

INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

**ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600**

UMI[®]

A

RECURSION CATEGORIES OF COALGEBRAS

by

FLORIAN LENGYEL

**A dissertation submitted to the Graduate Faculty in Mathematics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy, The City University of New York**

2002

UMI Number: 3047238

**Copyright 2002 by
Lengyel, Florian**

All rights reserved.

UMI[®]

UMI Microform 3047238

**Copyright 2002 by ProQuest Information and Learning Company.
All rights reserved. This microform edition is protected against
unauthorized copying under Title 17, United States Code.**

**ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346**

©2002

FLORIAN LENGYEL

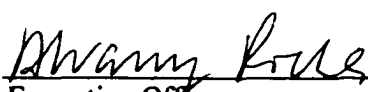
All Rights Reserved

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

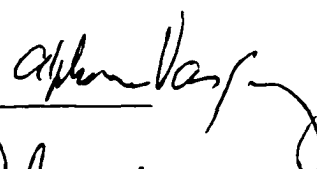
30 April 2002
Date


Chair of Examining Committee

April 30th 2002
Date

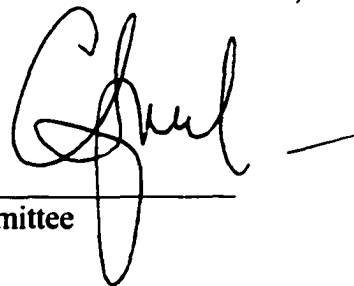

Executive Officer

Alphonse Vasquez



Sergei Artemov

Supervisory Committee



THE CITY UNIVERSITY OF NEW YORK

Abstract

RECURSION CATEGORIES OF COALGEBRAS

by

Florian Lengyel

Adviser: Professor Alex Heller

Let F be a nontrivial endofunctor on the category of sets that preserves weak pullbacks and such that the category $\mathbf{Set}_F$ of F -coalgebras has products. The category $\mathbf{Set}_F$ may be embedded in the category $\mathbf{Pfn}_F$ of F -coalgebras and partial morphisms, which is an iteration category that is not dominical in general, and which need not be locally connected. If $\mathbf{Pfn}_F$ is locally connected, then Heller's existence theorem for recursion categories may be applied to show that $\mathbf{Pfn}_F$ contains many recursion categories. We extend Heller's existence theorem for recursion categories to non-locally connected iteration categories that admit an embedding into a locally connected iteration category by a functor preserving some of the structure of an iteration category.

In memory of Robert A. Di Paola

Acknowledgements

I am indebted to my committee, especially to my adviser Professor Alex Heller for introducing me to recursion categories and for numerous suggestions and support. I thank Professor Alphonse Vasquez for scholarly mathematical advice. I thank Professor Sergei Artemov for his suggestions and encouragement.

The Department of Information Resources at the CUNY Graduate Center deserves special mention for providing me with a stimulating means of supporting my math habit throughout most of my career as a doctoral student.

Without the administrative and moral support of the former and current Executive Officers of the CUNY Graduate Center's Ph.D. Program in Mathematics, Professor Józef Dodziuk and Professor Alvany Rocha, this thesis would not have been written. I thank Professor Janos Pach for providing me the opportunity to work with him in probabilistic and enumerative combinatorics. I thank Professor Burton Randol for sage advice. I thank Professor Noson Yanofsky for his careful reading of drafts of this thesis.

A decade ago my stepfather, Dorsey Davidoff, showed up at my apartment to invite me to attend a lecture at the New York Academy of Sciences that was about to begin. I went with him and met Nancy Beverly, who was finishing her doctorate at the time; she inspired and helped me to pursue a lifelong dream.

New York, New York
April 30, 2002

Florian Lengyel

Contents

Acknowledgements	vi
Introduction	1
1 Coalgebras	6
1.1 The category $\mathbf{Set}_F$	6
1.2 The coalgebra topology	9
1.3 Conditions on $\mathbf{Set}$ endofunctors	11
1.4 Diagram lemmas and factorizations	13
2 P-categories of coalgebras	16
2.1 P -categories	16
2.2 P - and $P\Sigma$ - categories of coalgebras	19
2.3 Prodominical categories	24
2.4 Ranges	32
3 The iteration lemma in $\mathbf{Pfn}_F$	37
3.1 The iteration lemma	37
3.2 Free semigroups and $B^\#$ -categories	40
3.3 Iteration categories revisited	45
4 The generalized existence theorem	47
4.1 Uniform lists	47

4.2	Isotypical objects, isotypes and frames	49
4.3	Uniform generation	50
4.4	Turing developments and locally faithful embeddings	57
4.5	Recursion categories	66
4.6	Examples	70
5	Conclusion	73
A	Proof of the Iteration Lemma via the free semigroup	76
B	Local connectedness in Top	78
	Bibliography	80

Introduction

Dominical categories, algebraic structures in which the notion of the domain of a partial function is definable, and Turing morphisms, which correspond to universal Turing machines, were invented by Alex Heller in a categorical account of computability theory, written after he gave a course in computability at the City University of New York in the 1980's. Later, at Di Paola's suggestion, Di Paola and Heller further developed Heller's theory of dominical categories as an element-free categorical algebraization of classical recursion theory, in which one has categorical analogs of the construction of the universal Turing machine, the s-m-n theorem, the Kleene recursion theorem, Rice's theorem and various analogs of the notion of a creative set [DPH87].¹

In his thesis, Rosolini generalized the dominical categories to the P -categories, which are categories with extra structure capturing the notion of a partial function [Ros86]; the P -categories have since gained acceptance as an appropriate setting for categorical developments of recursion theory and programming semantics [Hel90, Tay93, Fio96].

The development of dominical categories (and later, P -categories) was intended eventually to include algebraic forms of generalized Gödel incompleteness theorems;

¹The term "recursion category" was proposed by Di Paola to designate dominical categories possessing a Turing morphism.

a survey of generalized incompleteness theorems within abstract formal systems is given in Odifreddi [Odi89].² Prior to the introduction of dominical categories, certain attempted approaches to the algebraization of incompleteness had been shown to be futile: Di Paola proved the impossibility of extending the De Jongh-Sambin fixed point theorem for the propositional logic of provability to a language permitting arithmetical terms and quantification; this meant, for example, the impossibility of proving an algebraic analog of the diagonalization theorem in any generalization, permitting arithmetical terms and quantification, of the theory of diagonalizable algebras [DP84, Ber84, Mag75].³

Montagna constructed two so-called syntactic P -categories (denoted by S_{PA} and S'_{PA} , respectively) generated by Peano Arithmetic [Mon89].⁴ In concluding his study of these categories, Montagna remarks that he had “...not been able to find a reasonable category-theoretic version of the incompleteness theorems...” and to that end suggested relating by categorical methods “...a recursion category, reflecting true arithmetic (or, equivalently, recursive functions in the real world)...” and “...a P -recursion category, reflecting formal arithmetic (or, equivalently, recursive functions considered with the eyes of PA)” [Mon89].

Heller adopted Rosolini’s definition of P -categories (in place of dominical categories) in his existence theorem for recursion categories, and gave examples of recursion categories to illustrate the application of the existence theorem to “contexts

²An example of a generalized Gödel incompleteness theorem is the statement that a consistent formal system is essentially undecidable, in the sense that the pair of sets (T, R) of codes of the provable and refutable sentences of the formal system is effectively inseparable [Tar53].

³Diagonalizable algebras are Boolean algebras with an operator corresponding to a provability term in Peano Arithmetic.

⁴The term “syntactic P -category” is due to A. Heller.

very distant from that of the natural or ordinal numbers and their subsets, the locus of the traditional theory” [Hel90]. Heller’s existence theorem applied to certain P -categories that are “locally connected.” Di Paola and Montagna subsequently produced generalizations of Montagna’s syntactic P -categories that are recursion categories (P -categories possessing a Turing morphism) that fail to be locally connected, and suggested the possibility of a more general existence theorem for recursion categories that would handle their examples [DPM91].

We give examples of P -categories of coalgebras exhibiting some of the properties of the P -categories of Di Paola and Montagna, such as the existence of weakly-total but non-total morphisms and the failure of local connectedness. Consideration of the examples of P -categories of coalgebras led us to a generalization Heller’s existence theorem which applies to non-locally connected iteration categories that can be embedded in a locally connected iteration category by a “locally” faithful functor preserving coproducts, the domain operator, and the iteration operator. We show that this generalization applies to our examples of non-locally connected P -categories of coalgebras; we also suggest that the study of the functors arising in the generalization of the existence theorem addresses the question posed by Montagna in his concluding remarks [Mon89].

The theory of universal coalgebras has recently undergone vigorous development as a descriptive study of automata, data structures, and the semantics of object oriented programming, among other applications [Rut00, Gum99]. Coalgebras as data structures include colorings, permutations, trees, partial self-mappings, generalized deterministic and nondeterministic automata, the real numbers, streams and formal

power series. Virtually every data structure arising in computer science can be realized coalgebraically [Gum99, PP99, PE98].

Let F be a nontrivial endofunctor on the category of sets such that F weakly preserves pullbacks, and such that the category $\mathbf{Set}_F$ of F -coalgebras has products. For example, if the forgetful functor $\mathbf{Set}_F \longrightarrow \mathbf{Set}$ has a right adjoint, then $\mathbf{Set}_F$ is complete [Kur00]. Under various mild assumptions on the endofunctor F , such as boundedness or accessibility, the right adjoint will exist [GS00c, Kur00, PW98]. The category $\mathbf{Set}_F$ may be embedded in the category $\mathbf{Pfn}_F$ of F -coalgebras and partial morphisms, which is a P -category that is prodominical but not dominical in general, and which may fail to be locally connected.

This thesis is organized as follows. In the first chapter we recall definitions and results needed from the theory of coalgebras that will be used to construct categories of coalgebras and partial morphisms.

In the second chapter, we review enough of G. Rosolini's theory of P -categories to construct P -categories of F -coalgebras, where F is an endofunctor on $\mathbf{Set}$ [Ros86]. We verify that subject to mild conditions on the endofunctor F , $\mathbf{Pfn}_F$ possesses certain algebraic properties required for the application of the generalization of Heller's existence theorem.

In the third chapter we reprove A. Heller's iteration lemma in the context $\mathbf{Pfn}_F$ using the property that contravariant representable functors for the coalgebra topology are sheaves. This allows us to simplify the definition of an iteration category given in Heller [Hel90]. We next review uniform generation and related notions (due to Heller) needed for the statement of the existence theorem.

In the fourth chapter, We state and prove a generalization of Heller's existence

theorem for recursion categories; this generalization will handle non-locally connected P -categories. The generalized existence theorem is applied to certain subcategories of $\mathbf{Pfn}_F$ to obtain examples of non-locally connected recursion categories of coalgebras.

The concluding chapter suggests open problems regarding the syntactic P -categories of Di Paola and Montagna; we discuss the problem of embedding of a certain syntactic P -category arising from Peano Arithmetic into the “classical” recursion category. If an embedding can be found, the generalized existence theorem will apply; if not, it is suggested that the obstruction is a non-trivial algebraic manifestation of incompleteness.

Chapter 1

Coalgebras

1.1 The category $\mathbf{Set}_F$

Let F be an endofunctor on the category $\mathbf{Set}$ of sets. An F -coalgebra (A, α) is a set A together with a map $\alpha : A \longrightarrow FA$ called its *structure map*. A morphism of F -coalgebras $\varphi : (A, \alpha) \longrightarrow (B, \beta)$ is a map $\varphi : A \longrightarrow B$ such that $\beta \circ \varphi = F\varphi \circ \alpha$. The F -coalgebras and their morphisms form a category, denoted by $\mathbf{Set}_F$. Coalgebras in this sense are more general than those associated with comonads [ML98]. The category $\mathbf{Set}_F$ is cocomplete, well-powered, co-well-powered, and is closed under homomorphic images [Gum99].

Let $U : \mathbf{Set}_F \longrightarrow \mathbf{Set}$ be the *forgetful* functor, which sends an F -coalgebra $\alpha : A \longrightarrow F(A)$ to the set A , and a morphism of coalgebras to itself as a morphism of $\mathbf{Set}$. Let (X, α) be a coalgebra. A subset A of UX has at most one coalgebra structure such that the inclusion $A \hookrightarrow UX$ induces a morphism of coalgebras. A *subcoalgebra* of a coalgebra (X, α) is a coalgebra (A, β) with $UA \subseteq UX$ and such that the inclusion $UA \hookrightarrow UX$ is a morphism. The functor U *creates colimits*; i.e., for each functor $G : \mathbf{J} \longrightarrow \mathbf{Set}_F$ from a small category $\mathbf{J}$ to $\mathbf{Set}_F$, if $\tau : UG \longrightarrow X$

is a colimit cone in **Set**, there is exactly one pair (A, σ) consisting of a coalgebra $A \in \mathbf{Set}_F$ with $UA = X$ and a colimit cone $\sigma : G \longrightarrow A$ in $\mathbf{Set}_F$ with $U\sigma = \tau$ [Gum99, ML98].

A commutative square in a category **C** is a *weak pullback* if it satisfies all of the conditions of a pullback except for the uniqueness requirement for induced morphisms. Analogously, one may define a *weak limit* of a diagram in **C**. A *sink* is a collection of morphisms of **C** with a common codomain. A *(weak) generalized pullback* is a (weak) limit of a sink. A functor *preserves weak (generalized) pullbacks* if it sends weak (generalized) pullbacks to weak (generalized) pullbacks. A functor *weakly preserves (generalized) pullbacks* if it sends (generalized) pullbacks to weak (generalized) pullbacks. If (generalized) pullbacks exist in **C**, and if $F : \mathbf{C} \longrightarrow \mathbf{Set}$ is a set valued functor, then F preserves weak (generalized) pullbacks if and only if F weakly preserves (generalized) pullbacks [Gum99]. More generally, if **C** has all limits of certain diagrams D , then F preserves weak limits of D -diagrams if and only if F weakly preserves limits of D -diagrams.

The following **Set** endofunctors preserve weak limits of sinks: the identity functor, all constant functors, functors of the form $X \longmapsto X \coprod Y$ for a fixed set Y , functors of the form $X \longmapsto X^Y$ for a fixed set Y , and the powerset functor. Moreover, sums, products and composites of **Set** endofunctors that preserve weak limits of sinks also preserve weak limits of sinks [GS00a].

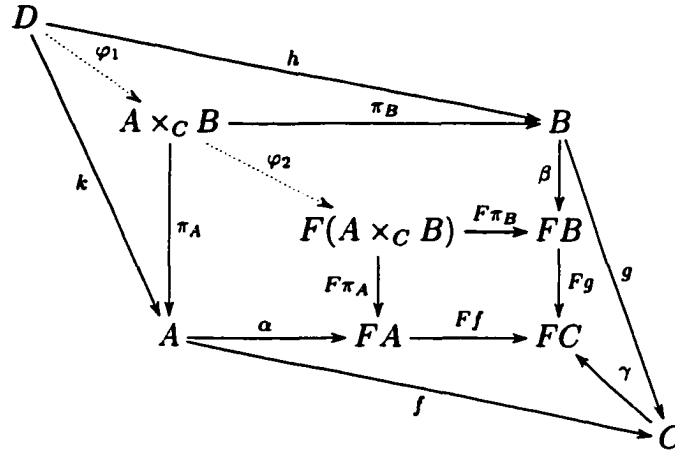
The forgetful functor $\mathbf{Set}_F \longrightarrow \mathbf{Set}$ preserves weak pullbacks (equivalently, weakly preserves pullbacks) if F does; this is Proposition 6.3 of Worrell [Wor98]. For convenience we supply our own proof.

Proposition 1.1.1 (Prop. 6.3 [Wor98]). *If the $\mathbf{Set}$ endofunctor F preserves weak pullbacks, then the forgetful functor $U : \mathbf{Set}_F \longrightarrow \mathbf{Set}$ also preserves weak pullbacks.*

Proof. Suppose that the following diagram is a weak pullback in $\mathbf{Set}_F$.

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

To show that this is a weak pullback in $\mathbf{Set}$, consider the following diagram in $\mathbf{Set}$.



The map φ_1 is induced by the universal property of the pullback in $\mathbf{Set}$ of f and g . Since F preserves weak pullbacks, the innermost square is a weak pullback. Since f and g are morphisms of coalgebras, the right hand triangles commute, and so there exists an induced map φ_2 , which defines a coalgebra structure on $A \times_C B$ such that π_A and π_B are morphisms of coalgebras. As $f\pi_A = g\pi_B$ in $\mathbf{Set}_F$, there exists an induced map $\varphi_3 : A \times_C B \longrightarrow P$; the composition $\varphi_3\varphi_1 : D \longrightarrow P$ does what we want. $\square$

1.2 The coalgebra topology

Let $\mathcal{L}X$ be the (partially ordered) set of subcoalgebras of the coalgebra X ; the set $\mathcal{L}X$ is a lattice with arbitrary joins; we follow Gumm's account [Gum99]. If $(A_i)_{i \in I}$ is a family of subcoalgebras of a coalgebra X , one can form their coproduct $\coprod_{i \in I} A_i$ in $\mathbf{Set}_F$; since the forgetful functor U preserves and creates colimits and since $\mathbf{Set}_F$ has ranges which are computed as they are in $\mathbf{Set}$, the map $\coprod_{i \in I} A_i \longrightarrow X$ from the coproduct into X induced by the inclusions $A_i \hookrightarrow X$ factors through the union $\bigcup_i U A_i$, which is the range of the induced map from the coproduct into X and which therefore must be a subcoalgebra of X ; this defines the join $\bigvee_i A_i$ in $\mathcal{L}X$. For any coalgebra X and any subset $S \subseteq UX$, the *coalgebra cogenerated by S* , denoted by $[S]$ is the join of all subcoalgebras A of X with $S \subseteq UA$. The meet in $\mathcal{L}X$ of a family $(A_i)_{i \in I}$ of subcoalgebras of a coalgebra X is defined by $\bigwedge_{i \in I} A_i = [\bigcap_{i \in I} U A_i]$.

Gumm shows that each topological space is a coalgebra for the filter functor F , which assigns to each set X the set FX of all filters on X ; the construction will be given in the sequel. The filter functor preserves weak pullbacks, but not generalized weak pullbacks [Gum01]. The example of topological spaces as coalgebras for the filter functor implies that the arbitrary intersection of subcoalgebras need not be a subcoalgebra, and one-cogenerated subcoalgebras need not exist. Gumm and Schröder give conditions on an endofunctor F which imply the existence of one-cogenerated subcoalgebras [GS00a].

Theorem 1.2.1 (Theorem 5.10 [GS00a]). *Let κ be a cardinal number. The $\mathbf{Set}$ endofunctor F weakly preserves κ -pullbacks of injections (monomorphisms in $\mathbf{Set}$) if and only if the intersection of a κ -family of subcoalgebras is a subcoalgebra.*

The set endofunctor F is *nontrivial* if for any set X , $FX = \emptyset$ implies $X = \emptyset$. If the **Set** endofunctor F is nontrivial, and if X is an F -coalgebra, then the set ULX of underlying sets of subcoalgebras of X forms a topology on X called the *coalgebra topology* [GS00c]. Since the coalgebra topology plays an important role in this work, we will sketch Gumm and Schröder's proof that ULX is closed under finite intersections.¹

Theorem 1.2.2 (Gumm, Schröder [GS00c]). *Let F be a nontrivial **Set** endofunctor. If A, B are subcoalgebras of the F -coalgebra C , then $U(A \wedge B) = UA \cap UB$.*

Proof. Since $U(A \wedge B) \subseteq UA \cap UB$, we may assume that $UA \cap UB \neq \emptyset$. Choose $c \in UA \cap UB$ and define maps $p : UA \longrightarrow UA \cap UB$ and $q : UB \longrightarrow UA \cap UB$ by

$$p(a) = \begin{cases} a, & \text{if } a \in UA \cap UB; \\ c, & \text{otherwise,} \end{cases} \quad \text{and} \quad q(b) = \begin{cases} b, & \text{if } b \in UB; \\ c, & \text{otherwise.} \end{cases}$$

Next, consider the following cubical diagram, in which the coalgebra structure β is defined on $UA \cap UB$ by means of the left hand face. The top face commutes in both directions; by functoriality so does the bottom face. The front face commutes since A is a subcoalgebra of C ; since B is a subcoalgebra, $\alpha\ell = F(\ell)(\alpha|B)$; moreover

¹Our version of Gumm and Schröder's is a paraphrase; their statement ends with "...if U and V are subcoalgebras of A , then so is $U \cap V$." [GS00c].

which are coalgebras for the **Set** endofunctor given by $X \longmapsto A \times X$, where A is a fixed nonempty set [Gum99, PE98, PP99]. In each of these examples, the type endofunctor preserves weak limits of sinks.

If F is nontrivial, each F -coalgebra has the coalgebra topology, so the forgetful functor U factors through the subcategory of topological spaces and continuous open maps of **Top**, the category of topological spaces and continuous maps. Kurz [Kur00] has shown that if the forgetful functor U has a right adjoint, then $\mathbf{Set}_F$ is complete. The existence of the right adjoint implies that the cofree coalgebra on any set exists; limits are constructed as certain subcoalgebras of cofree coalgebras. Gumm and Schröder construct such a right adjoint to the forgetful functor U in the case when the endofunctor F is bounded; hence if F is bounded, $\mathbf{Set}_F$ is complete [GS00c].

