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FLAT POINTS ON CONTINUOUS ISOMETRIC DEFORMATIONS OF CONNECTED PLANAR REGIONS

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FLAT POINTS ON CONTINUOUS ISOMETRIC DEFORMATIONS OF CONNECTED PLANAR REGIONS

by

Martin Lewinter

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Abstract

FLAT POINTS ON CONTINUOUS ISOMETRIC DEFORMATIONS OF CONNECTED PLANAR REGIONS

by

Martin Lewinter

Adviser: Dr. Richard Sacksteder

The thesis demonstrates that the punctured unit disk whose rays are to be generators can't be embedded in R^3 as a developable surface consisting entirely of parabolic points. The method of proof involves the spherical image curve of the generators of the developable and the relationship of the occurence of flat points on the developable and the convexity of the spherical image curve of the generators.

The basic results are then generalized, after some modification, to augmented and diminished unit disks, i.e. disks whose rays emanating from their origins rotate through angles larger than and smaller than 2π, respectively.

The paper closes with a discussion of the embedding of an annulus. Three cases are discussed: monotonic, strictly monotonic, and non-monotonic rotation in the planer annulus of the pre-images of the generators.

Acknowledgements

I would like to thank my wife, Jane, for her constant encouragement. I wish to express my gratitude to Dr. Chavel for his helpful comments and corrections. Dr. Richard Sacksteder, my thesis adviser, gave me the necessary guidance and inspiration without which this paper could not have been written.

Finally, I would like to dedicate this paper to my parents, Bernhard and Esther, as a down payment on a debt that I can never fully repay.

Table of Contents

I. Introduction

In this paper we will present several theorems belonging to Differential Geometry in the large. The hypotheses and proofs of these theorems will make use of facts pertaining also to Differential Geometry in the small.

We will concern ourselves with the distribution of flat points on certain developable surfaces. In particular, we will employ the convexity (or non-convexity) of the generator spherical image curves of these surfaces to demonstrate the existence (or absence) of flat points on them.

On a developable surface, the Gaussian curvature K is identically zero. Put another way, if we denote the co-efficients of the second fundamental form by L, M, and N, as is done in the traditional manner (see [5] p. 86) , then $LN-M^2= 0$.

A flat point of a developable surface is a point at which L, M, and N are zero. As is well known (see [5] p. 124) , if one point of a generator is a flat point, then all of the points of that generator are flat points.

Consequently, we will in fact be searching for <u>generators</u> on the various developables consisting entirely of flat points.

One of the interesting features of flat points on developable surfaces is that they are the only possible umbilical points. This is easily seen by noting that the normal curvature at a point in any given direction is given by the second fundamental form divided by the first fundamental form. Thus $L = M = N = 0$ at a point implies that all the normal curvatures are zero there.

In particular, these normal curvatures are all equal. At such a point, we have a singularity in the direction field for the lines of curvature of the surface. This direction field is given by a system of differential equations involving the co-efficients of the first and second fundamental forms. See [5] Chapter IV, Section 16 for further details.

As stated above, our methods of proof in this paper will often involve the convexity (or non-convexity) of the generator spherical image curve of the surface under consideration.

The generator spherical image curve is obtained in the following manner. Consider a region R of a developable surface such that each point of R lies on a unique generator. Let $P(t)$ be the vector representation of a regular curve in R which crosses each generator exactly once. We want the curve P(t) to cross these generators transversally, i.e. P'(t) shall never lie in a generator direction.

Now construct a continuous vector field in the surface. Denote this field of vectors by Z(t). Let these vectors all be unit length. Furthermore, at each point of P(t), let Z(t) point in the direction of the generator through that point. Our surface can now be represented in R by

$$X(t,s) = P(t) + sZ(t)$$

Now if all the unit vectors Z(t) are transported to the origin of co-ordinates through a parallel motion, they will determine a curve on the surface of the unit sphere. This curve Z(t) is the generator spherical image curve of R.

We will often prefer to parametrize the generator spherical image curve with its arc length. The arc length parameter will be denoted u. As we shall see later, if Z'(t) = 0 or fails to exist for some t value, or if Z'(t) vanishes identically throughout some t interval, then the spherical image curve of the generators may contain irregular or singular points.

Note that at a regular point of $Z(u)$, the tangent $Z'(u)$ is a unit vector. This follows from u being an arc length parameter. Furthermore, since $Z(u)$ is a vector of constant length (unity) , $Z'(u)$ is orthogonal to $Z(u)$.

When $u = u(t)$ is a strictly monotone function, there is a one-to-one correspondence between the points of $Z(u)$ and the generators of R. Now since each generator of R has a unique unit surface normal vector $X_3(t)$ associated with it (once we orient the surface), we may through parallel transport in R^3 affix it to the point $Z(t)$ on the surface of the unit sphere. After a parameter change from t to u, we may speak of $X_3(u)$. Thus at every point $Z(u)$ on the generator spherical image curve we have an orthonormal basis in R^3 consisting of $Z(u)$, $Z'(u)$, and $X_3(u)$.

A word is in order about the uniqueness of the vector X_3. Every generator on the surface has associated with it a unique normal direction; so in fact there are two possibilities for the surface unit normal vector X_3. However, if the generator spherical image curve is convex, we may select an orientation of the surface so that the implied direction of $X_3(u)$ coincides with the direction of the projection of $Z''(u)$ into the tangent plane of the unit sphere at $Z(u)$.

The reader is reminded that $Z''(u)$ is the curvature vector of $Z(u)$. So the geodetic curvature of $Z(u)$ is readily seen to be $Z''(u)\cdot X_3(u)$.

When we discuss spherical curves with positive geodetic curvature at all of its points, we will use the terms "support great circle arc" and "tangent great circle arc" interchangeably. The two terms refer to the same great circle arc in view of the strong convexity of such curves.

The surfaces under consideration in this paper will be assumed to have partial derivatives of up to third order in the parameters s and t. (See page 3).

The developable surfaces dealt with in this paper will be isometric deformations of various connected planar regions. Put another way, they will be surfaces obtained by bending but not stretching or tearing planar regions. The following connected planar regions will be embedded in R^3:

a) the punctured unit disk - in polar co-ordinates, the region given by $0 < r \leq 1$ and $0 \leq \theta \leq 2\pi$.

b) the diminished (punctured) unit disk - $0 < r \leq 1$ and $0 \leq \theta \leq 2\pi - \epsilon$. ϵ is a small positive angle. The rays $\theta = 0$ and $\theta = 2\pi - \epsilon$ are attached in the embedding.

c) the augmented (punctured) unit disk - $0 < r \leq 1$ and $0 \leq \theta \leq 2\pi + \epsilon$. Here a sector of angle ϵ is attached by its radial boundaries to the rays $\theta = 0$ and $\theta = 2$ of a punctured unit disk cut along the ray $\theta = 0$.

d) an annulus - $a \leq r \leq b$ and $0 \leq \theta \leq 2\pi$.

The various lemmas presented in this paper have been known for some time. Never-the-less we present them with proofs consistent with the general methods of this paper.

II. Basic Lemmas

Lemma I: Given a region R of a developable surface whose generator spherical image curve $Z(u)$ is regular, then the geodetic curvature $k_g(u_1)$ of the generator spherical image curve vanishes if and only if the generator in R corresponding to u_1 consists entirely of flat points.

Proof: Let $X_3(u)$ be the unit normal vector (on the developable surface) corresponding to the generator $Z(u)$. Then

$$X_3(u)\cdot Z(u) = 0.$$

Upon differentiation with respect to u we have

$$X_3'(u)\cdot Z(u) + X_3(u)\cdot Z'(u) = 0.$$

Now on a developable surface, $Z'(u)$ lies in the tangent plane containing $Z(u)$, so

$$Z'(u)\cdot X_3(u) = 0.$$

Then

$$X_3'(u)\cdot Z(u) = 0. \tag{1}$$

Now since $X_3(u)$ is a unit vector it follows that

$$X_3'(u)\cdot X_3(u) = 0.$$

Consequently

$$X_3'(u) = a(u)Z(u) + b(u)Z'(u) \tag{2}$$

where $a(u)$ and $b(u)$ are scalar functions.

