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A study of the output process of a queueing system and its covariance effect in tandem queues

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A STUDY OF THE OUTPUT PROCESS OF A QUEUEING SYSTEM AND ITS COVARIANCE EFFECT IN TANDEM QUEUES

by

Su-chao Chiou

A Dissertation submitted to the
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Abstract

A STUDY OF THE OUTPUT PROCESS OF A QUEUEING SYSTEM
AND ITS COVARIANCE EFFECT IN TANDEM QUEUES

by

Su-chao Chiou

Advisor: Professor Georghios P. Sphicas

A wide range of Erlang phases is considered on the interarrival time distribution and the processing time distribution in the Erlangian queues (E_j /E_K /1) to represent a more flexible queueing system. The primary focus of this study is on the departure process of such system and its covariance effect on the next station in tandem queues.

This study uses computer simulation as the primary methodology incorporating a set of numerical procedures. The interdeparture intervals of the consecutive departing transactions are first collected for the purpose of analysis. These are fitted into a hypothesized Gamma distribution with estimated scale parameter and shape parameter. These estimated parameters are then tested by

Chi-square and Kolmogorov-Smirnov tests of the goodness-of-fit testing procedures. The test statistics are found to be very statistically significant.

These estimated parameters of the output process are then fitted in a power function of the input and processing time parameters with strong empirical evidence. The characteristics of lag-1 covariance and correlation structure of the departure process are also obtained and fitted into a quadratic function of the input and processing time parameters with very significant empirical support.

The covariances of the consecutive interdeparture times are found to be non-zero. They are positive on some cases, and are negative on some other cases. The t-test is then employed to test the significance level and the sign of the correlation of interdeparture intervals. The degree of significance and the sign of the correlation is found to be mainly determined by the system utilization and Erlang phases. The effect of these significant positive and negative covariances on the waiting time measurement in the next station of a tandem queue is then examined. It is found that the positive lag-1 covariance results in an underestimate on the waiting time measurement; and negative lag-1 covariance results in an overestimate.

The results of this kind of study are useful in the design and control of a serial production system. With the knowledge of the underestimate or overestimate on the performance measurement, the control process of the serial production system can be simplified.

Acknowledgements

At the completion of this work, I would like to acknowledge the contributions of those who have made this research possible.

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CHAPTER I

INTRODUCTION

Problem Statement

Queueing is a practical problem as long as the efficiency of the system is concerned. System here may mean any transformation system which contains three major components: inputs, a transformation center, and outputs. Transactions which request services are inputs. The transformation center performs services on the transactions which may have one or more stages of process. Outputs represent the transactions which already received required services and left the system. The efficiency of the system here may mean the measurement of the length of customer waiting time, the length of waiting line or the utilization of facility in the transformation center. Queues arise due to the changeability of the transaction input rate and/or the variability of service rate. The system is considered to be inefficient if the customer waiting time is longer than an acceptable level, or if the idle time in the transformation center is higher than a prespecified level. Through analytical techniques, some prescriptions have been provided in literature for improving the efficiency of waiting lines and to balance the cost of service with the cost of waiting. Some studies obtain descriptive results on the behavior of queueing system by employing simulation techniques.

There are analytical results for single stage queueing problems under a variety of assumptions. These assumptions range from constant time with zero variability to exponential time with high variability; from single server to multiple or infinite number of servers; from no waiting space to finite or infinite waiting spaces. Some consider dependent arrival and/or service rates. Some further include the behavioral aspects of arrivals and servers in their analysis. However, there are very few results about the structure of the output process of a queueing system and its role in tandem queues.

Serial queueing systems are encountered in various fields. Examples include a wide range of real life systems. They may range from a simple assembly line to a more complex computer network system or a communication network system. In these multistage production/service systems, a transaction must move through a sequence of stations, the departures from one station form arrivals for the next station. The output process of each stage is therefore crucial to the system evaluation. The efficiency measurement of the entire system will not be accurate enough if the input pattern of each successive stage is not known. To avoid an unacceptable discrepancy level in the performance measurement, the output process of a queueing stage has to be studied to determine its effect on the following stage performance. And the value of covariance of the interdeparture time will indicate the degree of dependence of the successive stage on the previous stage and the manner of the dependency relationship.

Most of the literature on the queueing output process is concerned with identifying the conditions necessary for a renewal output process. If the output is a renewal process, the analysis of tandem queue is much more amenable because each stage can be treated independently. Unfortunately, most queueing systems are not in this category. Some researchers move one more step further to find the correlation structure of a non-renewal output process. Only very few of them pay attention to whether the covariance is positive or negative or to its value. As far as we know, only one paper moved beyond the correlation analysis and proceeded to investigate its effect on the performance of the next stage in tandem queue. Some papers neglect the covariance effect and measure efficiency using the assumption of independence or some ways of approximation. This may lead to an overestimate or underestimate on the actual performance measurement of the entire system when queues are in series.

The output process is a renewal process in only a few simple systems with severe assumptions. If one or more assumptions are relaxed, the sytems become analytically intractable. In other cases, the analytical results are not practical because of the high level of complexity of the model produced and/or because of unrealistic assumptions. Computer simulation is usually called for to study the behavior of these systems. In this study, the primary methodology considered is computer simulation supplemented by some numerical procedures.

