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ON GL $(3,\mathbb{R})$.

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FOURIER SERIES AND HECKE OPERATORS ON GL(3,$\mathbb{R}$)

by

UMESH P. NAGARKATTE

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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TABLE OF CONTENTS

		Page Number
Acknowledgements		iii
§0.	Introduction	1
§1.	Preliminaries on Operators	2
§2.	Fourier Expansion of a Cusp-form	8
	2.1. First Expansion	8
	2.2. Second Expansion	10
§3.	Action of Hecke Operators on a Cusp-form	14
§4.	The Main Theorem	20
§5.	Converse	24
Bibliography		28

0. Introduction

Hecke operators and their relation to the Fourier expansion of automorphic forms have been studied extensively for $GL(2,\mathbb{R})$. Here we extend the theory to $GL(3,\mathbb{R})$.

Our main result can be formulated as follows. Let φ be a cusp form on $GL(3,\mathbb{R})$ (for the group $\Gamma = SL(3,\mathbb{Z})$). Suppose it is an eigenfunction of the Hecke algebra. Then

$$\varphi(g) = \sum_{\substack{\alpha \geq 1 \\ d \geq 1}} h_{\alpha d,\alpha} W(\gamma g) .$$

Here W is a fixed function with the property

$$W\left[\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} g\right] = \exp(2i\pi(x+y))W[g] ,$$

γ varies on an appropriate set in $GL(2,\mathbb{Z})$ (identified to a subgroup of $GL(3,\mathbb{R})$) and $h_{u,v}$, for $v|u$, is defined by

$$S_1(u,v)\varphi = h_{u,v}\varphi ,$$

where $S_1(u,v)$ is an appropriate element of the Hecke algebra. We prove also a converse theorem.

This paper has been to a very large extent inspired by the work of Gelfand, Jacquet, Kajdan, Pyatetskii-Shapiro, Shalika. Their work is expressed in the language of ideles, adels, group-representations. We have however tried to stay as close as possible to the "classical" point of view; thus our methods and language are very different, and in a sense elementary. We hope therefore this will be of some use to others.

§1. Preliminaries On Operators

Let $GL(3,\mathbb{R})$ be the multiplicative group of 3×3 matrices with real coefficients. Let Γ be $SL(3,\mathbb{Z})$. It is the discrete subgroup of $GL(3,\mathbb{R})$ consisting of integral matrices with determinant one.

By the Hecke Algebra we mean the Q-algebra generated by double cosets $\Gamma\alpha\Gamma$, where α is an integral matrix with positive determinant. As a consequence of the elementary divisor theorem on matrices, we may take for generators of this algebra the double cosets of the form $T(a,b,c) = \Gamma\begin{pmatrix} a & & \\ & b & \\ & & c\end{pmatrix}\Gamma$, where a,b,c are positive and $a|b|c$. For an integer $m \geq 1$, $T(m)$ is the sum of all double cosets $T(a,b,c)$ such that $abc = m$.

For $v|u$ we define an element $S(u,v)$ of the Hecke algebra by the following conditions.

1) $S(u,p^m) = T(u)T(p^m) - T(up)T(p^{m-1})$ if $p^m | u$, $m \geq 1$;

2) $S(u,1) = T(u)$

3) S is multiplicative, i.e.,

$$S(u,v)\cdot S(u',v') = S(uu',vv')$$

if $v|u$, $v'|u'$, $(u,u') = 1$.

We also define $S_1(u,v)$ by $S_1(u,v) = S(u,v)u^{-2}\cdot v^{-1}$ if $v|u$.

It will be convenient to set $S(u,v) = 0$, $S_1(u,v) = 0$ if $v \nmid u$ and similarly $T(m) = 0$ if $m > 0$ but not an integer.

Theorem 1: For $v|u$,

$$T(m)S(u,v) = \sum r^3 T(r,r,r)\, S\left(\frac{ut}{r}, \frac{vs}{r}\right)$$

the summation being for all positive integers r,s,t such that

$$rst = m,\ r|v,\ s\left|\frac{u}{v}\right. .$$

We divide the proof into several steps.

Step (i): Suppose $(m,u) = 1$. Then the formula to be proved reads

$$T(m)S(u,v) = S(um,v) \cdot$$

Since $S(m,1) = T(m)$, this is just a consequence of the multiplicativity of S .

Step (ii): Suppose the formula has been proved for 2 triples (m,u,v), (m',u',v') with $v|u$, $v'|u'$, $(m,m') = 1$, $(v,v') = 1$.

The multiplicativity of the operators T,S as well as $T(a,a,a)$ implies the formula for the triple (mm',uu',vv') .

Step (iii): Thus it suffices to prove the formula when u is a power of a prime p and m a power of a prime q . If $p \neq q$, one can apply Step (i). Thus we may assume $p = q$ in which case the formula reads:

Theorem 2: For all $m \geq 0$, $v \leq u$

$$(1.1.1) \qquad T(p^m)S(p^u,p^v) = \sum p^{3r}T(p,p,p)^r S(p^{u+m-2r-s},p^{v+s-r})$$

where the sum is for all pairs (r,s)

$$r \leq v \ , \ s \leq u - v \ , \ r + s \leq m \ .$$

Proof: By Theorem 3.21 of [6] we know that

$$\sum_{m\geq 0} T(p^m)X^m = (1 - T(p)X + pT(1,p,p)X^2 - p^3T(p,p,p)X^3)^{-1} \ .$$

We take 3 indeterminates α,β,γ such that

$$1 - T(p)X + pT(1,p,p)X^2 - p^3T(p,p,p)X^3 = (1 - \alpha X)(1 - \beta X)(1 - \gamma X)$$

or

$$\alpha + \beta + \gamma = T(p) = T(1,1,p) \ ,$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = pT(1,p,p) \ ,$$

$$\alpha\beta\gamma = p^3T(p,p,p) \ .$$

Then

$$\sum_{m\geq 0} T(p^m)X^m = [(1-\alpha X)(1-\beta X)(1-\gamma X)]$$

$$= \frac{A}{1-\alpha X} + \frac{B}{1-\beta X} + \frac{C}{1-\gamma X}$$

where

$$A = \frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)}, \quad B = \frac{\beta^2}{(\beta-\gamma)(\beta-\alpha)}, \quad C = \frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)} .$$

Thus we get two expressions for $T(p^m)$,

$$T(p^m) = \sum_{i+j+k=m} \alpha^i\beta^j\gamma^k$$

$$= -\frac{\alpha^{2+m}(\beta-\gamma) + \beta^{2+m}(\gamma-\alpha) + \gamma^{2+m}(\alpha-\beta)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} .$$

Using the second expression in the definition of

$$S(p^u,p^v) = T(p^u)T(p^v) - T(p^{u+1})T(p^{v-1}) , \quad (u \geq v) ,$$

we easily get

$$S(p^u,p^v) = -\frac{\alpha^{2+u}\beta^{1+v} - \alpha^{1+v}\beta^{2+u} + \beta^{2+u}\gamma^{1+v} - \beta^{1+v}\gamma^{2+u} + \gamma^{2+u}\alpha^{1+v} - \gamma^{1+v}\alpha^{2+u}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} .$$

Let π denote the circular permutation $(\alpha\beta\gamma)$. Then the symmetric polynomial as well as all other expressions in α,β,γ for $T(p^m)$ or $S(p^u,p^v)$ are invariant under π . $S(p^u,p^v)$ can be written in terms of α,β and π as follows:

$$S(p^u,p^v) = -\frac{(1+\pi+\pi^2)(\alpha^{2+u}\beta^{1+v} - \alpha^{1+v}\beta^{2+u})}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} .$$

The <u>right hand side</u> of the formula (1.1.1) to be proved is

$$= -\frac{(\alpha\beta\gamma)^r(1+\pi+\pi^2)(\alpha^{2+u+m-2r-s}\beta^{1+v+s-r} - \alpha^{1+v+s-r}\cdot\beta^{2+u+m-2r-s})}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

where $0 \leq r \leq v$, $0 \leq s \leq u-v$ and $r + s \leq m$.