Now $a(u) = 0$ by equation (1).

So we have

$$X_3'(u) = b(u)Z'(u) \tag{3}$$

where

$$b(u) = X_3'(u) \cdot Z'(u). \tag{4}$$

Now

$$Z''(u) = p(u)Z(u) + q(u)Z'(u) + r(u)X_3(u).$$

To evaluate p(u) we have

$$Z(u) \cdot Z'(u) = 0.$$

Differentiating with respect to u yields

$$Z'(u) \cdot Z'(u) + Z(u) \cdot Z''(u) = 0$$

$$1 + p(u) = 0$$

so

$$p(u) = -1.$$

Since Z'(u) is a unit vector (u is arc length on Z(u)),

$$Z''(u) \cdot Z'(u) = 0$$

so

$$q(u) = Z'(u) \cdot Z''(u) = 0.$$

Consequently we have

$$Z''(u) = -Z(u) + r(u)X_3(u) \tag{5}$$

with

$$r(u) = X_3(u) \cdot Z''(u). \tag{6}$$

Now

$$X_3(u) \cdot Z'(u) = 0$$

so

$$X_3'(u)\cdot Z'(u) + X_3(u)\cdot Z''(u) = 0.$$

By (4) and (6) above, this last equation can be written

$$b(u) + r(u) = 0. \tag{7}$$

Note that at a point u_1 on the generator spherical image curve $Z(u)$ for which the geodetic curvature is zero,it follows that $Z''(u_1)$ lies exclusively in the direction of $Z(u_1)$. Then by equation (5) we see that $r(u_1) = 0$. By equation (7) it follows that $b(u_1)$ is also zero. Then from equation (3), $X_3'(u_1) = 0$.

This means that the surface R contains a generator corresponding to u_1 along which the surface normal $X_3(u)$ is stationary, i.e. we have a generator consisting entirely of flat points.

On the other hand, if the surface R contains a generator, parametrized by u_1, consisting entirely of flat points, then $X_3'(u_1) = 0$. Then since by hypothesis $Z'(u)$ is never zero (u is arc length on the generator spherical curve), we see from equation (3) that $b(u)$ must be zero.Then $r(u)$ is zero also, by equation (7).

Setting $r(u)$ equal to zero in equation (5) shows that $Z''(u_1) = -Z(u_1)$. Consequently $k_g(u_1) = 0$, and we are done with the proof of our lemma.

We now state and prove Lemma II.

Lemma II: A closed simple curve on the unit sphere with positive geodetic curvature at all of its points

a) is confined to a hemisphere, and

b) has the globally convex property - i.e. any two points on the curve can be connected by a great circle minor arc wholly contained in the curve's interior (suitably defined).

Proof of part (a): We shall begin the proof by orienting the curve so that as its arc length parameter increases, the tangent great circles at each point of the curve are on the right of an observer standing on the outside surface of the unit sphere and facing the direction of increasing parameter values at the point of tangency of the curve. See Figure 1.

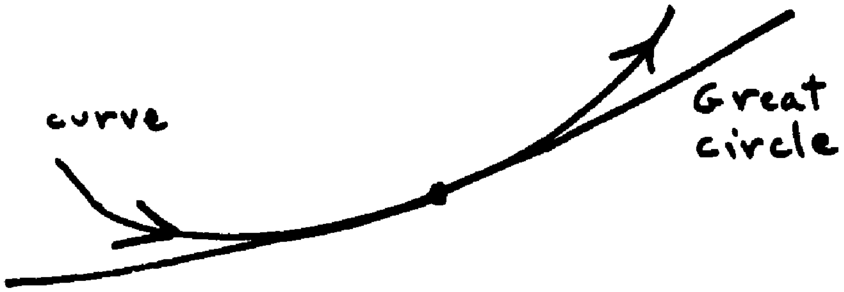

Figure 1

Now suppose that R and S are two points on our curve C of Lemma II such that some great circle G intersects C at R and S, but at no point between R and S are there any intersections (i.e. say R corresponds to u_1 of the curve's arc length parameter, and $u_2 > u_1$ corresponds to S. Then for any u such that $u_1 < u < u_2$, the point corresponding to u doesn't intersect the great circle.).

Let us assume, also, that R and S are not antipodal points. Then we can pick two antipodal points P and P' on great circle G so that R and S are on the same semicircle of G determined by P and P', as shown in Figure 2.

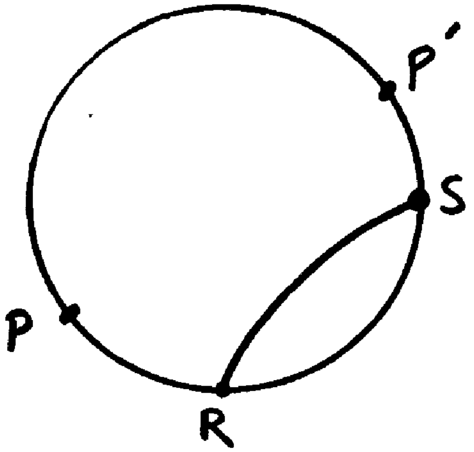

Figure 2

Now consider the family of great semi-circles through P and P' and note that one member of this family must intersect curve C between R and S tangentially at at least one point, as shown in Figure 3. Then the orientation of curve C must be from S to R. This fact will now be used to prove part (a) of Lemma II.

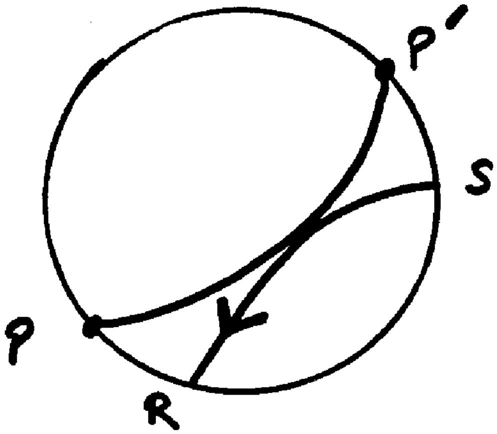

Figure 3

Suppose, then, that we have a point P on curve C of Lemma II, and we denote by G the support great circle of C at P. That G is a local support line of our curve C at P follows immediately from our hypothesis that $k_g > 0$ at all points of C, in particular at P.

We will show that G is in fact a global support for C - i.e. the whole curve C lies inside G. This would prove part (a) of Lemma II.

Say curve C is oriented in accordance with our agreement above that the support great circles are on the right and the interior on the left as the arc length parameter u increases. Say, also, that P corresponds to the value u_1.

Consider the point S corresponding to the next $u > u_1$ at which C intersects G, as shown in Figure 4. The orientation of the curve is indicated in the diagram.

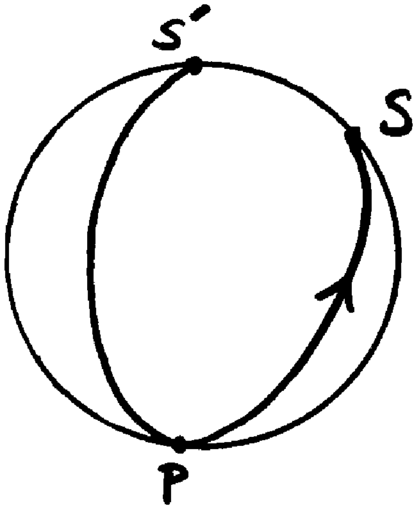

Figure 4

As was shown above, arc $\overset{\frown}{PS}$ of great circle G can't be minor, or else $\overset{\frown}{PS}$ on curve C would have an orientation going from S to P, which would contradict our assumption earlier that the arc length parameter value of S is strictly greater than u_1 , the parameter value of P. Consequently, the arc length of great circle arc PS is greater than or equal to π.

Similarly, let S' (see Figure 4) be the first intersection of C and G before P, i.e. the greatest $u < u_1$ for which C intersects G. Then by the same argument, arc $\overset{\frown}{S'}$P along G can't be minor.

