if a point in $S_i \cap (S^2 \setminus D)$ being mapped on the unit circle S^1 . Then the number of the points in $S_{i+1} \cap S^1$ will be bigger than that of $S_i \cap S^1$. It is impossible. If a point in $S_i \cap (S^2 \setminus D)$ being mapped into the open unit disk Δ . Then its image will be a critical point of F_n . this contradicts that F_n on every $D(\gamma)$ for $\gamma \in \mathcal{I}$ is homeomorphic.

Remember be the set of critical points of F_n is Ω_f . Now we decompose Ω_f into two sets. One (denoted as O) is the set of those points on the unit circle. The other (denoted as Q) is the set of those points outside the closed unit disk. The set of critical points of F_n is $O \cup Q \cup Q^$ where Q^* is the symmetric*

set of Q about the unit circle. For each point in O , it is a periodic point of F_n of period q_n . For each point in Q , its orbit under F_n has three types. The first type is that its orbit is contained in the outside of the closed unit disk. This orbit is also the same orbit under f . So its length is bounded by a constant M independent of n . The second type is that its orbit eventually lands on the unit circle. In this case, points on the unit circle is a periodic orbit of period q_n . The third type is that its orbit will enter the open unit disk. By our assumption, the first entering point is a critical point of F_n .

Let k be the number of curves in the Levy cycle \mathcal{I} . Let γ be in \mathcal{I} . If $D(\gamma)$ contains $w \in P_{F_n}$ outside the closed unit disk D , then $k \leq M$ since $F_n^i(w)$ is outside D and in some $D(\gamma)$ and $\{D(\gamma) | \gamma \in \mathcal{I}\}$ are all disjoint (innermost condition). Let $O_n = \cup_{n \geq 0} F_n(O)$. Then it is finite set and cuts S^1 into intervals $\{K_l\}$. Let $I_\gamma = D(\gamma) \cap S^1$ for $\gamma \in \mathcal{I}$. Since $D(\gamma)$ contains no critical point of F_n , $I_\gamma \subseteq K_l$ for some l . Since the orbit of I_γ under F_n has period k and $F_n : K_l \rightarrow K_{l'}$ is homeomorphic, the orbit of K_l has period k too. This implies the boundary points of K_l has period k . But the boundary points of K_l are in Q_n which have period q_n . Since $q_n \rightarrow \infty$ as $n \rightarrow \infty$. This is a contradiction when n large enough.

If $D(\gamma)$ contains only points from $P_{F_n} \cap S^1$, let z and z' be two such

point. Denote $J_\gamma \subset D(\gamma)$ be the closed interval in $D(\gamma)$ bounded by z and z' . Then its orbit $\{F_n^l(J_\gamma)\}$ consists of intervals with pairwise disjoint interiors and cover the unit circle. This implies that some $D(F_n^l(\gamma))$ contains a critical point of F_n in O . This is again a contradiction to the fact that F_n on $D(F^l(\gamma))$ is homeomorphic. Now we complete the proof of the lemma.

§3.4. Compactness

By Lemma 3.5 and Thurston's Theorem in Chapter 1, we have a sequence of Blaschke products G_n such that F_n is combinatorially equivalent to G_n for all $n = 1, 2, \dots$. By Lemma 3.2, we know $\{G_n\}$ is compact. By taking a subsequence, we can assume that $G_n \rightarrow G$. Clearly G is a Blaschke product and $G|_{\partial D}$ is a diffeomorphism with rotation number θ . To complete the proof of Theorem 3.1, we have to show that the configuration of the post-critical points of G_n has a bounded geometry in the following sense: there is a uniform $\delta > 0$ such that $d(x, y) > \delta$ and $d(x, \partial D) > \delta$ for any $x \neq y \in P_{G_n} \setminus \partial D$, where d means the spherical metric.

