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On Extrema in Space-Time

by

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Preface

As indicated by Fred Supnick in his paper Extreme Hamiltonian Lines (Annals of Mathematics, July 1957, pp. 179 - 201), a class of "metric-topologic" problems, as yet unsolved, has received attention because of the importance of their applications. Among such problems is that of determining the minimal Hamiltonian circuit, sometimes referred to as the Traveling Salesman Problem, as salesmen are often confronted with the necessity of determining the shortest circuit passing through n points starting and finishing at the home base. Of course, the same questions arise in the routing of vehicles of transportation, or of machines in industrial processes (cf. G. Dantzig, R. Fulkerson, S. Johnson, Solution of a large scale traveling salesman problem, Journal of the Operations Research Soc. of Amer., Vol. 2 (1954), pp. 393-410; M. Flood, The traveling salesman problem, ibid., Vol. 4 (1956), pp. 61-75; K. Menger, Ergebnisse eines Mathematischen Kolloquiums, Heft 2 (1932), p. 12 (the "Botenproblem").

The following representative of this class of problems was the subject of the above-mentioned paper Extreme Hamiltonian Lines: Consider the rectilinear graph whose vertices are n (distinct) points of E_r $(r > 0)$, and whose edges are rectilinear segments joining all distinct pairs of vertices. We permit singularities to enter, such as edges overlapping, or vertices falling on edges. We call such a graph a complete rectilinear graph and consider the problem of determining the shortest and longest (i.e., the extreme) Hamiltonian circuits. (A Hamiltonian circuit of a graph is a closed, connected edge-chain, each vertex of the graph being an endpoint of 2 and only 2 edges of the chain). There are $(n-1)!/2$

Hamiltonian circuits in a complete rectilinear graph. Except for very small n, examination of the lengths of all Hamiltonian circuits and picking out the shortest and longest is a prohibitive operation (even with mechanical aid). The problem is therefore one of minimizing the number of operations needed to obtain the extremes.

In their paper Shortest Hamiltonian Lines (Amer. Math. Monthly, 72 (1965), pp. 977-980) Louis V. Quintas and Fred Supnick proved that for the class of point distributions which fall on the boundary, B, of their convex hull (in the Euclidean plane or on the 2-sphere) the minimal circuit (the circuit having least length) of all circuits with these points as vertices is B, the boundary of the convex hull. The maximal circuits (the circuits having greatest length) for this class of point distributions were given by the authors in the papers Extreme Hamiltonian Circuits: Resolution of the Convex-Odd Case and Extreme Hamiltonian Circuits: Resolution of the Convex Even Case which appeared in Proceedings of the American Mathematical Society, Volumes 15 and 16 (1964 and 1965), respectively.

Other results of the same type, namely, classes of point distributions in Euclidean space for which there is an explicit procedure for ordering the vertices to yield extrema were given by Fred Supnick in the paper Extreme Hamiltonian Lines, mentioned above, as realizations of the Four-Point Condition (Theorem III) and of Theorem I. If the search for classes of point distributions of this type (classes of distributions for which a procedure yielding the extreme circuits immediately can be given), initiated by Fred Supnick in Extreme Hamiltonian Lines, were to result in the classes obtained being jointly exhaustive of the arbitrary case, i.e., if an arbitrary point distribution were to fall into one of these classes, the result would be tantamount to a solution of the general problem of finding the extreme circuits on any finite point distribution. Until the recent development by David Sanders of a criterion yielding extrema (On Extremel Circuits; dissertation submitted to the Graduate Faculty in Mathematics, CUNY, Spring, 1968), essentially the only classes of point distributions with known procedures making extrema immediately obtainable were the "convex case" and the classes given in Supnick's Extreme Hamiltonian Lines.

In the present paper we continue the investigation initiated by Louis V. Quintas and Fred Supnick in their paper Extrema in Space-Time [1]. We recall here some of their considerations:

"Consider an astronomer and his observation field, i.e., the set of observable (light or radio) signal-emitting loci of the universe. Let the observation field be ordered by attaching a date to each observable locus indicating the time in the history of the universe that the signal was emitted from its source. Whereas both the astronomer and his observation field age with time, the observations of the astronomer may trace a sequence of loci whose time labels proceed forward or backward in time." (Extrema in Space-Time, p. 678)

We ask the reader to note the definitions in §1 of the present paper, and we observe that Quintas and Supnick obtained both classes of distributions of events in space-time for which the Four Point Condition yields extrema and classes of event distributions analogous to the realizations for Theorem I of Supnick's Extreme Hamiltonian Lines. The distributions of events considered by Quintas and Supnick were required to constitute timelike distributions (distributions in which each event lies in the time cone of every other event) since the Four-Point Condition in space-time is meaningful only in such a context, i.e., only for a timelike distribution of events.

In the present paper we consider procedures yielding extrema for certain classes of event distributions in space-time which are analogous to the "convex case" in Euclidean space and exhibit classes of event distributions for which Theorems 1.1 and 4.1 of [3] provide criteria yielding extrema.

It is not our purpose to develop possible applications to physics of the extreme circuits obtainable by the processes given in this paper. However, it does not seem inconceivable that in the future such applications might be possible. Indeed, Feynman Diagrams represent the operators corresponding to the creation or destruction of an electron or positron at the space-time point x by a directed line segment leaving or coming to the point x upward or downward; and closed circuits have been used to represent vacuum processes. (Silvan S. Schweber, An Introduction to Relativistic Quantum Field Theory, Row, Peterson and Company, Elmsford, New York, 1961). Moreover, apart from any applications to physics which might be possible, the results of the present paper can be interpreted in terms of the record spaces defined in Extrema in Space-Time (pp. 690-691) by Quintas and Supnick,

1. Definitions and Summary of Results. In this paper we continue the investigation initiated by Louis V. Quintas and Fred Supnick concerning extrema in space-time [1].

Let S be a finite set of events in L^n, n-dimensional space-time.[1] A rectilinear world-line segment with endpoints in S will be called a rectilinear connection in S and will be denoted E_iE_j where $E_i(x_i^1,\ldots,x_i^{n-1},t_i)$ and $E_j(x_j^1,\ldots,x_j^{n-1},t_j)$, events in L^n, are the endpoints of the segment. A rectilinear connection E_iE_j in L^n will be called timelike if

$$(t_i - t_j)^2 > \sum_{k=1}^{n-1} (x_i^k - x_j^k)^2 \quad .$$

This inequality holds when each of the events E_i and E_j lies in the time cone of the other. The time-separation of a rectilinear connection with endpoints $E_i(x_i^1,\ldots,x_i^{n-1},t_i)$ and $E_j(x_j^1,\ldots,x_j^{n-1},t_j)$ is denoted $s(E_i,E_j)$ and is equal to

$$\left[(t_i - t_j)^2 - \sum_{k=1}^{n-1} (x_i^k - x_j^k)^2\right]^{\frac{1}{2}} \quad .$$

Where the symbol $s(E_i,E_j)$ is used below it is to be understood that E_i and E_j are positioned so that the expression within the brackets is non-negative. A set of rectilinear connections that form a polygon with vertex set S will be called a polygonal connection of S or a circuit on S. The clock time or length of a polygonal connection, P, is defined to be the sum of all the time separations of its rectilinear connections

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[1] Riemannian n-space having the fundamental form

$$\Phi = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^{n-1})^2 - (dt)^2, \; n \geq 2 \; .$$

and will be denoted by L(P) . A polygonal connection, P , having either the least or the greatest clock time of all polygonal connections in the covertex class of P (all polygonal connections with the same vertex set as P) will be referred to as a minimal or maximal circuit, respectively, on S . (These constitute the extreme circuits on S .)

The distributions of events considered in [1] were required to constitute timelike distributions, distributions in which each event lies in the time cone of every other event, since, for example, the Four-Point Condition ([1], §2) is meaningful only for a timelike distribution of events.

In the present paper we are concerned with semi-timelike distributions of events: A semi-timelike distribution of events is a distribution that admits at least one circuit in which each rectilinear connection is timelike (referred to below as a semi-timelike circuit).

All distributions of events will be assumed to be finite and all circuits will be assumed to be semi-timelike unless otherwise specified.

Definition. Suppose the open timelike edge (rectilinear connection) P_aP_b and the closed timelike edge P_cP_d intersect in L^2 .

(i) Suppose the time coordinates of their endpoints $P_a(x_a,t_a)$, $P_b(x_b,t_b)$, $P_c(x_c,t_c)$, $P_d(x_d,t_d)$ satisfy either

$$\max(t_a,t_c) < \min(t_b,t_d)$$

or

$$\max(t_b,t_d) < \min(t_a,t_c) \quad .$$

If $\{P_a,P_b,P_c,P_d\}$ constitutes a timelike distribution, the intersection will be called timelike of Type I. If $\{P_a,P_b,P_c,P_d\}$ does not constitute a timelike distribution, the intersection will be called <u>semi-timelike of Type I</u>.

(ii) Suppose the time coordinates of P_a, P_b, P_c and P_d satisfy either

$$\max(t_a, t_d) < \min(t_b, t_c)$$

or

$$\max(t_b, t_c) < \min(t_a, t_d) \quad .$$

If P_aP_b and P_cP_d intersect in a single point, the intersection will be said to be <u>of Type II</u>.

<u>Theorem 1</u>. A necessary condition that a circuit on a distribution of events in L^2 be minimal of all semi-timelike circuits in its covertex class is that the circuit contain no timelike intersection of Type I. There exist distributions of events for which this condition is sufficient.

<u>Theorem 2</u>. A necessary condition that a circuit on a distribution of events in L^2 be maximal of all semi-timelike circuits in its covertex class is that the circuit contain no intersection of Type II. There exist distributions of events for which this condition is sufficient.

A pseudodisk, H, with center $P(h^1, \ldots, h^{n-1}, k)$ and radius r in L^n is a subset of L^n of the form

$$H = \{E(x^1, \ldots, x^{n-1}, t) : s(E,P) \leq r\} \quad .$$

One may visualize a pseudodisk in L^2 as the connected component of the plane bounded by an equilateral hyperbola with center (h,k) and the lines $t = x + (k-h)$ and $t = -x + (k+h)$.

If H_i and H_j are pseudodisks with centers P_i and P_j and radii r_i and r_j in L^2, their intersection, $H_i \cap H_j$, is clearly a non-empty set of points in L^2. If their <u>line of centers</u>, the line segment joining P_i and P_j, has points in common with $H_i \cap H_j$, the pseudodisks will be said to <u>L-overlap</u> effectively. If their line of centers has exactly one point in common with $H_i \cap H_j$, the pseudodisks will be said to <u>L-overlap critically</u>. The pseudodisks will simply be said to L-overlap if they either L-overlap effectively or L-overlap critically. An <u>A-collection</u> (<u>B-collection</u>) of pseudodisks in L^2 is a finite collection of pseudodisks in which each pseudodisk L-overlaps, either effectively or critically, exactly two others (all but two others). A semi-timelike circuit on n events in L^2 will be called an <u>A-circuit</u> (<u>B-circuit</u>) if it admits an A-collection (B-collection) of pseudodisks, with centers the vertices, in a natural way, i.e., the center of each pseudodisk is joined by an edge of the circuit to the centers of the two pseudodisks which it L-overlaps (the two pseudodisks which it does not L-overlap).

<u>Theorem 3</u>. A-circuits on n vertices exist for every $n \geq 3$, and any A-circuit is uniquely minimal of all semi-timelike circuits in its covertex class.

<u>Theorem 4</u>. Any B-circuit is uniquely maximal of all semi-timelike circuits in its covertex class. The existence of B-circuits on n vertices is not independent of n.

The set of integers for which B-circuits exist contains the integers 3,4,6 and 8; the set of integers for which B-circuits do not exist contains the integer 5 (cf. §6). The problem of completely determining these sets is as yet unresolved.

In addition to the A-collections (B-collections) of pseudodisks, we may distinguish another subclass of the class of finite collections of pseudodisks, the <u>A′-collections</u> (<u>B′-collections</u>): those collections in which each pseudodisk, H, contains the center of exactly two others (contains the centers of all but two others) and these two others contain the center of H (all but these two others contain the center of H). A semi-timelike circuit will be called an <u>A'-circuit</u> (<u>B'-circuit</u>) if it admits an A'-collection (B'-collection) of pseudodisks with centers the vertices of the circuit, in such a way that the center of each pseudodisk is joined by an edge of the circuit to the two centers which the pseudodisk contains (to the two centers which the pseudodisk does not contain). If we interpret the center of each pseudodisk in an A'-collection (B'-collection) as a transmitter-receiver, with range represented by the pseudodisk, then providing realizations of the following theorem is equivalent to placing a finite number of transmitter-receivers so that each lies within the range (outside the range) of the two adjacent to it in a particular ordering, with the circuit defined by this ordering being minimal (maximal).

<u>Theorem 5</u>. A'-circuits (B'-circuits) on n vertices exist for any $n \geq 3$ (for $n = 3$ and $n = 4$ only), and any A'-circuit (B'-circuit) is uniquely minimal (maximal) of all semi-timelike circuits in its covertex class.