Objectives

This study attempts to find some properties of the output process for a class of queueing systems and to examine its covariance effect in a tandem queue so as to fill up some gaps in the literature and make some extensions in the current state of knowledge of such systems. The objectives are:

1. To reveal the underlying structure of the departure process for a class of queueing systems. More specifically, to derive an approximate functional relationship between the parameters of the output interdeparture process and the parameters of interarrival time and transformation time process. This includes determining characteristics of covariance as a function of the parameters of interarrival time and service time distributions. These approximate functions can then lead to a better approximation of the output process.

2. To investigate the effect of lag-1 covariance of consecutive interdeparture times on the average waiting time of the next station in tandem queues, to find the discrepancy resulting from the independence assumption, and

to examine the level of the covariance effect on this measurement.

3. To derive a set of simulation models and numerical procedures which are not only applicable to the queueing systems considered in this study, but also are applicable to complex real world systems.

The ultimate goal of this type of studies is to provide usable results for the design of multistage production/service systems, and for the purpose of controlling such systems so that the efficiency of the entire operating system can be improved. This is beyond the scope of this study.

To obtain these objectives, some hypotheses are formulated and tested through a set of numerical statistical procedures.

The Scope

The scope of this study is limited by some assumptions on the nature of problems considered as well as by some applicable statistical techniques and the availability of computer simulation techniques. The assumptions on the queueing problems are as follows:

1. It is assumed that the arrival rate is smaller than the service rate (i.e., $\lambda/\mu < 1$) so that the system under consideration is capable of reaching steady state.

2. It is assumed that the interarrival time and service time follow Erlang distribution with a single server. This is more general than the exponential time or constant time usually assumed in literature as these are special cases.

3. The arrival rate and service rate are assumed to be independent of the state of the system. Here, we also preclude the balking or reneging behavior of arrivals.

4. The input sources and waiting rooms are assumed to be infinite, and customers are served in a first-come

first-served manner.

The assumption of Erlang distribution for both interarrival time and service time is appropriate in queueing systems. Figure 1 depicts a queueing system with Erlang arrival process and Erlang service time with infinitely waiting rooms before the service station.

Consider that there are an infinite source of supply to the service system and a transaction which has to move through a series of identical process before it can enter the service facility, and each phase of process requires exponential time duration, then a sequence of j tasks would have an Erlang distribution. In the transformation center, also consider that each transaction requires a sequence of identical operations, and the performance duration for each operation is exponential, then it also have an Erlang distribution. In the arrival channel and service channel, the new unit can enter the phase 1 only after the previous unit is discharged in the last phase.

Another reason of the Erlangian assumption comes from the properties of Erlang distribution. The variability of Erlang distribution ($0 < \sigma < 1/\mu$) represents a general case of various distributions which may range from exponential distribution with great variability ($\sigma = 1/\mu$)

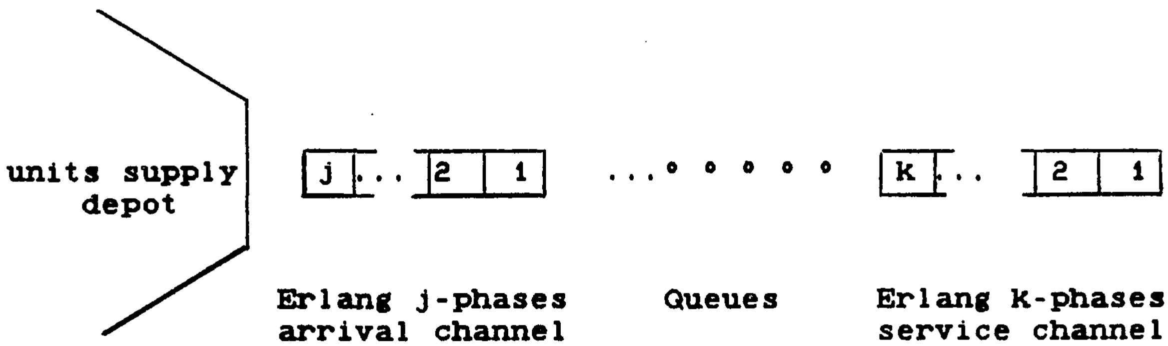


Figure 1: Representation of an $E_j/E_k/1$ queueing system with infinite waiting spaces before the service facility

to constant distribution with zero variability ($\sigma = 0$).

Suppose that a random variable X has an Erlang distribution with parameters α and k, then its probability density function is

$$f(x) = \frac{\alpha^{k} x^{k-1} e^{-\alpha x}}{(k-1)!}$$

Where α is a positive number represents the scale of the function and k is a positive integer represents number of phases. Its mean is k/α , and variance is k/α^2. When $k=1$, this distribution reduces to the exponential distribution; when $k > 1$, it is unimodal with unimode = $(k-1)/\alpha k$, as k increases it becomes more symmetrical; whereas as $k \rightarrow \infty$, the Erlang distribution approaches the constant distribution. It can therefore be considered a more flexible distribution for modeling than the exponential. Different levels of fluctuation can be produced by choosing different value of k. Also, k need not be restricted to be an integer. If k is not an integer value, the Erlang becomes Gamma distribution which can be viewed as the generalization of the k-phase Erlangian. And in this study, the Gamma distribution is considered to be the hypothesized distribution for the output interdeparture process of the queueing system under study because the estimated parameters from the observed data are no longer integers.