(1.1.2)
$$= -\frac{\sum(1+\pi+\pi^2)(\alpha^{2+u+m-r-s}.\beta^{1+v+s}-\alpha^{1+v+s}.\beta^{2+u+m-r-s})\gamma^r}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

where $0 \le r \le v$, $0 \le s \le u-v$ and $r + s \le m$.

The <u>left hand side</u> of (1.1.1) is $T(p^m)S(p^u,p^v)$. The <u>numerator</u> of this is

$$= \Big(\sum_{i+j+k=m} \alpha^i\beta^j\gamma^k\Big)(1+\pi+\pi^2)(\alpha^{2+u}\beta^{1+v}-\alpha^{1+v}\beta^{2+u})$$

$$= (1+\pi+\pi^2)\Big[\Big(\sum_{i+j+k=m} \alpha^i\beta^j\gamma^k\Big)(\alpha^{2+u}\beta^{1+v}-\alpha^{1+v}\beta^{2+u})\Big] .$$

But we have

$$\Big(\sum_{i+j+k=m} \alpha^i\beta^j\gamma^k\Big)(\alpha^{2+u}\beta^{1+v}-\alpha^{1+v}\beta^{2+u})$$

$$= \Big(\sum_{r+s\le m} \alpha^{m-r-s}\beta^s\gamma^r\Big)\alpha^{2+u}\beta^{1+v}-\Big(\sum_{r+s\le m} \alpha^s\beta^{m-r-s}\gamma^r\Big)\alpha^{1+v}\beta^{2+u}$$

$$= \sum_{r+s\le m} (\alpha^{2+u+m-s-r}\beta^{1+v+s}-\alpha^{1+v+s}\beta^{2+u+m-s-r})\gamma^r \quad .$$

Hence the <u>left side</u> of (1.1.1) is

(1.1.3)
$$= -\frac{\sum(1+\pi+\pi^2)(\alpha^{2+u+m-r-s}\beta^{1+v+s}-\alpha^{1+v+s}\beta^{2+u+m-r-s})\gamma^r}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

where $r + s \le m$.

If $v > m$ and $u - v > m$, then the required conditions on r and s are trivially satisfied and the Theorem 2 follows from (1.1.2) and (1.1.3).

On the other hand, if $v < m$ or $u - v < m$ then the <u>part</u> of the above sum (1.1.3) where $r \ge v + 1$ or $s \ge u - v + 1$ respectively, vanishes. This is seen as follows.

Assume $v < m$ and $r \geq v + 1$.

The terms like $\alpha^{2+u+i}\beta^{1+v+j}\gamma^{r}$, $i + j + r = m$ appear in pairs with opposite signs in (1.1.3) and hence cancel.

For

$$\alpha^{2+u+i}\beta^{1+v+j}\gamma^{r} = (\alpha^{i}\beta^{j}\gamma^{r})(\alpha^{2+u}\beta^{1+v}) \ ;$$

$$\frac{\alpha^{2+u+i}\beta^{1+v+j}\gamma^{r}}{\alpha^{2+u}\gamma^{1+v}} = \alpha^{i}\beta^{(1+v)+j}\gamma^{r-(1+v)}$$

and $\alpha^{i}\beta^{(1+v)+j}\gamma^{r-(1+v)}$ is a term in the symmetric polynomial for $T(p^{m})$ while $-\gamma^{1+v}\alpha^{2+u}$ is a term in $S(p^{u},p^{v})$ which together give rise to

$$\alpha^{i}\beta^{(1+v)+j}\gamma^{r-(1+v)}(-\alpha^{2+u}\gamma^{1+v}) = -\alpha^{2+u+i}\beta^{1+v+j}\gamma^{r} \ .$$

Now assume that $u - v < m$ and $s \geq u - v + 1$.

Again, the terms like $\alpha^{2+u+i}\beta^{1+v+s}\gamma^{k}$, $i + s + k = m$ appear in pairs with opposite signs in (1.1.3) and hence cancel.

For

$$\alpha^{2+u+i}\beta^{1+v+s}\gamma^{k} = (\alpha^{i}\beta^{s}\gamma^{k})\alpha^{2+u}\beta^{1+v} \ , \ i + s + k = m \ ;$$

$$\frac{\alpha^{2+u+i}\beta^{1+v+s}\gamma^{k}}{\alpha^{1+v}\beta^{2+u}} = \alpha^{(1+u-v)+i}\beta^{s-(u-v+1)}\gamma^{k}$$

and $\alpha^{(1+u-v)+i}\beta^{s-(u-v+1)}\gamma^{k}$ is one of the terms in the expression for $T(p^{m})$ while $(-\alpha^{1+v}\beta^{2+u})$ is a term in the expression for $S(p^{u},p^{v})$ which together give rise to

$$\alpha^{(1+u-v)+i}\beta^{s-(u-v+1)}\gamma^{k}(-\alpha^{1+v}\beta^{2+u}) = -\alpha^{2+u+i}\beta^{1+v+s}\gamma^{k} \ .$$

Thus the formula (1.1.1) holds.

Q.E.D.

<u>Corollary</u>: <u>For</u> $v \mid u$, <u>we obtain the following relation for</u> $S_{1}(u,v)$:

$$T(m)S_{1}(u,v)u^{2}v = \sum r^{3}T(r,r,r)S_{1}\left(\frac{ut}{r} , \frac{vs}{r}\right)\left(\frac{ut}{r}\right)^{2}\left(\frac{vs}{r}\right)$$

<u>where</u> $rst = m$, $r \mid v$ and $s \mid \frac{u}{v}$. I.e., we have

$$(1.1.4) \qquad T(m)S_1(u,v) = \sum st^2 T(r,r,r) S_1\left(\frac{ut}{r}, \frac{vs}{r}\right)$$

where $rst = m$, $r \mid v$ and $s \mid \frac{u}{v}$.

§2. Fourier Series of a Cusp Form

2.1. First Expansion

Let φ be a C^∞-function on $GL(3,\mathbb{R})$ invariant by $\Gamma = SL(3,\mathbb{Z})$ on the left.

I.e., $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$.

We further assume that φ is invariant under the center of $GL(3,\mathbb{R})$.

I.e., $\varphi(\alpha g) = \varphi(g)$, $\alpha = \begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix}$, $a \in \mathbb{R}$.

Remark: Any 3×3 matrix in $GL(3,\mathbb{Z})$ be written as $\pm\gamma$, $\gamma \in SL(3,\mathbb{Z})$. So φ is invariant under $\Gamma' = GL(3,\mathbb{Z})$.