We consider next the problem of enumerating the semi-timelike circuits on a given semi-timelike distribution. Specific classes of event distributions (as, for example, those of §3 and §4) may permit immediate enumeration. The general problem we consider in §8 is the following:

Let G be the complete graph on n vertices (every two distinct vertices are joined by exactly one edge and every edge has two distinct endpoints). Suppose k edges of the graph, G, are distinguished in some way, for example, for a graph in L^2, the k spacelike edges might be colored red and the remaining (timelike) edges green. Suppose there is at least one Hamiltonian circuit on the n vertices which does not contain any of the k red edges, i.e., a green Hamiltonian circuit. Enumerate the green Hamiltonian circuits.

The following algorithm provides a method of enumerating such circuits.

Theorem 6. Let G be a complete graph on n vertices with k edges colored red and the remaining edges colored green. Denote the red edges by $e_1, e_2, \ldots, e_k$ and let A_i denote the set of Hamiltonian circuits on the n vertices each of which contains the edge e_i, $i = 1,2,\ldots,k$. Let $\{i_1, i_2, \ldots, i_j\}$ be a subset of $\{1,2,\ldots,k\}$. If $\bigcap_{m=1}^{j} A_{i_m}$ is non-empty, and P is a circuit in $\bigcap_{m=1}^{j} A_{i_m}$, let p denote the number of maximal chains formed by the edges $e_1, e_2, \ldots, e_j$ in P (maximal in the sense that no two of these chains have an endpoint in common). Then the number of green Hamiltonian circuits on the n vertices of G is

$$\text{(1.1)} \qquad \tfrac{1}{2}(n-1)! + \sum_{j=1}^{k} \left[(-1)^j \sum_{\{i_1,\ldots,i_j\}\subset\{1,\ldots,k\}} 2^{p-1}(n-(j+1))!\right]$$

where the $(j+1)^{st}$ term of (1.1) is summed over those subsets $\{i_1, i_2, \ldots, i_j\}$ of $\{1,2,\ldots,k\}$ for which $\bigcap_{m=1}^{j} A_{i_m}$ is non-empty.

Note that (1.1) is of the form

$$\text{(1.2)} \qquad \tfrac{1}{2}(n-1)! - \sum_{j=1}^{k} (-1)^{j-1} c_j(G)(n-(j+1))!$$

where $c_j(G)$ is the sum of $\binom{k}{j}$ or fewer integers of the form

2^{p-1} , $1 \le p \le j$, with each of the occuring values of p depending upon the configuration of the k red edges in G and upon the choice of the subset $\{i_1, i_2, \ldots, i_j\}$ of $\{1,2,\ldots,k\}$. We shall show in §8 that if, in the complete green-red graph G , none of the k red edges have an endpoint in common, then $c_j(G) = \binom{k}{j} 2^{j-1}$ and the number of green Hamiltonian circuits is

$$(1.3) \qquad \tfrac{1}{2}(n-1)! + \sum_{j=1}^{k} (-1)^j \binom{k}{j} 2^{j-1}(n-(j+1))!$$

If the k red edges all have a common endpoint, then the number of green Hamiltonian circuits is

$$(1.4) \qquad \tfrac{1}{2}(n-1)! \; - \; k(n-2)! \; + \; \binom{k}{j}(n-3)!$$

In §8 we will give a further illustration of the method of enumeration for a specific configuration of five red edges in a graph on n vertices.

Note that not every green-red graph on n vertices represents a graph in L^2 with timelike edges corresponding to the green edges and spacelike edges corresponding to the red edges of the abstract graph (cf. §8 for an example).

Problem: Classify those green-red abstract graphs on n vertices that correspond to graphs in L^2 with timelike edges corresponding to the green edges and spacelike edges corresponding to the red edges.

We conclude with the following question: Is it possible to embed any abstract green-red graph in L^3 so that the graph is non-singular (no two edges have interior points in common) with the green edges of the abstract graph corresponding to timelike edges, the red edges to spacelike edges?

2. Necessary Conditions of Theorems 1 and 2. The symbol $[P_1P_2 \ldots P_n]$ will be used to denote the circuit P whose vertices are the events labelled $P_1, P_2, \ldots, P_n$ and whose edges are $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$, P_nP_1 . The symbol for a circuit is understood to be cyclic and symmetric. The symbol

$$[P_1 \ldots P_{i-1}(P_iP_{i+1} \ldots P_{j-1}P_j)\ P_{j+1} \ldots P_n]$$

will denote the circuit

$$P' = [P_1 \ldots P_{i-1}P_jP_{j-1} \ldots P_{i+1}P_iP_{j+1} \ldots P_n]$$

obtained from the circuit

$$P = [P_1 \ldots P_{i-1}P_iP_{i+1} \ldots P_{j-1}P_jP_{j+1} \ldots P_n]$$

and this operation will be referred to as an arc-inversion.

Thus if $E(P)$ denotes the set of edges of P , then $E(P) - \{P_{i-1}P_i, P_jP_{j+1}\} = E(P') - \{P_{i-1}P_j, P_iP_{j+1}\}$.

We will make use of the following lemma.

Lemma 1. Let S be a semi-timelike distribution of noncollinear events in L^2 . Suppose $P = [\ldots P_aP_b \ldots P_cP_d \ldots]$ is a semi-timelike circuit containing intersecting edges P_aP_b (open) and P_cP_d (closed).

(i) If the intersecting edges are timelike of Type I, and if $P' = [\ldots P_a(P_b \ldots P_c)P_d \ldots]$, then $L(P) > L(P')$, i.e., the clock time of P is greater than the clock time of P' .

(ii) If the intersecting edges of Type II, then $L(P) < L(P')$.

Proof. Suppose $P = [\ldots P_aP_b \ldots P_cP_d \ldots]$ is a semi-timelike circuit containing intersecting edges P_aP_b (open) and P_cP_d (closed). By the remark before the statement of the lemma, $L(P) > L(P')$ if and only if $s(P_a,P_b) + s(P_c,P_d) > s(P_a,P_c) + s(P_b,P_d)$ and $L(P) < L(P')$ if and only if $s(P_a,P_b) + s(P_c,P_d) < s(P_a,P_c) + s(P_b,P_d)$.

(i) Suppose the intersecting edges are timelike of Type I. Without loss of generality, we may assume $\max(t_a,t_c) < \min(t_b,t_d)$.

Either the edges P_aP_b and P_cP_d intersect in a single point, or their intersection is a line segment, and when P_aP_b and P_cP_d are thought of as directed arcs, P_aP_b and P_cP_d have the same direction.

Let us assume first that the edges intersect in a single point, $Q(x_Q,t_Q)$, which is different from both P_c and P_d . Note that $\max(t_a,t_c) < t_Q < \min(t_b,t_d)$. Suppose we have $t_a = \min(t_a,t_c)$. Applying the triangle inequality of L^2 to P_a, P_c and Q , we have $s(P_a,Q) > s(P_a,P_c) + s(P_c,Q)$. If $t_b = \max(t_b,t_d)$, we may apply the triangle inequality to P_b, P_d and Q to obtain $s(Q,P_b) > s(Q,P_d) + s(P_d,P_b)$. Since $s(P_a,P_b) = s(P_a,Q) + s(Q,P_b)$ and $s(P_c,P_d) = s(P_c,Q) + s(Q,P_d)$, we have $s(P_a,P_b) > s(P_a,Q)$ and $s(P_c,P_d) > s(Q,P_d)$ so that $s(P_a,P_b) + s(P_c,P_d) > s(P_a,Q) + s(Q,P_d) > s(P_a,P_c) + s(P_c,Q) + s(Q,P_b) + s(P_b,P_d) > s(P_a,P_c) + s(P_b,P_d)$. Hence $L(P) > L(P')$, as asserted. If $t_d = \max(t_b,t_d)$, we apply the triangle inequality to P_b , P_d and Q to obtain $s(Q,P_d) > s(Q,P_b) + s(P_b,P_d)$. Since $s(P_a,P_b) > s(P_a,Q)$ and $s(P_c,P_d) > s(Q,P_d)$, we have $s(P_a,P_b) + s(P_c,P_d) > s(P_a,Q) + s(Q,P_d) > s(P_a,P_c) + s(P_c,Q) + s(Q,P_b) + s(P_b,P_d)$ and, a fortiori, $s(P_a,P_b) + s(P_c,P_d) > s(P_a,P_c) + s(P_b,P_d)$. Thus $L(P) > L(P')$. Clearly the same inequality would be obtained by assuming $t_c = \min(t_a,t_c)$ and

$t_b = \max(t_b, t_d)$ or $t_d = \max(t_b, t_d)$. Thus, as asserted, $L(P) > L(P')$ when P_aP_b and P_cP_d intersect in the single point Q .

Suppose now that the intersection point is P_c . Then if $t_d = \min(t_b, t_d)$ we may apply the triangle inequality to P_b, P_c and P_d to obtain $s(P_b,P_c) > s(P_b,P_d) + s(P_d,P_c)$. Noting that $s(P_a,P_b) = s(P_a,P_c) + s(P_c,P_b)$, we have $s(P_a,P_b) > s(P_a,P_c) + s(P_b,P_d) + s(P_d,P_c) > s(P_a,P_c) + s(P_b,P_d)$. Then, a fortiori, we have $s(P_a,P_b) + s(P_c,P_d) > s(P_a,P_c) + s(P_b,P_d)$ and $L(P) > L(P')$ as asserted. If $t_b = \min(t_b, t_d)$, applying the triangle inequality to P_b , P_c and P_d will result in the inequality $s(P_c,P_d) > s(P_c,P_b) + s(P_b,P_d)$ and, a fortiori, $s(P_c,P_d) > s(P_b,P_d)$. Since $s(P_a,P_b) = s(P_a,P_c) + s(P_c,P_b)$, we have $s(P_a,P_b) > s(P_a,P_c)$. Thus $s(P_a,P_b) + s(P_c,P_d) > s(P_a,P_c) + s(P_b,P_d)$ and $L(P) > L(P')$.

If the intersection point is P_d , and if $t_a = \min(t_a, t_c)$, we apply the triangle inequality to P_a , P_c and P_d to obtain $s(P_a,P_d) > s(P_a,P_c) + s(P_c,P_d)$. We note that $s(P_a,P_b) = s(P_a,P_d) + s(P_d,P_b)$ and conclude that $L(P) > L(P')$ by arguments similar to those above. If $t_c = \min(t_a, t_c)$, the application of the triangle inequality to P_a , P_c and P_d yields the inequality $s(P_c,P_d) > s(P_c,P_a) + s(P_a,P_d)$. Since $s(P_a,P_b) = s(P_a,P_d) + s(P_d,P_b)$ implying $s(P_a,P_b) > s(P_b,P_d)$, again we have $s(P_a,P_b) + s(P_c,P_d) > s(P_a,P_c) + s(P_b,P_d)$ and $L(P) > L(P')$.

Thus if P_aP_b and P_cP_d intersect in a single point, $L(P) > L(P')$, as asserted.

Now suppose the intersection of P_aP_b and P_cP_d is a line segment. Suppose $t_a = \min(t_a, t_c)$ and $t_b = \min(t_b, t_d)$. Then $s(P_a,P_b) = s(P_a,P_c) + s(P_c,P_b)$ and $s(P_c,P_d) = s(P_c,P_b) + s(P_b,P_d)$, implying $s(P_a,P_b) > s(P_a,P_c)$ and $s(P_c,P_d) > s(P_b,P_d)$. Thus $s(P_a,P_b) +$

$s(P_c,P_d) > s(P_a,P_c) + s(P_b,P_d)$. If $t_d = \min(t_b,t_d)$, then $s(P_a,P_b) = s(P_a,P_c) + s(P_c,P_d) + s(P_d,P_b)$. Hence $s(P_a,P_b) > s(P_a,P_c) + s(P_b,P_d)$ and, a fortiori, $s(P_a,P_b) + s(P_c,P_d) > s(P_a,P_c) + s(P_b,P_d)$.

By similar arguments, if $t_c = \min(t_a,t_c)$, we have $L(P) > L(P')$.

Hence if the intersection of P_aP_b and P_cP_d is a line segment, $L(P) > L(P')$, as asserted.

Thus part (i) of Lemma 1 is established.

(ii) Suppose the intersection of P_aP_b and P_cP_d is of Type II with Q , the point of intersection, different from both P_c and P_d . Assuming that $\max(t_a,t_d) < \min(t_b,t_c)$, we have $\max(t_a,t_d) < t_Q < \min(t_b,t_c)$. Applying the triangle inequality to P_a , Q and P_c , we have $s(P_a,P_c) > s(P_a,Q) + s(Q,P_c)$. Applying the triangle inequality to P_b,Q and P_d , we have $s(P_b,P_d) > s(P_b,Q) + s(Q,P_d)$. Combining these inequalities we have $s(P_a,P_c) + s(P_b,P_d) > s(P_a,Q) + s(Q,P_c) + s(P_b,Q) + s(Q,P_d)$. Since $s(P_a,Q) + s(Q,P_b) = s(P_a,P_b)$ and $s(P_c,Q) + s(Q,P_d) = s(P_c,P_d)$, we have $s(P_a,P_b) + s(P_c,P_d) < s(P_a,P_c) + s(P_b,P_d)$ and $L(P) < L(P')$, as desired.

