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RATIONAL HOMOTOPY THEORY: THE GENERAL NILPOTENT CASE

City University of New York PH.D. 1987

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Rational Homotopy Theory: The General Nilpotent Case

by

James E. Searl

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements of the degree of Doctor of Philosophy, City University of New York.

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Abstract

Let S_o be the category of reduced simplicial sets and C the category of commutative DG coalgebras over k (k a field of characteristic 0). A pair of adjoint functors $S_o \underset{\mathcal{S}}{\overset{|\ |}{\rightleftarrows}} C$ is constructed (| | is the realization and $\mathcal{S}$ the singular functor) which induce an adjunction between the homotopy categories. With k = Q, the unit for the adjunction when restricted to nilpotent spaces gives a Q-localization and when restricted further to rational, nilpotent spaces gives an isomorphism. On the homotopy categories, the realization functor when restricted to the replete subcategory of rational, nilpotent spaces (and corestricted to its image) gives an equivalence of categories. This generalizes the equivalence result of Quillen for simply connected spaces. It also removes the homology finiteness conditions which were imposed by Neisendorfer to relate the homotopy category of spaces to that of coalgebras (via the Bousfield-Gugenheim equivalence for spaces and algebras). The realization functor $|\ |: S_o \longrightarrow C$ is characterized by three basic properties in the sense that any functor $F: S_o \longrightarrow C$ which has these basic properties will also yield a Q-localization and give an equivalence of categories as was obtained using | |.

Acknowledgements and Preface

It is a genuine pleasure to thank Professor Heller for his help and encouragement concerning this dissertation. As my thesis advisor, his patience, accessibility, and informed enthusiasm have been heartfelt. His concern that basic properties and essentials be given a central role in ones considerations is manifest throughout this thesis-- as seen, for example, in the natural characterization of the realization functor, which is constructed in this dissertation.

I also want to express my appreciation to Professor Alphonse Vasquez for his carefully presented courses on topology at the Graduate School and who introduced me to Sullivan's ideas concerning rational homotopy theory in his lectures, and for his continued interest in the subject. I also appreciated Professor Roitberg's seminars on localization and homotopy groups of spheres here. Finally I wish to thank the other members of the Department, particularly Professors Eldon Dyer, Burton Randol, Edgar Feldman, and Gilbert Baumslag, for their contributions to making my period at The Graduate School a satisfying one.

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Introduction

In this paper we construct a model for the rational homotopy category of nilpotent reduced simplicial sets using the category of DG commutative coalgebras over Q. Before we discuss the results obtained or methods used, we want to summarize some of the approaches to rational homotopy theory which relate directly to ours.

Quillen [17] considered a sequence of six pairs of adjoint functors:

$$\tau_2 \rightleftarrows S_2 \rightleftarrows \cdots \rightleftarrows (\mathcal{L})_1 \underset{\mathcal{C}}{\overset{\mathcal{L}}{\rightleftarrows}} (C)_2$$

where τ_2 is the category of 1-connected pointed topological spaces, S_2 is the category of simply connected simplicial sets (i.e., X_1 and X_0 are one point sets), $(\mathcal{L})_1$ is the category of connected DG (chain) Lie algebras over Q (i.e., $L_0 = 0$), and $(C)_2$ is the category of simply connected DG (chain) coalgebras over Q (i.e., $C_1 = 0$, $\varepsilon_0 : C_0 \xrightarrow{\cong} Q$). Each of the categories in the above sequence has a closed model structure (except τ_2), and the above functors induce adjoint pairs of functors between the consecutive homotopy categories which give equivalences of the homotopy categories. In particular, equivalences are obtained between the rational homotopy category of simply connected simplicial sets and the homotopy categories of connected Lie algebras and simply connected coalgebras. However, since the compositions of the functors in the above sequence involve both left and right adjoint functors, the resultant compositions do not necessarily preserve either limit or colimit constructions.

Sullivan introduced the notion of a minimal algebra in his work on rational homotopy, and his ideas are presented in many papers and larger works, including, e.g., [19] and [7][1]. Part of his work concerns simplicial complexes: given a simply connected simplicial complex X, a simply connected commutatuve DG (cochain) algebra (over Q) A^*X is constructed (the algebra of P. L. forms of X). A minimal model for A^*X is then constructed; i.e., a DG algebra map $M(A^*X) \longrightarrow A^*X$ which gives a cohomology isomorphism and where $M(A^*X)$ is a minimal algebra (i.e., free as a graded commutative algebra and with a differential which has decomposable values). Finally, a rational space X(M) is associated with each minimal algebra M, with the space $X(MA^*X)$ representing the Q-localization of X. The first construction $X \longmapsto A^*X$ is functorial; however, the second $M \longmapsto X(M)$ is not. These ideas are then also applied to the case of the C^∞ forms of a simply connected smooth maniford. Sullivan's papers are rich with ideas, and we have mentioned a portion which relates directly to the treatment in our paper. His notion of a minimal model has been studied extensively, and we refer the reader to the work and bibliography of Tanré [21].

For X a simplicial set and k a field of characteristic 0, Swan [20] constructed a commutative cochain algebra AX over k such that $H^*AX \simeq H^*(X;k)$ which was based on an unpub-

[1]This work is based on lecture notes from a course taught by the authors and Eric Friedlander in Florence during 1972.

lished construction of Thom (1957) for the case $k = \mathbb{R}$.

Bousfield and Gugenheim [2] express the Sullivan-de Rham approach to rational homotopy theory in a completely functorial setting. For S the category of simplicial sets and $\mathcal{A}$ the category of commutative DG (cochain) algebras over k (k a field of characteristic 0), they construct a pair of adjoint contravariant functors $S \underset{F}{\overset{A}{\rightleftarrows}} \mathcal{A}$ where AX has the property that $H^*AX \simeq H^*(X;k)$ with X in S. Both categories have a closed model structure, and the functors induce an adjoint pair of contravariant functors $hoS \underset{F}{\overset{M}{\rightleftarrows}} ho\mathcal{A}$ on the homotopy categories where MX satisfies $j:MX \longrightarrow AX$ in $\mathcal{A}$ with MX a cofibrant algebra and j gives a cohomology isomorphism. With k = Q, these functors give an equivalence of homotopy categories when restricted (and corestricted) to the full subcategories of rational, nilpotent, connected simplicial sets of finite Q-type (i.e., $H_n(X;\mathbb{Z})$ are uniquely divisible and finite dimensional $(n \geq 1)$) and homologically connected algebras of finite Q-type (i.e., algebras whose minimal models have finite type cohomology).

Neisendorfer [15] generalized part of Quillen's treatment by showing that the adjoint functors $C \underset{\mathcal{C}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{L}$ gave an equivalence between the homotopy category hoC and a subcategory of $ho\mathcal{L}$. He then imposed "finiteness conditions" to combine this result with that of Bousfield and Gugenheim to obtain an equivalence between the two subcategories of the homotopy categories of simplicial sets and DG Lie algebras.

We also mention that Unsöld [22] obtained an equivalence between the homotopy categories of 1-reduced rational simplicial sets and 1-connected minimal topological algebras over Q.

With S_* the category of pointed, connected simplicial sets, let Nh_oS_* (respectively, QNh_oS_*) be the full subcategory of h_oS_* determined by the nilpotent (respectively, rational and nilpotent) objects. A Q-localization functor is a left adjoint to the inclusion $QNh_oS_* \hookrightarrow Nh_oS_*$ (equivalently, such a functor is given by universal maps $u: X \longrightarrow X_Q$ from X to QNh_oS_* (X in Nh_oS_*)). Bousfield and Kan [3] use a very general setting to show the existence of a Q-localization functor. Bousfield, in [1], shows that such a Q-localization also arises from the localization of pointed simplicial sets with respect to $H_*(\ ;Q)$-isomorphisms and then restricting to nilpotent, connected objects. In the Bousfield and Gugenheim work discussed above, the adjoint map $X \longrightarrow FMX$ in Nh_oS_* gives a Q-localization where X is nilpotent of finite Q-type (this uses the "pointed" version of their equivalence).

Let S_o be the category of reduced simplicial sets and C the category of commutative, connected DG coalgebras over k (k a field of characteristic 0). In this paper we construct a pair of adjoint functors $S_o \underset{\mathcal{S}}{\overset{|\ |}{\rightleftarrows}} C$ ($|\ |$ is the realization and $\mathcal{S}$ the singular functor) which induce an adjunction between the homotopy categories. With k = Q, the unit for the adjunction when restricted to nilpotent spaces gives a Q-localization and when restricted further to rational, nilpotent spaces gives an isomorphism. On the homotopy categories, the realization functor when restricted to the replete subcategory of rational, nilpotent spaces (and corestricted to its image) gives an equivalence of categories. This generalizes the equivalence result of Quillen for simply connected spaces. It also removes the homology finiteness conditions which were imposed by Neisendorfer to relate the homotopy category of spaces to that of coalgebras (via the Bousfield-Gugenheim equivalence for spaces and algebras). The realization functor $| \; |: S_0 \longrightarrow C$ is characterized by three basic properties in the sense that any functor $F: S_0 \longrightarrow C$ which has these basic properties will also yield a Q-localization and give an equivalence of categories as was obtained using $| \; |$.

After summarizing the general conventions of this paper in Section 1, we consider the category S of simplicial sets and its subcategory of reduced simplicial sets in Section 2. It is shown that the latter subcategory can be viewed as the pointed functor category: $S_0 = \mathrm{Sets}_*^{\cdot \, \bar{\Delta}^{op}}$ where $\bar{\Delta}$ is a quotient of the indexing category Δ for simplicial sets (two maps in Δ are identified if they factor through [0] in Δ) and Sets_* is the category of pointed sets.

In Section 3, the categories C of commutative, connected DG (chain) coalgebras over k and M of DG k-modules (where k is a field of characteristic 0) are considered (and also the related categories $C^{\#}$ and $M^{\#}$ of graded coalgebras and graded k-modules, respectively), and the basic properties of the forgetful functors between these categories are examined.

In Section 4, we use the functor $\mathcal{L} \xrightarrow{\mathcal{C}} C$ which was considered by Quillen and Neisendorfer to define the realization functor $|\ |: S_o \longrightarrow C$; for X in S_o, $|X|$ is given by the coend

$$|X| = \int^{[n]} X_n \cdot \gamma_n \quad \text{in } C$$

where γ is the composition $\bar{\Delta} \longrightarrow \mathcal{L} \xrightarrow{\mathcal{C}} C$ (the first functor in this composition is itself the composition $F_L s^{-1} C_*^N \bar{\Delta}$ of the free Lie algebra, desuspension, normalized chain complex, and reduced standard n-simplex functors). It is shown that $|\ |: S_o \longrightarrow C$ has a right adjoint given by the singular functor $\mathcal{S}: C \longrightarrow S_o$ (for Y in C, $\mathcal{S}Y: \bar{\Delta}^{op} \longrightarrow Sets_*$ is given by the composition $\bar{\Delta} \xrightarrow{\gamma} C \xrightarrow{C(\_,Y)} Sets_*$). After proving that $|\bar{\Delta}_n| = \gamma_n$, the realization functor is then shown to have the following basic properties (in addition to having a right adjoint): 1) $|\ |$ preserves monomorphisms and 2) there is a natural transformation $\lambda: |\ | \xrightarrow{\sim} C_*^N$ such that $(\lambda_X)_*: H_*|X| \longrightarrow H_*(X;k)$ is an isomorphism of coalgebras (X in S_o). The existence of a natural map $\mathcal{C} F_L s^{-1} V \longrightarrow V$ in M (where V has $V_0 = k$, $d_1 = 0$) which gives a homology isomorphism motivates the definition of the realization functor and is the basis for the construction of the natural transformation $\lambda: |\ | \xrightarrow{\sim} C_*^N$.

Quillen's axioms for a closed model category $\tilde{C}$ are recalled in Section 5, and some homotopy lemmas which are used in this paper are proved. The two homotopy categories $Ho\tilde{C}$ (the category of fractions with respect to the class of weak equivalences of $\tilde{C}$) and $ho\tilde{C}$ (the category with the same objects as $\tilde{C}_{cf}$-- the full subcategory of $\tilde{C}$ determined by the cofibrant-fibrant objects-- and whose morphisms are the homotopy classes of maps) are discussed briefly, and it is pointed out that ho$\tilde{C}$ is also a category of fractions (with respect to the weak equivalences in $\tilde{C}_{cf}$). It is presumably well-known that the pullback of a weak equivalence along a fibration, which has a fibrant codomain, is also a weak equivalence (theorem 5.9). Perhaps less well-known or appreciated is the fact that the pullback of a weak equivalence along a fibration, which is induced from a fibration having a fibrant codomain, is a weak equivalence (proposition 5.10).

In Section 6, we recall Quillen's closed model structure for S_o: the cofibrations are the injective maps, weak equivalences are the maps whose topological realizations give isomorphisms on all homotopy groups, and fibrations are those maps which have the right lifting property (RLP) with respect to the class of trivial cofibrations. We also recall the Quillen-Neisendorfer closed model structure for C: the cofibrations are the injective maps, weak equivalences are the maps which give homology isomorphisms, and fibrations are those maps which have the RLP with respect to the class of trivial cofibrations.

In Section 7, we show that $| \;|: S_o \longrightarrow C$ preserves both cofibrations and weak equivalences and that $\mathcal{S}: C \longrightarrow S_o$ preserves fibrations and takes a weak equivalence between fibrant objects in C to a weak equivalence in S_o. The pair of adjoint functors, thus, gives a pair of adjoint functors $HoS_o \underset{\mathcal{S}'}{\overset{|\ |}{\leftrightarrows}} HoC$ on the homotopy categories, and we show that one can take $\mathcal{S}'(X) = X'$ where $i:X \longrightarrow X'$ is a weak equivalence in C with X' fibrant. On the more concrete homotopy categories, it is the realization functor which must be modified (since it does not necessarily preserve fibrant objects); here the pair of adjoint functors $hoS_o \underset{\mathcal{S}}{\overset{|\ |'}{\leftrightarrows}} hoC$ has $|X|' = X'$ where $i:|X| \longrightarrow X'$ is a weak equivalence in C with X' fibrant. Finally the relations between the realization functors and between the singular functors on the two homotopy categories are given for reference.

In Section 8, we recall the familiar principal fibration $\tilde{h}:\tilde{E}(\pi,n) \longrightarrow \tilde{K}(\pi,n)$ in S, where π is an abelian group and $\tilde{K}(\pi,n)$ is a $K(\pi,n)$ space in S, and observe that $\tilde{h}$ can be viewed as a fibration in S_o. We also recall Neisendorfer's extension sequence $K(\pi,n-1) \xhookrightarrow{i} E(\pi,n) \xrightarrow{h} K(\pi,n)$ in C where π is a k-module and $K(\pi,n)$ is the cofree coalgebra generated by π which is concentrated in degree n. We then show that there is a commutative diagram:

$$\begin{array}{ccc} |\tilde{E}(\pi,n)| & \xrightarrow{\phi_E} & E(\pi,n) \\ {\scriptstyle |\tilde{h}|}\downarrow & & \downarrow{\scriptstyle h} \\ |\tilde{K}(\pi,n)| & \xrightarrow{\phi_K} & K(\pi,n) \end{array} \qquad \text{in C,} \qquad (0.1)$$

where ϕ_E and ϕ_K are weak equivalences; i.e., the realization of $\tilde{h}$ is weakly equivalent to h. We also introduce the notion that a fibration $f:X \longrightarrow B$ in C is a <u>fiber bundle</u> if f, when viewed in $C^{\#}$, is "essentially" a projection $B \otimes F \longrightarrow B$ where F is the fiber of f (this is in agreement with Neisendorfer's notion of an <u>extension sequence</u> in C), and show that fiber bundles are closed under pullbacks.

The main result of Section 9 is to show that the realization of a fibration in S_o, which is induced from $\tilde{h}:\tilde{E}(\pi,n) \longrightarrow \tilde{K}(\pi,n)$, is weakly equivalent to a fiber bundle in C which is induced from $h:E(\pi,n) \longrightarrow K(\pi,n)$ $(n>1)$. This result (theorem 9.2) is expected to be valid in the more general setting of a fibration $\tilde{h}:\tilde{E} \longrightarrow \tilde{K}$ in S_o which is related (as in diagram (0.1)) to a fiber bundle $h:E \longrightarrow K$ in C provided some additional hypothesis is imposed on the fibration $\tilde{h}$. This section concludes by showing that the realization of a tower of fibrations under X in S_o gives a tower of maps under $|X|$ in C which is weakly equivalent to a tower of fiber bundles in C (where in S_o $X_n \longrightarrow X_{n-1}$ is induced from the principal fibration $\tilde{h}:\tilde{E}_{n+1} \longrightarrow \tilde{K}_{n+1}$ with $\tilde{h}$ related to the fiber bundle $h:E_{n+1} \longrightarrow K_{n+1}$ in C).

The Main Theorem of this paper is proved in Section 10. We now assume that the ground field is the rationals $k = Q$. This result is that when X is nilpotent in S_o and $i:|X| \longrightarrow X'$ is a weak equivalence in C with X' fibrant, then the adjoint map $\tilde{i}:X \longrightarrow \mathcal{S}X'$ in S_o gives a Q-homology isomorphism. A corollary of this result is that when X is nilpotent and fibrant in S_o, then the X-component of the unit $\eta_X = \tilde{\tilde{i}}:X \longrightarrow \mathcal{S}X'$ in hoS_o is a Q-localization. A second corollary is that when restricted to rational and nilpotent spaces in hoS_o, then η_X is an isomorphism in hoS_o. We also show that on the homotopy categories the realization functor when restricted to the replete subcategory of rational, nilpotent spaces (and corestricted to its image) gives an equivalence of categories. Finally, since the treatment in this paper uses only the basic properties (mentioned above) of the realization functor, we obtain the result that any functor $F:S_0 \longrightarrow C$, which has the same basic properties as $| \ |:S_0 \longrightarrow C$, will necessarily provide a Q-localization and give an equivalence of categories as was obtained using $| \ |$.

In Section 11 of the Appendix, we show that the cofree coalgebra $K(\pi,n)$ classifies cohomology; i.e., for X in C, there is a natural isomorphism $[X,K(\pi,n)] \longrightarrow H^n(X;\pi)$. This result is used in Section 8 to show that $\mathcal{S}K(\pi,n)$ in S_0 is an Eilenberg-MacLane space of type (π,n).

The basic definitions, background material, and background references are incorporated into the sections as the subject is developed.

1. General conventions.

The notation $f:X \hookrightarrow Y$ is used to imply that the map f is a monomorphism.

The notation $H \xrightarrow{\cdot} K$ is used to denote a natural transformation from the functor H to the functor K (as in [11]).

A *null* object in a categoty is an object which is both an initial and terminal object; a *pointed* category is a category with a null object. In a pointed category $\tilde{C}$, the coproduct of A with B is denoted by $A \vee B$.

Let X be a set with an equivalence relation $\sim$, then $\bar{x}$ is often used to denote the equivalence class in $X/_{\sim}$ which contains x.

Let $f:X \longrightarrow Y$ be a map in Sets, then $f(x) = y$ is sometimes denoted by $x \longmapsto y$. Similarly for a functor $F:\tilde{C} \longrightarrow D$, then $Fc = d$ is sometimes denoted by $c \longmapsto d$ (c an object in $\tilde{C}$, d an object in D).

Let $f:A \longrightarrow C$, $g:B \longrightarrow C$ be maps in a category $\tilde{C}$, then the unique map $A \amalg B \longrightarrow C$ out of a coproduct which is determined by f and g is denoted by (f,g).

Let $f:C \longrightarrow A$, $g:C \longrightarrow B$ be maps in $\tilde{C}$, then the unique map $C \longrightarrow A \prod B$ into a product which is determined by f and g is denoted by $\langle f,g \rangle$.

2. Simplicial sets and reduced simplicial sets. Background references include [3], [4], [6], [8], [12], and [16].

Let $S = \mathrm{Sets}^{\Delta^{op}}$ denote the category of simplicial sets (Δ is the familiar indexing category: the objects of Δ are the ordered sets $[n] = \{0,1,2,\ldots,n\}$ and $\Delta([m],[n])$ equals all the nondecreasing functions: $[m] \longrightarrow [n]$).

Now consider the hom sets $\Delta([m],[n])$ of Δ, and identify two maps $f,g:[m] \longrightarrow [n]$ if they factor through the object $[0]$ in Δ. This identification is clearly preserved by pre- and post-composition with maps from Δ and, thus, gives a congruence. Let $\bar{\Delta}$ be the quotient category of Δ; and let $\pi: \Delta \longrightarrow \bar{\Delta}$ be the quotient functor. Note that $\bar{\Delta}$ is a pointed category ($[0]$ is both an initial and terminal object).

Let Sets_* denote the category of pointed sets; Sets_* is also a pointed category (the one-point set $\{*\}$ gives $(\{*\},*)$ in Sets_* which is both an initial and terminal object). Let $S_o \dot{=} \mathrm{Sets}_*^{\bar{\Delta}^{op}}$ denote the pointed functor category (where a _pointed_ functor is a functor between two pointed categories which preserves the null object). Note for a pointed functor $X: \bar{\Delta}^{op} \longrightarrow \mathrm{Sets}_*$, $X([0])$ is a pointed one-point set.

For X in S_o or S, $X([n])$ is usually denoted by X_n ($[n]$ in $\bar{\Delta}$ or Δ).

Now consider the functor $S_o \longrightarrow S$ determined by $X \longmapsto$ the composotion: $\Delta^{op} \xrightarrow{\pi^{op}} \bar{\Delta}^{op} \xrightarrow{X} \mathrm{Sets}_* \longrightarrow \mathrm{Sets}$ where X is in S_o and where the last functor is the obvious forgetful functor. Let S_R denote the full subcategory of S of reduced simplicial sets (i.e., those X in S which have only one vertex

($X_o = \{x_o\}$). The functor $S_o \longrightarrow S$ clearly corestricts to S_R.

Proposition 2.1: The functor $S_o \longrightarrow S_R$ is an isomorphism of categories.

Proof: We determine a functor $S_R \longrightarrow S_o$. For this, let X be in S_R with $X_o = \{x_o\}$. Using the o^{th} degeneracy maps $s_o^k : X_k \longrightarrow X_{k+1}$ of X, the points $x_n = (\prod_{k=o}^{n-1} s_o^k)(x_o) \in X_n$ for $n > o$ give a subsimplicial set $x_* \subset X$. And so the pair $\tilde{X} = (X, x_*)$ can be viewed as a simplicial pointed set $\tilde{X}: \Delta^{op} \longrightarrow Sets_*$. But since $(\tilde{X})_o = (\{x_o\}, x_o)$ is a point object in $Sets_*$, $\tilde{X}$ factors uniquely through $\bar{\Delta}^{op}$ (via π^{op}) to obtain $\hat{X}: \bar{\Delta}^{op} \longrightarrow Sets_*$ which is necessarily a pointed functor. The desired functor $S_R \longrightarrow S_o$ is then determined by $X \longmapsto \hat{X}$. That the functors compose correctly (i.e., that $S_R \to S_o \to S_R = id_{S_R}$ and $S_o \to S_R \to S_o = id_{S_o}$) is readily verified which completes the proof.

And so the corestriction of the functor $S_o \longrightarrow S$ gives an isomorphism between the category S_o and the full subcategory of reduced simplicial sets of S. We will also refer to S_o as the category of reduced simplicial sets.

For X in S (or S_R), many notions (such as the dimension of X, the n-skeleton of X, a degenerate n-simplex $x \in X_n$, etc.) have analogous definitions for objects in S_o. We will be explicit about some basic notions but will assume the reader is familiar with others.

We will find the next property to be a useful one.

Proposition 2.2: The forgetful functor $S_o \longrightarrow$ graded Sets preserves, reflects, and creates both pullbacks and pushouts.

Proof: We consider first the forgetful functor $S \longrightarrow$ graded Sets. Pullbacks in S are constructed degree-wise by using the pullback in Sets for each degree. Further the required face and degeneracy maps of the pullback are uniquely determined by this construction. This implies that $S \longrightarrow$ graded Sets preserves, reflects, and (using the uniqueness of the simplicial structure) also creates pullbacks. Similarly, $S \longrightarrow$ graded Sets preserves, reflects, and creates pushouts since pushouts in S are also constructed degree-wise. The construction of pullbacks and pushouts in S remains valid for the subcategory $S_R \hookrightarrow S$ (objects differ at most only in degree 0); and so the forgetful functor $S_R \longrightarrow$ graded Sets has the same properties which we stated for $S \longrightarrow$ graded Sets. Finally since the functor $S_o \longrightarrow$ graded Sets can be given by the composition $S_o \xrightarrow{\simeq} S_R \longrightarrow$ graded Sets, we can conclude that $S_o \longrightarrow$ graded Sets has the desired properties.

We next observe that $J: S_R \hookrightarrow S$ has a left adjoint $L: S \longrightarrow S_R$; i.e., S_R is a reflective subcategory of S. For this, let $LX = X/X^0$ where X^0 is the 0-skeleton of X (X in S). The adjunction for $L \dashv J$ is then given by: for $g \in S(X,JY)$, let $\psi^{-1}(g) \in S_R(LX,Y)$ be defined by $\psi^{-1}(g) = \bar{g}: X/X^0 \longrightarrow Y/Y^0 = Y$.

Note that the composition $S_o \xrightarrow{\simeq} S_R \hookrightarrow S$, thus, has a left adjoint, and so $S_o \hookrightarrow S$ preserves limits.

We remark that in S_o the pullback of a surjection is a surjection (since $S_o \longrightarrow$ graded Sets preserves pullbacks, and Sets has the property we are asserting for S_o), and dually the pushout of an injection is an injection.

<u>Proposition</u> 2.3: Let $\tilde{C} = S_0$ or Sets. Let the following diagram be a pushout in $\tilde{C}$:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & B \arrow[d] \\ A \arrow[r, hook] & P \end{tikzcd}$$

, then the square is bicartesian in $\tilde{C}$.

Proof: To prove this proposition for $\tilde{C} = S_0$, note that it is sufficient to show that this diagram is bicartesian in each degree since the forgetful functor $S_0 \longrightarrow$ graded Sets reflects pullbacks. And so we need only show this lemma for $\tilde{C}$ = Sets. To this end we show that the diagram has the universal mapping property of a pullback. For this, consider the diagram:

$$\begin{tikzcd} W \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\ & C \arrow[r, hook, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\ & A \arrow[r, hook, "i'"'] & P \end{tikzcd}$$

in Sets.

Since P is a pushout, the only relations in P are between im(f) and im(i). And so, since $f'h = i'k$, we can conclude that $h(W) \subset im(i)$ (and $k(W) \subset im(f)$). Now use $h(W) \subset im(i)$ to define $W \xrightarrow{\phi} C$ by $\phi(w) = c$ where $i(c) = h(w)$. Note that ϕ is well-defined since i is monic. This gives $i\phi = h$. To show we also have $k = f\phi$, note that $i'k = f'h = f'i\phi = i'f\phi$. And so the result follows since i' is monic. The required uniqueness property is immediate since any map ϕ' which satisfies $i\phi' = h$ must agree with our definition of ϕ.

Let $/\!\bar{\Delta}_n : /\!\bar{\Delta}^{op} \longrightarrow Sets_*$ denote the <u>standard reduced n-sim-</u>

__plex__ in S_o which is determined by $\bar{\Delta}(\_,[n]) : \bar{\Delta}^{op} \longrightarrow Sets$. This gives a functor $\bar{\Delta}: \Delta \longrightarrow S_o$ $([n] \mapsto \bar{\Delta}_n)$.

For X in S_o and $n \geq 0$, the __n-skeleton of X__, denoted X^n, is the subobject of X generated by the graded pointed set: X_p with $p \leq n$. Let $\dot{\bar{\Delta}}_n$ denote the (n-1)-skeleton of $\bar{\Delta}_n$ $(n > 0)$.