Let $X_i, i = 0, 1$ be F -coalgebras. A *bisimulation* between X_0 and X_1 is a relation $R \subseteq UX_0 \times UX_1$ equipped with an F -coalgebra structure $\rho : R \longrightarrow F(R)$, called a *bisimulation structure*, such that the projections $\pi_i : R \longrightarrow X_i, i = 0, 1$ are morphisms of coalgebras [Gum99, Rut00]. A bisimulation structure need not be unique if it exists. If X is a coalgebra, and if $\varphi_i : X \longrightarrow Y_i, i = 0, 1$ are coalgebra morphisms, then $(U\varphi_0 \times U\varphi_1)UX$ is a bisimulation between Y_0 and Y_1 [Gum99]. Let X, Y be coalgebras and let $\varphi : UX \longrightarrow UY$ be a map in **Set**. Then φ defines a coalgebra morphism $X \longrightarrow Y$ if and only if its graph is a bisimulation; it follows that for any coalgebra X , the diagonal $\Delta_X = \{(x, x) : x \in UX\}$ is a bisimulation [Rut00].

1.4 Diagram lemmas and factorizations

The following so-called diagram lemmas are useful.

Lemma 1.4.1 ([Rut00]). *Let A, B, C be coalgebras, let $\phi : A \longrightarrow C$ be a coalgebra morphism, and let $f : A \longrightarrow B$, $g : B \longrightarrow C$ be maps in **Set** such that the following diagram commutes.*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & C \\ & \searrow f & \nearrow g \\ & B & \end{array}$$

1. *If f is a surjective coalgebra morphism, then g is a morphism of coalgebras.*
2. *If g is an injective coalgebra morphism, then f is a morphism of coalgebras.*

Lemma 1.4.2 (First Diagram Lemma. [Gum99]). *Let A, B and $C \neq \emptyset$ be coalgebras and let $\phi : A \longrightarrow B$, $\psi : A \longrightarrow C$ be coalgebra morphisms. If ϕ is an epimorphism, then there is a unique coalgebra morphism $\nu : B \longrightarrow C$ with $\nu \circ \phi = \psi$ if and only if for each $x, y \in A$, $\phi x = \phi y$ implies $\psi x = \psi y$.*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \searrow \psi & \downarrow \nu \\ & & C \end{array}$$

Lemma 1.4.3 (Second Diagram Lemma. [Gum99]). *Let A, B and C be coalgebras and $\phi : B \longrightarrow A$, $\psi : C \longrightarrow A$ coalgebra morphisms. If ϕ is a monomorphism, then there is a unique coalgebra morphism $\nu : C \longrightarrow B$ with $\phi \circ \nu = \psi$ if and only if $\psi[C] \subseteq \phi[B]$.*

$$\begin{array}{ccc} A & \xleftarrow{\phi} & B \\ & \nwarrow \psi & \uparrow \nu \\ & & C \end{array}$$

Although a morphism in $\mathbf{Set}_F$ that is injective in $\mathbf{Set}$ is a monomorphism in $\mathbf{Set}_F$, the converse does not hold in general [Gum99, Rut00], and so monomorphisms do not necessarily correspond to subcoalgebras in $\mathbf{Set}_F$. To obtain a workable notion of a partial map in the sequel, we will want monomorphisms to correspond to subcoalgebras; we use the following result on factorization systems in $\mathbf{Set}_F$ due to Kurz, who introduced in his thesis the technique of lifting factorization systems from the base category to obtain a characterization of subcoalgebras in general categories of coalgebras [Kur00].²

Let $\mathbf{C}$ be a category. A monomorphism is *regular* if it is an equalizer. A monomorphism m in $\mathbf{C}$ is *extremal* if and only if $m = fe$ for some f and for some epi e implies that e is an isomorphism. A monomorphism m is *strong* if and only if for all epis e and for all f, g such that the following square commutes, there is a unique d such that the triangles in the following diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ f \downarrow & \swarrow d & \downarrow g \\ C & \xrightarrow{m} & D \end{array}$$

Strong monomorphisms are closed under composition, intersection and left cancellation [AR94, Bor94]. The classes of monomorphisms and of regular, strong and extremal monomorphisms of $\mathbf{C}$ will be denoted by $\mathbf{Mono}$, $\mathbf{RegMono}$, $\mathbf{StrongMono}$ and $\mathbf{ExtrMono}$, respectively. We have the inclusions

$$\mathbf{ExtrMono} \subseteq \mathbf{StrongMono} \subseteq \mathbf{RegMono} \subseteq \mathbf{Mono}.$$

Theorem 1.4.4 (Theorem 1.3.9 [Kur00]). *The following assertions hold in $\mathbf{Set}_F$.*

²It is sufficient for our application to cite Proposition 4.7 of [Rut00], which states that if F preserves weak pullbacks, then monomorphisms of $\mathbf{Set}_F$ are precisely the injective morphisms; however, the use of factorization systems applies to more general categories of coalgebras.

1. *(Epi, StrongMono) is a factorization system in $\mathbf{Set}_F$. Moreover, Epi contains the surjective coalgebra morphisms, StrongMono contains precisely the injective morphisms, and (Epi, StrongMono)-factorizations are calculated as (Epi, Mono)-factorizations in $\mathbf{Set}$.*
2. *The classes ExtrMono, StrongMono and RegMono coincide.*
3. *If F preserves weak pullbacks, then Mono coincides with StrongMono and hence with ExtrMono and RegMono.*

We note that the class of epimorphisms coincides with the class of strong epimorphisms if and only if the class of monomorphisms coincides with the class of strong monomorphisms; in this case we may unambiguously speak of strong epi-mono factorizations.

Chapter 2

P -categories of coalgebras

In this chapter, a wide class of categories and partial morphisms will be constructed that will satisfy conditions needed for the application of Heller's existence theorem for recursion categories, and its generalization. Let F denote an endofunctor on **Set**. Under certain restrictions on F , the category $\mathbf{Set}_F$ can be extended to the category of F -coalgebras and partial morphisms, denoted by $\mathbf{Pfn}_F$, which we will obtain by following the procedure in Rosolini's thesis [Ros86] for constructing a category of partial maps.

2.1 P -categories

Let $\mathbf{C}$ be a category with finite limits. A *dominion* is a class $\mathcal{M}$ of monomorphisms of $\mathbf{C}$ that is closed under identities, composition and pullbacks along morphisms of $\mathbf{C}$ [Ros86]. If $\mathcal{M}$ is a dominion of $\mathbf{C}$ and if X and Y are objects of $\mathbf{C}$, a *partial map from X to Y defined in $\mathcal{M}$* is a pair (m, φ) , where $m : U \longrightarrow X$ is in $\mathcal{M}$, and where $\varphi : U \longrightarrow Y$ is a morphism of $\mathbf{C}$. Two such pairs (m, φ) and (m', φ') are *equivalent*

if there exists an isomorphism θ making the following obvious diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & Y \\ m \downarrow & \theta & \uparrow \varphi' \\ X & \xleftarrow{m'} & U' \end{array}$$

The equivalence class of the pair (m, φ) is denoted by $\{m, \varphi\}$ and is called a *partial morphism from X to Y* , also denoted by $X \dashrightarrow Y$. Composition of partial morphisms $A \dashrightarrow B$ and $B \dashrightarrow C$ is defined by the following diagram in which the square is a pullback.

$$\begin{array}{ccccc} W & \longrightarrow & V & \xrightarrow{\varphi'} & C \\ \downarrow & & \downarrow m' & \nearrow \{m', \varphi'\} & \\ U & \xrightarrow{\varphi} & B & & \\ \downarrow m & \nearrow \{m, \varphi\} & & & \\ A & & & & \end{array}$$

The equivalence class $\{m'', \varphi''\}$ of the composite is determined by the monomorphism $m'' : W \longrightarrow A$ obtained as the composite of the morphisms of the left vertical column, and the morphism $\varphi'' : W \longrightarrow C$ obtained as the composite of the morphisms of the top horizontal row. We can then express the composite by the equation

$$\{m'', \varphi''\} = \{m', \varphi'\} \circ \{m, \varphi\}.$$

The composite is associative. For each object A , the equivalence class $\{1_A, 1_A\}$ acts as an identity for composition. Given a category $\mathbf{C}$ with finite products together with a dominion $\mathcal{M}$, the *category of partial maps* $P(\mathbf{C}, \mathcal{M})$ is the category whose objects are those of $\mathbf{C}$ and whose morphisms are the equivalence classes of partial maps defined in $\mathcal{M}$.

We will define the category $\mathbf{Pfn}_F$ as $P(\mathbf{Set}_F, \mathcal{M})$ by taking the dominion $\mathcal{M}$ to be the class of monomorphisms of $\mathbf{Set}_F$ subject to conditions on the endofunctor F

such that $\mathbf{Set}_F$ will have products and such that monomorphisms will correspond to subcoalgebras.

For the first requirement, we note that if the forgetful functor $\mathbf{Set}_F \longrightarrow \mathbf{Set}$ has a right adjoint, which happens, for example, if F is bounded or accessible, then $\mathbf{Set}_F$ is complete [GS00c, Kur00, PW98]; in particular, $\mathbf{Set}_F$ has products.

For the second requirement, we rely on Kurz's Theorem 1.4.4 in the preceding chapter and suppose that F preserves weak pullbacks (equivalently, that F weakly preserves pullbacks).

In virtue of the preceding, unless stated otherwise we will assume from now on that the $\mathbf{Set}$ endofunctor F is nontrivial, preserves weak pullbacks, is such that $\mathbf{Set}_F$ has products and that the class $\mathcal{M}$ of monomorphisms of $\mathbf{Set}_F$ is closed under pullbacks along arbitrary morphisms. Fixing such an F , we define the dominion $\mathcal{M}$ in $\mathbf{Set}_F$ to be the class of monomorphisms of $\mathbf{Set}_F$ and we set $\mathbf{Pfn}_F = P(\mathbf{Set}_F, \mathcal{M})$. Next, we extend the product $\times : \mathbf{Set}_F^2 \longrightarrow \mathbf{Set}_F$ to $\boxtimes : \mathbf{Pfn}_F^2 \longrightarrow \mathbf{Pfn}_F$ by defining it on objects X, Y by $X \boxtimes Y = X \times Y$ and on partial morphisms by the formula

$$\{m, \varphi\} \boxtimes \{m', \varphi'\} = \{m \times m', \varphi \times \varphi'\}.$$

This assignment makes sense since $m \times m' \in \mathcal{M}$ and is independent of the choice of representatives. The reason for introducing the box product notation is to distinguish it among the four products in use, namely, the product of categories, the product in $\mathbf{Set}$, the product in $\mathbf{Set}_F$ and its extension to a *near product*, to be defined, on $\mathbf{Pfn}_F$. In general, with these definitions, one has the following proposition.

Proposition 2.1.1 (Rosolini). *Let $\mathbf{A}$ be a category with finite products, and let $\mathcal{M}$ be a dominion. Then there exists an embedding $J : \mathbf{A} \longrightarrow P(\mathbf{A}, \mathcal{M})$; moreover the product defined on the image of $\mathbf{A}$ extends to a bifunctor on $P(\mathbf{A}, \mathcal{M})$.*

With these definitions, a coalgebraic version of Rosolini's Proposition 2.1.1 is immediate [Ros86].

Proposition 2.1.2. *There is a faithful embedding of $\mathbf{Set}_F$ into $\mathbf{Pfn}_F$ which sends each F -coalgebra to itself, and which sends the F -coalgebra morphism $\varphi : A \longrightarrow B$ to the equivalence class $\{1_A, \varphi\}$. The product $\times$ on $\mathbf{Set}_F$ extends to the bifunctor $\boxtimes$ on $\mathbf{Pfn}_F$.*

Since the forgetful functor $U : \mathbf{Set}_F \longrightarrow \mathbf{Set}$ creates colimits, it follows that the category $\mathbf{Pfn}_F$ is cocomplete.

2.2 P - and $P\Sigma$ - categories of coalgebras

The category $\mathbf{Pfn}_F$ with the product $\boxtimes$ has the structure of a P -category, a notion, due to Rosolini, which generalizes the dominical categories of Di Paola and Heller and the B -categories (categories with a binary product) of Heller [Ros86, DPH87, Hel90]. The category $\mathbf{Pfn}$ of sets and partial functions is the prototypical P -category. We recall the relevant definitions from Heller with slight changes in notation and with some additional remarks [Hel90].

Let $F, G : \mathbf{C} \longrightarrow \mathbf{D}$ be functors. An *infranatural transformation* $\phi : F \longrightarrow G$ is a family of morphisms $\phi_X : FX \longrightarrow GX$, called components, for each object X of $\mathbf{C}$. The *naturalizer* of ϕ , denoted by $\text{nat}\phi$, is the largest subcategory of $\mathbf{C}$ containing all the objects of $\mathbf{C}$ such that ϕ is a natural transformation $F|_{\text{nat}\phi} \longrightarrow G|_{\text{nat}\phi}$.

Let $\mathbf{C}$ be a category. The *diagonal functor* $\delta_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}$ is given on objects by $A \longmapsto (A, A)$ and on morphisms by $f \longmapsto (f, f)$.

A P -category consists of a category $\mathbf{C}$ together with a functor $\boxtimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$,

called a *near product*, a natural transformation $\Delta : 1_{\mathbf{C}} \longrightarrow \boxtimes \circ \delta_{\mathbf{C}}$, and infranatural transformations $p_i : \boxtimes \longrightarrow \pi_i$, where $\pi_i : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ is the projection onto the i -th factor for $i = 0, 1$. For morphisms $f : X \longrightarrow Y$, $g : X \longrightarrow Z$ of $\mathbf{C}$, we set $\langle f, g \rangle = (f \boxtimes g) \Delta_X : X \longrightarrow Y \boxtimes Z$. These functors and transformations are subject to the following four conditions:

i) For objects X in $\mathbf{C}$ the following diagrams commute.

$$\begin{array}{ccc} X & & X \boxtimes X \xrightarrow{\Delta_{X \boxtimes X}} (X \boxtimes X) \boxtimes (X \boxtimes X) \\ \Delta_X \downarrow & \searrow 1 & \searrow 1 \downarrow p_{0X, X} \boxtimes p_{1X, X} \\ X \boxtimes X & \xrightarrow{p_{iX, X}} & X \boxtimes X \end{array}$$

where $i = 0, 1$.

ii) If $\mathbf{P} \subseteq \mathbf{C}$ is the smallest subcategory closed under $\boxtimes$ containing all components of $p_i, i = 0, 1$, then $\mathbf{C} \times \mathbf{P} \subseteq \text{nat} p_0$ and $\mathbf{P} \times \mathbf{C} \subseteq \text{nat} p_1$. This implies that that projections satisfy certain identities; e.g., as in the following naturality square diagram, where we have used the property that the component $p_{0Y, Z} : Y \boxtimes Z \longrightarrow Y$ is in $\mathbf{P}$, and therefore $(1_X, p_{0Y, Z})$ is in $\text{nat} p_0$.

$$\begin{array}{ccccc} (X, Y \boxtimes Z) & & X \boxtimes (Y \boxtimes Z) & \xrightarrow{p_{0X, Y \boxtimes Z}} & X \\ (1_X, p_{0Y, Z}) \downarrow & & 1_X \boxtimes p_{0Y, Z} \downarrow & & \downarrow 1_X \\ (X, Y) & & X \boxtimes Y & \xrightarrow{p_{0X, Y}} & X \end{array}$$

Moreover, for any morphism $\phi : X \longrightarrow Y$ and for any object Z of $\mathbf{C}$, the pair $(\phi, 1_Z) \in \mathbf{C} \times \mathbf{P}$, and so the following diagram commutes.

$$\begin{array}{ccccc} (X, Z) & & X \boxtimes Z & \xrightarrow{p_{0X, Z}} & X \\ (\phi, 1_Z) \downarrow & & \phi \boxtimes 1_Z \downarrow & & \downarrow \phi \\ (Y, Z) & & Y \boxtimes Z & \xrightarrow{p_{0Y, Z}} & Y \end{array}$$

The essential point is that the corresponding diagram with p_1 does not commute in a P -category, unless ϕ is total (to be defined).¹

iii) There is a natural isomorphism

$$\text{ass}_{\boxtimes} : ((- \boxtimes -) \boxtimes -) \longrightarrow (- \boxtimes (- \boxtimes -))$$

of functors $\mathbf{C}^3 \longrightarrow \mathbf{C}$ whose component $\text{ass}_{\boxtimes X, Y, Z}$ is given by

$$\langle p_{0X, Y} p_{0X \boxtimes Y, Z}, \langle p_{1X, Y} p_{0X \boxtimes Y, Z}, p_{1X \boxtimes Y, Z} \rangle \rangle : (X \boxtimes Y) \boxtimes Z \rightarrow X \boxtimes (Y \boxtimes Z). \quad (2.2.1)$$

iv) Let tr_X be the endofunctor on $\mathbf{C}^2$ given by $(X, Y) \longmapsto (Y, X)$. There is a natural isomorphism

$$\text{tr}_{\boxtimes} : \boxtimes \longrightarrow \boxtimes \circ \text{tr}_X \quad (2.2.2)$$

of functors $\mathbf{C}^2 \longrightarrow \mathbf{C}^2$ whose component $\text{tr}_{\boxtimes X, Y}$ is given by

$$\langle p_{1X, Y}, p_{0X, Y} \rangle : X \boxtimes Y \longrightarrow Y \boxtimes X.$$

The natural isomorphisms $\text{ass}_{\boxtimes}$ and $\text{tr}_{\boxtimes}$ must make $\boxtimes$ coherently associative and commutative; i.e., the natural isomorphism $\text{tr}_{\boxtimes}$ must satisfy the condition $\text{tr}_{\boxtimes Y, X} \circ \text{tr}_{\boxtimes X, Y} = 1_{X \boxtimes Y}$ and the hexagonal coherence condition; the natural isomorphism $\text{ass}_{\boxtimes}$ must satisfy the pentagonal coherence condition [ML98, ML63].

In Rosolini's original definition, a P -category is a category $\mathbf{C}$ with a bifunctor $\boxtimes$ and a natural transformation Δ as above, but instead of infranatural transformations $p_i : \boxtimes \longrightarrow \pi_i$ of functors $\mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ (for $i = 0, 1$), Rosolini specifies for each object X of $\mathbf{C}$ natural transformations $p_{0(), X} : () \boxtimes X \longrightarrow 1_{\mathbf{C}}$ and

¹This is also to reassure the reader that $\mathbf{C} \times \mathbf{P} \subseteq \text{nat}_{p_0}$, and not $\mathbf{P} \times \mathbf{C} \subseteq \text{nat}_{p_0}$, was intended.

$p_{1X,(\)} : X \boxtimes (\) \longrightarrow 1_{\mathbf{C}}$ such that the following identities hold and such that conditions iii) and iv) above are satisfied, along with the coherence requirement [Ros86].

$$\begin{aligned} p_{0X,X}\Delta_X &= 1_X = p_{1X,X}\Delta_X, & (p_{0X,Y} \boxtimes p_{1X,Y})\Delta_{X\boxtimes Y} &= 1_{X\boxtimes Y}, \\ p_{0X,Y}(1_X \boxtimes p_{0Y,Z}) &= p_{0X,Y\boxtimes Z}, & p_{0X,Z}(1_X \boxtimes p_{1Y,Z}) &= p_{0X,Y\boxtimes Z}, \\ p_{1X,Z}(p_{0X,Y} \boxtimes 1_Z) &= p_{1X\boxtimes Y,Z}, & p_{1Y,Z}(p_{1X,Y} \boxtimes 1_Z) &= p_{1X\boxtimes Y,Z}. \end{aligned}$$

Proposition 2.2.1. *Heller's and Rosolini's definitions of P -categories are equivalent.*

Proof. Starting with Rosolini's definition, condition i) of Heller's definition is satisfied; noting that the collection of natural transformations $p_{0(\),X}, p_{1X,(\)}$ for $X \in \mathbf{C}_0$ define infranatural transformations $p_i : \boxtimes \longrightarrow \pi_i$ (for $i = 0, 1$) such that ii) is satisfied, it follows that Rosolini's definition implies Heller's. On the other hand, the infranatural transformations of Heller's definition together with conditions i) and ii) above imply the existence of natural transformations $p_{0(\),X}, p_{1X,(\)}$ for each $X \in \mathbf{C}_0$ such that the identities of Rosolini's definition hold. $\square$

The preceding equations can be interpreted to mean that the projections of a P -category are “as close to natural” as one might expect in a category of partially defined morphisms. In this connection, one has the following psychologically satisfying proposition. It most likely belongs to the folklore of the subject; however, the author has been unable to locate its statement in the literature; the proof has been included since it seems to require a combinatorial trick.