Suppose now that the intersection point is P_c . Applying the triangle inequality to P_b,P_c and P_d , we have $s(P_b,P_d) > s(P_b,P_c) + s(P_c,P_d)$. Then $s(P_a,P_c) + s(P_b,P_d) > s(P_a,P_c) + s(P_b,P_c) + s(P_c,P_d)$; or, since $s(P_a,P_b) = s(P_a,P_c) + s(P_b,P_c)$, $s(P_a,P_c) + s(P_b,P_d) > s(P_a,P_b) + s(P_c,P_d)$ and $L(P) < L(P')$.

If the intersection point is P_d , we apply the triangle inequality to P_a,P_c and P_d to obtain $s(P_a,P_c) \geq s(P_a,P_d) + s(P_d,P_c)$. Thus $s(P_a,P_c) + s(P_b,P_d) > s(P_a,P_d) + s(P_d,P_c) + s(P_b,P_d) = s(P_a,P_b) + s(P_c,P_d)$ since $s(P_a,P_b) = s(P_a,P_d) + s(P_d,P_b)$, and therefore $L(P) < L(P')$.

Hence part (ii) of Lemma 1 is established.

Proof of Necessary Conditions of Theorems 1 and 2. It follows directly from Lemma 1 that if a semi-timelike circuit, P , contains a timelike intersection of Type I, then P is not minimal in its covertex class. Thus a necessary condition that a circuit on a distribution of events in L^2 be minimal of all semi-timelike circuits in its covertex class is that the circuit contain no timelike intersection of Type I. Similarly, if a semi-timelike circuit, P , contains an intersection of Type II, then, by Lemma 1, P is not maximal in its covertex class. Thus a necessary condition that a circuit on a distribution of events in L^2 be maximal of all semi-timelike circuits in its covertex class is that the circuit contain no intersection of Type II.

3. Proof of Existence Assertion of Theorem 1. The following class of semi-timelike distributions demonstrates the existence assertion of Theorem 1.

Let C denote the class of semi-timelike circuits in L^2 whose vertices lie on the boundary of their convex hull.

A semi-timelike circuit, P , in C belongs to C_0 if

(i) P contains open timelike edges, say P_1P_2 and P_3P_4 , which intersect in a single point and have the properties:

> (a) P_1P_3 and P_2P_4 are not timelike rectilinear connections, and
>
> (b) if $P_i(x_i, t_i)$ is any other vertex of P, $\max(t_1, t_3) < t_i < \min(t_2, t_4)$;

(ii) Assuming without loss of generality that $x_3 > x_1$ and $x_2 > x_4$

the remaining vertices of P lie in either the region

$$R_{14} = \{(x,t): -x + (t_4 + x_4) < t < x + (t_1 - x_1)\}$$

or the region

$$R_{23} = \{(x,t): x + (t_2 - x_2) < t < -x + (t_3 + x_3)\} \quad .$$

(iii) The portion of the convex hull containing P_1, P_4 and the vertices in R_{23} and the portion of the convex hull containing P_2, P_3 and the vertices in R_{14} each consist of a chain of timelike edges.

(iv) $P = [P_1 P_2 A_1 A_2 \ldots A_p P_3 P_4 B_1 B_2 \ldots B_{q-1} B_q]$ where $A_i \in R_{14}$, $B_j \in R_{23}$, $t_{A_i} > t_{A_{i+1}}$ and $t_{B_j} > t_{B_{j+1}}$, $i = 1,2,\ldots,p-1$ and $j = 1,2,\ldots,q-1$.

To see that for any $n \geq 4$ there is a circuit on n vertices in C_0, note that we can choose the n vertices as follows:

Let $A(x_A, t_A)$ be any point in L^2. Let r be a positive real number. Each of the lines $L^+ = \{(x,t): x = x_A + r\}$, $L^- = \{(x,t): x = x_A - r\}$ intersects the lines $D^+ = \{(x,t): t - t_A = x - x_A\}$ and $D^- = \{(x,t): t - t_A = -(x - x_A)\}$ in a single point. Let $P(x_P, t_P) = L^+ \cap D^+$, $Q(x_Q, t_Q) = L^+ \cap D^-$, $R(x_R, t_R) = L^- \cap D^+$, and $S(x_S, t_S) = L^- \cap D^-$. Take P_1, P_2, P_3 and P_4 to be the points with coordinates $P_1(x_A - r,\ t_R - \epsilon)$, $P_2(x_A + r,\ t_P + \epsilon)$, $P_3(x_A + r,\ t_Q - \epsilon)$, $P_4(x_A - r,\ t_S + \epsilon)$, where $0 < \epsilon < \frac{1}{2}(t_P - t_Q)$, i.e., ϵ is small enough so that $L^+ \cap R_{14}$ and $L^- \cap R_{23}$ are non-empty sets (R_{14} and R_{23} defined above). If we choose each of $n - 4$ points to lie on $L^+ \cap R_{14}$ or on $L^- \cap R_{23}$, then these $n - 4$ points together with P_1, P_2, P_3 and P_4 constitute a suitable vertex set for the (obvious) circuit, P, in C_0.

A less trivial example of a semi-timelike circuit in C_0 is

$$P = [P_1P_2A_1A_2 \ldots A_5P_3P_4B_1B_2]$$

where the coordinates of the vertices are $P_1(-7,4)$, $P_2(-2,18)$, $A_1(3,12)$, $A_2(3.5,11)$, $A_3(3,9)$, $A_4(2.5,7)$, $A_5(2,5)$, $P_3(0,0)$, $P_4(-8,14)$, $B_1(-11,10\frac{2}{3})$, $B_2(-11\frac{1}{3},9\frac{2}{3})$.

We now show that if a semi-timelike circuit P belongs to C_0 , then the vertex set, S , of P is a distribution of events demonstrating the existence assertion of Theorem 1. That is, a sufficient condition that a circuit on S be minimal of all semi-timelike circuits on S is that the circuit contain no timelike intersection of Type I. We show that P is minimal in its covertex class of semi-timelike circuits and that any such circuit, Q , which is different from P contains a time-like intersection of Type I. Beginning with any such Q we will obtain a sequence $Q, Q', Q'', \ldots, P$ of semi-timelike circuits with the property that $L(Q) > L(Q') > L(Q'') > \ldots > L(P)$.

Indeed, let Q be a semi-timelike circuit on S which is different from P . We show first that Q must contain the edges P_1P_2 and P_3P_4. Since, by definition of R_{14} , P_1 cannot be joined by a timelike edge to any A_i and since P_1P_3 is not timelike (by assumption), Q must contain two edges from the set $\{P_1P_2, P_1P_4, P_1B_j: j = 1,2,\ldots,q\}$. Suppose P_1B_i and P_1B_j, $i,j \in \{1,2,\ldots,q\}$, are edges of Q . Let $B_m \ldots B_iP_1B_j \ldots B_n$ be the largest chain of Q containing the chain $B_iP_1B_j$ and having all its vertices, except P_1 , in R_{23} . Either B_m or B_n , say B_m , must be joined to P_4 since no event in R_{23} may be joined by a timelike segment to P_2, P_3 or any event in R_{14} . Then it is impossible to complete a semi-timelike circuit on S since there is no event to which B_n may be joined by a timelike segment. Hence Q can contain at most one edge of the form P_1B_i , $i \in \{1,2,\ldots,q\}$. Suppose now that Q contains the edge P_1P_4 . If Q also contains an edge of the form P_1B_j , let

$$B_m \ldots B_jP_1P_4 \ldots B_n$$

be the largest chain of Q which contains the chain $B_jP_1P_4$ and has all its vertices, except P_1 , in R_{23} . Then it is impossible to complete a semi-timelike circuit on S since none of the vertices $P_2,P_3,A_1,\ldots,A_p$ may be joined to B_m or B_n by a timelike segment. Thus Q must contain the edge P_1P_2 since Q must contain two edges from the set

$$\{P_1P_2,\ P_1P_4,\ P_1B_j\colon\ j = 1,2,\ldots,q\}$$

but cannot contain two edges of the form P_1B_j and cannot contain P_1P_4 and an edge of the form P_1B_j . By a similar argument, Q must contain P_3P_4 . Thus if Q is a semi-timelike circuit on S , Q is of the form

$$Q = [P_1P_2(A_{i_1}\ A_{i_2}\ \ldots\ A_{i_p})\ P_3P_4(B_{j_1}\ B_{j_2}\ \ldots\ B_{j_q})]$$

where $(A_{i_1}\ A_{i_2}\ \ldots\ A_{i_p})$ and $(B_{j_1}\ B_{j_2}\ \ldots\ B_{j_q})$ are permutations of $\{A_1,A_2,\ldots,A_p\}$ and $\{B_1,B_2,\ldots,B_q\}$ respectively.

If Q contains no self-intersections, then Q = P . But, by assumption, $Q \neq P$.

Consider the edge P_1B_i of Q where $B_i \in R_{23}$. If P_1B_i does not intersect any edge of Q , then no vertex, B_j , lies between P_1 and B_i on the convex hull of S and hence $B_i = B_q$. In this case, consider the edge B_qB_k of Q. If this edge does not intersect any edge of Q, then $B_k = B_{q-1}$ by similar reasoning. If, continuing this process, we obtain the chain $P_1B_qB_{q-1} \ldots B_2B_1P_4P_3$ we proceed to consider the edge P_3A_i of Q where $A_i \in R_{14}$. We must eventually reach intersecting edges since $Q \neq P$.

Suppose, however, that B_iB_j is the edge of Q having the property that $t_{B_i} = \min\{t_{B_k} : B_k$ an endpoint of an edge of Q which intersects some other edge$\}$.

Note that the chain $P_1B_qB_{q-1} \ldots B_{i+1}B_i$ lies along the convex hull of S. Let $B_mB_n \ldots B_sP_4$ be the chain of Q in which B_mB_n is the only edge which intersects B_iB_j. We show below that the arc-inversion $[P_1B_qB_{q-1} \ldots B_i(B_j \ldots B_m)\ B_n \ldots B_sP_4 \ldots]$ defines a semi-timelike circuit Q' with $L(Q') < L(Q)$.

By Lemma 1 (§2), it is sufficient to show that the intersection of B_iB_j and B_mB_n is timelike of Type I. That is, we must show that $\{B_i,B_j,B_m,B_n\}$ constitutes a timelike distribution and that $\max(t_{B_i},t_{B_m}) < \min(t_{B_j},t_{B_n})$.

Clearly by choice of B_iB_j we have $t_{B_i} < \min(t_{B_j},t_{B_m},t_{B_n})$ and by choice of B_mB_n we have $t_{B_m} < t_{B_n}$. Since the edges intersect we have $t_{B_m} < t_{B_j}$. Thus $\max(t_{B_i},t_{B_m}) < \min(t_{B_j},t_{B_n})$. Since the portion of the convex hull of S containing P_1,P_4 and the vertices in R_{23} consists of a chain of timelike edges and since the relation "A lies in the future portion of the time cone of B" is clearly a transitive relation, $\{B_i,B_j,B_m,B_n\}$ constitutes a timelike distribution. We now consider $Q' = [P_1B_qB_{q-1} \ldots B_iB_m \ldots B_jB_n \ldots B_sP_4 \ldots]$. If $Q' \neq P$, Q' contains intersecting edges. If $B_m \neq B_{i-1}$ then B_iB_m intersects some other edge of Q' and we repeat the above procedure, obtaining Q'' with $L(Q') > L(Q'')$. If $B_m = B_{i-1}$ we ask whether the edge $B_{i-1}B_c$ of Q' intersects any other edge.

By a repetition of the above process we obtain a sequence $Q, Q', Q'', \ldots, P$ in which $L(Q) > L(Q'') > \ldots > L(P)$; and the assertion that P is minimal is established.

Remark. If $N(S)$ denotes the number of semi-timelike circuits on S , then $N(S) = p!q!$ where p is the number of events of S in R_{14} and q is the number of events of S in R_{23} .

Indeed, as noted above, we may represent any semi-timelike circuit on S by $[P_1P_2(A_{i_1} A_{i_2} \ldots A_{i_p})P_3P_4(B_{j_1} B_{j_2} \ldots B_{j_q})]$ where $(A_{i_1} A_{i_2} \ldots A_{i_p})$ and $(B_{j_1} \ldots B_{j_q})$ are permutations of $\{A_1, \ldots, A_p\}$ and $\{B_1, \ldots, B_q\}$ respectively. Since there are $p!$ permutations of $\{A_1, \ldots, A_p\}$ and $q!$ permutations of $\{B_1, \ldots, B_q\}$, there are $p!q!$ semi-timelike circuits on S .

Problem: Classify all semi-timelike distributions, S , in L^2 having the property that any semi-timelike circuit, P , on S is minimal if and only if P contains no timelike intersections of Type I.

Remark. Note that the class C_2 of semi-timelike circuits defined in the next paragraph has the property that if P belongs to C_2 , then the vertex set, S , of P is a semi-timelike distribution on which any semi-timelike circuit is either minimal or contains a timelike intersection of Type I.