For X in S, one has the familiar Yoneda lemma bijection $\Psi: S(\Delta_n, X) \longrightarrow X_n$ in Sets where $\Psi(\lambda) = \lambda_{[n]}(id_{[n]})$ with $\Psi^{-1}(x)$ usually denoted by $\tilde{x}$ $(x \in X_n)$; one also has the basic pushout diagram (see, e.g., [6]):

$$\begin{array}{ccc} \coprod_\alpha \dot{\Delta}_n & \hookrightarrow & \coprod_\alpha \Delta_n \\ \downarrow & & \downarrow (\tilde{x}) \\ X^{n-1} & \hookrightarrow & X^n \end{array} \qquad \text{in } S \qquad (2.1)$$

where for the right vertical map one includes all maps $\tilde{x}$ for which x is non-degenerate.

__Proposition__ 2.4: $\Psi: S_o(\bar{\Delta}_n, X) \longrightarrow X_n$ where $\Psi(\lambda) = \lambda_{[n]}(id_{[n]})$ is a bijection in $Sets_*$ (where the hom-set of S_o has the point $\bar{\Delta}_n \to * \to X$).

Proof: We have seen that the composition $J: S_o \xrightarrow{\cong} S_R \hookrightarrow S$ has a left adjoint L; the sequence of bijections $S_o(L\Delta_n, X) \xrightarrow{\cong} S(\Delta_n, JX) \xrightarrow{\cong} X_n$ then gives the desired pointed map Ψ (using $L\Delta_n = \bar{\Delta}_n$).

Next let X be in S_o and using the bijection in the above proposition form the diagram which is analogous to the one above:

$$\begin{tikzcd} \bigvee_{\alpha} \dot{\bar{\Delta}}_n \arrow[r, hook] \arrow[d] & \bigvee_{\alpha} \bar{\Delta}_n \arrow[d, "(\bar{x})"] \\ X^{n-1} \arrow[r, hook] & X^n \end{tikzcd} \qquad \text{in } S_o. \qquad (2.2)$$

__Proposition__ 2.5: Diagram (2.2) is a pushout in S_o.

Proof: Note that $\pi\colon \Delta \longrightarrow \bar{\Delta}$ gives a map $\pi\colon \Delta_n \longrightarrow \bar{\Delta}_n$ in S (where $\bar{\Delta}_n$ is viewed as an object in S). When we consider diagram (2.1) for $X = \bar{\Delta}_n$, we obtain the pushout:

$$\begin{tikzcd} \dot{\Delta}_n \arrow[r, hook] \arrow[d] & \Delta_n \arrow[d, "\pi"] \\ \dot{\bar{\Delta}}_n \arrow[r, hook] & \bar{\Delta}_n \end{tikzcd} \qquad \text{in } S$$

(note π corresponds to the one non-degenerate n-simplex of $\bar{\Delta}_n$). Next view diagram (2.2) as a diagram in S (using $S_o \xrightarrow{\simeq} S_R \hookrightarrow S$) and form the following diagram:

$$\begin{tikzcd} \coprod_{\alpha} \dot{\Delta}_n \arrow[r, hook] \arrow[d] & \coprod_{\alpha} \Delta_n \arrow[d] \\ \bigvee_{\alpha} \dot{\bar{\Delta}}_n \arrow[r, hook] \arrow[d] & \bigvee_{\alpha} \bar{\Delta}_n \arrow[d] \\ X^{n-1} \arrow[r, hook] & X^n \end{tikzcd} \qquad \text{in } S.$$

The outer rectangle of this diagram is a pushout since the composition of the vertical maps agree with those of diagram (2.1). The top square is clearly a pushout; and so we can conclude that the bottom square is a pushout in S and, hence, in S_o.

Gabriel and Zisman ([6], p. 31) give the following categorical presentation of $\dot{\Delta}_n$:

$$\begin{tikzcd} \coprod\limits_{0 \leqslant i < j \leqslant n} \Delta_{n-2} \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \coprod\limits_{0 \leqslant i \leqslant n} \Delta_{n-1} \arrow[r, "j"] \arrow[rr, bend left, "p"] & Z \arrow[r, dashed, "p'"] & \Delta_n \end{tikzcd} \qquad \text{in } S \qquad (2.3)$$

where p is determined by the i^{th} face operators $\Delta_{n-1} \longrightarrow \Delta_n$

($0 \leqslant i \leqslant n$) and where u and v are determined by the $j-1^{th}$ and i^{th} face operators $\Delta_{n-2} \longrightarrow \Delta_{n-1}$ with codomain indices i and j, respectively ($0 \leqslant i < j \leqslant n$). Z is the coequalizer of u and v, p is onto $\dot{\Delta}_n$, and the map $p': Z \longrightarrow \Delta_n$ determined by p corestricts to an isomorphism between Z and $\mathrm{im}(p') = \dot{\Delta}_n$.

By applying the functor $L: S \longrightarrow S_o$, which is left adjoint to $J: S_o \longrightarrow S$, to the above diagram (and retaining the same symbols for the maps), we obtain:

$$\begin{tikzcd} \bigvee\limits_{0 \leqslant i < j \leqslant n} \bar{\Delta}_{n-2} \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \bigvee\limits_{0 \leqslant i \leqslant n} \bar{\Delta}_{n-1} \arrow[r, "j"] \arrow[rr, bend left, "p"] & Z \arrow[r, dashed, "p'"] & \bar{\Delta}_n \end{tikzcd} \quad \text{in } S_o. \qquad (2.4)$$

Here the coequalizer Z in S_o gives a similar factorization $p = p'j$ with p' corestricting to an isomorphism between Z and $\mathrm{im}(p') = \dot{\bar{\Delta}}_n$.

The normalized k-chain functor $C^N_*: S_o \longrightarrow M$ (k a field of characteristic 0) preserves colimits and monomorphisms (for this, consider C_* as the composition $S_o \xrightarrow{F_*} \text{k-modules}^{\Delta^{op}} \longrightarrow M$ where F:Sets$\longrightarrow$k-modules is the free k-module functor).

For a map $f: X \longrightarrow Y$ in S_o, the map $f_* = H_*(f;k): H_*(X;k) \longrightarrow H_*(Y;k)$ is called a k-homology map.

Proposition 2.6: $H_k \bar{\Delta}_n \approx \mathbb{Z}$, $\mathbb{Z}^n$, or 0 for k = 0, 1, or >1, respectively.

Proof: The result for k = 0 is clear. Next note that $C^N_k \bar{\Delta}_n = C^N_k \Delta_n$ for $k > 0$ (since $\bar{\Delta}_n$ arises from Δ_n by the identification of all vertices, and higher dimensional identifications involve only degenerate simplices of Δ_n). We, thus, have the above result for $k > 1$ (since Δ_n is acyclic). Finally we also have $H_1 \overline{/\!\Delta}_n = H_1(/\!\Delta_n, (/\!\Delta_n)^0)$ where $(/\!\Delta_n)^0$ is the 0-skeleton of $/\!\Delta_n$. Considering the long exact sequence in homology for the pair $(/\!\Delta_n, (/\!\Delta_n)^0)$, we then obtain the desired result for $k = 1$ (using $H_0(/\!\Delta_n)^0 = \mathbb{Z}^{n+1}$, $H_0 /\!\Delta_n = \mathbb{Z}$). Further, the n edges of a reduced n-horn freely generate $H_1 \overline{/\!\Delta}_n$ (using the definition of the connecting homomorphism in the long exact sequence).

The __n-sphere__ in S_0 is defined by $\overline{S^n} = \overline{/\!\Delta}_n / \dot{/\!\Delta}_n$ $(n > 0)$. $\overline{S^n}$ has only one non-degenerate p-simplex when $p = n, 0$ (all other values of p give degenerate p-simplices); and so $H_p\overline{S^n} \simeq \mathbb{Z}$ if $p = n, 0$, and $\simeq 0$ otherwise.

3. Coalgebras. Background references include [13], [15], and [17].

For k a field of characteristic 0, let $M^{\#}$ denote the category of graded k-modules where the grading is given by the nonnegative integers. And let M denote the category of differential graded k-modules where the differential has degree = -1.

We note several useful properties of the forgetful functor $\tilde{U}: M \longrightarrow M^{\#}$. Firstly it is readily verified that $\tilde{U}$ preserves, reflects, and creates both kernels and cokernels. Also $\tilde{U}$ preserves and reflects both pullbacks and pushouts. For this result, with M and $M^{\#}$ abelain categories, we can combine the above preservation and reflection properties of $\tilde{U}$ with the following characterization of pullbacks and pushouts in terms of exact sequences.

<u>Proposition</u> 3.1 (see, e.g.,[5]): For an abelian category, given a square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "k"] \\ A \arrow[r, "h"'] & D \end{tikzcd} ,$$

consider the sequence $C \xrightarrow{\langle f,g \rangle} A \oplus B \xrightarrow{(h,-k)} D$.

Then the square is

a pullback $\Leftrightarrow$ $0 \longrightarrow C \longrightarrow A \oplus B \longrightarrow D$ is exact,

a pushout $\Leftrightarrow$ $C \longrightarrow A \oplus B \longrightarrow D \longrightarrow 0$ is exact, and

bicartesian $\Leftrightarrow$ $0 \longrightarrow C \longrightarrow A \oplus B \longrightarrow D \longrightarrow 0$ is exact.

We also give a useful consequence of this characterization:

<u>Corollary</u> 3.2: If the following square is a pushout in an abelian category:

$$\begin{tikzcd} C \arrow[r, hook, "g"] \arrow[d, "f"'] & B \arrow[d] \\ A \arrow[r, hook] & P \end{tikzcd}$$

, then the square is bicartesian.

Proof: This result is immediate since in any category the map, $C \xrightarrow{\langle f,g \rangle} A \amalg B$, is monic if f or g is monic.

We remark that a similar result for S_0 and Sets was proved in the previous section.

Let C denote the category of differential, commutative, connected, and associative graded coalgebras over k. A <u>coalgebra</u> refers to an object in C. And let $C^{\#}$ denote the category of commutative, connected, and associative graded coalgebras over k. C and $C^{\#}$ are complete and cocomplete; for A and B in C (or $C^{\#}$), the product is given by the tensor product $A \otimes B$.

The forgetful functor $U: C \longrightarrow C^{\#}$ preserves pullbacks. For the proof of this property, it suffices to show that U preserves products and equalizers. The preservation of products is immediate (since the product is given by the tensor product in both C and $C^{\#}$). To show the preservation of equalizers, let $Z \longrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be an equalizer diagram in C; note that we can take Z to be the largest subcoalgebra of X in C which equalizes f and g (i.e., Z is the internal sum of all subcoalgebras of X which equalize f and g). Clearly UZ equalizes U(f) and U(g). To show that UZ is the largest subcoalgebra of UX in $C^{\#}$ which has this property, consider the following: let X^a be a subcoalgebra of UX which equalizes U(f) and U(g), and note that $d(X^a)$ equalizes U(f) and U(g). The internal sum $X^a + d(X^a)$ in graded k-modules gives a DG subcoalgebra of X which equalizes f and g. We then have that $X^a \subset U(X^a + d(X^a)) \subset UZ$ in $C^{\#}$. Hence, UZ is an equalizer of U(f) and U(g).

The forgetful functor $\hat{U}:C \longrightarrow M$ has a right adjoint $S':M \longrightarrow C$. For V in M, S'(V) is the <u>symmetric</u> coalgebra on V. Letting T'(V) be the tensor coalgebra on V, then S'(V) can be constructed as the subspace of T'(V) comprised of those elements which are invariant under the action of the permutation groups (which permute the tensors). Let $j:\hat{U}S' \xrightarrow{\cdot} id_M$ denote the counit for $\hat{U} \dashv S'$; and for $g:\hat{U}X \longrightarrow V$ in M, let the adjoint map be denoted by $\hat{g}:X \longrightarrow S'(V)$. We will make use of the fact that $\hat{U}$, being a left adjoint, preserves colimits.

Similarly, the forgetful functor $\hat{U}^{\#}:C^{\#} \longrightarrow M^{\#}$ has a right adjoint which we also denote by $S':M^{\#} \longrightarrow C^{\#}$. It will be clear from the context whether the intended domain of S' is M or $M^{\#}$.

The commutativity property $\hat{U}^{\#}\ U = \tilde{U}\ \hat{U}$ is immediate.

The next lemma concerns the replacement of an isomorphism on homology by a coalgebra map (this technical lemma is used in the proof of lemma 8.3).

<u>Lemma</u> 3.3: Let V be a k-module, and let X be a coalgebra such that the map $\hat{\theta}:H_*X \longrightarrow S'(V_n)$ is an isomorphism in $C^{\#}$ where V_n in $M^{\#}$ has V concentrated in degree n. Then there is a map $\hat{\beta}:X \longrightarrow S'(V_n)$ in C which gives a homology isomorphism

and is such that $\hat{j}_*(\hat{\beta})_* = \hat{\Theta}$ where $\hat{j}_*: H_*S'(V_n) \xrightarrow[\cong]{} S'H_*V_n = S'(V_n)$. Moreover, the construction of $\hat{\beta}$ has the following commutativity property: let $f: X \hookrightarrow X'$ be a map in C with both f and $f_*: H_*X \hookrightarrow H_*X'$ injective, and let $i: V_n \longrightarrow V_n'$ be a map in $M^{\#}$;

if
$$\begin{tikzcd} H_*X \arrow[r, "f_*"] \arrow[d, "\hat{\Theta}"', "\cong"] & H_*X' \arrow[d, "\hat{\Theta}'", "\cong"'] \\ S'(V_n) \arrow[r, "S'(i)"'] & S'(V_n') \end{tikzcd}$$
commutes in $C^{\#}$,

then
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\hat{\beta}"'] & X' \arrow[d, "\hat{\beta}'"] \\ S'(V_n) \arrow[r, "S'(i)"'] & S'(V_n') \end{tikzcd}$$
commutes in C.

Proof: Let $\tilde{V} \subset X_n$ be a sub k-module of cycles such that $a_n =$ the composition $\tilde{V} \subset \ker d_n \longrightarrow H_nX$ is an isomorphism; i.e., $\tilde{V}$ is given by free generators in X_n of H_nX. Define $g_n: X_n \longrightarrow \tilde{V}$ by $g_n(x) = x$ for $x \in \tilde{V}$ ($=0$ otherwise); this gives a chain map $g: X \longrightarrow \tilde{V}_n$ (since for $x \in \tilde{V}$, x is not a boundary ($x \neq 0$)) which in turn gives a map $\hat{g}: X \longrightarrow S'(\tilde{V}_n)$ in C. Next consider the composition $\tilde{\Theta} = \tilde{V}_n \xrightarrow{a} H_*X \xrightarrow{\Theta} V_n$ where $\tilde{\Theta}_n = \tilde{V} \xrightarrow[\cong]{a_n} H_nX \xrightarrow[\cong]{\Theta_n} V$. Noting that $(g_n)_* = (a_n)^{-1}: H_nX \xrightarrow[\cong]{} H_n\tilde{V} = \tilde{V}$, we obtain $\tilde{\Theta}g_* = \Theta$. Finally, the composition $\hat{\beta} = \widehat{g\tilde{\Theta}}: X \xrightarrow{\hat{g}} S'(\tilde{V}_n) \xrightarrow[\cong]{S'(\tilde{\Theta})} S'(V_n)$ in C has the desired property that $(\hat{\beta})_* = (\hat{g})_*(S'(\tilde{\Theta}))_*: H_*X \longrightarrow H_*S'(\tilde{V}_n) \xrightarrow[\cong]{} H_*S'(V_n)$ is an isomorphism (this is an exercise in the universal mapping property of cofree coalgebras and also uses the fact that H_* and S' commute (see [15], lemma 1.11)). That $\hat{\beta}$ satisfies $\hat{j}_*(\hat{\beta})_* = \hat{\Theta}$ is a similar exercise.

We next prove the commutativity property in the statement of this lemma. We show that there is a k-module map $\tilde{f}_n: \tilde{V}$

$\longrightarrow \tilde{V}'$ (and, thus, a map $\tilde{f}:\tilde{V}_n \longrightarrow \tilde{V}_n'$ in M) such that the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\hat{g}"'] \arrow[dd, bend right=60, "\hat{\beta}"'] & X' \arrow[d, "\hat{g}'"] \arrow[dd, bend left=60, "\hat{\beta}'"] \\ S'(\tilde{V}_n) \arrow[r, "S'(\tilde{f})"] \arrow[d, "S'(\tilde{\Theta})"'] & S'(\tilde{V}_n') \arrow[d, "S'(\tilde{\Theta}')"] \\ S'(V_n) \arrow[r, "S'(i)"'] & S'(V_n') \end{tikzcd} \quad \text{in } C$$

commutes (note that the left and right vertical maps compose to give $\hat{\beta}$ and $\hat{\beta}'$, respectively). For this map, note that if for $\tilde{V}'$ we take an extension of $f_n(\tilde{V})$, then the restriction to $\tilde{V}$ of $f_n:\ker d_n \longrightarrow \ker d_n'$ corestricts to $\tilde{V}'$; i.e., we simply let $\tilde{f}_n:\tilde{V} \longrightarrow \tilde{V}'$ be the restriction and corestriction of f_n. For the commutivity of the upper square in the above diagram, note that it suffices to show that $g'f = \tilde{f}g$; i.e., to show that $g_n'f_n = \tilde{f}_n g_n$. Using the definition of g_n (and g_n'), this in turn reduces to showing that $x \in \tilde{V} \iff f_n(x) \in \tilde{V}'$. We have left to show $x \in \tilde{V} \Leftarrow f_n(x) \in \tilde{V}'$ (i.e., $x \notin \tilde{V} \Rightarrow f_n(x) \notin \tilde{V}'$): a) note that $x \in X_n - \ker d_n \Rightarrow f_n(x) \in X_n' - \ker d_n'$ (using f_{n-1} is injective) and b) $x \in \underline{\ker d_n - \tilde{V}} \Rightarrow f_n(x) \in f_n(\ker d_n) - \tilde{V}'$ (since $\overline{f_n(x)} \in H_nX'$ is in $\overline{f_n(\tilde{V})} \subset H_nX'$, and so $f_n(x) \notin \tilde{V}'$). The commutativity of the lower square in the above diagram is immediate since $\tilde{\Theta}'\tilde{f} = i\tilde{\Theta}$ (using the definition of $\tilde{f}$ and $\Theta' f_* = i\Theta$).

4. **The realization and singular functors.**

For k a field of characteristic 0, let $\mathcal{L}$ denote the category of differential graded Lie algebras over k where the grading is given by the nonnegative integers and where the differential has degree = -1. We adopt the convention that ungraded Lie algebras are objects of $\mathcal{L}$ when concentrated in degree 0.

Quillen (in [17]) defined a pair of adjoint functors $C \underset{\mathcal{C}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{L}$ with $\mathcal{L} \dashv \mathcal{C}$, and showed that the components of the unit and counit for the adjunction gave homology isomorphisms when the coalgebras X were simply connected ($X_1 = 0$) and when the Lie algebras L were connected ($L_0 = 0$), respectively. Neisendorfer (in [15]) showed that the components gave homology isomorphisms without these restrictions.

We give Neisendorfer's very efficient construction of $\mathcal{C}:\mathcal{L} \dashrightarrow C$. The underlying non-differential coalgebra of $\mathcal{C}L$ is given by $U(\mathcal{C}L) = S'(sL)$ where L is in $\mathcal{L}$ and where $s:M^{\#} \dashrightarrow M^{\#}$ is the suspension ($V \mapsto \tilde{V}$ with $\tilde{V}_{n+1} = V_n$). $S'(sL)$ has a bigraded structure, and the differential on $S'(sL)$ is given as a sum $d = d_I + d_{[\ ]}$. Also the differential is uniquely determined by the composition $\bar{d} = S'(sL) \xrightarrow{d} S'(sL) \xrightarrow{i} sL$. With $\bar{d} = \bar{d}_I + \bar{d}_{[\ ]}$, $\bar{d}$ is then defined on the generators sl, $(sl')(sl'')$ ($= sl' \otimes sl'' + (-1)^{\deg l' \deg l''} sl'' \otimes sl'$), ... of $S'(sL)$ ($\subset T'(sL)$) by $\bar{d}_I(sl) = -sdl$, $\bar{d}_{[\ ]}(sl')(sl'') = s[l',l'']$, and $\bar{d}_I$ and $\bar{d}_{[\ ]}$ are zero on the other generators.

For what follows, we consider two notions: with c in a pointed category $\tilde{C}$ and (X,x_0) in Sets_*, the pointed copower is defined by $X\cdot c = \bigvee_{x \in X-\{x_0\}} c$ and the pointed power by $c^X = \prod_{x \in X-\{x_0\}} c$ in $\tilde{C}$. Note that for $\tilde{C} = \mathrm{Sets}_*$, $X^Y = \mathrm{Sets}_*(X,Y)$. Also $\tilde{C}(X\cdot c, c') \longrightarrow \tilde{C}(c,c')^X$ is a bijection in Sets_*-- a result which is similar to that for the unpointed case.

Now consider the composite functor $\bar{\Delta} \longrightarrow \mathcal{L}$ ($[n] \longmapsto F_L(s^{-1}C_*^N\bar{\Delta}_n)$) where $F_L: M \longrightarrow \mathcal{L}$ is the free Lie algebra functor and where $s^{-1}: M \longrightarrow M$ is the desuspension ($V \longmapsto \tilde{V}$ with $\tilde{V}_{n-1} = V_n$). And let γ be the composition $\bar{\Delta} \longrightarrow \mathcal{L} \xrightarrow{\mathcal{C}} C$. Now note for X in S_0, we have $\bar{\Delta}^{op} \times \bar{\Delta} \xrightarrow{X \times \gamma} \mathrm{Sets}_* \times C \longrightarrow C$ where the last functor is given by the pointed copower. And so we can consider the <u>realization</u> functor $|\ |: S_0 \longrightarrow C$ defined (on objects) by the coend

$$|X| = \int^{[n]} X_n \cdot \gamma_n \quad (X \text{ in } S_0).$$

Coends (and ends) are discussed in [11]; the universal mapping property of coends defines $|\ |$ on the morphisms of S_0.

Let the <u>singular</u> functor $\mathcal{S}: C \longrightarrow S_0$ be defined (on objects) as follows: for Y in C, let $\mathcal{S}Y$ be given by the composition $\bar{\Delta} \xrightarrow{\gamma} C \xrightarrow{C(\_,Y)} \mathrm{Sets}_*$ (since $\gamma_0 = k$, the composition determines an object in S_0).

<u>Lemma</u> 4.1: Let X,Y be in S_0, then $S_0(X,Y) = \int_{[n]} \mathrm{Sets}_*(X_n,Y_n) = \int_{[n]} Y_n^{X_n}$ in Sets_* (where the right end involves the pointed power in Sets_*).

Proof: This is the analogue for S_0 of the end formula for S. The hom-sets of a functor category have the universal mapping property of an end: let $H, K: \tilde{C} \longrightarrow D$ be functors, then $D^{\tilde{C}}(H,K)$

$= \int^{c} D(Hc,Kc)$. Further, for D a pointed category, this formula gives an isomorphism in $Sets_*$. And so we have our desired result for $S_o(X,Y)$.

Theorem 4.2: The functors $S_o \underset{\mathcal{S}}{\overset{|\ |}{\rightleftarrows}} C$ are an adjoint pair: $|\ | \dashv \mathcal{S}$.

Proof: Consider the following sequence of isomorphsims in $Sets_*$:

$$\begin{aligned}
C(|X|,Y) &= C(\textstyle\int^{[n]} X_n \cdot \gamma_n, Y) && \text{(definition of } |X|) \\
&\simeq \textstyle\int_{[n]} C(X_n \cdot \gamma_n, Y) && \text{(since } C(_,Y) \text{ is continuous)} \\
&\simeq \textstyle\int_{[n]} C(\gamma_n, Y)^{X_n} && \text{(relation between pointed copower and pointed power)} \\
&= \textstyle\int_{[n]} (\mathcal{S}Y)_n{}^{X_n} && \text{(definition of } \mathcal{S}Y) \\
&\simeq S_o(X,\mathcal{S}Y) && \text{(by above lemma).}
\end{aligned}$$

Finally since each isomorphism is natural in X and Y, so is the resulting composition.

We remark that $|\ |$, being a left adjoint, preserves colimits and $\mathcal{S}$, being a right adjoint, preserves limits.

Lemma 4.3: Let D and $\tilde{C}$ be pointed categories, and let $F:D \longrightarrow \tilde{C}$ be a pointed functor. Assume that the pointed copowers $D(d',d_o)\cdot Fd$ exist in $\tilde{C}$ for all d', d_o, d in D and that the coend in the following formula exists for all d_o in D.

Then $\int^{d} D(d,d_o)\cdot Fd = Fd_o$ (where the copower is pointed).

Proof: The above formula is a special case of the following coend formula for a left Kan extension (see [11], p. 236): $(Lan_K F)(a) = \int^{d} A(Kd,a)\cdot Fd$ where $K:D \longrightarrow A$ (provided the copowers $A(Kd',a)\cdot Fd$ exist for all d', d in D and all a in A and that the coend exists for all a in A). Note that for K =

$id_D: D \longrightarrow D$, this formula gives $(\mathrm{Lan}_{id_D}F)(d_o) = \int^{d} D(d,d_o)\cdot Fd$. But $(\mathrm{Lan}_{id_D}F)(d_o) = Fd_o$; and so the desired result follows (when F is a pointed functor, this formula is valid provided the pointed copower is used).

We remark that, in a similar fashion, if one applies the end formula for a right Kan extension to the case $K = id_D$, then one obtains an end formula (which is analogous to the coend formula of this lemma) which generalizes the Yoneda lemma. This observation is the note upon which Mac Lane ends [11].

<u>Proposition</u> 4.4: $|\bar{\Delta}_n| = \gamma_n$ in C.

Proof: By definition we have $|\bar{\Delta}_n| = \int^{[k]} \bar{\Delta}([k],[n])\cdot\gamma_k$; and then applying the above lemma with $F = \gamma: \bar{\Delta} \longrightarrow C$ we can conclude that this coend equals γ_n $(= \gamma[n])$.

Thus, $|\bar{\Delta}|$ and $\gamma: \bar{\Delta} \longrightarrow C$ are isomorphic functors.

The basic properties of the realization functor are the following (in addition to having a right adjoint): 1) $|\ |$ preserves monomorphisms and 2) there is a natural transformation $\lambda: \hat{U}|\ | \xrightarrow{\cdot} C^N_*$ such that $(\lambda_X)_*: H_*|X| \longrightarrow H_*(X;k)$ is an isomorphism of coalgebras (X in S_o). We now proceed to prove these properties.

<u>Theorem</u> 4.5: $|\ |$ preserves monomorphisms.

Proof: The approach taken is similar to that given in [6] for $|\ |: S \longrightarrow \mathrm{Top}$, with the geometric standard n-simplex $\Delta_n = |\Delta_n|$ in Top and $\gamma_n = |\bar{\Delta}_n|$ in C being treated analogously. We first show that $i: \dot{\bar{\Delta}}_n \hookrightarrow \bar{\Delta}_n$ in S_o realizes to a monomorphism $|i|: |\dot{\bar{\Delta}}_n| \longrightarrow |\bar{\Delta}_n|$ in C. By applying $|\ \ |: S_o \longrightarrow C$ to the presentation of $\dot{\bar{\Delta}}_n$ given by diagram (2.4), we obtain the top row in the following diagram:

$$\begin{tikzcd}
\bigvee\limits_{0\le i<j\le n} |\bar{\Delta}_{n-2}| \arrow[r, shift left, "|u|"] \arrow[r, shift right, "|v|"'] \arrow[d] & \bigvee\limits_{0\le i\le n} |\bar{\Delta}_{n-1}| \arrow[r, "|j|"] \arrow[d] \arrow[rr, bend left, "|p|"] & |Z| \arrow[r, dashed, "|p'|"] \arrow[d, dashed] & |\bar{\Delta}_n| \arrow[d] \\
\bigvee\limits_{0\le i<j\le n} \gamma_{n-2} \arrow[r, shift left, "\tilde{u}"] \arrow[r, shift right, "\tilde{v}"'] & \bigvee\limits_{0\le i\le n} \gamma_{n-1} \arrow[r, "\tilde{j}"] \arrow[rr, bend right, "\tilde{p}"'] & \tilde{Z} \arrow[r, dashed, "\tilde{p}'"] & \gamma_n
\end{tikzcd} \quad \text{in } C;$$

further, this row is a coequalizer (since $|\ \ |$ preserves colimits). The bottom row is obtained using the functor $\gamma: \bar{\Delta} \longrightarrow C$ in a similar fashion as was done for the coequalizer diagram (2.3). The coequalizer $\tilde{Z}$ is the qoutient coalgebra with relations which attach the faces of γ_n. Recall that $\gamma_n = \mathcal{C}(F_L(s^{-1}C_*^N \bar{\Delta}_n))$ where $U\mathcal{C}(L) = S'(sL)$. I.e., when we view γ_n in $C^{\#}$, γ_n has generators which are the permuted tensor products of the free Lie brackets on the non-degenerate simplices of $\bar{\Delta}_n$. Viewing the free generators of $\tilde{Z}$ as a subset of those of γ_n-- there are missing Lie brackets, we then have that $\tilde{p}': \tilde{Z} \longrightarrow \gamma_n$ is monic (since all the functors which are involved preserve monics). In the above diagram, the two left vertical maps are isomorphisms (since γ and $|\bar{\Delta}|: \bar{\Delta} \longrightarrow C$ are naturally isomorphic) and the left hand squares commute (using the naturality). There is, thus, a canonical map $|Z| \longrightarrow \tilde{Z}$ such that the middle square commutes; further, this map is necessarily an isomorphism. Also the right hand square commutes (using the naturality and the fact that $|j|$ is an epimorphism). This implies that $|p'|:|Z| \longrightarrow |\bar{\Delta}_n|$ is a monomorphism (since $\tilde{p}'$ is); hence, $|i|:|\dot{\bar{\Delta}}_n| = |\mathrm{im}(p')| \longrightarrow |\bar{\Delta}_n|$ is a monomorphism.