We also assume that the function φ satisfies the following conditions of "cuspidality".

$$\int_{(\mathbb{R}/\mathbb{Z})^2} \varphi\left[\begin{pmatrix} 1 & x & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} g\right] dxdz = 0 \quad ,$$

and

$$\int_{(\mathbb{R}/\mathbb{Z})^2} \varphi\left[\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix} g\right] dydz = 0 \quad .$$

A "cusp form" φ on $GL(3,\mathbb{R})$ is essentially a function satisfying these requirements.

Consider the subgroup U of $GL(3,\mathbb{R})$ defined by

$$U = \left\{\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix} : z,y \in \mathbb{R}\right\} \quad .$$

The function $u \to \varphi(ug)$ may be regarded as a function on the group $U/\Gamma\cap U \cong (\mathbb{R}/\mathbb{Z})^2$. We expand φ as a double Fourier Series.

We get

$$\varphi(g) = \sum_{(m,n)\neq(0,0)} W_{(m,n)}[g] \quad , \tag{2.1.1}$$

where

$$W_{(m,n)}[g] = \int_{(\mathbb{R}/\mathbb{Z})^2} \varphi\left[\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix} g\right] \exp(-2i\pi(mz+ny))\,dydz\ .$$

By direct computation we obtain

$$W_{(m,n)}\left[\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix} g\right] = \exp(2i\pi(mz+ny))W_{(m,n)}[g]\ .$$

Since $\varphi \in C^\infty(GL(3,\mathbb{R}))$, the Fourier Series converges absolutely and uniformly on compact sets to $\varphi(g)$, $g \in GL(3\mathbb{R})$. (See [5]).

Let $(m,n) = d$. I.e., $m = dc_1$, $n = dd_1$, $(c_1,d_1) = 1$ and $d \geq 1$.

Let Γ_2 be the group of 2X2 integral matrices of determinant one.

If $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ such that $a_1d_1 - b_1c_1 = 1$, $\gamma \in \Gamma_2$, then $(m,n) = d(0,1)\gamma$. Note that $d(0,1)\gamma = d(0,1)\gamma'$ if and only if $\gamma' = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\gamma$; i.e., $\gamma' \in (N \cap \Gamma_2)\gamma$ where $N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\right\}$.

Hence we have

$$\varphi(g) = \sum_{\gamma \in N \cap \Gamma_2 \backslash \Gamma_2} W_{d(0,1)\gamma}[g]$$

where

$$W_{d(0,1)\gamma}[g] = \int_{(\mathbb{R}/\mathbb{Z})^2} \varphi\left[\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix} g\right] \exp\left(-2i\pi d(0,1)\gamma\begin{pmatrix} z \\ y \end{pmatrix}\right) dzdy\ .$$

We replace $\begin{pmatrix} z \\ y \end{pmatrix}$ by $\gamma^{-1}\begin{pmatrix} z \\ y \end{pmatrix}$. Then dzdy is not changed and we have the substitution:

$$\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \gamma^{-1} & \begin{pmatrix} z \\ y \end{pmatrix} \\ & 1 & & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \gamma^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix}\begin{pmatrix} \gamma & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

If there is no risk of confusion, we identify γ and $\begin{pmatrix} \gamma & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for all $\gamma \in \Gamma_2$.

Since φ is invariant under $\Gamma_2 \subset \Gamma$,

$$\varphi\left[\gamma^{-1}\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix}\gamma g\right] = \varphi\left[\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix}\gamma g\right] .$$

Thus

$$(2.1.2) \qquad \varphi(g) = \sum_{\substack{\gamma\in N\cap\Gamma_2\backslash\Gamma_2 \\ d\geq 1}} W_{(0,d)\gamma}[g] = \sum_{\substack{\gamma\in N\cap\Gamma_2\backslash\Gamma_2 \\ d\geq 1}} W_{(0,d)}[\gamma g]$$

where

$$W_{(0,d)}[g] = \int_{(\mathbb{R}/\mathbb{Z})^2} \varphi\left[\begin{pmatrix}1 & 0 & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right] \exp(-2i\pi dy)\,dz\,dy$$

and

$$W_{(0,d)}[\gamma g] = W_{(0,d)\gamma}[g] \ , \quad \gamma \in N\cap\Gamma_2\backslash\Gamma_2 \quad .$$

Note that

$$W_{(0,d)}\left[\begin{pmatrix}-1 & & \\ & -1 & \\ & & 1\end{pmatrix} g\right] = W_{(0,d)\begin{pmatrix}-1 & \\ & -1\end{pmatrix}}[g] = W_{(0,-d)}[g] \ .$$

Remark: We could also use a Fourier expansion on the group

$$V = \left\{\begin{pmatrix}1 & x & z\\ & 1 & 0\\ & & 1\end{pmatrix} , \quad x,z \in \mathbb{R}\right\} \quad .$$

2.2 Second Expansion

We have

$$W_{(0,d)}\left[\begin{pmatrix}1 & n & 0\\ & 1 & 0\\ & & 1\end{pmatrix} g\right] = W_{(0,d)}[g] \ , \quad n \in \mathbb{Z} \quad .$$

Let $V' = \left\{\begin{pmatrix}1 & x & 0\\ & 1 & 0\\ & & 1\end{pmatrix} , \ x \in \mathbb{R}\right\}$. Then $v \rightarrow W_{(0,d)}[vg]$ is a function on the group $V'/\Gamma\cap V' \cong \mathbb{R}/\mathbb{Z}$; and $W_{(0,d)}$ has also a Fourier expansion

$$W_{(0,d)}[g] = \sum_{\alpha\neq 0} W_d^\alpha[g]$$

where

$$W_d^\alpha[g] = \int_{\mathbb{R}/\mathbb{Z}} W_{(0,d)}\left[\begin{pmatrix}1 & x & 0\\ & 1 & 0\\ & & 1\end{pmatrix} g\right] \exp(-2i\pi\alpha x)\,dx ;$$ $W_d^0[g] = 0$ by the first cuspidality condition.

Now for the upper triangular group

$$\mathcal{N} = \left\{\begin{pmatrix}1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix} : \ x,y,z \in \mathbb{R}\right\}$$

dxdydz is an invariant Haar measure; while

$$\begin{pmatrix}1&x&0\\&1&y\\&&1\end{pmatrix}\begin{pmatrix}1&m&0\\&1&n\\&&1\end{pmatrix}=\begin{pmatrix}1&x+m&nx\\&1&y+n\\&&1\end{pmatrix}, \quad m,n\in\mathbb{Z}$$

and

$$\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}\begin{pmatrix}1&0&p\\&1&0\\&&1\end{pmatrix}=\begin{pmatrix}1&x&z+p\\&1&y\\&&1\end{pmatrix}, \quad p\in\mathbb{Z}$$

imply that the domain defined by

$$0\le x<1 \qquad 0\le y<1 \qquad 0\le z<1$$

is a fundamental domain for the $\mathcal{N}\cap\Gamma$-cosets of $\mathcal{N}$.