Proposition 2.2.2. *The projections p_i of a P -category $\mathbf{C}$ are natural if and only if for each pair $\varphi_i : Y \longrightarrow X_i$ (for $i = 0, 1$) of morphisms in $\mathbf{C}$, $p_i\langle\varphi_0, \varphi_1\rangle = \varphi_i$.*

Proof. We sketch the proof. That naturality of the projections implies $p_i\langle\varphi_0, \varphi_1\rangle = \varphi_i$ ($i = 0, 1$) is straightforward; for the converse, one can use the hypothesis to show

that for any objects A and B and for any morphism $\mu : A \longrightarrow B$ of $\mathbf{C}$, one has the following.

$$\begin{array}{ccc} A \boxtimes A & \xrightarrow{p_0} & A \\ 1_A \boxtimes \mu \downarrow & & \downarrow 1_A \\ A \boxtimes B & \xrightarrow{p_0} & A \end{array}$$

Taking $A := X \boxtimes Y$, $B := X \boxtimes Z$ and $\mu := 1_X \boxtimes \phi$, in the preceding diagram, one obtains the center square of the following cubical diagram.

$$\begin{array}{ccccc} X \boxtimes Y & & \xrightarrow{p_0} & & X \\ & \searrow p_1 & & \nearrow p_0 & \\ & (X \boxtimes Y) \boxtimes (X \boxtimes Y) & \xrightarrow{p_0} & X \boxtimes Y & \\ & \downarrow (1_{X \boxtimes Y}) \boxtimes (1_X \boxtimes \phi) & & \downarrow 1_{X \boxtimes Y} & \\ & (X \boxtimes Y) \boxtimes (X \boxtimes Z) & \xrightarrow{p_0} & X \boxtimes Y & \\ & \nearrow p_1 & & \searrow p_0 & \\ X \boxtimes Z & & \xrightarrow{p_0} & & X \end{array}$$

$1_X \boxtimes \phi$ (left vertical arrow), 1_X (right vertical arrow)

The left trapezoid commutes in the downward direction (in the direction of the diagonal) by the assumption and the remaining trapezoids commute in any P -category. A diagram chase then shows that the outer square commutes. Pasting the outer square on top of another square (valid in any P -category) of the form

$$\begin{array}{ccc} X \boxtimes Z & \longrightarrow & X \\ \psi \boxtimes 1_Y \downarrow & & \downarrow \psi \\ W \boxtimes Z & \longrightarrow & W \end{array}$$

and composing, we obtain finally the naturality of the projection p_0 ; by symmetry, p_1 is also natural. $\square$

With the preceding definitions, we paraphrase Rosolini's Theorem 2.1.9, which gives conditions for a category $\mathbf{C}$ to be a P -category; essentially one verifies the preceding identities on the subcategory $\mathbf{C}_t$ of total maps of $\mathbf{C}$, to be defined [Ros86].

Theorem 2.2.3 (Rosolini). *Let $\mathbf{C}$ be a category with a bifunctor $\boxtimes$, a natural transformation Δ and infranatural transformations p_i , $i = 0, 1$ as above. The category $\mathbf{C}$ is a P -category if and only if there exists a wide subcategory $\mathbf{B}$ of $\mathbf{C}$ such that the bifunctor $\boxtimes$ and the infranatural transformations p_i define a cartesian product on $\mathbf{B}$, and the derived associativity and commutativity isomorphisms are natural and coherent with respect to maps in $\mathbf{C}$.²*

Applying the preceding theorem with $\mathbf{C} = \mathbf{Pfn}_F$ and $\mathbf{B} = \mathbf{Set}_F$, we have the following.

Theorem 2.2.4. *The category $\mathbf{Pfn}_F$ is a P -category.*

2.3 Prodominical categories

A *system of zero morphisms* is a collection of morphisms $0_{X,Y} : X \longrightarrow Y$ for each pair of objects X and Y of $\mathbf{C}$ such that for objects W, Z and morphisms $f : W \longrightarrow X$ and $g : Y \longrightarrow Z$ of $\mathbf{C}$, one has $g0_{X,Y}f = 0_{W,Z}$. A system of zero morphisms is unique if it exists. For any coalgebra A , the empty map $0_{\emptyset,A} : \emptyset \longrightarrow A$ is vacuously a morphism in $\mathbf{Set}_F$. The equivalence classes $\{0_{\emptyset,A}, 0_{\emptyset,B}\}$ for coalgebras A and B form a system of zeros of $\mathbf{Pfn}_F$.

A *prodominical category* $\mathbf{C}$ is a P -category that is *pointed*; i.e., $\mathbf{C}$ contains a system of zero morphisms and, for any $\phi : A \longrightarrow B$, $\phi \boxtimes 0_{C,D} = 0_{A \boxtimes C, B \boxtimes D}$. In a P -category $\mathbf{C}$, the *domain* $\text{dom}\phi$ of a morphism $\phi : X \longrightarrow Y$ is the composite $p_{0X,Y} \circ \langle 1_X, \phi \rangle : X \longrightarrow X$. For an object X of $\mathbf{C}$, let $\text{dom}(X)$ denote the set of domains $\text{dom}\phi$ for morphisms $\phi : X \longrightarrow Y$. The set $\text{dom}(X)$ is a meet semilattice with

²A subcategory $\mathbf{B}$ of a category $\mathbf{C}$ is a *wide subcategory* if $\mathbf{B}_0 = \mathbf{C}_0$; i.e., if $\mathbf{B}$ and $\mathbf{C}$ have the same class of objects.

meet defined by composition; the domain operator on morphisms satisfies algebraic properties that will be recalled as needed [DPH87, Hel90, Ros86]. If $\alpha, \beta \in \text{dom}(X)$, we write $\alpha \leq \beta$ if $\alpha\beta = \alpha$; we write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

A morphism $f : X \longrightarrow Y$ of a P -category is *total* if $\text{dom}\phi = 1_X$. A morphism $f : X \longrightarrow Y$ of a pointed category is *weakly total* if for all $\phi : W \longrightarrow X$, $f\phi = 0_{W,Y}$ implies that $\phi = 0_{W,X}$.

We take the following proposition from Heller [Hel90] as the definition of a dominical category. A prodominical category is *dominical* if every weakly total morphism is total and if $\phi \boxtimes \psi = 0$ implies $\phi = 0$ or $\psi = 0$. In Di Paola and Heller, weak totality was taken to be the definition of totality; in a dominical category, these notions coincide [DPH87]. In a general P -category, these notions diverge; however, P -categories retain much of the phenomena of classical recursion theory as do the dominical categories [Mon89, Ros86].

The category $\mathbf{Pfn}_F$ can fail to be dominical due to certain pathologies of the product in $\mathbf{Set}_F$; Gumm and Schröder [GS00c] give examples of three finite coalgebras A, B, C for the finite powerset functor such that $A \times A \cong A$, $A \times B = \emptyset$ and such that $C \times C$ is infinite. These possibilities will necessitate some care regarding products in the sequel.

Proposition 2.3.1. *There exists a nontrivial, bounded $\mathbf{Set}$ endofunctor F that weakly preserves pullbacks such that $\mathbf{Pfn}_F$ is prodominical but not dominical.*

Proof. The finite powerset functor P_ω is nontrivial, preserves weak pullbacks and is bounded [GS00c, GS00a, PW98], hence $\mathbf{Pfn}_{P_\omega}$ is prodominical. Moreover there exist two nonempty finite coalgebras A and B for the finite powerset functor such that $A \times B = \emptyset$ [GS00c]. Let $i_A : A \longrightarrow A \amalg B$ be the coproduct injection in $\mathbf{Set}_{P_\omega}$,

and let 1_B be the identity on B . By previous remarks,

$$i_A \times 1_B : A \times B \longrightarrow (A \amalg B) \times B = 0_{\emptyset, B \times B},$$

and therefore there exist nonzero $\phi, \psi \in \mathbf{Pfn}_{P_{\mathcal{L}}}$ with $\phi \boxtimes \psi = 0$. $\square$

It is straightforward to produce a category of coalgebras and partial morphisms which possesses a non-total weakly total morphism. In $\mathbf{Pfn}_F$, weak totality can be interpreted by means of the coalgebra topology. The topological criterion given below for a weakly total morphism to be total can be used to show that $\mathbf{Pfn}_F$ is not dominical in general; Montagna makes a related observation about certain P -categories obtained from Peano Arithmetic [Mon89]. The following statements are immediate.

Proposition 2.3.2. *Let $\varphi : X \longrightarrow Y = \{ m : W \rightharpoonup X, \psi : W \longrightarrow Y \}$ be a morphism in $\mathbf{Pfn}_F$, with m mono.*

i) The morphism φ is weakly total if and only if the image $m[W]$ of m is dense in the coalgebra topology on X .

ii) If the subcoalgebra U of X is dense in X , then the partial morphism $\{i, 1_U\}$ defined by

$$\begin{array}{ccc} U & \xrightarrow{1_U} & U \\ i \downarrow & \nearrow \{i, 1_U\} & \\ X & & \end{array}$$

is weakly total.

iii) The coalgebra X is irreducible in the coalgebra topology if and only if every morphism $\varphi : X \longrightarrow Y$ other than $0_{X,Y}$ is weakly total.

iv) Every weakly total morphism $\varphi : X \longrightarrow Y$ is total if and only if X is the only nonempty open dense subset of X .

v) If X is Hausdorff, then every weakly total morphism $\varphi : X \longrightarrow Y$ is total.

To obtain a non-total weakly total morphism in $\mathbf{Pfn}_F$, it suffices to find a coalgebra with a proper nonempty subcoalgebra that is dense in the coalgebra topology.

Proposition 2.3.3. *Let $1_{\mathbf{Set}}$ be the identity functor on $\mathbf{Set}$. Then $\mathbf{Pfn}_{1_{\mathbf{Set}}}$ is prodominical but not dominical.*

Proof. A coalgebra for the identity functor is given by a self map of a set. Let $X = \{x, y\}$ be a set with $x \neq y$, and define $\alpha : X \longrightarrow X$ by $\alpha(x) = y$ and $\alpha(y) = y$. Then $U = \{y\}$ is a proper subcoalgebra of X that is open and dense in the coalgebra topology on X . It follows from Proposition 2.3.2, statement iv) that the partial map $X \longrightarrow U$ of statement ii) is weakly total but not total. $\square$

We record some properties of the product in $\mathbf{Set}_F$ that will be needed in the sequel. Fortunately, despite the example of two finite nonempty coalgebras A, B for which $A \times B = \emptyset$, we have the following fact; this justifies the existence of the free semigroup (to be defined) in $\mathbf{Set}_F$.

Proposition 2.3.4. *If $\mathbf{Set}_F$ has finite products, and if A is a nonempty F -coalgebra, then for each $n \geq 1$, A^n is nonempty.*

Proof. Let $U : \mathbf{Set}_F \longrightarrow \mathbf{Set}$ be the forgetful functor. If A is nonempty, so is the diagonal $\Delta_A = \{(a, a) : a \in U(A)\}$, which is a bisimulation on A [Rut00]. Lemma 8.1 of [GS00c] states that if $A_0 \times A_1$ is a product in $\mathbf{Set}_F$ with projections $\pi_i, i = 0, 1$, then $(U\pi_0 \times U\pi_1)U(A_0 \times A_1)$ is the largest bisimulation between A_0 and A_1 ; taking $A_0 = A_1 = A$, we have that $\emptyset \neq \Delta_A \subseteq (U\pi_0 \times U\pi_1)U(A \times A)$ so that $A \times A$ cannot be empty. It follows that $A^n \neq \emptyset$ for any $n \geq 1$. $\square$

The product in $\mathbf{Set}_F$ distributes over the coproduct.³ This follows essentially from theorem 6.4 of Worrel [Wor98], which states that if F is a bounded endofunctor which preserves weak pullbacks, then $\mathbf{Set}_F$ is a full reflective subcategory of a Grothendieck topos. Moreover, assuming only that F is nontrivial and preserves weak pullbacks, Worrel's proof implies that in $\mathbf{Set}_F$, coproducts are disjoint and stable under pullback, every epimorphism is the coequalizer of its kernel pair, and epimorphisms are stable under pullback.

Proposition 2.3.5. *Let F be a nontrivial $\mathbf{Set}$ endofunctor that weakly preserves pullbacks and such that $\mathbf{Set}_F$ has products and a terminal object. Let X be an F -coalgebra, let I be a set, and let $\{Y_\alpha\}_{\alpha \in I}$ be a family of F -coalgebras. There is a canonical isomorphism*

$$X \times \coprod_I Y_\alpha \longrightarrow \coprod_I (X \times Y_\alpha)$$

Proof. By a result of Gumm and Schröder [GS00a], the preimage of a subcoalgebra is a subcoalgebra if and only if F weakly preserves pullbacks along monomorphisms; hence pullbacks of subcoalgebras are subcoalgebras in $\mathbf{Set}_F$. Paraphrasing Worrel [Wor98], since the forgetful functor $U : \mathbf{Set}_F \longrightarrow \mathbf{Set}$ creates colimits and since U weakly preserves pullbacks if F does, coproducts are disjoint and stable under pullback. Fix coalgebras A and B ; it follows by standard arguments [MLM92] that the pullback operation $- \times_A B$ distributes over coproducts; taking A to be the terminal object, it follows that products distribute over coproducts. $\square$

Domains in $\mathbf{Pfn}_F$ correspond precisely to subcoalgebras.

³Necessary and sufficient conditions for distributivity in categories of coalgebras were announced in joint work of H. Peter Gumm and Tobias Schröder of the University of Marburg, Germany, and Jesse Hughes of Carnegie Mellon University [GHS00].

Proposition 2.3.6. *Let X be a coalgebra in $\mathbf{Pfn}_F$. There is a meet semilattice isomorphism from $\text{dom}(X)$ to the lattice $\mathcal{L}_X$ of sub-coalgebras of X (considered as a meet semilattice).*

Proof. The map $\mathcal{L}_X \longrightarrow \text{dom}(X)$ is defined by sending the sub-coalgebra U of X to the domain $p_{0X,Y} \circ \langle 1_X, \phi \rangle$, where $\phi = \{U \hookrightarrow X, U \hookrightarrow X\}$. The inverse map is defined as follows. Given a domain $\phi \in \text{dom}(X)$, we may write $\phi = p_{0X,Y} \circ \langle 1_X, \{m, \varphi\} \rangle$, where $m : V \twoheadrightarrow X$ is a monomorphism of the coalgebras, and where $\varphi : V \longrightarrow Y$ is a morphism in $\mathbf{Set}_F$. By definition of composition in $\mathbf{Pfn}_F$ one obtains the following diagram in which the squares are pullbacks. We adopt the convention of not showing the structure maps and induced morphisms of the coalgebras occurring in commutative diagrams of coalgebras.

$$\begin{array}{ccccccc}
 V & \longrightarrow & X \times V & \xrightarrow{1_X \times \varphi} & X \times Y & \xrightarrow{p_{0X,Y}} & X \\
 \downarrow 1 & & \downarrow 1 & & \downarrow 1 & \nearrow p_{0X,Y} & \\
 V & \longrightarrow & X \times V & \xrightarrow{1_X \times \varphi} & X \times Y & & \\
 \downarrow & & \downarrow & & \downarrow & \nearrow 1_X \boxtimes \varphi & \\
 X & \xrightarrow{\Delta_X} & X \times X & & & & \\
 \downarrow 1 & \nearrow \Delta_X & & & & & \\
 X & & & & & &
 \end{array}$$

We use this diagram to define the coalgebra $U \in \mathcal{L}_X$ corresponding to the domain ϕ as the image of V under the left vertical column of the diagram. Showing that the meets are preserved involves certain large diagrams such as those occurring in [DPH87]. $\square$

Let $\mathbf{C}$ be a P -category. A morphism $\phi : X \longrightarrow Y$ of $\mathbf{C}$ is *total* if $\text{dom}\phi = 1_X$. The collection of total morphisms of $\mathbf{C}$ generate its subcategory $\mathbf{C}_T$ of total morphisms. The near-product and infranatural transformations of a P -category $\mathbf{C}$

become a product and natural transformations, respectively, on its subcategory $\mathbf{C}_T$, which has the structure of a B -category; for convenience we include the definition from Heller [Hel90]. A B -category is a category $\mathbf{C}$ with a bifunctor $\times : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ and natural transformations $p_0, p_1, \Delta, \text{ass}_\times, \text{tr}_\times$ satisfying the conditions i), iii) and iv) for P -categories above, with $\times$ replacing $\boxtimes$. It should be emphasized that the projections of a B -category are required to be natural and not merely infranatural transformations; we write $\times$ for the product of a B -category and $\boxtimes$ for the near product of a P -category. Dually, one may speak of a category with a binary coproduct, together with natural transformations $i_0, i_1, \nabla, \text{ass}_\amalg, \text{tr}_\amalg$ satisfying the duals of the conditions 1), iii) and iv) in which the *injections* i_0, i_1 replace the *projections* p_0, p_1 , the *codiagonal* ∇ replaces the *diagonal* Δ , where $\text{ass}_\amalg$ and $\text{tr}_\amalg$ replace $\text{ass}_\times$ and $\text{tr}_\times$, respectively, and where $\amalg$ replaces $\boxtimes$. A B -category with a binary coproduct that has a natural isomorphism called *dist* inverse to the natural transformation

$$(X \times Y) \amalg (X \times Z) \longrightarrow X \times (Y \amalg Z)$$

is called a B^+ -category. A B^+ -category with a countable coproduct $\amalg_{\mathbf{N}}$ such that $\times$ distributes over $\amalg_{\mathbf{N}}$ is called a $B\Sigma$ -category.

By analogy with B^+ - and $B\Sigma$ -categories, one may define P^+ and $P\Sigma$ -categories, in which the coproduct injections are required to be natural (and not merely infranatural) transformations. Under the embedding $\mathbf{Set}_F \longrightarrow \mathbf{Pfn}_F$, the canonical isomorphism of proposition 2.3.5 defines the natural isomorphism

$$\text{dist}_{X, (Y_n)} : X \boxtimes \amalg_{\mathbf{N}} Y_n \longrightarrow \amalg_{\mathbf{N}} X \boxtimes Y_n$$

of functors $\mathbf{Pfn}_F \times \mathbf{Pfn}_F^{\mathbf{N}} \longrightarrow \mathbf{Pfn}_F$.

Proposition 2.3.7. *The category $\mathbf{Pfn}_F$ is a $P\Sigma$ -category.*

Proof. Let $f : X \dashrightarrow W$ in $\mathbf{Pfn}_F$ be given by $f = \{\mu, \phi\}$, where $\mu : U \twoheadrightarrow X$ and $\phi : U \longrightarrow W$ are in $\mathbf{Set}_F$, with μ mono, and for $n \in \mathbb{N}$ let $g_n : Y_n \dashrightarrow Z_n$ in $\mathbf{Pfn}_F$ be given by $g_n = \{\nu_n, \psi_n\}$, where $\nu_n : V_n \twoheadrightarrow Y_n$ and $\psi_n : V_n \longrightarrow Z_n$ are in $\mathbf{Set}_F$, with ν_n mono. Consider the following diagram.

$$\begin{array}{ccccc}
 U \times \coprod_{\mathbb{N}} V_n & \xrightarrow{\text{dist}_{U, (V_n)}} & \coprod_{\mathbb{N}} (U \times V_n) & & \\
 \downarrow \phi \times \coprod_{\mathbb{N}} \psi_n & \searrow \mu \times \coprod_{\mathbb{N}} \nu_n & \downarrow \coprod_{\mathbb{N}} (\mu \times \nu_n) & & \\
 & & \coprod_{\mathbb{N}} (U \times Y_n) & \xrightarrow{\text{dist}_{X, (Y_n)}} & \coprod_{\mathbb{N}} (X \times Y_n) \\
 & & \downarrow \coprod_{\mathbb{N}} (\phi \times \psi_n) & & \downarrow \coprod_{\mathbb{N}} (f \boxtimes g_n) \\
 W \times \coprod_{\mathbb{N}} Z_n & \xrightarrow{\text{dist}_{W, (Z_n)}} & \coprod_{\mathbb{N}} (W \times Z_n) & & \\
 \uparrow f \boxtimes \coprod_{\mathbb{N}} g_n & \swarrow \mu \times \coprod_{\mathbb{N}} \nu_n & \uparrow \coprod_{\mathbb{N}} (\mu \times \nu_n) & & \\
 & & X \times \coprod_{\mathbb{N}} Y_n & \xrightarrow{\text{dist}_{X, (Y_n)}} & \coprod_{\mathbb{N}} (X \times Y_n)
 \end{array}$$

We claim that the bottom parallelogram commutes in $\mathbf{Pfn}_F$. Observe that the top parallelogram is a pullback in $\mathbf{Set}_F$. It follows that the composite

$$\coprod_{\mathbb{N}} (f \boxtimes g_n) \circ \text{dist}_{X, (Y_n)}$$

in $\mathbf{Pfn}_F$ is represented by the pair

$$\left\{ \mu \times \coprod_{\mathbb{N}} \nu_n, \coprod_{\mathbb{N}} (\phi \times \psi_n) \circ \text{dist}_{U, (V_n)} \right\}. \quad (2.3.1)$$

On the other hand, the composite

$$\text{dist}_{W, (Z_n)} \circ \left(f \boxtimes \coprod_{\mathbb{N}} g_n \right)$$

in $\mathbf{Pfn}_F$ is represented by the pair

$$\left\{ \mu \times \coprod_{\mathbb{N}} \nu_n, \text{dist}_{W, (Z_n)} \circ \left(\phi \times \coprod_{\mathbb{N}} \psi_n \right) \right\},$$

which equals (2.3.1) since the (back) rectangle commutes, as *dist* is natural in $\mathbf{Set}_F$.