We next show the general case. Let $f:X \longrightarrow Y$ be a monomorphism in S_o; we can assume that X is a subsimplicial set with f the inclusion. Let $\{\alpha'\}$ be the set of non-degenerate n-simplices of Y which are not in X, and consider the commutative diagram:

$$\begin{tikzcd} \bigvee_{\alpha'} \dot{\bar{\Delta}}_n \arrow[r, hook] \arrow[d] & \bigvee_{\alpha'} \bar{\Delta}_n \arrow[d] \\ Y^{n-1} \cup X \arrow[r, hook] & Y^n \cup X \end{tikzcd} \qquad \text{in } S_o$$

where, e.g., Y^n is the n-skeleton of Y. This diagram in S, with $\dot{\Delta}_n$ and Δ_n (instead of $\dot{\bar{\Delta}}_n$ and $\bar{\Delta}_n$), is shown in [6] to be a pushout. The same approach which was used to show that diagram (2.2) is a pushout can also be used to show that the above diagram is a pushout in S_o. By applying $|\ \ |$ to this diagram, we then obtain the pushout diagram:

$$\begin{tikzcd} \bigvee_{\alpha'} |\dot{\bar{\Delta}}_n| \arrow[r, hook] \arrow[d] & \bigvee_{\alpha'} |\bar{\Delta}_n| \arrow[d] \\ |Y^{n-1} \cup X| \arrow[r, hook] & |Y^n \cup X| \end{tikzcd} \qquad \text{in } C$$

where the bottom map is necessarily monic (since one can view this diagram as a pushout in the abelian category M). Finally by considering the directed colimits, we obtain that $|f|:|X| = |Y^0 \cup X| \longrightarrow \varinjlim_{n} |Y^n \cup X| = Y$ is a monomorphism.

Let $M_c \subset M$ be the full subcategory determined by the objects V such that $V_0 = k$ and $d_1 = 0$.

Lemma 4.6: There is a natural transformation $\vec{\hat{\lambda}}:\hat{U}\mathcal{C}F_L s^{-1} \xrightarrow{\cdot}$

id_{M_c} such that $(\overset{*}{\lambda}_V)_*: H_*(\mathcal{C}F_L s^{-1}V) \longrightarrow H_*V$ is an isomorphism of coalgebras (V in M_c).

Proof: This is an immediate consequence of Neisendorfer's result ([15], prop. 4.2 b): for a Lie algebra L in $\mathcal{L}$, if $L = F_L(V)$ in $\mathcal{L}^{\#}$ = graded Lie algebras, then $\bar{H}_*\mathcal{C}L$ is isomorphic to sH_*QL as a coalgebra where $QL = L/[L,L]$ is the abelianization of L. The isomorphism is given by the composition $\mathcal{C}(L) = S'(sL) \xrightarrow{j} sL \xrightarrow{s(\pi)} sQL$ (where $L \xrightarrow{\pi} QL$) which is a chain map. With $L = F_L(\tilde{V})$, then $QL = \tilde{V}$, and this map becomes $\mathcal{C}F_L(\tilde{V}) \longrightarrow s\tilde{V}$ which is clearly natural in $\tilde{V}$. Further, for V in M_c, the isomorphism is also true for the unreduced homology on $\mathcal{C}L$, and the application of this proposition to $\tilde{V} = s^{-1}V$ gives our desired result.

And so applying $\overset{*}{\lambda}$ to the functor $C^N_*\bar{\Delta}: /\bar{\Delta} \longrightarrow M$ we obtain a natural transformation $\tilde{\lambda} = \overset{*}{\lambda}_{C^N_*\bar{\Delta}}: \hat{U}\gamma \xrightarrow{\cdot} C^N_*\bar{\Delta}$ such that $(\tilde{\lambda}_n)_*: H_*\gamma_n \longrightarrow H_*(/\bar{\Delta}_n;k)$ is an isomorphism in $C^{\#}$.

Now apply the coend bifunctor $\int^{[n]} -_n \cdot -_n$ to $\tilde{\lambda}: \hat{U}\gamma \xrightarrow{\cdot} C^N_*\bar{\Delta}$ to obtain a natural transformation $\lambda: \hat{U}|\;| \xrightarrow{\cdot} C^N_*$ which gives

$$\hat{U}|X| = \int^{[n]} X_n \cdot \hat{U}\gamma_n \xrightarrow{\lambda_X = \int^{[n]} X_n \cdot \tilde{\lambda}_n} \int^{[n]} X_n \cdot C^N_* /\bar{\Delta}_n = C^N_* X \quad .$$

__Theorem__ 4.7: For the natural transformation $\lambda: \hat{U}|\;| \xrightarrow{\cdot} C^N_*$, the map $(\lambda_X)_*: H_*|X| \longrightarrow H_*(X;k)$ is an isomorphism in $C^{\#}$ (X in S_o).

Proof: We first give an inductive proof that the map $\lambda_{X^n}: |X^n| \longrightarrow C^N_* X^n$ in M (X^n is the n-skeleton of X) gives a homology isomorphism.

For $n = 0$, the 0-skeleton $X^0 = *$ in S_o, both $|X^0|$ and

$C^N_*X^0$ equal k in M, and we have $\lambda_X0 = id_k : |X^0| \longrightarrow C^N_*X^0$ in M.

We now assume that $\lambda_Xn\text{-}1 : |X^{n-1}| \longrightarrow C^N_*X^{n-1}$ gives a homology isomorphism and show it follows that λ_Xn gives one as well. The application of $|\ |$ to the basic pushout diagram (2.2) of S_0 yields the following diagram (recall that $|\ |$ preserves coproducts and monomorphisms):

$$\begin{tikzcd} \vee_\alpha |\dot{\Delta}_n| \arrow[r, hook] \arrow[d] & \vee_\alpha |\bar{\Delta}_n| \arrow[d] \\ |X^{n-1}| \arrow[r, hook] & |X^n| \end{tikzcd} \quad \text{in } C.$$

This diagram is a pushout in C (since $|\ |$ preserves pushouts) and is, thus, a pushout in M; further, it is also a pullback in M (since two opposite maps are monomorphisms) and is, thus, bicartesian. Note if we apply $C^N_* : S_0 \longrightarrow M$ to diagram (2.2), we obtain an analogous pushout diagram in M which is also bicartesian. Next consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \vee_\alpha |\dot{\Delta}_n| \arrow[r] \arrow[d, "\amalg_\alpha \lambda_{\dot{\Delta}_n}"] & |X^{n-1}| \oplus \vee_\alpha |\bar{\Delta}_n| \arrow[r] \arrow[d, "\lambda_Xn\text{-}1 \oplus \amalg_\alpha \lambda_{\bar{\Delta}_n}"] & |X^n| \arrow[r] \arrow[d, "\lambda_Xn"] & 0 \\ 0 \arrow[r] & \amalg_\alpha C^N_* \dot{\Delta}_n \arrow[r] & C^N_*X^{n-1} \oplus \amalg_\alpha C^N_* \bar{\Delta}_n \arrow[r] & C^N_*X^n \arrow[r] & 0 \end{tikzcd} \quad \text{in } M.$$

Each row is obtained from a bicartesian diagram and is, thus, exact. To show that the left square commutes consider the following: let $\tilde{x} : \bar{\Delta}_n \longrightarrow X^n$ be a map in S_0 and note that this gives a map $\dot{\Delta}_n = (\bar{\Delta}_n)^{n-1} \longrightarrow (X^n)^{n-1} = X^{n-1}$ on the $(n-1)$-skeletons; then applying the naturality of λ to this map we obtain $|\dot{\Delta}_n| \to |X^{n-1}| \to C^N_*X^{n-1} \; = \; |\dot{\Delta}_n| \to C^N_* \dot{\Delta}_n \to C^N_*X^{n-1}$. Also applying the naturality of λ to $\dot{\Delta}_n \hookrightarrow \bar{\Delta}_n$ we obtain $|\dot{\Delta}_n| \hookrightarrow |\bar{\Delta}_n| \to C^N_* \bar{\Delta}_n = |\dot{\Delta}_n| \to C^N_* \dot{\Delta}_n \hookrightarrow C^N_* \bar{\Delta}_n$. These two results imply the commutativity of the left square. The commutativity of the right hand square follows from applying the naturality of λ to the maps $X^{n-1} \hookrightarrow X^n$ and $\tilde{x}: \bar{\Delta}_n \longrightarrow X^n$.

We claim that λ_{X^n} gives a homology isomorphism. For this, note that $\coprod_a \lambda_{\dot{\Delta}_n}$ and $\phi_1 = \coprod_a \lambda_{\bar{\Delta}_n}$ give homology isomorphisms (since $\lambda_{\dot{\Delta}_n}$ and $\lambda_{\bar{\Delta}_n}$ do, and homology commutes with arbitrary coproducts), and so $\lambda_{X^{n-1}} \oplus \phi_1$ does as well. Next considering the two Mayer-Vietoris long exact sequences in homology, and applying the 5-lemma, we can conclude that λ_{X^n} gives a homology isomorphism.

Now X is the union of its n-skeletons: $X = \bigcup_n X^n$; i.e., X is given by the colimit: $X = \underset{\overrightarrow{n}}{\lim} X^n$. Also C_*^N and $\hat{U}|\ |$ preserve colimits; and so, after applying the naturality of λ to the maps $X^n \hookrightarrow \underset{\overrightarrow{n}}{\lim} X^n = X$ $(n \geq 0)$, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\underset{\overrightarrow{n}}{\lim} |X^n| & \xrightarrow{\underset{\overrightarrow{n}}{\lim} \lambda_{X^n}} & \underset{\overrightarrow{n}}{\lim} C_*^N X^n \\
\downarrow \cong & & \downarrow \cong \\
|X| = |\underset{\overrightarrow{n}}{\lim} X^n| & \xrightarrow[\lambda_X]{} & C_*^N(\underset{\overrightarrow{n}}{\lim} X^n) = C_*^N X
\end{array} \quad \text{in } M.$$

We claim that λ_X gives a homology isomorphism. For this, apply homology $H_*: M \longrightarrow M^{\#}$ to the above diagram and note that $(\underset{\overrightarrow{n}}{\lim} \lambda_{X^n})_*$ is an isomorphism (since $(\lambda_{X^n})_*$ is an isomorphism $(n \geq 0)$ and homology commutes with directed colimits (see, e.g., [18])); and so we can conclude that $(\lambda_X)_*$ is an isomorphism.

<u>Theorem</u> 4.8: The canonical map $|X \amalg Y| \longrightarrow |X| \otimes |Y|$ in C induces a homology isomorphism.

Proof: Note that $\langle |pr_X|_*, |pr_Y|_* \rangle : H_*|X \pi Y| \longrightarrow H_*|X| \otimes H_*|Y|$ in $C^\#$ is an isomorphism (using the naturality of $\lambda : \hat{U}|\ | \xrightarrow{\cdot} C^N_*$ and the fact that the analogous map $H_*(X \pi Y;k) \longrightarrow H_*(X;k) \otimes H_*(Y;k)$ in $C^\#$ is an isomorphism). It follows that the map $H_*|X \pi Y| \longrightarrow H_*(|X| \otimes |Y|)$, which is induced by the canonical map, is an isomorphism (using the Künneth theorem for coalgebras; i.e., $H_* : C \longrightarrow C^\#$ preserves products).

5. Closed model categories and some lemmas.

<u>Definition</u> (Quillen, [17]): A <u>closed model category</u> is a category $\tilde{C}$ together with three distinguished classes of maps called <u>fibrations</u>, <u>cofibrations</u>, and <u>weak equivalences</u> which satisfy the following:

CM0. $\tilde{C}$ is closed under finite limits and finite colimits.

CM1. Given a commutative diagram in $\tilde{C}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad , \qquad (5.1)$$

then a dotted map exists which gives commutative triangles if either a) i is a cofibration and p is a fibration and weak equivalence or b) i is a cofibration and weak equivalence and p is a fibration.

CM2. Any map f has two factorizations: a) f = pi where i is a cofibration and weak equivalence and p is a fibration, and b) f = p'i' where i' is a cofibration and p' is a fibration and weak equivalence.

CM3. Let f:X--->Y, and g:Y--->Z. If any two of f, g, and gf are weak equivalences, then so is the third.

CM4. Each of the classes: fibrations, cofibrations, and weak equivalences is closed under retracts.

A <u>trivial</u> fibration (<u>trivial</u> cofibration) is a fibration (cofibration) which is also a weak equivalence.

If a dotted map exists in diagram (5.1), p has the <u>right lifting property</u> (RLP) with respect to i, and i has the <u>left lifting property</u> (LLP) with respect to p.

Note that with these definitions axiom CM1 says that fibrations have the RLP with respect to the class of trivial cofibrations, and cofibrations have the LLP with respect to the class of trivial fibrations.

Quillen (in [17], p. 234) gives a concise discussion of closed model categories, and we refer the reader to this work and, of course, also to [16] for the treatment of homotopy theory in the setting of closed model categories.

In [16] Quillen defines the notion of a model category and then defines the notion of a closed model category. The axioms for the latter are then reformulated (in [17]). It is this reformulation which we have given above, and, in the following, statements which are made for model categories can be interpreted by the reader as statements regarding closed model categories.

We do want to mention the following: for $\tilde{C}$ a model category, the relation of left (respectively, right) homotopy $\sim_L$ (respectively, $\sim_R$) is an equivalence relation on $\tilde{C}(X,Y)$ if X is cofibrant (respectively, Y is fibrant). Let $v:W \longrightarrow X$, f and $g:X \longrightarrow Y$, and $u:Y \longrightarrow Z$ be maps in $\tilde{C}$. If $f \sim_L g$ (respectively, $f \sim_R g$), then $uf \sim_L ug$ (respectively, $fv \sim_R gv$), $fv \sim_L gv$ when Y is fibrant (respectively, $uf \sim_R ug$ when X is cofibrant), and $f \sim_R g$ when X is cofibrant (respectively, $f \sim_L g$ when Y is fibrant). Thus, for X cofibrant and Y fibrant, the relations $\sim_L$ and $\sim_R$ agree; and then this relation on $\tilde{C}(X,Y)$ is denoted simply $\sim$, and the set of equivalence classes is denoted $[X,Y]$.

Let $\tilde{C}_{cf}$ denote the full subcategory of $\tilde{C}$ determined by the objects of $\tilde{C}$ which are both cofibrant and fibrant; then $\sim$ is a congruence on $\tilde{C}_{cf}$, and the quotient category is denoted $ho\tilde{C}$. Thus, $ho\tilde{C}$ has the same objects as $\tilde{C}_{cf}$, and $ho\tilde{C}(X,Y) = [X,Y]$. Finally let $\pi:\tilde{C}_{cf} \longrightarrow ho\tilde{C}$ denote the quotient functor ($ho\tilde{C}$ is denoted $\pi\tilde{C}_{cf}$ in [16] and [17]).

Let $f,g:X \longrightarrow Y$ be maps in $\tilde{C}$. Quillen (see [16], §1 lemma 1), in effect, defines the relation $\sim_L$ on $\tilde{C}(X,Y)$: $f \sim_L g$ if there exists a left homotopy $H:\tilde{X} \longrightarrow Y$ from f to g where $\tilde{X}$ is a cylinder object for X. The question naturally arises as to the extent to which the relation $\sim_L$ depends on the choice of $\tilde{X}$. The next lemma addresses this question and is intended as a remark.

<u>**Lemma**</u> 5.1: Let X be cofibrant and Y be fibrant in the model category $\tilde{C}$. Then the relation $\sim$ ($= \sim_L$) on $\tilde{C}(X,Y)$ is independent of the choice of cylinder object.

I.e.,, let $\tilde{X}$ and $\tilde{X}'$ be two cylinder objects for X, and let $f,g:X \longrightarrow Y$. Then in the following commutative diagram:

$$\begin{tikzcd}
X \amalg X \arrow[r, "(i_0,i_1)"] \arrow[d, "(i_0',i_1')"'] \arrow[dr, "(f,g)"] & \tilde{X} \arrow[d, dashed, "H"] \\
\tilde{X}' \arrow[r, dashed, "H'"'] & Y
\end{tikzcd},$$

one has that the map H exists $\Longleftrightarrow$ H' exists.

Proof: Let P be a pushout in $\tilde{C}$ of $X \amalg X \longrightarrow \tilde{X}$ along $X \amalg X \longrightarrow \tilde{X}'$; the projection maps p and p' of the cylinder objects give a map $\tilde{p}:P \longrightarrow X$ as indicated in the commutative diagram:

$$\begin{tikzcd}
X \amalg X \arrow[r, "{(i_0, i_1)}"] \arrow[d, "{(i_0', i_1')}"'] & \tilde{X} \arrow[d, "v"] \arrow[dd, bend left, "p"] \\
\tilde{X}' \arrow[r, "v'"] \arrow[dr, bend right, "p'"'] & P \arrow[d, dashed, "\tilde{p}"] \\
& X
\end{tikzcd}$$
.

Note that v and v' are cofibrations. Now consider the factorization $\tilde{p}$ = the composition $P \xrightarrow{i} \bar{X} \xrightarrow{\bar{p}} X$ where i is a cofibration and $\bar{p}$ is a trivial fibration. It is readily shown that the maps $u = iv$ and $u' = iv'$ are trivial cofibrations (e.g., since ui_0 is a trivial cofibration).

Now assume $f,g:X \longrightarrow Y$ are left homotopic via $H:\tilde{X} \longrightarrow Y$, and consider the commutative diagram:

$$\begin{tikzcd}
X \amalg X \arrow[r, "{(i_0, i_1)}"] \arrow[d, "{(i_0', i_1')}"'] \arrow[rdd, bend left=60, "{(f,g)}"] & \tilde{X} \arrow[d, "u"] \arrow[dd, bend left, "H"] \\
\tilde{X}' \arrow[r, "u'"] & \bar{X} \arrow[d, dashed, "\bar{H}"'] \\
& Y
\end{tikzcd}$$
.

Note that there is a map $\bar{H}$ such that $\bar{H}u = H$ (since Y is fibrant and u is a trivial cofibration). Then $H' = \bar{H}u'$ is a left homotopy for $f \sim g$ using the cylinder object $\tilde{X}'$. Note that the symmetry of $\bar{X}$ gives that H' exists => H exists.

Similarly, using the construction which is dual to that given for $\bar{X}$, one can show that $\sim$ ($= \sim_R$) is independent of the choice of path object (with X cofibrant and Y fibrant).

Let $\tilde{C}$ be a category and let S be a subclass of morphisms of $\tilde{C}$. A <u>category of fractions of $\tilde{C}$ with respect to S</u> is a category, denoted $\tilde{C}[S^{-1}]$, together with a functor $\gamma:\tilde{C} \longrightarrow$

$\tilde{C}[S^{-1}]$ such that a) $\gamma(s)$ is an isomorphism for all s in S, and b) if a functor $F:\tilde{C}\longrightarrow D$ has the property that $F(s)$ is an isomorphism in D for all s in S, then there is a unique functor $F':\tilde{C}[S^{-1}]\longrightarrow D$ such that $F'\gamma = F$.

For $\tilde{C}$ a model category, the <u>homotopy category</u> of C, denoted $\gamma:\tilde{C}\longrightarrow Ho\tilde{C}$, is a category of fractions of $\tilde{C}$ with respect to the class of weak equivalences. Quillen (in [16]) proves that $Ho\tilde{C}$ always exists. We remark that $Ho\tilde{C}$ and $\tilde{C}$ have the same objects (i.e., $\gamma X = X$ for X in $\tilde{C}$).

The following lemma is useful and also one which is easily proved.

<u>Proposition</u> 5.2: Let $\tilde{C}$ be a model category. Let $f:X\longrightarrow X'$ be a weak equivalence in $\tilde{C}$ with X and X' cofibrant, and let Y be fibrant. Then $f^*:[X',Y]\longrightarrow [X,Y]$ is a bijection in Sets.

Proof: For A cofibrant and B fibrant in $\tilde{C}$, there is a natural bijection $[A,B]\longrightarrow Ho\tilde{C}(A,B)$ in Sets (see [16]). Let $\gamma:\tilde{C}\longrightarrow Ho\tilde{C}$ be the canonical functor, and consider the commutative diagram:

$$\begin{array}{ccc} [X',Y] & \longrightarrow & Ho\tilde{C}(X',Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (\gamma(f))^*} \\ [X,Y] & \longrightarrow & Ho\tilde{C}(X,Y) \end{array} \quad \text{in Sets.} \qquad (5.2)$$

Note that $\gamma(f)$ is an isomorphism in $Ho\tilde{C}$, and so $(\gamma(f))^*$ is a bijection in Sets; and since the horizontal arrows in this diagram are bijections, we can conclude that f^* is a bijection in Sets.

The next corollary and subsequent proposition show that for $f:X \longrightarrow Y$ in the closed model category $\tilde{C}$ with X and Y both cofibrant and fibrant, then f is a homotopy equivalence $\Leftrightarrow$ f is a weak equivalence.

Corollary 5.3: Let $\tilde{C}$ be a model category. Let $f:X \longrightarrow X'$ be a weak equivalence in $\tilde{C}$ with X and X' cofibrant as well as fibrant. Then f is a homotopy equivalence.

Proof: By the above lemma, $f^*:[X',X] \longrightarrow [X,X]$ is a bijection in Sets. In particular, f^* is surjective; so there is a map $g:X' \longrightarrow X$ in $\tilde{C}$ with $gf \sim id_X$. But $f^*:[X',X'] \longrightarrow [X,X']$ is also a bijection. Noting that this map gives $f^*(\overline{fg}) = \overline{(fg)f} = \overline{f(gf)} = \overline{f} = f^*(\overline{id_{X'}})$, we have $\overline{fg} = \overline{id_{X'}}$ (since f^* is injective); i.e., we also have $fg \sim id_{X'}$. And so f is a homotopy equivalence (with g a homotopy inverse).

Proposition 5.4: Let $\tilde{C}$ be a closed model category. Let $f:X \longrightarrow Y$ be a homotopy equivalence in $\tilde{C}$ with X and Y cofibrant as well as fibrant. Then f is a weak equivalence in $\tilde{C}$.

Proof: As in the proof of the previous proposition, we consider the bijection $[A,B] \longrightarrow Ho\tilde{C}(A,B)$ ($\bar{h} \longmapsto \gamma(h)$) where A is cofibrant and B is fibrant and where $\gamma:\tilde{C} \longrightarrow Ho\tilde{C}$ is the canonical functor. Let $g:Y \longrightarrow X$ be a homotopy inverse for f; and so $gf \sim id_X$, and $fg \sim id_Y$. But note that since the above bijections are well-defined and γ is a functor, we obtain $\gamma(g)\gamma(f) = id_{\gamma X}$ and $\gamma(f)\gamma(g) = id_{\gamma Y}$. Thus, $\gamma(f)$ is an isomorphism in $Ho\tilde{C}$; and so f is a weak equivalence (since $\tilde{C}$ is a closed model category).

A map in $\tilde{C}_{cf}$ is called a weak equivalence if it is such in $\tilde{C}$.

__Proposition__ 5.5: Let $\tilde{C}$ be a closed model category. Then the quotient functor $\pi:\tilde{C}_{cf} \longrightarrow ho\tilde{C}$ makes $ho\tilde{C}$ a category of fractions with respect to the weak equivalences in $\tilde{C}_{cf}$.

Proof: The previous two results show that for f in $\tilde{C}_{cf}$, then f is a homotopy equivalence $\Leftrightarrow$ f is a weak equivalence; and so $\pi(f)$ is an isomorphism if f is a weak equivalence. The following is easily shown (see [16], §1 lemma 8): assume that a functor $F:\tilde{C}_{cf} \longrightarrow D$ takes weak equivalences in $\tilde{C}_{cf}$ into isomorphisms in D, then $f \sim g \Rightarrow F(f) = F(g)$ (f,g in $\tilde{C}_{cf}$). F, thus, factors uniquely through $ho\tilde{C}$ via π.

Since the composition $\tilde{C}_{cf} \xrightarrow{i} \tilde{C} \xrightarrow{\gamma} Ho\tilde{C}$ takes weak equivalences to isomorphisms, γ induces a unique functor $\bar{\gamma}:ho\tilde{C} \longrightarrow Ho\tilde{C}$ such that $\bar{\gamma}\pi = \gamma i$.

__Theorem__ 5.6 (Quillen, [16]): $\bar{\gamma}:ho\tilde{C} \longrightarrow Ho\tilde{C}$ is an equivalence of categories.

We remark, for later reference, that Quillen defines a functor $\overline{RQ}:\tilde{C} \longrightarrow ho\tilde{C}$ which takes weak equivalences in $\tilde{C}$ to isomorphisms in $ho\tilde{C}$. The unique functor $Ho\tilde{C} \longrightarrow ho\tilde{C}$, induced by $\overline{RQ}$, is an equivalence which is inverse to $\bar{\gamma}$.

__Cylinder mapping lemma__ 5.7: Let $\tilde{C}$ be a model category. Let $f:X \longrightarrow Y$ be a map in $\tilde{C}$; and let $\tilde{X}$ be a cylinder object for X. Then there is a cylinder object $\tilde{Y}$ for Y and a map $\tilde{f}:\tilde{X} \longrightarrow \tilde{Y}$ such that the following diagram:

$$\begin{tikzcd} X \amalg X \arrow[r, "(i_0,i_1)"] \arrow[d, "f\amalg f"'] & \tilde{X} \arrow[r, "p"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ Y \amalg Y \arrow[r, "(i_0',i_1')"'] & \tilde{Y} \arrow[r, "p'"'] & Y \end{tikzcd}$$

commutes. Moreover, our construction of $\tilde{Y}$ has the property that $\tilde{f}$ is a cofibration if f is such.

Proof: Let P be a pushout in $\tilde{C}$ of $f\amalg f$ along (i_0,i_1). And let $\beta:P\longrightarrow Y$ be the unique map out of P such that the composition $Y\amalg Y\longrightarrow P\xrightarrow{\beta}Y = (id_Y,id_Y)$ and $\tilde{X}\xrightarrow{u}P\xrightarrow{\beta}Y = \tilde{X}\xrightarrow{p}X\xrightarrow{f}Y$. Next consider the factorization $\beta = P\xrightarrow{i}\tilde{Y}\xrightarrow{p'}Y$ where i is a cofibration and p' is a trivial fibration. Then $\tilde{Y}$ is a cylinder object for Y, and the map $\tilde{f} = iu$ has the required properties.

The dual assertion is the <u>path object mapping lemma</u> (which is obtained by the dualization of this proof).

<u>Theorem</u> 5.8: Let $\tilde{C}$ be a model category, and let $f:X\longrightarrow Y$ be a cofibration with X cofibrant. Let the following diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & A \arrow[d] \\ Y \arrow[r, "g'"'] & P \end{tikzcd}$$

be a pushout in $\tilde{C}$. If g is a weak equivalence, then so is g'.