Let (ψ_n, ϕ_n) be a combinatorial equivalence, i.e., ϕ_n is isotopic to ψ_n rel to P_{F_n} and $G_n = \psi_n \circ F_n \circ \phi_n^{-1}$. On the unit circle,

$$F_n(z) = e^{2\pi i \theta_n} z \rightarrow R_\theta = e^{2\pi i \theta}$$

and

$$G_n \rightarrow G.$$

Since G is an analytic critical circle map with rotation θ , G is topological conjugate to the rigid rotation R_θ . By the Hermann-Swiatek theorem in §3.5 ϕ is a quasi-symmetric circle homeomorphism. We have that

Lemma III.23 $\phi_n \rightarrow \phi$ as n goes to ∞ .

Proof 16 For any $\delta > 0$, there is an integer $M > 0$ such that for any x in the unit circle, there are two integers $0 < k_1, k_2 < M$ such that

$$e^{2\pi i k_1 \theta} \in (x + \delta, x + 2\delta) \quad \text{and} \quad e^{2\pi i k_2 \theta} \in (x - 2\delta, x - \delta).$$

Take N large such that when $n > N$,

$$|\theta_n - \theta| < \frac{\delta}{M}.$$

So for M large enough, we have

$$e^{2\pi i k_1 \theta_n} \in (x, x + 3\delta) \quad \text{and} \quad e^{2\pi i k_2 \theta_n} \in (x - 3\delta, x).$$

This implies

$$|\phi_n(x) - \phi(x)| \leq |\phi_n(e^{2\pi i k_1 \theta_n}) - \phi_n(x)| + |\phi_n(e^{2\pi i k_1 \theta_n}) - \phi(e^{2\pi i k_1 \theta})|$$

$$\begin{aligned}
& + |\phi(e^{2\pi k_1 \theta}) - \phi(x)| \\
\leq & |\phi_n(e^{2\pi i k_1 \theta_n}) - \phi_n(e^{2\pi i k_2 \theta_n})| + |\phi_n(e^{2\pi i k_1 \theta_n}) - \phi(e^{2\pi i k_1 \theta})| + |\phi(e^{2\pi i k_1 \theta}) - \phi(x)| \\
& = |G_n^{k_1}(1) - G_n^{k_2}(1)| + |G_n^{k_1}(1) - G^{k_1}(1)| + |\phi(e^{2\pi i k_1 \theta}) - \phi(x)| \\
\leq & |G_n^{k_1}(1) - G^{k_1}(1)| + |G^{k_1}(1) - G^{k_2}(1)| + |G^{k_2}(1) - G_n^{k_2}(1)| + |G_n^{k_1}(1) - G^{k_1}(1)| \\
& + |\phi(e^{2\pi i k_1 \theta}) - \phi(x)| \leq 20\delta.
\end{aligned}$$

Lemma III.24 *There exists a $\delta > 0$ independent of n such that for any $x \in P_{G_n}$ and $G_n^k(x) = x$, we have $d(x, \partial D) > \delta$ and $1 \leq k \leq N$.*

Proof 17 *Let N be the maximal period of the periodical cycle in $P_{F_n} \setminus \partial D$. By lemma 3.2, we know G_n converges to G uniformly in a neighborhood of ∂D . Since the rotation number of $G|_{\partial D}$ is θ , there exists $\delta > 0$ such that $G_n^k(x) \neq x$ whenever $d(x, \partial D) < \delta$.*

Lemma III.25 *There exists a $\delta > 0$ such that for any $n \geq 1$ and $x \in P_{G_n} \setminus D$ and $G_n(x) \in \partial D$, we have $d(x, \partial D) > \delta$.*

Proof 18 *Suppose it is not true. Then we can find a x_{n_k} for a subsequence of natural numbers (let us still assume this subsequence is all natural numbers) such that*

$$1) \ x_{n_k} \notin \partial D,$$

2) $G_n(x_n) \in \partial D$, and

3) $d(x_n, \partial D) \rightarrow 0$ as $n \rightarrow \infty$.