For testing the hypothesized distribution, goodness of fit testing is made through Chi-square test and Kolmogorov-Smirnov test.

Review of Relevant Literature

Renewal vs. Non-renewal Output Process

A renewal process is the statistical definition of a sequence of independent and identically distributed nonnegative random variables; whereas a non-renewal process implies dependency. If the interdeparture times of consecutive customers of a queueing system are correlated, it is considered to be a non-renewal output process. On the other hand, if they are uncorrelated, it is a renewal output process.

In the queueing literature, the question about the nature of output process was brought up by the study of queueing networks. In a serial queueing network, the queueing stages are arranged in series, and the output of a particular stage forms the input of the next stage. The nature of the output process, dependency or independency, is therefore crucial to the performance measurement for control purpose. The independent process allows the serial queueing network to be treated in a decomposition manner, and thus leads to tractable computation. If independency between stages is assumed for the non-renewal process, it will lead to incorrect results.

The study of output process and tandem queues was initiated by R.R.P. Jackson(1954). In the study of a tandem queueing system of two stages (M/M/1 --> ./M/1), he found that the probability of finding x customers in the first queue and y customers in the second queue is simply the joint state distribution. That is,

$$P(x,y) = (1-\rho_1)\rho_1^{x}(1-\rho_2)\rho_2^{y}$$

Where ρ_1 and ρ_2 are respective utilization factors in stage 1 and stage 2. This result implies that the second queue is also M/M/1, and is independent of the first stage.

Burke (1956) formally proved that the output of an M/M/c queue is a poisson process with the same rate as the input process, and the consecutive interdeparture times are independent. Based on this result, R. R. P. Jackson's finding can be generalized for more than two stages with M/M/c queues. It becomes [see R. R. P. Jackson (1957)]

$$P(x_1, x_2, \ldots x_k) = \prod_{i=1}^{k} P_i(x_i)$$

Therefore, a stage-by-stage decomposition analysis would be easier than the simultaneous analysis on the entire system.

J.R. Jackson (1957), in his study on network with feedback loops, also showed the same product solution form for Markovian queues in equilibrium although the combined input to each stage is apparently nonpoisson. However, he showed that each stage in the network behaves as if it were an independent poisson process. The literature in queueing networks once had a serious confusion which was caused by the so-called "Jackson's Theorem". Some researchers in their analysis of queueing networks seriously took Jackson's result and proceeded to treat each stage independently although input to each stage is nonpoisson. [see Kleinrock (1975) ,Disney (1981)]

In the M/M/c queue, Reich (1957) also obtained the same renewal result as Burke's. He even moved one step further about the more generalized queue $E_j/E_k/c$, and concluded that the input-output equivalence property of M/M/c queue can not be generalized for $E_j/E_k/c$ except for the extreme cases when $j=k=1$ (this is M/M/c) and $j=k=\infty$ (this actually is D/D/c). As to the more general cases with $j=k$, he proved by contradiction that the output of $E_2/E_2/1$ is not E_2. This result is relevant to this study, since we wish to find the underlying structure of the output process with various values of j, k and other Erlang parameters through numerical procedures.

In a non-markovian queue, the input process and service time need not to be both poisson. It is considered to be more general than M/M/1 queue. Researchers proceeded to find whether the independence property still holds for these systems, and try to identify the required conditions for a renewal output process.

Finch (1958) in his study of M/G/1 queue, proved that the renewal output process can be formed only in the case of exponential service and unlimited waiting space. Chang (1962) derived an expression for the interdeparture time distribution for GI/G/1 queue by applying the complex-variable analysis. By letting $D_{n-1} = I_n + S_n$, the distribution of D_{n-1} is the convolution of I_n and S_n. Where, D_{n-1} is the interdeparture time between the n^{th} customer and the $n-1^{th}$ customer. I_n is the server idle time right after the n^{th} customer departed, and S_n is the service time of the n^{th} customer. His result is possible only when the Laplace transforms of the distributions exist and P_0 is also known. P_0 is the probability of having no customer in the system.

On the M/G/1 queue, Disney et al (1973) using methods from semi-markov process summarized some conditions for the renewal departure processes. These are M/O/1/N (the service times are all 0 with probability 1), M/G/1/O, M/D/1/1 [also shown by King (1971)], and M/M/1/∞ [also shown by Finch (1959)]. For other M/G/1/N system, the output process have not been found to be independent.

Covariance Structure

From the literature being reviewed, we know that most systems with poisson input do not generate a renewal output due to either non-exponential service time or limited waiting rooms. For those non-renewal output queueing systems, the degree of autocorrelation between consecutive interdeparture time are measured by value of covariance which can be positive or negative.