Proof: Heller ([9], Lemma 1.2) proved that for an h-c category, the pushout of a homotopy equivalence along a cofibration is a homotopy equivalence. The proof given, although based on a different axiomatic system for a homotopy theory, remains valid for a model category $\tilde{C}$. A crucial part of the proof concerns the treatment of a cylinder object for X (called a cylinder over X in [9]) which, in essence, is our cylinder mapping lemma 5.7. Recall that the map $\tilde{f}:\tilde{X}\longrightarrow\tilde{Y}$ between cylinder objects is given by $\tilde{f} = iu$ where i is a cofibration (the analogues of $\tilde{f}$ and i in [9] are denoted z and $(i_0 z i_1)$, respectively). Next consider the pullback P' in $\tilde{C}$ of $(i_0,i_1):X\amalg X\longrightarrow\tilde{X}$ along $f\amalg id_X$; let $a:P'\longrightarrow\tilde{Y}$ be the map determined by $\tilde{f}$ and the composition $Y\amalg X\xrightarrow{id_Y\amalg f}Y\amalg Y\xrightarrow{(i_0',i_1')}\tilde{Y}$. It is easily shown that a is a cofibration (using i is a cofibration). The proof in [9] involves the construction of a diagram which consists of three concentric pushout squares--a construction which can also be done in $\tilde{C}$-- with the analogue of the cofibration a (denoted $i_0 z$) occurring as part of the diagram. Properties of an h-c category-- with analogous ones valid for $\tilde{C}$-- are then used to obtain the desired conclusion. We remark that in order for the proof to work for $\tilde{C}$, X is assumed cofibrant in order that the map $i_j:X\longrightarrow\tilde{X}$ ($j = 0,1$) be a trivial cofibration (in an h-c category, all objects are cofibrant).

__Theorem__ 5.9: In a model category, the pullback of weak equivalence along a fibration which has a fibrant codomain is a weak equivalence.

Proof: The proof for theorem 5.8 dualizes.

We remark that there are closed model categories where the pullback of a weak equivalence along a fibration (whose codomain is not fibrant) need not be a weak equivalence; Quillen ([17], II Remark 2.9) gives an example of such a pullback in S_R.[1] For these situations the following lemma may be a useful one for it shows that a fibration, which is induced from a fibration having a fibrant codomain, has the desired weak equivalence pullback property.[2]

<u>Proposition</u> 5.10: Let $\tilde{C}$ be a model category, and let $f:X \longrightarrow Y$ be a fibration in $\tilde{C}$ with Y fibrant. Let the following be a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \arrow[r, "g"'] & Y' \arrow[r] & Y \end{tikzcd}$$

where both squares are pullback diagrams. Then g' is a weak equivalence if g is such.

Proof: Consider the construction of the following commutative diagram:

$$\begin{tikzcd} \tilde{P} \arrow[rr] \arrow[ddd] & & X \arrow[ddd, "f"] \\ & X' \arrow[ul, dashed] \arrow[ur] \arrow[d] & \\ & Y' \arrow[dl] \arrow[dr] & \\ \tilde{Y}' \arrow[rr] & & Y \end{tikzcd}$$

Here $Y' \longrightarrow \tilde{Y}'$ is a trivial cofibration with $\tilde{Y}'$ fibrant (from a factorization of $Y' \longrightarrow e$, where e is a terminal object);

[1] A. K. Bousfield and E. M. Friedlander call a closed model category <u>proper</u> if weak equivalences are closed under pullbacks of fibrations and are closed under pushouts of cofibrations-- S and S_* are examples; they then derive some basic properties for such categories (see their paper: Homotopy theory of Γ-spaces, spectra, and bisimplicial sets; <u>Geometric Applications of Homotopy Theory</u>, L.N.M. Vol. 658 (Springer-Verlag, New York/Berlin, 1978) 80-130).

[2] We remark that D. Husemoller, J. C. Moore, and J. Stasheff call a map $f:X \longrightarrow Y$ in S a <u>weak fibration</u> if it is surjective and if homotopy equivalences in S are closed under pullbacks of the type described in prop. 5.10 (with Y not assumed fibrant); see their paper (p. 164): Differential homological algebra and homogeneous spaces, <u>Jour. Pure and Appl. Alg.</u> 5 (1974) 113-185.

and, hence, one has the bottom triangle (since Y is fibrant). $\tilde{P}$ is a pullback of f along $\tilde{Y}' \longrightarrow Y$ giving an obvious map $X' \longrightarrow \tilde{P}$. Note that the left trapezoid is a pullback (using the basic properties of pullbacks in a category); and so $X' \longrightarrow \tilde{P}$ is a weak equivalence (since it is the pullback of a weak equivalence along a fibration which has a fibrant codomain). Next consider the commutative diagram:

$$\begin{tikzcd}
P \arrow[r, "g'"] \arrow[d] & X' \arrow[r] \arrow[d] & \tilde{P} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
A \arrow[r, "g"'] & Y' \arrow[r] & \tilde{Y}' \arrow[r] & Y
\end{tikzcd}$$

where all vertical maps are fibrations. $P \longrightarrow A$ is a pullback along the weak equivalence $A \xrightarrow{g} Y' \longrightarrow \tilde{Y}'$, and so the composition $P \xrightarrow{g'} X' \longrightarrow \tilde{P}$ is a weak equivalence (since $\tilde{Y}'$ is fibrant). And so we can conclude that g' is a weak equivalence (since $X' \longrightarrow \tilde{P}$ is such).

<u>**Lemma**</u> 5.11 (picture frame): Let $\tilde{C}$ be a model category. Let the following be a commutative diagram:

$$\begin{tikzcd}
A' \arrow[rrr] \arrow[dr] \arrow[ddd] & & & X' \arrow[dl] \arrow[ddd] \\
& A \arrow[r] \arrow[d] & X \arrow[d] & \\
& B \arrow[r] & Y & \\
B' \arrow[rrr] \arrow[ur] & & & Y' \arrow[ul]
\end{tikzcd}$$

where the inner and outer squares are both pullbacks with Y and Y' fibrant. Assume that a) $B' \longrightarrow Y'$ and $X \longrightarrow Y$ are fibrations or b) $X' \longrightarrow Y'$ and $X \longrightarrow Y$ are fibrations. Then $A' \longrightarrow A$ is a weak equivalence if the other three diagonal maps in the diagram are such.

Proof: For part a), we will apply prop. 5.10 to two pullback diagrams which are adjacent horizontally and then apply the same lemma to two pullbacks which are adjacent vertically. Consider the construction of the following commutative diagram:

$$\begin{tikzcd}
A' \arrow[rrrrr] \arrow[dr] \arrow[dddd] & & & & & X' \arrow[dl] \arrow[dddd] \\
& P' \arrow[rrr] \arrow[dr] \arrow[dddl] & & & P \arrow[dl] \arrow[dddr] & \\
& & A \arrow[r] \arrow[d] & X \arrow[d] & & \\
& & B \arrow[r] & Y & & \\
B' \arrow[rrrrr] \arrow[urr] & & & & & Y' \arrow[ull]
\end{tikzcd} \quad .$$

P is a pullback of the fibration X--->Y along the weak equivalence Y'--->Y; and so P--->X is a weak equivalence (since Y is fibrant). There is an obvious map X'--->P into the pullback P; further this map is a weak equivalence (since X'--->X and P--->X are such). P' is a pullback of P--->Y' along B'--->Y' giving an obvious map P'--->A into the pullback A. Note that P' is also a pullback of A--->B along the weak equivalence B'--->B (using the basic properties of pullbacks in a category); and so P'--->A is a weak equivalence (using prop. 5.10). There is an obvious map A'--->P' into the pullback P'. And note that A' is also a pullback of P'--->P along the weak equivalence X'--->P (again using the basic properties of pullbacks in a category); and so A'--->P' is a weak equivalence (since B'--->Y' is a fibration with Y' fibrant and using prop. 5.10). And so we can conclude that A'--->A is a weak equivalence (since A'--->P' and P'--->A are such).

For part b), consider the following commutative diagram:

$$\begin{tikzcd}
A' \arrow[rrrrr] \arrow[dr] \arrow[ddddd] & & & & & X' \arrow[dl] \arrow[ddddd] \\
& A \arrow[rrr] \arrow[dr] \arrow[ddd] & & & X \arrow[dl, "id_X"'] \arrow[ddd] & \\
& & P \arrow[r] \arrow[d] & X \arrow[d] & & \\
& & \tilde{B} \arrow[r] & Y & & \\
& B \arrow[rrr] \arrow[ur] & & & Y \arrow[ul, "id_Y"'] & \\
B' \arrow[rrrrr] \arrow[ur] & & & & & Y' \arrow[ul]
\end{tikzcd}$$

where $B \longrightarrow Y$ gives a factorization $B \longrightarrow \tilde{B} \longrightarrow Y$ of a weak equivalence followed by a fibration, and where P is a pullback of $X \longrightarrow Y$ along $\tilde{B} \longrightarrow Y$; $A \longrightarrow P$ is the obvious map into the pullback P. Note that we can conclude that $A \longrightarrow P$ is a weak equivalence (by applying part a) to the fibration pair $X \longrightarrow Y$, $\tilde{B} \longrightarrow Y$). A similar application of part a) to the fibration pair $X' \longrightarrow Y'$, $\tilde{B} \longrightarrow Y$ allows us to conclude that the composition $A' \longrightarrow A \longrightarrow P$ is a weak equivalence; and, hence, $A' \longrightarrow A$ is a weak equivalence.

The following lemma is a useful form of the homotopy lifting property for fibrations.

<u>Lemma</u> 5.12 (<u>homotopy lifting</u>): Let $\tilde{C}$ be a model category. Let $f: X \longrightarrow Y$ be a fibration and let B be cofibrant. Let $\beta: B \longrightarrow Y$ and $g: B \longrightarrow X$ with $fg \sim_L \beta$. Diagramatically this gives:

$$\begin{tikzcd}
& X \arrow[d, "f"] \\
B \arrow[ur, "g"] \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

Then there is a map $g': B \longrightarrow X$ such that $g \sim_L g'$ and $fg' = \beta$.

Proof: Let $h: \tilde{B} \longrightarrow Y$ be a left homotopy for $fg \sim_L \beta$. And consider the commutative diagram:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "i_o"'] & X \arrow[d, "f"] \\ \tilde{B} \arrow[r, "h"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

The map i_o is a trivial cofibration (since B is cofibrant), and so there is a map H which completes the diagram. Then the map $g' = Hi_1$ has the required properties.

The dual to this lemma is the <u>homotopy extension</u> lemma.

<u>Theorem</u> 5.13: Let $\tilde{C}$ be a model category. Let $p:X \longrightarrow B$ be a fibration in $\tilde{C}$ with B fibrant, and let $g_o, g_1 : A \longrightarrow B$ with $g_o \sim_L g_1$. Let $X_j' \longrightarrow A$ be a pullback in $\tilde{C}$ of p along g_j ($j = o,1$). Now let $f:C \longrightarrow A$ with C cofibrant and let $k_o:C \longrightarrow X$ be a lift of g_of over p (i.e., $g_of = pk_o$). Finally let $\phi_o:C \longrightarrow X_o'$ be the map into the pullback X_o' determined by f and k_o. Then a) there is a lift $k_1:C \longrightarrow X$ of g_1f over p with $k_o \sim_L k_1$, and let $\phi_1:C \longrightarrow X_1'$ be the map into the pullback X_1' determined by f and k_1; thus, we have the following commutative diagrams:

$$\begin{tikzcd} C \arrow[dr, "\phi_j"] \arrow[drr, bend left, "k_j"] \arrow[ddr, bend right, "f"'] & & \\ & X_j' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ & A \arrow[r, "g_j"'] & B \end{tikzcd} \qquad j = o,1.$$

And b) if ϕ_o is a weak equivalence, then so is ϕ_1.

Proof: Let $g_o \sim_L g_1$ via a homotopy $H':\tilde{A}' \longrightarrow B$ (where $\tilde{A}'$ is a cylinder object for A). Recall (Quillen in [16]) one can obtain a homotopy $\tilde{H}:g_of \sim_L g_1f$, by first replacing the original cylinder object $\tilde{A}'$ by a cylinder object $\tilde{A}$ (which has a trivial fibration for the projection map $\tilde{A} \longrightarrow A$) and replacing H' by a new homotopy $H:\tilde{A} \longrightarrow B$ (using B is fibrant). One next finds a map between cylinder objects $\tilde{f}:\tilde{C} \longrightarrow \tilde{A}$ which satisfies $C \amalg C \longrightarrow \tilde{C} \xrightarrow{\tilde{f}} \tilde{A} = C \amalg C \xrightarrow{f \amalg f} A \amalg A \longrightarrow \tilde{A}$ (using $\tilde{A} \longrightarrow A$ is a trivial fibration) and which, thus, gives $\tilde{H} = H\tilde{f}:g_o f \sim_L g_1 f$ for the desired homotopy.

Now use the homotopy lifting property as indicated in the following diagram (here i_o is a trivial cofibration since C is cofibrant)

$$\begin{tikzcd} C \arrow[r, "k_o"] \arrow[d, "i_o"'] & X \arrow[d, "p"] \\ \tilde{C} \arrow[r, "\tilde{H}"'] \arrow[ur, dashed, "\tilde{H}'"] & B \end{tikzcd}$$

to lift $\tilde{H}:\tilde{C} \longrightarrow B$ to a homotopy $\tilde{H}':\tilde{C} \longrightarrow X$; and let $k_1 = \tilde{H}' i_1$: $C \longrightarrow X$. Now let $P \longrightarrow \tilde{A}$ be a pullback of $p:X \longrightarrow B$ along $H:\tilde{A} \longrightarrow B$. And note that $p\tilde{H}' = \tilde{H} = H\tilde{f}$ determines a unique map $\tilde{C} \longrightarrow P$. And finally let $X_j' \longrightarrow A$ be a pullback of the fibration $P \longrightarrow \tilde{A}$ along the weak equivalence $A \xrightarrow{i_j'} \tilde{A}$ ($j = o,1$). Note that $X_j' \longrightarrow P$ is a weak equivalence ($j = o,1$) using prop. 5.10. Note that $X_j' \longrightarrow A$ is also a pullback of $p:X \longrightarrow B$ along $Hi_j' = g_j$ ($j = o,1$). And consider the map $\phi_j:C \longrightarrow X_j'$ determined by f and $\tilde{H}' i_j$ ($\tilde{H}' i_o = k_o$, $\tilde{H}' i_1 = k_1$). Note that ϕ_j ($j = o,1$) also satisfies $C \xrightarrow{\phi_j} X_j' \longrightarrow P = C \xrightarrow{i_j} \tilde{C} \longrightarrow P$ (using $\tilde{f}$ satisfies $\tilde{f} i_j = i_j' f$). Now consider this equality for $j = o$. Note that if ϕ_o is a weak equivalence, we can conclude that $\tilde{C} \longrightarrow P$ is a weak equivalence (since three out of the four maps are weak equivalences). But then considering the equality for $j = 1$, with $\tilde{C} \longrightarrow P$ known to be a weak equivalence, the same reasoning allows us to conclude that ϕ_1 is a weak equivalence.

6. Closed model structures for S_o and C.

Quillen [16] has shown that S is a closed model category where cofibrations are the injective maps, weak equivalences are the maps whose topological realizations give isomorphisms on all homotopy groups, and fibrations are those maps which have the RLP with respect to the class of trivial cofibrations (i.e., fibrations are the Kan fibrations).

Theorem 6.1 (Quillen, [17], II theorem 2.2 (with r =1)): This definition also gives a closed model category structure for $S_R \subset S$ (i.e., cofibrations and weak equivalences in S_R are the maps which are such when viewed as maps in S, and fibrations in S_R are those maps which have the RLP with respect to the class of trivial cofibrations of S_R).

We remark that the weak equivalence axiom CM3 is immediate. It is also immediate that cofibrations of S_R and weak equivalences of S_R are closed under retracts. It is easily verified that, in general, if a map f in an arbitrary category $\tilde{C}$ has the RLP with respect to a given class of maps, then so does a retract of f. It follows from this that the class of fibrations of S_R is also closed under retracts. Axiom CM4 is, thus, satisfied. Part b) of axiom CM1 is immediate by the definition of a fibration. It is the verification of CM1 a) and the factorizations of axiom CM2 which are non-trivial in the proof of this theorem.

Quillen ([17], II prop. 2.4 (with r =1)) also shows that for a map f in S_R, then f is a fibration in S_R and $\pi_1 f$ is surjective $\Leftrightarrow$ f is a fibration in S. In particular, this implies that a surjective fibration in S_R is a fibration in S.

Using the isomorphism $S_0 \longrightarrow S_R$ we obtain a closed model category structure for S_0 from that for S_R.

Neisendorfer [15] has shown that C is a closed model category where cofibrations are the injective maps, weak equivalences are the maps which give isomorphisms on homology, and fibrations are those maps which have the RLP with respect to the class of trivial cofibrations.

Quillen [17] had previously shown that this definition makes the subcategory of simply connected coalgebras a closed model category.

7. The realization and singular functors on the homotopy categories.

Proposition 7.1: $|\ |:S_o \longrightarrow C$ preserves cofibrations and preserves weak equivalences.

Proof: The first property is immediate since cofibrations in S_o and C are the injections and the former are preserved by $|\ |$. For the second property, letting $f:X \longrightarrow Y$ be a weak equivalence in S_o, then f induces a homology isomorphism; in particular, $f_* = H_*(f;k)$ is an isomorphism. Recall that the natural transformation $\lambda:\hat{U}|\ | \xrightarrow{\ \cdot\ } C_*^N$ gives an isomorphism $(\lambda_Z)_*:H_*|Z| \longrightarrow H_*(Z;k)$ (Z in S_o); and using the naturality of λ, we obtain $f_*(\lambda_X)_* = (\lambda_Y)_*|f|_*$. And so $|f|$ is a weak equivalence in C.

We remark that the above proof shows that any map f in S_o which induces an isomorphism $H_*(f;k)$ is taken by $|\ |$ to a weak equivalence $|f|$ in C.

The next propositions show that $\mathcal{S}:C \longrightarrow S_o$ preserves fibrations and takes a weak equivalence between fibrant objects in C to a weak equivalence in S_o.

Proposition 7.2: Let $f:X \longrightarrow Y$ be a fibration in C. Then $\mathcal{S}(f):\mathcal{S}X \longrightarrow \mathcal{S}Y$ is a fibration in S_o.

Proof: We show that $\mathcal{S}(f)$ has the right lifting property with respect to trivial cofibrations. For this, let the following be a commutative diagram with g a trivial cofibration:

$$\begin{array}{ccc} A & \xrightarrow{k} & \mathcal{S}X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(f)} \\ B & \xrightarrow[h]{} & \mathcal{S}Y \end{array} \qquad \text{in } S_o.$$

With $|\ | \dashv \mathcal{S}$, consider the following conventions for the adjunction: let $\tilde{p}: W \longrightarrow \mathcal{S}D$ denote the adjoint of the map $p: |W| \longrightarrow D$; also let $\tilde{q}: |Z| \longrightarrow E$ denote the adjoint of the map $q: Z \longrightarrow \mathcal{S}E$. Note that $f\tilde{k} = \widetilde{\mathcal{S}(f)k} = \tilde{h}|g|$ (using the natuarlity of the adjunction), and so we obtain the following commutative diagram:

$$\begin{tikzcd} {|A|} \arrow[r, "\tilde{k}"] \arrow[d, "{|g|}"'] & X \arrow[d, "f"] \\ {|B|} \arrow[r, "\tilde{h}"'] \arrow[ur, dashed, "\phi"] & Y \end{tikzcd} \quad \text{in } C.$$

Note that $|g|$ is a trivial cofibration (since $|\ |$ preserves trivial cofibrations), and so there is a map $\phi: |B| \longrightarrow X$ as indicated since f has the right lifting property with respect to trivial cofibrations. Now use the naturality of the adjunction again to obtain $\tilde{\phi}g = \widetilde{\phi|g|} = \tilde{\tilde{k}} = k$ (i.e., $\tilde{\phi}g = k$); also $\mathcal{S}(f)\tilde{\phi} = \widetilde{f\phi} = \tilde{\tilde{h}} = h$ (i.e., $\mathcal{S}(f)\tilde{\phi} = h$). Note that $\tilde{\phi}$, thus, gives a desired lifting in our original diagram. And so $\mathcal{S}(f)$ is a fibration in S_o.

__Corollary__ 7.3: Let X be fibrant in C. Then $\mathcal{S}X$ is fibrant in S_o.

__Lemma__ 7.4: Let $f: X \longrightarrow Y$ be a trivial fibration in C. Then $\mathcal{S}(f): \mathcal{S}X \longrightarrow \mathcal{S}Y$ is a trivial fibration in S_o.

Proof: Recall that in a closed model category a map is a trivial fibration if and only if it has the right lifting property with respect to cofibrations. But note that since $|\ |$ preserves cofibrations, the same idea which was used to show that $\mathcal{S}$ preserves fibrations can be used to show that $\mathcal{S}$ preserves trivial fibrations.

__Lemma__ 7.5: Let $f,g:X \longrightarrow Y$ in C with Y fibrant. If $f \sim_R g$, then $\mathcal{S}(f) \sim_R \mathcal{S}(g)$ in S_o.

Proof: The basic idea for this proof is that $\mathcal{S}$ preserves a path object for a fibrant coalgebra. Let h be a right homotopy for $f \sim_R g$ via a path object $Y \pi Y \xleftarrow{\langle d_o, d_1 \rangle} \tilde{Y} \xleftarrow{s_Y} Y$ (here $\langle d_o, d_1 \rangle$ is a fibration and s_Y is a weak equivalence). Note that since Y is fibrant, d_j is a trivial fibration ($j = o,1$) and the path object $\tilde{Y}$ is fibrant. Now recall that $\mathcal{S}$ preserves products and apply $\mathcal{S}$ to this path object to obtain $\mathcal{S}Y \pi \mathcal{S}Y \xleftarrow{\langle \mathcal{S}(d_o), \mathcal{S}(d_1) \rangle} \mathcal{S}\tilde{Y} \xleftarrow{\mathcal{S}(s_Y)} \mathcal{S}Y$. Note that $\langle \mathcal{S}(d_o), \mathcal{S}(d_1) \rangle$ is a fibration (using $\mathcal{S}$ preserves fibrations). But also $\mathcal{S}(d_j)$ is a trivial fibration (since $\mathcal{S}$ preserves trivial fibrations), and so considering $\mathcal{S}(d_j)\mathcal{S}(s_Y) = id_{\mathcal{S}Y}$, we can conclude that $\mathcal{S}(s_Y)$ is a weak equivalence. And so $\mathcal{S}\tilde{Y}$ is a path object for $\mathcal{S}Y$. And, thus, $\mathcal{S}(f) \sim_R \mathcal{S}(g)$ via the right homotopy $\mathcal{S}(h)$: $\mathcal{S}X \longrightarrow \mathcal{S}\tilde{Y}$.

__Proposition__ 7.6: Let $f:X \longrightarrow Y$ be a weak equivalence in C with X and Y fibrant. Then $\mathcal{S}(f):\mathcal{S}X \longrightarrow \mathcal{S}Y$ is a weak equivalence in S_o.

Proof: Note that f is a homotopy equivalence (using the model category cor. 5.3), and let $g:Y \longrightarrow X$ be a homotopy inverse for f; so that $gf \sim id_X$ and $fg \sim id_Y$. But $\mathcal{S}$ preserves the relation of right homotopy between maps in C which have a fibrant codomain. Thus, $\mathcal{S}(g)$ is a homotopy inverse for $\mathcal{S}(f)$; and so $\mathcal{S}(f)$ is a homotopy equivalence. And so we can conclude that $\mathcal{S}(f)$ is a weak equivalence in S_o (using the closed model category prop. 5.4).

The next theorem allows us to conclude that the adjoint functors $|\ |$ and $\mathcal{S}$ induce a pair of adjoint functors between the homotopy categories HoS_0 and HoC.

Theorem 7.7 (Quillen, [16]): Let $\tilde{C}$ and $\tilde{C}'$ be model categories, and let $\tilde{C} \underset{G}{\overset{F}{\rightleftarrows}} \tilde{C}'$ be a pair of adjoint functors, with $F \dashv G$. Assume that F preserves cofibrations and takes weak equivalences between cofibrant objects in $\tilde{C}$ to weak equivalences in $\tilde{C}'$. Assume that G preserves fibrations and takes weak equivalences between fibrant objects in $\tilde{C}'$ to weak equivalences in $\tilde{C}$. Then the total derived functors $Ho\tilde{C} \underset{R(G)}{\overset{L(F)}{\rightleftarrows}} Ho\tilde{C}'$ are canonically adjoint, with $L(F) \dashv R(G)$.

Note that since $|\ |$ preserves all the weak equivalences from S_0, it is immediate that $|\ |$ induces a canonical functor $|\ |: HoS_0 \longrightarrow HoC$ (where we use the same symbol for the latter functor); in this situation, one has that the left derived functor is then given simply by $L(|\ |) = |\ |$.

Since all objects in C are cofibrant, Quillen's construction of a right derived functor R(G) for the case $G = \mathcal{S}$ can be modified slightly and which we now describe. For each X in C, choose a weak equivalence $i_X: X \longrightarrow R(X)$ in C where R(X) is fibrant (when X is fibrant, let $i_X = id_X$). For each map $f: X \longrightarrow Y$ in C, using $(i_X)^*: [R(X), R(Y)] \longrightarrow [X, R(Y)]$ is a bijection (since i_X is a weak equivalence and R(Y) is fibrant), choose a map $R(f): R(X) \longrightarrow R(Y)$ in C such that $R(f)i_X \sim i_Y f$; and note that R(f) is unique up to homotopy. One observes that $R(gf) \sim R(g)R(f)$ and $R(id_X) \sim id_{R(X)}$. We thus obtain a functor $\bar{R}: C \longrightarrow hoC$ ($X \longmapsto R(X)$, $f \longmapsto \overline{R(f)}$). The functor $\pi: C_{cf} \longrightarrow hoC$ makes hoC a category of fractions with respect to the class of weak equivalences; and so, since the restriction $\mathcal{S}: C_{cf} \longrightarrow S_o$ preserves weak equivalences, there is a unique functor $hoC \longrightarrow HoS_o$ such that

$$\begin{tikzcd} C_{cf} \arrow[r, "\mathcal{S}"] \arrow[d, "\pi"'] & S_o \arrow[d, "\gamma"] \\ hoC \arrow[r] & HoS_o \end{tikzcd}$$

commutes. Finally $R(f)$ is a weak equivalence if f is (homotopic maps in C give the same map on homology), and so $\bar{R}$ takes weak equivalences to isomorphisms; and $R(\mathcal{S})$ is obtained using the universal mapping property of $\gamma: C \longrightarrow HoC$ as the following factorization:

$$\begin{tikzcd} C \arrow[dr, "\bar{R}"] \arrow[dd, "\gamma"'] & & \\ & hoC \arrow[dr] & \\ HoC \arrow[rr, "R(\mathcal{S})"'] & & HoS_o \end{tikzcd} .$$

Note that on objects we have simply $R(\mathcal{S})(X) = \mathcal{S}R(X)$ for X in HoC (since $\gamma(Z) = Z$ where $\gamma: \tilde{C} \longrightarrow Ho\tilde{C}$). And so letting $\mathcal{S}' = R(\mathcal{S})$, we have $\mathcal{S}': HoC \longrightarrow HoS_o$ with $\mathcal{S}'X = \mathcal{S}X'$ where $i: X \longrightarrow X'$ is a weak equivalence in C with X' fibrant. We remark that when X is fibrant, $\mathcal{S}'X = \mathcal{S}X$. We also remark that if the objects of C were not necessarily cofibrant, then Quillen's construction of $\mathcal{S}'$ would require that i also be a cofibration.

In a moment we will show that $\mathcal{S}$ induces a functor $\mathcal{S}: hoC \longrightarrow hoS_o$; it is then easily shown that the functor $hoC \longrightarrow HoS_o$ which occurs in the above two diagrams, in fact, equals the composition $\mathrm{hoC} \xrightarrow{\mathcal{S}} \mathrm{hoS}_0 \xrightarrow{\bar{\gamma}} \mathrm{HoS}_0$ (using the universal mapping property of hoC).