□

2.4 Ranges

In a P -category, if $\phi : X \longrightarrow Y$ is a morphism and if $\varepsilon \in \text{Dom}Y$, we write $\phi \prec \varepsilon$ if and only if $\varepsilon\phi = \phi$; we say that ε *receives* ϕ . If ε receives ϕ , and in addition, ε satisfies for all appropriate ψ, ψ' , $\psi\phi = \psi'\phi$ implies $\psi\varepsilon = \psi'\varepsilon$, then ε is the least domain in $\text{Dom}Y$ receiving ϕ , since if $\delta \in \text{dom}Y$ satisfies $\delta\phi = \phi$, then $\delta\phi = \varepsilon\phi$, which implies that $\delta\varepsilon = \varepsilon\varepsilon = \varepsilon$ and therefore $\varepsilon \leq \delta$. In this case we say that ϕ has range ε and we write $\text{ran}\phi = \varepsilon$.

The receives relation $\prec$, which holds between a morphism and a domain, is to be distinguished from the partial order $\leq$ on domains. We record the following seemingly innocuous but necessary fact.

Proposition 2.4.1. *If $\phi : X \longrightarrow Y$ is a morphism and β, γ are domains in Y such that $\phi \prec \beta$ and $\beta \leq \gamma$, then $\phi \prec \gamma$.*

Proof. We have $\phi = \beta\phi = (\beta\gamma)\phi = \gamma(\beta\phi) = \gamma\phi$, so that $\phi \prec \gamma$. $\square$

If each morphism of the P -category $\mathbf{C}$ has a range, we say that $\mathbf{C}$ *has ranges*. In this case, for a morphism $\phi : X \longrightarrow Y$ we define $\phi_* : \text{dom}(X) \longrightarrow \text{dom}(Y)$ by $\phi_*\alpha = \text{ran}(\phi\alpha)$ for $\alpha \in \text{dom}(X)$. Also, in any P -category we define, for any morphism $\phi : X \longrightarrow Y$, a map $\phi^* : \text{dom}(Y) \longrightarrow \text{dom}(X)$, given by $\phi^*\beta = \text{dom}(\beta\phi)$ for $\beta \in \text{dom}(Y)$. For total $\phi : X \longrightarrow Y$ we have that ϕ_* is left adjoint to ϕ^* .

Proposition 2.4.2. *If $\phi : X \longrightarrow Y$ is total, then for each $\alpha \in \text{dom}(X)$ and for each $\beta \in \text{dom}(Y)$, $\phi_*\alpha \leq \beta$ if and only if $\alpha \leq \phi^*\beta$.*

Since $\mathbf{Set}_F$ has (Epi, StrongMono)-factorizations, the image of a coalgebra under a morphism is a subcoalgebra of the codomain; conversely every subcoalgebra is an image. If $\phi = \{m, \psi\}$ is a morphism of $\mathbf{Pfn}_F$, where $m : U \twoheadrightarrow X$ and $\psi : U \longrightarrow Y$

are morphisms of $\mathbf{Set}_F$ with m mono, we define the image $\text{im}\phi$ of ϕ by $\text{im}\phi = \psi[U]$. Under the semilattice isomorphism of Proposition 2.3.6, images of morphisms in $\mathbf{Pfn}_F$, which are coalgebras, correspond with ranges, which are morphisms.

Proposition 2.4.3. *The category $\mathbf{Pfn}_F$ has ranges.*

If every morphism of the P -category $\mathbf{C}$ has a range, and if for morphisms ϕ, ψ of $\mathbf{C}$, $\text{ran}(\phi \boxtimes \psi) = \text{ran}\phi \boxtimes \text{ran}\psi$, then one says [Hel90] that $\mathbf{C}$ is an rP -category; such a category has a calculus of ranges [DPH87]. In a B^+ -category, if f and g are morphisms with the same codomain Y , then we define $[f, g] = \nabla_Y(f \amalg g)$. In an rP^+ - ($rP\Sigma$ -) category, the meet semilattice $\text{dom}(X)$ becomes a distributive lattice if one defines the join by $\varepsilon \vee \delta = \text{ran}[\varepsilon, \delta]$ for $\varepsilon, \delta \in \text{dom}(X)$; in that case the maps ϕ^* preserve $\vee$ and $\wedge$ and the maps ϕ_* preserve $\vee$. [DPH87, Hel90].

To show that $\mathbf{Pfn}_F$ is an $rP\Sigma$ -category, we show that the epi-mono factorization of a product of morphisms in $\mathbf{Set}_F$ is the product of their epi-mono factorizations; aside from a technical result of Gumm and Schröder (that inclusions commute with products to within isomorphism), the proof relies on folklore arguments. The lack of a simple description of the product in $\mathbf{Set}_F$ and the possibility of pathologies seems to necessitate the use of categorical technique in the proof.

Proposition 2.4.4. *For morphisms ϕ, ψ of $\mathbf{Pfn}_F$,*

$$\text{ran}(\phi \times \psi) = \text{ran}\phi \times \text{ran}\psi.$$

Proof. It suffices to prove the result for ϕ, ψ in $\mathbf{Set}_F$. We first show that if $f : A \longrightarrow B$

is (isomorphic to) a subcoalgebra of $B_0 \times B_1$. If $\phi : X \longrightarrow Y$ and $\psi : W \longrightarrow Z$ are coalgebra morphisms, then by previous remarks, one has two canonically isomorphic epi-mono factorizations of $\phi \times \psi$ as indicated in the following commutative diagram.

$$\begin{array}{ccc}
 X \times W & \xrightarrow{\quad} & \text{im}(\phi \times \psi) \\
 \downarrow & \nearrow \text{dotted} & \downarrow \\
 \text{im}\phi \times \text{im}\psi & \xrightarrow{\quad} & Y \times Z
 \end{array}$$

Under the isomorphism of Proposition 2.3.6, the conclusion follows. $\square$

Proposition 2.4.5. *The category $\mathbf{Pfn}_F$ is an $rP\Sigma$ -category. Moreover, for any F -coalgebra X , the meet semilattice isomorphism of Proposition 2.3.6 is a lattice isomorphism from $\text{dom}(X)$ to the lattice $\mathcal{L}_X$ of sub-coalgebras of X .*

The following definitions are taken from Heller [Hel90]. In a P -category, a *section* of a morphism $\phi : X \longrightarrow Y$ is a morphism $\sigma : Y \longrightarrow X$ such that $\phi\sigma = \text{dom}\sigma$ and $\phi\sigma\phi = \phi$. A P -category satisfies the *axiom of choice* if every morphism has a section. A morphism ϕ is a *partial monomorphism* if $\phi\theta = \phi\theta'$ implies $(\text{dom}\phi)\theta = (\text{dom}\phi)\theta'$. A P -category satisfies the *weak axiom of choice* if each partial monomorphism has a section.

Proposition 2.4.6. *The category $\mathbf{Pfn}_F$ satisfies the weak axiom of choice.*

Proof. Let $\phi : X \longrightarrow Y$ be a partial monomorphism in $\mathbf{Pfn}_F$. Then $\phi = \{\mu, \psi\}$, where $\mu, \psi : U \longrightarrow X$ are monomorphisms in $\mathbf{Set}_F$, and where U is a subcoalgebra of X . The category $\mathbf{Set}_F$ has (Epi-StrongMono) factorizations, hence one has the

following diagram in which ψ has the epi-mono factorization $\psi = me$.

$$\begin{array}{ccc}
 Q & \xrightarrow{m} & Y \\
 \uparrow e & \downarrow f & \downarrow 1 \\
 U & \xrightarrow{\psi} & Y \\
 \downarrow \mu & \nearrow \phi = \{\mu, \psi\} & \\
 X & &
 \end{array}$$

A morphism in $\mathbf{Set}_F$ is epi if and only if it is surjective in $\mathbf{Set}$. The morphism e is a monomorphism since m and ψ are. Therefore e is invertible in $\mathbf{Set}$ with inverse f which, since the image of m is contained in the image of ψ , is a morphism in $\mathbf{Set}_F$ by the second diagram lemma (Lemma 1.4.3).

The partial morphism σ defined by $\sigma = \{m, \mu f\}$ gives the required section. The required properties $\phi\sigma = \text{dom}\sigma$ and $\phi\sigma\phi = \phi$ can be verified in the following commutative diagram, in which the squares are pullbacks in $\mathbf{Set}_F$.

$$\begin{array}{ccccc}
 U & \xrightarrow{e} & Q & \xrightarrow{f} & U & \xrightarrow{\psi} & Y \\
 \downarrow 1 & & \downarrow 1 & & \downarrow \mu & \nearrow \phi & \\
 U & \xrightarrow{e} & Q & \xrightarrow{\mu f} & X & & \\
 \downarrow 1 & & \downarrow m & \nearrow \sigma & & & \\
 U & \xrightarrow{\psi} & Y & & & & \\
 \downarrow \mu & \nearrow \phi & & & & & \\
 X & & & & & &
 \end{array}$$

□

Chapter 3

The iteration lemma in $\mathbf{Pfn}_F$

In this chapter we reprove A. Heller's iteration lemma in $\mathbf{Pfn}_F$ using the property that contravariant representable functors for the coalgebra topology are sheaves. This will allow us to simplify the definition of an iteration category given in Heller [Hel90]. The original statement of the iteration lemma relied on the notion of a $B^\#$ -category and its associated formally free semigroup monad. In $\mathbf{Pfn}_F$, however, there is enough structure to give a conceptual proof of the iteration lemma without mentioning either the formally free semigroup or $B^\#$ categories. These notions will be needed for the existence theorem; they will be treated in the following section.

3.1 The iteration lemma

The coalgebra topology can be used to define certain sheaves of sets, as follows. Let X, Y be F -coalgebras. Let $\mathcal{O}(X)$ denote the category of open subsets of X , considered as a topological space with the coalgebra topology. The assignment

$$\begin{aligned}\mathcal{F} : \mathcal{O}(X)^{\text{op}} &\longrightarrow \mathbf{Set} \\ U &\longmapsto \mathbf{Set}_F(U, Y)\end{aligned}$$

defines a presheaf of sets in a natural way, since each open subset U of X inherits a unique coalgebra structure from that of X .

Proposition 3.1.1. *The presheaf $\mathcal{F}$ is a sheaf.*

Proof. We show that for each open subset U of X , and for each open covering $U = \bigcup_{i \in I} U_i$ of U by a family $(U_i)_{i \in I}$ of open subsets of X , the following diagram is an equalizer in **Set**.

$$\mathcal{F}(U) \xrightarrow{e} \prod_i \mathcal{F}(U_i) \xrightarrow[p]{q} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

The map e is clearly a monomorphism. Moreover, given a compatible family of coalgebra morphisms $f_i : U_i \longrightarrow Y$, there exists a unique continuous map $f : U \longrightarrow Y$ in the category **Top** of topological spaces such that $f|_{U_i} = f_i$ for each $i \in I$; it remains to show that f is a morphism of coalgebras. This is a straightforward diagram chase; consider the following diagram.

$$\begin{array}{ccccc}
 U_i & & & & Y \\
 \alpha|_{U_i} \downarrow & & f_i \xrightarrow{\quad} & & \downarrow \beta \\
 & U & \xrightarrow{f} & & Y \\
 & \downarrow \alpha & & & \\
 FU_i & \xrightarrow{Ff_i} & & & FY \\
 & \downarrow Fj & & & \\
 & FU & \xrightarrow{Ff} & &
 \end{array}$$

The top triangle commutes in **Set** by definition, the bottom triangle commutes by functoriality and the back and left squares commute by definition of f_i and j , respectively. Chasing an element $x \in U$, which we may assume to lie in U_i , we have that the right front square commutes, so that $f \in \mathbf{Set}_F(U, Y)$. $\square$

We give a translation of the iteration lemma from Heller [Hel90] into $\mathbf{Pfn}_F$. The iteration lemma and its application to Turing data in the sequel is an alternative to the use of fixed point semantics for iteration; one such use is exemplified by Manes' application of the Tarski fixed point theorem to iterative specifications in ω -complete categories [Man92].

Lemma 3.1.2 (Iteration Lemma [Hel90]). *Let X be a coalgebra, let $i : U \hookrightarrow X$ be the inclusion of a sub-coalgebra U into X , and let $f : X \longrightarrow X$ be a morphism in $\mathbf{Set}_F$ such that $f \circ i = i$. Then there is a unique morphism $\text{It}(f, U) : X \longrightarrow X$ in $\mathbf{Pfn}_F$ with domain $\cup_n f^{-n}[U]$ such that for each $n \geq 0$, $\text{It}(f, U)[f^{-n}[U]] = f^n[f^{-n}[U]]$. Moreover, $\text{im } \text{It}(f, U) \subseteq U$ and, for appropriate g and V ,*

$$\text{It}(f \times g, U \times V) = \text{It}(f, U) \boxtimes \text{It}(g, V),$$

$$\text{It}(f \amalg g, U \amalg V) = \text{It}(f, U) \amalg \text{It}(g, V).$$

Proof. Let $n \geq 1$ be a positive integer, and consider the following pullback diagram in $\mathbf{Set}_F$.¹

$$\begin{array}{ccc} (f^n)^{-1}[U] & \xrightarrow{i_n} & X \\ \downarrow f_n & & \downarrow f^n \\ U & \xrightarrow{i} & X \end{array}$$

The family of maps $if_n : (f^n)^{-1}[U] \longrightarrow X$ is compatible; i.e., for positive integers m and n , $(if_m)|_{(f^m)^{-1}[U] \cap (f^n)^{-1}[U]} = (if_n)|_{(f^m)^{-1}[U] \cap (f^n)^{-1}[U]}$. To show this, let $m < n$ and suppose that $x \in (f^m)^{-1}[U] \cap (f^n)^{-1}[U]$. Then

$$if_n(x) = f^n i_n(x) = f^{n-m} (f^m i_n(x)) = f^{n-m} (f^m i_m(x)) = f^m i_m(x) = if_m(x),$$

¹For the iteration lemma, it is sufficient for $\mathbf{Set}_F$ to have products and for F to preserve weak pullbacks (to ensure the existence of pre-images and strong epi-mono factorizations); in that case, the displayed diagram will be a pullback even if pullbacks do not exist in general.

since $i_n(x) = i_m(x)$ and $f^m i_m(x) \in U$. By the previous proposition, there exists a morphism $\bigcup_n (f^n)^{-1}[U] \xrightarrow{g} X$ in $\mathbf{Set}_F$ such that $g|(f^n)^{-1}[U] = f^n i_n$ for each $n \geq 1$.

To see that the iteration operator commutes with products, we make the following observations. In a P -category $\mathbf{C}$, $\text{dom}(\phi \boxtimes \phi') = \text{dom}(\phi) \boxtimes \text{dom}(\phi')$ (see Rosolini [Ros86]). It follows that for domains $\varepsilon, \varepsilon'$, $(\phi \boxtimes \phi')^*(\varepsilon \boxtimes \varepsilon') = \phi^* \varepsilon \boxtimes (\phi')^* \varepsilon'$. Moreover, inclusions and images commute with products, and so by definition of the iteration operator (by gluing), the following calculation is justified.²

$$\begin{aligned} \text{It}(f \times g, U \times V) | ((f \times g)^n)^*(U \times V) &= (f \times g)^n | ((f \times g)^n)^*(U \times V) \\ &= (f^n \times g^n) | (f^n)^* U \times (g^n)^* V = (f^n | (f^n)^* U) \times (g^n | (g^n)^* V) \\ &= (\text{It}(f, U) | (f^n)^* U) \times (\text{It}(g, V) | (g^n)^* V) = (\text{It}(f, U) \times \text{It}(g, V)) | (f^n)^* U \times (g^n)^* V \\ &= (\text{It}(f, U) \times \text{It}(g, V)) | ((f \times g)^n)^*(U \times V). \end{aligned}$$

The desired equality follows by Proposition 3.1.1.

Similarly, since pullbacks of coproducts are disjoint, the iteration operator commutes with coproducts.

Our calculations have been in $\mathbf{Set}_F$, however, the existence of the morphism $\text{It}(f, U) : X \longrightarrow X$ in $\mathbf{Pfn}_F$ with the required properties is immediate.

□

3.2 Free semigroups and $B^\#$ -categories

If $\mathbf{C}$ is a $B\Sigma$ - or $P\Sigma$ -category, one may construct the *free semigroup* $X^\# = \coprod_{n \in \mathbb{N}} X^{n+1}$ of an object X of $\mathbf{C}$. In view of the possibility of pathological products of coalgebras, we remark that Proposition 2.3.4 implies that the free semigroup of a nonempty

²We take the restriction of a morphism to a subcoalgebra to be the composition with an appropriate inclusion.

coalgebra is nonempty. The associative multiplication $m : X^\# \times X^\# \longrightarrow X^\#$ of the free semigroup is the composite

$$(\coprod_{\mathbb{N}} X^{n+1}) \times (\coprod_{\mathbb{N}} X^{n+1}) \longrightarrow \coprod_{\mathbb{N} \times \mathbb{N}} X^{n+1} \times X^{m+1} \longrightarrow \coprod_{\mathbb{N}} X^{k+1}$$

where the left map is obtained from two applications of the natural transformation *dist*, and where the right map is, to within a natural isomorphism, the coproduct of the codiagonal maps

$$\nabla_k : \coprod_{n+m+1=k} X^{n+1} \times X^{m+1} \longrightarrow X^{k+1}$$

for $k \geq 1$ (compare with the construction of the free monoid in [ML98], page 172). In particular, the categories **Set**_F and **Pfn**_F possess these canonical coalgebra morphisms.

More general than $B\Sigma$ -categories are the $B^\#$ -categories, whose definition will be needed for the statement of Heller's existence theorem. A $B^\#$ -category has an associated monad, denoted by $\#$, called the *formally-free semigroup* functor which, although lacking the universal property of the free semigroup, has enough of the structure of the free semigroup to define by categorical means operations corresponding to the manipulation of finite sequences. Such operations were used in Heller [Hel90] to define the iteration of a morphism and prove an iteration lemma in the context of an iteration category, to be defined.

The definition of a $B^\#$ -category, taken verbatim from Heller, follows. For this purpose some additional notation is needed. The formally-free semigroup functor $\#$ comes with an associative multiplication m ; for morphisms f, g with common codomain $X^\#$, one defines $f \cdot g = m(f \times g)$. We define $j_n : X^n \longrightarrow X^\#$ by $j_1 = j$ and $j_{n+1} = j \cdot j_n$ for $j > 1$. When $X^\#$ is the free semigroup, the j_n are (to within

canonical isomorphism) the coproduct injections.

A $B^\#$ -category is a B^+ -category equipped with seven natural transformations

$$\begin{aligned} m : X^\# \times X^\# &\longrightarrow X^\# \\ j : X &\longrightarrow X^\# \\ e : X^{\#\#} &\longrightarrow X^\# \\ l : X^\# &\longrightarrow X \amalg (X \times X^\#) \\ r : X^\# &\longrightarrow (X^\# \times X) \amalg X \\ par : X \times Y^\# &\longrightarrow (X \times Y)^\# \end{aligned}$$

$$\begin{aligned} wd : (X \amalg Y)^\# &\longrightarrow X^\# \amalg Y^\# \amalg (X^\# \times Y^\#)^\# \amalg (Y^\# \times X^\#)^\# \\ &\amalg (Y^\# \times (X^\# \times Y^\#)^\#) \amalg (X^\# \times (Y^\# \times X^\#)^\#) \end{aligned}$$

subject to the following conditions.

i) m is associative, hence $X^\#$ is a semigroup and, if f is a morphism, then $f^\#$ is a semigroup homomorphism.

ii) $\#$ is a monad with unit j and multiplication e , and each component of e is a semigroup homomorphism.

iii) l and r are respectively the inverses of

$$\begin{aligned} [j, j \cdot X^\#] : X \amalg (X \times X^\#) &\longrightarrow X^\#, \\ [X^\# \cdot j, j] : (X^\# \times X) \amalg X &\longrightarrow X^\#. \end{aligned}$$

iv) The following diagram commutes.

$$\begin{array}{ccc} X \times Y^\# & \xrightarrow{par} & (X \times Y)^\# \\ & \searrow p_{1,X,Y^\#} & \downarrow p_{1,X,Y}^\# \\ & & Y^\# \end{array}$$

v) wd is inverse to the morphism

$$[i_0^\#, i_1^\#, e(i_0^\# \cdot i_1^\#)^\#, e(i_1^\# \cdot i_0^\#)^\#, i_1^\# \cdot e(i_0^\# \cdot i_1^\#)^\#, i_0^\# \cdot e(i_1^\# \cdot i_0^\#)^\#].$$

A functor between $B^\#$ -categories is called a $B^\#$ -functor if it preserves the functors $\times$, Π , and $\#$ and the fourteen natural transformations

$$\Delta, p_0, p_1, \nabla, i_0, i_1, \text{dist}, m, j, e, l, r, \text{par}, \text{wd}$$

defined above. A $B^\#$ -functor F preserves the natural transformations $\text{ass}_\times$ and $\text{tr}_\times$ as well, which form part of the structure of a $B^\#$ category as derived transformations introduced by the formulae (2.2.1) and (2.2.2) (see Chapter 2, section 2.2, numbers iii) and iv)), which involve only compositions of the projections p_0, p_1 and the diagonal Δ , subject to equations asserting coherence conditions.

Let $\mathbf{C}$ be a $B^\#$ -category, and let S be a set of morphisms of $\mathbf{C}$. The $B^\#$ -subcategory generated by S is the smallest $B^\#$ -category of $\mathbf{C}$ containing S ; this subcategory is denoted by $B^\#(S)$. If $\mathbf{C}$ is a small $B^\#$ -category, then the set $\mathbf{C}_0$ of objects of $\mathbf{C}$ is an algebra with signature $(\times, \Pi, \#)$; such an algebra is called a $(\times, \Pi, \#)$ -algebra. In the sequel, such algebras will be obtained from certain objects of a $B^\#$ -category called isotypical objects, to be defined. In particular, such an algebra will generate a $B^\#$ -category, in the following sense.