A semi-timelike circuit, P , in C belongs to the class C_2 if

(i) The convex hull of the vertices of P is a chain of timelike segments in which no three vertices are collinear;

(ii) If P_m and P_M represent the vertices of P having the least and the greatest time-coordinates, respectively, and if B represents the convex hull of the vertices of P, then the set $B - \{P_m, P_M\}$ consists of two connected components B_1 and B_2 with the property that no vertex on B_1 lies in the time cone of any vertex on B_2 and no vertex on B_2 lies in the time cone of any vertex on B_1. Every vertex of P lies in the intersection of the time cones of P_m and P_M;

(iii) $P = [P_1 P_2 \ldots P_n]$ is the circuit whose vertex ordering is that induced by the convex hull.

Example: The semi-timelike circuit $P = [P_1P_2P_3P_4P_5P_6P_7P_8P_9]$ where $P_1(0,0)$, $P_2(5,11)$, $P_3(5,13)$, $P_4(4,15)$, $P_5(-3,23)$, $P_6(-7,17)$, $P_7(-8,14)$, $P_8(-7,10)$, $P_9(-6,8)$, belongs to the class C_2.

The verification that if P belongs to C_2, then P is minimal in its covertex class of semi-timelike circuits is similar to that given above for C_0. One first demonstrates that if $Q = [\ldots P_k P_m P_1 \ldots P_i P_M P_j \ldots]$ is a semi-timelike circuit in the covertex class of P, then the vertices in the chain $P_1 \ldots P_i$ must all lie on either B_1 or B_2, and if all these vertices lie on B_i, then the vertices in the chain $P_j \ldots P_k$ must all lie on B_{3-i}, $i = 1,2$. One then shows that if $Q \neq P$, Q contains a timelike intersection of Type I, and using methods similar to those for $P \in C_0$, one obtains a sequence $Q, Q', Q'', \ldots, P$ with $L(Q) > L(Q') > L(Q'') > \ldots > L(P)$.

4. Proof of Existence Assertion of Theorem 2. The following class of semi-timelike distributions demonstrates the existence assertion of Theorem 2.

Let T denote the class of distributions, S, with the property that S can be expressed as $S = \{A_1, A_2, \ldots, A_k\} \cup \{B_1, B_2, \ldots, B_k\}$ with $x_{A_1} < x_{A_2} < \ldots < x_{A_k}$, $x_{B_1} < x_{B_2} < \ldots < x_{B_k}$ and $\max\{t_{A_i} : i = 1,\ldots,k\} < \min\{t_{B_j} : j = 1,\ldots,k\}$, where no edge of the form A_iA_j is timelike, no edge of the form B_iB_j is timelike and each edge of the form A_iB_j is timelike.

For any $n = 2k$ there is such a distribution of n events. Indeed, let $P(x_p, t_p)$ be any point in L^2 and let r be a positive real number. Let $U = L^+ \cap D^-$, $V = L^+ \cap D^+$, $W = L^- \cap D^+$ and $Z = L^- \cap D^-$ where $L^+ = \{(x,t): t = t_p + r\}$, $L^- = \{(x,t): t = t_p - r\}$, $D^+ = \{(x,t): t - t_p = x - x_p\}$ and $D^- = \{(x,t): t - t_p = -(x - x_p)\}$. Then if S is the distribution consisting of k points on L^+ (strictly) between U and V and k points on L^- (strictly) between W and Z, S belongs to the class T.

The distribution $S = \{A_1(3,4),\ A_2(5,3),\ A_3(8,4),\ A_4(10,3),\ A_5(12,2),\ B_1(0,15),\ B_2(2,16),\ B_3(5,14),\ B_4(8,15),\ B_5(11,17)\}$ provides another example of a distribution in T. Not all the points in S fall on the boundary of their convex hull.

We shall prove below that if S belongs to T, then S satisfies the existence assertion of Theorem 2, and

$$P = [A_kB_{k-1}A_{k-2}B_{k-3} \cdots A_3B_2A_1B_1A_2B_3 \cdots A_{k-3}B_{k-2}A_{k-1}B_k]$$

is maximal in its covertex class of semi-timelike circuits.

Suppose that Q is a semi-timelike circuit on S which is different from P. Then Q is of the form $Q = [A_{i_1}B_{j_1}A_{i_2}B_{j_2} \ldots A_{i_k}B_{j_k}]$ for some permutations $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_k)$ of $\{1,2,\ldots,k\}$ since all timelike edges on S are of the form A_iB_j. Since the representation of Q is cyclic, we may assume $A_{i_1} = A_k$, i.e., $Q = [A_kB_{j_1}A_{i_2}B_{j_2} \ldots A_{i_k}B_{j_k}]$. We will obtain P from Q by a sequence of arc-inversions which will give $L(Q) < L(Q') < L(Q'') < \ldots < L(P)$. Note that to obtain a semi-timelike circuit from Q by an arc inversion $[A_{i_1}B_{j_1} \ldots P_a(P_b \ldots P_c)P_d \ldots A_{i_k}B_{j_k}]$ either both P_b and P_c must belong to $\{A_1,\ldots,A_k\}$ or both must belong to $\{B_1,\ldots,B_k\}$.

If $B_{j_1} = B_k$ so that $Q = [A_kB_kA_{i_2}B_{j_2} \ldots A_{i_k}B_{j_k}]$ we will write $Q = [A_kB_{j_k}A_{i_k} \ldots B_{j_2}A_{i_2}B_k]$ since the representation for Q is symmetric. Suppose, however, that $B_{j_1} \neq B_k$ and $B_{j_k} \neq B_k$. Then the edge $A_kB_{j_1}$ intersects the edges $A_{i_p}B_k$ and $B_kA_{i_q}$ where i_p and i_q are less than k. We perform the arc-inversion $[A_k(B_{j_1} \ldots A_{i_p}B_k)A_{i_q}B_{j_q} \ldots A_{i_k}B_{j_k}]$ to obtain $Q' = [A_kB_kA_{i_p} \ldots B_{j_1}A_{i_q}B_{j_q} \ldots A_{i_k}B_{j_k}]$ or $Q' = [A_kB_{j_k}A_{i_k} \ldots A_{i_p}B_k]$, using symmetry. Since $\max(t_{A_i}) < \min(t_{B_j})$ by our choice of S, $L(Q') > L(Q)$ by Lemma 1 (ii). We ask whether, in Q', $B_{j_k} = B_{k-1}$. If so, we ask whether $A_{i_p} = A_{k-1}$. If so, we ask whether $A_{i_k} = A_{k-2}$ and proceed in this way until we find an edge of Q' which is not an edge of P.

Suppose $C = P_y \ldots A_{k-2}B_{k-1}A_kB_kA_{k-1}B_{k-2} \ldots P_x$ is the longest chain containing A_kB_k which $Q' = [A_kB_{k-1} \ldots P_y \ldots P_x \ldots A_{k-1}B_k]$ has in common with P . We have four cases to consider:

(i) P_x and P_y are in $\{A_1, \ldots, A_{k-1}\}$

(ii) P_x is in $\{A_1, \ldots, A_{k-1}\}$ and P_y is in $\{B_1, \ldots, B_{k-1}\}$

(iii) P_x is in $\{B_1, \ldots, B_{k-1}\}$ and P_y is in $\{A_1, \ldots, A_{k-1}\}$

(iv) P_x and P_y are in $\{B_1, \ldots, B_{k-1}\}$.

(i) Suppose $P_x = A_{i_s}$ and $P_y = A_{i_t}$ with $i_t < i_s$, so that $Q' = [A_k B_{k-1} \ldots A_{i_t} \ldots B_{j_m} A_{i_s} \ldots A_{k-1} B_k]$. Then $x_{B_{j_m}} < x_{B_{i_{s-1}}}$ and $B_{i_{s-1}}$ is not a vertex in the chain C . We perform the arc-inversion

$$[A_k B_{k-1} \ldots A_{i_t} \ldots A_{i_n} (B_{i_{s-1}} \ldots B_{j_m}) A_{i_s} \ldots A_{k-1} B_k]$$

to obtain $Q'' = [A_k B_{k-1} \ldots A_{i_t} \ldots A_{i_n} B_{j_m} \ldots B_{i_{s-1}} A_{i_s} \ldots A_{k-1} B_k]$. Again by Lemma 1 (ii), $L(Q'') > L(Q')$.

(ii) Suppose $P_x = A_{i_s}$ and $P_y = B_{j_t}$ with $i_s < j_t$ so that $Q' = [A_k B_{k-1} \ldots B_{j_t} A_{i_m} \ldots A_{i_s} \ldots A_{k-1} B_k]$. Then $x_{A_{i_m}} < x_{A_{j_{t-1}}}$ and $A_{j_{t-1}}$ is not a vertex in the chain C . We perform the arc-inversion

$$[A_k B_{k-1} \ldots B_{j_t} (A_{i_m} \ldots A_{j_{t-1}}) B_{j_n} \ldots A_{i_s} \ldots A_{k-1} B_k]$$

to obtain $Q'' = [A_k B_{k-1} \ldots B_{j_t} A_{j_{t-1}} \ldots A_{i_m} B_{j_n} \ldots A_{i_s} \ldots A_{k-1} B_k]$. By Lemma 1, $L(Q'') > L(Q')$.

Clearly if $j_t < i_s$ a similar argument will yield the desired result.

Cases (iii) and (iv) can also be resolved by arguments like those for (ii) and (i), respectively.

We can continue this process, obtaining a sequence of semi-timelike circuits, Q, Q', Q'',..., with monotonically increasing clock time. To see that this sequence terminates with P, assume that we have reached $Q^{(v)} = [A_k B_{k-1} \ldots A_5 B_4 A_3 B_x A_y B_u A_v B_3 A_4 B_5 \ldots A_{k-1} B_k]$ in the sequence of circuits. Either $B_x = B_1$ or $B_x = B_2$. Suppose $B_x = B_1$. Then $B_u = B_2$ and $Q^{(v)} = [A_k B_{k-1} \ldots A_3 B_1 A_y B_2 A_v B_3 \ldots A_{k-1} B_k]$. The edge $A_3 B_1$ intersects both $A_y B_2$ and $B_2 A_v$. Thus the arc-inversion $[A_k B_{k-1} \ldots A_3 (B_1 A_y B_2) A_v B_3 \ldots A_{k-1} B_k]$ yields $Q^{(v+1)} = [A_k B_{k-1} \ldots A_3 B_2 A_y B_1 A_v B_3 \ldots A_{k-1} B_k]$ with $L(Q^{(v+1)}) > L(Q^{(v)})$ by the same arguments as those seen above. If $A_y = A_1$, then $A_v = A_2$ and $Q^{(v+1)} = P$. If $A_y = A_2$, then $A_v = A_1$ and $A_1 B_3$ intersects $B_2 A_2$ and $A_2 B_1$. Since, as before, the intersection of $B_2 A_2$ and $A_1 B_3$ is of Type II, the arc-inversion $[A_k \ldots A_3 B_2 (A_2 B_1 A_1) B_3 \ldots B_k]$, which produces P, gives us $L(P) > L(Q^{(v)})$. If $B_x = B_2$, then $B_u = B_1$ and we can obtain P by the arc-inversions above which affect the positions of A_y and A_v. Thus the existence assertion of Theorem 2 is established, and P is maximal, as asserted.

Remark. If $N(S)$ denotes the number of semi-timelike circuits on S, then $N(S) = \frac{k!(k-1)!}{2}$.

Indeed, as remarked above, any semi-timelike circuit on S is of the form $[A_{i_1} B_{j_1} A_{i_2} B_{j_2} \ldots A_{i_k} B_{j_k}]$ where $(i_1 i_2 \ldots i_k)$ and $(j_1 j_2 \ldots j_k)$ are permutations of the numbers $1,2,\ldots,k$. Since the symbol representing a circuit is considered to be cyclic, we may assume $A_{i_1} = A_1$. Thus $A_2,\ldots,A_k$ can occur in $(k-1)!$ ways and $B_1,\ldots,B_k$ can occur in $k!$ ways giving $k!(k-1)!$ representations of the form $[A_1 B_{j_1} A_{i_2} B_{j_2} \ldots A_{i_k} B_{j_k}]$. Since such circuit representations are symmetric, the number of semi-timelike circuits on S is $\frac{k!(k-1)!}{2}$, as asserted.

There is a certain dualism between the distributions S in C_0 and S' in T. If S belongs to C_0, there are two distinguished subdistributions of S: the set A of events, $A_1, A_2, \ldots, A_p$, that lie in R_{14} and the set B of events, $B_1, B_2, \ldots, B_q$, which lie in R_{23}. Any edge with both endpoints in A or both endpoints in B is timelike, whereas no edge with one endpoint in A and one endpoint in B is timelike. If S' belongs to T, $A' = \{A_1, \ldots, A_k\}$ and $B' = \{B_1, \ldots, B_k\}$, no edge with both endpoints in A' or both endpoints in B' is timelike, and every edge with one endpoint in A' and one endpoint in B' is timelike.