By Fubini's theorem and using the integral expression for $W_{(0,d)}$ from (2.1.2) we write

$$W_d^{\alpha}[g] = \int_{\mathbb{R}/\mathbb{Z}}\left(\int_{(\mathbb{R}/\mathbb{Z})^2}\varphi\left[\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}g\right]\exp(-2i\pi dy)\,dydz\right)\exp(-2i\pi\alpha x)\,dx$$

$$= \int_{(\mathbb{R}/\mathbb{Z})^3}\varphi\left[\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}g\right]\exp(-2i\pi(dy+\alpha x))\,dxdydz$$

$$= \int_{\mathcal{N}/\mathcal{N}\cap\Gamma}\varphi\left[\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}g\right]\exp(-2i\pi(dy+\alpha x))\,dxdydz \quad .$$

By direct computation,

$$W_d^{\alpha}\left[\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}g\right]=\exp(2i\pi(\alpha x+dy))\,W_d^{\alpha}[g] \ .$$

Note that if $\epsilon_1=\pm 1$, $\epsilon_2=\pm 1$, then

$$W_d^{\alpha}\left[\begin{pmatrix}\epsilon_1&&\\&\epsilon_2&\\&&1\end{pmatrix}g\right]=W_{\epsilon_2 d}^{\epsilon_1\epsilon_2\alpha}[g] \ .$$

For

$$W_d^{\alpha}\left[\begin{pmatrix}\epsilon_1&&\\&\epsilon_2&\\&&1\end{pmatrix}g\right]=\int_{(\mathbb{R}/\mathbb{Z})^3}\varphi\left[\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}\begin{pmatrix}\epsilon_1&&\\&\epsilon_2&\\&&1\end{pmatrix}g\right]\exp(-2i\pi(\alpha x+dy))\,dxdydz \ .$$

But

$$\begin{pmatrix}1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix}\begin{pmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & 1\end{pmatrix} = \begin{pmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & \epsilon_1\epsilon_2 x & \epsilon_1 z\\ & 1 & \epsilon_2 y\\ & & 1\end{pmatrix}$$

and

$$\begin{aligned}\varphi\left[\begin{pmatrix}1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix}\begin{pmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & 1\end{pmatrix} g\right] &= \varphi\left[\begin{pmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & \epsilon_1\epsilon_2 x & \epsilon_1 z\\ & 1 & \epsilon_2 y\\ & & 1\end{pmatrix} g\right]\\ &= \varphi\left[\begin{pmatrix}1 & \epsilon_1\epsilon_2 x & \epsilon_1 z\\ & 1 & \epsilon_2 y\\ & & 1\end{pmatrix} g\right]\end{aligned}$$

by the invariance of φ under Γ'. We replace $\epsilon_1\epsilon_2 x$ by x, $\epsilon_1 z$ by z and $\epsilon_2 y$ by y. Then dxdydz remains unchanged. Hence

$$W_d^\alpha\left[\begin{pmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & 1\end{pmatrix} g\right] = \int_{(\mathbb{R}/\mathbb{Z})^3} \varphi\left[\begin{pmatrix}1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right]\exp(-2i\pi(\epsilon_1\epsilon_2\alpha x + \epsilon_2 dy))\,dxdydz\,.$$

Thus

$$W_{(0,d)}[g] = \sum_{\substack{\alpha\geq 1\\ \epsilon=1,-1}} W_d^\alpha\left[\begin{pmatrix}\epsilon & & \\ & 1 & \\ & & 1\end{pmatrix} g\right]$$

Combining this with the previous expansion (2.1.2) we get

$$\text{(2.2.1)}\qquad \varphi(g) = \sum W_d^\alpha\left[\begin{pmatrix}\epsilon & & \\ & 1 & \\ & & 1\end{pmatrix}\gamma g\right]$$

where $\alpha \geq 1$, $d \geq 1$, $\epsilon = \pm 1$, $\gamma \in N \cap \Gamma_2\backslash\Gamma_2$.

Let $\Gamma'_2 = GL(2,\mathbb{Z})$. We identify it with a subgroup of Γ. Then this is also

$$\text{(2.2.2)}\qquad \varphi(g) = \sum_{\substack{\alpha\geq 1\\ d\geq 1}} W_d^\alpha[\gamma g] \qquad \gamma \in N \cap \Gamma'_2\backslash\Gamma'_2\,.$$

We know that the multiple series is uniformly convergent on compact sets. See [5].

Remark: Such an expansion is unique. Suppose φ is also given by:

$$\varphi(g) = \sum W'^{\alpha}_{d}[\gamma g]$$

where W'^{α}_{d} satisfies

$$W'^{\alpha}_{d}\left[\begin{pmatrix}1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right] = \exp(2i\pi(\alpha x+dy))W'^{\alpha}_{d}[g]\ , \quad \text{and}$$

$\alpha \geq 1$, $d \geq 1$, $\gamma \in N \cap \Gamma'_2\backslash\Gamma'_2$. Then for $b > 0$

$$W_{(0,b)}[g] = \int \varphi\left[\begin{pmatrix}1 & 0 & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right] \exp(-2i\pi\ by)dydz.$$

But

$$W'^{\alpha}_{d}\left[\gamma\begin{pmatrix}1 & 0 & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right] = \exp\left(2i\pi(0,d)\gamma\begin{pmatrix}z\\ y\end{pmatrix}\right)\ W'^{\alpha}_{d}[\gamma g]$$

$$= \exp(2i\pi(\gamma_3 dz + \gamma_4 dy))\ W'^{\alpha}_{d}[\gamma g]$$

where $\gamma = \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{pmatrix}$. Replacing φ by its second expansion in the integral,

$$W_{(0,b)}[g] = \int_{(\mathbb{R}/\mathbb{Z})^2} \sum W'^{\alpha}_{d}\left[\gamma\begin{pmatrix}1 & 0 & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right] \exp(-2i\pi\ by)dydz$$

$$= \int_{(\mathbb{R}/\mathbb{Z})^2} \sum W'^{\alpha}_{d}[\gamma g]\exp(2i\pi\gamma_3 dz)\exp(2i\pi(\gamma_4 d-b)y)dydz$$

$$= \sum_{\alpha\geq 1, d\geq 1} \ \sum_{\gamma \in N \cap \Gamma'_2\backslash\Gamma'_2} W'^{\alpha}_{d}[\gamma g] \int_{(\mathbb{R}/\mathbb{Z})^2} \exp(2i\pi\gamma_3 dz)\exp(2i\pi(\gamma_4 d-b)y)dydz\ .$$

By the property of exp , all terms vanish except those for which

$$\gamma_3 = 0 \quad \text{and} \quad \gamma_4 d = b\ .$$

It follows that $\gamma_4 > 0$ and hence $\gamma_4 = 1$. We may take γ to be 1. We also have $d = b$. Thus

$$W_{(0,b)}[g] = \sum_{\alpha\neq 0} W'^{\alpha}_{b}[g]$$

and then $W^{\alpha}_{b} = W'^{\alpha}_{b}$ by the Fourier theory on the subgroup V' .

§3. Action of the Hecke Operators on a Cusp-Form

We fix an integer $m \geq 1$.

We recall that the Hecke operator $T(m)$ is the sum of all double cosets $\Gamma\begin{pmatrix} a & & \\ & b & \\ & & c\end{pmatrix}\Gamma$ where a,b,c are positive, $a|b|c$ and $abc = m$. We can express a double coset as a union of disjoint cosets of the form $\Gamma\alpha$. The following lemma is well known. For lack of a convenient reference we give a proof.