Proposition 3.2.1 (Proposition 2.2 [Hel90]). *If $\mathbf{C}$ is a small $B^\#$ -category, $\mathbf{D}_0$ is a $(\times, \Pi, \#)$ -algebra and $F_0 : \mathbf{D}_0 \longrightarrow \mathbf{C}_0$ is a homomorphism, then there exists, uniquely to within canonical isomorphism, a $B^\#$ -category $\mathbf{D}$ with object algebra $\mathbf{D}_0$, supplied with a full and faithful $B^\#$ -functor $F : \mathbf{D} \longrightarrow \mathbf{C}$ extending F_0 .*

Proof. Following the procedure in [Hel90] for obtaining the smallest $B^\#$ category containing a given set of morphisms, we think of the elements of $\mathbf{D}_0$ as the objects of an as yet unspecified $B^\#$ -category $\mathbf{D}$, and construct the set $\mathbf{D}_1$ of morphisms accordingly. For each $A \in \mathbf{D}_0$, we adjoin an identity morphism 1_A to $\mathbf{D}_1$, subject to

the functoriality relations $1_{A \times B} = 1_A \times 1_B$, and so on. Next, adjoin the values of the fourteen natural transformations above with arguments in $\mathbf{D}_0$ to $\mathbf{D}_1$, subject to the relations i) through v) above and subject to the relations that hold in a B^+ -category. Finally, we close the set of morphisms that results under $\times, \amalg, \#$ and composition. This produces the category $\mathbf{D}$. The homomorphism F_0 is extended to $\mathbf{D}_1$ in the only way possible, following the three step construction of $\mathbf{D}_1$. Identities in $\mathbf{D}_1$ must be preserved by F . The image in $\mathbf{C}$ under F of a value of one of the fourteen natural transformations is completely determined by functoriality and by definition of F_0 . For example, the value of F on $\nabla_A : A \amalg A \longrightarrow A$ is $\nabla_{FA} : FA \amalg FA \longrightarrow FA$. Finally, we require that F commute with $\times, \amalg, \#$ and composition; for example, $F(\nabla_X \times 1_{Y\#}) = F\nabla_X \times 1_{F(Y\#)} = \nabla_{FX} \times 1_{(FY)\#}$. $\square$

Let $\mathbf{C}$ be a $B^\#$ -category. An object X of $\mathbf{C}$ generates by Proposition 3.2.1 a $B^\#$ -category denoted by $\mathbf{C}\langle X \rangle$, whose object algebra is free on the generator X , together with a $B^\#$ -functor $F : \mathbf{C}\langle X \rangle \longrightarrow B^\#(1_X)$. In case the $(\times, \amalg, \#)$ -algebra generated by X , namely $B^\#(1_X)_0$, is free, then F gives an identification of $\mathbf{C}\langle X \rangle$ with the full subcategory $B^\#(1_X)$ of $\mathbf{C}$ and, following Heller, we use the notation

$$\mathbf{C} = \mathbf{C}\langle X \rangle \tag{3.2.1}$$

to say that $\mathbf{C}_0$ is freely generated by X as a $(\times, \amalg, \#)$ -algebra.

The definition of an iteration category requires the notion of a stable union, a categorical notion which needs no special definition in $\mathbf{Set}_F$. Let $\mathbf{C}$ be a P -category, and let X be an object of $\mathbf{C}$, and let $\varepsilon_n \in \text{dom}(X)$ be a countable family of domains. A domain δ is a *union* of $\{\varepsilon_n\}$ if for each n , $\varepsilon_n \leq \delta$ and if, for any ϕ and ϕ' , $\phi\varepsilon_n = \phi'\varepsilon_n$ implies $\phi\delta = \phi'\delta$. This determines δ uniquely, for if δ' is any other union of $\{\varepsilon_n\}$,

then $\delta\varepsilon_n = \varepsilon_n = \delta'\varepsilon_n$, so $\delta = \delta\delta = \delta'\delta$ and also $\delta' = \delta'\delta' = \delta\delta'$. Since composition of domains is commutative, $\delta = \delta'$. In this case. we set $\delta = \vee\varepsilon_n$. If for any domain δ of $\mathbf{C}$,

$$\delta \boxtimes \bigvee \varepsilon_n = \bigvee (\delta \boxtimes \varepsilon_n),$$

the union is called *stable* [Hel90]. In a $P\Sigma$ -category with ranges, every countable family of domains $\{\varepsilon_n\}$ has the stable union $\text{ran}(\nabla \Sigma_{\mathbb{N}} \varepsilon_n)$.³

An *iteration category* is a prodominical P^+ -category with ranges that satisfies the weak axiom of choice, such that the B^+ -category structure of $\mathbf{C}_T$ has been extended to a $B^\#$ -category so that for each object X , the formally free semigroup $X^\#$ is the stable union of $\{\text{ran} j_n\}$.

Proposition 3.2.2. *$\mathbf{Pfn}_F$ is an iteration category.*

Proof. Since $\mathbf{Pfn}_F$ is a $P\Sigma$ -prodominical category with ranges that satisfies the weak axiom of choice, it is an iteration category. $\square$

3.3 Iteration categories revisited

The proof of the iteration lemma we gave avoided the use of the formally free semigroup monad and any mention of stable unions. Since Heller's development of the existence theorem depends on the truth of the iteration lemma in an iteration category, which is a P -category with certain additional structure, but not on any specific proof using the formally free semigroup and properties of stable unions, we may relax the definition of an iteration category to that of a P -category with enough structure to express the iteration lemma and in which it holds; the statement of Heller's

³Here the natural transformation ∇ is the countable codiagonal; see Heller [Hel90] for further details.

iteration lemma—adapted to our needs—follows.

Lemma 3.3.1 (Iteration Lemma [Hel90]). *Let $\mathbf{C}$ be a prodominical P -category with ranges and coproducts that satisfies the weak axiom of choice. Suppose that $f : X \longrightarrow X$ is a total morphism (i.e., is in $\mathbf{C}_T$), $\varepsilon \in \text{dom} X$ and $f \circ \varepsilon = \varepsilon$. Then the family of domains $\{(f^n)^* \varepsilon\}$ has a union in $\text{dom} X$ and there exists a unique morphism $\text{It}(f, \varepsilon) : X \longrightarrow X$ in $\mathbf{C}$ with domain $\bigvee_n (f^n)^* \varepsilon$ such that for each $n \geq 0$, $\text{It}(f, \varepsilon)_* ((f^n)^* \varepsilon) = f_*^n ((f^n)^* \varepsilon)$. Moreover, $\text{It}(f, \varepsilon) \preceq \varepsilon$ and, for appropriate f' and ε' ,*

$$\text{It}(f \times f', \varepsilon \times \varepsilon') = \text{It}(f, \varepsilon) \boxtimes \text{It}(f', \varepsilon'),$$

$$\text{It}(f \amalg f', \varepsilon \amalg \varepsilon') = \text{It}(f, \varepsilon) \amalg \text{It}(f', \varepsilon').$$

In the original iteration lemma, the union of the family $\{(f^n)^* \varepsilon\}$ was assumed to be stable; however, if the preceding statement of the iteration lemma is assumed, the union is necessarily stable.

Proposition 3.3.2. *Suppose that the iteration lemma holds in the P -category $\mathbf{C}$. In the notation of the iteration lemma above, the union of the family $\{(f^n)^* \varepsilon\}$ is stable.*

Proof. If $\delta \in \text{dom}(1_Y)$, then by the iteration lemma and by previous remarks,

$$\text{dom}(\text{It}(1_Y \times f, \delta \boxtimes \varepsilon)) = \text{dom}(\text{It}(1_Y, \delta)) \boxtimes \text{dom}(\text{It}(f, \varepsilon)) = \delta \boxtimes \text{dom}(\text{It}(f, \varepsilon)),$$

and therefore

$$\bigvee_n (\delta \boxtimes (f^n)^* \varepsilon) = \bigvee_n ((1_Y \boxtimes f)^n)^* (\delta \boxtimes \varepsilon) = \delta \boxtimes \left(\bigvee_n (f^n)^* \varepsilon \right).$$

□

Accordingly we will redefine an iteration category to be a prodomonical category with coproducts and ranges, which satisfies the weak axiom of choice and in which the iteration lemma holds.

Chapter 4

The generalized existence theorem

In this chapter, a generalization of Heller's existence theorem will be stated, proved and applied to a class of examples to which the original existence theorem does not apply. We first review the notions of uniform list, isotypes, frames and uniform generation, needed for the statement of Heller's existence theorem, and its generalization.

4.1 Uniform lists

In this section we define the notion of a uniform list, adapted from Heller, and reprove the facts needed for the proof of the existence theorem.

If $\mathbf{C}$ is a P -category and $t : W \times X \longrightarrow X$ is a morphism in $\mathbf{C}$, then an *index* of $f : X \longrightarrow X$ relative to the *catalog* t is a total morphism $g : X \longrightarrow W$ such that the following diagram commutes.

$$\begin{array}{ccc} X \times X & \xrightarrow{g \times 1} & W \times X \\ p_1 \downarrow & & \downarrow t \\ X & \xrightarrow{f} & X \end{array} \quad (4.1.1)$$

If $\mathbf{B}$ is a subcategory of the P -category $\mathbf{C}$, the *uniform list catalogued by t with indices in $\mathbf{B}$* is the set $\mathcal{L}(\mathbf{B}, t)$ of morphisms $f \in \mathbf{C}(X, X)$ for which there exists a

total $g \in \mathbf{B}(X, W)$ such that the diagram (4.1.1) commutes.

In Heller's original definition, the categories $\mathbf{C}$ and $\mathbf{B}$ were assumed to be B -categories, and so for Heller, the notion of a uniform list was an abstraction of the notion of a Gödel numbering of the total recursive functions. Our notion of a uniform list is an abstraction of the notion of a numbering of the partial recursive functions; we follow the development in Heller using the more general notion [Hel90].

Lemma 4.1.1 (Lemma 4.1 [Hel90]). *If $\mathbf{C}$ is a P^+ -category and $\mathbf{B}$ is a P^+ -subcategory of $\mathbf{C}$, then*

$$i) \mathcal{L}(\mathbf{B}, fp_1) = \{f\};$$

ii) *If $t : W \times X \longrightarrow X$ and $t' : W' \times X \longrightarrow X$ are in $\mathbf{C}$, then*

$$\mathcal{L}(\mathbf{B}, t) \cup \mathcal{L}(\mathbf{B}, t') \subseteq \mathcal{L}(\mathbf{B}, [t, t']dist').$$

Proof. For i), note that if $h : X \longrightarrow X$ is in $\mathcal{L}(\mathbf{B}, fp_1)$, there exists a total morphism $g : X \longrightarrow X$ in $\mathbf{B}$ such that the square in the following diagram commutes; the triangle commutes by definition of a P -category.

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_X} & X \times X & \xrightarrow{g \times 1} & X \times X \\ & \searrow 1 & \downarrow p_1 & & \downarrow fp_1 \\ & & X & \xrightarrow{h} & X \end{array}$$

Since $\text{dom } g = 1_X$, $f = f(p_1(g \times 1)\Delta_X) = hp_1\Delta_X = h$; taking $h = f$, it follows that $f \in \mathcal{L}(\mathbf{B}, fp_1) \subseteq \{f\}$. The proof of ii) is as in Heller [Hel90]. $\square$

Similarly, we have the following useful proposition.

Proposition 4.1.2. *If $\mathbf{C}$ is a P -category, and if $t : W \times X \longrightarrow X$ is in the P -subcategory $\mathbf{B}$ of $\mathbf{C}$, then $\mathcal{L}(\mathbf{B}, t) \subseteq \mathbf{B}$.*

Proof. If $f \in \mathcal{L}(\mathbf{B}, t)$, then there exists a total $g : X \longrightarrow W$ in $\mathbf{B}$ such that the following diagram commutes.

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_X} & X \times X & \xrightarrow{g \times 1} & W \times X \\ & \searrow 1 & \downarrow p_1 & & \downarrow t \\ & & X & \xrightarrow{f} & X \end{array}$$

Since $fp_1 = t(g \times 1_X) \in \mathbf{B}$, it follows that $f = fp_1\Delta_X \in \mathbf{B}$. □

4.2 Isotypical objects, isotypes and frames

The following is a paraphrase of Heller [Hel90]. A category $\mathbf{C}$ is *isotypical* if any two of its objects are isomorphic. For example, if $\mathfrak{M}$ is an infinite cardinal number, then the full subcategory $\mathbf{Set}_{\mathfrak{M}}$ of $\mathbf{Set}$ containing the sets of cardinality $\mathfrak{M}$ can be given the structure of a $B\Sigma$ -category, hence of an isotypical $B^\#$ -category.

Let $\mathbf{C}$ be a $B^\#$ category. An object X of $\mathbf{C}$ is *isotypical* if it is isomorphic to each of $X \times X$, $X \amalg X$, and $X^\#$. Such objects are used in the construction of recursion categories in the sequel. If X is an isotypical object of a $B^\#$ -category $\mathbf{C}$, then $\mathbf{C}\langle X \rangle$ (cf. equation 3.2.1) is an isotypical $B^\#$ -category.

If $\mathbf{C}$ is a $B^\#$ -category and $\mathbf{B} = \mathbf{B}\langle X \rangle$ is an isotypical $B^\#$ -subcategory, then $\mathbf{B}$ contains a *frame* b at X , namely a collection of isomorphisms $b_\times : X \longrightarrow X \times X$, $b_\amalg : X \longrightarrow X \amalg X$, $b_\# : X \longrightarrow X^\#$, along with their inverses.

The category $\mathbf{Set}$ is a $B^\#$ -category, and the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is an isotypical object of $\mathbf{Set}$. We give a motivating example of a frame b at $\mathbb{N}$.

The isomorphism $b_\times : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ is the inverse of the standard pairing function

$$(x, y) \longmapsto (x^2 + 2xy + y^2 + 3x + y)/2.$$

The isomorphism $b_{\sqcup} : \mathbb{N} \longrightarrow \mathbb{N} \amalg \mathbb{N}$ is given by

$$n \longmapsto \begin{cases} i_0(n/2) & \text{if } n \text{ is even;} \\ i_1((n-1)/2) & \text{otherwise,} \end{cases}$$

where $i_j : \mathbb{N} \longrightarrow \mathbb{N} \amalg \mathbb{N}$ is the coproduct injection for $j = 0, 1$.

The isomorphism $b_{\#} : \mathbb{N} \longrightarrow \coprod_{k=1}^{\infty} \mathbb{N}^k$ is given by

$$n \longmapsto (a_0, \dots, a_{k-1} - 1)$$

where $a_j = \text{ord}_{p_j}(n+2)$ is largest power of p_j dividing $n+2$ for $0 \leq j \leq k$, p_j denotes the j -th prime number, and where k is the largest integer such that $\text{ord}_{p_k}(n+2)$ is nonzero. The map $b_{\#}$ is injective by unique factorization. If $k \geq 1$ and $\alpha = (a_0, \dots, a_{k-1}) \in \mathbb{N}^k$, then $n = p_0^{a_0} \cdots p_{k-1}^{a_{k-1}+1} - 2 \geq 0$, so that $b_{\#}(n) = \alpha$ and hence $b_{\#}$ is surjective.

4.3 Uniform generation

The isotypical $B^{\#}$ -subcategory $\mathbf{B}$ is called a *uniformly generated $B^{\#}$ -subcategory* of $\mathbf{C}\langle X \rangle$ provided for some $t : W \times X \longrightarrow X$ in $\mathbf{B}$ and for some frame b at X ,

$$\mathbf{B} = B^{\#}(b \cup \mathcal{L}(\mathbf{B}, t)) \quad (4.3.1)$$

(cf. Proposition 3.2.1 and preceeding remarks, and section 4.2).

Uniformly generated categories are independent of the choice of frame in the following sense.

Proposition 4.3.1. *Let $\mathbf{C}$ be a $B^{\#}$ -category and let $\mathbf{B} = \mathbf{B}\langle X \rangle \subseteq \mathbf{C}$ be an isotypical $B^{\#}$ -subcategory with frame b at X . If for some $t : W \times X \longrightarrow X$ in $\mathbf{B}$ and for*

some frame b at X , the condition (4.3.1) holds, then for any frame b' at X in $\mathbf{B}$, there exists $t' : W \times X \longrightarrow X$ in $\mathbf{B}$ such that $\mathbf{B} = B^\#(b' \cup \mathcal{L}(\mathbf{B}, t'))$.

Proof. Since $b_x^{-1}b'_x p_1 : X \times X \longrightarrow X \in \mathbf{B}$, by Lemma 4.1.1, number i) and by Proposition 4.1.2 we have $\{b_x^{-1}b'_x\} = \mathcal{L}(\mathbf{B}, b_x^{-1}b'_x p_1) \subseteq \mathbf{B}$. Moreover, $[t, b_x^{-1}b'_x p_1] \text{dist}' \in \mathbf{B}$, so by Lemma 4.1.1, number ii) and by Proposition 4.1.2 we have

$$\mathcal{L}(\mathbf{B}, t) \cup \{b_x^{-1}b'_x\} \subseteq \mathcal{L}(\mathbf{B}, [t, b_x^{-1}b'_x p_1] \text{dist}') \subseteq \mathbf{B}.$$

Repeating this argument, it follows that there exists $t' : W' \times X \longrightarrow X$ in $\mathbf{B}$ such that

$$\mathcal{L}(\mathbf{B}, t) \cup \{b_x^{-1}b'_x, b'_x b_x, b_{\Pi}^{-1}b'_{\Pi}, b'_{\Pi} b_{\Pi}, b_{\#}^{-1}b'_{\#}, b'_{\#} b_{\#}\} \subseteq \mathcal{L}(\mathbf{B}, t') \subseteq \mathbf{B},$$

and therefore

$$\mathbf{B} = B^\#(b \cup \mathcal{L}(\mathbf{B}, t)) \subseteq B^\#(b \cup \mathcal{L}(\mathbf{B}, t')) = B^\#(b' \cup \mathcal{L}(\mathbf{B}, t')) \subseteq \mathbf{B}.$$

Since $\mathbf{B}$ is isotypical, the morphism t' can be taken of the form $W \times X \longrightarrow X$. $\square$

Uniformly generated isotypical categories have the following properties.

Proposition 4.3.2 (Proposition 4.3 [Hel90]). *Let $\mathbf{C}$ be a $B^\#$ -category and X an isotypical object of $\mathbf{C}$.*

- i) *Any finitely generated isotypical $B^\#$ -subcategory $\mathbf{B} = \mathbf{B}\langle X \rangle \subset \mathbf{C}\langle X \rangle$ is uniformly generated.*
- ii) *The uniformly generated $\mathbf{B} = \mathbf{B}\langle X \rangle \subset \mathbf{C}\langle X \rangle$ are directed by inclusion.*
- iii) *If b is a frame at X , $t : W \times X \longrightarrow X$ and $\{\mathbf{B}_\alpha\}$ is a family of categories satisfying (4.3.1), then so does $\mathbf{B} = B^\#(\cup_\alpha \mathbf{B}_\alpha)$.*
- iv) *For any $t : W \times X \longrightarrow X$ in $\mathbf{C}\langle X \rangle$ there is a uniformly generated $\mathbf{B} \subseteq \mathbf{C}$ containing t and $\mathcal{L}(\mathbf{B}, t)$.*

For any frame b and for any catalogue t such that (4.3.1) holds, there exists a maximal uniformly generated category $\mathbf{B}$ satisfying (4.3.1).

Lemma 4.3.3 below plays a crucial role in the construction of recursion categories. We reprove the lemma it depends on, Lemma 3.4 of Heller, correcting some typographical errors appearing there [Hel90].

Lemma 4.3.3 (Lemma 4.4 [Hel90]). *If $\mathbf{C}$ is a $B^\#$ -category, then the isotypical category $\mathbf{B} = \mathbf{B}\langle X \rangle \subseteq \mathbf{C}\langle X \rangle$ is uniformly generated if and only if for some $s : X \times X \longrightarrow X$, $\mathbf{B}(X, X)$ is generated, as a semigroup, by $\mathcal{L}(\mathbf{B}, s)$.*

Introducing Lemma 3.4 involves the reduction of a monogenic isotypical $B^\#$ -category $\mathbf{C} = \mathbf{C}\langle X \rangle$ to $B^\#$ -category with a single object, and from there to a $B^\#$ -algebra. For convenience, we outline the the reduction from Heller [Hel90].

Given a monogenic isotypical $B^\#$ -category $\mathbf{C} = \mathbf{C}\langle X \rangle$, fix a frame b at X in $\mathbf{C}$. The subcategory $\mathbf{D}$ of $\mathbf{C}$ generated by the frame b and closed under $\times$, $+$ and $\#$ is an connected (indiscrete) groupoid; for any object Y of $\mathbf{D}$, there is a unique isomorphism $b_Y : X \longrightarrow Y$; for example, for $Y = X^\# \amalg ((X \times X) \amalg X)$, b_Y is the composite

$$X \xrightarrow{b_\amalg} X \amalg X \xrightarrow{b_\# \amalg ((b_\times \amalg 1_X) b_\amalg)} X^\# \amalg ((X \times X) \amalg X).$$

Proposition 4.3.4 (Heller [Hel90]). *Given a monogenic isotypical $B^\#$ -category $\mathbf{C} = \mathbf{C}\langle X \rangle$ and a frame b at X , the monoid $\mathbf{C}(X, X)$ has a unique structure as a $B^\#$ -category such that the functor $R : \mathbf{C} \longrightarrow \mathbf{C}(X, X)$ defined by*

$$(f : Y \longrightarrow Z) \longmapsto b_Z^{-1} f b_Y$$

is a $B^\#$ -functor.