5. <u>Existence of A-circuits and A'-circuits.</u> We will prove that A'-circuits exist for $n \geq 3$ and show that any semi-timelike circuit which admits an A'-collection of pseudodisks also admits an A-collection. (However, not every semi-timelike circuit which admits an A-collection of pseudodisks also admits an A'-collection. For example, the circuit $P = [P_1 P_2 P_3 P_4 P_5 P_6]$, where $P_1 = (0,0)$, $P_2 = (8,20)$, $P_3 = (2,31)$, $P_4 = (4,36)$, $P_5 = (-2,54)$, and $P_6 = (-26,28)$, admits an A-collection of pseudodisks $\{H_i: i = 1,2,\ldots,6\}$ where H_i has center P_i and radius r_i: $r_1 = 13$, $r_2 = 8$, $r_3 = 2$, $r_4 = 3$, $r_5 = 14$ and $r_6 = 5$. In any A'-collection of pseudodisks on the circuit P, we would have to have $s(P_4,P_5) \leq r_4'$. But $s(P_4,P_2) < s(P_4,P_5)$. Hence H_4', the pseudodisk centered at P_4, would contain P_2, P_3 and P_5, and the collection of

pseudodisks would not be an A'-collection.)

It is possible to treat the case where n is even and the case where n is odd simultaneously. We show first that it is possible to construct a semi-timelike circuit on four vertices and one on five vertices, each of which admits an A'-collection of pseudodisks. We then show that from any A'-circuit on m vertices we can obtain an A'-circuit on $(m+2)$ vertices, and thus we can obtain an A'-circuit on n vertices for any $n \geq 4$. We treat the case $n = 3$ separately because, as will be seen below, we cannot obtain an A'-circuit on 5 vertices from one on 3 vertices by the methods given below for proceeding from m to $(m+2)$.

<u>Construction of an A'-circuit on four vertices.</u>

Let $P_1(x_1,t_1)$ be a point in L^2, and let $\varepsilon_a, \varepsilon_b, \varepsilon_c, \varepsilon_d$, be "small" positive real numbers. Through P_1 construct a line L_a with slope $1 + \varepsilon_a$ and a line L_b with slope $-(1 + \varepsilon_b)$. Choose a point $P_2(x_2,t_2)$ with $t_2 > t_1$ on L_a and a point $P_4(x_4,t_4)$ with $t_4 > t_1$ on L_b such that the slope of the line through P_2 and P_4 is, in absolute value, less than 1. Through P_2 construct a line L_c with slope $-(1 + \varepsilon_c)$ and through P_4 construct a line L_d with slope $1 + \varepsilon_d$. Denote by $P_3(x_3,t_3)$ the point of intersection of L_c and L_d. Note that we have $x_4 < x_3 < x_2$ and $t_3 > t_2$, $t_3 > t_4$ by this construction. Then $P = [P_1P_2P_3P_4]$ is obviously a semi-timelike circuit. We show that P admits an A'-collection of pseudodisks H_1,H_2,H_3,H_4, centered at P_1,P_2,P_3,P_4, satisfying the conditions that H_i contains P_{i-1} and P_{i+1}, but not P_{i+2}, $i = 1,\ldots,4 \pmod 4$.

By the triangle inequality in L^2, we have $s(P_1,P_3) \geq s(P_1,P_2) + s(P_2,P_3)$ and $s(P_1,P_3) \geq s(P_1,P_4) + s(P_4,P_3)$. Thus $s(P_1,P_3) >$

$\max(s(P_1,P_2),s(P_1,P_4))$ and $s(P_1,P_3) > \max(s(P_2,P_3),s(P_3,P_4))$. So for the radius of the pseudodisk H_1 , we can choose r_1 such that $\max(s(P_1,P_2),s(P_1,P_4)) < r_1 < s(P_1,P_3)$; and for the radius of H_3 , we can choose r_3 so that $\max(s(P_2,P_3),s(P_3,P_4)) < r_3 < s(P_1,P_3)$. Since the connection P_2P_4 is not timelike, we have P_2 outside of H_4 and P_4 outside of H_2 whatever the radii of H_2 and H_4 . Thus we can choose r_2 so that $\max(s(P_1,P_2),s(P_2,P_3)) < r_2$ and r_4 so that $\max(s(P_1,P_4),s(P_3,P_4)) < r_4$. Then the collection of pseudodisks H_1,H_2,H_3,H_4 is an A'-collection, and $P = [P_1P_2P_3P_4]$ is an A'-circuit.

Construction of an A'-circuit on five vertices.

Let $P_1(x_1,t_1)$ be a point in L^2 and let a be a positive real number. Let $P_5(x_5,t_5)$ be the point with coordinates $x_5 = x_1$, $t_5 = t_1 + a$. Then $s(P_1,P_5) = a$. The region

$$R = \{P(x,t): a < s(P_1,P) < 2a\} \cap \{P(x,t): x_5 < x, t_5 < t < t_5 + (x-x_5)\}$$

is clearly non-empty. Choose a point $P_2(x_2,t_2)$ in this region. (For example, P_2 with $x_2 = x_1 + 3a$, $t_2 = t_1 + 3.5a$).

Then we may choose a real number r_1 , with $s(P_1,P_2) < r_1 < 2a$, for the radius of a pseudodisk H_1 centered at P_1 . Note that P_5 lies in H_1 .

Since $x_2 > x_5$ and $t_2 > t_5$, we may choose a point $P_4(x_4,t_4)$ in the region $S = \{(x,t): \min((x-x_5),-(x-x_5)) < (t-t_5)$ and $(t-t_2) < -(x-x_2)\}$ (i.e., P_4 lies outside the timecone of P_2 and inside the timecone of P_5) such that $x_4 < x_5$ and $s(P_4,P_5) > a$. For example, (x_1-a,t_1+3a) is such a point. By the triangle inequality in L^2 ,

$s(P_1,P_4) \geq s(P_1,P_5) + s(P_5,P_4)$; so we can choose a positive number r_4 in such a way that the pseudodisk H_4 with radius r_4 centered at P_4 contains P_5 but not P_1 .

Center a pseudodisk H_2 with radius r_2, with $r_2 > s(P_1,P_2)$, at P_2 . Note that H_2 will contain neither P_4 nor P_5 since these points do not lie in the time cone of P_2 .

The pseudodisks H_2 and H_4 intersect in two regions, one of which has the property that for $P(x,t)$ in this region, $t > \max(t_2,t_4)$. Denote this region by T . That we can choose in T a point $P_3(x_3,t_3)$ having the property that $s(P_3,P_2) < s(P_3,P_5)$ can be seen as follows: The intersection point Q of the lines $D_2^- = \{(x,t)\colon t - t_2 = -(x - x_2)\}$ and $D_4^+ = \{(x,t)\colon t - t_4 = x - x_4\}$ lies in the region T and has the property that $s(Q,P_2) = 0$ and $s(Q,P_5)$ is some positive number. By the continuity of the function $s(P,P_2)$ for $|t_P - t_2| \geq |x_P - x_2|$ we can find a disk of radius δ centered at Q with the property that if P lies in the intersection of this disk with the region T, then $s(P,P_2) < s(Q,P_5)$. Let P_3 be such a point. Since $s(P_3,P_2) < s(Q,P_5)$ by choice of P_3 and since, by the triangle inequality in L^2, we have $s(P_3,P_5) \geq s(P_3,Q) + s(Q,P_5)$, we see that $s(P_3,P_2) < s(P_3,P_5)$. Also using the triangle inequality, we have $s(P_3,P_4) < s(P_3,P_5)$ since $s(P_3,P_5) \geq s(P_3,P_4) + s(P_4,P_5)$.

Choose a positive real number r_3 with the property that $\max(s(P_3,P_2),s(P_3,P_4)) < r_3 < s(P_3,P_5)$ as the radius of a pseudodisk H_3 centered at P_3 .

That we can choose a positive real number r_5 for the radius of a pseudodisk H_5 centered at P_5 so that $\max(s(P_5,P_1),s(P_5,P_4)) < r_5 < s(P_5,P_3)$ can be seen as follows: As seen above, $s(P_4,P_5) < s(P_5,P_3)$

and we chose P_4 so that $s(P_4,P_5) > a = s(P_1,P_5)$.

$P = [P_1P_2P_3P_4P_5]$ is obviously a semi-timelike circuit, and $\{H_i: i = 1,\ldots,5\}$ is an A'-collection of pseudodisks, making P an A'-circuit.

Let P be a non-self-intersecting semi-timelike circuit on m vertices, $P_1,P_2,\ldots,P_m$, which admits an A'-collection of pseudodisks, $H_1,H_2,\ldots,H_m$. We give below a method for constructing from P a semi-timelike circuit, P' , with $m+2$ vertices.

We note first that every non-self-intersecting semi-timelike circuit, P , has a diagonal, $P_{i-1}P_{i+1}$, with P_{i-1} and P_{i+1} the endpoints of a polygonal subarc of P having two edges, and $P_{i-1}\,P_i\,P_{i+1}$ a triangle all of whose interior points are in the interior of P . Suppose $P_{i-1}P_{i+1}$ is a diagonal of this type in the given circuit P . To simplify the discussion we assume $x_{i-1} < x_{i+1} < x_i$ and $t_{i-1} < t_i < t_{i+1}$. We will replace the vertex P_i by the point $Q(x_Q,t_Q)$ where $x_Q = x_i - c$, $t_Q = t_i$ with the positive number c chosen so that $x_{i-1} < x_Q < x_{i+1}$ and so that none of the m vertices of P lies on either of the lines $D_Q^+ = \{(x,t): t - t_Q = x - x_Q\}$, $D_Q^- = \{(x,t): t - t_Q = -(x - x_Q)\}$. We replace the pseudodisk H_i by a pseudodisk H_Q centered at Q , choosing the radius r_Q in such a way that no vertex of P lies in H_Q. This can be done by choosing r_Q less than $v = \min\{s(P_k,Q): k = 1,\ldots,m$ where $|t_Q - t_k| > |x_Q - x_k|\}$.

Let R_{i-1} be the closed region consisting of the intersection $H_Q \cap H_{P_{i-1}} \cap \{(x,t): x_Q \le x,\ t_{i-1} < t < t_Q\}$. We can choose a point Q' in R_{i-1} and a positive real number $r_{Q'}$ for the radius of a pseudodisk $H_{Q'}$ centered at Q' so that $H_{Q'}$ contains Q and P_{i-1} but none of the vertices $P_1, \ldots P_{i-2}, P_{i+1}, \ldots P_m$ as follows: The point T of intersection of the lines D_Q^- and $D_{i-1}^+ = \{(x,t): t - t_{i-1} = x - x_{i-1}\}$ has the property that $s(T,Q) = s(T,P_{i-1}) = 0$. By the continuity of $s(P,Q)$ for $P(x,t)$ in the region $W = \{(x,t): |t - t_Q| \geq |x - x_Q|\}$ we can find a disk D_Q of radius d_Q centered at T so that for P in the intersection of D_Q with the region W, $s(P,Q)$ is less than v. Similarly, by the continuity of $s(P,P_{i-1})$ for P in the closed time cone of P_{i-1}, we can find a disk D_{i-1} of radius d_{i-1} centered at T so that for P in the intersection of D_{i-1} with the closed time cone of P_{i-1}, $s(P,P_{i-1})$ is less than $\min\{s(Q,P_{i-1}), s(P_k,P_{i-1}):$ $k = 1,\ldots,i-2, i+1,\ldots,m$ and $|t_{i-1} - t_k| > |x_{i-1} - x_k|\}$.

By taking Q' in the intersection $R_{i-1} \cap D_Q \cap D_{i-1}$ we can choose $r_{Q'}$ for the radius of $H_{Q'}$ so that $H_{Q'}$ contains Q and P_{i-1} but none of the vertices $P_1,\ldots,P_{i-2}, P_{i+1},\ldots,P_m$.

Similarly, if we let R_{i+1} be the region consisting of the intersection $H_Q \cap H_{P_{i+1}} \cap \{(x,t): x_{i+1} \leq x,\ t_Q < t < t_{i+1}\}$, we can choose a point Q'' in R_{i+1} and a positive real number $r_{Q''}$ so that the pseudodisk $H_{Q''}$ of radius $r_{Q''}$ centered at Q'' contains Q and P_{i+1} but not Q' or any of the vertices $P_1,\ldots,P_{i-1}, P_{i+2},\ldots,P_m$.

Then the semi-timelike circuit $P' = [P_1,\ldots,P_{i-1}, Q',Q,Q'',P_{i+1},\ldots,P_m]$ obtained by replacing the chain $P_{i-1}P_iP_{i+1}$ of P by the chain $P_{i-1}Q'\,Q\,Q''\,P_{i+1}$ is an A'-circuit on $m+2$ vertices where the A'-collection of pseudodisks is $H_1,\ldots,H_{i-1}, H_{Q'}, H_Q, H_{Q''}, H_{i+1},\ldots,H_m$.

Thus for $n \geq 4$, we can obtain an A'-circuit on n vertices. That we can obtain an A'-circuit on three vertices is obvious. It is also easily seen that the entire circuit will lie within each of two of the three pseudodisks centered at the vertices. For this reason it is not possible to construct from it, by the method given above, a circuit on five vertices. Indeed, if we label the three vertices of the original circuit P_1, P_2, P_3 so that $P = [P_1 P_2 P_3]$ lies entirely within $H_1 \cap H_3$ and choose Q as described above, then Q will lie in the interior of the triangular region bounded by P. For the circuit $P' = [P_1 Q' Q\, Q'' P_3]$ to be an A'-circuit, we must have H_1 containing Q' and P_3 but not Q or Q''. This is impossible because if H_1 contains P_3 it will also contain Q.