Assume $x_n \rightarrow x_0 \in \partial D$. Since x_n^* also tends to 0, and $G_n(x_n^*) \in \partial D$. This implies that G_n is many-to-one in a neighborhood of x_0 for n large enough. So $G'(x_0) = 0$. Since $G_n \rightarrow G$ uniformly, there is a neighborhood U of x_0 such that $G_n : U \rightarrow V$ is a branched covering map and onto. From Lemma 3.6, we can assume V contains only at most one critical value of G_n on the unit circle. Let m be the topological degree of $G_n|U$. We claim that there is a c_0 in U but not in ∂D such that $G_n(c_0)$ is not on ∂D . Let us prove it. First V must contain some critical values of $G_n|U$. If none of these critical values are on ∂D , then we get it. If $G_n(x_n)$ is not a critical value, we may assume $\partial D \cap V$ contains no critical value from G_n (following Lemma 3.6 and critical values of F_n are the same for all n). This becomes the above case. Now we suppose $G_n(x_n)$ is a critical value. It then must be the only critical value on ∂D . We call this value v . Since

$$m = \sum_{G_n(y_i)=v} \deg_{y_i} G_n = \sum_{G_n(y_i)=v} (\deg_{y_i} G_n - 1) + 1,$$

we get $\#(G_n^{-1}(v)) = 1$. But $G_n(x_n) = G_n(x_n^*) = v$. This is a contradiction.

Using Lemma 3.7, the orbit of c_0 must fall on ∂D . Following the argument

in the previous paragraph, we can find a critical point c_1 of G_n such that $G_n(c_1)$ is not on the unit circle. So on we can find a sequence of critical points $\{c_k\}$ closed to the unit circle such that c_k and $G_n(c_k)$ are all not on the unit circle. This sequence is a shadowing periodic sequence and the period is bounded by a integer independent of n . But it is also close to the unit circle and $G_n \rightarrow G$, and on the unit circle G is a circle diffeomorphism with irrational rotation number θ , so the period has to be large when n is large. This contradiction implies the lemma.

By Lemma 3.7 and 3.8 we can get

Lemma III.26 *There exists a $\delta > 0$ such that $d(x, \partial D) > \delta$ for all $x \in P_{G_n} \setminus \partial D$.*

Lemma III.27 *There is a $\delta > 0$ such that $d(x, y) \geq \delta$ for all $x \neq y \in P_{G_n} \setminus \partial D$.*

Proof 19 *For any $\delta > 0$ small. Let $\Gamma_{\delta, n}$ be the set of all simple closed ϕ_n short geodesic in $S^2 \setminus \partial D$ such that $\|\gamma\|_{\phi_n, P_{F_n}} \leq \delta$. If $\Gamma_{\delta, n}$ is not empty, then it must be F_n -stable. The number of elements in $\Gamma_{\delta, n}$ is bounded above by a constant independent of n . Therefore, we have an integer $m > 0$ independent*

of n such that $\|A_n^m\| \leq 1/2$ where A_n is the matrix of the Thurston linear transformation.

Let

$$P_0 = (\cup_{k=1}^m F_n^k(\Omega_{F_n})) \cap \partial D.$$

Consider $P_n = (P_{F_n} \setminus \partial D) \cup P_0$. Let $G_{\delta,n}$ be the set of all simple closed ϕ_n short geodesic in $S^2 \setminus \partial D$ such that $\|\gamma\|_{\phi_n, P_n} \leq \delta$. By Lemma 3.7 and Lemma 1.10, $G_{\delta,n} = \Gamma_{\delta,n}$. So when δ is small, $G_{\delta,n}$ is still a F_n -stable family.

$$S_n = \sum_{\gamma \in G_{\delta,n}} \frac{1}{\|\gamma\|_{\phi_n, P_n}}.$$

Let $G'_{\delta,n}$ be the set of all non-peripheral components of $F_n^{-m}(G_{\delta,n})$. Then

$$S_n \leq \sum_{\gamma \in G'_{\delta,n}} \frac{1}{\|\gamma\|_{\phi_n, F_n^{-m}(P_n)}} + M = \sum_j \sum_{\gamma_i \in G_{\delta,n}} \frac{b_{ij}}{\|\gamma_i\|_{\phi_n, P_n}} + M \leq \frac{1}{2} S_n + M.$$

So S_n is uniformly bounded above for all n . Similar to Lemma 1.3, $d(x, y) > \delta > 0$ for some δ independent of n .