Jenkins (1966) derived explicit expressions of autocorrelations of lag 1 and lag 2 for the $M/E_k/1$ queue. In that,

$$Cor(D_n, D_{n+1}) = \left[\frac{(1-\rho)k}{\rho^2(1-k)+k}\right]\left\{\frac{(k-1)\rho-k}{\rho+k}+\left(\frac{k}{\rho+k}\right)^k\right\}$$

Where D_n is the interdeparture time between the $n+1^{th}$ customer and the n^{th} customer.

ρ is the system utilization

k is the number of Erlang phases

King (1971) also produced the same structure for $M/E_k/1$ queue. Shimshak & Sphicas (1982) found the lag 1 correlation is positive, and also examined its effect on the waiting time at the second station. The waiting time is found to be underestimated if independence assumed in the $M/E_k/1$ ->. /M/1 queueing systems.

In the general class of M/G/1/N queues, Disney and Cherry (1974) proved that

$$Cov(D_n, D_{n+1}) = \frac{P_0}{\lambda P}\left[P' \frac{P^2}{\lambda} + \left(\frac{1}{\mu} + \frac{P_0}{\lambda}\right) P\right]$$

$$\text{Where, } P = \int_0^\infty e^{-\lambda x} dH(x), \text{ and } P' = \frac{dP}{d\lambda}$$

λ: the expected arrival rate

μ: the expected service rate

P_0: probability of an empty system

In M/E_k /1/N queues, Disney and deMorais (1976) showed that the covariance can be positive, negative, or zero depending on ρ, N and Erlang parameters.

King (1971), in his study of M/G/1/N queues, proved that the lag 1 covariance is negative if N=1. Disney (1982) considered the M/M/1 queues with limited waiting space, and found that for M/M/1/N queues $(0< N < \infty)$, the lag-1 covariance is always negative.

Daley (1968) also found some correlation properties of the departure process in a stationary GI/G/1 queue,

$$Var(D_n) + 2\sum_{\tau=1}^{\infty} Cov(D_n, D_{n+\tau}) = Var(T_n)$$

Where, D_n: the interdeparture interval between the n^{th} customer and the $n+1^{th}$ customer

T_n: the interarrival interval between the n^{th} customer and the $n+1^{th}$ customer

$D_{n+\tau}$: the interdeparture interval separated by τ number of customers

In GI/M/1 queue, the $Cov(D_n, D_{n+\tau})$ are of the same sign for all n, and converges to zero monotonically. In the M/G/1 queue, he found that there exist T_n and S_n such that the $Cov(D_n, D_{n+\tau})$ are not the same sign. Where, T_n is the interarrival time between the n^{th} customer and the $n+1^{th}$ customer, and S_n is the service time of the n^{th} customer.

For the effect of the covariance on the variance of departure time, Disney (1982) asserted that the true variance of the departure process can be underestimated, overestimated, or correctly estimated by the variance estimated under the assumption of independence. A negative covariance produces an overestimate in the true variance and a positive one produces underestimate.

As to the degree of covariance effect on the level of discrepancy in the variance measurement, Disney did not make any comment on it. None of these researchers proceeds to investigate the covariance effect on the waiting time of the next station except Shimshak and Sphicas (1982). The covariance effect is one of the problems considered in this study.

In the literature, the assumptions on the interarrival time and service time distribution are either too general (e.g. GI/G/1) or too restricted (e.g. M/M/1). For the class of Erlangian cases, most papers considered only the service time to have an Erlang distribution. There are very few research results on queueing system when both interarrival time and service time are Erlangian. It is this class of queueing systems considered by this study.

In this study, some hypotheses on the queueing output process of the Erlangian queueing system and on its covariance effect in tandem queues are formulated and stated in Chapter II. The simulation methodology on the queueing system is developed and also is given in Chapter II. Chapter III describes the procedures for the output analysis and for the hypotheses testing. It also includes the procedure for approximating the waiting time when independence is assumed on the arrival process of the second stage in a tandem queues. Chapter IV presents the results of the hypotheses testing and the chacteristics of the departure process and its effect in tandem queues. The conclusions of this study and some suggestions for future reaserch are given in Chapter V.

CHAPTER II

HYPOTHESES AND SIMULATION METHODOLOGY

Hypotheses

After reviewing the relevant literature in the area of queueing output process and related works, we recognize the significance of the study on queueing output process and identify the gaps to be filled and extensions needed to be made. There are two hypotheses to be tested:

1. The output process of $E_j/E_k/1$ queueing system is approximately in the Gamma distribution family, and the estimated parameters of the Gamma distribution - are functions of the parameters of the interarrival time and service time distributions. The covariance of interdeparture time is also a function of the parameters of the interarrival time and service time distributions.

2. The covariance of the output process has a significant effect on the waiting time of the next stage in the tandem queues. The positive or negative values of the covariance determine the underestimate or overestimate on

actual waiting time when independence is assumed. And the size of the discrepancy in evaluation corresponds to the value of covariance which is determined by the parameters of Erlang distribution in the $E_j/E_K/1$ queue.

The formulations on these hypotheses are stated in Chapter IV and are then tested through a set of numerical procedures. The goodness of fit testing is used to test the hypothesized distribution. In that, Chi-square and Kolmogorov-Smirnov tests are used to decide whether to accept or to reject the null hypotheses. The parameters of the fitted distribution are estimated by method of moments. The functional relationships between the output parameters and input parameters are to be fitted by power approximation.