We next consider the functors induced by $|\ |$ and $\mathcal{S}$ on the categories hoS_0 and hoC. Our treatment is independent of the previous development involving the categories HoS_0 and HoC primarily because we are interested in the explicit adjunction which it affords and also because the results which we currently have at our disposal make this treatment fairly immediate. We will then relate the two pairs of functors.

The restriction of $\mathcal{S}$ to C_{cf} corestricts to $(S_0)_{cf}$ (since $\mathcal{S}$ preserves fibrant objects) and also preserves weak equivalences; and so the composition $C_{cf} \xrightarrow{\mathcal{S}} (S_0)_{cf} \xrightarrow{\pi} \mathrm{hoS}_0$ takes weak equivalencees to isomorphisms. We. thus, obtain a unique functor $\mathcal{S}:\mathrm{hoC} \to \mathrm{hoS}_0$ for which

$$\begin{array}{ccc} C_{cf} & \xrightarrow{\mathcal{S}} & (S_0)_{cf} \\ \pi\downarrow & & \downarrow\pi \\ \mathrm{hoC} & \xrightarrow[\mathcal{S}]{} & \mathrm{hoS}_0 \end{array}$$

commutes.

The functor $|\ |$ does not preserve fibrant objects; however, the composition $(S_0)_{cf} \xrightarrow{|\ |} C \xrightarrow{\bar{R}} \mathrm{hoC}$ takes weak equivalences to isomorphisms (recall that $\bar{R}$ occurs in our treatment of the derived functor $R(\mathcal{S})$). And so we obtain a unique functor $|\ |':\mathrm{hoS}_0 \to \mathrm{hoC}$ for which

$$\begin{array}{ccc} (S_0)_{cf} & \xrightarrow{|\ |} & C \\ \pi\downarrow & & \downarrow\bar{R} \\ \mathrm{hoS}_0 & \xrightarrow[|\ |']{} & \mathrm{hoC} \end{array}$$

commutes. Note that on objects, we have $|X|' = X'$ where

$i:|X| \longrightarrow X'$ is a weak equivalence in C with X' fibrant.

Lemma 7.8: Let Ψ be the adjunction for $|\ | \dashv \mathcal{S}$, and let Y be fibrant in C. Then the obvious map $\bar{\Psi}:[|X|,Y] \longrightarrow [X,\mathcal{S}Y]$ is is a bijection which is natural in X and Y.

Proof: We first show that $\bar{\Psi}$ is well-defined. Let $f,g:|X| \longrightarrow Y$ be in C with $f \sim_R g$, then $\mathcal{S}(f) \sim_R \mathcal{S}(g)$ (by lemma 7.5); and so $\tilde{f} = \mathcal{S}(f)\eta_X \sim_R \mathcal{S}(g)\eta_X = \tilde{g}$ where η is the unit for $|\ | \dashv \mathcal{S}$ and the adjoint of, e.g., f is denoted $\tilde{f}$ (= $\Psi(f)$). In a similar fashion one shows that $\overline{\Psi^{-1}}$ is well-defined (by first showing that $|\ |$ preserves a cylinder object for a (co-fibrant) object in S_o and then considering left homotopic maps in $S_o(X,\mathcal{S}Y)$). $\bar{\Psi}$ is, thus, a bijection with inverse $\overline{\Psi^{-1}}$. The naturality of $\bar{\Psi}$ is immediate.

Theorem 7.9: The functors $\mathrm{ho}S_o \underset{\mathcal{S}}{\overset{|\ |'}{\rightleftarrows}} \mathrm{ho}C$ are an adjoint pair with $|\ |' \dashv \mathcal{S}$.

Proof: For X in $(S_o)_{cf}$, let $|X|' = X'$ where $i:|X| \longrightarrow X'$ is a weak equivalence in C with X' fibrant. And so for Y fibrant in C, $i^*:[|X|',Y] \longrightarrow [|X|,Y]$ is a bijection which is natural in X (using the definition of $\bar{R}$ on maps) and Y. And so the composition of natural bijections $[|X|',Y] \xrightarrow{i^*} [|X|,Y] \xrightarrow{\bar{\Psi}} [X,\mathcal{S}Y]$ gives an adjunction for $|\ |' \dashv \mathcal{S}$.

It is easily shown that the unit for the adjunction is given by $\bar{\tilde{i}} \in [X,\mathcal{S}|X|']$ where $\tilde{i}:X \longrightarrow \mathcal{S}X'$ in S_o is the adjoint map of the weak equivalence $i:|X| \longrightarrow X'$ in C.

Recall that the functor $\bar{\gamma}:\mathrm{ho}\tilde{C} \longrightarrow \mathrm{Ho}\tilde{C}$ is an equivalence of categories. When the objects of $\tilde{C}$ are all cofibrant, one can use the unique functor $Ho\tilde{C} \longrightarrow ho\tilde{C}$, induced by $\bar{R}:\tilde{C} \longrightarrow ho\tilde{C}$ defined as above when $\tilde{C} = C$ (versus Quillen's functor $\overline{RQ}$), for the equivalence which is inverse to $\bar{\gamma}$. Our two pairs of adjoint functors are then related by the following commutative diagrams:

$$
\begin{tikzcd}
hoS_o \arrow[r, "|\ |'"] \arrow[d, "\bar{\gamma}"'] & hoC \\
HoS_o \arrow[r, "|\ |"'] & HoC \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
hoC \arrow[r, "\mathcal{S}"] \arrow[d, "\bar{\gamma}"'] & hoS_o \arrow[d, "\bar{\gamma}"] \\
HoC \arrow[r, "\mathcal{S}'"'] & HoS_o
\end{tikzcd}
$$

by a straight forward application of the universal mapping property of hoS_o and hoC, respectively.

8. Principal bundles in S_0 and in C; fiber bundles in C.

We first recall several notions and constructions for S. Let π be a group and n a positive integer, then an __Eilenberg-MacLane__ (E-M) __space of type (π,n)__ is a fibrant K in S such that its homotopy groups satisfy: $\pi_k(K,\phi) \simeq \pi$ if $k = n$, and $=$ 0 if $k \neq n$. Such an object K is called a __$K(\pi,n)$__ if it is minimal (where a fibrant X in S is __minimal__ if the face maps $d_j:X_p \longrightarrow X_{p-1}$ satisfy: $d_i x = d_i x'$ $i \neq k$ implies $d_k x = d_k x'$).

Let Ab denote the category of abelian groups. When π is in Ab, we also recall the familiar construction of $\tilde{K}(\pi,n)$ in S which is a $K(\pi,n)$ (see, e.g., [12]): Consider the normalized cochain complex $C_N^*(/\Delta_p;\pi) = Ab(C_*^N/\Delta_p,\pi)$. Let $\tilde{E}(\pi,n+1)$: $/\Delta^{op} \longrightarrow Ab$ be determined by $(\tilde{E}(\pi,n+1))_p = C_N^n(/\Delta_p;\pi)$ where $p \geqslant 0$ with the simplicial structure obtained using the functor Δ: $/\Delta \longrightarrow S$. Let $\tilde{K}(\pi,n)$: $/\Delta^{op} \longrightarrow Ab$ be determined by the normalized cocycles: $(\tilde{K}(\pi,n))_p = Z_N^n(/\Delta_p;\pi) \subset C_N^n(/\Delta_p;\pi)$ with the simplicial structure such that $\tilde{K}(\pi,n) \subset \tilde{E}(\pi,n+1)$ in ${}_{Ab}/\Delta^{op}$. The coboundary operators δ of $C_N^*(/\Delta_p;\pi)$ with $p \geqslant 0$ give a map $\tilde{h}:\tilde{E}(\pi,n) \longrightarrow \tilde{K}(\pi,n)$ in ${}_{Ab}/\Delta^{op}$; $\tilde{h}$ is surjective (since $H^n(/\Delta_p;\pi) = 0$ for $n > 0$), and $\ker(\tilde{h}) = \tilde{K}(\pi,n-1)$. Further, $\tilde{h}$ is a principal fibration.

Let $(\pi \otimes k)_n$ in $M^{\#}$ have $\pi \otimes k$ concentrated in degree n. We will make use of the fact that there is an isomorphism $H_*(\tilde{K}(\pi,n);k) \longrightarrow S'((\pi \otimes k)_n)$ in $C^{\#}$ (see, e.g., [14], Theorem p. 199; and using $S'(V) = \Gamma(V)$ where V with $V_0 = 0$ is in $M^{\#}$ over a field of char. 0). In particular, for π a k-module the codomain of this isomorphism becomes $S'(\pi_n)$ ($= K(\pi,n)$ in later notation).

Of course $\tilde{E}(\pi,n)$ and $\tilde{K}(\pi,n)$ can be viewed as objects in S (using Ab--->Sets). We note that $\tilde{E}(\pi,n)$ is in S_R for $n > 0$ (since $C_n^N/\Delta_0 = 0$ for $n > 0$; and so $(\tilde{E}(\pi,n))_0 = 0$ in Ab), and so $\tilde{K}(\pi,n)$ is also in S_R. And so we can view $\tilde{h}:\tilde{E}(\pi,n)\longrightarrow \tilde{K}(\pi,n)$ as a map in S_R; further, $\tilde{h}$ is a fibration in S_R and $\tilde{K}(\pi,n)$ fibrant in S_R, and, hence, both also in S_o.

We next consider several constructions in C. Let V be a k-module. Let $K(V,n) = S'(V_n)$ where V_n in M has V concentrated in degree n. Now let W_n in M be given by $(W_n)_p = V$ if $p = n, n-1$ (and $= 0$ otherwise) and with differential having $d_n = id_V$. And let $E(V,n) = S'(W_n)$. $E(V,n)$ and $K(V,n)$ are both fibrant ($S'(\tilde{V})$ is fibrant for any $\tilde{V}$ in M). There is an obvious map $\phi:W_n\longrightarrow V_n$ in M; and the map $h = S'(\phi):E(V,n)\longrightarrow K(V,n)$ in C is a fibration (see [15]). There is also an obvious map $\beta:V_{n-1}\hookrightarrow W_n$ in M; let $i = S'(\beta)$, and note that the diagram:

$$\begin{array}{ccc} K(V,n-1) & \overset{i}{\hookrightarrow} & E(V,n) \\ \downarrow & & \downarrow h \\ k & \hookrightarrow & K(V,n) \end{array} \qquad \text{in C}$$

is a pullback (since it arises by applying S' to the bicartesian diagram $V_{n-1}\hookrightarrow W_n\longrightarrow V_n = V_{n-1}\longrightarrow 0\longrightarrow V_n$ in M). And so $K(V,n-1)$ is the fiber of h.

We next show that $E(\pi,n)$ is contractible (see section 11 for the notion of an object being <u>contractible</u> in a model category). $E(\pi,n) = S'(W_n)$; and using cor. 5.3, it suffices to show that $S'(W_n)\longrightarrow k$ is a weak equivalence. But this is immediate since H_* and S' commute and $H_*W_n = k$.

Now consider the obvious map $\tilde{\beta}: W_n \longrightarrow V_{n-1}$ in $M^{\#}$; and note that $s = S'(\tilde{\beta}): E(V,n) \longrightarrow K(V,n-1)$ in $C^{\#}$ satisfies $si = id_{K(V,n-1)}$ and the map $\langle h,s\rangle : E(V,n) \longrightarrow K(V,n) \otimes K(V,n-1)$ is an isomorphism in $C^{\#}$ (since $W_n = V_n \oplus V_{n-1}$ in $M^{\#}$ and S' preserves products). And so the following diagram (where we view the coalgebras as objects in $C^{\#}$):

$$\begin{tikzcd} E(V,n) \arrow[r, "\langle h,s\rangle", "\cong"'] \arrow[dr, "h"'] & K(V,n) \otimes K(V,n-1) \arrow[d, "pr"] \\ & K(V,n) \end{tikzcd} \qquad \text{in } C^{\#}$$

commutes. In the terminology of Neisendorfer [15], the sequence of coalgebras $K(V,n-1) \overset{i}{\rightarrowtail} E(V,n) \xrightarrow{h} K(V,n)$ is an example of an extension sequence in C (i.e., there is a (splitting) map s in $C^{\#}$ such that $si = id_{K(V,n-1)}$ and $\langle h,s\rangle$ is an isomorphism in $C^{\#}$).

A fibration $f: X \longrightarrow B$ in C with fiber F ($\overset{i}{\rightarrowtail} X$) is a fiber bundle if there is a splitting $s: X \longrightarrow F$ in $C^{\#}$ (i.e., $si = id_F$ and $\langle f,s\rangle$ is an isomorphism in $C^{\#}$). And so, viewed as a map in $C^{\#}$, f is "essentially" a projection; i.e., the following diagram:

$$\begin{tikzcd} X \arrow[r, "\langle f,s\rangle", "\cong"'] \arrow[dr, "f"'] & B \otimes F \arrow[d, "pr"] \\ & B \end{tikzcd} \qquad \text{in } C^{\#}$$

commutes.

We remark that this diagram gives f as a pullback in $C^{\#}$ of the projection pr along id_B (using the fact that $\langle f,s\rangle$ is an isomorphism).

__Proposition__ 8.1: Let $f':X'\longrightarrow B'$ be a pullback in C of a fiber bundle $f:X\longrightarrow B$ along a map $g:B'\longrightarrow B$. Then f' is a fiber bundle (i.e., fiber bundles in C are closed under pullbacks). Moreover, there is a splitting $s':X'\longrightarrow F$ in $C^{\#}$ of $F \overset{i'}{\hookrightarrow} X' \xrightarrow{f'} B'$ such that the following diagram:

$$\begin{tikzcd}
X' \arrow[rrr, "g'"] \arrow[dr, "\langle f',s'\rangle", "\simeq"'] \arrow[ddr, bend right, "f'"'] & & & X \arrow[dl, "\langle f,s\rangle"', "\simeq"] \arrow[ddl, bend left, "f"] \\
& B'\otimes F \arrow[r, "g\otimes F"'] \arrow[d, "pr"'] & B\otimes F \arrow[d, "pr"] & \\
& B' \arrow[r, "g"'] & B &
\end{tikzcd} \quad \text{in } C^{\#} \qquad (8.1)$$

commutes. I.e., in $C^{\#}$ f' is "essentially" given by the pullback of a projection.

Proof: The fiber F ($\overset{i'}{\hookrightarrow} X'$) of f' is the pullback of f' along $k\longrightarrow B'$. And so we can take $i = g'i':F\hookrightarrow X$ for the inclusion of the fiber of f. Let $s:X\longrightarrow F$ in $C^{\#}$ be a splitting for $F \overset{i}{\hookrightarrow} X \xrightarrow{f} B$. And let $s' = sg':X'\longrightarrow F$ in $C^{\#}$. Note that $s'i' = id_F$. We next show that $\langle f',s'\rangle:X'\longrightarrow B'\otimes F$ is an isomorphism in $C^{\#}$. First note that the rectangle in the above diagram is a pullback and observe that $\langle f',s'\rangle$ is the unique map into $B'\otimes F$ such that the left triangle and top trapezoid commute. But the outer rectangle in the following diagram:

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] & X \arrow[r, "\langle f,s\rangle"] \arrow[d, "f"] & B\otimes F \arrow[d, "pr"] \\
B' \arrow[r, "g"'] & B \arrow[r, "id_B"'] & B
\end{tikzcd} \quad \text{in } C^{\#}$$

is a pullback (since each square is a pullback and $U:C\longrightarrow C^{\#}$ preserves pullbacks for the left square). Thus, X' and $B'\otimes F$

are both pullbacks of the same maps into B. Now $\langle f', s'\rangle : X' \longrightarrow B' \circledS F$ is the canonical map between these pullbacks and is, hence, an isomorphism in $C^{\#}$.

We now continue our study of the maps $\tilde{h}$ in S_0 and h in C and show that $|\tilde{h}|$ is weakly equivalent to h.

Proposition 8.2: Let π be a k-module. Then $\mathcal{S}K(\pi,n)$ in S_0 is an E-M space of type (π,n).

Proof: We first note that $\mathcal{S}K(\pi,n)$ is fibrant (since $K(\pi,n)$ is fibrant and $\mathcal{S}$ preserves fibrations) and recall that $\mathcal{S}'X = \mathcal{S}X$ when X is fibrant in C (where $\mathcal{S}' : \mathrm{HoC} \longrightarrow \mathrm{HoS}_0$). We next observe that it suffices to show that $\mathcal{S}'K(\pi,n)$ represents cohomology for reduced simplicial sets and consider the following sequence of natural isomorphisms of functors $\mathrm{HoS}_0 \longrightarrow \mathrm{Ab}$:

$$\begin{aligned}
\mathrm{HoS}_0(_,\mathcal{S}'K(\pi,n)) &\simeq \mathrm{HoC}(|_|,K(\pi,n)) && \text{(using } |\ | \dashv \mathcal{S}'\text{)}\\
&\simeq H^n(|_|;\pi) && \text{(using theorem 11.4)}\\
&\simeq H^n(_;\pi) && \text{(using } \lambda : \hat{U}|\ | \longrightarrow C_*^N \text{ and the universal coefficient theorem for M).}
\end{aligned}$$

Lemma 8.3: There is a weak equivalence $\phi_K : |\tilde{K}(\pi,n)| \longrightarrow K(\pi,n)$ in C such that the adjoint map $\tilde{\phi}_K : \tilde{K}(\pi,n) \longrightarrow \mathcal{S}K(\pi,n)$ in S_0 is a weak equivalence.

I.e., the component of the unit $\eta_{\tilde{K}(\pi,n)} = \bar{\phi}_K : \tilde{K}(\pi,n) \longrightarrow \mathcal{S}K(\pi,n)$ (for $|\ |' \dashv \mathcal{S}$) is an isomorphism in hoS_0.

Proof: Consider the following composition of isomorphisms:

$$H_*|\tilde{K}(\pi,n)| \xrightarrow[\simeq]{\lambda_*} H_*(\tilde{K}(\pi,n);k) \xrightarrow{\simeq} K(\pi,n) \quad \text{in } C^{\#}.$$

Using lemma 3.3, there is a weak equivalence $\phi_K : |\tilde{K}(\pi,n)| \longrightarrow K(\pi,n)$ in C which agrees with the above composition in homology.

We next show that the adjoint map $\tilde{\phi}_K:\tilde{K}(\pi,n)\longrightarrow \mathcal{S}K(\pi,n)$ in S_0 is a weak equivalence when π is a finite dimensional k-module (this assumption is used in only one step in this part of the proof). As a preliminary, note that $H_*(\mathcal{S}K(\pi,n);k)$ is isomorphic to $K(\pi,n) = S'(\pi_n)$: $\mathcal{S}K(\pi,n)$ is an E-M space of type (π,n), and so there is a homotopy equivalence $f:\tilde{K}(\pi,n) \longrightarrow \mathcal{S}K(\pi,n)$ in S_0 (with homotopy inverse denoted by g). The following composition then gives our desired isomorphism:

$$H_*(\mathcal{S}K(\pi,n);k)\xrightarrow[\cong]{g_*}H_*(\tilde{K}(\pi,n);k)\xrightarrow[\cong]{}K(\pi,n) \quad \text{in } C^{\#}.$$

Now consider the component of the unit $\eta_{\tilde{K}(\pi,n)} = \tilde{\tilde{\phi}}_K:\tilde{K}(\pi,n) \longrightarrow \mathcal{S}K(\pi,n)$ (for $|\ |'\dashv\mathcal{S}$) and note that $|\eta_{\tilde{K}(\pi,n)}|':|\tilde{K}(\pi,n)|' \longrightarrow |\mathcal{S}K(\pi,n)|'$ in hoC has a left inverse $\varepsilon_{|\tilde{K}(\pi,n)|'}$ (using one of the triangle identities for a unit and counit). This implies that $H_*|\tilde{\phi}_K|':H_*|\tilde{K}(\pi,n)|'\longrightarrow H_*|\mathcal{S}K(\pi,n)|'$ has a left inverse and is, thus, injective. Then considering the following commutative diagram:

$$\begin{tikzcd}
H_*(\tilde{K}(\pi,n);k) \arrow[d, "H_*(\tilde{\phi}_K;k)"'] & H_*|\tilde{K}(\pi,n)| \arrow[l, "\lambda_*"', "\cong"] \arrow[r, "\cong"'] \arrow[d, "H_*(|\tilde{\phi}_K|)"] & H_*|\tilde{K}(\pi,n)|' \arrow[d, hook, "H_*(|\tilde{\phi}_K|')"] \\
H_*(\mathcal{S}K(\pi,n);k) & H_*|\mathcal{S}K(\pi,n)| \arrow[l, "\lambda_*"', "\cong"] \arrow[r, "\cong"'] & H_*|\mathcal{S}K(\pi,n)|'
\end{tikzcd},$$

we can conclude that $H_*(|\tilde{\phi}_K|)$ is injective and, thus, so is $H_*(\tilde{\phi}_K;k)$. But both $H_*(\tilde{K}(\pi,n);k)$ and $H_*(\mathcal{S}K(\pi,n);k)$ are isomorphic to the cofree coalgebra $S'(\pi_n)$ which has the module of primitives $PS'(\pi_n) = \pi_n$. And so (viewing each isomorphism as an equality) we obtain an injective map $PH_*(\tilde{\phi}_K;k):\pi\longrightarrow\pi$ which is, thus, an isomorphism (since π is finite dimensional); which allows us to conclude that $H_*(\tilde{\phi}_K;k)$ is an isomorphism. Finally since $\tilde{K}(\pi,n)$ and $\mathcal{S}K(\pi,n)$ are both rational (this notion is defined in section 10) and fibrant (using prop. 10.2 for the latter), prop. 10.1 implies that $\tilde{\phi}_K$ is a homotopy equivalence and, thus, a weak equivalence in S_o.

We now show that the adjoint map $\tilde{\phi}_K$ in S_o is a weak equivalence for arbitrary π. With π a directed colimit of its finite dimensional sub k-modules π_α and the isomorphism $H_*|\tilde{K}(\pi,n)| \longrightarrow K(\pi,n)$ from the beginning of this proof natural in π, we apply the commutativity property of lemma 3.3 to conclude that the family of maps $\{\phi_K(\pi_\alpha): |\tilde{K}(\pi_\alpha,n)| \longrightarrow K(\pi_\alpha,n)\}$ gives a morphism between directed systems. It follows that the family of adjoint maps $\{\tilde{\phi}_K(\pi_\alpha): \tilde{K}(\pi_\alpha,n) \longrightarrow \mathcal{S}K(\pi_\alpha,n)\}$ is also a morphism between directed systems. Now $\tilde{K}$ and $\mathcal{S}K$ preserve directed colimits in π, and, since $\tilde{\phi}_K(\pi_\alpha)$ give k-homology isomorphisms, we can conclude that $\tilde{\phi}_K: \tilde{K}(\pi,n) \longrightarrow \mathcal{S}K(\pi,n)$ does as well; hence, $\tilde{\phi}_K$ is a weak equivalence (using the same argument as above for finite dimensional π).

__Proposition__ 8.4: Let π be a k-module. There is a commutative diagram:

$$\begin{array}{ccc} |\tilde{E}(\pi,n)| & \xrightarrow{\phi_E} & E(\pi,n) \\ {\scriptstyle |\tilde{h}|}\downarrow & & \downarrow{\scriptstyle h} \\ |\tilde{K}(\pi,n)| & \xrightarrow{\phi_K} & K(\pi,n) \end{array} \qquad \text{in } C, \qquad (8.2)$$

where the horizontal maps ϕ_K, ϕ_E are weak equivalences in C and the related adjoint maps $\tilde{\phi}_K: \tilde{K}(\pi,n) \longrightarrow \mathcal{S}K(\pi,n)$, $\tilde{\phi}_E: \tilde{E}(\pi,n) \longrightarrow \mathcal{S}E(\pi,n)$ are weak equivalences in S_o (and where $\tilde{K}(\pi,n)$ and $\tilde{E}(\pi,n)$ use only the abelian group structure of π).

Proof: We need only show that there is a map ϕ_E with the required properties. To this end, note that the contractibility of $\tilde{E}(\pi,n)$ implies that $|\tilde{E}(\pi,n)|$ is left contractible (recall from the proof of lemma 7.8 that $|\ |$ preserves left homotopic maps). Now $|\tilde{h}|$, being a map out of $|\tilde{E}(\pi,n)|$, is left null homotopic; and so the composition $\phi_K|\tilde{h}|$ is null homotopic. And, thus, by lemma 11.1 there is a map ϕ_E which makes the above diagram commutative. But ϕ_E is necessarily null homotopic (being a map with contractible codomain) and, hence, is a weak equivalence in C. That the adjoint map $\tilde{\phi}_E:\tilde{E}(\pi,n) \longrightarrow \mathcal{S}E(\pi,n)$ in S_o is a weak equivalence is also immediate: here the contractibility of $\mathcal{S}E(\pi,n)$ follows from that of $E(\pi,n)$ (using lemma 7.5), and so $\tilde{\phi}_E$ is null homotopic and, hence, a homotopy equivalence; which in turn implies that it is a weak equivalence.

9. The realization of induced fibrations.

The main result of this section is theorem 9.2. With the fibration $\tilde{h}:\tilde{E}(\pi,n)\longrightarrow\tilde{K}(\pi,n)$ in S_0 related (by diagram (8.2)) to the fiber bundle $h:E(\pi,n)\longrightarrow K(\pi,n)$ in C, using this theorem we will be able to conclude that the realization of a fibration in S_0 induced from $\tilde{h}$ is weakly equivalent to a fiber bundle in C induced from h. In theorem 9.7, the basic theorem is then applied to a tower of fibrations under X in S_0.

In our proof of theorem 9.2, we will make use of the following lemma.

First we recall the notion of a comma category. Let $\tilde{C}$ be an arbitrary category, and let B be an object in $\tilde{C}$. The category of <u>objects over B</u>, denoted $(\tilde{C}\downarrow B)$, is the category with each object being a map $f:D\longrightarrow B$ in $\tilde{C}$ and a morphism $h:f\longrightarrow f'$ being a map $h:D\longrightarrow D'$ in $\tilde{C}$ for which $f'h = f$.

We remark that it is easily shown that the obvious forgetful functor $(\tilde{C}\downarrow B)\longrightarrow\tilde{C}$ preserves, reflects, and creates pushouts. In particular, we will make use of the fact that a pushout in $(\tilde{C}\downarrow B)$ is given simply by the pushout in $\tilde{C}$.

<u>**Lemma**</u> 9.1: Let $f:B'\longrightarrow B$ be a map in $\tilde{C}$ and consider the functors $(\tilde{C}\downarrow B')\underset{f^*}{\overset{f_*}{\rightleftarrows}}(\tilde{C}\downarrow B)$, where f_* on objects is given by $f_*(X\xrightarrow{h}B') = fh$, and f^* on objects is given by the pullback along f. It is easily shown that $f_*\dashv f^*$. When $\tilde{C}$ = Sets, f^* also has a right adjoint $f_!$; i.e., $f_*\dashv f^*\dashv f_!$.

For this last property of Sets, see, e.g., [10] (in this reference different notations for f_* and $f_!$ are used). We will make use of the fact that since f^* has a right adjoint, it, in particular, preserves pushouts.

This lemma is used in showing that the outer square of diagram (9.4), which occurs in the proof of the next theorem, is bicartesian in S_o.

Let $\tilde{h}:\tilde{E}\longrightarrow\tilde{K}$ be a fibration in S_o, and let $h:E\longrightarrow K$ be a fiber bundle in C; then $\underline{\tilde{h}}$ __is related to__ $\underline{h}$ if there is a commutative square of the form:

$$\begin{tikzcd} |\tilde{E}| \arrow[r, "\phi_E"] \arrow[d, "|\tilde{h}|"'] & E \arrow[d, "h"] \\ |\tilde{K}| \arrow[r, "\phi_K"] & K \end{tikzcd} \qquad \text{in C,}$$

where the horizontal maps are weak equivalences.

We remark that if $\tilde{h}:\tilde{E}\longrightarrow\tilde{K}$ is a fibration in S_o which is related to the fiber bundle $h:E\longrightarrow K$ in C with K fibrant (and it is assumed that the obvious map $|\tilde{F}|\longrightarrow F$ is a weak equivalence, with $\tilde{F}$ and F the fibers of $\tilde{h}$ and h, respectively), then the proof of the following basic theorem remains valid except for the treatment given in lemma 9.4 for the special case where the base is $\bar{\Delta}_1$. This basic theorem is expected to be true in more general situations with some additional hypothesis imposed on the fibration $\tilde{h}$.