Lemma 1: A set of representatives of disjoint cosets $\Gamma\alpha$ for $T(m)$ is

$$S_m = \left\{\begin{pmatrix} r & u & w \\ & s & v \\ & & t\end{pmatrix} : rst = m \ ; \ u \in \mathbb{Z}/s \ ; \ v,w \in \mathbb{Z}/t\right\} \ .$$

Proof: Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{pmatrix} \quad , \quad \det \alpha = m \ .$$

We can express α as

$$\alpha = \gamma \cdot \begin{pmatrix} r & u & w \\ & s & v \\ & & t\end{pmatrix} \quad , \quad rst = m \text{ and } \gamma \in \Gamma \ .$$

For instance,

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33}\end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{pmatrix} = \begin{pmatrix} * & * & * \\ a_1x_{21}+b_1x_{22}+c_1x_{23} & * & * \\ a_1x_{31}+b_1x_{32}+c_1x_{33} & a_2x_{31}+b_2x_{32}+c_2x_{33} & *\end{pmatrix}$$

Here we can choose $x_{21} = 0$, $x_{31} = 1$. We find x_{22}, x_{23} such that $b_1x_{22}+c_1x_{23} = 0$ and coprime, while x_{32}, x_{33} as solutions of the system of equations

$$b_1x_{32} + c_1x_{33} = -a_1 \ ; \ b_2x_{32} + c_2x_{33} = -a_2 \ .$$

The remaining coefficients x can be found such that the matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 1 & x_{32} & x_{33} \end{pmatrix} \quad \text{has determinant } 1 .$$

We may take γ to be the inverse of this matrix.

This shows that we can take upper triangular matrices of the form $\begin{pmatrix} r & u & w \\ & s & v \\ & & t \end{pmatrix}$, $rst = m$ for representatives for the cosets $\Gamma\alpha$ of $T(m)$.

Furthermore we note that

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}\begin{pmatrix} r & u & w \\ & s & v \\ & & t \end{pmatrix} = \begin{pmatrix} r & u+xs & w+xv+zt \\ & s & v+yt \\ & & t \end{pmatrix} ,$$

i.e., u varies modulo s, v varies modulo t and w varies modulo t.

Thus a full set of representatives for disjoint cosets $\Gamma\alpha$ is given by the set S_m and Lemma 1 follows.

Let φ be a cusp-form as defined in §2. Let $\varphi' = T(m)\varphi$, defined by

$$\varphi'(g) = \sum \varphi(\Gamma\alpha g)$$

where

$$T(m) = \underset{\substack{\alpha \\ \text{(disjoint)}}}{\cup} \Gamma\alpha .$$

The number of disjoint right cosets being finite the above definition makes sense. Moreover, φ' is a cusp form. We can also write

$$\text{(3.1.1)} \qquad \varphi'(g) = \sum_{\substack{rst=m \\ u\in\mathbb{Z}/s \;;\; v,w\in\mathbb{Z}/t}} \varphi\left[\begin{pmatrix} r & u & w \\ & s & v \\ & & t \end{pmatrix} g\right] .$$

We replace the <u>right hand side</u> of (3.1.1) by the first expansion for φ as in (2.1.1).

$$\varphi'(g) = \sum_{\substack{rst=m \\ u,v,w}} \sum_{(c,d)\neq(0,0)} W_{(c,d)} \left[\begin{pmatrix} r & u & w \\ & s & v \\ & & t \end{pmatrix} g \right]$$

$$= \sum\sum W_{(c,d)} \left[\begin{pmatrix} 1 & 0 & w/t \\ & 1 & v/t \\ & & 1 \end{pmatrix} \begin{pmatrix} r & u & 0 \\ & s & 0 \\ & & t \end{pmatrix} g \right]$$

$$= \sum\sum W_{(c,d)} \left[\begin{pmatrix} r & u & 0 \\ & s & 0 \\ & & t \end{pmatrix} g \right] \exp(2i\pi(\tfrac{w}{t}\, c + \tfrac{v}{t}\, d))$$

$$= \sum_{(c,d)\neq(0,0)} \sum_{\substack{rst=m \\ u\in\mathbb{Z}/s}} W_{(c,d)} \left[\begin{pmatrix} r & u & 0 \\ & s & 0 \\ & & t \end{pmatrix} g \right] \Big(\sum_{v,w\in\mathbb{Z}/t} \exp(2i\pi(c\, \tfrac{w}{t} + d\, \tfrac{v}{t})) \Big) .$$

But $\sum \exp(2i\pi(c\,\frac{w}{t} + d\,\frac{v}{t})) = t^2$ if $t\,|\,(c,d)$

$= 0$ otherwise.

We get

(3.1.2) $$\varphi'(g) = \sum_{\substack{rst=m \\ u\in\mathbb{Z}/s \\ t|(c,d)}} t^2 W_{(c,d)} \left[\begin{pmatrix} r & u & 0 \\ & s & 0 \\ & & t \end{pmatrix} g \right] .$$

Since φ' is a cusp form, it has a Fourier series expansion of the form (2.1.2). Thus

$$\varphi'(g) = \sum_{\substack{\gamma\in N\cap\Gamma_2\backslash\Gamma_2 \\ b\geq 1}} W'_{(0,b)} [\gamma g]$$

where

$$W'_{(0,b)}[g] = \int_{(\mathbb{R}/\mathbb{Z})^2} \varphi' \left[\begin{pmatrix} 1 & 0 & z \\ & 1 & y \\ & & 1 \end{pmatrix} g \right] \exp(-2i\pi by)dydz .$$

Using (3.1.2) in this integral we have

$$W'_{(0,b)}[g] = \int_{(\mathbb{R}/\mathbb{Z})^2} \sum t^2 W_{(c,d)}\left[\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix}\begin{pmatrix} 1 & 0 & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right] \exp(-2i\pi by)\,dy\,dz \;. \tag{3.1.3}$$

But

$$\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix}\begin{pmatrix} 1 & 0 & z\\ & 1 & y\\ & & 1\end{pmatrix} = \begin{pmatrix} r & u & rz+uy\\ & s & sy\\ & & t\end{pmatrix} = \begin{pmatrix} 1 & 0 & (rz+uy)/t\\ & 1 & sy/t\\ & & 1\end{pmatrix}\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix}$$

and

$$W_{(c,d)}\left[\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix}\begin{pmatrix} 1 & 0 & z\\ & 1 & y\\ & & 1\end{pmatrix} g\right] = \exp(2i\pi((rz+uy)ct^{-1} + sydt^{-1}) \, W_{(c,d)}\left[\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix} g\right] \;.$$

On account of uniform convergence of the series, we may interchange summation and integration in (3.1.3) and obtain

$$W'_{(0,b)}[g] = \sum_{\substack{rst=m\\ u\in\mathbb{Z}/s\\ t\mid(c,d)}} t^2 W_{(c,d)}\left[\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix} g\right]\cdot\int_{(\mathbb{R}/\mathbb{Z})^2} \exp(2i\pi(\frac{rzc}{t} + (\frac{uc}{t} + \frac{sd}{t} - b)y))\,dy\,dz.$$

All terms vanish except those for which $c = 0$ and $\frac{sd}{t} = b$.