Thus A'-circuits exist for $n \geq 3$, as asserted.

To see that any semi-timelike circuit which admits an A'-collection of pseudodisks also admits an A-collection, we note that if a semi-timelike circuit $P = [P_1 P_2 \ldots P_n]$ is an A'-circuit with the pseudodisks H_i' centered at P_i and having radius R_i, $i = 1,2,\ldots,n$, then $s(P_i, P_{i+1}) \leq \min(R_i, R_{i+1})$ for $i = 1,2,\ldots,n \pmod n$ and $s(P_i, P_j) > \max(R_i, R_j)$ for $j \neq i-1, \neq i+1 \pmod n$. If $P = [P_1 P_2 \ldots P_n]$ is an A'-circuit, then P admits an A-collection of pseudodisks $\{H_i\}$ where the radius r_i of H_i is equal to $\frac{1}{2} R_i$, the radius of H_i'. Indeed to see that H_i and H_{i+1} L-overlap for $i = 1,2,\ldots,n \pmod n$, let L_i denote the line of centers of H_i and H_{i+1}. We must show that

(1) L_i has some point(s) in common with $H_i \cap H_{i+1}$ and

(2) if $j \neq i-1$ and $j \neq i+1$ the line of centers of H_i and H_j does not have any point in common with $H_i \cap H_j$.

(1) Clearly there is a point Q_i on L_i such that $s(P_i, Q_i) = r_i$. We show that $Q_i \in H_{i+1}$, i.e., that $s(P_{i+1}, Q_i) \leq r_{i+1}$. Note that $s(P_i, P_{i+1}) = s(P_i, Q_i) + s(Q_i, P_{i+1})$. This implies

$\min(R_i, R_{i+1}) \geq r_i (= \frac{1}{2}R_i) + s(Q_i, P_{i+1})$, since $\min(R_i, R_{i+1}) \geq s(P_i, P_{i+1})$, or

$$(5.1) \qquad \min(R_i, R_{i-1}) - \tfrac{1}{2} R_i \geq s(Q_i, P_{i+1}) \quad .$$

If $R_i = R_{i+1}$, the left hand side of (5.1) becomes $\frac{1}{2}R_i = \frac{1}{2}R_{i+1} = r_{i+1}$ and $Q_i \in H_{i+1}$. If $R_i < R_{i+1}$, again the left hand side of (5.1) becomes $\frac{1}{2}R_i$ which is less than $\frac{1}{2}R_{i+1} = r_{i+1}$ and again $Q_i \in H_{i+1}$. If $R_i > R_{i+1}$, the left hand side of (5.1) becomes $R_{i+1} - \frac{1}{2}R_i$ which is less than $R_{i+1} - \frac{1}{2}R_{i+1} = \frac{1}{2}R_{i+1} = r_{i+1}$, and $Q_i \in H_{i+1}$.

(2) Note that if $j \neq i-1$ and $j \neq i+1$, $s(P_i, P_j) > \max(R_i, R_j) \geq \frac{1}{2}R_i + \frac{1}{2}R_j = r_i + r_j$. Thus $H_i \cap H_j = \emptyset$ and the line of centers of H_i and H_j has no points in common with $H_i \cap H_j$.

Thus the assertion that A-circuits exist for $n \geq 3$ is established.

6. Existence of B-circuits and B′-circuits. Note that if a distribution of events admits either a B-circuit or a B′-circuit, $P = [P_1 P_2 \ldots P_n]$, the distribution must be timelike. For if P_i is an event in the distribution, the line of centers which joins P_i to P_j where $j \neq i-1$, $i+1$ will be timelike since the line of centers will have some point(s) in common with the intersection of the two pseudospheres H_i and H_j . The line segments joining P_i to P_{i-1} and to P_{i+1} must be timelike since, by definition, they are edges of the B-circuit or of the B′-circuit.

For $n = 3$ we can obviously obtain a B′-circuit on n vertices. We show below that B′-circuits exist for $n = 4$ and, as in §5, we can obtain from any B′-circuit a B-circuit on the same vertex set.

To obtain a B′-circuit on four vertices, let $P_1(x_1, t_1)$ be a point in L^2 and let $P_3(x_3, t_3)$, $x_3 > x_1$, $t_3 > t_1$, be a point in the time cone of P_1. Let r_1 and r_3 be positive real numbers with $r_1 > s(P_1,P_3)$ and $r_3 > s(P_1,P_3)$. At P_i center a pseudodisk H_i of radius r_i, $i = 1,3$.

Let $P_2(x_2,t_2)$, $t_2 > t_3$, be a point in the intersection of the time cones of P_1 and P_3 which has the property that $s(P_2,P_1) > r_1$ and $s(P_2,P_3) > r_3$. Let r_2 be a positive real number with $r_2 < \min[s(P_2,P_1),s(P_2,P_3)]$. Take r_2 as the radius of a pseudodisk H_2 centered at P_2. Let $P_4(x_4,t_4)$ be a point inside the pseudodisk H_2 having the properties $s(P_1,P_4) > r_1$, $s(P_3,P_4) > r_3$ and $s(P_2,P_4) < \min[s(P_1,P_4),s(P_3,P_4)]$. Then we can choose a positive real number r_4 with $s(P_2,P_4) < r_4 < \min[s(P_1,P_4),s(P_3,P_4)]$ as the radius of a pseudodisk H_4 centered at P_4.

The circuit $P = [P_1P_2P_3P_4]$ admits the B'-collection of pseudodisks H_1,H_2,H_3,H_4 and is thus a B'-circuit. An argument similar to that of §5 demonstrates that a collection of pseudodisks centered at $\{P_i\}$ having radii equal to one-half the radii of the corresponding pseudodisks in the B'-collection constitutes a B-collection and makes P a B-circuit.

We show below that for $n = 5$ no semi-timelike distribution of n events admits a B-circuit. (As a result, no such distribution admits a B'-circuit since B'-circuits give rise to B-circuits, as we have seen.)

We attempt to choose distinct events $P_i(x_i,t_i)$, $i = 1,2,\ldots,5$, in L^2 and center a pseudodisk H_i with radius r_i at the event P_i so that the circuit $P = [P_1P_2P_3P_4P_5]$ is a B-circuit. We may choose P_1 arbitrarily. Let P_3 be a point in L^2 with $x_3 > x_1$ and $t_3 > t_1$. Then we must choose positive numbers r_1 and r_3 for radii of the pseudodisks H_1 and H_3 to be centered at P_1 and P_3 so that

$s(P_1,P_3) \leq r_1 + r_3$. P_4 must lie in the time cone of P_3 and r_4 must be chosen so that $s(P_1,P_4) \leq r_1 + r_4$ but $s(P_3,P_4) > r_3 + r_4$.

If $t_4 > t_3 > t_1$, $s(P_1,P_4) \geq s(P_1,P_3) + s(P_3,P_4)$ by the triangle inequality in L^2 . Then r_1, r_3 and r_4 must satisfy the inequalities $r_1 + r_4 \geq s(P_1,P_4) \geq s(P_1,P_3) + s(P_3,P_4) > s(P_1,P_3) + r_3 + r_4$. Hence $r_1 - r_3 > s(P_1,P_3)$. Since $s(P_1,P_3) > 0$, we must have $r_1 > r_3$. Thus $s(P_1,P_3) < r_1$ and P_3 must lie in H_1 .

If $t_3 > t_4 > t_1$, then $s(P_1,P_3) \geq s(P_1,P_4) + s(P_4,P_3)$ by the triangle inequality. So r_1, r_3 and r_4 must satisfy the inequalities $r_1 + r_3 \geq s(P_1,P_3) \geq s(P_1,P_4) + s(P_4,P_3) > s(P_1,P_4) + r_3 + r_4$. This implies $s(P_1,P_4) < r_1 - r_4$, and P_4 must lie in H_1 .

We may also choose $t_3 > t_1 > t_4$. This does not impose any additional condition on $s(P_1,P_3)$ or $s(P_1,P_4)$ like that imposed when $t_4 > t_3 > t_1$ or when $t_3 > t_4 > t_1$.

We have noted that the vertices must be distinct and must constitute a timelike distribution. Hence we cannot have $t_4 = t_3$ or $t_4 = t_1$.

P_5 must lie in the intersection of the time cones of P_1, P_3 and P_4 and r_5 must be chosen so that $s(P_1,P_5) > r_1 + r_5$, $s(P_4,P_5) > r_4 + r_5$ and $s(P_3,P_5) \leq r_3 + r_5$.

Consider the case where $t_4 > t_3 > t_1$. Suppose $t_5 > t_4 > t_3 > t_1$. Then $s(P_3,P_5) \geq s(P_3,P_4) + s(P_4,P_5)$. This requires that $r_4 < 0$ since r_3, r_4 and r_5 must satisfy $r_3 + r_5 \geq s(P_3,P_5), s(P_3,P_4) > r_3 + r_4$ and $s(P_4,P_5) > r_4 + r_5$. Suppose either $t_4 > t_5 > t_3 > t_1$ or $t_4 > t_3 > t_5 > t_1$. Then $s(P_1,P_4) \geq s(P_1,P_5) + s(P_5,P_4)$ which requires $r_5 < 0$ by similar reasoning. Suppose $t_4 > t_3 > t_1 > t_5$. Then $s(P_3,P_5) \geq s(P_3,P_1) + s(P_1,P_5)$. This requires $r_3 - r_1 > s(P_3,P_1) > 0$.

However, we already have the requirement $r_3 < r_1$. As before, we cannot have $t_5 = t_i$, $i = 1,3,4$, since P_1, P_3, P_4, P_5 are distinct and P_5 must lie in the time cones of P_1, P_3 and P_4.

Consider the case where $t_3 > t_4 > t_1$. Clearly we can choose P_5 and r_5 so that $t_5 > t_3 > t_4 > t_1$ and $s(P_3,P_5) \leq r_3 + r_5$, $s(P_1,P_5) > r_1 + r_5$ and $s(P_4,P_5) > r_4 + r_5$. Suppose $t_3 > t_5 > t_4 > t_1$. Then $s(P_3,P_1) \geq s(P_3,P_5) + s(P_5,P_1)$ which implies r_3 and r_5 must satisfy $r_3 - r_5 > s(P_3,P_5)$ so that P_5 must lie in H_3. Suppose $t_3 > t_4 > t_5 > t_1$. Then $s(P_1,P_4) \geq s(P_1,P_5) + s(P_5,P_4)$ requiring $r_5 < 0$. Suppose $t_3 > t_4 > t_1 > t_5$. Then $s(P_3,P_5) \geq s(P_3,P_4) + s(P_4,P_5)$ requiring $r_4 < 0$.

Consider the case where $t_3 > t_1 > t_4$. Clearly the conditions on P_5 and r_5 can be met if $t_5 > t_3 > t_1 > t_4$. If $t_3 > t_5 > t_1 > t_4$, then the triangle inequality applied to P_5, P_3 and P_1 requires $r_3 - r_5 > s(P_3,P_5)$ and P_5 must lie in H_3. If $t_3 > t_1 > t_5 > t_4$, an application of the triangle inequality to P_1, P_5 and P_4 requires $r_5 < 0$. If $t_3 > t_1 > t_4 > t_5$, an application of the triangle inequality to P_3, P_4 and P_5 requires $r_4 < 0$.

P_2 must lie in the intersection of the time cones of P_1 and P_3 and r_2 must be chosen so that $s(P_1,P_2) > r_1 + r_2$, $s(P_2,P_3) > r_2 + r_3$, $s(P_2,P_4) \leq r_2 + r_4$, and $s(P_2,P_5) \leq r_2 + r_5$.

The cases we must consider are those in which

(i) $t_5 > t_3 > t_4 > t_1$ (P_4 lies in H_1).

(ii) $t_3 > t_5 > t_4 > t_1$ (P_4 lies in H_1 and P_5 lies in H_3).

(iii) $t_5 > t_3 > t_1 > t_4$

(iv) $t_3 > t_5 > t_1 > t_4$ (P_5 lies in H_3).

We consider the possible orderings of the time coordinates of P_1, P_2, P_3, P_4, P_5:

(i) $t_2 > t_5 > t_3 > t_4 > t_1$
$t_5 > t_2 > t_3 > t_4 > t_1$
$t_5 > t_3 > t_2 > t_4 > t_1$
$t_5 > t_3 > t_4 > t_2 > t_1$
$t_5 > t_3 > t_4 > t_1 > t_2$

(ii) $t_2 > t_3 > t_5 > t_4 > t_1$
$t_3 > t_2 > t_5 > t_4 > t_1$
$t_3 > t_5 > t_2 > t_4 > t_1$
$t_3 > t_5 > t_4 > t_2 > t_1$
$t_3 > t_5 > t_4 > t_1 > t_2$

(iii) $t_2 > t_5 > t_3 > t_1 > t_4$
$t_5 > t_2 > t_3 > t_1 > t_4$
$t_5 > t_3 > t_2 > t_1 > t_4$
$t_5 > t_3 > t_1 > t_2 > t_4$
$t_5 > t_3 > t_1 > t_4 > t_2$

(iv) $t_2 > t_3 > t_5 > t_1 > t_4$
$t_3 > t_2 > t_5 > t_1 > t_4$
$t_3 > t_5 > t_2 > t_1 > t_4$
$t_3 > t_5 > t_1 > t_2 > t_4$
$t_3 > t_5 > t_1 > t_4 > t_2$.