§3.5 Quasi-Conformal Surgery

By Hermann-Sweatek theorem, we know $G = \phi \circ R_\theta \circ \phi^{-1}$ on the unit circle where ϕ is quasi-symmetric. Let

$$X = \{z \in \Omega_f \mid \text{Orbit}(f, z) \text{ enters into } D^\circ \text{ and does not contain } 0\}.$$

and

$$X' = \{z \in \Omega_G \mid \text{Orbit}(G, z) \text{ enters into } D^\circ \text{ and does not contain } 0\}.$$

For $z \in X'$, let y be the first moment of $\text{Orbit}(G, z)$ that enters into D° . Since ϕ is quasi-symmetric, by Ahlfors-Bers quasi-conformal extension theorem, we can extend ϕ into $H : D \rightarrow D$ such that $H(x) = y$ for the correspondent critical orbit and $H(0) = 0$.

Now let

$$G_0(z) = G(z) \quad \text{for } z \notin D$$

and

$$G_0(z) = H \circ R_\theta \circ H(z)^{-1}$$

In this section, we will prove that f and G_0 are combinatorially Equivalent.

Now let's define the critical finite quotient of F_n and G_n as follows: Let $f_n(z) = |z|F_n(\frac{z}{|z|})$ for $z \in D$, and $f_n(z) = e^{2\pi i\theta_n}z$ for $z \notin D$. Similarly, let $g_n(z) = |z|G_n(\frac{z}{|z|})$ for $z \in D$ and $g_n(z) = G_n(z)$ for $z \notin D$.

For $k \geq 0$, let E_0, \dots, E_r be the connected components of $f^{-(k+1)}(D)$ and L_0, \dots, L_s be the the connected components of $f^{-k}(D)$. Let E_0, L_0 be the component containing 0 respectively. Take k large enough such that

1. $\cup_{0 \leq i < \infty} (f^{-i}(D) \cap P_{f_n}) = f^{-k}(D) \cap P_{f_n}$ and

2. If $E_i \cap L_j \cap P_{f_n} \neq \emptyset$, then $E_i \cap P_{f_n} = L_j \cap P_{f_n}$.

Let E'_i and L'_j be simply connected open topological disks such that

i. $E_i \subset E'_i, L_j \subset L'_j$;

ii. E'_i s are disjoint;

iii. L'_j s are disjoint.

Define quotient maps $r, s : S^2 \rightarrow S^2$ such that

a. $r|_{S^2 \setminus \cup E'_i}, s|_{S^2 \setminus \cup L'_j}$ are injective, and

b. $r(z) = z$ if $z \notin \cup_{1 \leq i \leq r} E'_i$;

c. $s(z) = z$ if $z \notin \cup_{1 \leq j \leq s} L'_j$;

d. $r(E_i) = s(L_j)$ is a single point if $E_i \cap L_j \neq \emptyset$.

we have

$$\begin{array}{ccc} S^2 & \xrightarrow{r} & S^2 \\ f_n \downarrow & & \downarrow qf_n \\ S^2 & \xrightarrow{s} & S^2 \end{array}$$

where $qf_n = s \circ f_n \circ r^{-1}$ is well defined. By construction we know qf_n is critically finite and we call qf_n the *critically finite quotient* of f_n . Similarly, we can construct the *critically finite quotient* qg_n, qf, qG_0 of G_n, f , and G_0

respectively. The quotient maps can be taken in such a way that $qf_n \rightarrow qf$ and $qg_n \rightarrow qG_0$ as $n \rightarrow \infty$. Since F_n is combinatorially equivalent to G_n , there is an induced combinatorial equivalence between qf_n and qg_n . We need the following elementary lemma:

Lemma III.28 *Let f_n be a sequence of critically finite branched covering map. If there is a critically finite branched covering map f such that*

1. $d(f_n(z), f(z)) \rightarrow 0$ uniformly where d is the spherical metric. and
2. $|P_f| = |P_{f_n}|$.