__Basic theorem__ 9.2: Let $\tilde{h}:\tilde{E}\longrightarrow\tilde{K}$ in S_o denote the fibration $\tilde{h}:\tilde{E}(\pi,m)\longrightarrow\tilde{K}(\pi,m)$, and let $h:E\longrightarrow K$ in C denote the fiber bundle $h:E(\pi,m)\longrightarrow K(\pi,m)$ where $m>1$ (and let $\tilde{F}$ and F denote the fibers of $\tilde{h}$ and h, respectively).

Let $f:X\longrightarrow B$ be a pullback in S_o of $\tilde{h}:\tilde{E}\longrightarrow\tilde{K}$ along a map

$g:B \longrightarrow \tilde{K}$. Let $Y \longrightarrow |B|$ be a pullback in C of $h:E \longrightarrow K$ along $|B| \xrightarrow{|g|} |\tilde{K}| \longrightarrow K$. Let $\phi: |X| \longrightarrow Y$ be the unique map into the pullback Y such that $|X| \xrightarrow{\phi} Y \longrightarrow E = |X| \longrightarrow |\tilde{E}| \longrightarrow E$ and $|X| \xrightarrow{\phi} Y \longrightarrow |B| = |f|$. And so we have

$$\begin{tikzcd} |X| \arrow[rr, "\phi"] \arrow[dr, "|f|"'] & & Y \arrow[dl] \\ & |B| & \end{tikzcd}$$

commutes; then ϕ is a weak equivalence.

We give the following diagrams for reference:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \tilde{E} \arrow[d, "\tilde{h}"] \\ B \arrow[r, "g"'] & \tilde{K} \end{tikzcd} \quad \text{in } S_o \qquad \xrightarrow{|\ |} \qquad \begin{tikzcd} |X| \arrow[rr] \arrow[dr, "\phi"] \arrow[ddr, bend right, "|f|"'] & & |\tilde{E}| \arrow[dr] & \\ & Y \arrow[rr] \arrow[d] & & E \arrow[d, "h"] \\ & |B| \arrow[r, "|g|"'] & |\tilde{K}| \arrow[r] & K \end{tikzcd} \quad \text{in C.}$$

The proof of this theorem will make use of two special cases:

__Lemma__ 9.3 (Special Case where the base is a coproduct):

Let $f:X \longrightarrow B_1 V B_2$ be a fibration in S_o which gives

$$\begin{tikzcd} |X| \arrow[rr, "\phi"] \arrow[dr, "|f|"'] & & Y \arrow[dl] \\ & |B_1| V |B_2| & \end{tikzcd}$$

as described in the statement of this theorem. Let ${}^XB_i$ $(i = 1,2)$ be given by a pullback of $f:X \longrightarrow B_1 V B_2$ along $B_i \hookrightarrow B_1 V B_2$. And let ${}^YB_i$ $(i = 1,2)$ be given by a pullback of $Y \longrightarrow |B_1| V |B_2|$ along $|B_i| \hookrightarrow |B_1| V |B_2|$. Now let $\phi_i: |{}^XB_i| \longrightarrow {}^YB_i$ be the obvious map into the pullback. If ϕ_i $(i = 1,2)$ are weak equivalences, then so is ϕ.

Proof: Note that the induced fibrations ${}^XB_i \longrightarrow B_i$ $(i = 1,2)$ and f have the same fiber $\tilde{F}$ as $\tilde{h}$; and consider the commutative diagram:

$$\begin{array}{ccc} \tilde{F} & \hookrightarrow & X_{B_2} \\ \downarrow & & \downarrow \\ X_{B_1} & \hookrightarrow & X \end{array} \qquad \text{in } S_o. \quad (9.1)$$

We claim that this diagram is a pushout in S_o. We remark that this diagram is then also a pullback (since it is a pushout in S_o with two injections as opposite maps) and is, thus, bicartesian.

The proof of our claim is, in essence, an exercise in coproducts in $Sets_*$ and pullbacks in Sets. For A and A' in S_o, the standard construction of the coproduct $A \vee A'$ in degree n is given by an identification of the disjoint union: $(A \vee A')_n = ((A_n \sqcup A'_n)/a_n \sim a'_n, \overline{a_n})$ where e.g., $(A)_n = (A_n, a_n)$ in $Sets_*$. And so for $B_1 \vee B_2$, let b_n^i be the point of the pointed set $(B_i)_n$ $(i = 1,2)$; and let b_n $(= \overline{b_n^1} = \overline{b_n^2})$ be the identified point of $(B_1 \vee B_2)_n$. Now for simplicity view the pullbacks X_{B_i} $(i = 1,2)$ and diagram (9.1) as being in S_R using the isomorphism $S_o \longrightarrow S_R$ (and for convenience retain the same symbols for them). It suffices to show that this diagram has in each degree the universal mapping property of a pushout in Sets. The pullbacks $\tilde{F}$ and X_{B_i} $(i = 1,2)$ give pullbacks in each degree, and, using the standard construction of pullbacks in Sets, our diagram in degree n is given by

$$\begin{array}{ccc} \{b_n\} \times f_n^{-1}(b_n) & \hookrightarrow & \displaystyle\bigcup_{b_2 \in (B_2)_n} \{b_2\} \times f_n^{-1}(\overline{b_2}) \\ \downarrow & & \downarrow \\ \displaystyle\bigcup_{b_1 \in (B_1)_n} \{b_1\} \times f_n^{-1}(\overline{b_1}) & \hookrightarrow & X_n \end{array} \qquad \text{in Sets.}$$

But note that $X_n = X_1 \sqcup f_n^{-1}(b_n) \sqcup X_2$ where $X_i = \bigcup_{b_i} f_n^{-1}(\overline{b_i})$ with the union over all $b_i \in (B_i)_n$ provided $\overline{b_i} \neq b_n$ $(i = 1,2)$. With X_n expressed as a coproduct, it is easily shown that this last diagram is a pushout in Sets using the universal mapping property for this coproduct. Finally, our use of the standard constructions in this proof has not resulted in any loss of generality.

Now consider the diagram which is analogous to diagram (9.1):

$$\begin{array}{ccc} F & \hookrightarrow & Y_{B_2} \\ \downarrow & & \downarrow \\ Y_{B_1} & \hookrightarrow & Y \end{array} \qquad \text{in } C. \qquad (9.2)$$

We will show that this diagram is a pushout in M. First note that all coalgebras in this square are the total spaces of fibrations induced from h. And so if we apply the forgetful functor $U: C \longrightarrow C^{\#}$ to this diagram in conjunction with diagrams of the type (8.1), we obtain the commutative diagram:

$$\begin{array}{ccc} k \otimes F & \hookrightarrow & |B_2| \otimes F \\ \downarrow & & \downarrow \\ |B_1| \otimes F & \hookrightarrow & (|B_1| \vee |B_2|) \otimes F \end{array} \qquad (9.3)$$

Note that, by neglecting the comultiplication structures, we can regard this as a square in $M^{\#}$. But if we view the coproduct $|B_1| \vee |B_2|$ as a pushout (in $M^{\#}$) and then apply the functor $\_\otimes F$ we also obtain the above diagram. But $\_\otimes F$ preserves pushouts (since it is an exact functor); and, thus, diagram (9.3) is a pushout in $M^{\#}$. And so diagram (9.2), when regarded in $M^{\#}$, is a pushout (using the isomorphisms of the type present in (8.1)). Finally, since the forgetful functor $\tilde{U}:M \longrightarrow M^{\#}$ reflects pushouts, we can conclude that diagram (9.2) is a pushout when regarded in M.

Now consider the commutative diagram:

$$\begin{tikzcd}
F \arrow[rrr, hook] \arrow[ddd, hook] & & & Y_{B_2} \arrow[ddd, hook] \\
& {|\tilde{F}|} \arrow[ul, "\phi_F"'] \arrow[r, hook] \arrow[d, hook] & {|X_{B_2}|} \arrow[ur, "\phi_2"'] \arrow[d, hook] & \\
& {|X_{B_1}|} \arrow[dl, "\phi_1"] \arrow[r, hook] & {|X|} \arrow[dr, "\phi"] & \\
Y_{B_1} \arrow[rrr, hook] & & & Y
\end{tikzcd} \qquad \text{in } C,$$

where the inner square is obtained by applying the realization functor $|\ |$ to diagram (9.1), and ϕ_F is the evident map. Since $|\ |$ preserves pushouts, the inner square is a pushout in C and is, thus, a pushout in M as is the outer square. We have assumed that the maps ϕ_1 and ϕ_2 give homology isomorphisms. ϕ_F does as well. Finally, since all vertical (or horizontal) maps in this diagram are monic, we can conclude that ϕ gives an isomorphism in homology and is, thus, a weak equivalence in C.

This lemma is valid where the base is an infinite coproduct and can be proved using a directed colimit argument.

<u>Lemma</u> 9.4 (Special case where base is $\bar{\Delta}_n$): Let $f:X \longrightarrow \bar{\Delta}_n$ be a fibration in S_o which gives

$$\begin{tikzcd}
{|X|} \arrow[rr, dashed, "\phi"] \arrow[dr, "|f|"'] & & Y \arrow[dl] \\
& {|\bar{\Delta}_n|} &
\end{tikzcd}$$

as described in the statement of this theorem. Then ϕ is a weak equivalence.

Proof: We first consider the case for $n = 1$ (recall that $\tilde{h}:\tilde{E} \longrightarrow \tilde{K}$ in S_o denoted the fibration $\tilde{h}:\tilde{E}(\pi,m) \longrightarrow \tilde{K}(\pi,m)$, and $h:E$

$\longrightarrow K$ in C denoted the fiber bundle $h:E(\pi,m)\longrightarrow K(\pi,m)$). Note that the map $g:\bar{\Delta}_1 \longrightarrow \tilde{K}(\pi,m)$ is the trivial map (since $(\tilde{K}(\pi,m))_1 = 0$ for $m>1$); and so the pullback of $\tilde{h}:\tilde{E}(\pi,m)\longrightarrow \tilde{K}(\pi,m)$ along the trivial map $\bar{\Delta}_1 \longrightarrow * \longrightarrow \tilde{K}(\pi,m)$ gives the projection $\bar{\Delta}_1 \Pi\, \tilde{K}(\pi,m-1) \longrightarrow \bar{\Delta}_1$ for the induced fibration f. But then the pullback of $h:E(\pi,m)\longrightarrow K(\pi,m)$ along the trivial map $\phi_K|g|:|\bar{\Delta}_1| \longrightarrow K(\pi,m)$ also gives the projection $|\bar{\Delta}_1| \otimes K(\pi,m-1) \longrightarrow |\bar{\Delta}_1|$ for the induced fibration $Y \longrightarrow |\bar{\Delta}_1|$. The unique map $\phi:|X|\longrightarrow Y$, thus, has the form $\phi:|\bar{\Delta}_1 \Pi\, \tilde{K}(\pi,m-1)| \longrightarrow |\bar{\Delta}_1| \otimes K(\pi,m-1)$. Finally, it is easily shown that $\phi = |\bar{\Delta}_1 \Pi\, \tilde{K}(\pi,m-1)| \longrightarrow |\bar{\Delta}_1| \otimes |\tilde{K}(\pi,m-1)| \xrightarrow{\overline{id}_{|\bar{\Delta}_1|} \otimes \phi_F} |\bar{\Delta}_1| \otimes K(\pi,m-1)$; and so ϕ is a weak equivalence using theorem 4.8.

Now consider the general case where $n > 1$. Let W equal the n edges of a reduced n-horn; i.e., $W = \bar{\Delta}_1 \vee \cdots \vee \bar{\Delta}_1$ (n co-products) in S_0. And let X_W be a pullback in S_0 of the map $f:X\longrightarrow \bar{\Delta}_n$ along the weak equivalence $W \hookrightarrow \bar{\Delta}_n$. Note that f is a surjective fibration in S_0 (since it is induced from $\tilde{h}:\tilde{E}(\pi,m)\longrightarrow \tilde{K}(\pi,m)$, which is surjective ($m>1$)) and, hence, is a fibration in S; and so viewing X_W as a pullback in S of f along the above weak equivalence, we can conclude that $X_W \longrightarrow X$ is a weak equivalence (recall that in S weak equivalences are preserved under pullbacks of arbitrary fibrations-- i.e., the bases need not be fibrant). Now let Y_W be a pullback in C of $Y \longrightarrow |\bar{\Delta}_n|$ along $|W| \hookrightarrow |\bar{\Delta}_n|$. Note that $|W| \hookrightarrow |\bar{\Delta}_n|$ is a weak equivalence; and so $Y_W \hookrightarrow Y$ is a weak equivalence (using prop. 5.10). Let $\phi_W:|X_W| \longrightarrow Y_W$ be the unique map into the pullback Y_W such that $|X_W| \xrightarrow{\phi_W} Y_W \longrightarrow |W| = |X_W| \longrightarrow |W|$ and $|X_W| \xrightarrow{\phi_W} Y_W \hookrightarrow Y = |X_W| \hookrightarrow |X| \xrightarrow{\phi} Y$. Using this last equality, note that once we have shown that ϕ_W is a weak equivalence, then so is ϕ which is our desired conclusion (since then three out of the four maps are weak equivalences). To this end in showing that ϕ_W is a weak equivalence, observe that $|W| = |/\bar{\Delta}_1| \vee \cdots \vee |/\bar{\Delta}_1|$, and so the base of $Y_W \longrightarrow |W|$ is a coproduct. The coproduct lemma 9.3, thus, reduces the proof to the case where $n = 1$.

<u>Proof of theorem 9.2</u>: The proof is by induction on the dimension of the base B.

A base which has dimension zero is necessarily a null object $*$ in S_o. But then a pullback in S_o of $\tilde{h}:\tilde{E} \longrightarrow \tilde{K}$ along $* \longrightarrow \tilde{K}$ gives the fiber $\tilde{F}$ of $\tilde{h}$. Recall that $|*| = k$, which is the null object in C. And so a pullback in C of $h:E \longrightarrow K$ along $k = |*| \longrightarrow |\tilde{K}| \longrightarrow K$ gives the fiber F of h. And so we obtain $\phi:|\tilde{F}| \longrightarrow F$, which is a weak equivalence.

We now consider the construction of the following commutative diagram:

$$\begin{tikzcd}
\hat{X} \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \tilde{X} \arrow[dl] \arrow[ddd] \\
& \vee /\dot{\Delta}_n \arrow[r, hook] \arrow[d] & \vee \bar{\Delta}_n \arrow[d] & \\
& B^{n-1} \arrow[r, hook] & B^n & \\
X^{(n-1)} \arrow[rrr, hook] \arrow[ur] & & & X^{(n)} \arrow[ul]
\end{tikzcd} \quad \text{in } S_o. \qquad (9.4)$$

For this, let B^k be the k-skeleton of B. And let $X^{(k)}$ be a pullback in S_o of $f:X \longrightarrow B$ along $B^k \hookrightarrow B$ $(k \geqslant 0)$. Note that there is an obvious map $X^{(n-1)} \hookrightarrow X^{(n)}$ into the pullback $X^{(n)}$; this makes the bottom trapezoid in diagram (9.4) commutative and also a pullback. The inner square is the basic pushout diagram (2.2) in S_o. Now let $\tilde{X}$ and $\hat{X}$ be pullbacks of the fibrations $X^{(n)} \longrightarrow B^n$ and $X^{(n-1)} \longrightarrow B^{n-1}$, respectively, along the maps indicated in the above diagram. Finally let $\hat{X} \longrightarrow \tilde{X}$ be the unique map into the pullback $\tilde{X}$ such that the outer square and top trapezoid both commute.

__Claim__: The outer square of diagram (9.4) is bicartesian in S_o.

Proof: The inner square, being a pushout in S_o of two opposite injections, is also a pullback. It necessarily follows that the outer square is a pullback using the basic properties of pullbacks in a category. To show that the outer square is also a pushout, it is sufficient to show that this square is a pushout in each degree since the forgetful functor from S_o to graded Sets reflects pushouts.

This claim is, thus, a consequence of the following lemma.

__Lemma__ 9.5: Let the following be a commutative diagram:

$$\begin{tikzcd}
X_C \arrow[rrr] \arrow[dr] \arrow[ddd] & & & X_B \arrow[dl] \arrow[ddd] \\
& C \arrow[r, hook] \arrow[d] & B \arrow[d] & \\
& A \arrow[r, hook] & P & \\
X_A \arrow[ur] \arrow[rrr] & & & X \arrow[ul]
\end{tikzcd} \qquad \text{in Sets,}$$

where the inner square is a pushout, where X_B and X_C are pullbacks of $X \longrightarrow P$ and $X_A \longrightarrow A$, respectively, along the maps indicated in the diagram, and where the bottom trapezoid is also a pullback. Then the outer square is a pushout in Sets.

We remark that the inner square, being a pushout in Sets of two opposite injections, is also a pullback. And so the inner square is bicartesian and, with this lemma, so is the outer square.

Proof: That the outer square is a pushout is an immediate consequence of lemma 9.1. To see this, let f equal the map $X \longrightarrow P$. Then applying the functor f^* to the inner square one obtains the outer square with $X_A = f^*(A)$, $X_B = f^*(B)$, and $f^*(C)$ = the pullback of $C \longrightarrow A \longrightarrow P$ along f. This last pullback is equivalent that $f^*(C)$ be the pullback of $C \longrightarrow A$ along $f^*(A) \longrightarrow A$. But since f^* is a left adjoint, it must preserve pushouts in $(\mathrm{Sets}{\downarrow}P)$ which ,in turn, are given simply by pushouts in Sets. And so we can conclude that the outer square is a pushout in Sets.

Now that diagram (9.4) is shown to be bicartesian, we can observe that the top map $\hat{X} \longrightarrow \tilde{X}$ is monic since it is a pullback of a monic.

Now consider the application of the realization functor $|\ |$ to diagram (9.4) to obtain the following commutative diagram:

$$\begin{tikzcd}
{|\hat{X}|} \arrow[rrr, hook] \arrow[dr] \arrow[ddd] & & & {|\tilde{X}|} \arrow[dl] \arrow[ddd] \\
& {|V/\dot{\bar{\Delta}}_n|} \arrow[r, hook] \arrow[d] & {|V/\bar{\Delta}_n|} \arrow[d] & \\
& {|B^{n-1}|} \arrow[r, hook] & {|B^n|} & \\
{|X^{(n-1)}|} \arrow[rrr, hook] \arrow[ur] & & & {|X^{(n)}|} \arrow[ul]
\end{tikzcd} \quad \text{in } C. \qquad (9.5)$$

Note that all the horizontal maps are monic since $|\ |$ preserves monomorphisms. But since $|\ |$ also preserves pushouts, the two squares are pushouts in C; and, hence, pushouts in M. But recall that a pushout diagram with opposite monomorphisms in the abelian category M is also a pullback; and so we can conclude that, when the above diagram is viewed in M, the two squares are bicartesian.

Now consider the analogue of diagram (9.4):

$$\begin{tikzcd}
\hat{Y} \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \tilde{Y} \arrow[dl] \arrow[ddd] \\
& {|V / \dot{\Delta}_n|} \arrow[r, hook] \arrow[d] & {|V / \bar{\Delta}_n|} \arrow[d] & \\
& {|B^{n-1}|} \arrow[r, hook] & {|B^n|} & \\
Y^{(n-1)} \arrow[rrr, hook] \arrow[ur] & & & Y^{(n)} \arrow[ul]
\end{tikzcd} \quad \text{in C.} \qquad (9.6)$$

Here, $Y^{(k)}$ is a pullback in C of the fibration $Y \longrightarrow |B|$ along $|B^k| \hookrightarrow |B|$ $(k \geqslant 0)$. Again note that there is an obvious map $Y^{(n-1)} \hookrightarrow Y^{(n)}$ into the pullback $Y^{(n)}$; this makes the bottom trapezoid commutative and also a pullback. Now let $\tilde{Y}$ and $\hat{Y}$ be pullbacks of the fibrations $Y^{(n)} \longrightarrow |B^n|$ and $Y^{(n-1)} \longrightarrow |B^{n-1}|$, respectively, along the maps as indicated in the above diagram. Finally, let $\hat{Y} \longrightarrow \tilde{Y}$ be the unique map into the pullback $\tilde{Y}$ such that the outer square and top trapezoid both commute.

Note that the top trapezoid in diagram (9.6) is a pullback in C using the basic properties of pullbacks in a category. And so the map $\hat{Y} \longrightarrow \tilde{Y}$ is monic since it is the pullback of a monic.

<u>**Claim**</u>: The outer square of (9.6) is bicartesian when re-

garded as a square in M.

Proof: The basic approach taken here is similar to that taken in proving part of the coproduct lemma 9.3 (with the outer square of (9.6) and diagram (9.2) serving analogous functions). First note that all coalgebras in the outer square of (9.6) are the total spaces of fibrations induced from h. And so if we apply the forgetful functor $U:C \longrightarrow C^{\#}$ to this square in conjunction with diagrams of the type (8.1), we obtain the commutative diagram:

$$\begin{tikzcd} {|V/\dot{\bar{\Delta}}_n| \otimes F} \arrow[r, hook] \arrow[d] & {|V/\bar{\Delta}_n| \otimes F} \arrow[d] \\ {|B^{n-1}| \otimes F} \arrow[r, hook] & {|B^n| \otimes F} \end{tikzcd} .$$

By neglecting the comultiplication structures, we can regard this as a diagram in $M^{\#}$. Note that if we apply the functor $\_\otimes F$ to the inner square of (9.6) (viewed as a square in $M^{\#}$), we also obtain this diagram. But recall we have shown that the inner square of (9.6) is bicartesian in M and, hence, bicartesian in $M^{\#}$; and so, since $\_\otimes F$ (being an exact functor) preserves bicartesian squares, we can conclude that the above diagram is bicartesian in $M^{\#}$. And so the outer square of (9.6), when regarded in $M^{\#}$, is bicartesian (using the isomorphisms of the type present in (8.1)). Finally, since the forgetful functor $\tilde{U}:M \longrightarrow M^{\#}$ reflects pushouts and pullbacks, we can conclude that the outer square of (9.6) is bicartesian when regarded in M.

We now consider the construction of the following commutative diagram:

$$\begin{tikzcd}
{|\hat{X}|} \arrow[rrrrrr, hook] \arrow[dr, "\hat{\phi}"'] \arrow[ddddd] & & & & & & {|\tilde{X}|} \arrow[dl, "\tilde{\phi}"] \arrow[ddddd] \\
& \hat{Y} \arrow[rrrr, hook] \arrow[dr] \arrow[ddd] & & & & \tilde{Y} \arrow[dl] \arrow[ddd] & \\
& & {|V/\dot{\Delta}_n|} \arrow[rr, hook] \arrow[d] & & {|V/\bar{\Delta}_n|} \arrow[d] & & \\
& & {|B^{n-1}|} \arrow[rr, hook] & & {|B^n|} & & \\
& Y^{(n-1)} \arrow[ur] \arrow[rrrr, hook] & & & & Y^{(n)} \arrow[ul] & \\
{|X^{(n-1)}|} \arrow[ur, "\phi^{n-1}"] \arrow[rrrrrr, hook] & & & & & & {|X^{(n)}|} \arrow[ul, "\phi^n"']
\end{tikzcd} \quad \text{in } C. \qquad (9.7)$$

For this, note that there are obvious maps $\phi^k: |X^{(k)}| \longrightarrow Y^{(k)}$ into the pullbacks $Y^{(k)}$ $(k \geqslant 0)$; these give commutativity to the bottom trapezoid in diagram (9.7). Also there are obvious maps $\tilde{\phi}: |\tilde{X}| \longrightarrow \tilde{Y}$ and $\hat{\phi}: |\hat{X}| \longrightarrow \hat{Y}$ into the pullbacks $\tilde{Y}$ and $\hat{Y}$, respectively, which give commutativity to the left and right trapezoids in diagram (9.7). Finally under composition the maps along the diagonal of this diagram give the corresponding diagonal maps of diagram (9.5).

We are now ready to apply the <u>induction</u> <u>hypothesis</u>: assume if $f': X' \longrightarrow B'$ is a pullback of $\tilde{h}: \tilde{E} \longrightarrow \tilde{K}$ along some map $g': B' \longrightarrow \tilde{K}$ where B' is $(n-1)$-dimensional in S_0, then ϕ' is a weak equivalence where
$$\begin{tikzcd}
{|X'|} \arrow[rr, "\phi'"] \arrow[dr, "|f'|"'] & & Y' \arrow[dl] \\
& {|B'|} &
\end{tikzcd}$$
commutes and where $Y' \longrightarrow |B'|$ is a pullback of $h: E \longrightarrow K$ along $|B'| \xrightarrow{|g'|} |\tilde{K}| \longrightarrow K$.

Note $X^{(n-1)} \longrightarrow B^{n-1}$ is a pullback of $\tilde{h}: \tilde{E} \longrightarrow \tilde{K}$ along $g' = B^{n-1} \hookrightarrow B \xrightarrow{g} \tilde{K}$, and
$$\begin{tikzcd}
{|X^{(n-1)}|} \arrow[rr, "\phi^{n-1}"] \arrow[dr] & & Y^{(n-1)} \arrow[dl] \\
& {|B^{n-1}|} &
\end{tikzcd}$$
commutes where $Y^{(n-1)} \longrightarrow |B^{n-1}|$ a pullback of $h: E \longrightarrow K$ along $|B^{n-1}| \xrightarrow{|g'|}$

$|\tilde{K}| \dashrightarrow K$. And so since B^{n-1} is $(n-1)$-dimensional, ϕ^{n-1} is a weak equivalence by the induction hypothesis.

Also note $\hat{X} \dashrightarrow V\, /\dot{\Delta}_n$ is a pullback of $\tilde{h}: \tilde{E} \dashrightarrow \tilde{K}$ along $g' = V\, /\dot{\Delta}_n \dashrightarrow B^{n-1} \hookrightarrow B \xrightarrow{g} \tilde{K}$, and
$$\begin{tikzcd} {|\hat{X}|} \arrow[rr, "\hat{\phi}"] \arrow[dr] & & \hat{Y} \arrow[dl] \\ & {|V\, /\dot{\Delta}_n|} & \end{tikzcd}$$
commutes where $\hat{Y} \dashrightarrow |V\, /\dot{\Delta}_n|$ a pullback of $h: E \dashrightarrow K$ along $|V\, /\dot{\Delta}_n| \xrightarrow{|g'|} |\tilde{K}| \dashrightarrow K$. But $V\, /\dot{\Delta}_n$ is $(n-1)$-dimensional (since $/\dot{\Delta}_n$ is); and so $\hat{\phi}$ is a weak equivalence by the induction hypothesis.

We now show that $|\tilde{X}| \xrightarrow{\tilde{\phi}} \tilde{Y}$ is a weak equivalence. Note that $\tilde{X} \dashrightarrow V\, /\bar{\Delta}_n$ is a pullback of $\tilde{h}: \tilde{E} \dashrightarrow \tilde{K}$ along $g' = V\, /\bar{\Delta}_n \dashrightarrow B^n \hookrightarrow B \xrightarrow{g} \tilde{K}$, and
$$\begin{tikzcd} {|\tilde{X}|} \arrow[rr, "\tilde{\phi}"] \arrow[dr] & & \tilde{Y} \arrow[dl] \\ & {|V\, /\bar{\Delta}_n|} & \end{tikzcd}$$
commutes where $\tilde{Y} \dashrightarrow |V\, /\bar{\Delta}_n|$ is a pullback of $h: E \dashrightarrow K$ along $|V\, /\bar{\Delta}_n| \xrightarrow{|g'|} |\tilde{K}| \dashrightarrow K$. Observe that the base is the coproduct $\underset{\alpha}{V} |/\bar{\Delta}_n| = |\underset{\alpha}{V} /\bar{\Delta}_n|$; and apply lemma 9.3 by considering
$$\begin{tikzcd} {|X_\alpha|} \arrow[rr, "\phi_\alpha"] \arrow[dr] & & Y_\alpha \arrow[dl] \\ & {|/\bar{\Delta}_n|} & \end{tikzcd}$$
where, e.g., the fibration $X_\alpha \dashrightarrow /\bar{\Delta}_n$ is given as a pullback of $\tilde{X} \dashrightarrow \underset{\alpha}{V} /\bar{\Delta}_n$ along $/\bar{\Delta}_n \xhookrightarrow{i_\alpha} \underset{\alpha}{V} /\bar{\Delta}_n$ (and is, hence, also induced from $\tilde{h}: \tilde{E} \dashrightarrow \tilde{K}$). But ϕ_α is a weak equivalence (for all α) since the base is $/\bar{\Delta}_n$; and so we can conclude that $\tilde{\phi}$ is a weak equivalence.