To test the second hypothesis, lag 1 covariance and correlation coefficients of consecutive interdeparture times are calculated. The Lag 1 covariance indicates the relationship between consecutive departures. Our attention is on the sign of the covariance and its value so that the relationship between the covariance and waiting time approximation error can be found.

Simulation Methodology

Due to the incapability of analytical techniques in dealing with some complex queueing systems; or sometimes, due to the low applicability of analytical results which comes from the high level of complexity of models and/or from unrealistic assumptions, this study will use computer simulation as the primary methodology. This is then incorporated with a set of numerical procedures to achieve some objectives of this study.

For the E_j /E_k /1 queueing systems under study, a computer simulation model is constructed. The customer departure times are collected for the purpose of covariance analysis and distribution approximation on the interdeparture process. Many simulation runs with various values of j and k and other distribution parameters of Erlang distribution are made. The lag-1 covariance and correlation for each case are calculated to show the degree of correlation of the consecutive interdeparture times. The mean and variance parameters for the hypothesized distribution are obtained using the method of moments. The goodness of fit between the hypothesized distribution and the data is then tested by Chi-square and Kolmogorov-Smirnov testing procedures. The power approximation technique is then used to estimate the functional relationship between the parameters of the fitted distribution and that of the input and process time distributions.

To investigate the covariance effect on the waiting time of the second stage in a tandem queue, a multistage computer simulation model is also constructed. The actual waiting time in the second stage is found. The logic is similar to that of the single stage model except that the output of the first stage forms the input process of the second stage. Therefore, in a tandem queue (E_j /E_k /1 -->. /M/1), the input process of the second stage is not generated by any independent identical distribution random deviates, but instead, is just the output of the first stage. The average waiting time of the second stage is calculated for each simulation run and is considered as the actual waiting time. This is then compared with the waiting time of GI/M/1 queue. In the GI/M/1 queue, the input process is assumed to be i.i.d. random variables, its waiting time is therefore considered as an approximation. In the comparison of the actual waiting time with the approximated waiting time, the various cases considered in the covariance analysis are considered again here, so that the covariance effect on the waiting time approximation error can be examined.

The simulation and numerical procedures is sketched in Figure 2.