On the other hand, arcs $\overset{\frown}{PS}$ and $\overset{\frown}{S'P}$ of great circle G can't both be major (actually neither can be major) by the "simple" property of curve C. See Figure 5.

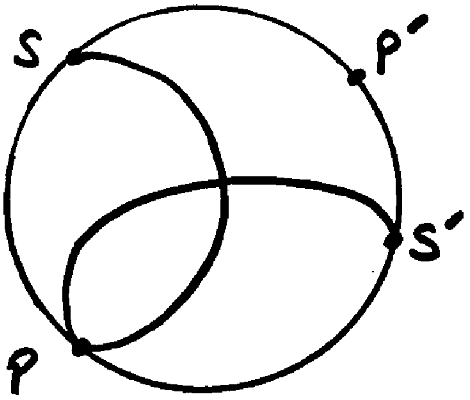

Figure 5

This leaves us with $\overset{\frown}{PS}$ and $\overset{\frown}{PS'}$ along G both having arc length π, or in other words, S and S' are the same point P' - the antipodal point of P.

We now show that even such a state of affairs is impossible. We now supposedly have a curve which is simple, closed, and having positive geodetic curvature at each of its points. Furthermore, this curve C is tangent at antipodal points P and P' to great circle G.

Draw a great semi-circle through P and P' running through the interior of C as shown in Figure 6. By the tangency of G and C, it follows that in some neighborhoods of P and P' this great semi-circle can be selected so that it will not intersect curve C in those neighborhoods.

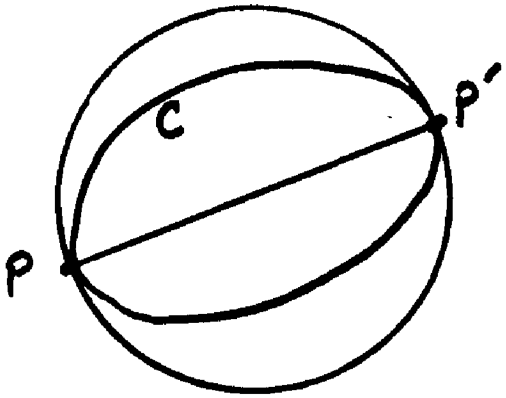

Figure 6

On the other hand, if this great semi-circle were to intersect the curve C at some other point S as shown in Figure 7, the curve would not have a consistent orientation as shown.

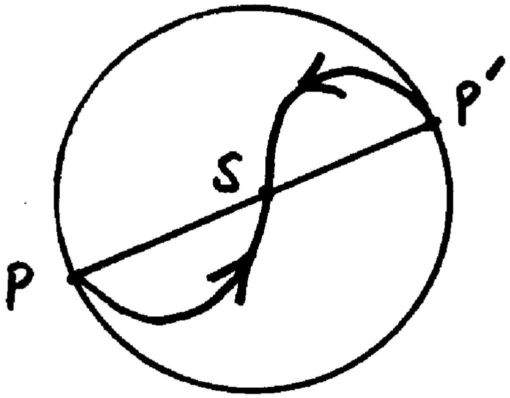

Figure 7

Now we rotate our great semi-circle till it intersects curve C tangentially at some point (other than P and P' of course!). (As we saw above, this intersection couldn't be a crossing point.)

But at this intersection, the great semi-circle would be a tangent on the wrong side of curve C!

So we conclude that curve C is confined to the interior of a hemisphere as Lemma II (a) maintains.

We now prove part (b) of this Lemma, that such a curve is globally convex, by projecting the hemisphere (in whose interior our simple closed curve is located) onto a plane M parallel to the equatorial plane of the hemisphere and tangent to it at the polar point of the equator.

The image P' of any point P on the hemisphere shall be the intersection of the straight line through P and the center of the hemisphere with the plane M, as shown in Figure 8. This mapping is obviously one-to-one.

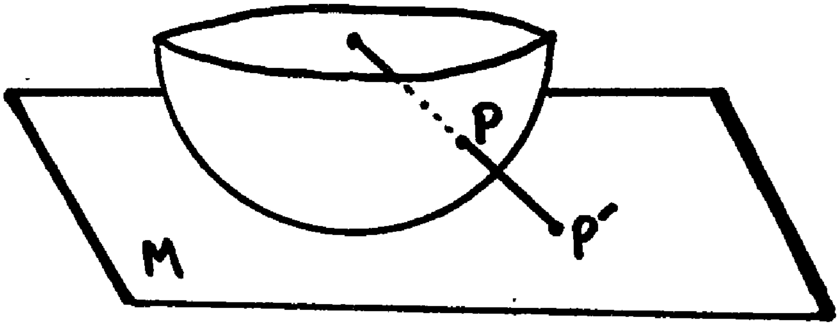

Figure 8

This well known technique maps the great semicircles of the hemisphere into infinitely long straight lines in M, and it maps simply connected regions on the interior of the surface of the hemisphere into simply connected regions of M.

In particular, our simple closed curve C gets mapped into some simple closed plane curve in M which is locally convex. Call this curve C'.

This last remark follows from the fact that the support great circle arcs exterior to C map into support lines exterior to C'.

Now it is a well known fact (see [5] p. 45) about plane closed simple curves that local convexity ($k_g > 0$) implies global convexity. Consequently, C' is globally convex. Now the pre-image of the unique line segment in M connecting any two boundary points A and B on C' (and wholly contained in the interior of C') will be a minor greatcircle arc wholly contained in the interior of C connecting the pre-images of A and B on spherical curve C. Therefore, curve C is globally convex as part (b) of Lemma II maintains.

We will need one more lemma before we state and prove the first theorem of this paper.

Lemma III: Let C be a simple closed curve on the unit sphere such that $k_g > 0$ at all its points. Let T be a geodetic polygon circumscribed around C, consisting of minor great circle arcs. Then the arc length of C is strictly less than the arc length of T.

Proof: We call the points on polygon T and in its interior B_1. Obviously, B_1 is a closed convex set (in fact weakly convex).

Now denote by P_i the points of tangency of curve C and polygon T. (If there are k points of tangency, then $1 \leq i \leq k$.) Let M_i be the midpoint of arc $\overset{\frown}{P_i P_{i+1}}$ of curve C.

At each of the points M_i, draw the support great circle arcs of C which are contained in B_1. Now delete those arcs of polygon T no longer needed to fully circumscribe C, thereby forming a new polygon T_2.

Clearly, the arc length of T_2 is strictly less than the arc length of T.

Now denote by B_2 the points on T_2 and in its interior. B_2 is clearly a closed convex set. Furthermore,

B_2 is strictly contained in B_1.

Continuing in this manner, we form closed convex set B_3 by bisecting the arcs of curve C between its points of tangency with T_2 , and drawing the support great circle arcs of C at these midpoints. We keep only those parts of the support lines which are contained in B_2. We also discard the arcs of T_2 no longer needed to fully circumscribe C.

Once again, B_3 is strictly contained in B_2 , and the boundary length of B_3 is strictly less than the boundary length of B_2.

Repeating this procedure ad infinitum, we get an infinite sequence of closed convex sets $[B_n]$ which is nested, i.e. B_{j+1} is (strictly) contained in B_j for all positive integers j.

Furthermore, the boundary lengths of these sets form a strictly decreasing sequence of positive real numbers. Our strategy will be to set up a metric on the space of closed convex sets on the unit sphere and then show that the sequence $[B_n]$ converges in this metric to the closed convex set B, where B is the set of points on and in the interior of curve C.

We will then use the fact that the boundary volumes of a convergent sequence of closed convex sets converges to the boundary volume of the limit closed convex set, provided that the dimension of the latter equals the dimension of the Riemannian manifold in which the sets are located. (See [2] p. 284)

It would then follow that the strictly monotonically decreasing sequence of boundary lengths of the B_n converge to the arc length of curve C (which is the boundary length of B). Then the arc length of C would of necessity be strictly less than the arc length of T as the Lemma maintains.