For example, the relations $R(f \times g) = R(f) \times R(g)$ for $f : Y \longrightarrow Z$, $g : U \longrightarrow V$ in $\mathbf{C}$ forces the product of $\phi, \psi : X \longrightarrow X$ in $\mathbf{C}(X, X)$ to be defined by $b_x^{-1}(\phi \times \psi)b_x$. Let τ be any one of the fourteen natural transformations

$$\Delta, p_0, p_1, \nabla, i_0, i_1, \text{dist}, m, j, e, l, r, \text{par}, \text{wd}$$

of $\mathbf{C}$. Since τ is natural, the components τ_Y, τ_Z of τ satisfy $R(\tau_Y) = R(\tau_Z)$. For example, we have the following commutative diagram, in which the outer squares commute by definition of R , and in which the inner square commutes by naturality.

$$\begin{array}{ccccccc} X & \xrightarrow{1} & X & \xrightarrow{b_Y} & Y & \xrightarrow{b_Y^{-1}} & X \\ R(\Delta_X) \downarrow & & \Delta_X \downarrow & & \Delta_Y \downarrow & & \downarrow R(\Delta_Y) \\ X & \xrightarrow{b_X} & X \times X & \xrightarrow{b_Y \times b_Y} & Y \times Y & \xrightarrow{b_{Y \times Y}^{-1}} & X \end{array}$$

This implies that the $B^\#$ -category $\mathbf{C}(X, X)$ could be given the structure of an algebra $\mathbf{C}_b$ (depending on the choice of frame b at X) with fifteen constants X (the unit), $\Delta, p_0, p_1, \nabla, i_0, i_1, \text{dist}, m, j, e, l, r, \text{par}$ and wd , one unary operation $\#$, and three binary operations $\circ, \times$ and $\amalg$, which satisfy a finite set of equations, such as $X^\# = X \times X = X \amalg X = X$.

The generation of $B^\#$ -categories reduces to the generation of $B^\#$ -algebras.

Proposition 4.3.5 (Heller [Hel90]). *Given a monogenic isotypical $B^\#$ -category $\mathbf{C} = \mathbf{C}\langle X \rangle$ and a subset $\mathfrak{W} \subseteq \mathbf{C}(X, X)$. The following assertions are equivalent.*

- (i) *For some (any) frame b at X , $\mathfrak{W}$ generates $\mathbf{C}_b$ as a $B^\#$ -algebra.*
- (ii) *For some (any) frame b at X , $b \cup \mathfrak{W}$ generates $\mathbf{C}$ as a $B^\#$ -category.*

Let $\mathbf{C}$ be a $B^\#$ -algebra and let $\mathfrak{W}$ be a subset of $\mathbf{C}$. We let $\text{alg}\mathfrak{W}$ ($\text{mon}\mathfrak{W}$) denote the $B^\#$ -algebra (monoid) generated by $\mathfrak{W}$. Let $\mathfrak{N}$ be the set consisting of the fifteen 0-ary operations of $\mathbf{C}$, as well as the derived 0-ary operations $\text{ass}_x, \text{ass}_x^{-1}$,

$\text{tr}_x, \text{ass}_\sqcup, \text{ass}_\sqcup^{-1}, \text{tr}_\sqcup, \text{dist}^{-1}, l^{-1}, \text{first} = [Xp_0]l, \text{dist}' = (\text{tr}_x \sqcup \text{tr}_x)\text{dist} \text{tr}_x, \text{dist}'^{-1}, l_2 = (X \sqcup (X \times l))l, \text{ and } l_2^{-1}.$

For a subset $\mathfrak{W}$ of $\mathbf{C}$, define $\mathfrak{W}^\dagger = \mathfrak{N} \cup \{(X \sqcup f)^\# | f \in \mathfrak{W} \cup \mathfrak{N}\}.$

Lemma 4.3.6 (Lemma 3.3 [Hel90]). $\text{alg}\mathfrak{W} = \text{mon}\mathfrak{W}^\dagger.$

Proof. The inclusion $\text{mon}\mathfrak{W}^\dagger \subseteq \text{alg}\mathfrak{W}$ is immediate. We follow the arrangement in Heller [Hel90] of the proof of the reverse inclusion. Let $M = \text{mon}\mathfrak{W}^\dagger$

1) If $f_0 \times f_1 \in M$, then $f_j = p_j(f_0 \times f_1)\Delta \in M$ for $j = 0, 1$. Dually, if $f_0 \sqcup f_1 \in M$, then $f_j = \nabla(f_0 \sqcup f_1)i_j \in M$ for $j = 0, 1$ and also $f_1 \sqcup f_0 = \text{tr}_\sqcup(f_0 \sqcup f_1)\text{tr}_\sqcup \in M$. If $f^\# \in M$ then $f = \text{first}f^\#j \in M$. It follows that $\mathfrak{W} \cup \mathfrak{N} \subseteq M$.

2) It follows from 1) that

$$\text{ass}_x, \text{ass}_\sqcup, \text{ass}_x \sqcup X, \text{ass}_\sqcup \sqcup X, X \sqcup \text{ass}_x, X \sqcup \text{ass}_\sqcup \quad (4.3.2)$$

and their inverses are in M . Let $f_0, \dots, f_n$ be a finite sequence of elements of $\mathbf{C}$. Conjugating any association of any permutation of $f_0 \sqcup \dots \sqcup f_n$ by composites of morphisms in the list 4.3.2 and by $\text{tr}_\sqcup$, it follows that if an association of any permutation of $f_0 \sqcup \dots \sqcup f_n$ is in M , then so are all the others. It follows from 1) that for any subset $I \subseteq \{0, \dots, n\}$ of the indices, the coproduct $\sqcup_{k \in I} f_k$ is also in M .

3) If $(f \sqcup g)^\# \in M$ then, conjugating by wd ,

$$f^\# \sqcup g^\# \sqcup (f^\# \times g^\#)^\# \sqcup (g^\# \times f^\#)^\# \sqcup (g^\# \times (f^\# \times g^\#)^\#) \sqcup (f^\# \times (g^\# \times f^\#)^\#)$$

is in M . It follows from 2) that the summands are in M .

4) By 2) and 3) it follows that $\text{ass}_\sqcup^\#$ and its inverse are in M . Therefore, if $(X \sqcup f)^\# \in M$, then since $X \sqcup X = X$, $(X \sqcup (X \sqcup f))^\# \in M$. Since $(\text{first} \sqcup X)^\#, (j \sqcup$

$X)^\# \in \mathfrak{W}^\dagger$, by 1), first $\amalg X, j \amalg X \in M$.¹ By 3), $X^\# \amalg (X \amalg f)^\# \in M$, so that $X \amalg (X \amalg f)^\# = (\text{first} \amalg X)(X^\# \amalg (X \amalg f)^\#)(j \amalg X) \in M$. Therefore, for any $f \in \mathfrak{W}^\dagger$, $X \amalg f \in M$; for if $f \in \mathfrak{N}$, then $X \amalg f \in \mathfrak{W}^\dagger$, and if $f = (X \amalg g)^\#$ for some $g \in \mathfrak{W} \cup \mathfrak{N}$, then $X \amalg f = X \amalg (X \amalg g)^\# \in M$.

5) If $g \in M$, then $X \amalg g \in M$, since $g = g_1 \circ \cdots \circ g_k$ with $g_j \in \mathfrak{W}^\dagger$, so that by 4) $X \amalg g_j \in M$ and hence $X \amalg g = (X \amalg g_1) \circ \cdots \circ (X \amalg g_k) \in M$. Consequently, if $f, g \in M$, then $X \amalg f \amalg X, X \amalg X \amalg g \in M$, so that the composition $X \amalg f \amalg g$ is also in M . By 1), it follows that M is closed under $\amalg$.

6) Conjugating an element of the form $(X \amalg f)^\# \in M$ with l_2 , we have

$$X \amalg f \amalg (X \amalg f) \times (X \amalg f \amalg (X \amalg f) \times (X \amalg f)^\#) \in M.$$

Conjugating this with $X \amalg \text{dist}$, we have

$$X \amalg f \amalg (X \amalg f) \times (X \amalg f) \amalg (X \amalg f) \times (X \amalg f \amalg (X \amalg f) \times (X \amalg f)^\#) \in M,$$

so that by 1) and 2), and conjugation with dist and dist' , it follows that $X \times f$ and $X \times (X \amalg f)^\#$ are in M .

As in 4), if $f \in \mathfrak{W}^\dagger$ then $X \times f \in M$; for if $f \in \mathfrak{N}$, then $(X \amalg f)^\# \in \mathfrak{W}^\dagger$ and hence $X \times f \in M$; otherwise if $f = (X \amalg g)^\#$ for some $g \in \mathfrak{W} \cup \mathfrak{N}$, then $X \amalg f = X \times (X \amalg g)^\# \in M$.

As in 5), if $g \in M$, then $X \times g \in M$, since $g = g_1 \circ \cdots \circ g_k$ with $g_j \in \mathfrak{W}^\dagger$, so by previous remarks, $X \times g_j \in M$ and hence $X \times g = (X \times g_1) \circ \cdots \circ (X \times g_k) \in M$. Consequently, if $f, g \in M$, then $X \times f \times X, X \times X \times g \in M$, so the composition $X \times f \times g$ is also in M . By 1), it follows that M is closed under $\times$.

¹In an isotypical monogenic $B^\#$ -category $\mathbf{B} = \mathbf{B}(X)$ with frame b , the relation $b_U^{-1}(1_X \amalg 1_X)b_U = 1_X$ implies that the unit X of a $B^\#$ -algebra satisfies $X \amalg X = X$. Similarly, the unit X satisfies $X \times X = X$ and $X^\# = X$.

7) If $(X \amalg f)^\#$ is in M , then so is $(X \amalg f)^{\#\#}$. By 3), $(X^\# \times f^\#)^\# \in M$. Moreover, $f^\# = p_1(X \times f^\#)\Delta$, so that $f^{\#\#} = p_1^\#(X \times f^\#)^\#\Delta^\# = p_1^\#(X^\# \times f^\#)^\#\Delta^\#$, since $X = X^\#$. Since p_1 and Δ are in $\mathfrak{N}$, $(X \amalg p_1)^\#$ and $(X \amalg \Delta)^\#$ are in $\mathfrak{W}^\dagger$. By 2) and 3), $p_1^\#$ and $\Delta^\#$ are in M . Therefore $(X \amalg f)^\# \in M$ implies $f^{\#\#} \in M$. Also, $(X \amalg f)^\# \in M$ implies $(X \amalg (X \amalg f))^\# \in M$, so by 3), $(X^\# \times (X \amalg f)^\#)^\# \in M$, and by the preceding argument it follows that $(X \amalg f)^{\#\#} \in M$.

As in 4), if $f \in \mathfrak{W}^\dagger$ then $f^\# \in \text{mon}\mathfrak{W}^\dagger$; for if $f \in \mathfrak{N}$, then $(X \amalg f)^\# \in \mathfrak{W}^\dagger$ and hence by 3), $f^\# \in \text{mon}\mathfrak{W}^\dagger$; otherwise if $f = (X \amalg g)^\#$ for some $g \in \mathfrak{W} \cup \mathfrak{N}$, then $f^\# = (X \amalg g)^{\#\#} \in M$.

As in 5), if $g \in M$, then $g^\# \in M$, since $g = g_1 \circ \dots \circ g_k$ with $g_j \in \mathfrak{W}^\dagger$, and by previous remarks, $g_j^\# \in M$, so that $g^\# = g_1^\# \circ \dots \circ g_k^\# \in M$.

8) By construction, $M = \text{mon}\mathfrak{W}^\dagger$ contains the set of 0-ary operations $\mathfrak{N}$, so it is a subalgebra containing $\mathfrak{W}$. □

Corollary 4.3.6.1 (Corollary 3.4 [Hel90]). *A $B^\#$ -algebra is finitely generated as an algebra if and only if it is finitely generated as a monoid.*

4.4 Turing developments and locally faithful embeddings

The iteration lemma 3.1.2 will be applied to total coalgebra maps $f : X \longrightarrow X$ that fix a sub-coalgebra U arising as a summand of a coproduct $X = U \amalg V$.

In the general case, we let $\mathbf{C}$ be an iteration category; the following nomenclature is due to Heller [Hel90]. A *Turing datum* in $\mathbf{C}$ is a diagram

$$X \xrightarrow{u} W \xrightarrow{v} W \amalg Y$$

in $\mathbf{C}_T$ (i.e., u and v are total); in our case $\mathbf{C} = \mathbf{Pfn}_F$ and $\mathbf{C}_T = \mathbf{Set}_F$. The map $[v, i_1] : W \amalg Y \longrightarrow W \amalg Y$ is total and satisfies $[v, i_1](\emptyset \amalg Y) = \emptyset \amalg Y$, so by the iteration lemma 3.1.2 there is a map

$$\text{It}([v, i_1], \emptyset \amalg Y) : W \amalg Y \longrightarrow W \amalg Y.$$

The *Turing development* $\text{Tur}\left(X \xrightarrow{u} W \xrightarrow{v} W \amalg Y\right) : X \longrightarrow Y$ of the given Turing datum is the composite

$$X \xrightarrow{i_0 u} W \amalg Y \xrightarrow{\text{It}([v, i_1], \emptyset \amalg Y)} W \amalg Y \xrightarrow{[0, 1_Y]} Y.$$

We sometimes denote such a Turing development by $\text{Tur}(u, v)$.

If $\mathbf{A}$ is a B^+ -subcategory of $\mathbf{C}_T$, then the *class of all Turing developments* of Turing data in $\mathbf{A}$ is denoted by $\text{Tur}(\mathbf{A})$. The class $\text{Tur}(\mathbf{A})$ is a directed graph containing $\mathbf{A}$ with the same objects as $\mathbf{A}$ and closed under $\boxtimes$ and $\amalg$; this follows from the from Lemma 8.1 of Heller [Hel90], whose statement and proof is included here for the convenience of the reader. Moreover, under the additional assumption that connected domains of $\mathbf{C}$ separate morphisms, it follows that the graph $\text{Tur}(\mathbf{A})$ is closed under

composition [Hel90]. A more general criterion for $\text{Tur}(\mathbf{A})$ to be a category is given in the sequel.

Lemma 4.4.1 (Lemma 8.1 Heller [Hel90]). *In an iteration category, the following statements hold.*

(i) *If $f : X \longrightarrow Y$ is total, then*

$$\text{Tur} \left(X \xrightarrow{1} X \xrightarrow{if} X \amalg Y \right) = f.$$

(ii) *The class of Turing developments is closed under coproduct:*

$$\begin{aligned} & \text{Tur} \left(X \xrightarrow{u} W \xrightarrow{v} W \amalg Y \right) \amalg \text{Tur} \left(X' \xrightarrow{u'} W' \xrightarrow{v'} W' \amalg Y' \right) \\ &= \text{Tur} \left(X \amalg X' \xrightarrow{u \amalg u'} W \amalg W' \xrightarrow{t(v \amalg v')t} (W \amalg W') \amalg (Y \amalg Y') \right), \end{aligned}$$

where t is the middle-term transposition derived from tr_+ .

(iii) *The class of Turing developments is closed under product:*

$$\begin{aligned} & \text{Tur} \left(X \xrightarrow{u} W \xrightarrow{v} W \amalg Y \right) \boxtimes \text{Tur} \left(X' \xrightarrow{u'} W' \xrightarrow{v'} W' \amalg Y' \right) \\ &= \text{Tur} \left(X \times X' \xrightarrow{u'''} W''' \xrightarrow{v'''} W''' \amalg (Y \amalg Y') = (W \amalg Y) \times (W' \amalg Y') \right), \end{aligned}$$

for certain morphisms u''' and v''' , where $W''' = (W \times W') \amalg (Y \times W') \amalg (W \times Y')$.

Proof. Statement (i) is immediate; statement (ii) follows from Lemma 3.1.2 (the coproduct of the iterates is the iterate of the coproduct); the third statement also depends on Lemma 3.1.2 (the product of the iterates is the iterate of the product) and the commutativity of the following naturality square, in which the vertical arrows are derived from distributivity and associativity isomorphisms, and where the maps $\alpha = [i_{Y \times W'}, 1_{Y \times Y'}]$ and $\beta = [i_{W \times Y'}, 1_{Y \times Y'}]$ are appropriate conjugates by associativity isomorphisms of the induced maps shown.

$$\begin{aligned} Y \times W' &\xrightarrow{1_Y \times v'} Y \times (W' \amalg Y') = (Y \times W') \amalg (Y \times Y') \xrightarrow{[i_{Y \times W'}, 1_{Y \times Y'}]} W''' \amalg (Y \times Y') \\ W \times Y' &\xrightarrow{v \times 1_{Y'}} (W \amalg Y) \times Y' = (W \times Y') \amalg (Y \times Y') \xrightarrow{[i_{W \times Y'}, 1_{Y \times Y'}]} W''' \amalg (Y \times Y') \end{aligned}$$

$$\begin{array}{ccc}
(W \amalg Y) \times (W' \amalg Y') & \xrightarrow{[v, i_Y] \times [v', i_{Y'}]} & (W \amalg Y) \times (W' \amalg Y') \\
\downarrow & & \downarrow \\
W''' \amalg (Y \times Y') & \xrightarrow{[[v \times v', \alpha(1_Y \times v'), \beta(v \times 1_{Y'})], i_{Y \times Y'}]} & W''' \amalg (Y \times Y')
\end{array}$$

We take u''' to be the composite $X \times X' \xrightarrow{u \times u'} W \times W' \xrightarrow{i_{W \times W'}} W'''$, and we take $v''' = [v \times v', \alpha(1_Y \times v'), \beta(v \times 1_{Y'})]$. $\square$

A domain α is *connected* if $\alpha = \beta \vee \gamma$ for domains β, γ with $\beta \wedge \gamma = 0$, then either $\beta = 0$ or $\gamma = 0$. The iteration category $\mathbf{C}$ is *locally connected* if for morphisms f, g of $\mathbf{C}$, $f = g$ whenever $f\alpha = g\alpha$ for each connected domain α . The category $\mathbf{Pfn}$ of sets and partial functions is a locally connected iteration category. In an appendix we compare the usual topological definition of local connectedness to its categorical analog, due to Heller; the terminology was suggested by an analogy with topological spaces and continuous maps [Hel90]. In $\mathbf{Set}_F$ we have the following.

Proposition 4.4.2. *If the Set endofunctor F weakly preserves generalized pullbacks of monomorphisms, then each F -coalgebra is locally connected in the coalgebra topology.*

Proof. By Theorem 1.2.1, for each F -coalgebra X and for each $x \in X$, the subcoalgebra cogenerated by x is nonempty; hence the 1-cogenerated subcoalgebras of X form a base for the coalgebra topology on X . Each 1-cogenerated subcoalgebra is connected in the coalgebra topology. $\square$

We make some notational abbreviations in the following. If $Y = V \amalg W$, there are domains $1_V \amalg 0_{W,W}, 0_{V,V} \amalg 1_W \in \text{dom}(Y)$ and we have $1_Y = (1_V \amalg 0_{W,W}) \vee (0_{V,V} \amalg 1_W)$. We write V for 1_V and $V \amalg W$ for $(V \amalg 0) \vee (0 \amalg W)$.

Proposition 4.4.3. (i) If $f : X \longrightarrow Y$ is total and $\alpha \in \text{dom}(X)$ is connected, then $f_*\alpha$ is connected.

(ii) If $f : X \longrightarrow Y$ is total, $\alpha \in \text{dom}(X)$ is connected and $V \amalg 0, 0 \amalg W$ are domains in $\text{dom}(Y)$ such that $f_*\alpha \leq V \amalg W$, then $f\alpha \leq V \amalg 0$ or $f\alpha \leq 0 \amalg W$.

(iii) If $f : X \longrightarrow Y$ is total, $\alpha \in \text{dom}(X)$ is connected and $V \amalg 0, 0 \amalg W$ are domains in $\text{dom}(Y)$ such that $f\alpha \prec V \amalg W$, then $f_*\alpha \prec V \amalg 0$ or $f_*\alpha \prec 0 \amalg W$.

Proof. For (ii), note that $f_*\alpha = f_*\alpha \wedge (V \amalg W) = f_*\alpha \wedge ((V \amalg 0) \vee (0 \amalg W)) = (f_*\alpha \wedge (V \amalg 0)) \vee (f_*\alpha \wedge (0 \amalg W))$; the conclusion follows from (i).

For (iii), note that $f_*\alpha$ is a range, so that $f\alpha \prec f_*\alpha$; i.e., $(f_*\alpha)f\alpha = f\alpha$. The result follows immediately from (ii) and Proposition 2.4.1. $\square$

Di Paola and Montagna produced examples of non-locally connected recursion categories, the so-called syntactic P -categories; we defer their discussion to the conclusion [Mon89, DPM91].