Since the point distribution is timelike the inequalities (a) $t_1 > t_2 > t_3$, (b) $t_2 > t_3 > t_4$, (c) $t_3 > t_4 > t_5$, (d) $t_2 > t_1 > t_5$ imply (respectively) the inequalities (a′) $s(P_1,P_3) \geq s(P_1,P_2) + s(P_2,P_3)$, (b′) $s(P_2,P_4) \geq s(P_2,P_3) + s(P_3,P_4)$, (c′) $s(P_3,P_5) \geq s(P_3,P_4) + s(P_4,P_5)$ and (d′) $s(P_2,P_5) \geq s(P_2,P_1) + s(P_1,P_5)$ by the triangle inequality. The latter inequalities together with the conditions on the pseudodisks require $r_2 < 0$, $r_3 < 0$, $r_4 < 0$ and $r_1 < 0$, respectively. However, each of the inequalities (i) - (iv) contains one of the inequalities (a) - (d). Thus we cannot find a set of five events in L^2 which admit a B-circuit. It follows that no set of five events in L^2 admits a B′-circuit.

The circuits $P = [P_1P_2P_3P_4P_5P_6]$ and $Q = [Q_1Q_2Q_3Q_4Q_5Q_6Q_7Q_8]$ on the events $P_1(-3,8)$, $P_2(-3,38)$, $P_3(-2,4)$, $P_4(4,30)$, $P_5(0,0)$, $P_6(0,23)$ and $Q_1(8,-10)$, $Q_2(3,26)$, $Q_3(12,-15)$, $Q_4(0,31)$, $Q_5(0,0)$, $Q_6(0,18)$, $Q_7(4,-5)$, $Q_8(1,22)$ are B-circuits on six vertices and eight vertices respectively. Indeed, at $P_1,P_2,\ldots,P_6$ we center pseudodisks $H_1,H_2,\ldots,H_6$ with radii $r_1 = 2$, $r_2 = \sqrt{1425} - 10.3$, $r_3 = 6.3$, $r_4 = \sqrt{435} - 2$, $r_5 = 10.3$, $r_6 = \sqrt{357} - 6.3$ and at $Q_1,Q_2,\ldots,Q_8$ we center pseudodisks $H_1,H_2,\ldots,H_8$ with radii $r_1 = 24.86$, $r_2 = \sqrt{960} - 20.45$, $r_3 = 29$, $r_4 = \sqrt{1617} - 24.86$, $r_5 = 15.64$, $r_6 = \sqrt{720} - 24.86$, $r_7 = 20.45$, $r_8 = \sqrt{483} - 15.64$. One verifies directly that P and Q are B-circuits.

B'-circuits, as asserted in Theorem 5, do not exist on n events for $n > 5$. Indeed, we place the vertices P_1,P_3,P_4 and P_5 and the pseudodisks H_1,H_3,H_4,H_5 so as to satisfy the necessary conditions and show that it is impossible to place the vertex P_2 to be consistent with the necessary conditions.

We may choose H_1 with center P_1 and radius r_1 arbitrarily. Let P_3 be a point in L^2 with the property that $t_3 > t_1$ and $s(P_1,P_3) \leq r_1$, and choose r_3 so that $s(P_1,P_3) \leq r_3$, P_4 must lie in the intersection of the time cone of P_3 with H_1, but so that $s(P_3,P_4) > r_3$. Since we need to choose r_4 so that $s(P_1,P_4) \leq r_4 < s(P_3,P_4)$, P_4 must also satisfy $s(P_1,P_4) < s(P_3,P_4)$. A necessary condition for this is that $t_3 > t_1 > t_4$. This is seen as follows:

By the triangle inequality, $t_4 > t_3 > t_1$ implies $s(P_3,P_4) \leq s(P_1,P_4)$, but we must have $s(P_3,P_4) > s(P_1,P_4)$. The condition $P_4 \in H_1$ together with $t_3 > t_4 > t_1$ implies either $P_4 \in H_3$ or P_4 is outside the time cone of H_3. Since the vertices in question are distinct and must constitute a timelike distribution, we cannot have $t_4 = t_3$ or $t_4 = t_1$.

P_5 must lie in the time cone of P_4 but with $s(P_4,P_5) > r_4$. P_5 must satisfy $s(P_1,P_5) \le r_1$ and $s(P_3,P_5) \le r_3$. Then as shown below, we must have either $t_5 > t_3 > t_1 > t_4$ or $t_3 > t_5 > t_1 > t_4$. Indeed, if $t_5 = t_i$, then either $P_5 = P_i$, $i = 1,3,4$ respectively, or P_5 lies outside the time cone of P_i violating one of the necessary conditions. If $t_3 > t_1 > t_5 > t_4$, $s(P_1,P_4) \ge s(P_4,P_5) + s(P_5,P_1)$. This implies $r_4 > r_4$ since we need $r_4 > s(P_1,P_4)$ and $s(P_4,P_5) > r_4$. If $t_3 > t_1 > t_4 > t_5$, $s(P_1,P_5) \ge s(P_1,P_4) + s(P_4,P_5)$. But r_5 will have to be chosen so that $s(P_1,P_5) \le r_5$ and $s(P_4,P_5) > r_5$. Thus we will be led to the contradiction $r_5 > r_5$. Hence either $t_5 > t_3 > t_1 > t_4$ or $t_3 > t_5 > t_1 > t_4$.

Suppose $t_5 > t_3 > t_1 > t_4$. P_2 must line in the intersection of the time cones of P_3 and P_1, but so that $s(P_2,P_3) > r_3$ and $s(P_2,P_1) > r_1$. P_2 must also lie in $H_4 \cap H_5$. As we show below, P_2 cannot be chosen to satisfy these conditions. Indeed, if $t_2 > t_3$, the triangle inequality implies $s(P_2,P_4) \ge s(P_2,P_3) + s(P_3,P_4)$. But P_2 must be chosen so that $s(P_2,P_4) \le r_4$ and hence we would have $s(P_3,P_4) < r_4$, which is not allowable. If $t_3 > t_2$, the triangle inequality implies $s(P_2,P_5) \ge s(P_2,P_3) + s(P_3,P_5)$. But r_2 must be chosen to that $r_2 \ge s(P_2,P_5)$ and $s(P_2,P_3) > r_2$. Such a choice of r_2 is in this case impossible.

Suppose $t_3 > t_5 > t_1 > t_4$. If $t_2 > t_3$, $s(P_2,P_5) \ge s(P_2,P_3) + s(P_3,P_5)$. But we must have $s(P_2,P_5) \le r_2$ and $s(P_2,P_3) > r_2$ which is, in this case, impossible. If $t_3 > t_2 > t_5$, then $s(P_3,P_5) \ge s(P_3,P_2) + s(P_2,P_5)$. But $s(P_3,P_5) \le r_3$ and we need $s(P_3,P_2) > r_3$, which we cannot have in this case. If $t_5 > t_2 > t_1$, then $s(P_1,P_5) \ge s(P_1,P_2) + s(P_2,P_5)$. But the inequalities $s(P_1,P_5) \le r_1$ and $s(P_1,P_2) > r_1$ must hold simultaneously and they cannot in this case. Similarly, the inequalities $s(P_1,P_4) \le r_1$ and $s(P_1,P_2) > r_1$ must hold simultaneously. They cannot if $t_1 > t_2 > t_4$ since this inequality implies $s(P_1,P_4) \ge s(P_1,P_2) + s(P_2,P_4)$. If $t_4 > t_2$, $s(P_2,P_5) \ge s(P_2,P_4) + s(P_4,P_5)$. But we must have $r_5 \ge s(P_2,P_5)$ and $s(P_4,P_5) > r_5$ simultaneously, which is impossible in this case.

Thus it is impossible to place the vertex P_2 so as to obtain a B'-circuit on $P = [P_1P_2 \ldots P_n]$ where $n > 5$, and since we saw above that B-circuits do not exist for $n = 5$, the assertion of Theorem 5 concerning the existence of B'-circuits is established.

7. Continuation of Proofs of Theorems 3, 4 and 5. We first show that any A-circuit is uniquely minimal of all semi-timelike circuits in its covertex class, as asserted in Theorem 3.

Let $P = [P_1P_2 \ldots P_n]$ be an A-circuit with the pseudodisks $H_1,H_2,\ldots,H_n$ having centers $P_1,P_2,\ldots,P_n$ and radii $r_1,r_2,\ldots,r_n$, respectively, constituting the A-collection of pseudodisks in question.

We associate the $n \times n$ matrix (a_{ij}) with the A-circuit P as follows:

$$a_{ij} = \begin{cases} s(P_i,P_j) & \text{if } i = j \text{ or } i \ne j \text{ and the edge } P_iP_j \text{ is timelike} \\ M = n \cdot \max\{r_i, |t_k - t_h| : i,k,h \in \{1,2,\ldots,n\}\} & \text{if the edge } P_iP_j \text{ is not timelike} \end{cases}$$

If $\bar{P} = [P_{i_1}P_{i_2} \ldots P_{i_n}]$ is a cyclic permutation of $P_1,P_2,\ldots,P_n$, let $\delta(\bar{P}) = \{\{P_{i_1},P_{i_2}\},\ldots,\{P_{i_{n-1}},P_{i_n}\},\{P_{i_n},P_{i_1}\}\}$ denote the set of edges of the ~~cir~~cuit corresponding to the permutation $\bar{P}$, and let $L(\bar{P})$ denote the sum

$$\sum_{\{P_i,P_j\}\in\delta(\bar{P})} a_{ij} = a_{i_1 i_2} + \dots + a_{i_{n-1} i_n} + a_{i_n i_1} .$$

Definition: A cyclic permutation P is "$[r_1 \dots r_n]$-consistent with the $n\times n$ real symmetric matrix (a_{ij})" abbreviated "$[r_k]$-consistent" if the matrix is understood, whenever (i) $a_{ij} \le r_i + r_j$ if $\{i,j\} \in \delta(P)$ and (ii) $a_{ij} \ge r_i + r_j$ if both $\{i,j\} \notin \delta(P)$ and $i \ne j$. (Cf. [3], p.8).

Theorem A: Given a real symmetric $n\times n$ matrix (a_{ij}) perhaps with unspecified diagonal entries, and real numbers $r_1,\dots,r_n$ such that there is at least one $[r_k]$-consistent cyclic permutation, then such and only such minimize L among all cyclic permutations. (Cf. [3], p.9).

The radii $r_1,\dots,r_n$ of the pseudodisks $H_1,\dots,H_n$ provide the real numbers required by Theorem A. Indeed, $a_{ij} \le r_i + r_j$ if $\{P_i,P_j\} \in \delta(P) = \{\{P_1,P_2\},\dots,\{P_{n-1},P_n\},\{P_n,P_1\}\}$ since the pseudodisks H_i and H_{i+1}(mod n) L-overlap. If $\{P_i,P_j\} \notin \delta(P)$, either P_iP_j is timelike or it is not. If P_iP_j is timelike, $s(P_i,P_j) = a_{ij} > r_i + r_j$ since the pseudodisks H_i and H_j do not L-overlap for $j \ne i-1, i+1$. If P_iP_j is not timelike, $a_{ij} = M > r_i + r_j$. (Note that if $a_{i_j i_{j+1}} = M$ for some $a_{i_j i_{j+1}}$ in the sum

(7.1) $$a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_n i_1} ,$$

then (7.1) is greater than any sum $a_{k_1k_2} + a_{k_2k_3} + \dots + a_{k_nk_1}$ where

$a_{k_j k_{j+1}}$ is different from M for all $j = 1,\ldots,n(\text{mod } n)$.)

Thus Theorem A applies and P is minimal. It is clear from the definition of A-circuit that P is the only $[r_k]$-consistent circuit on $\{P_1,P_2,\ldots,P_n\}$.

If $P = [P_1P_2\ldots P_n]$ is an A'-circuit with the pseudodisks $H_1,H_2,\ldots,H_n$ having centers $P_1,P_2,\ldots,P_n$ and radii $R_1,R_2,\ldots,R_n$, respectively, constituting the A'-collection of pseudodisks, we associate with P the $n\times n$ matrix (a_{ij}) with entries

$$a_{ij} = \begin{cases} s(P_i,P_j) & \text{if } i = j \text{ or } i \neq j \text{ and the edge } P_iP_j \text{ is timelike} \\ M = n \cdot \max\{R_i, |t_k - t_h|, \; i,k,h \in \{1,2,\ldots,n\}\} & \text{if the edge } P_iP_j \text{ is not timelike.} \end{cases}$$

The real numbers $r_i = \frac{1}{2}R_i$, $i = 1,2,\ldots,n$, are the real numbers required by Theorem A. Indeed, $a_{ij} \leq r_i + r_j$ if $\{P_i,P_j\} \in \delta(P)$ since $a_{i,i+1} = s(P_i,P_{i+1}) \leq \min(R_i,R_{i+1}) \leq \frac{1}{2}R_i + \frac{1}{2}R_{i+1} = r_i + r_{i+1}$. Similarly, if $\{P_i,P_j\} \not\in \delta(P)$, $a_{ij} > r_i + r_j$ since $a_{ij} = s(P_i,P_j) > \max(R_i,R_j) \geq \frac{1}{2}R_i + \frac{1}{2}R_j = r_i + r_j$. Thus any A'-circuit is minimal and is the only $[r_k]$-consistent circuit on its vertices.