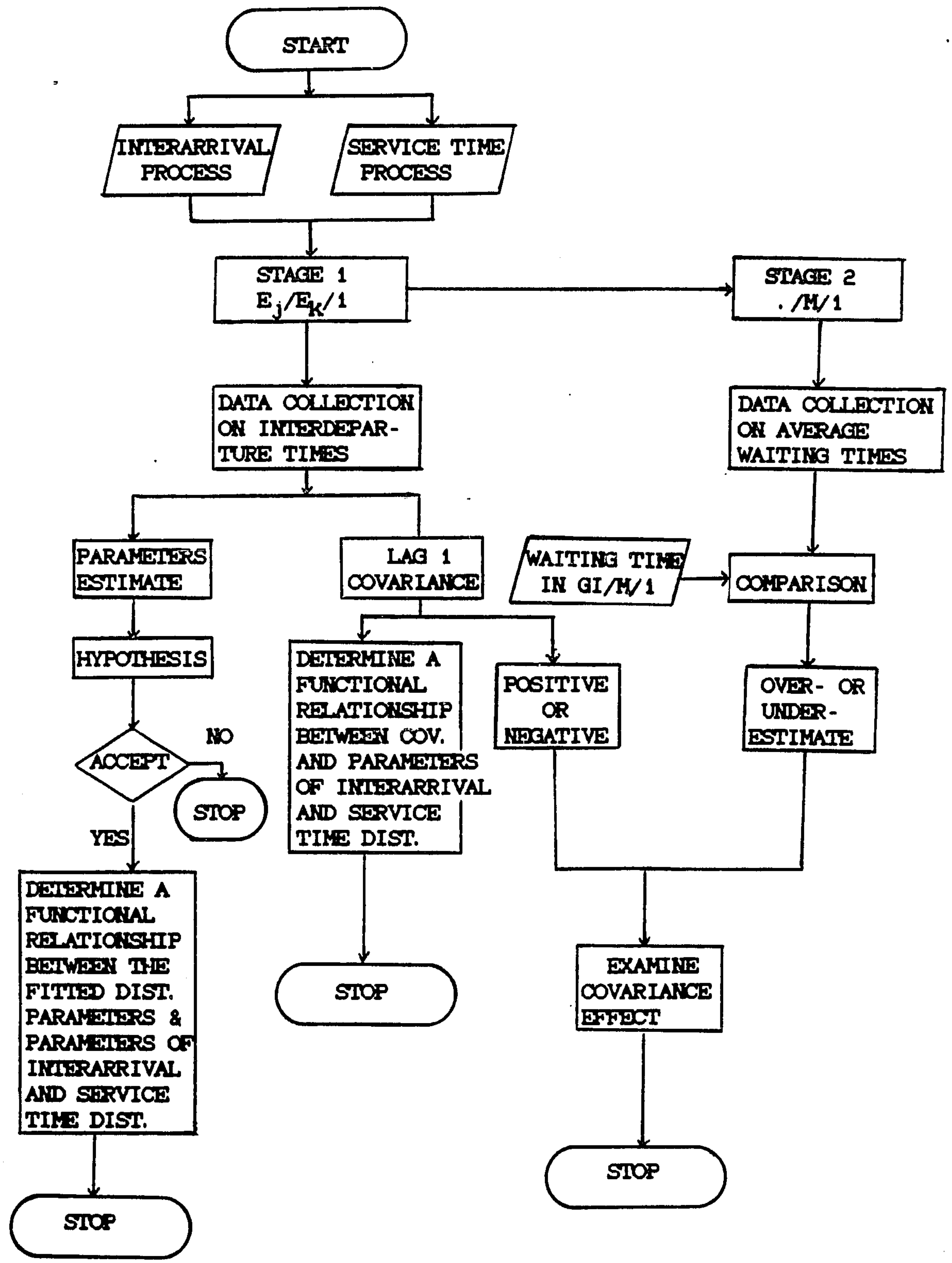


Figure 2: Flow Chart for the Simulation Procedures

Queueing Simulation Models

This study uses a general purpose language FORTRAN, to model the queueing systems under study for several reasons. First, this study focuses on particular outputs of the queueing systems. These outputs are usually not internally provided in the specially designed simulation language fixed output format. The simulation languages are designed to model a wide variety of systems with a set of blocks, and provide a standard set of output which is sometimes not usable in a specific study on output of the system. It therefore has lower flexibility and also takes longer execution time than a general purpose language which can be designed to a particular application.

Second, the simulation model in this study requires to generate a stream of nonuniform random variables and to perform extensive numerical calculations. GPSS (General Purpose Simulation System) is known to have limited numerical calculation functions and has no exact automatic nonuniform random variable generators. The Erlang random deviates which are assumed in this study would have to be approximated by a sequence of straight line segments on the inverse of the distribution function if GPSS were used.

Third, in GPSS, the simulation clock is advanced in integer increments, the simulator would specify a small basic unit of time (e.g. second) to represent all activity times so as to obtain a certain level of accuracy. In this manner, a high value of time unit has to be carried throughout the simulation.

Finally, the queueing system under this study is a basic type (single server queueing model), the FORTRAN program can therefore be coded without much difficulty.

This model uses the next-event time-advance for the discrete event simulation where the initial system condition is "empty and idle". The logical flow chart of the E_j /E_K /1 queueing system is shown in Appendix A1. This is constructed based on the basic one developed by Law and Kelton (1982).

This basic model is then extended to a two-stage single server model. In a two stage model, as the transactions depart from the first stage of service they join the queue of the second stage. The logic relationship is shown in Appendix A2.

Simulation Model Validation and Run Length

To determine whether the simulation model accurately represents the system under study, a confidence interval approach is used. By comparing the performance of the simulation model with known analytical steady-state results (for example, the mean waiting time in a M/M/1 system), the model can be validated.

The process is first simulated for some arbitrary run length n and note whether the confidence interval formed covers the steady-state value μ. Let the steady state mean of the stochastic process $(X_i,\ i = 1, 2, \ldots n)$ be

$$\mu = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} X_i}{n}$$

To construct a $100(1-\alpha)\%$ confidence interval for μ, let the point estimator of μ be $\bar{X}(n)$ and the estimator of σ^2 be $S^2(n)$, where

$$\bar{X}(n) = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$S^2(n) = \frac{\sum_{i=1}^{n}[X_i - \bar{X}(n)]^2}{n-1}$$

Based on the central limit theorem,

$$Z = \frac{\bar{X}(n) - \mu}{\sqrt{\frac{S^2(n)}{n}}} \dot{\sim} N(0,1) \text{ for } n \to \infty$$

Where N(0,1) is the standardized Normal random variable with mean=0, and variance=1.

It then follows for large n that

$$P\left\{-Z_{1-\alpha/2} < \frac{\bar{X}(n) - \mu}{\sqrt{\frac{S^2(n)}{n}}} < Z_{1-\alpha/2}\right\} \sim 1-\alpha$$

Therefore, an approximate 100(1-α)% confidence interval for μ is given by

$$X(n) \pm Z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}$$

To resolve the problem of the initial transient in the simulation run, we start data collection after the simulation runs for K observations. As to the size of K required to lead the simulation run into a more stable state, there is no definitive procedure established in literature as far as we know. However, the length of simulation run can also be determined by taking the confidence interval of μ from a fixed number of replications.

The valid M/M/1 simulation model is then transformed into the $E_j/E_k/1$ model. And the validation of the $E_j/E_k/1$ simulation model is further made by comparing the expected simulation results with those values obtained by Hiller and Yu (1981) since an analytical result is not available.