We have

$$W'_{(0,b)}[g] = \sum_{\substack{rst=m\\ u\in\mathbb{Z}/s\\ t\mid d:\ sd/t = b}} t^2\, W_{(0,d)}\left[\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix} g\right] \;.$$

We now use the Fourier series for $W_{(0,d)}$. We get

$$\sum_{u\in\mathbb{Z}/s} W_{(0,d)}\left[\begin{pmatrix} r & u & 0\\ & s & 0\\ & & t\end{pmatrix} g\right] = \sum_{\substack{\alpha\neq 0\\ u\in\mathbb{Z}/s}} W_d^{\alpha}\left[\begin{pmatrix} 1 & ru/s & 0\\ & 1 & 0\\ & & 1\end{pmatrix}\begin{pmatrix} r & & \\ & s & \\ & & t\end{pmatrix} g\right]$$

$$= \sum_{\alpha\neq 0}\ \sum_{u\in\mathbb{Z}/s} \exp(2i\pi \frac{ru}{s}\,\alpha) W_d^{\alpha}\left[\begin{pmatrix} r & & \\ & s & \\ & & t\end{pmatrix} g\right] \;.$$

Since $$\sum_{u\in\mathbb{Z}/s} \exp(2i\pi \frac{ru}{s}\,\alpha) \begin{array}{ll} = s & \text{if } s|\alpha \\ = 0 & \text{otherwise,} \end{array}$$

the following relation holds.

$$(3.1.4) \qquad W'_{(0,b)}[g] = \sum_{\substack{rst=m\\ t|d:sd/t = b\\ s|\alpha}} st^2\, W_d^\alpha\left[\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix} g\right] .$$

We now consider the Fourier Series for $W'_{(0,b)}$.

$$W'^{a}_{b}[g] = \int_{\mathbb{R}/\mathbb{Z}} W'_{(0,b)}\left[\begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} g\right] \exp(-2i\pi ax)dx \quad .$$

In this integral we use (3.1.4) and then interchange the order of summation and integration. Thus

$$W'^{a}_{b}[g] = \sum_{r,s,t} st^2 \int W_d^\alpha\left[\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} g\right]\exp(-2i\pi ax)dx \; .$$

But

$$\begin{pmatrix} r & 0 & 0 \\ & s & 0 \\ & & t \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & rx/s & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix}$$

and

$$W_d^\alpha\left[\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} g\right] = W_d^\alpha\left[\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix} g\right]\cdot\exp(2i\pi \frac{r}{s}\, x\alpha) \; .$$

This gives

$$W'^{a}_{b}[g] = \sum_{\substack{rst=m\\ t|d:\ sd/t = b\\ s|\alpha}} st^2\, W_d^\alpha\left[\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix} g\right] \int_{\mathbb{R}/\mathbb{Z}} \exp(2i\pi(\frac{rx}{s}\,\alpha - ax))dx$$

$$= \sum_{\substack{r.s.t=m\\ t|d:sd/t=b \text{ and } s|\alpha:\ \frac{\alpha}{s}r = a}} st^2\, W\,{}_d^\alpha\left[\begin{pmatrix} r & & \\ & s & \\ & & t \end{pmatrix} g\right] \qquad ,$$

since other terms vanish. Thus we have

Theorem 3: *If* $\varphi' = T(m)\varphi$; *then the corresponding functions* W'^a_b *is given by*

$$W'^a_b[g] = \sum st^2 W^\alpha_d\left[\begin{pmatrix} r & s \\ & t \end{pmatrix} g\right] \tag{3.1.5}$$

the sum being for $rst = m$; $t|d$, $s\frac{d}{t} = b$; *and* $s|\alpha$, $r \cdot \frac{\alpha}{s} = a$.

Corollary: *We assume that* φ *is an eigenfunction of* $T(m)$, i.e., *there exists a complex number* c_m *such that* $T(m)\varphi(g) = c_m\varphi(g)$. *Then we have*

$$c_m W^a_b[g] = \sum_{\substack{rst=m \\ t|d\ :\ (\frac{d}{t})\cdot s = b \\ s|\alpha\ :\ (\frac{\alpha}{s})\cdot r = a}} st^2 W^\alpha_d\left[\begin{pmatrix} r & s \\ & t \end{pmatrix} g\right] . \tag{3.1.6}$$

Proof: follows immediately from the fact that

$$W'^a_b[g] = c_m W^a_b[g] ;$$

and from (3.1.5).

Remark: The following 2 particular cases of (3.1.6) will be used in the later sections.

Case i): *If* $a = 1$, *then* $r = 1$, $\alpha = s$ *and*

$$c_m W^1_b[g] = \sum_{\substack{st=m \\ t|d\ :\ (\frac{d}{t})\cdot s = b}} st^2 W^s_d\left[\begin{pmatrix} 1 & s \\ & t \end{pmatrix} g\right] . \tag{3.1.7}$$

Case ii): *If* $(a,m) = 1$, *then* $r = 1$, $\alpha = sa$ *and*

$$c_m W^a_b[g] = \sum_{\substack{st=m \\ t|d\ :\ (\frac{d}{t})\cdot s = b}} st^2 W^{sa}_d\left[\begin{pmatrix} 1 & s \\ & t \end{pmatrix} g\right] . \tag{3.1.8}$$

§4. The Main Theorem

4.1. Action of the Operator $S_1(ab,a)$

We assume that the cusp form φ is a common eigenfunction for all the operators $T(m)$, $m \geq 1$. Because these operators generate the full Hecke algebra, φ is also seen to be an eigenfunction of $S_1(ab,a)$. Let c_m, $h_{ab,a}$ be the eigenvalues, i.e.,

$$S_1(ab,a)\varphi(g) = h_{ab,a}\varphi(g) \ , \qquad h_{ab,a} \in \mathbb{C} \ .$$

Using the identity (1.1.4) for $\varphi(g)$ we obtain

$$c_m h_{ab,a}\varphi(g) = \sum st^2 \ h_{abt/r,as/r} \ T(r,r,r)\varphi(g) \ .$$

By the invariance of φ under the center of $GL(3,\mathbb{R})$,

$$c_m h_{ab,a}\varphi(g) = \sum st^2 \ h_{abt/r,as/r} \ \varphi(g) \quad ,$$

where the sum is taken over all triples (r,s,t) such that $rst = m$, $r|a$ and $s|b$.

Recall that $S_1(m,1) = \frac{T(m)}{m^2}$. Hence $h_{m,1} = \frac{c_m}{m^2}$. Let $\alpha = as/r$ and $d = bt/s$. Then the above identity can be written as follows.

$$(4.1.1) \quad m^2 h_{m,1} \ h_{ab,a} = \sum \ st^2 \ h_{\alpha d,\alpha}$$

where $rst = m$, $t|d$: $(d/t)\cdot s = b$, $s|\alpha$: $(\alpha/s)r = a$.

4.2. The Main Theorem

We are now able to prove:

The Main Theorem:

$$W_b^a[g] = h_{ab,a} \ W_1^1\left[\begin{pmatrix} ab & & \\ & b & \\ & & 1 \end{pmatrix} g\right] \ .$$

Proof: For $a = 1$ the statement is

$$W_b^1[g] = h_{b,1} \ W_1^1\left[\begin{pmatrix} b & & \\ & b & \\ & & 1 \end{pmatrix} g\right] \ .$$

In (3.1.7) we set $b = 1$. Then we get $s = 1$, $t = m$ and

$$c_m W_1^1[g] = m^2 W_m^1\left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & m \end{pmatrix} g\right] .$$

Hence,

$$c_m W_1^1\left[\begin{pmatrix} m & & \\ & m & \\ & & 1 \end{pmatrix} g\right] = m^2 W_m^1\left[\begin{pmatrix} m & & \\ & m & \\ & & m \end{pmatrix} g\right] .$$

Since W_m^1 is invariant under the center and $c_m/m^2 = h_{m,1}$, the theorem follows for $a = 1$.