Then then there exist homeomorphisms $h_n, h'_n : (S^2, P_f) \rightarrow (S^2, P_{f_n})$ which fix $0, 1, \infty$ such that

- a. h_n, h'_n are isotopic to each other rel P_f , and
- b. $f = h_n^{-1} \circ f_n \circ h'_n$, and
- c. $h_n \rightarrow id, h'_n \rightarrow id$ as $n \rightarrow \infty$.

By assumption, there exist homeomorphisms $h_n : (S^2, P_f) \rightarrow (S^2, P_{f_n})$ such that $d(h_n(z), z) \rightarrow 0$ uniformly as $n \rightarrow \infty$. We make a triangle partition Λ_1 of the sphere S^2 with P_f as all the vertexes. The map h_n induces a triangle partition Λ_2 of S^2 with P_{f_n} as all the vertexes. Let Λ'_1 be the Pull back of

Λ_1 by f and Λ'_2 be the pull back of Λ_2 by f_n . When n large enough, there exists $h'_n : S^2 \rightarrow S^2$ such that

$$\begin{array}{ccc} \Lambda'_1 & \xrightarrow{h'_n} & \Lambda'_2 \\ f \downarrow & & \downarrow f_n \\ \Lambda_1 & \xrightarrow{h_n} & \Lambda_2 \end{array}$$

commutes. Clearly h_n, h'_n are isotopic rel P_f and $h_n, h'_n \rightarrow id$.

By Lemma 3.28, we know qf_n is combinatorially equivalent to qf , and qg_n is combinatorially equivalent to qG_0 for n large enough. Because qf_n and qg_n are combinatorially equivalent, so we have

Lemma III.29 *qf is combinatorially equivalent to qG_0 . Furthermore, the combinatorial equivalence between qf and qG_0 is unique.*

The first part is obvious. Because qf, qG_0 have no Thurston obstruction, they are combinatorially equivalent to a polynomial g_0 . By Thurston's theorem, the combinatorial equivalence between g_0 and qf is unique. Similarly, the combinatorial equivalence between qG_0 and g_0 is also unique. This implies the combinatorial equivalence between qf and qG_0 is unique. By Lemma 3.28, we can see the combinatorial equivalence between qf_n and qg_n has a limit. As $n \rightarrow \infty$.

To recover the equivalence map between f and G_0 , we need to blow up

the quotient equivalence at $f^{-k}(D)$. First we define $\circ = \psi = H$ on D . On $f^{-k}(D)$, Let $\phi = \psi$ be the pull back of H defined as follows:

$$\begin{array}{ccc} f^{-k}(D) & \xrightarrow{\phi} & G_0^{-k}(D) \\ f^k \downarrow & & \downarrow G_0^k \\ D & \xrightarrow{H} & D \end{array}$$

Let $\gamma \subset S^2 \setminus (P_{F_n} \cup F_n^{-k}(D))$ be a curve connecting ∞ and $1 \in D$. $\phi_n(\gamma)$ is a curve connecting ∞ and $\phi_n(1)$. We want to show that $[\phi_n(\gamma)]$ has a limit. By Lemma 3.29, we need only to prove the winding number of γ around 0 is fixed when n is large enough. In fact, in the following diagram, if the top winding number is changed by k , then the bottom winding number is changed by $d_0 k$ where $d_0 \geq 2$ be the degree of the qf at 0. This is a contradiction. So when n is large enough, the winding number is a constant. By this way, we proved that f and G_0 are combinatorially equivalent to each other. In next section, we will prove G_0 is combinatorially equivalent to a polynomial with a Siegel disk.

Now we can pull back the structure of H by G_0 and get a G_0 -invariant line field $\mu(z)$ on the whole Riemann sphere. Let ϕ be the Beltrami solution for μ . then $g = \phi \circ G_0 \circ \phi^{-1}$ is the polynomial which is combinatorially equivalent to f .