We procede analogously to show any B-circuit (respectively, B'-circuit) is uniquely maximal in its covertex class. (Recall that the vertices of a B-circuit or a B'-circuit constitute a timelike distribution.)

In the case of B-circuits or B'-circuits, we make use of the following:

<u>Theorem B</u>. Given a real symmetric $n\times n$ matrix (a_{ij}), perhaps with unspecified diagonal entries, and real numbers $r_1,\ldots,r_n$ such that there is at least one cyclic permutation P satisfying

(i) $a_{ij} \geq r_i + r_j$ if $\{i,j\} \in \delta(P)$ and (ii) $a_{ij} \leq r_i + r_j$ if both $\{i,j\} \notin \delta(P)$ and $i \neq j$; then these and only these maximize L among all cyclic permutations. (Cf. [3], p.25).

If $P = [P_1P_2 \ldots P_n]$ is a B-circuit with the pseudodisks $H_1, H_2, \ldots, H_n$ having centers $P_1, P_2, \ldots, P_n$ and radii $r_1, r_2, \ldots, r_n$ constituting the B-collection of pseudodisks, we can associate the $n \times n$ matrix (a_{ij}) with the B-circuit P by letting $a_{ij} = s(P_i, P_j)$, $i,j \in \{1,2,\ldots,n\}$. As in the argument for A-circuits the radii $r_1, r_2, \ldots, r_n$ of the pseudodisks $H_1, \ldots, H_n$ provide the real numbers required by the theorem: Indeed, $a_{ij} > r_i + r_j$ if $\{i,j\} \in \delta(P)$ since the pseudodisks H_i and $H_{i+1 \pmod n}$ do not L-overlap and $a_{ij} \leq r_i + r_j$ if $\{i,j\} \notin \delta(P)$ since H_i and H_j L-overlap for $j \neq i-1, i+1$. Thus the theorem applies and P is maximal. As for A-circuits, the definition of B-circuit gives the uniqueness of this maximal P.

The proof that any B'-circuit is uniquely maximal in its covertex class is the same as the above argument, except that the real numbers $r_1, \ldots, r_n$ required by Theorem B are, as in the argument for A'-circuits, given by $r_i = \frac{1}{2}R_i$, $i = 1,2,\ldots,n$.

8. Proof of Enumeration Assertions. We enumerate the green Hamiltonian circuits in a complete green and red graph G on n vertices which admits at least one green Hamiltonian circuit as follows: We label the k red edges $e_1, e_2, \ldots, e_k$ and let A_i denote the set of those Hamiltonian circuits on the n vertices (referred to below as "n-circuits") which contain the red edge e_i for $i = 1,2,\ldots,k$.

Theorem C. If of N objects, $N(a)$ have property a, $N(b)$ property b, ..., $N(ab)$ both a and b, ..., $N(abc)$ a, b, and c, and so on, the number $N(a'b'c'\dots)$ with none of these properties is given by

$$N(a'b'c'\dots) = N - N(a) - N(b) - \dots + N(ab) + N(ac) + \dots - N(abc) - \dots + \dots \qquad ([2], \text{p.}51).$$

We apply Theorem C in the following form: Of the $\frac{1}{2}(n-1)!$ Hamiltonian circuits on the complete green-red graph G, the number of green n-circuits is

$$(8.1) \qquad \frac{(n-1)!}{2} - \sum_{i=1}^{k} |A_i| + \sum_{\{i_1,i_2\}\subset\{1,2,\dots,k\}} |A_{i_1} \cap A_{i_2}| - \dots$$

$$+ (-1)^j \sum_{\{i_1,i_2,\dots,i_j\}\subset\{1,2,\dots,k\}} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_j}|$$

$$+ \dots + (-1)^k \left| \bigcap_{i=1}^{k} A_i \right| ,$$

where $|A|$ denotes the cardinality of the set A and the $(j+1)^{st}$ term of (8.1) is summed over the $\binom{k}{j}$ possible intersections of the k sets, $A_1, A_2, \dots, A_k$, taken j at a time.

We give below a formula for $\left| \bigcap_{m=1}^{j} A_{i_m} \right|$ which, when substituted in (8.1) yields (1.1).

Suppose $\{i_1, i_2, \dots, i_j\}$ is a subset of $\{1,2,\dots,k\}$ for which $\bigcap_{m=1}^{j} A_{i_m}$ is non-empty. Every circuit, P, in $\bigcap_{m=1}^{j} A_{i_m}$ will contain the red edges $e_{i_1}, e_{i_2}, \ldots, e_{i_j}$. Let p denote the number of maximal chains in P formed by the edges

(8.2) $$e_{i_1}, e_{i_2}, \ldots, e_{i_j}$$

(maximal in the sense that no two of these chains have a common endpoint): say a chain of t_1 red edges from (8.2), a chain of t_2 edges from (8.2),..., and a chain of t_p edges from (8.2) (the t_i not necessarily distinct integers) with $t_1 + t_2 + \ldots + t_p = j$. The number of such chains and the number of edges in each chain depend upon the geometric configuration of the k red edges in the complete green-red graph and upon the particular subset $\{i_1, i_2, \ldots, i_j\}$ of $\{1,2,\ldots,k\}$.

Of the p chains of red edges which occur in each circuit in $\bigcap_{m=1}^{j} A_{i_m}$ we choose one and denote the number of edges in the chain by t_1. We choose one of the endpoints of this chain to occupy the first place in the symbol for any circuit in $\bigcap_{m=1}^{j} A_{i_m}$, with the first (t_1+1) places in each circuit representation occupied by the (t_1+1) vertex symbols which specify the chosen chain. In the following we will represent any circuit in $\bigcap_{m=1}^{j} A_{i_m}$ in this "standard form". Note that we may regard each of the orderings of vertex symbols which define the chains in question as a single symbol. Thus for any circuit in $\bigcap_{m=1}^{j} A_{i_m}$ we have p symbols denoting these chains and $n - \sum_{i=1}^{p} (t_i+1) = n - (j+p)$ symbols representing the vertices which do not occur in any of the "chain symbols" in question. Note that wherever a chain symbol occurs, it may occur in either of two ways, representing the two directions in which the chain can be traversed (with the exception of the chain of t_1 red edges whose position and direction we have fixed). Thus each circuit in $\bigcap_{m=1}^{j} A_{i_m}$ corresponds to a permutation of $n - (j+p)$ vertex symbols and $(p-1)$ chain symbols with each of the latter occurring in one of two possible ways. Hence we have

$$(8.3) \qquad \left| \bigcap_{m=1}^{j} A_{i_m} \right| = 2^{p-1}(n - (j+p) + (p-1))! = 2^{p-1}(n - (j+1))!$$

As noted above, the substitution of (8.3) in (8.1) yields (1.1) and Theorem 6 is established.

If, in the complete green-red graph, none of the red edges have an endpoint in common, then each circuit P in $\bigcap_{m=1}^{j} A_{i_m}$ will contian j "chains" each consisting of one red edge for each choice of j elements, $i_1, i_2, \ldots, i_j$, from the set $\{1,2,\ldots,k\}$. Thus for each such choice,

$$\left| \bigcap_{m=1}^{j} A_{i_m} \right| = 2^{j-1}(n - (j+1))!$$

and

$$c_j(G) = \binom{k}{j} 2^{j-1} \quad .$$

Substituting this expression in (1.2), we have (1.3) true, as asserted.

If, in the complete green-red graph, all the red edges have a common endpoint, then $|A_{i_1} \cap A_{i_2}| = (n-3)!$ for each subset $\{i_1, i_2\}$ of $\{1,2,\ldots,k\}$ since every two red edges have an endpoint in common. That is, we have one chain of two edges for each choice of i_1 and i_2, and

$c_2(G) = \binom{k}{2}$. Clearly, $|\bigcap_{m=1}^{j} A_{i_m}| = 0$ for $j \geq 3$. Since for any graph, G, $c_1(G) = k$, (1.4) is established.

The following is a further illustration of the method of enumeration. If G is a complete green-red graph containing the five red edges $e_1 = P_1P_2$, $e_2 = P_3P_4$, $e_3 = P_4P_5$, $e_4 = P_5P_6$ and $e_5 = P_5P_7$ (with all the other edges green), then the number of green circuits on G will be given by determining the coefficients $c_j(G)$ of (1.2).

Since, from the configuration of the red edges in G we have

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_1 \cap A_5| = |A_2 \cap A_4| = |A_2 \cap A_5| = 2(n-3)!$$

and

$$|A_2 \cap A_3| = |A_3 \cap A_4| = |A_3 \cap A_5| = |A_4 \cap A_5| = (n-3)!$$

we have $c_2(G) = 6(2) + 4(1) = 16$.

Since $|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_2 \cap A_5| = 2^2(n-4)!$,

$$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_3 \cap A_4| = |A_1 \cap A_3 \cap A_5| = |A_1 \cap A_4 \cap A_5| =$$

$$= |A_2 \cap A_4 \cap A_5| = 2(n-4)! \ ,$$

$$|A_2 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_5| = (n-4)! \ ,$$

and

$$|A_3 \cap A_4 \cap A_5| = 0 \ ,$$

we have $c_3(G) = 2(2^2) + 5(2) + 2(1) + 0 = 20$. We also have $|A_1 \cap A_2 \cap A_4 \cap A_5| = 2^2(n-5)!$, $|A_1 \cap A_2 \cap A_3 \cap A_4| = |A_1 \cap A_2 \cap A_3 \cap A_5| = 2(n-5)!$ and $|A_1 \cap A_3 \cap A_4 \cap A_5| = |A_2 \cap A_3 \cap A_4 \cap A_5| = 0$, so that $c_4(G) = 2^2 + 2(2) + 0 = 8$. Since $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 0$, $c_5(G) = 0$. Thus the number of green Hamiltonian circuits on G is

$$\frac{1}{2}(n-1)! -5(n-2)! + 16(n-3)! -20(n-4)! + 8(n-5)! \quad .$$

Remark: Not every green-red abstract graph on n vertices represents a graph in L^2 with timelike edges corresponding to the green edges and spacelike edges corresponding to the red edges of the abstract graph.

Indeed, consider the following green-red abstract graph on the vertices of a regular pentagon: Label the vertices A,B,C,D,E consecutively following the boundary of their convex hull (the pentagon). Let the edges AB,BC,CD,DE,EA be green and the remaining edges AC,AD,BD,BE,CE be red. No graph in L^2 corresponds, in the sense described above, to this green-red abstract graph. For suppose $A(x_A,t_A)$ is any event in L^2. Since AB is to be timelike, $B(x_B,t_B)$ must lie in the time cone of A. We can assume without loss of generality that $t_B > t_A$. Since BC is to be timelike and AC is to be spacelike, we must have $C(x_C,t_C)$ inside the time cone of B and outside the time cone of A; that is, in either of the regions

$$R_1 = \{(x,t): x - x_A < t - t_A < -(x - x_A),\ t - t_B < (x - x_B)\}$$

or

$$R_2 = \{(x,t): -(x - x_A) < t - t_A < (x - x_A),\ t - t_B < -(x - x_B)\} \quad .$$

Again without loss of generality we may make the assumption that C lies in R_2. Since CD is to be timelike and AD and BD spacelike, $D(x_D,t_D)$ must lie in the time cone of C and outside the time cones of A and B. Thus D must lie in the region $\{(x,t): t-t_A < x-x_A,\ t-t_B > -(x-x_B),\ t-t_C > x-x_C\}$. Since DE and EA are to be timelike and BE and CE spacelike, $E(x_E,t_E)$ must lie in the time cones of A and D and outside the time cones of B and C. Since D is outside the time cone of A, the intersection of T(A), the time cone of A, and T(D), the time cone of D, consists of two disconnected components. Every event P(x,t) in one of these components has the property that $t > t_A$ and $t > t_D$; every event in the other has the property that $t < t_A$ and $t < t_D$. E must lie in one of the two components of $T(A) \cap T(D)$. Suppose $t_E > t_D$. Then since D is in the time cone of C and $t_D > t_C$, we have E in the time cone of C. But CE must be spacelike. Suppose $t_E < t_A$. Since A is in the time cone of B and $t_A < t_B$, we have E in the time cone of B. But BE must be spacelike.

Thus it is not possible to choose events A,B,C,D,E in L^2 so that the graph with these events as vertices corresponds to the given green-red abstract graph.

As noted in §1, the problem of classifying those green-red abstract graphs that correspond, in the sense above, to graphs in L^2 awaits resolution.

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Autobiographical Statement

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