In general, the formula to be proved is:

$$W^v_{uv^{-1}}[g] = h_{u,v} W_1^1\left[\begin{pmatrix} u & 0 & 0 \\ 0 & u/v & 0 \\ 0 & 0 & 1 \end{pmatrix} g\right] , \text{ where } v|u .$$

We will prove this by induction on the number of primes which divide v. In other words, let u,v be two integers, p a prime: suppose $v|u$, $p \nmid u$, $p \nmid v$. We may assume that for all $\alpha \geq 0$,

$$W^v_{uv^{-1}p^\alpha}[g] = h_{up^\alpha,v} W_1^1\left[\begin{pmatrix} up^\alpha & & \\ & uv^{-1}p^\alpha & \\ & & 1 \end{pmatrix} g\right] .$$

We have to show that for all β, $0 \leq \beta \leq \alpha$,

$$W^{vp^\beta}_{uv^{-1}\cdot p^{\alpha-\beta}}[g] = h_{up^\alpha, vp^\beta} W_1^1\left[\begin{pmatrix} up^\alpha & & \\ & uv^{-1}p^{\alpha-\beta} & \\ & & 1 \end{pmatrix} g\right] . \tag{4.2.1}$$

This we prove by induction on β. The formula being true for $\beta = 0$, we may assume $\alpha \geq \beta > 0$ and the formula true for $0 \leq i \leq \beta - 1$.

From (3.1.8) we get:

$$\begin{aligned} c_{p^\beta} W^v_{uv^{-1}p^\alpha}[g] &= \sum_{\substack{st=p^\beta \\ (d/t)\cdot s = uv^{-1}p^\alpha}} st^2 W_d^{sv}\left[\begin{pmatrix} 1 & & \\ & s & \\ & & t \end{pmatrix} g\right] \\ &= \sum_{i=0}^{\beta-1} p^i (p^{\beta-i})^2 W^{vp^i}_{uv^{-1}p^{\alpha+\beta-2i}}\left[\begin{pmatrix} 1 & & \\ & p^i & \\ & & p^{\beta-i} \end{pmatrix} g\right] \\ &\quad + p^\beta W^{vp^\beta}_{uv^{-1}p^{\alpha-\beta}}\left[\begin{pmatrix} 1 & & \\ & p^\beta & \\ & & 1 \end{pmatrix} g\right] . \end{aligned} \tag{4.2.2}$$

Now, we have by induction hypothesis on β

$$W^{vp^i}_{uv^{-1}p^{\alpha+\beta-2i}}[g] = h_{up^{\alpha+\beta-i},vp^i}\, W_1^1\left[\begin{pmatrix} & & uvp^{\alpha+\beta} \\ & up^{\alpha+\beta-i} & \\ vp^i & & \end{pmatrix} g\right]$$

and

$$W^{v}_{uv^{-1}p^{\alpha}}[g] = h_{up^{\alpha},v}\, W_1^1\left[\begin{pmatrix} & & uvp^{\alpha} \\ & up^{\alpha} & \\ v & & \end{pmatrix} g\right] .$$

Since $(u,p) = 1$

$$h_{up^{\alpha+\beta-i},vp^i} = h_{u,v}\, h_{p^{\alpha+\beta-i},p^i} , \qquad i \ge 0 .$$

Also by definition of h,

$$h_{p^{\alpha+\beta-i},p^i} = \frac{c_{p^{\alpha+\beta-i}}\, c_{p^i} - c_{p^{\alpha+\beta-i+1}}\, c_{p^{i-1}}}{p^{2(\alpha+\beta)-i}} .$$

Thus,

$$\sum_{i=0}^{\beta-1} p^{2\beta-i}\, W^{vp^i}_{uv^{-1}p^{\alpha+\beta-2i}}\left[\begin{pmatrix} 1 & & \\ & p^i & \\ & & p^{\beta-i} \end{pmatrix} g\right]$$

$$= \sum_{i=0}^{\beta-1} p^{2\beta-i}\, h_{u,v}\, \frac{c_{p^{\alpha+\beta-i}}\cdot c_{p^i} - c_{p^{\alpha+\beta-i+1}}\cdot c_{p^{i-1}}}{p^{2(\alpha+\beta)-i}}$$

$$\cdot\, W_1^1\left[\begin{pmatrix} & & uvp^{\alpha} \\ & up^{\alpha} & \\ v & & \end{pmatrix} g\right] ,$$

using invariance under the center,

$$= \frac{h_{u,v}}{p^{2\alpha}}\left[\, c_{p^{\alpha+1}}\, c_{p^{\beta-1}}\right] W_1^1\left[\begin{pmatrix} & & uvp^{\alpha} \\ & up^{\alpha} & \\ v & & \end{pmatrix} g\right] .$$

From (4.2.2) we have

$$p^{\beta}\, W^{vp^{\beta}}_{uv^{-1}p^{\alpha-\beta}}\left[\begin{pmatrix} 1 & & \\ & p^{\beta} & \\ & & 1 \end{pmatrix} g\right] = c_{p^{\beta}} \cdot h_{u,v}\, \frac{c_{p^{\alpha}}}{p^{2\alpha}}\, W_1^1\left[\begin{pmatrix} & & uvp^{\alpha} \\ & up^{\alpha} & \\ v & & \end{pmatrix} g\right]$$

$$- h_{u,v}\, \frac{c_{p^{\alpha+1}}}{p^{2\alpha}}\, c_{p^{\beta-1}}\, W_1^1\left[\begin{pmatrix} & & uvp^{\alpha} \\ & up^{\alpha} & \\ v & & \end{pmatrix} g\right] .$$

Hence, $$W^{vp^\beta}_{uv^{-1}p^{\alpha-\beta}}\left[\begin{pmatrix}1 & p^\beta \\ & 1\end{pmatrix} g\right] = h_{u,v}\,\frac{c_{p^\alpha}\,c_{p^\beta} - c_{p^{\alpha+1}}\,c_{p^{\beta-1}}}{p^{2\alpha+\beta}} \cdot W^1_1\left[\begin{pmatrix}uvp^\alpha & up^\alpha \\ & v\end{pmatrix} g\right]$$

i.e., $$W^{vp^\beta}_{uv^{-1}p^{\alpha-\beta}}[g] = h_{u,v}\,h_{p^\alpha,p^\beta}\,W^1_1\left[\begin{pmatrix}uvp^{\alpha+\beta} & up^\alpha \\ & vp^\beta\end{pmatrix} g\right]$$

$$= h_{up^\alpha,vp^\beta}\,W^1_1\left[\begin{pmatrix}up^\alpha & uv^{-1}p^{\alpha-\beta} \\ & 1\end{pmatrix} g\right] ,$$

by using the multiplicativity of h and invariance under the center. Hence we have formula (4.2.1), in other words

$$W^v_{uv^{-1}}[g] = h_{u,v}\,W^1_1\left[\begin{pmatrix}u & uv^{-1} \\ & 1\end{pmatrix} g\right] , \text{ for all } v|u ,$$

i.e., $$W^a_b[g] = h_{ab,a}\,W^1_1\left[\begin{pmatrix}ab & b \\ & 1\end{pmatrix} g\right] .$$

Q.E.D.