The metric we shall employ is a variation of the Hausdorff metric. Let A and B be two closed sets on the unit sphere. Let a and b denote variable points of A and B respectively. Denote by a' the supremum of the distances from a to B, written sup d(a,B), $a \in A$. Similarly, let b' = sup d(b,A), $b \in B$. Then the distance from A to B, written d(A,B) shall be max(a',b').

If $d(A,B) = 0$, then $A = B$, in light of the fact that A and B are closed sets. For further details on the Hausdorff metric, see [4] p. 59 ff. or [3] p. 59 and 60.

Note that if A is strictly contained in B, $a' = 0$. Consequently, $d(A,B) = b'$.

We shall now show that the sequence $[B_n]$ converges to B, where B is the closed convex set whose boundary is curve C. Observe that B is strictly contained in each B_n (for all positive integers n). Consequently, B must be contained in the intersection of the closed convex B_n. (Note that the intersection is closed and convex). Now since the sequence $[B_n]$ is nested, its limit is its intersection. So $B \subseteq \lim[B_n]$.

Suppose x belongs to the limit set of the B_n but does not belong to B. Let $d(x,B) = \varepsilon > 0$. From point x which clearly belongs to the exterior of B, draw two tangents to curve C, as shown in Figure 9.

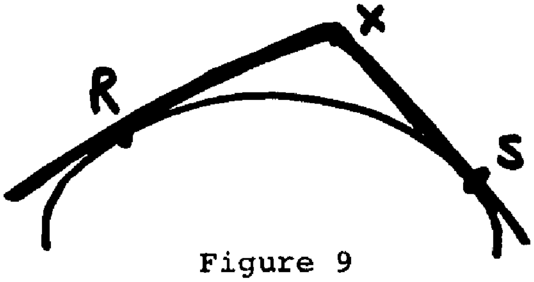

Figure 9

Call the points of tangency of curve C and the great circle arcs drawn from x, R and S. Call the length of arc $\frown{RS}$ along C, s. Now it is easily seen from the method of construction of the B_n that for some integer j, there exists a set B_j of our sequence $[B_n]$ such that a point P exists on the arc PS of curve C (other than R or S) such that a boundary great circle arc of B_j is tangent to C at point P. See Figure 10.

Clearly, x can't belong to B_j. Consequently x can't belong to the limit of the sequence $[B_n]$ since the latter is contained in each B_n, and is contained in B_j in particular. Therefore $\lim[B_n] \subseteq B$, and we see that $\lim[B_n]$ equals B. This completes the proof of Lemma III in light of the convergence of the boundary volumes of $[B_n]$ to the boundary volume of $\lim[B_n]$ discussed above.

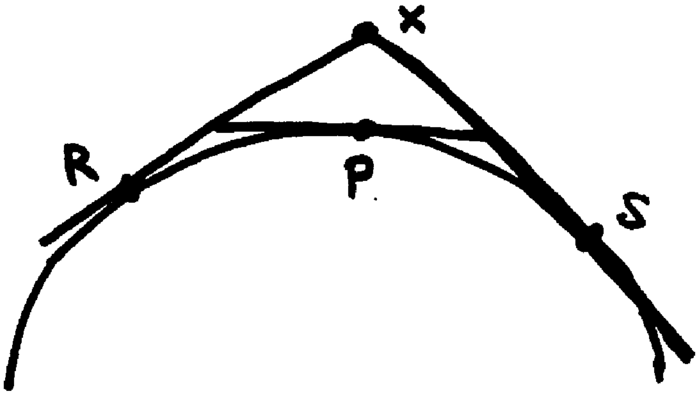

Figure 10

<u>Corollary</u>: The arc length of a closed simple curve on the unit sphere such that k_g is positive at all of its points is strictly less than 2π.

In proving this corollary to Lemma III we will use the fact that such a curve is confined to the interior of a hemisphere. We also assume that the curve encloses a strongly convex body.

Pick two points P and P'(its antipodal point) on the great circle boundary of the hemisphere, and draw the two great semi-circles C_1 and C_1' through P_1 and P_1' tangent to our curve C, as shown in Figure 11, at points T_1 and T_1'. Note that the sum of the arc lengths of great semi-circles C_1 and C_1' is 2π.

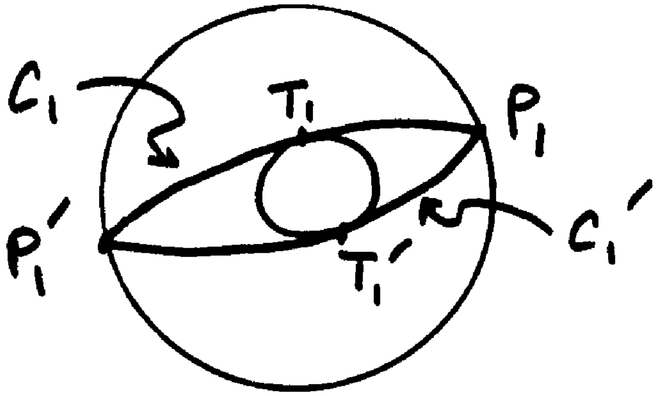

Figure 11

We now draw two new antipodal points P_2 and P_2' on the hemisphere's boundary, and select the two great semicircles through P_2 and P_2' which are tangent to curve C at points T_2 and T_2'. Denote these tangents by C_2 and C_2'. See Figure 12.

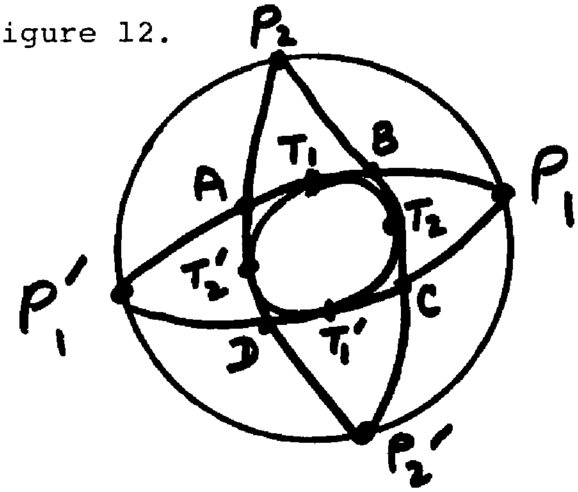

Figure 12

Now discard all but the great circle polygon $\widehat{ABCD}$ that is circumscribed around our curve C. We have

$$|BC| < |BP_1| + |CP_1|$$
$$|AB| = |AB|$$
$$|DC| = |DC|$$
$$|AD| < |AP_1'| + |P_1'D|$$

Adding both sides of these equations and inequalities yields

$$|\widehat{ABCD}| < |C_1| + |C_1'| = 2\pi.$$

Now by Lemma III, the arc length of curve C is strictly less than the arc length of polygon $\overset{\frown}{ABCD}$ which in turn is strictly less than 2π, so the corollary is seen to be true.

III. Embedding the Unit Disk.

We shall now present the first theorem of this paper.

Theorem I: The punctured unit disk whose rays are to be generators can't be embedded isometrically (as a developable surface) in R^3 without flat points.

Proof: We will proceed by contradiction. Let us assume we have an embedding without flat points. Consider the generator spherical image curve of our surface. We will show that it is a simple closed curve with positive geodetic curvature everywhere, and that its arc length is 2π.

It would then follow by our Lemmas and the corollary above that such a curve can't exist. This is seen to be so by recalling that the generator curve of such a surface as we are considering would be confined to a hemisphere and its arc length would of necessity be strictly *less* than 2π.

Simplicity follows from the fact that all the generators emanate from the same point. This can be seen as follows.

Consider the family of circles on the unit disk centered at the (punctured) center of the disk. Clearly, each such circle crosses every ray emanating from the center exactly once. Now the image on the embedding of this circle is a closed curve which crosses every generator exactly once.

Let the radius of the circles on the disk go to zero, and consider their images on the embedding. Since the mapping is an isometry, the arc lengths of these curves must go to zero. Consequently, they approach or "shrink" down to a point (not on the embedding).

Now as these curves shrink to a point, their points of intersection on each generator move toward the same point, namely the point from which all the generators emanate.