Chapter 4. Further Discussion

§4.1 Zero Measure of Julia Set

In this section, we will present an idea to attack the zero measure problem:

Let f be a topological polynomial of simple Siegel disk type without Thurston obstruction, does the Julia set of the canonical polynomial of f has zero Lebesgue measure?

The main idea is as follows. We need only to prove that the Julia set has no Lebesgue point. We start from the Blanschek model G constructed in Theorem 1. Let $z \in J_G$. Since the forward orbit of the the Lebesgue point of the Julia set will approach to P_G , we assume that $G^n(z) \rightarrow \partial\Delta$. By Herman-Swiatek's linearization theorem on the critical circle mapping, $\partial\Delta$ has an induced metric σ . Let $I \subset \partial\Delta$ be an arc. Consider the hyperbolic space $\Omega = \mathbb{P}^1 \setminus (P_G \setminus I)$. We use $U_d(I)$ to denote the hyperbolic neighborhood of I in Ω :

$$U_d(I) = \{z \in \Omega : d_\Omega(z, I) \leq d\}$$

If I is small enough, $U_d(I)$ looks like the hyperbolic neighborhood of a segment J in the slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus J)$. Fix a suitable $d > 0$.

For each n , let $m(n) \leq n$ be the number such that

- there exists $I \in \partial\Delta$ such that $f^{m(n)}(z) \in U_d(I)$ and
- for any $k \leq n$ and $J \in \partial\Delta$ such that $f^k(z) \in U_d(J)$, we always have $\sigma(I) < \sigma(J)$.

Such I must contain a critical value of G when n is large enough. Otherwise we can pull back $U_d(I)$ and get a smaller neighborhood which is a contradiction with the minimal property of $U_d(I)$. Furthermore, we can prove that $f^{m(n)}(z)$ has to be on the boundary of $U_d(I)$ and can not approach the two end points too much. Then $f^{m(n)-1}(z)$ must lie in the strong contraction region which is bounded by two half rays starting from some critical point on the unit circle.

The sequence $\{m(n)\}$ obtained in this way is an infinite sequence. For n large enough, we can take a ball B_n containing $f^{m(n)-1}(z)$ with bounded deformation such that

- the density of the Fatou points in B_n has a universal positive lower bound, and
- the hyperbolic diameter of B_n is uniformly bounded above.

By taking B_n smaller, we can assure that the univalent inverse branch of G is defined in a larger area Ω_n such that the modulus of $\Omega_n \setminus B_n$ has a positive

lower bound. The pull back of B_n along $Orbit(G, z)$ will produce a sequence of nested neighborhoods around z with bounded deformation such that the density of the Fatou points has a positive lower bound. Therefore z is not a Lebesgue point. We complete the proof.

§4.2 Rigidity Problem

In this section, we will present an idea to attack the rigidity problem:

Let f be a topological polynomial of simple Siegel disk type without Thurston obstruction. Then up to a conformal conjugation, the canonical polynomial of f is the unique polynomial which is combinatorially equivalent to f such that the boundary of the Siegel disk is a quasi-circle.

The main idea is as follows. Let g be the canonical polynomial of f . Suppose h be a polynomial which is combinatorially equivalent to f such that the boundary of the Siegel disk of h is a quasi-circle. Then we can show that h and g are quasi-conformal equivalent. We know if two rational maps are quasi-conformal equivalent, then they are quasi-conformal conjugate. (See C. McMullen's theorem, Acta Math 180(1998), 247-292). Therefore, g and h are quasi-conformal conjugate. By our zero measure statement, the Julia set of g has zero Lebesgue measure. Therefore, g has no trivial invariant line

field. So g and h must be conformally conjugate to each other. We complete the proof.

Remark III.2 *The Rigidity statement implies that the canonical polynomial g constructed in Theorem 3.1 does not depend on the construction of the Blanschke model G and the qiasi-conformal surgery. By Shishikura's recent result, which says that the boundary of the Siegel disk of constant type of any polynomial must be a quasi-circle, we get the rigidity completely*

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