Now using the fact that the two outer squares of diagram (9.7) are bicartesian in M, we have the following diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 \longrightarrow & |\hat{X}| & \longrightarrow & |X^{(n-1)}| \oplus |\tilde{X}| & \longrightarrow & |X^{(n)}| & \longrightarrow 0 \\ & \downarrow \hat{\phi} & & \downarrow \phi^{n-1} \oplus \tilde{\phi} & & \downarrow \phi^{n} & \\ 0 \longrightarrow & \hat{Y} & \longrightarrow & Y^{(n-1)} \oplus \tilde{Y} & \longrightarrow & Y^{(n)} & \longrightarrow 0 \end{array} \quad \text{in } M.$$

Note that the two left vertical maps are weak equivalences. And so considering the two Mayer-Vietoris long exact sequences in homology, and applying the 5-lemma, we can conclude that ϕ^n is a weak equivalence.

And so we have the diagram

$$\begin{array}{ccccccccccc} |X^{(0)}| & \hookrightarrow & |X^{(1)}| & \hookrightarrow & \cdots & \hookrightarrow & |X^{(n)}| & \hookrightarrow & \cdots & |X| \\ \downarrow \phi^0 & & \downarrow \phi^1 & & & & \downarrow \phi^n & & & \downarrow \phi \\ Y^{(0)} & \hookrightarrow & Y^{(1)} & \hookrightarrow & \cdots & \hookrightarrow & Y^{(n)} & \hookrightarrow & \cdots & Y \end{array} \quad \text{in } C,$$

where the vertical maps ϕ^k $(k \geq 0)$ are weak equivalences. It remains to show that ϕ is a weak equivalence. Now B is a sequential colimit of its n-skeletons; i.e., $B = \varinjlim_{n} B^n$. Recalling that $X^{(n)}$ is a pullback in S_0 of $f: X \longrightarrow B$ along $B^n \hookrightarrow B$, we then obtain $X = \varinjlim_{n} X^{(n)}$ (in the colimit, the pullback of f is along id_B). We, thus, have $|B| = \varinjlim_{n} |B^n|$ and $|X| = \varinjlim_{n} |X^{(n)}|$ (since $|\ |$ preserves colimits). Next recalling that $Y^{(n)}$ is a pullback in C of $Y \longrightarrow |B|$ along $|B^n| \hookrightarrow |B|$, we also obtain $Y = \varinjlim_{n} Y^{(n)}$. And so we have that $\phi: |X| \longrightarrow Y$ is given by $\varinjlim_{n} \phi^n : \varinjlim_{n} |X^{(n)}| \longrightarrow \varinjlim_{n} Y^{(n)}$. Now homology commutes with directed colimits. Thus, $H_*(\phi) = H_*(\varinjlim_{n} \phi^n) = \varinjlim_{n} H_*(\phi^n)$; which allows us to conclude that ϕ is a weak equivalence (since each ϕ^n is a weak equivalence $(n \geq 0)$).

The following theorem extends the result of our basic theorem 9.2 to an induced fibration in C which is a pullback of h along any map which is homotopic to $|B| \xrightarrow{|g|} |\tilde{K}| \longrightarrow K$.

Theorem 9.6: As in the above theorem, let $f:X \longrightarrow B$ be a pullback in S_o of $\tilde{h}:\tilde{E} \longrightarrow \tilde{K}$ along a map $g:B \longrightarrow \tilde{K}$ and also let $\tilde{h}$ be related to the fiber bundle $h:E \longrightarrow K$ in C. Let $\tilde{g}:|B| \longrightarrow K$ be a map in C which is homotopic to $|B| \xrightarrow{|g|} |\tilde{K}| \longrightarrow K$. Let $\tilde{Y} \longrightarrow |B|$ be a pullback in C of $h:E \longrightarrow K$ along $\tilde{g}$. Then a) there is a map $|X| \longrightarrow E$ which is a lift of $\tilde{g}|f|:|X| \longrightarrow K$ over h with $|X| \longrightarrow E \sim |X| \longrightarrow |\tilde{E}| \longrightarrow E$, and b) Let $\tilde{\phi}:|X| \longrightarrow \tilde{Y}$ be the unique map into the pullback $\tilde{Y}$, such that $|X| \xrightarrow{\tilde{\phi}} \tilde{Y} \longrightarrow E = |X| \longrightarrow E$ and $|X| \xrightarrow{\tilde{\phi}} \tilde{Y} \longrightarrow |B| = |f|$. And so we have

$$\begin{tikzcd} {|X|} \arrow[rr, "\tilde{\phi}"] \arrow[dr, "|f|"'] & & \tilde{Y} \arrow[dl] \\ & {|B|} & \end{tikzcd}$$

commutes; then $\tilde{\phi}$ is a weak equivalence.

Proof: From the basic theorem 9.2 we can conclude that ϕ is a weak equivalence; the conclusion of this theorem is then a consequence of theorem 5.13 for an arbitrary model category.

We next apply our basic theorem to a tower of fibrations under X in S_o.

Theorem 9.7: Let $\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow *$ be a tower of fibrations under X in S_o where $X_n \longrightarrow X_{n-1}$ is a pullback of $\tilde{h}:\tilde{E}_{n+1} \longrightarrow \tilde{K}_{n+1}$ along a map $X_{n-1} \longrightarrow \tilde{K}_{n+1}$ $(n > o)$ with $\tilde{h}$ related to the fiber bundle $h:E_{n+1} \longrightarrow K_{n+1}$ in C (with K_{n+1} fibrant) as in Theorems 9.2 and 9.6. Also assume $\tilde{K}_{n+1}$ is fibrant. Then applying the realization functor $|\ |$ to this tower in S_o gives a tower of maps under $|X|$ in C which is weakly equivalent to a tower of fiber bundles in C.

This is made more explicit by the following diagrams:

$$\begin{tikzcd}[row sep=small] & \cdots \arrow[d] \\ & X_n \arrow[d] \\ X \arrow[ur] \arrow[r] & X_{n-1} \arrow[d] \\ & \cdots \arrow[d] \\ & X_1 \arrow[d] \\ & * \end{tikzcd}$$

where

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \tilde{E}_{n+1} \arrow[d, "\tilde{h}"] \\ X_{n-1} \arrow[r] & \tilde{K}_{n+1} \end{tikzcd}$$

in S_o,

$| \ |$ $\longmapsto$

$$\begin{tikzcd}[row sep=small] & \cdots \arrow[d] & \cdots \arrow[d] \\ & {|X_n|} \arrow[r] \arrow[d] & Y_n \arrow[d] \\ {|X|} \arrow[ur] \arrow[r] & {|X_{n-1}|} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ & \cdots \arrow[d] & \cdots \arrow[d] \\ & {|X_1|} \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ & {|*|} \arrow[r] & k \end{tikzcd}$$

where

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d] & E_{n+1} \arrow[d, "h"] \\ Y_{n-1} \arrow[r] & K_{n+1} \end{tikzcd}$$

in C,

where $|X_n| \longrightarrow Y_n$ is a weak equivalence $(n > o)$,

and where the two separate squares are pullbacks in S_o and C, respectively.

Proof: At the beginning of the construction when $n = 1$, note that a pullback in S_o of the fibration $\tilde{h}:\tilde{E}_2 \longrightarrow \tilde{K}_2$ along $* \longrightarrow \tilde{K}_2$ gives the fiber $\tilde{F}_2 \hookrightarrow \tilde{E}_2$. Similarly, a pullback in C of $h:E_2 \longrightarrow K_2$ along $k \longrightarrow K_2$ gives the fiber $F_2 \hookrightarrow E_2$. And so for $|X_1| \longrightarrow Y_1$ we can take the weak equivalence $|\tilde{F}_2| \longrightarrow F_2$. Now assume that this tower of fiber bundles in C has been constructed up to and including stage $n-1$. That the construction of this tower can be extended to stage n follows from the next lemma; and, thus, the construction is complete.

We remark that Y_n is fibrant for all n since the composition $Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow k$ is the composition of fibrations.

In the following lemma, we retain the same setting and notation of the previous theorem.

Lemma 9.8: Let $f_n : X_n \dashrightarrow X_{n-1}$ be a pullback in S_0 of $\tilde{h} : \tilde{E}_{n+1} \dashrightarrow \tilde{K}_{n+1}$ along a map $g_{n-1} : X_{n-1} \dashrightarrow \tilde{K}_{n+1}$ with $\tilde{h}$ related to the fiber bundle $h : E_{n+1} \dashrightarrow K_{n+1}$ in C. And assume we have a weak equivalence $\psi_{n-1} : |X_{n-1}| \longrightarrow Y_{n-1}$ in C with Y_{n-1} fibrant. Then a) the map g_{n-1} can be replaced by a cofibration g'_{n-1} in such a way that f_n is a pullback in S_0 of a fibration $\hat{h}$ along $g'_n{-1}$ and where $\hat{h}$ is related to h as indicated below:

$$\begin{tikzcd} X_n \arrow[r, hook, "g''_{n-1}"] \arrow[d, "f_n"'] & \hat{E}_{n+1} \arrow[d, "\hat{h}"] \\ X_{n-1} \arrow[r, hook, "g'_{n-1}"'] & \hat{K}_{n+1} \end{tikzcd} \qquad \begin{tikzcd} {|\hat{E}_{n+1}|} \arrow[r, "\phi''_E"] \arrow[d, "|\hat{h}|"'] & E_{n+1} \arrow[d, "h"] \\ {|\hat{K}_{n+1}|} \arrow[r, "\phi''_K"'] & K_{n+1} \end{tikzcd} \quad , \tag{9.8}$$

b) there is a map $g' : Y_{n-1} \longrightarrow K_{n+1}$ for which $\phi''_E|g''_{n-1}|$ is a lift of $g'\psi_{n-1}|f_n| : |X_n| \longrightarrow K_{n+1}$ over h, and c) the obvious map $\psi_n : |X_n| \longrightarrow Y_n$ into the pullback Y_n is a weak equivalence.

We give the following diagram for reference:

$$\begin{tikzcd} {|X_n|} \arrow[rr, hook] \arrow[r, dashed, "\psi_n"'] \arrow[d, "|f_n|"'] & Y_n \arrow[r] \arrow[d] & {|\hat{E}_{n+1}|} \arrow[r, "\phi''_E"] & E_{n+1} \arrow[d, "h"] \\ {|X_{n-1}|} \arrow[r, "\psi_{n-1}"'] & Y_{n-1} \arrow[rr, "g'"'] & & K_{n+1} \end{tikzcd} \quad \text{in C,} \tag{9.9}$$

and remark that g' satisfies $g'\psi_{n-1} = \phi''_K|g'_{n-1}|$.

Proof: We first show that we can replace g_{n-1} by a cofibration. For this consider the construction of the following diagram:

$$\begin{tikzcd} X_n \arrow[r, hook, "g''_{n-1}"] \arrow[d, "f_n"'] & \hat{E}_{n+1} \arrow[r, "p'"] \arrow[d, "\hat{h}"] & \tilde{E}_{n+1} \arrow[d, "\tilde{h}"] \\ X_{n-1} \arrow[r, hook, "g'_{n-1}"'] & \hat{K}_{n+1} \arrow[r, "p"'] & \tilde{K}_{n+1} \end{tikzcd} \quad \text{in } S_0.$$

Here g_{n-1} = the composition $X_{n-1} \overset{g'_{n-1}}{\hookrightarrow} \hat{K}_{n+1} \xrightarrow{p} \tilde{K}_{n+1}$ of a cofibration g'_{n-1} followed by a trivial fibration p (note that

$\hat{K}_{n+1}$ is necessarily fibrant since $\tilde{K}_{n+1}$ is). $\hat{E}_{n+1}$ is a pullback in S_o of $\tilde{h}$ along p; and so p' is a trivial fibration. There is an obvious map $X_n \xrightarrow{g''_{n-1}} \hat{E}_{n+1}$ into the pullback $\hat{E}_{n+1}$ such that the original map $X_n \longrightarrow \tilde{E}_{n+1}$ = the composition $X_n \xrightarrow{g''_{n-1}} \hat{E}_{n+1} \xrightarrow{p'} \tilde{E}_{n+1}$. Note that the left square in the above diagram is a pullback (using the basic properties of pullbacks in a category). And so f_n is a pullback of the fibration $\hat{h}$ along the cofibration g'_{n-1}. We now show that $\hat{h}$ in S_o is related to h in C. For this consider the commutative diagram:

$$\begin{tikzcd}
{|\hat{E}_{n+1}|} \arrow[r, "|p'|"] \arrow[d, "|\hat{h}|"'] & {|\tilde{E}_{n+1}|} \arrow[r, "\phi_E"] \arrow[d, "|\tilde{h}|"'] & E_{n+1} \arrow[d, "h"] \\
{|\hat{K}_{n+1}|} \arrow[r, "|p|"'] & {|\tilde{K}_{n+1}|} \arrow[r, "\phi_K"'] & K_{n+1}
\end{tikzcd} \qquad \text{in C.}$$

Let $\phi'_K = \phi_K|p|$ and $\phi'_E = \phi_E|p'|$; and since ϕ'_K and ϕ'_E are weak equivalences (each map is the composition of weak equivalences), we have that $\hat{h}$ is related to h (via ϕ'_K and ϕ'_E).

We now continue the proof. Note that $\overline{\phi'_K|g'_{n-1}|} \in [|X_{n-1}|, K_{n+1}]$. But $(\psi_{n-1})^*: [Y_{n-1}, K_{n+1}] \longrightarrow [|X_{n-1}|, K_{n+1}]$ is a bijection in Sets since ψ_{n-1} is a weak equivalence and K_{n+1} is fibrant. In particular, $(\psi_{n-1})^*$ is surjective; and so there is a map $g': Y_{n-1} \longrightarrow K_{n+1}$ such that $g'\psi_{n-1} \simeq \phi'_K|g'_{n-1}|$. Using the homotopy extension dual to lemma 5.12, there is a map $\phi''_K: |\hat{K}_{n+1}| \longrightarrow K_{n+1}$ with $\phi''_K \simeq \phi'_K$ such that $g'\psi_{n-1} = \phi''_K|g'_{n-1}|$. Note that ϕ''_K is a weak equivalence since ϕ'_K is (i.e., since homotopic maps in C induce the same map on homology and $H_*(\phi'_K)$ is an isomorphism). Now since ϕ'_E is a lift of $\phi'_K|\hat{h}|$ over h, there is a lift $\phi''_E:|\hat{E}_{n+1}|\longrightarrow E_{n+1}$ of $\phi''_K|\hat{h}|$ over h with $\phi''_E \simeq \phi'_E$ (using theorem 5.13, part a). And note that ϕ''_E is a weak equivalence since ϕ'_E is; and so $\hat{h}$ is also related to h via ϕ''_K and ϕ''_E. Now let $\tilde{g} = g'\psi_{n-1}$, and let $\tilde{Y}_n \longrightarrow |X_{n-1}|$ be a pullback of $h:E_{n+1}\longrightarrow K_{n+1}$ along $\tilde{g}:|X_{n-1}| \longrightarrow K_{n+1}$. Since we also have $\tilde{g} = \phi''_K|g'_{n-1}|$, the composition $|X_n| \xrightarrow{|g''_{n-1}|} |\hat{E}_{n+1}| \xrightarrow{\phi''_E} E_{n+1}$ is a lift of $\tilde{g}|f_n|$ over h giving an obvious map $\tilde{\phi}_n:|X_n|\longrightarrow \tilde{Y}_n$ into the pullback $\tilde{Y}_n$, which makes

$$\begin{tikzcd} |X_n| \arrow[rr, "\tilde{\phi}_n"] \arrow[dr, "|f_n|"'] & & \tilde{Y}_n \arrow[dl] \\ & |X_{n-1}| & \end{tikzcd}$$

commutative. Note that we can now apply our basic theorem 9.2 to conclude that $\tilde{\phi}_n$ is a weak equivalence.

Now let $Y_n \longrightarrow Y_{n-1}$ be a pullback of $h:E_{n+1}\longrightarrow K_{n+1}$ along $g':Y_{n-1}\longrightarrow K_{n+1}$. Using our map $|X_n| \xrightarrow{|g''_{n-1}|} |\hat{E}_{n+1}| \xrightarrow{\phi''_E} E_{n+1}$, we obtain an obvious map $\psi_n:|X_n| \dashrightarrow Y_n$ into the pullback Y_n. We now show that ψ_n is a weak equivalence. For this, using the fact that $\tilde{g} = g'\psi_{n-1}$, we obtain an obvious map $\tilde{Y}_n \dashrightarrow Y_n$ into the pullback Y_n. We also have that $\tilde{Y}_n \dashrightarrow |X_{n-1}|$ is a pullback of the fibration $Y_n \dashrightarrow Y_{n-1}$ along $\psi_{n-1}:|X_{n-1}| \dashrightarrow Y_{n-1}$ using the basic properties of pullbacks in a category. And so $\tilde{Y}_n \dashrightarrow Y_n$ is a weak equivalence since it is a pullback of the weak equivalence ψ_{n-1} along a fibration which has a fibrant codomain. Finally, we note that $\psi_n = |X_n| \xrightarrow{\tilde{\phi}_n} \tilde{Y}_n \dashrightarrow Y_n$ (since this composition satisfies the same conditions which determine ψ_n). But these two maps are weak equivalences, and so we can conclude that ψ_n is a weak equivalence.

We remark that the replacement of the map g_{n-1} by a cofibration g'_{n-1} is not necessary in order that lemma 9.8 complete the proof of theorem 9.7. However, in the proof of the Main Theorem 10.3 the adjoint maps $\tilde{\gamma}_n : X_n \longrightarrow SY_n$ are shown (in a special case) to give weak equivalences (see diagram (10.1)), and for this we will make use of the fact that the γ_n complete commutative diagrams which have the form of (9.9).

10. <u>The Main Theorem</u>.

Let W be in S_0. W is <u>rational</u>-- also <u>Q-local</u> (see [1], § 2)-- if $f:X \longrightarrow X'$ in S_0 is any map such that $f_*:H_*(X;Q) \longrightarrow H_*(X';Q)$ is an isomorphism, then f induces a bijection $f^*:HoS_0(X',W) \longrightarrow HoS_0(X,W)$.

Note that when W is also fibrant, we have $f^*:HoS_0(X',W) \longrightarrow HoS_0(X,W)$ is a bijection $\Leftrightarrow$ $f^*:[X',W] \longrightarrow [X,W]$ is a bijection (recall diagram (5.2) in the proof of prop. 5.2).

<u>Proposition</u> 10.1: Let $f:X \longrightarrow Y$ in S_0 with X and Y both rational and fibrant. If $f_*:H_*(X;Q) \longrightarrow H_*(Y;Q)$ is an isomorphism, then f is a homotopy equivalence.

Proof: Since X is rational, f induces a bijection $f^*:[Y,X] \longrightarrow [X,X]$. And so there is a map $g:Y \longrightarrow X$ with $gf \sim id_X$. We now show that g is also a right homotopy inverse for f. For this, first note that since $H_*(\_):S_0 \longrightarrow$ graded abelian groups takes weak equivalences to isomorphisms, $H_*(\_)$ takes homotopic maps in S_0 to the same map on homology. And so since $gf \sim id_X$, we obtain $(g)_*(f)_* = (gf)_* = (id_X)_* = id_{H_*X}:H_*X \longrightarrow H_*X$; i.e., g_* is a left inverse for $(f)_*$. The same is true when we consider $H_*(\_;Q)$; but then $f_* = H_*(f;Q)$ is an isomorphism, and so $g_* = H_*(g;Q)$ is an inverse for f_* and is, thus, an isomorphism. Now using Y rational, we note that g induces a bijection $g^*:[X,Y] \longrightarrow [Y,Y]$. And so there is a map $h:X \longrightarrow Y$ with $hg \sim id_Y$. But we then have $h \sim h(gf) = (hg)f \sim id_Y f = f$; i.e., $h \sim f$. But then $id_Y \sim hg \sim fg$; i.e., $id_Y \sim fg$. And so we can conclude that g is a homotopy inverse for f.

From now on we will assume that the ground field is the rationals: $k = Q$.

Proposition 10.2: If Y is fibrant in C, then $\mathcal{S}Y$ is rational in S_0.

Proof: Let $f:X \longrightarrow X'$ be in S_0 such that $H_*(f;Q):H_*(X;Q) \longrightarrow H_*(X';Q)$ is an isomorphism. Recall that if $H_*(f;Q)$ is an isomorphism, then $|f|$ is a weak equivalence in C. Now consider the commutative diagram:

$$\begin{tikzcd} HoS_0(X',\mathcal{S}Y) \arrow[r, "f^*"] & HoS_0(X,\mathcal{S}Y) \\ HoC(|X'|,Y) \arrow[r, "|f|^*"'] \arrow[u] & HoC(|X|,Y) \arrow[u] \end{tikzcd} \quad \text{in Sets.}$$

The adjunction for $| \quad | \dashv \mathcal{S}'$ gives the vertical bijections (recall for Y fibrant, $\mathcal{S}'Y = \mathcal{S}Y$). Since $|f|$ is a weak equivalence in C, the bottom map $|f|^*$ is a bijection. We can, thus, conclude that f^* is a bijection; and so $\mathcal{S}Y$ is rational in S_0.

Let $\tilde{C}$ be a subcategory of D, and let d be an object in D. A <u>universal map from d to $\tilde{C}$</u> is an object c in $\tilde{C}$ and a map $u:d \longrightarrow c$ such that given any map $f:d \longrightarrow c'$ (where c' is in $\tilde{C}$) in D, then there is a unique map $f':c \longrightarrow c'$ in $\tilde{C}$ such that $f'u = f$. Diagramatically this gives the commutative diagram:

$$\begin{tikzcd} d \arrow[r, "u"] \arrow[dr, "f"'] & c \arrow[d, dashed, "f'"] \\ & c' \end{tikzcd} \quad \text{in } D.$$

We remark that in the terminology of [11], this is a universal map from d to the inclusion functor $\tilde{C} \longrightarrow D$.

Let $\tilde{C}$ be a subcategory of D. $\tilde{C}$ is a <u>reflective</u> (or <u>reflexive</u>) subcategory if the inclusion functor $J:\tilde{C} \longrightarrow D$ has a left adjoint $F:D \longrightarrow \tilde{C}$. We remark that when $\tilde{C}$ is also a full subcategory of D, then each c-component of the counit $\varepsilon_c:Fc \longrightarrow c$ is an isomorphism.

These two notions are equivalent in the following sense: first note that the adjunction for $F \dashv J$ (with unit $\eta:id_D \xrightarrow{\cdot} JF$) when applied to a map $f':Fd \longrightarrow c'$ in $\tilde{C}$ gives an adjoint map $\widetilde{f'}$ in D which can be expressed as $\widetilde{f'} = J(f')\eta_d = f'\eta_d:d \longrightarrow Jc'$ $(=c')$. And so for each d in D, the d-component of the unit $\eta_d:d \longrightarrow JFd$ $(=Fd)$ for the adjunction (when corestricted to $\tilde{C}$) is a universal map from d to $\tilde{C}$. Conversely, if there is a universal map $u:d \longrightarrow c$ from d to $\tilde{C}$ for each d in D, then there is an obvious functor $F:D \longrightarrow \tilde{C}$ (determined by $Fd = c$) which is left adjoint to the inclusion functor $J:\tilde{C} \hookrightarrow D$; i.e., $\tilde{C}$ is a reflective subcategory of D.

Now let $NhoS_o$ (respectively, $QNhoS_o$) denote the full subcategory of hoS_o determined by the nilpotent (respectively, rational and nilpotent) objects.

A <u>Q-localization</u> functor is a left adjoint $L:NhoS_o \longrightarrow QNhoS_o$ to the inclusion functor $J:QNhoS_o \hookrightarrow NhoS_o$.

As mentioned in the introduction, Bousfield and Kan [3] show the existence of a related Q-localization $L_*:NhoS_* \longrightarrow QNhoS_*$ where S_* is the category of pointed, connected simplicial sets. We claim that S_* and S_R (and, hence, S_o) have equivalent homotopy categories. The existence of L_* then implies the existence of a Q-localization $L: NhoS_o \longrightarrow QNhoS_o$. The proof of the next theorem uses the existence of such a Q-localization.

Main Theorem 10.3: Let X be nilpotent and fibrant in S_o, and let $i: |X| \longrightarrow X'$ be a weak equivalence in C with X' fibrant.

Then the adjoint map $\tilde{i}: X \longrightarrow \mathcal{S}X'$ in S_o gives an isomorphism $H_*(X;Q) \longrightarrow H_*(\mathcal{S}X';Q)$.

Proof: We consider the natural Postnikov system of X (see, e.g., [8], [12]) and observe that this tower of maps is in S_o (since each Postnikov section $X^{(k)}$ has $((X^{(k)})_q = X_q$ if $q \leq k$). First note that it suffices to prove this theorem where X is also assumed to be rational (by considering a Q-localization $u: X \longrightarrow LX$ -- which necessarily gives a Q-homology isomorphism -- and observing that $|u|$ is a weak equivalence in C; one also uses the fact that if $i_j: |X| \longrightarrow X_j'$ is a weak equivalence in C with X_j' fibrant ($j = 1, 2$), then there is a homotopy equivalence $h: X_1' \longrightarrow X_2'$ such that $hi_1 \sim i_2$). For such an X there is a tower of principal fibrations under X with a finite number of fibrations, which are induced from principal fibrations of the form $\tilde{h}: \tilde{E}(\pi,n) \longrightarrow \tilde{K}(\pi,n)$ (where $n>1$ and π is a Q-module), occurring between any two consecutive Postnikov sections of X ([3], III, prop. 5.3). We can, thus, apply the construction of theorem 9.7 to obtain a tower of fiber bundles in C which is weakly equivalent to the realization of this tower from S_o. In what follows we retain most of the notation of this theorem.

Let $Y = \lim\limits_{\overleftarrow{n}} Y_n$ in C (note Y is fibrant), and observe that there is a unique map $\Psi: |X| \dashrightarrow Y$ into the limit Y such that $|X| \xrightarrow{\Psi} Y \dashrightarrow Y_n = |X| \dashrightarrow |X_n| \xrightarrow{\Psi_n} Y_n$ for all n. Recall if $q_{n_*}: X \dashrightarrow X_{n_*}$ is an $n_*{}^{th}$ Postnikov section map, then $(q_{n_*})_*: H_k X \dashrightarrow H_k X_{n_*}$ is an isomorphism for all $k < n_*$ (since $(q_{n_*})_*: \pi_k X \dashrightarrow \pi_k X_{n_*}$ is an isomorphism for all $k \leq n_*$). Further, any $n_*'{}^{th}$ Postnikov section map $q_{n_*'}$, where $n_*' > n_*$, gives a homology isomorphism as above which includes all $k < n_o$, and the intermediate section maps q_n, where $n > n_*$, of the refined Postnikov system do as well. And so for k fixed, q_n-- and, hence, $|q_n|$-- induce homology isomorphisms as above for n sufficiently large (namely equal to or beyond the $k+1^{st}$ Postnikov section in the tower). Next using the weak equivalences $\Psi_n: |X_n| \dashrightarrow Y_n$, we have that $H_k Y_n$ also stabilize as n increases; and so we have that $Y \dashrightarrow Y_n$ gives a homology isomorphism in degree k (and smaller) again for n sufficiently large. We can, thus, conclude that $\Psi: |X| \dashrightarrow Y$ is a weak equivalence.