Local connectedness can be used to show that the directed graph $\text{Tur}(\mathbf{A})$ is a category, where A is a subcategory of the category $\mathbf{C}_T$ of total morphisms of the iteration category $\mathbf{C}$. This follows from Lemma 8.2 of Heller; its generalization in the sequel relies heavily on the proof, so we include it here.

Lemma 4.4.4 (Lemma 8.2 [Hel90]). If $\mathbf{C}$ is a locally connected iteration category, and

$$X \xrightarrow{u} W \xrightarrow{v} W \amalg Y \text{ and } Y \xrightarrow{s} V \xrightarrow{t} V \amalg Z$$

are Turing data, then

$$\begin{aligned} \text{Tur} \left(Y \xrightarrow{s} V \xrightarrow{t} V \amalg Z \right) \text{Tur} \left(X \xrightarrow{u} W \xrightarrow{v} W \amalg Y \right) \\ = \text{Tur} \left(X \xrightarrow{i_W u} W \amalg V \xrightarrow{r} W \amalg V \amalg Z \right), \end{aligned}$$

where $W \amalg V \xrightarrow{r} W \amalg V \amalg Z$ is the composition

$$W \amalg V \xrightarrow{v \amalg 1_V} W \amalg Y \amalg V \xrightarrow{1_W \amalg [i_V s, t]} W \amalg V \amalg Z.$$

The proof of this lemma depends on the following propositions, which pertain to the iteration of a endomorphism of a coproduct in an iteration category; they are implicit in Heller [Hel90]. Their role is to allow the verification of the commutativity of certain diagrams by reasoning as if the morphisms of those diagrams behave like continuous maps on connected subspaces, and they introduce a categorical notion of the length of a computation.

Proposition 4.4.5. *Let $\mathbf{C}$ be an iteration category, let $f : X \amalg Y \longrightarrow X \amalg Y$ be a total morphism such that $f(0 \amalg Y) = 0 \amalg Y$. If $\alpha \leq \text{dom}(\text{It}(f, 0 \amalg Y))$ and for all $n \geq 1$, $f_*^n \alpha \leq X \amalg 0$, then $\alpha = 0$.*

Proof. By hypothesis (since f_* is left adjoint to f^*), for each $n \geq 1$, $\alpha \leq (f^n)^*(X \amalg 0)$, and so $\alpha \wedge (f^n)^*(0 \amalg Y) \leq (f^n)^*(X \amalg 0) \wedge (f^n)^*(0 \amalg Y) = (f^n)^*(X \amalg 0 \wedge 0 \amalg Y) = (f^n)^*(0) = 0$, since f is total. It follows from the iteration lemma that

$$\alpha = \alpha \wedge \bigvee_n (f^n)^*(0 \amalg Y) = \bigvee_n \alpha \wedge (f^n)^*(0 \amalg Y) \leq \bigvee_n 0.$$

□

Proposition 4.4.6. *Let $\mathbf{C}$ be an iteration category, let $f : X \amalg Y \longrightarrow X \amalg Y$ be a total morphism such that $f(0 \amalg Y) = 0 \amalg Y$, and suppose that $\alpha \in \text{dom}(X \amalg 0)$ is connected and nonempty.*

- (i) *If $\alpha \leq \text{dom}(\text{It}(f, 0 \amalg Y))$, then there exists $n \geq 1$ such that $f_*^n \alpha \leq 0 \amalg Y$.*
- (ii) *If $\alpha \leq \text{dom}(\text{It}(f, 0 \amalg Y))$, then there exists $n \geq 1$ such that $f^n \alpha < 0 \amalg Y$.*

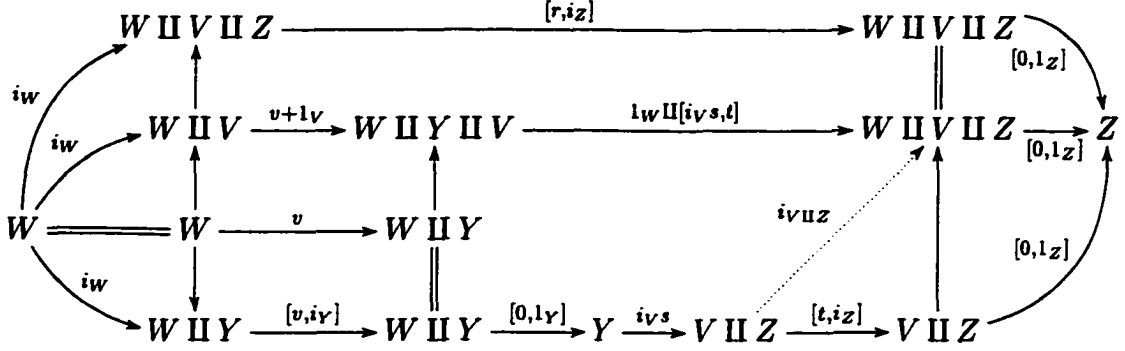
Proof. For (i), by the preceding proposition, there exists $n \geq 1$ such that $f_*^n \alpha \not\leq X \amalg 0$. Since f is total, α is connected, $f_*^n \alpha$ is a connected domain in $X \amalg Y$ and therefore $f_*^n \alpha \leq 0 \amalg Y$.

Statement (ii) follows immediately from (i), the statement $f^n \alpha \prec f_*^n \alpha$ (equivalent to $f_*^n \alpha f^n \alpha = f^n \alpha$, which holds by definition of ranges) and Proposition 2.4.1. $\square$

The hypothesis that the domain and codomain of f is a coproduct cannot be relaxed. A counterexample can be obtained by taking F to be the identity functor on **Set**, and by defining an F -coalgebra $\varphi : \omega \longrightarrow \omega$ by $\varphi(0) = 0$ and $\varphi(n+1) = n$. The map φ is also a morphism from this coalgebra to itself such that $\varphi\{0\} = \{0\}$; the map $\text{It}(\varphi, \{0\})$ given by the iteration lemma—the zero map—has domain $\bigvee_n n = \omega$. Moreover, the domain ω is connected and there is no positive integer n such that $\varphi^n \omega \prec \{0\}$.

Proof of Lemma 4.4.4. We may assume that $X = W$ and that u is the identity, as in the following diagram, which is intended to organize a proof by “diagram chasing” that $\text{Tur}(i_W u, r) = \text{Tur}(s, t)\text{Tur}(u, v)$ using local connectedness. Let α be a non-empty connected domain in $\text{dom}(W)$. For simplicity we also write α for $i\alpha$, where i is a coproduct injection; for “horizontal” maps ϕ, ψ , we write $\phi\alpha \sim \psi\alpha$ if there are coproduct injections (or identities) i and j with $i\phi\alpha = j\psi\alpha$. In virtue of Propositions 4.4.3, 4.4.6 and 2.4.1, we may argue as if α were a point-like object that may occupy

at most one summand of a coproduct at any vertex.



Suppose that $0 \neq \alpha \leq \text{dom}(\text{Tur}(1_W, r))$. Then $\alpha \leq \text{dom}(\text{It}([r, i_Z], 0 \amalg 0 \amalg Z))$. By Proposition 4.4.6, there exists a least positive integer $n > 1$ such that $[r, i_Z]^{n+1}\alpha \prec 0 \amalg 0 \amalg Z$, and therefore $[r, i_Z]^n\alpha \prec W \amalg V \amalg 0$. It cannot happen that $[r, i_Z]^n\alpha \prec W \amalg 0 \amalg 0$; otherwise, a diagram chase would yield $[r, i_Z]^{n+1}\alpha \prec W \amalg V \amalg 0$. By Proposition 4.4.3, $[r, i_Z]^n\alpha \prec 0 \amalg V \amalg 0$, so there is a least $m < n$ such that $[r, i_Z]^m\alpha \prec W \amalg 0 \amalg 0$. By a diagram chase, $[r, i_Z]^m\alpha \sim [v, i_Y]^m\alpha$, so that $[r, i_Z]^{m+1}\alpha \sim i_V s \text{Tur}(W, v)\alpha \prec 0 \amalg V \amalg Z$. If β is a connected domain in $0 \amalg V \amalg W$, then $[r, i_Z]\beta \sim 1_W \amalg [t, i_Z]\beta$, so that $[r, i_Z]^{n+1}\alpha \sim (1_W \amalg [t, i_Z])^{n-m} i_V s \text{Tur}(1_W, v)\alpha$. It follows that $\text{Tur}(s, t)\text{Tur}(1_W, v)\alpha = \text{Tur}(1_W, r)\alpha$.

Conversely, if $0 \neq \alpha \leq \text{dom}(\text{Tur}(s, t)\text{Tur}(1_W, v))$, then for some k , $[v, i_Y]\alpha \prec 0 \amalg Y$, $[r, i_Z]^{k+1}\alpha \prec 0 \amalg V \amalg Z$ and for some l , $[r, i_Z]^l[r, i_Z]^{k+1}\alpha \sim (1_W \amalg [t, i_Z])^l i_V s \text{Tur}(1_W, r)\alpha \prec 0 \amalg 0 \amalg Z$ and therefore $\text{Tur}(s, t)\text{Tur}(1_W, v)\alpha = \text{Tur}(1_W, r)\alpha$. $\square$

We generalize the notion of the local connectedness of a single category $\mathbf{C}$ to a relative notion of local connectedness; this will allow us to extend the application of the existence theorem.

Let $\mathbf{C}, \mathbf{D}$ be iteration categories. An *iteration* functor from $\mathbf{C}$ to $\mathbf{D}$ is a functor $L : \mathbf{C} \longrightarrow \mathbf{D}$ that preserves zero morphisms, coproducts, domains, and the iteration

operator; specifically:

- a) $L0_{X,Y} = 0_{LX,LY}$;
- b) $L(A \amalg B) = (LA) \amalg (LB)$;
- c) for each morphism ϕ in $\mathbf{C}$, $L(\text{dom}\phi) = \text{dom}L\phi$; and
- d) if $\phi : X \longrightarrow X$ is a total morphism of $\mathbf{C}$ and if ε is a domain in $\text{dom}X$ such that $\phi\varepsilon = \varepsilon$, then $L(\text{It}(\phi, \varepsilon)) = \text{It}(L\phi, L\varepsilon)$.

An iteration functor need not preserve products. For example, if $\mathbf{Pfn}_F$ is an iteration category, then the forgetful functor to $\mathbf{Pfn}$ is an iteration functor, but it does not preserve products in general.

Let $\mathbf{C}$ and $\mathbf{D}$ be prodominical categories with coproducts and ranges. A functor $L : \mathbf{C} \longrightarrow \mathbf{D}$ is *locally faithful* if for any morphisms $\phi, \psi : X \longrightarrow Y$ in $\mathbf{C}$, $\phi = \psi$ if and only if for every connected domains $d \in \text{dom}L(X)$, $(L\phi)d = (L\psi)d$. In case $\mathbf{D}$ is locally connected, the functor L is locally faithful if and only if it is faithful.

Let $\mathbf{C}$ and $\mathbf{D}$ be iteration categories. The iteration category $\mathbf{C}$ is *locally connected* in $\mathbf{D}$ if there exists a locally faithful iteration functor $L : \mathbf{C} \longrightarrow \mathbf{D}$.

For example, taking $\mathbf{D} = \mathbf{C}$ and taking L to be the identity functor, we have that $\mathbf{C}$ is locally connected in itself if and only if it is locally connected.

Next, we generalize Heller's Lemma 8.2.

Lemma 4.4.7. *If $\mathbf{C}$ and $\mathbf{D}$ are iteration categories with $\mathbf{C}$ locally connected in $\mathbf{D}$, and if $X \xrightarrow{u} W \xrightarrow{v} W \amalg Y$ and $Y \xrightarrow{s} V \xrightarrow{t} V \amalg Z$ are Turing data in $\mathbf{C}$, then*

$$\begin{aligned} \text{Tur} \left(Y \xrightarrow{s} V \xrightarrow{t} V \amalg Z \right) \text{Tur} \left(X \xrightarrow{u} W \xrightarrow{v} W \amalg Y \right) \\ = \text{Tur} \left(X \xrightarrow{i_0 u} W \amalg V \xrightarrow{r} W \amalg V \amalg Z \right), \end{aligned}$$

where $W \amalg V \xrightarrow{r} W \amalg V \amalg Z$ is the composition

$$W \amalg V \xrightarrow{v \amalg 1_V} W \amalg Y \amalg V \xrightarrow{1_W \amalg [i_0 s, t]} W \amalg V \amalg Z.$$

Proof. By hypothesis, there exists a locally faithful iteration functor $L : \mathbf{C} \longrightarrow \mathbf{D}$. The functor L sends total morphism in $\mathbf{C}$ to total morphisms in $\mathbf{D}$: if $\text{dom} \phi = 1_X$, then $\text{dom} L\phi = L\text{dom} \phi = 1_{LX}$. In particular, L sends a Turing datum in $\mathbf{C}$ to a Turing datum in $\mathbf{D}$; moreover, L commutes with Turing developments; for example,

$$L \left(\text{Tur} \left(X \xrightarrow{i_W u} W \amalg V \xrightarrow{r} W \amalg V \amalg Z \right) \right) = \text{Tur} \left(LX \xrightarrow{i_{LW} \circ Lu} LW \amalg LV \xrightarrow{Lr} LW \amalg LV \amalg LZ \right).$$

Applying the iteration functor to everything in sight, we may assume that L is the identity functor; under this reduction, the proof of Lemma 8.2 applies. $\square$

Proposition 4.4.8. *If F is a nontrivial **Set** endofunctor that weakly preserves pullbacks, and if $\mathbf{Set}_F$ has products, then $\mathbf{Pfn}_F$ is locally connected in $\mathbf{Pfn}$.*

Proof. The forgetful functor $U : \mathbf{Pfn}_F \longrightarrow \mathbf{Pfn}$ preserves coproducts, domains, and iterates; moreover, the only nonempty connected domains in $\mathbf{Pfn}$ are singletons. $\square$

We have the following generalizations of Heller's Lemma 8.3 and Proposition 8.4, respectively [Hel90].

Lemma 4.4.9. *If $\mathbf{C}, \mathbf{D}$ are iteration categories, with $\mathbf{C}$ locally connected in $\mathbf{D}$, and if the Turing development $t = \text{Tur} \left(X \xrightarrow{u} W \xrightarrow{v} W \amalg (X \amalg Y) \right)$ is total, then*

$$\begin{aligned} \text{Tur} \left(X \xrightarrow{1_X} X \xrightarrow{\text{Tur}(u,v)} X \amalg Y \right) &= \\ \text{Tur} \left(X \xrightarrow{i_X} X \amalg W \amalg X \xrightarrow{[i_W i_X u, i_W v, i_X]} X \amalg (W \amalg (X \amalg Y)) \cong (X \amalg W \amalg X) \amalg Y \right). \end{aligned}$$

Proposition 4.4.10. *If $\mathbf{C}, \mathbf{D}$ are iteration categories, if $\mathbf{C}$ is locally connected in $\mathbf{D}$, and if $\mathbf{B}$ is a B^+ -subcategory of $\mathbf{C}$, then*

- (i) $\text{Tur}(\mathbf{B})$ is a $+$ -prodominical subcategory of $\mathbf{C}$ containing $\mathbf{B}$.
- (ii) $\text{Tur}(\mathbf{B})_T = \text{Tur}(\mathbf{B}) \cap \mathbf{C}_T$, and
- (iii) $\text{Tur}(\text{Tur}(\mathbf{B})_T) = \text{Tur}(\mathbf{B})$.

In view of the preceding proposition, $\text{Tur}(\mathbf{B})$ is sometimes called the *Turing completion* of $\mathbf{B}$.

We remark that if F is nontrivial, bounded and preserves generalized pullbacks, then $\mathbf{Pfn}_F$ is a locally connected iteration category. If the $\mathbf{Set}$ endofunctor F is nontrivial and preserves weak pullbacks, and if $\mathbf{C}$ is an iteration category contained in $\mathbf{Pfn}_F$, then $\mathbf{C}$ is locally connected in $\mathbf{Pfn}$. This extends Heller's results on iteration categories to a wide class of non-locally connected iteration categories.

4.5 Recursion categories

In this section we give the statement of Heller's existence theorem and discuss certain aspects of its proof. We then state a generalization of the existence theorem to the relatively locally connected case and derive some corollaries for categories of coalgebras and partial morphisms. In the sequel, we will define the filter functor on $\mathbf{Set}$ and exhibit all topological spaces as coalgebras for the filter functor. Using the property that the filter functor is nontrivial and preserves weak pullbacks, we will show that the subcategory of $\mathbf{Set}_F$ consisting of topological spaces and continuous open maps gives rise to iteration categories (and recursion categories) that are not locally connected, but which are locally connected in $\mathbf{Pfn}$; moreover, the generalized

existence theorem for recursion categories will apply.

A *Turing morphism* in a prodominical isotype is a morphism $\tau : W \boxtimes X \longrightarrow Y$ such that for any $\phi : V \boxtimes X \longrightarrow Y$ there exists a total $g : V \longrightarrow W$ such that the following diagram commutes.

$$\begin{array}{ccc} V \boxtimes X & \xrightarrow{g \times 1_X} & W \boxtimes X \\ & \searrow \phi & \downarrow \tau \\ & & Y \end{array}$$

A *recursion category* is a prodominical isotype in which there is a Turing morphism. With this definition, we can state Heller's existence theorem for recursion categories.

Theorem 4.5.1 (Heller [Hel90]). *Let $\mathbf{C}$ be a locally connected iteration category. If $\mathbf{B} = \mathbf{B}\langle X \rangle$ is a uniformly generated isotypical $B^\#$ -subcategory of $\mathbf{C}\langle X \rangle$, then $\text{Tur}(\mathbf{B})$ is a recursion category.*

The proof proceeds by constructing, in the category $\mathbf{B} = \mathbf{B}\langle X \rangle$, a *weak Turing morphism*; i.e., a map $\sigma : W \boxtimes X \longrightarrow Y$ such that for any $\phi : X \longrightarrow Y$ in $\mathbf{B}$ there exists a total $g : W \longrightarrow W$ such that the following diagram commutes.

$$\begin{array}{ccc} W \boxtimes X & \xrightarrow{g \boxtimes 1_X} & W \boxtimes X \\ \downarrow p_1 & & \downarrow \sigma \\ X & \xrightarrow{\phi} & Y \end{array}$$

The existence of a weak Turing morphism in $\mathbf{B}$ implies that $\mathbf{B}$ is a recursion category.

Lemma 4.5.2 (Lemma 9.1 Heller [Hel90]). *If a prodominical isotype has a weak Turing morphism, then it has a Turing morphism.*

For the construction of a weak Turing morphism $\sigma : X \boxtimes X \longrightarrow X$ in the prodominical isotype $\text{Tur}(\mathbf{B})$ (which is a prodominical isotype in virtue of Proposition

4.4.10, statement (i)), one notes that since $\mathbf{B} = \mathbf{B}\langle X \rangle$ is an isotypical uniformly generated $B^\#$ -subcategory of $\mathbf{C}\langle X \rangle$, Lemma 4.3.3 implies that for some $s : X \times X \longrightarrow X$, the uniform list $\mathcal{L}(\mathbf{B}, s)$ cataloged by s generates $\mathbf{B}(X, X)$ as a semigroup. This fact will allow the representation of the (total) morphism $f : X \longrightarrow X \amalg X$ occurring in a typical element $\phi = \mathbf{Tur}(i, f) : X \longrightarrow X$ (with i and f suitably normalized and i independent of ϕ) of the Turing completion $\mathbf{Tur}(\mathbf{B})$ of $\mathbf{B}$ as a composite of morphisms $b_+^{-1}f = f_1 \circ \cdots \circ f_n$, with $f_i \in \mathcal{L}(\mathbf{B}, s)$. This means that the f_i each have an “index” with respect to s ; i.e., a total $g_i : X \longrightarrow X$ such that $f_i p_1 = s(g_i \boxtimes X)$. The algebraic operations of the formally free semigroup $X^\#$ are used to “encode” the factorization of $b_+^{-1}f$ into an “index” g for ϕ , which an appropriately constructed weak Turing morphism σ can “decode” into the factors f_j , using the catalog s and the operations of the formally free semigroup. The verification that g is an index for ϕ with respect to the weak Turing morphism σ uses local connectedness to verify the equality $f p_1 = \sigma(g \boxtimes X)$; for further details we refer the reader to Heller [Hel90].

Theorem 4.5.3. *Let $\mathbf{C}$ and $\mathbf{D}$ iteration categories, and suppose there exists a locally faithful embedding $L : \mathbf{C} \longrightarrow \mathbf{D}$ that preserves products. If $\mathbf{B} = \mathbf{B}\langle X \rangle$ is a uniformly generated isotypical $B^\#$ -subcategory of $\mathbf{C}\langle X \rangle$, then $\mathbf{Tur}(\mathbf{B})$ is a recursion category.*

Proof. An examination of the proof of Heller’s existence theorem reveals that a similar reduction to the case of local connectedness as in the proof of Lemma 4.4.7 is possible, after one makes the observation that for the reduction it is enough for the iteration preserving functor $\mathbf{C} \longrightarrow \mathbf{D}$ to preserve products; it need not preserve the additional structure of a $B^\#$ -category on the category $\mathbf{B}$. □

By the preceding theorem and Proposition 4.4.8 we have the following.