We may now deduce the simplicity of the generator spherical image curve by noting that if this curve were to cross through itself, then the image of the punctured disk, i.e. the resulting cone (minus its vertex) would of necessity be compelled to break through its lateral boundary.

If on the other hand, the spherical image curve was merely tangent to itself at some point, as depicted in Figure 13a, a global support great circle wouldn't be able to exist at the point of tangency. But this would contradict the assumed convexity of the generator spherical image curve.

If at the point of tangency discussed in the above paragraph, both arcs of the spherical image curve were on the same side of the support great circle, the implied orientation of these arcs would bring us back to the case discussed above. The arcs of the spherical image curve connecting the two mutually tangent arcs would have to cross through one another. See Figure 13b.

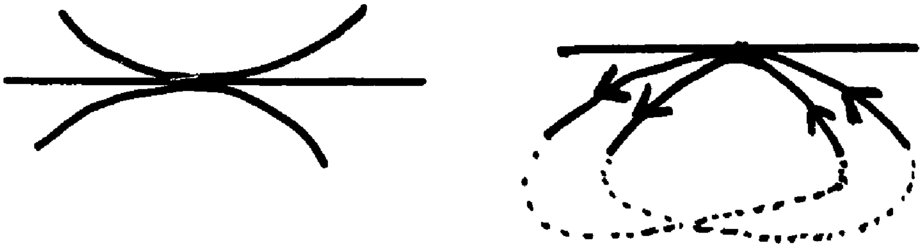
(a)
(b)

Figure 13

We assume that the generator spherical image curve is closed since it is the image under a continuous mapping of the circumference of the unit disk.

The generator spherical image curve has positive geodetic curvature at all of its points in light of Lemma I and the continuity of k_g.

Finally to show that the generator spherical image curve has arc length equal to 2π, observe that the circumference of the punctured unit disk is 2π, and in our embedding, the boundary curve of the image surface is also 2π. But this boundary curve is the spherical image map of our generators since the generators on the embedding are all unit length.

So our contradiction is established and the proof of Theorem I is complete.

IV. Embedding the Augmented Unit Disk.

Our next theorem concerns itself with the same basic situation, namely a cone that is the image of the punctured unit disk. However, before we embed the punctured unit disk, we make a cut along a generator and add a sector of angle ε by adjoining its radial boundaries to the two radial boundaries of the cut unit disk, thereby forming an augmented punctured unit disk whose rays will be the generators of a cone. We state and prove the following theorem.

Theorem II: A cone formed from an augmented punctured unit disk whose rays are generators must have flat points.

Proof: As in the proof of Theorem I, consider the generator spherical image curve of the cone. It is a closed simple curve of always positive geodetic curvature if the cone has no flat points, and its arc length is obviously $2\pi + \varepsilon$.

But by the corollary to Lemma III this is impossible. The arc length of the generator spherical image curve would of necessity be strictly less than 2π. So we must conclude that there are flat points on the cone.

V. Embedding the Diminished Unit Disk.

Before we state and prove our next theorem, we make the following observation.

Let C_e be a circle on the unit sphere other than a great circle. Call the interior of C_e the smaller of the two regions it divides the surface of the sphere into. Then if C is a closed simple curve whose geodetic curvature is always positive, and which is located wholly in the interior of C_e, then the arc length of C is strictly less than the arc length of C_e.

This will become obvious when we circumscribe a geodetic polygon around C whose arc length is strictly less than the arc length of C_e. We will then be able to invoke Lemma III to get our result.

Select a point P_1 on C and draw the segment of the tangent circle at P_1 which is on and in the interior of C_e as shown in Figure 14. Call the points of intersection of this tangent and C_e , S_1 and S_1' as shown.

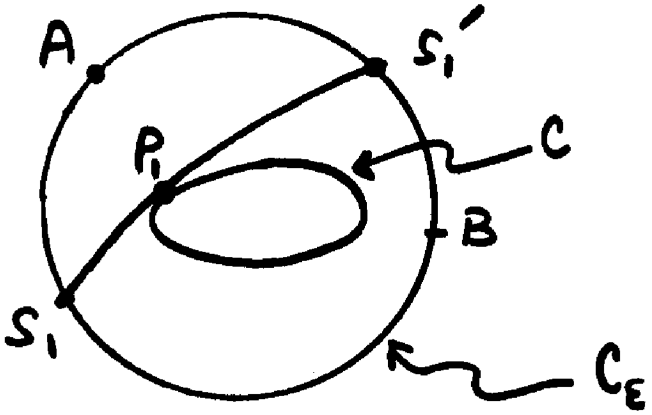

Figure 14

Now remove arc $\overset{\frown}{S_1AS_1'}$ of C_e. Then the arc length of $\overset{\frown}{P_1S_1BS_1'}$ is strictly less than the arc length of C_e, since $\overset{\frown}{S_1AS_1'}$ is not a geodetic path between S_1 and S_1' whereas $\overset{\frown}{S_1P_1S_1'}$ is.

Now select point P_2 on C close enough to P_1 so that their tangents intersect inside C_e (as shown in Figure 15) at S_2'. Let S_2 be the intersection of the tangent to C (at P_2) and C_e.

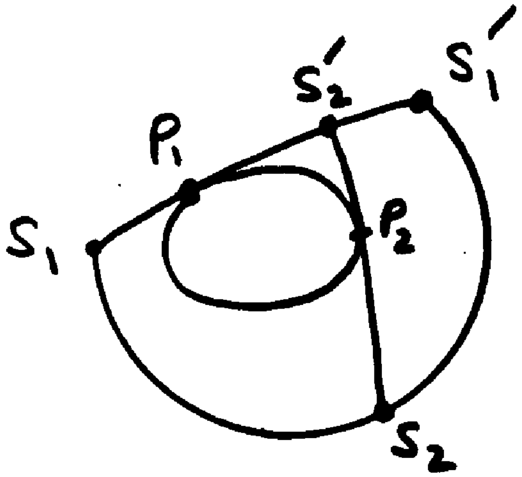

Figure 15

Now remove arc $\overset{\frown}{S_1'S_2}$ of C_e and arc $\overset{\frown}{S_2'S_1'}$ of the tangent at P_1. The remaining figure $\overset{\frown}{S_1P_1S_2'P_2S_2}$ has strictly less arc length than $\overset{\frown}{S_1P_1S_2'S_1'S_2}$ using the same reasoning.

Continuing this process finitely many times, we get a geodetic polygon circumscribed around C whose length is strictly less than the length of C_e.

Then by Lemma III, the arc length of curve C is strictly less than the arc length of this polygon. It then follows that the arc length of C is strictly less than the arc length of C_e.

The observation just stated will enable us to present a theorem on the occurence of flat points on certain embeddings of the diminished unit disk (described in the introduction of this paper).

We now state and prove Theorem III.

Theorem III: Given a cone formed from the punctured unit disk from which a sector of angle e has been deleted, with the two boundary rays joined, such that its rays are to be generators, then there exists a circle C_e on the unit sphere such that whenever the spherical image curve of the generators lies wholly in the interior of C_e , the cone must have flat points. The radius in R^3 of C_e is $1-\frac{e}{2\pi}$.

Proof: Say there were no flat points. Then we would have a simple closed curve of always positive geodetic curvature whose arc length would be $2\pi - e$.

Now if such a curve were in the interior of C_e whose arc length is exactly $2\pi - e$, its arc length would have to be strictly less than $2\pi - e$, clearly an absurd situation. So there must be flat points and the theorem is proven.

Note that the diminished unit disk can, needless to say, be embedded as a cone without flat points. For example, it can be embedded as a right circular cone, in which case the circle at the base of this cone is the "C_e" of Theorem III.

VI. Embedding the Annulus.

We will now consider a slightly more complicated situation. Instead of embedding the punctured unit disk, we will embed an annulus via a "small" continuous isometric deformation. We will soon deal precisely with what we mean by small.

The question will again involve the occurrence of of flat points on the embedding. Once again, by an isometric embedding we mean one which preserves the length, area, and angle measurements on the surface (i.e. all measurements dependent on the first fundamental form, or put another way, all measurements invariant with respect to bending but not stretching the surface).