§5. Converse

Theorem: *Suppose* φ *is a cusp form and has a Fourier expansion*

$$\varphi(g) = \sum_{\substack{\alpha\geq 1, d\geq 1 \\ \gamma\in N\cap\Gamma_2'\backslash\Gamma_2'}} h_{\alpha d,\alpha}\, W\left[\begin{pmatrix} \alpha d & & \\ & d & \\ & & 1\end{pmatrix}\gamma g\right]$$

where W *is a fixed function satisfying*

$$W\left[\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1\end{pmatrix} g\right] = \exp(2i\pi(x+y))\, W[g] ,$$

and $h_{u,v}$ *are some constants satisfying*

$$(*) \quad m^2\, h_{m,1}\, h_{ab,a} = \sum st^2\, h_{\alpha d,\alpha}$$

where $rst = m$, $t\,|\,d$: $(d/t)s = b$, $s\,|\,\alpha$: $(\alpha/s)\, r = a$,

then

$$T(m)\varphi = m^2 h_{m,1}\, \varphi \quad .$$

Proof: By the remark on uniqueness of Fourier expansion of §2 ,

$$W_d^\alpha\,[g] = h_{\alpha d,\alpha}\, W\left[\begin{pmatrix} \alpha d & & \\ & d & \\ & & 1\end{pmatrix} g\right] .$$

On the other hand, let $\varphi' = T(m)\varphi$. Then for the corresponding functions W'^a_b , by (3.1.5)

$$W'^a_b\,[g] = \sum st^2\, W_d^\alpha\left[\begin{pmatrix} r & & \\ & s & \\ & & t\end{pmatrix} g\right]$$

where $rst = m$, $t\,|\,d$: $(d/t)\cdot s = b$, $s\,|\,\alpha$: $(\alpha/s)\cdot r = a$

$$= \sum st^2\, h_{\alpha d,\alpha}\, W\left[\begin{pmatrix} \alpha d r & & \\ & ds & \\ & & t\end{pmatrix} g\right]$$

$$= \sum st^2\, h_{\alpha d,\alpha}\, W\left[\begin{pmatrix} (\alpha r/s)\cdot(ds/t) & & \\ & ds/t & \\ & & 1\end{pmatrix} g\right]$$

$$= \sum st^2\, h_{\alpha d,\alpha}\, W\left[\begin{pmatrix} ab & & \\ & b & \\ & & 1\end{pmatrix} g\right]$$

By (*) $\sum st^2 h_{\alpha d,\alpha} = h_{ab,a} h_{m,1} m^2$. Thus

$$W'^{a}_{b}[g] = m^2 h_{m,1} h_{ab,a} W\left[\begin{pmatrix} ab & \\ & b \\ & & 1 \end{pmatrix} g\right] .$$

φ' and $m^2 h_{m,1}\varphi$ have the same Fourier expansion and we conclude that $\varphi' = m^2 h_{m,1}\varphi$.

Q.E.D.

Remark: The condition (*) is equivalent to the following conditions (obviously satisfied in the direct part):

1) $h_{u,v}$ is multiplicative ,

2) $h_{p^u,p^v} = \dfrac{c_{p^u} c_{p^v} - c_{p^{u+1}} c_{p^{v-1}}}{p^{2u+v}}$, $u \geq v$

where $c_{p^u} = p^{2u} h_{p^u,1}$, $c_{p^{-1}} = 0$ and $h_{1,1} = 1$,

3) $$\sum_{m\geq 0} h_{p^m,1} \cdot p^{-m(s-2)} = \frac{1}{1-h_{p,1}\cdot p^{-(s-2)}+h_{p,p}\cdot p^{-2s+3}-p^{-3s+3}} .$$

Proof: <== Assume 1), 2) and 3).

Let λ be the homomorphism from the Hecke algebra to $\mathbb{C}$ such that

$T(1,1,p) \to p^2 h_{p,1}$, $T(1,p,p) \to p^2 h_{p,p}$ and $T(p,p,p) \to 1$.

Then λ: $T(p^m) \to p^{2m} h_{p^m,1}$ by 3) and λ: $S_1(u,v) \to h_{u,v}$ by 1) and 2).

The condition (*) now follows from the corollary of Theorem 1.

==> Assume (*) holds.

We first show that 2) follows from (*) by induction on v.

For v = 0 , 2) trivially follows by definition of c_{p^u} .

Substituting $m = p^v$, $a = 1$ and $b = p^u$ in (*) we obtain

$$c_{p^v} h_{p^u,1} = \sum st^2 h_{\alpha d,\alpha}$$

where $rst = p^v$, $t|d$: $(d/t)\cdot s = p^u$, $s|\alpha$: $(\alpha/s)\cdot r = 1$, i.e.,

$$c_{p^v} h_{p^u,1} = \sum_{i=0}^{v-1} p^i(p^{v-i})^2 h_{p^{u+v-i},p^i} + p^v h_{p^u,p^v}$$

$$h_{p^u,p^v} = \frac{1}{p^v}\left[c_{p^v} \cdot \frac{c_{p^u}}{p^{2u}} - \sum_{i=0}^{v-1} p^{2v-i} \cdot \frac{c_{p^{u+v-i}} c_{p^i} - c_{p^{u+v-i+1}} c_{p^{i-1}}}{p^{2(u+v)-i}}\right]$$

$$= \frac{c_{p^u} c_{p^v} - c_{p^{u+1}} \cdot c_{p^{v-1}}}{p^{2u+v}} .$$

Thus 2) follows for all $v \geq 0$, $u \geq v$.

1) follows by induction on the number of primes dividing v and 2). This proof is similar to that of the main theorem.

Assuming (*),let us prove that

$$\left(\sum_{m\geq 0} p^{2m} h_{p^m,1} X^m\right)\left(1 - h_{p,1} p^2 X + p^3 h_{p,p} X^2 - p^3 X^3\right) = 1$$

which establishes 3). In fact, the constant term is 1 ; the coefficient of X^1 is

$$p^2 h_{p,1} - p^2 h_{p,1} = 0 ;$$

the coefficient of X^2 is

$$p^4 h_{p^2,1} - p^4 h_{p,1} h_{p,1} + p^3 h_{p,p} = 0 \quad \text{by 2) for } u = v = 1 .$$

The coefficient of X^m , $m \geq 3$, is

$$p^{2m} h_{p^m,1} - p^{2m} \cdot h_{p^{m-1},1} h_{p,1} + p^{2m-1} h_{p^{m-2},1} h_{p,p} - p^{2m-3} h_{p^{m-3},1} .$$

Replacing m by p^{m-2}, a by p and b by 1 in (*) we obtain

$$p^{2(m-2)} h_{p^{m-2},1} \, h_{p,p} = \sum st^2 \, h_{\alpha d,\alpha}$$

where $rst = p^{m-2}$, $s|\alpha$: $(\alpha/s)\cdot r = p$, $t|d$: $(d/t)\cdot s = 1$; i.e.,

$$p^{2m-4} \, h_{p^{m-2},1} \, h_{p,p} = \sum_{\substack{rt=p^{m-2} \\ r|p}} t^2 \, h_{(tp/r),(p/r)}$$

$$= p^{2(m-2)} h_{p^{m-1},p} + p^{2(m-3)} \, h_{p^{m-3},1} \quad ;$$

i.e.,

$$p^{2m-1} \, h_{p^{m-2},1} \, h_{p,p} = p^{2m-1} \, h_{p^{m-1},p} + p^{2m-3} \, h_{p^{m-3},1}$$

$$= p^{2m} \, h_{p^{m-1},1} \, h_{p,1} - p^{2m} \, h_{p^m,1}$$

$$+ p^{2m-3} \, h_{p^{m-3},1} \quad .$$

This shows that the coefficient of X^m , $m \geq 3$, is zero.

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