Theorem 4.5.4. *Let F be a nontrivial endofunctor on $\mathbf{Set}$ that weakly preserves pullbacks. Suppose that $\mathbf{Set}_F$ has products and that the forgetful functor $U : \mathbf{Set}_F \longrightarrow \mathbf{Set}$ preserves products. Then for any uniformly generated isotypical subcategory $\mathbf{C}$ of $\mathbf{Pfn}_F$, $\mathbf{Tur}(\mathbf{C})$ is a recursion category.*

Corollary 4.5.4.1. *Let F be a nontrivial bounded endofunctor on $\mathbf{Set}$ that weakly preserves pullbacks. Suppose that $\mathbf{Set}_F$ has products and that the forgetful functor $U : \mathbf{Set}_F \longrightarrow \mathbf{Set}$ preserves products. If X is an isotypical coalgebra of $\mathbf{Pfn}_F$ and if b is a frame at X , then for any finite collection of (total) morphisms $S \subset \mathbf{Set}_F(X, X)$, $\mathbf{Tur}(B^\#(b \cup S))$ is a recursion category.*

4.6 Examples

Taking the **Set** endofunctor F to be the identity functor and each coalgebra structure map $\alpha : X \longrightarrow FX$ to be the identity morphism 1_X , one obtains the category **Pfn** of sets and partial functions as a subcategory of **Pfn** _{F} ; hence **Pfn** _{F} yields all the examples of recursion categories that come from sets and partial functions, as in [Hel90].

Let $\mathbf{C}$ be a subcategory of **Set** _{F} such that the class $\mathcal{M}$ of monomorphisms of $\mathbf{C}$ is closed under pullbacks along arbitrary morphisms of $\mathbf{C}$, that $\mathbf{C}$ has products and that contains a terminal object 1 . For any F -coalgebra X of $\mathbf{C}$, let $Y = \coprod_{\mathbb{N}} X$ be a countable copower, let $Z = \coprod_{\mathbb{N}} Y^n = 1 \amalg Y \amalg Y^2 \amalg \cdots$ and let $W = Z^\#$. By proposition 1.1 of [Hel90], W is an isotypical object of $\mathbf{C}$ and hence of $P(\mathbf{C}, \mathcal{M})$. Moreover, $\mathbf{C}$ is an iteration category that is relatively locally connected in **Pfn**, so by the theorem its isotypical subcategory $\mathbf{C}\langle W \rangle$ contains many recursion categories.

If F is a bounded **Set** endofunctor that preserves generalized weak pullbacks, then **Pfn** _{F} is a locally connected iteration category, which must therefore contain many recursion categories.

To obtain examples of non-locally connected recursion categories, we first show there exists a faithful embedding of the category **Topen** of topological spaces and continuous open maps as into **Set** _{F} , where F is the filter functor; the representation of topological spaces as coalgebras for the filter functor, a **Set** endofunctor that preserves weak pullbacks but not generalized pullbacks, is due to Gumm [Gum01].

Let X be a set. A set $\mathcal{D}$ of subsets of X is *directed by containment* if $A, B \in \mathcal{D}$ implies $A \cap B \supseteq C$ for some $C \in \mathcal{D}$. A set $\mathcal{D}$ of subsets of X is a *filter* if it is directed by containment and if for each $A \in \mathcal{D}$ and for each subset $S \subseteq X$ of X , $A \subseteq S$

implies $S \in \mathcal{D}$. If $\mathcal{D}$ is a set of subsets of X directed by containment, then the filter generated by $\mathcal{D}$ is denoted and defined by $\mathcal{F}(\mathcal{D}) = \{S \subseteq X : \exists A \in \mathcal{D}, A \subseteq S\}$. The following properties are immediate: if $\mathcal{D}$ is directed by containment, then so is $\varphi\mathcal{D}$ (the set of images of elements of $\mathcal{D}$ under the Set map φ); if $\mathcal{D}$ and $\mathcal{E}$ are filters, then $\mathcal{D} \subseteq \mathcal{E}$ if and only if for each $A \in \mathcal{D}$ there exists $B \in \mathcal{E}$ with $B \subseteq A$; if $\mathcal{D}$ is directed by containment, then $\mathcal{F}(\varphi\mathcal{D}) = \mathcal{F}(\varphi\mathcal{F}(\mathcal{D}))$.

The filter functor $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ is defined by assigning to each set X the set FX of filters on X , and to each map $\varphi : X \longrightarrow Y$ the map $F\varphi : FX \longrightarrow FY$ which sends the filter $\mathcal{D}$ on X to the filter $\mathcal{F}(\varphi\mathcal{D})$ on Y generated by the image of $\mathcal{D}$ under φ . The preceding properties imply that F is a functor.

Theorem 4.6.1 (Theorem 3.4 [Gum01]). *The filter functor preserves weak pull-backs.*

Any topological space X can be represented as a coalgebra for the filter functor. If $\mathcal{O}(X)$ is the lattice of open subsets of X , an F -coalgebra whose coalgebra topology coincides precisely with that of X can be given by a map $\mathcal{N} : X \longrightarrow FX$, where

$$\mathcal{N}(x) = \{A \subseteq X : \exists U \in \mathcal{O}(X), x \in U \subseteq A\}.$$

That is, the coalgebra structure corresponding to the topological space X associates to each point $x \in X$ the filter $\mathcal{N}(x)$ of neighborhoods at x .

It follows that **Topen** is a wide subcategory of $\mathbf{Set}_F$, and by previous remarks, $\mathbf{Pfn}_F$ contains iteration categories that are not locally connected: the Cantor set K is a topological space for which there exist at least two distinct continuous open maps defined on it that agree on its only connected open subset: the empty set. The Cantor set K gives rise to non-locally connected iteration categories that are relatively locally

connected in $\mathbf{Pfn}$; by the generalized existence theorem, they contain many recursion categories.

Chapter 5

Conclusion

In this chapter, we suggest some directions for future research.

Di Paola and Montagna gave examples of recursion categories that are not locally connected, which suggested the possibility of an existence theorem for recursion categories in which the requirement of local connectedness is relaxed; our work was partly motivated by a desire to formulate and prove such a theorem [Mon89, DPM91]. An attempt to apply the generalized existence theorem to these categories, called syntactic P -categories, leads to an algebraic embedding problem; we turn to their construction.

Let T be a consistent extension of Peano Arithmetic; the syntactic P -categories S_T and S'_T are certain wide subcategories of the subcategory $\mathbf{C}$ of $\mathbf{Set}$ generated by ω and closed under Cartesian product $\times$ and disjoint union $\amalg$; we set $S_{T_0} = S'_{T_0} = \mathbf{C}_0$. For objects X, Y of $\mathbf{C}$, define $G(X, Y)$ to be the set of Gödel numbers of partial recursive functions $X \longrightarrow Y$. We set $S_T(X, Y) = G(X, Y) / \sim$, where $m \sim n$ if and only if

$$\vdash_T \forall x ((\phi_m(x) \downarrow \Leftrightarrow \phi_n(x) \downarrow) \wedge (\phi_m(x) \downarrow \Rightarrow \phi_m(x) = \phi_n(x)));$$

we set $S'_T(X, Y) = G(X, Y) / \approx$, where $m \approx n$ if and only if

$$\bigwedge_{x \in X} \vdash_T ((\phi_m(\bar{x}) \downarrow \Leftrightarrow \phi_n(\bar{x}) \downarrow) \wedge (\phi_m(\bar{x}) \downarrow \Rightarrow \phi_m(\bar{x}) = \phi_n(\bar{x}))).$$

An equivalence class of Gödel numbers in either category is denoted by $[e]$ for $e \in \omega$. Di Paola and Montagna show that the S_T are never locally connected, and that S'_T is locally connected if and only if T is Σ_1 -sound [DPM91].¹ Di Paola and Montagna showed that S_T and S'_T are recursion categories that are not locally connected in general, and raised the possibility of finding a generalization of Heller's existence theorem that would apply to them. In view of the generalization proved here, the problem reduces to finding a locally faithful iteration functor $F : S_T \longrightarrow \mathbf{C}$ (or $F : S'_T \longrightarrow \mathbf{C}$) to some iteration category $\mathbf{C}$.

A special case of this embedding problem appears to be of proof theoretic interest. Let R denote the "classical" recursion category, whose sole object is ω , and whose morphisms are the partial recursive functions $\varphi_e : \omega \longrightarrow \omega$ [DPH87, Hel90]. There is an iteration functor $F : S_{PA} \longrightarrow R$ given by sending the equivalence class $[e]$ to the partial recursive function φ_e . Although R is locally connected, the functor F is not a faithful embedding; otherwise, various non-recursively enumerable index sets would become weakly representable in PA . For example, the index set of total recursive functions is Π_2^0 -complete and is 1-1-reducible to the set index set I of the identity function; if F were an embedding, then I would be weakly representable in PA , which cannot happen.² Alternatively, if F were an embedding, the index set of the empty function, a Π_1^0 -complete set, would be weakly representable in PA .

The question of whether there is any faithful embedding F of S_{PA} into R arises.

¹A theory T extending Peano Arithmetic (PA) is Σ_1 -sound if each Σ_1 formula φ provable in T is true in the standard model of PA.

²The set of recursive functions that are provably total in PA is the set of ϵ_0 -recursive functions.

Such an embedding must take domains to domains, and it must be monotonic on domains, so it will induce a strict order preserving embedding of the domains of S_{PA} into the lattice $\mathcal{E}$ of recursively enumerable subsets under inclusion. The following theorem of Di Paola and Montagna imposes restrictions on possible embeddings.

Theorem 5.0.2 (Di Paola, Montagna [DPM91]). *There exists recursive functions f and g of two variables such that, for all e_1 and e_2 , if $[e_1] \prec [e_2]$ in S_T , then $[e_1] < [f(e_1, e_2)] < [e_2]$ in S_T , and if $[e_1] \prec [e_2]$ in S'_T , then $[e_1] < [g(e_1, e_2)] < [e_2]$ in S'_T .*

The proof makes use of the essential undecidability of PA . On account of the theorem of Di Paola and Montagna, whenever $[e]$ and $[e']$ are two domains in S_{PA} with $[e] \prec [e']$, the interval between $F[e]$ and $F[e']$ must be dense, $F[e] \setminus F[e']$ must be infinite and whenever $F[e]$ is finite, it is a minimal element of the image of F in $\mathcal{E}$.

We would like to relax the condition in Theorem 4.5.3 that the locally faithful iteration functor preserve products.

Finally, we observe that the results of Chapter 2 imply that when F is a **Set** endofunctor that preserves weak pullbacks and is such that **Set** $_F$ has products (e.g., bounded) we have the following embedding result of Rosolini, which appears as Proposition 3.3.6 of his thesis. For definitions and further details, see [Ros86].

Proposition 5.0.3 (Rosolini [Ros86]). *Suppose $\mathbf{C}$ is a P -category with ranges, zero maps and coproducts that are stable under products. Suppose also that $\mathbf{C}$ satisfies the weak axiom of choice. Then there exists a topos $\mathcal{D}$ with a dominance Υ such that $\mathbf{C}$ is fully embedded into $P(\mathcal{D}, \mathcal{M})$, where $\mathcal{M}$ is the family of monomorphisms classified by Υ , by a functor of P -categories preserving all the additional structure. Moreover, Υ is closed under unions.*

Appendix A

Proof of the Iteration Lemma via the free semigroup

The following proof of the iteration lemma in $\mathbf{Pfn}_F$ is adapted from the one given in Heller [Hel90]; it does not take advantage of the conceptual simplifications made possible by the coalgebra topology. Moreover, proof we give in $\mathbf{Pfn}_F$ presupposes that the endofunctor F preserves products, since we assume that products are constructed as they are in $\mathbf{Set}$; in general, products in $\mathbf{Set}_F$ are not so well behaved, so either one relies on the conceptual approach we offer, or else one relies on the use of the formally free semigroup, following Heller [Hel90].

We first state a generalization of the first diagram lemma (Lemma 1.4.2) of $\mathbf{Set}_F$ that holds in $\mathbf{Pfn}_F$; this lemma is a consequence of the weak axiom of choice in $\mathbf{Pfn}_F$ (Proposition 2.4.6). In Heller's nomenclature, one says that $\mathbf{Set}_F$ and $\mathbf{Pfn}_F$ are *factorial* [Hel90].

Lemma A.0.4. *Let X, Y and Z be coalgebras, let $\phi : X \longrightarrow Y$ and $\psi : X \longrightarrow Z$ be partial morphisms in $\mathbf{Pfn}_F$ with $\text{dom}\phi \subseteq \text{dom}\psi$, and suppose that for each $x, y \in \text{dom}\phi$, $\phi x = \phi y$ implies that $\psi x = \psi y$ (in Heller's terminology one says that ϕ divides ψ). Then there is a unique partial map, denoted by ψ/ϕ , such that $\text{dom}\psi/\phi = \text{ran}\phi$*

and such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & \searrow \psi & \downarrow \psi/\phi \\ & & Z \end{array}$$

Proof of the Iteration Lemma. Let $X^\# = \Sigma_{\mathbb{N}} X^{n+1}$ be the free semigroup generated by X and define the set

$$W = \{(x, fx, \dots, f^n x) \in X^\# : n \geq 1, x \in X, f^n x \in U\}.$$

The set W is a sub-coalgebra of $X^\#$ since, translating from Heller [Hel90] into this context,

$$W = \Delta_{X^\#}^{-1} \left[((j \circ \text{first} \cdot f^\#) \times (X^\# \cdot j \circ \text{last}))^{-1} [\text{im} \Delta_{X^\#}] \right] \cap \text{last}^{-1}[U],$$

where $\text{first} = [X, p_0]l_X$, $\text{last} = [p_1, X]r_X$, and where the dot $\cdot$ denotes concatenation of sequences in $X^\#$; i.e., $(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) = (x_1, \dots, x_m, y_1, \dots, y_n)$.

Next we apply Lemma A.0.4, the diagram lemma in $\mathbf{Pfn}_F$. There exists a unique induced map in the following diagram

$$\begin{array}{ccc} X^\# & \xrightarrow{\text{first}|W} & X \\ & \searrow \text{last}|W & \downarrow \text{last}|W \\ & & X \end{array}$$

We may suppose that $n > m$. If $\text{first}(x, fx, \dots, f^m x) = \text{first}(y, fy, \dots, f^n y)$, then $x = y$, and as both of the indicated sequences are in W , both $f^m x$ and $f^n x$ are in U , and therefore $f^{n-m} \circ f^m x = f^n x$. We take

$$\text{It}(f, U) = \frac{\text{last}|W}{\text{first}|W}.$$

The remaining assertions follow from Heller [Hel90]. □

Appendix B

Local connectedness in $\mathbf{Top}$

In this appendix we give conditions for local connectedness in the category $\mathbf{Top}$ of topological spaces and continuous maps for the purpose of comparison with Heller's categorical notion of local connectedness [Hel90].

Proposition B.0.5. *The topological space X is locally connected if and only if for every topological space Y , for every open subset U of X and for any continuous maps $f, g : U \longrightarrow Y$, if $f|_V = g|_V$ for every connected open subset V of U , then $f = g$.*

Proof. Necessity is clear, since X is locally connected if and only if there exists a basis for the topology of X consisting of connected open sets. For sufficiency, suppose that X is not locally connected; then there exists an open set $U \subseteq X$ and a point $x \in U$ such that for each open connected subset $V \subseteq U$, $x \notin V$; moreover, for each connected open subset V of U , $V \subseteq U \setminus \overline{\{x\}}$, where $\overline{\{x\}}$ is the closure of $\{x\}$.

Let $Y = \{a, b, c\}$ be a set with the open set topology $\{\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Define a function $f : U \longrightarrow Y$ by

$$f(u) = \begin{cases} a, & \text{if } u \in U \setminus \overline{\{x\}}; \\ b, & \text{if } u \in \overline{\{x\}}. \end{cases}$$

It is immediate that f is continuous, that the transposition $(b\ c) : Y \longrightarrow Y$ which interchanges b and c is a homeomorphism of Y , and that f and $(b\ c)f$ are distinct continuous maps that agree on every connected open subset of U . $\square$

Bibliography

- [AR94] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, Cambridge University Press, Cambridge, 1994.
- [Ber84] Claudio Bernardi, *A shorter proof of a recent result by R. Di Paola*, Notre Dame J. Formal Logic **25** (1984), no. 4, 390–393.
- [Bor94] Francis Borceux, *Handbook of categorical algebra. 1*, Cambridge University Press, Cambridge, 1994, Basic category theory.
- [DP84] Robert A. Di Paola, *A uniformly, extremely nonextensional formula of arithmetic with many undecidable fixed points in many theories*, Proc. Amer. Math. Soc. **91** (1984), no. 2, 291–297.
- [DPH87] Robert A. Di Paola and Alex Heller, *Dominical categories: recursion theory without elements*, J. Symbolic Logic **52** (1987), no. 3, 594–635.
- [DPM91] Robert A. Di Paola and Franco Montagna, *Some properties of the syntactic p -recursion categories generated by consistent, recursively enumerable extensions of Peano arithmetic*, J. Symbolic Logic **56** (1991), no. 2, 643–660.
- [Fio96] Marcelo P. Fiore, *Axiomatic domain theory in categories of partial maps*, Cambridge University Press, Cambridge, 1996, Dissertation, University of Edinburgh, Edinburgh, 1994.

- [GHS00] H. Peter Gumm, Jesse Hughes, and Tobias Schröder, *Distributivity of classes of coalgebras*, Techn. Report, FG Informatik, Philipps-Universität Marburg **25** (2000).
- [GS00a] H. Peter Gumm and Tobias Schröder, *Coalgebraic structure from weak limit preserving functors*, CMCS'2000: coalgebraic methods in computer science (Berlin), Elsevier, Amsterdam, 2000, p. 21 pp. (electronic).
- [GS00b] ———, *Coalgebras of bounded type*, Techn. Report, FG Informatik, Philipps-Universität Marburg **25** (2000).
- [GS00c] ———, *Products of coalgebras*, Algebra Universalis (2000), to appear.
- [Gum99] H. Peter Gumm, *Elements of the general theory of coalgebras*, LUATCS'99, Rand Afrikaans University, Johannesburg, South Africa (1999).
- [Gum01] ———, *Functors for coalgebras*, Algebra Universalis (2001), no. 45, 135–147.
- [Hel90] Alex Heller, *An existence theorem for recursion categories*, J. Symbolic Logic **55** (1990), no. 3, 1252–1268.
- [Kur00] A. Kurz, *Logics for coalgebras and applications to computer science*, Ph.D. thesis, Ludwig-Maximilians-Universität, München, 2000.
- [Mag75] Roberto Magari, *The diagonalizable algebras*, Boll. Un. Mat. Ital. (4) **12** (1975), no. 3, suppl., 117–125, The algebraization of the theories which express Theor, II, Collection in memory of Enrico Bompiani.
- [Man92] Ernest G. Manes, *Predicate transformer semantics*, Cambridge University Press, Cambridge, 1992.
- [ML63] Saunders Mac Lane, *Natural associativity and commutativity*, Rice Univ. Studies **49** (1963), no. 4, 28–46.

- [ML98] ———, *Categories for the working mathematician*, second ed., Springer-Verlag, New York, 1998.
- [MLM92] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Springer-Verlag, New York, 1992.
- [Mon89] Franco Montagna, “Pathologies” in two syntactic categories of partial maps, *Notre Dame J. Formal Logic* **30** (1989), no. 1, 105–116.
- [Odi89] Piergiorgio Odifreddi, *Classical recursion theory*, North-Holland Publishing Co., Amsterdam, 1989, The theory of functions and sets of natural numbers, With a foreword by G. E. Sacks.
- [PE98] D. Pavlović and M. H. Escardó, *Calculus in coinductive form*, Thirteenth Annual IEEE Symposium on Logic in Computer Science (Indianapolis, IN, 1998), IEEE Computer Soc., Los Alamitos, CA, 1998, pp. 408–417.
- [PP99] D. Pavlović and V. Pratt, *On coalgebra of real numbers*, CMCS’99 Coalgebraic Methods in Computer Science (Amsterdam, 1999), Elsevier, Amsterdam, 1999, p. 15 pp. (electronic).
- [PW98] John Power and Hiroshi Watanabe, *An axiomatics for categories of coalgebras*, First Workshop on Coalgebraic Methods in Computer Science (CMCS ’98) (Lisbon), Elsevier, Amsterdam, 1998, p. 18 pp. (electronic).
- [Ros86] G. Rosolini, *Continuity and effectiveness in topoi*, D.phil. thesis, University of Oxford, 1986.
- [Rut00] J. J. M. M. Rutten, *Universal coalgebra: a theory of systems*, Theoret. Comput. Sci. **249** (2000), no. 1, 3–80, Modern algebra and its applications (Nashville, TN, 1996).

- [Tar53] Alfred Tarski, *Undecidable theories. In collaboration with Andrzej Mostowski and Raphael M. Robinson*, North-Holland Publishing Company, Amsterdam, 1953.
- [Tay93] Paul Taylor, *An exact interpretation of while*, Theory and Formal Methods 1993: Proceedings of the First Imperial College, Department of Computing, Workshop on Theory and Formal Methods (G.L. Burn, S.J. Gay, and M.D. Ryan, eds.), Springer-Verlag Workshops in Computer Science, Isle of Thorns Conference Centre, Chelwood Gate, Sussex, UK, 29–31 March 1993, pp. 302–313.
- [Wor98] James Worrell, *Toposes of coalgebras and hidden algebras*, First Workshop on Coalgebraic Methods in Computer Science (CMCS '98) (Lisbon), Elsevier, Amsterdam, 1998, p. 19 pp. (electronic).