Our efforts with the generator spherical image map will be hampered by several factors.

Firstly, since the generators do not necessarily emanate from a single point, we have no a priori assurance that the generator spherical image curve will be simple.

Secondly, finding the arc length of the generator spherical image curve will require a more complicated calculation. We won't be able to simply take the arc length of the inner or outer circumference of the annulus, as we did in the embedding of the punctured unit disk.

Thirdly, we can't be certain that on the generator spherical image curve, $Z'(t) \neq 0$, or for that matter that $Z'(t)$ always exists, i.e. we can't be sure that the curve $Z(t)$ is regular at all its points.

Consequently, Lemma I might not help us here in the sense that on some region of our embedding there may be a flat point whose existence might not be detected by the geodetic curvature vanishing on the curve Z(t) because $\|Z'(t)\| = 0$ for all the generators of that region.

Put another way, the generators might be parallel for some region, i.e. part of the developable surface might be a cylinder, in which case that entire portion of the surface will be represented by just one point of the generator spherical image curve. Surely Lemma I would be of no use to us on the cylindrical portion of our surface.

To begin with, we will limit ourselves to embeddings for which the generators all go from one boundary to the other. We then rule out the flat-point-free embedding obtained by wrapping the annulus around some cylinder consisting entirely of parabolic points.

The pre-image annulus with the generator pre-images retained would look, in this case, like Figure 16 on the next page. Generators such as AB are being ruled out by our request that the generators of the embedding go from one boundary to the other. See Figure 16.

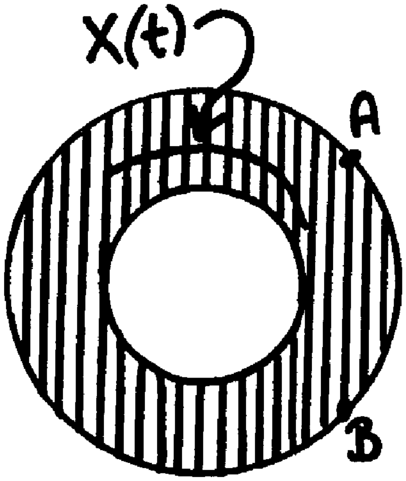

Figure 16

Note, however, that some portions of the developable surface could still be cylindrical (as a glance at Figure 16 shows) with $|Z'(t)| = 0$, t being the arc length parameter of some curve in the surface which crosses each of the generators of the cylindrical portion transversally. See Figure 16 - curve X(t) is transverse to the generators of the cylindrical portion whose generators do obey our request that they go from boundary to boundary; yet if $Z(t)$ is a unit vector along the generator crossing $X(t)$ at a particular t value, then for some t interval, $Z'(t)$ vanishes identically and our flat point test of Lemma I would fail to reveal the existence of flat points on a generator in this cylindrical portion.

Another difficulty is shown in the annulus with the pre-images of the generators retained shown in Figure 17.

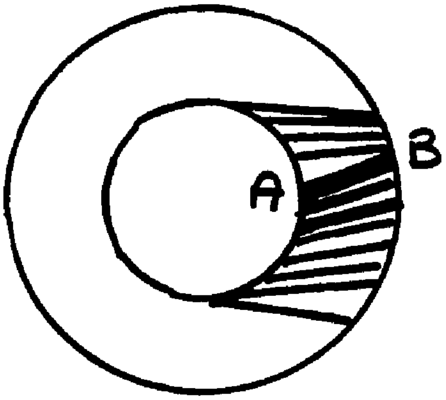

Figure 17

The generators below AB are rotating counterclockwise around the annulus (i.e. their angles with the positive x-axis is increasing). Above AB, the generators rotate clockwise.

If we call the t value at the intersection of $X(t)$ and generator AB t_1, then $Z'(t_1)$ will either be zero or fail to exist, and the generator spherical image curve $Z(t)$ will be jagged (i.e. not smooth) at the point $Z(t_1)$.

In fact, $Z'(t)$, the tangent to the spherical image curve of the generators, will reverse direction at $Z(t_1)$. Of course there is no geodetic curvature there and we certainly can't employ the corollary of Lemma III to show that the arc length of the generator spherical image curve is strictly less than 2π.

To avoid the difficulties discussed above, we are initially restricting ourselves to the case in which if we let a closed regular curve $X(t)$ in the surface cross all the generators exactly once and transversally, then $Z'(t)$ always exists and is never zero. On the annulus, the pre-images of the generators would appear to rotate strictly monotonically through a total angle of 2π.

Under this condition, we have a regular closed generator spherical image curve which <u>if</u> we assume our developable surface has no flat points, is everywhere at least locally convex, i.e. k_g is strictly greater than zero at all of its points.

We claim that the arc length of the generator spherical image curve under the above assumptions is 2π. We shall evaluate the arc length as follows.

Partition the spherical image curve of the generators of our embedded surface into finitely many arcs such that each one of these arcs shall represent a portion of the embedded surface small enough to contain an orthogonal trajectory of the generators of that portion of the surface.

Let us call the orthogonal trajectory we select on the portion of our surface corresponding to the i-th arc of our partitioned generator spherical image curve $P_i(t_i)$, wherein t_i is an arc length parameter for this i-th orthogonal trajectory. Then the generators of the i-th section of our surface in which $P_i(t_i)$ is contained can be parametrized by t_i.

In particular, the unit vectors orthogonal to the unit tangent vectors $P_i'(t_i)$, and lying in the generator directions can be represented by $Z(t_i)$. It is also possible to represent the i-th partition arc of the generator spherical image curve by $Z(t_i)$.

It then follows that the arc length of the i-th partition arc of the generator spherical image curve is given by $\int \left| Z'(t_i) \right| dt_i$.

Let us consider the integrand in this expression. The curvature vector of the curve $P_i(t_i)$ is $P_i''(t_i)$. We must project this last vector into the surface tangent plane at $P_i(t_i)$ to get the geodetic curvature of curve P_i.

Now since $P_i'(t_i)$ is a vector of constant length (unity), $P_i''(t_i)$ will be orthogonal to it. Then P_i'' will project into the tangent plane of the surface in the direction of $Z(t_i)$. Consequently, we may calculate the geodesic curvature of $P_i(t_i)$ as follows.

$$k_g(t_i) = -P_i''(t_i) \cdot Z(t_i).$$

But since

$$P_i'(t_i) \cdot Z(t_i) = 0$$

we have, upon differentiating,

$$P_i'(t_i) \cdot Z'(t_i) + P_i''(t_i) \cdot Z(t_i) = 0$$

or

$$\left| Z'(t_i) \right| = k_g(t_i)$$

since $P_i'(t_i)$ is a unit vector in the same direction as $Z'(t_i)$.

Then the arc length of the i-th arc of the generator spherical image curve is given by $\int k_g(t_i)dt_i$. This integral is clearly an intrinsic quantity which may, therefore, be evaluated on the section of the planar annulus corresponding to the i-th section of the surface.

In the planar annulus, the integral above is the total angle change of the pre-image of vector $P_i'(t_i)$. Now observe that the angle between the pre-image of $P_i'(t_i)$ and the pre-image of $Z(t_i)$ is the same as the angle between $P_i'(t_i)$ and $Z_i(t_i)$ on the surface since the mapping of the annulus to the surface is an isometry. This last angle is always by definition of $P_i(t_i)$, 90°. Consequently, the angle between the pre-images of $P_i(t_i)$ and $Z(t_i)$ in the planar annulus is always 90°.

Then the total angle change of the tangent to the pre-image of curve $P_i(t_i)$ on the i-th section of the planar annulus is equal to the total angle change of the pre-images of the generators in that section.

Then it is easily seen that the total arc length of the generator spherical image curve is the total angle through which the pre-images of the generators turn on the annulus which is 2π.

We will now deal with the issue of the simplicity of the generator spherical image curve. As was noted earlier, without the simple property of the generator spherical image curve, we can't use Lemma II or the corollary of Lemma III.