Random Number Generator and Erlang Random Deviates

Random Number Generator

To carry out the simulation, the random input stream and service time of the stochastic system have to be generated using a random number generator. The method used in this study is the Prime Modulus Multiplicative Linear Congruential Method. This method was developed by Hutchinson(1966) by taking the primitive elements in choosing the parameters used in the multiplicative linear congruential method. The linear congruential method was first introduced by Lehmer (1951). It is by far the most widely used random number generator. However, the level of statistical reliability and satisfaction depend on the parameters chosen in this recursive formula

$$Z_i = (aZ_{i-1} + c) \pmod{m}$$

Where, m (the modulus), a (the multiplier), c (the increment), and Z_i (the seeds) are nonnegative integers. When $c > 0$, the method is called mixed Linear Congruential Generators, and when $c=0$, it is called Miltiplicative

Linear Congruential. The choice of a and m are critical in the multiplicative case since they determine the length of a cycle and also assure every integer within a period (p) will occur exactly once so that it confirms to the law of uniform distribution. For $m = 2^b$ (b is the number of available bits for computation), Fishman (1973) proved the method will result in a short period of $m/4$, with large gaps in the Z_i. And For $a = 2^l + j$, it will induce poor statistical properties due to a shift that occurs in the successive random numbers.

Hutchinson (1966) suggested to choose m to be the largest prime number instead of letting $m = 2^b$. If m is prime, and a is a primitive element modulo m, it will permit the longest possible cycle length, and also assure the maximum period within this longest possible period. The choice of $a = 7^5$ (which is a primitive element modulo $2^{31}-1$), was found by Lewis, Goodman and Miller(1969) to produce very satisfactory results.

Based on this results found in literature and utilization of IBM 360/370 facility, this study adopts Schrage's (1979) FORTRAN Generator which uses $a = 16807$ (i.e. 7^5), $m = 2147483647$ (i.e. $2^{31}-1$), to generate a stream of i.i.d. random numbers between (0,1). These random numbers are then transformed into the random numbers having the desired distribution by means of the inverse probability integral, as discussed in the next section.

Generation of Erlang Deviates

The Erlang random variables can be generated through convolution method on the inverse transform because K-Erlang random variables with mean ß is the sum of K numbers of Exponential random variables with parameter ß/K.

That is, if $X = Y_1 + Y_2 + \ldots + Y_K$, and Y_i are exponential random variables generated by the general inverse transform method,

$$X = \sum_{i=1}^{K} \frac{-\beta}{K} \ln U_i \quad \text{if } Y_i = \frac{-\beta}{K} \ln U_i$$

The method's efficiency can be improved by noting that

$$X = \sum_{i=1}^{K} Y_i = \sum_{i=1}^{K} \frac{-\beta}{K} \ln U_i$$

$$= \frac{-\beta}{K} \ln\left(\prod_{i=1}^{K} U_i\right)$$

In this way, we need to compute only one logarithm rather than k logarithm. If k is large, general Gamma generation is more efficient because the k-Erlang distribution is a special case of the Gamma distribution with shape parameter equals to the integer k.

CHAPTER III

OUTPUT ANALYSIS AND HYPOTHESES TESTING

Goodness of Fit Testing

For the queueing systems, $E_j / E_k / 1$, under study, their output interdeparture times are to be fitted into a hypothesized distribution with estimated parameters so as to test the first hypothesis in this study. The fitted distribution, $\hat{F}$ is considered to be Gamma, instead of Erlang, because the estimated parameters from the set of observed data are no longer integers to meet the Erlangian requirement.

To estimate the parameters of the Gamma distribution,

let α be the shape parameter

β be the scale parameter,

and $\beta = \mu/\alpha$,

$\sigma^2 = \mu^2/\alpha$,

it then follows that

$$\alpha = \mu^2/\sigma^2 \text{ and}$$

$$\beta = \sigma^2/\mu,$$

where μ and σ^2 will be estimated from $\bar{X}$ and S_x^2.

Goodness of fit tests were performed to determine how well the fitted Gamma distribution fits the observed data. That is, the null hypothesis

$$H_o : X_i \sim \hat{F}$$

Several goodness-of-fit hypothesis tests have been developed in the literature. They differ in their procedure used, size of data required and level of power of test. In this study, the Chi-square and Kolmogorov- Smirnov tests were used to compare an empirical distribution function with the distribution function $\hat{F}$ of the hypothesized distribution. A histogram of the data will also be constructed so that the visual assessment can be made on the shape of distribution although it does not provide any information about the parameters of the distribution. The procedure of the goodness of fit testing is sketched in Figure 3.