We next show that the adjoint map $\tilde{\Psi}: X \dashrightarrow \mathcal{S}Y$ is a weak equivalence. For the weak equivalences $\Psi_n: |X_n| \dashrightarrow Y_n$ in C, consider the adjoint maps $\tilde{\Psi}_n: X_n \dashrightarrow \mathcal{S}Y_n$ in S_o. Using induction on n, we now show that these maps are also weak equivalences. Note that for $n = 1$, in the construction of theorem 9.7, $\Psi_1: |X_1| \dashrightarrow Y_1$ is the weak equivalence $\phi_K: |\tilde{K}(\pi_1,1)| \dashrightarrow K(\pi_1,1)$ between the fibers which has $\tilde{\phi}_K$ a weak equivalence. We next assume $\tilde{\Psi}_{n-1}$ is a weak equivalence and show that $\tilde{\Psi}_n$ is one as well. To this end, consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{S}Y_n \arrow[rrr, "\mathcal{S}(g'')"] \arrow[ddd, "\mathcal{S}(h')"'] & & & \mathcal{S}E(\pi_n, j_n+1) \arrow[ddd, "\mathcal{S}(h)"] \\
& X_n \arrow[ul, "\widetilde{\gamma}_n"'] \arrow[r, hook, "g''_{n-1}"] \arrow[d, "f_n"'] & \hat{E}_{n+1} \arrow[ur, "\widetilde{\phi}''_E"] \arrow[d, "\hat{h}"] & \\
& X_{n-1} \arrow[dl, "\widetilde{\gamma}_{n-1}"] \arrow[r, hook, "g'_{n-1}"] & \hat{K}_{n+1} \arrow[dr, "\widetilde{\phi}''_K"'] & \\
\mathcal{S}Y_{n-1} \arrow[rrr, "\mathcal{S}(g')"'] & & & \mathcal{S}K(\pi_n, j_n+1)
\end{tikzcd} \quad \text{in } S_o, \qquad (10.1)$$

where $j_n > 0$; we remark that j_n remains constant when the maps f_n are between two consecutive Postnikov sections of X. With $Y_n \longrightarrow Y_{n-1}$ a pullback of $h: E(\pi_n, j_n+1) \longrightarrow K(\pi_n, j_n+1)$ in C, the outer square is obtained as the image under $\mathcal{S}$ of this pullback and is a pullback in S_o (since $\mathcal{S}$ preserves pullbacks). The diagonal maps in this diagram are adjoints of weak equivalences in C (recall diagrams (9.8) and (9.9) of lemma 9.8); the naturality of the adjunction for $|\ | \dashv \mathcal{S}$ gives commutativity to the four trapezoids.

We claim that $\widetilde{\phi}''_K: \hat{K}_{n+1} \longrightarrow \mathcal{S}K(\pi_n, j_n+1)$ and $\widetilde{\phi}''_E: \hat{E}_{n+1} \longrightarrow \mathcal{S}E(\pi_n, j_n+1)$ are weak equivalences in S_o. For this, recall from the proof of lemma 9.8 that $\phi''_K \sim \phi'_K: |\hat{K}_{n+1}| \longrightarrow K_{n+1}$ where ϕ'_K is the composition $|\hat{K}_{n+1}| \xrightarrow{|p|} |\tilde{K}_{n+1}| \xrightarrow{\phi_K} K_{n+1}$ in C (with $\tilde{K}_{n+1} = \tilde{K}(\pi_n, j_n+1)$ in S_o and $K_{n+1} = K(\pi_n, j_n+1)$ in C). And so the adjoint map $\widetilde{\phi}'_K$ is the composition $\hat{K}_{n+1} \xrightarrow{p} \tilde{K}_{n+1} \xrightarrow{\widetilde{\phi}_K} \mathcal{S}K_{n+1}$ in S_o. With $\widetilde{\phi}_K$ a weak equivalence (by lemma 8.3) and p also such, we have that $\widetilde{\phi}'_K$ is a weak equivalence; and, thus, so is $\widetilde{\phi}''_K$ (since $\widetilde{\phi}''_K \sim \widetilde{\phi}'_K$ by lemma 7.8). The treatment of $\widetilde{\phi}''_E$ is similar. Noting that $\mathcal{S}(h)$ and $\hat{h}$ are fibrations with fibrant bases, we can apply the picture frame lemma 5.11 to conclude that $\tilde{\gamma}_n : X_n \longrightarrow \mathcal{S}Y_n$ is a weak equivalence.

We next consider the limits $\underset{\longleftarrow n}{\lim}(\tilde{\gamma}_n) : \underset{\longleftarrow n}{\lim} X_n \longrightarrow \underset{\longleftarrow n}{\lim} \mathcal{S}Y_n$ and note that this is the limit of weak equivalences between two towers of surjective fibrations in S_o; and so using the fact that $S_o \hookrightarrow S$ preserves limits and takes surjective fibrations in S_o to fibrations in S, we can conclude that this limit is a weak equivalence in S and, hence, in S_o. Next observe that $\mathcal{S}Y = \mathcal{S}(\underset{\longleftarrow n}{\lim} Y_n) = \underset{\longleftarrow n}{\lim} \mathcal{S}Y_n$. Then considering the composition $X \longrightarrow \underset{\longleftarrow n}{\lim} X_n \longrightarrow \underset{\longleftarrow n}{\lim} \mathcal{S}Y_n = \mathcal{S}Y$ in S_o (the first map is a homotopy equivalence), we can conclude that $X \longrightarrow \mathcal{S}Y$ is a weak equivalence; further, it is easily shown that this map is the adjoint of the map $\gamma : |X| \longrightarrow Y$ which we considered previously.

Finally, using this result, it follows that the adjoint map of any weak equivalence $i : |X| \longrightarrow X'$, where X' is fibrant, is a weak equivalence (as in the reduction step at the beginning of this proof, we use the fact that the two fibrant models for $|X|$-- X' and Y-- have the same homotopy type in C and then consider the application of the adjunction on the resulting triangle).

<u>Proposition</u> 10.4: Let X be nilpotent and fibrant in S_o, and let $|X| \longrightarrow X'$ be a weak equivalence in C with X' fibrant. Then $\mathcal{S}X'$ is nilpotent.

Proof: The proof of this proposition uses the same construction which is used in proving the Main Theorem. As in the proof of that theorem, it suffices to prove this proposition where X is also assumed to be rational (again by considering a Q-localization $u:X \longrightarrow LX$). Using that construction we obtain a weak equivalence $\gamma:|X| \longrightarrow Y$ in C which has an adjoint map $\tilde{\gamma}:X \longrightarrow \mathcal{S}Y$ in S_0 which is also a weak equivalence. With X nilpotent, we can, thus, conclude that $\mathcal{S}Y$ is nilpotent.

Finally, using this result, it follows that if $|X| \longrightarrow X'$ is any weak equivalence in C, where X' is fibrant, then $\mathcal{S}X'$ is nilpotent.

<u>**Corollary 1**</u>: Let X be in $NhoS_0$, and let $i:|X| \longrightarrow X'$ be a weak equivalence in C with X' fibrant.

Then the X-component of the unit $\eta_X = \tilde{\tilde{i}}:X \longrightarrow \mathcal{S}X'$ in hoS_0 is a universal map from X to $QNhoS_0$.

And, hence, the functor $L:NhoS_0 \longrightarrow QNhoS_0$, determined by $L(X) = \mathcal{S}X'$, is a Q-localization.

Proof: Note that $\mathcal{S}X'$ is rational and nilpotent (and also fibrant) using props. 10.2 and 10.4. Now let W be in $QNhoS_0$, and let $f:X \longrightarrow W$. By the Main Theorem, the adjoint map $\tilde{i}$: $X \longrightarrow \mathcal{S}X'$ gives a Q-homology isomorphism; and since W is rational, $\tilde{i}$ induces a bijection $(\tilde{i})^*:[\mathcal{S}X',W] \longrightarrow [X,W]$. And so there is a map $g:\mathcal{S}X' \longrightarrow W$, which is unique up to homotopy, with $g\tilde{i} \sim f$. And so when viewed as a map in $NhoS_0$, f factors uniquely through $\mathcal{S}X'$ via $\tilde{i}$; i.e., $\tilde{i}:X \longrightarrow \mathcal{S}X'$ gives a universal map in $NhoS_0$ from X to $QNhoS_0$.

Let S be a subclass of morphisms of a category $\tilde{C}$. A <u>**weak**</u> category of fractions of $\tilde{C}$ with respect to S is a category, denoted $w\tilde{C}[S^{-1}]$, together with a functor $\gamma:\tilde{C} \longrightarrow$

$w\tilde{C}[S^{-1}]$ which satisfies a) $\gamma(s)$ is an isomorphism for all s in S, and b) if a functor $F:\tilde{C} \longrightarrow D$ has the property that $F(s)$ is an isomorphism in D for all s in S, then there is a functor $F':w\tilde{C}[S^{-1}] \longrightarrow D$, which is unique up to natural isomorphism, such that $F'\gamma$ is naturally isomorphic to F.

Proposition 10.5: Let $\tilde{C}$ be a full subcategory of D with $J:\tilde{C} \hookrightarrow D$ the inclusion functor. Assume that $\tilde{C}$ is a reflective subcategory with $L:D \longrightarrow \tilde{C}$ left adjoint to J. Then $L:D \longrightarrow \tilde{C}$ makes the subcategory $\tilde{C}$ a weak category of fractions of D with respect to ker(L), where ker(L) = {f in D| L(f) is an isomorphism}.

Proof: Let $F:D \longrightarrow E$ be a functor which inverts ker(L). Define $F':\tilde{C} \longrightarrow E$ by $F' = FJ$ (where $L \dashv J$); and so we have the diagram:

$$\begin{tikzcd} D \arrow[d, "L"'] \arrow[dr, "F"] & \\ \tilde{C} \arrow[u, bend left=60, "J"] \arrow[r, dashed, "F'"'] & E \end{tikzcd} .$$

We claim that $F'L$ is naturally isomorphic to F. For this, note that since $\tilde{C}$ is a full subcategory of D, the inclusion functor J is full and faithful; and, thus, the adjunction for $L \dashv J$ has a counit $\varepsilon:LJ \xrightarrow{\cdot} id_{\tilde{C}}$ which is a natural isomorphism. Now let X be in D, and, using one of the triangle identities, we obtain $\varepsilon_{LX}L(\eta_X) = id_{LX}$ in $\tilde{C}$; but ε_{LX} is an isomorphism, and so $L(\eta_X)$ is an isomorphism. Thus, η_X is in Ker(L); and so η_X is inverted by F (for any X in D). Hence, since $F\eta:FJL \xleftarrow{\cdot} F$ is a natural isomorphism, we have that $F'L = FJL \xleftarrow{\approx} F$ is a natural isomorphism between $F'L$ and F.

We now show that F' is unique up to natural isomorphism. For this, assume that we have another functor $F'': \tilde{C} \longrightarrow E$ and a natural isomorphism $\lambda'': F''L \xrightarrow{\approx} F$. Note that $\lambda''_J: F''LJ \xrightarrow{\approx} FJ$ is also a natural isomorphism. Now recall that the counit $\varepsilon: LJ \xrightarrow{\approx} id_{\tilde{C}}$ is a natural isomorphism; and letting $\sigma = \varepsilon^{-1}$, we obtain the natural isomorphism $F''\sigma: F'' \xrightarrow{\approx} F''LJ$. Thus, the composition $F'' \xrightarrow{\approx} F''LJ \xrightarrow{\approx} FJ = F'$ is a natural isomorphism between F'' and F'.

Note, using the above proposition, we have that the Q-localization functor $L: NhoS_o \longrightarrow QNhoS_o$ makes $QNhoS_o$ a weak category of fractions of $NhoS_o$ with respect to ker(L); further, we have that $\ker(L) = \{\bar{f}$ in $NhoS_o \mid f$ gives a Q-homology isomorphism$\}$ (for this result, apply the unit to $\bar{f}$ and use prop. 10.1-- recall that $u: X \longrightarrow LX$ gives a Q-homology isomorphism).

<u>Corollary 2</u>: Let X be nilpotent, rational, and fibrant in S_o. Let $i: |X| \longrightarrow X'$ be a weak equivalence in C with X' fibrant.

Then the adjoint map $\tilde{i}: X \longrightarrow \mathcal{S}X'$ in S_o is a homotopy equivalence.

Proof: By the Main Theorem, the map $\tilde{i}: X \longrightarrow \mathcal{S}X'$ gives a Q-homology isomorphism. X is assumed to be rational; and since X' is fibrant, $\mathcal{S}X'$ is fibrant and also (by prop. 10.2) rational. And so by prop. 10.1, we can conclude that $\tilde{i}$ is a homotopy equivalence.

We remark that this corollary occurred as a special case in the proof of the Main Theorem, and we have treated it as a formal corollary by providing another proof.

We will use cor. 2 to show that the functors $hoS_0 \underset{S}{\overset{\perp\perp'}{\rightleftarrows}} hoC$ when restricted and corestricted to appropriate subcategories give an equivalence of categories. For this, we first recall a familiar result.

__Theorem__ 10.6 (see, e.g., [11]): Let $\tilde{C} \underset{G}{\overset{F}{\rightleftarrows}} D$ be a pair of adjoint functors with $F \dashv G$. Then G is full and faithful $\Leftrightarrow$ the counit $\varepsilon: FG \xrightarrow{\cdot} id_D$ is a natural isomorphism.

__Corollary__ 10.7: For F and G as in the above theorem, then F is full and faithful $\Leftrightarrow$ the unit $\eta: id_{\tilde{C}} \xrightarrow{\cdot} GF$ is a natural isomorphism.

Proof: This result is immediate: by considering the opposite categories, one observes that for $\tilde{C}^{op} \underset{G^{op}}{\overset{F^{op}}{\rightleftarrows}} D^{op}$ one has $G^{op} \dashv F^{op}$, and the unit $\eta: id_{\tilde{C}} \xrightarrow{\cdot} GF$ for $F \dashv G$ induces the counit $\eta^{op}: G^{op}F^{op} \xrightarrow{\cdot} id_{\tilde{C}^{op}}$ for $G^{op} \dashv F^{op}$. The result then follows from the above theorem (since F^{op} is full (faithful) $\Leftrightarrow$ F is full (faithful)).

A subcategory $\tilde{C}' \subset \tilde{C}$ is __replete__ if $c' \longrightarrow c$ is an isomorphism in C (where c' is in $\tilde{C}'$) implies that c is in $\tilde{C}'$.

__Proposition__ 10.8: Let $\tilde{C}' \subset \tilde{C}$ be a full subcategory, and let $\tilde{C}'' \subset \tilde{C}$ be the full replete subcategory of $\tilde{C}$ generated by the objects of $\tilde{C}'$. Then the inclusion functor $J: \tilde{C}' \hookrightarrow \tilde{C}''$ is an equivalence of categories.

Proof: The functor $F: \tilde{C}'' \longrightarrow \tilde{C}'$ determined by $F(c'') = c'$

(where $c' \xrightarrow{\simeq} c''$ in $\tilde{C}$) is left adjoint to J, and the unit and counit for $F \dashv J$ are easily shown to be natural isomorphisms.

The next lemma is useful to insure (if necessary) that the condition in prop. 10.10 concerning the unit is satisfied.

__Lemma__ 10.9: Let $\tilde{C} \underset{G}{\overset{F}{\rightleftarrows}} D$ be a pair of adjoint functors with $F \dashv G$. Assume $\tilde{C}' \subset \tilde{C}$ is a full subcategory such that each c'-component of the unit $\eta_{c'}: c' \longrightarrow GFc'$ is an isomorphism (c' in $\tilde{C}'$). Then $\eta_{c''}: c'' \longrightarrow GFc''$ is an isomorphism for all c'' in $\tilde{C}''$ (= the full replete subcategory of $\tilde{C}$ generated by $\tilde{C}'$).

Proof: This follows immediately from the naturality of η applied to the map $c' \xrightarrow{\simeq} c''$ in $\tilde{C}$.

__Proposition__ 10.10: Let $\tilde{C} \underset{G}{\overset{F}{\rightleftarrows}} D$ be a pair of adjoint functors with $F \dashv G$. Assume that $\tilde{C}' \subset \tilde{C}$ is a full replete subcategory such that $\eta_{c'}: c' \longrightarrow GFc'$ is an isomorphism for all c' in $\tilde{C}'$. Then the restriction $F: \tilde{C}' \longrightarrow D$ is full and faithful, and the d'-component of the counit $\varepsilon_{d'}: FGd' \longrightarrow d'$ is an isomorphism for all d' in $D' = F(\tilde{C}')$. Hence, the adjoint pair $\tilde{C}' \underset{G}{\overset{F}{\rightleftarrows}} F(\tilde{C}')$ gives an equivalence of categories.

Proof: First let $F(\tilde{C}')_{fu}$ be the full subcategory of D generated by $F(\tilde{C}')$, and consider the restriction $G: F(\tilde{C}')_{fu} \longrightarrow \tilde{C}$. Note that G also corestricts to $\tilde{C}'$ (since $\tilde{C}'$ is replete and the unit η gives an isomorphism in $\tilde{C}'$). We, thus, have that $\tilde{C}' \underset{G}{\overset{F}{\rightleftarrows}} F(\tilde{C}')_{fu}$ is an adjoint pair of functors. And so $F: \tilde{C}' \longrightarrow F(\tilde{C}')_{fu}$ is full and faithful (since the unit for this adjunction is a natural isomorphism); in particular, $F(\tilde{C}')_{fu} = F(\tilde{C}')$. Next consider the triangle identity $\varepsilon_{Fc'} F(\eta_{c'}) = id_{Fc'}$ (c' in $\tilde{C}'$), and, noting that $F(\eta_{c'})$ is an isomorphism, we obtain our desired conclusion that $\varepsilon_{Fc'}$ is an isomorphism (c' in $\tilde{C}'$).

In the proof of the following theorem, we will use the observation that $QNhoS_0$ is a replete subcategory of hoS_0.

__Theorem__ 10.11: The adjoint functors $hoS_0 \underset{\mathcal{S}}{\overset{|\;|'}{\rightleftarrows}} hoC$ restrict and corestrict to give an equivalence of categories: $QNhoS_0 \underset{\mathcal{S}}{\overset{|\;|'}{\rightleftarrows}} |QNhoS_0|'$.

Proof: Let X be in $QNhoS_0$, and let $i:|X| \longrightarrow X'$ be a weak equivalence in C with X' fibrant. Then by cor. 2 the X-component of the unit $\eta_X = \tilde{i}:X \longrightarrow \mathcal{S}X'$ is an isomorphism in hoS_0. With $QNhoS_0$ a replete subcategory of hoS_0, we can apply prop. 10.10 to obtain the above equivalence of categories.

The treatment in this paper which is given for $|\;|:S_0 \longrightarrow C$ uses only the three basic properties of the realization functor (these properties are discussed in Section 4), and, hence, the results of this development are valid for any functor $F:S_0 \longrightarrow C$ which also has these properties. This characterization of $|\;|$ is stated in the next theorem.

__Theorem__ 10.12: Let $F:S_0 \longrightarrow C$ be a functor which has the same basic properties as does $|\;|:S_0 \longrightarrow C$; i.e., 0) F has a right adjoint, 1) F preserves monomorphisms, and 2) there is a natural transformation $\tau:F \xrightarrow{\cdot} C_*^N$ such that $(\tau_X)_*:H_*(FX) \longrightarrow H_*(X;k)$ is an isomorphism of coalgebras (X in S_0). Then the

Main Theorem 10.3, cor. 1, cor. 2, and theorem 10.11 are valid when $|\ |$ is replaced by F.

Appendix

11. The classification of cohomology by $K(\pi,n)$ in C.

In this section we show that there is a natural isomorphism $[X,K(\pi,n)] \longrightarrow H^n(X;\pi)$ in abelian groups where π is a k-module and X is in C. This is the analogue in C of the familiar result for spaces (see, e.g., [12]).

Let $\tilde{C}$ be a pointed model category with null object $*$. Let the null map $X \longrightarrow * \longrightarrow Y$ between any two objects be denoted by 0_*. A map which is right (left) homotopic to 0_* is <u>right (left) null homotopic</u>. If X is cofibrant and Y fibrant, then $\sim_R \Leftrightarrow \sim_L$, and such a map is simply called <u>null homotopic</u>. X is <u>right (left) contractible</u> if $id_X \sim_R 0_*$ ($id_X \sim_L 0_*$). When X is both fibrant and cofibrant, the two notions agree, and when either is satisfied, X is <u>contractible</u>.

The next two statements are both familiar and immediate: X is right (left) contractible (with X fibrant) $\Leftrightarrow$ any map into X is right (left) null homotopic. This implies that if X is right (left) contractible and also fibrant, then any two maps into X (with cofibrant domain) are right (left) homotopic. One can also characterize a contractible object X in terms of maps out of X: X is left (right) contractible (with X cofibrant) $\Leftrightarrow$ any map out of X is left (right) null homotopic; and so if X is left (right) contractible and also cofibrant, then any two maps out of X (with fibrant codomain) are left (right) homotopic.

<u>Lemma</u> 11.1: Let $\tilde{C}$ be a pointed model category, and let X be cofibrant. Let $f:X \longrightarrow Y$ be left (right) null homotopic (with

Y fibrant), then for any fibration $h:W \longrightarrow Y$ there is a map $g:X \longrightarrow W$ such that $hg = f$; i.e., the following diagram:

$$\begin{tikzcd} & W \arrow[d, "h"] \\ X \arrow[ur, dashed, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$ commutes.

Proof: Let $f:X \longrightarrow Y$ be left null homotopic; i.e., $f \sim_L 0_*$. But note $0_* = h0_*$ (i.e., $0_* = X \longrightarrow * \longrightarrow W \xrightarrow{h} Y$), and so $f \sim_L h0_*$. Now using our homotopy lifting lemma 5.12, there is a map $g:X \longrightarrow W$ such that $0_* \sim_L g$ and $hg = f$.

Let $\tilde{C}$ be a category with finite products and a terminal object e. Let $Gp(\tilde{C})$ denote the subcategory of group objects in $\tilde{C}$ (a <u>group object</u> in $\tilde{C}$ has a multiplication $\mu:G \pi G \longrightarrow G$, a unit $\eta:e \longrightarrow G$, and an inverse $i:G \longrightarrow G$ which make the familiar diagrams commutative (see, e.g., [11])). Let $Ab(\tilde{C}) \subset Gp(\tilde{C})$ denote the subcategory of abelian group objects. For G in $Gp(\tilde{C})$, it is easily shown that $\tilde{C}(X,G)$ is in $Gp(Sets)$ (the structure maps are given by $fg = X \xrightarrow{\langle f,g \rangle} G \pi G \xrightarrow{\mu} G$, $1 = X \longrightarrow e \xrightarrow{\eta} G$, and $f^{-1} = X \xrightarrow{f} G \xrightarrow{i} G$) and $\tilde{C}(X,\_):Gp(\tilde{C}) \longrightarrow Gp(Sets)$ is a functor. Also for G in $Ab(\tilde{C})$, then $\tilde{C}(X,G)$ is in $Ab(Sets)$.

<u>Lemma</u> 11.2: Let $\tilde{C}$ be a pointed model category (and so has finite products). Let G be in $Gp(\tilde{C})$. Assume G is fibrant and X cofibrant. Then $[X,G]$ has a group structure (induced by that of $\tilde{C}(X,G)$), and the obvious map $\pi:\tilde{C}(X,G) \longrightarrow [X,G]$ is a group map.

The proof of this lemma is straight forward (to show that the multiplication on $[X,G]$ is well-defined: note that for $g' \sim_L g:X \longrightarrow G$ via a homotopy $H:\tilde{X} \longrightarrow G$, then $f'g' \sim_L f'g$

via the homotopy $\tilde{H}$ = the composition $\tilde{X} \xrightarrow{\langle p, id_{\tilde{X}} \rangle} X \times \tilde{X} \xrightarrow{f' \times H} G \times G \xrightarrow{\mu}$ G; similarly $f'g \sim_L fg$ (where $f' \sim_L f$), and so $f'g' \sim_L fg$).

Lemma 11.3: Let $\tilde{C}$ be a pointed model category. Let $f:E \longrightarrow G$ be a fibration in $\tilde{C}$ with E left or right contractible and G fibrant. Also assume that E and G are in $Ab(\tilde{C})$ and that f is a morphism in $Ab(\tilde{C})$. Finally let X be cofibrant. Then

$$\tilde{C}(X,E) \xrightarrow{f_*} \tilde{C}(X,G) \xrightarrow{\pi} [X,G] \longrightarrow 0$$

is exact in abelian groups.

Proof: Since π is surjective, we need only show that $im(f_*) = ker(\pi)$. To see that $im(f_*) \subset ker(\pi)$, let $h \in im(f_*)$ and so $h = fg$ for some $g:X \longrightarrow E$. Note when E is, e.g., left contractible, we have $g \sim_L 0_*$; and so $fg \sim_L f0_*$. But $f0_* = 0_*$; i.e., h (=fg) is left null homotopic, and so $h \in ker(\pi)$. To see that $im(f_*) \supset ker(\pi)$, let $h \in ker(\pi)$; i.e., h is null homotopic. And so by lemma 11.1, $h = fg$ for some map $g:X \longrightarrow E$.

Let π be a k-module, and recall that $K(\pi,n)$ is the cofree coalgebra $S'(\pi_n)$ where π_n has π concentrated in degree n; and so a typical map in $C(X,K(\pi,n))$ is given by a coextension $\hat{g}$ which is uniquely determined by the chain map $g:X \longrightarrow \pi_n$.

Theorem 11.4: Let X be a coalgebra. Then the map

$$[X,K(\pi,n)] \xrightarrow{\Psi} H^n(X;\pi)$$

given by $\Psi(\bar{\hat{g}}) = \overline{\overline{g_n}}$ is an isomorphism in abelian groups which is natural in X.

Proof: a) We first observe that the map

$$C(X,K(\pi,n)) \xrightarrow{\Psi_1} Z^n(X;\pi)$$

given by $\psi_1(\hat{g}) = g_n$ is an isomorphism in abelian groups, where Z^n denotes the group of n-cocycles. For this note that, since π is concentrated in degree n, the chain map condition on g gives $g_n d_{n+1} = 0$ (where d is the differential on X). Thus, g_n is an n-cocycle. That ψ_1 is surjective is immediate, and that it is injective uses $\hat{g}$ is uniquely determined by the chain map, g.

b) We next observe that the map

$$C(X,E(\pi,n)) \xrightarrow{\psi_2} C^{n-1}(X;\pi)$$

given by $\psi_2(\hat{g}) = g_{n-1}$ is an abelian group isomorphism, where C^{n-1} denotes the group of (n-1)-cochains. Recall that $E(\pi,n)$ is the cofree coalgebra $S'(W_n)$ where W_n has π concentrated in degrees n and n-1 with the differential given by the identity. Thus, the chain map condition on g gives $g_n d_{n+1} = 0$ and $g_{n-1} d_n = g_n$. And so an (n-1)-cochain defines a unique chain map which gives the required surjective property of ψ_2. Again that a chain map determines a unique coextension $\hat{g}$ gives the required injective property of ψ_2.

c) We now apply the above lemma to the fibration $h:E(\pi,n) \longrightarrow K(\pi,n)$ to obtain an exact top row in the following diagram:

$$\begin{tikzcd}
C(X,E(\pi,n)) \arrow[r, "h_*"] \arrow[d, "\psi_2"] & C(X,K(\pi,n)) \arrow[r, "\pi_1"] \arrow[d, "\psi_1"] & {[X,K(\pi,n)]} \arrow[r] \arrow[d, dashed] & 0 \\
C^{n-1}(X;\pi) \arrow[r, "\delta_{n-1} = d_n^*"] & Z^n(X;\pi) \arrow[r, "\pi_2"] & H^n(X;\pi) \arrow[r] & 0
\end{tikzcd}$$

This diagram is in abelian groups, and note that the bottom row is also exact. The left two vertical arrows are from parts a) and b), and the left square is easily shown to be commutative. Since the top row is exact, we have $[X,K(\pi,n)] = \mathrm{cok}(h_*)$. And so, since the two maps in the bottom row compose to zero, there is a unique abelian group map $\Psi:[X,K(\pi,n)] \longrightarrow H^n(X;\pi)$ such that the right square commutes. But using exactness of the bottom row, we also have $H^n(X;\pi) = \mathrm{cok}(\delta_{n-1})$; and so, since Ψ_1 and Ψ_2 are isomorphisms, we can conclude that Ψ is an isomorphism. The naturality of Ψ is easy to show.

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