To begin with, we shall set up the mutually orthogonal x,y, and z co-ordinate axes for R^3 such that the planar annulus is in the x-y plane, and the center of the annulus is at the origin. Furthermore, let the positive z direction co-incide with the direction of the pre-image of <u>all</u> the surface normals. Let N represent a unit vector in the positive z direction. N is clearly normal to the annulus. Note that N is also the position vector of the "north pole" of the unit sphere in this co-ordinate system.

We now define a "small" continuous deformation of the annulus as one whose <u>surface normal</u> spherical image curve is wholly contained in some ϵ-neighborhood of N on the unit sphere,i.e. the normal spherical image curve $X_3(t)$ is confined to the interior of a small circle of latitude near the north pole of the unit sphere whose radius on the unit sphere's surface is a small positive number ε .

It then follows that the spherical image curves of $Z(t)$ and $Z'(t)$ on the unit sphere are confined to a band of width 2ε around the equator of the unit sphere, in light of the orthogonality of these vectors to $X_3(t)$. (By "equator" we mean the great circle whose polar points are N and -N.)

Under the above assumptions, the generator spherical image curve $Z(t)$ will be a <u>simple</u> curve contained in the band around the equator described above. This can be seen by noting that for the curve $Z(t)$, the angle that its tangent great circle arc can make with the circle of latitude (at the point of tangency of $Z(t)$ and this great circle arc) is bounded in view of the fact that the tangent vector $Z'(t)$ can't make an angle greater than ε with any horizontal plane. In particular, the spherical image curve $Z(t)$ can never be orthogonal to a circle of latitude, and, therefore, can't intersect itself inside the band in which it is confined.

We are now ready to state the next theorem of this paper. The proof will follow with ease in light of the discussion of the past several pages.

Theorem IV: Given a developable surface formed by a continuous isometric deformation of an annulus such that:

a) the normal spherical image curve of the surface is wholly contained in a small neighborhood on the surface of the unit sphere and centered around the point on the unit sphere representing the normal vector of the annulus,

b) the generators of the surface all go from the inner boundary to the outer boundary, and

c) all the pre-images of the generators in the planer annulus rotate strictly monotonically around some fixed ray in the plane of the annulus,

then such a developable surface must contain flat points.

Proof: As we have shown above, the generator spherical image curve of such a surface, if it were free of flat points, would be a simple closed and convex curve, whose arc length is 2π. But by the corollary of Lemma

III, such a curve has arc length strictly less than 2π, and it therefore follows that the surface must contain flat points.

VII. Cylindrical Sections of an Embedded Annulus.

We turn our attention, now, to small continuous deformations of an annulus for which the generator spherical image curve is not regular at all its points.

We begin with the case where for some t interval $t_1 \leq t \leq t_2$, $|Z'(t)| = 0$. Consequently, $Z(t)$ is constant on this t interval. Put another way, all the generators of some section of the embedding are parallel, implying that this section is in fact cylindrical. Note that this entire cylindrical portion of the embedding is represented by one point on the generator spherical image curve, namely the point $Z(t_1)$ (or $Z(t_2)$ for that matter). We will assume that the rest of the generator spherical image curve is regular and simple.

We claim that the surface just described must contain flat points. This will be a little bit more difficult to prove than Theorem IV was! One problem is that since k_g is undefined at the point $Z(t_1)$ on the (gen.) spherical image curve, we have no a priori assurance that the global convexity of this curve is maintained. Of course, in the attempt to show the necessity of the occurence of flat points on the embedding with a cylindrical portion, we will proceed by contradiction and assume there are no flat points thus implying that the geodetic curvature is positive (except at the point representing the cylindrical portion).

In this case, however, we only have local convexity for $Z(t)$ at all its <u>regular</u> points. We can only establish global convexity for $Z(t)$ if we have local convexity at <u>all</u> its points.

One task then is to establish the local convexity of $Z(t)$ at the point $Z(t_1)$. To this end, consider a curve $P(t)$ in the surface orthogonal to the generators of the cylindrical portion and which extends to some portions of the surface adjacent to the cylindrical part (on both sides). See Figure 18 for a view of the pre-image of $P(t)$ on the planar annulus. Observe that the pre-images of the generators still rotate monotonically around the annulus. However, the pre-images of the parallel generators are obviously still parallel on the annulus (by the isometric property of the mapping), so the monotonicity of the generators is no longer strict. See Figure 18.

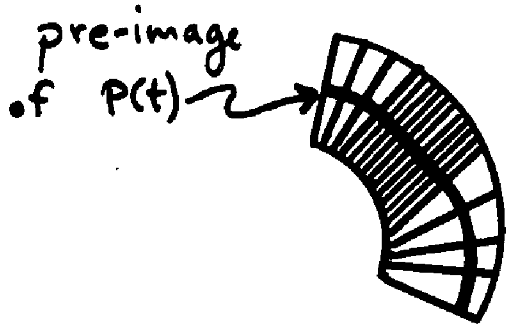

Figure 18

Let t be an arc length parameter of P(t), so that P'(t) shall be a unit vector co-directional with Z'(t), when the latter exists. Note that P'(t) is the unit tan-vector of the curve Z(t), after parallel transport in R^3.

We will require an understanding of the connection between the generator spherical image curve Z(t) and the unit surface normal vectors $X_3(t)$ affixed through parallel transport in R^3 to the point on the unitsphere represented by Z(t).

Clearly, $X_3(t)$ would then lie in the tangent plane of the unit sphere at Z(t), since $X_3(t) \cdot Z(t) = 0$.

By choosing $X_3(t)$ which will make the vectors $Z(t)$ $P'(t)$ and $X_3(t)$ a right handed moving frame in R^3, $X_3(t)$ will point into the interior of the curve $Z(t)$. Furthermore, since the curve $P(t)$ is a line of curvature of our surface, i.e. a curve tangent at every point to a principal curvature direction, it follows using the formula of Rodrigues (see [6] p. 94) that

$$X_3'(t) + k_n(t)P'(t) = 0$$

where $k_n(t)$ is a scalar function representing the normal curvature of the surface at the point $P(t)$ in the $P'(t)$ direction. (Note that k_n is one of the two principal or extreme normal curvatures at $P(t)$, the other being zero.) Now if our surface is to be flat-point-free, then $k_n(t)$ can never vanish, since its vanishing would imply that $X_3'(t) = 0$ which in turn signifies a flat point.

Along a regular arc of $Z(t)$ along which the geodetic curvature is always positive, $X_3'(t)$ by virtue of its being a scalar multiple of $P'(t)$, is located in the tangent plane of the unit sphere at $Z(t)$ and points in the direction opposite to $P'(t)$. See Figure 19 to visualize this situation.

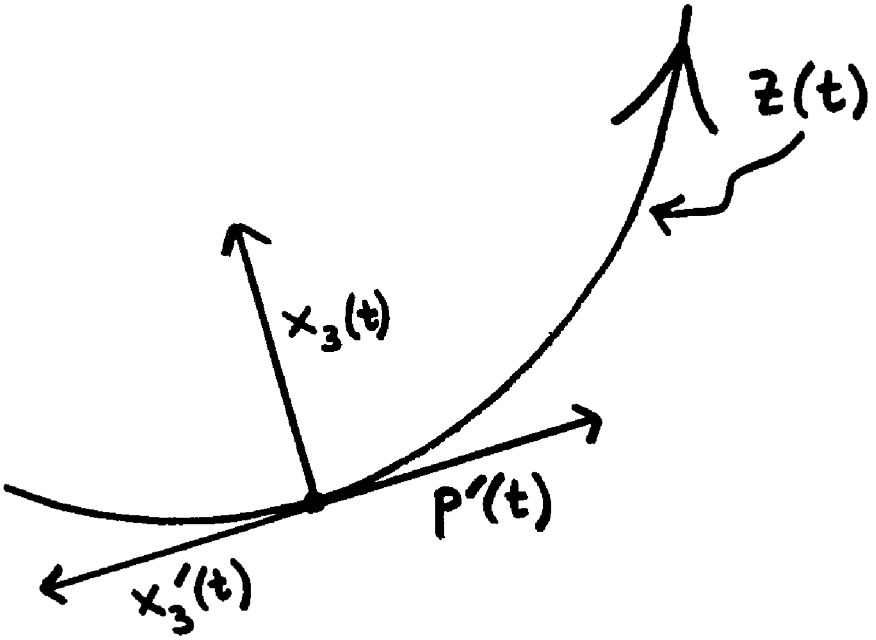

Figure 19

Now if we construct a parallel field along the curve Z(t) on the surface of the unit sphere (in the sense of Levi-Civita), then since the geodetic curvature of Z(t) is assumed positive on this regular arc, the angle between P'(t) and this parallel field will increase strictly monotonically. But in light of the fact that $X_3(t)$ is orthogonal to P'(t), its angle with respect to the parallel field will also increase strictly monotonically.

See Figure 19 again to realize that an observer walking along $Z(t)$ in the increasing parameter direction and on the outside of the sphere would regard $X_3(t)$ as if it were rotating counterclockwise smoothly.

Now as we consider the point $Z(t_1)$, we come to the heart of the matter. If we assume the surface has no flat points, then in the t interval from t_1 to t_2 , k_n can not vanish. Consequently $X_3'(t)$ is never zero and we see that $X_3(t)$ must continue its rotation in the tangent plane of the unit sphere at $Z(t_1)$ without changing the <u>counterclockwise</u> sense of the rotation.

Then the rotation from $X_3(t_1)$ to $X_3(t_2)$ in the tangent plane at $Z(t_1)$ is as presented in Figure 20 necessarily counterclockwise. (Note that $X_3'(t_1)$ exists, and points in the direction opposite to $P'(t_1)$ in the tangent plane of the unit sphere at $Z(t_1)$.) The angle between $X_3(t_1)$ and $X_3(t_2)$ is bounded by the ε which defines the smallness of the deformation: (See Theorem IV, (a).) $\theta < 2\varepsilon$.

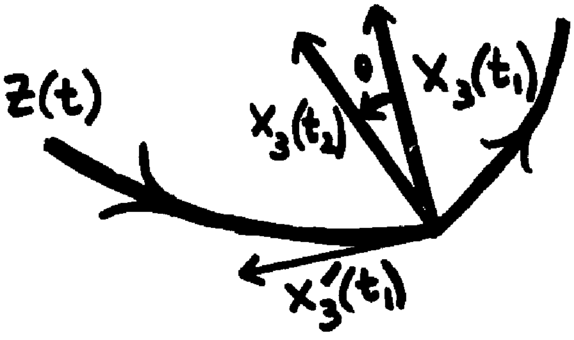

Figure 20

Note that $P'(t)$ is also rotating in this tangent plane (perpendicular to the vector $Z(t_1)$) but always having a fixed ($90°$) angle with $X_3(t)$. Consequently, the rotation in this plane of $P'(t)$ from $P'(t_1)$ to $P'(t_2)$ as viewed by our observer described above is also seen to be counterclockwise, thus preserving the convexity of the curve $Z(t)$ as shown in Figure 21.

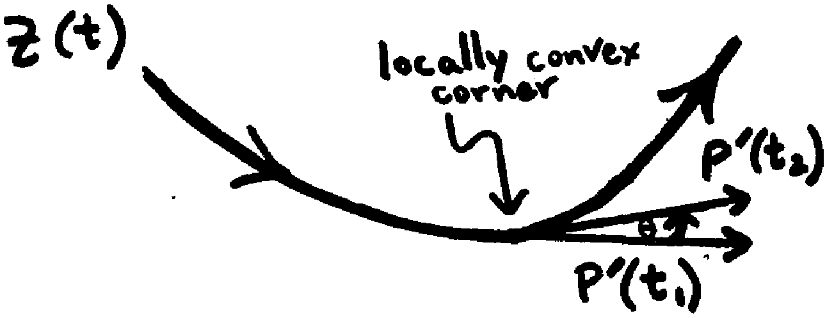

Figure 21

Since we have a small deformation, as in the case of Theorem IV, $Z(t)$ is located in an ε-band around the equator, and its tangent can't be orthogonal to any circle of latitude. Consequently, (and in view of Fig. 21)* $Z(t)$ is simple.

Observe that the pre-images of the generators in the annulus rotate monotonically through an angle of 2π, and therefore, the arc length of $Z(t)$ is exactly 2π.

---

*i.e. θ, the angle change of the tangent at the "corner", is small;

The corollary of Lemma III may be applied here after realizing that one corner in the globally convex curve will not harm the proof if the corner is a vertex of each circumscribed polygon. Consequently we can show by contradiction that the embedding with a cylindrical portion must have flat points.

Theorem IV': Given a developable surface formed by a continuous isometric deformation of an annulus such that:

a) the normal spherical image curve of the surface is wholly contained in a small neighborhood on the surface of the unit sphere and centered around the point on the unit sphere representing the normal vector of the annulus,

b) the generators of the surface all go from the inner boundary to the outer boundary, and

c) the pre-images of the generators in the planar annulus rotate strictly monotonically around some fixed ray in the plane of the annulus except for the pre-images of the generators on the section of the annulus which is the pre-image of a cylindrical portion of the surface (these pre-images of the generators are parallel and therefore

don't rotate at all),
then the developable surface must have flat points.

Proof: If the surface of this theorem had no flat points, its generator spherical image curve would be a closed simple and globally convex curve of arc length 2π, as was shown above. Then by the corollary of Lemma III, such a curve would not be able to exist, as its arc length would necessarily be strictly less than 2π.

VIII. Closing Remarks.

The methods of this paper will be inadequate for the case in which the pre-images of the generators in the annulus rotate non-monotonically in the plane of the annulus. Consider the generator spherical image curve in the vicinity of a point at which the geodetic curvature of P(t) (on our surface) changes sign. This is the same as Z'(t) changing direction (by 180°).

Since P'(t) does not change direction at this point, the orthogonal frame consisting of Z(t), P'(t), and $X_3(t)$ (which is right handed) compels $X_3(t)$ to point to the exterior of the curve Z(t). (Z'(t) and P'(t) now have opposite directions after the change in direction of Z'(t))

The state of affairs above suggests, in light of the previous discussion concerning the smooth rotation of X_3 as seen by an observer on the surface of the unit sphere if we assume there are no flat points, the picture in Figure 22.

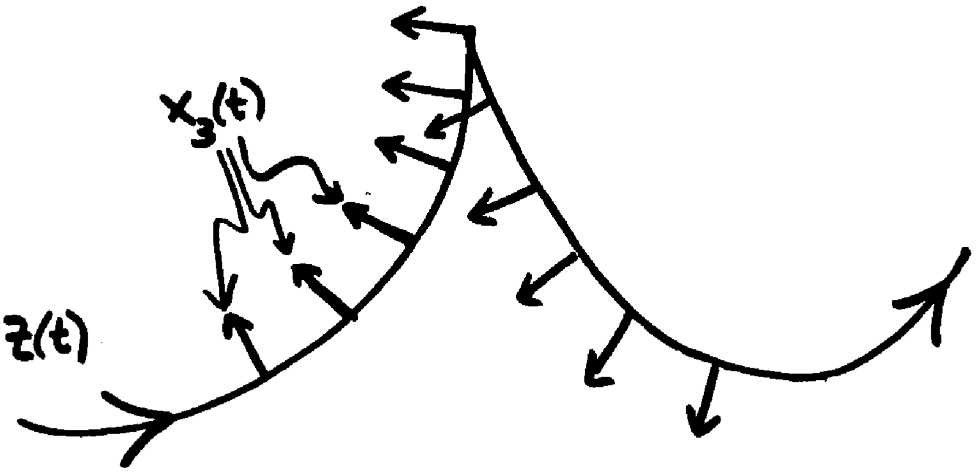

Figure 22

Obviously, the generator spherical image curve is no longer globally convex, thereby rendering useless the various Lemmas and the corollary of this paper.

It may be interesting to note that an annulus can be embedded in R^3 as a developable surface free of flat points if we throw out the condition in Theorems IV and IV' that the isometric deformation be small.(condition (a))

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