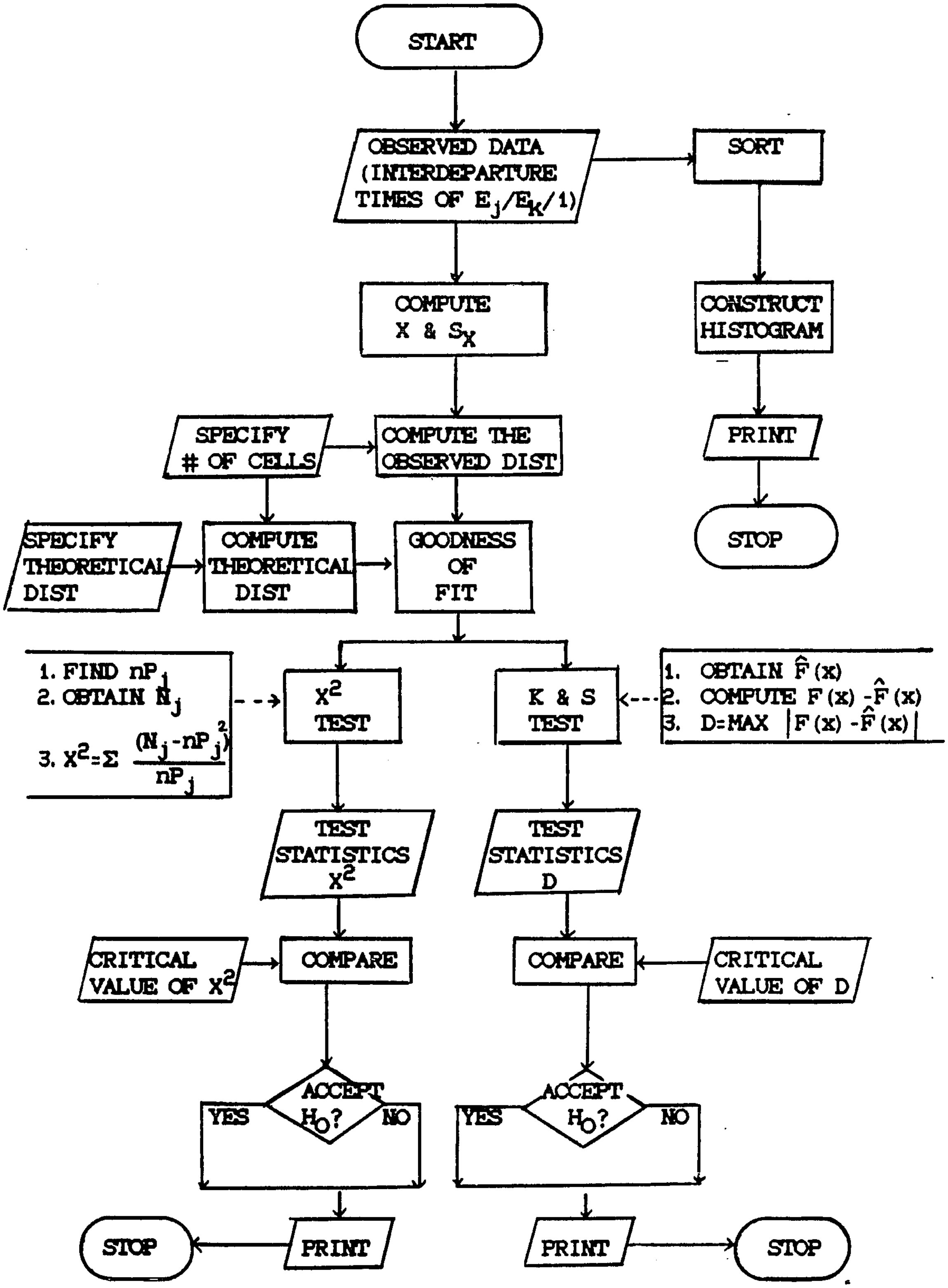


Figure 3: Flow Chart for Goodness-of-fit Testing Procedures

Chi-Square Test

The Chi-Square test was first developed by K. Pearson in 1903 to test whether a set of actual observations, X_i, are consistent with the assumption that they are obtained from a given theoretical distribution. It remains in wide use because it can be used for many different types of tests. To perform Chi-Square test on the fitted distribution, the sample data are first divided into K intervals as it is required in constructing a histogram of the data. The test statistic is then calculated as

$$\chi^2 = \sum_{j=1}^{K} \frac{(N_j - nP_j)^2}{nP_j}$$

Where, K: number of intervals

N_j: number of observed data points in the j^{th} interval

n is the total number of observations

P_j is the expected proportion of the X_i's that would fall in the j^{th} interval if we were sampling from the theoretical distribution.

nP_j: expected number of observations that would fall in the j^{th} interval

The distribution function of X^2 converges to a Chi-Square distribution with k-m-1 degrees of freedom, where m is the number of parameters for the theoretical distribution that are estimated from the sample data. H_0 will be rejected if X^2 exceeds the critical value for the desired significance level and sample size.

The main difficulty in using the Chi-Square goodness of fit test is to decide the number of intervals K. No definitive rule has been established. However, there are guidelines have been suggested such as P be chosen so that they are equal or nearly equal, and the numbers of observations in each interval are not too small. In this study, the observed data are grouped into k intervals based on these two suggestions in literature.

Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov test differs from the Chi-Square test in that we need not specify the number of intervals. This test compares the sample cumulative distribution function $\hat{F}(x_i)$ with the theoretical cumulative distribution function $F(x_i)$ specified in the null hypothesis using each sample observation so that no information is lost through grouping. The test statistic is the maximum absolute deviation between the two functions. That is

$$D = \max_{\text{all } i} \left| F(x_i) - \hat{F}(x_i) \right|$$

The rule of the test is to reject H_0 if D exceeds the critical value in the Kolmogorov-Smirnov table which is determined by the sample size and a chosen significance level.

The Kolmogorov- Smirnov test was originally designed for continuous probability distributions. It therefore is not applicable to discrete data. It was also originally developed under the assumption that the population mean and variance are known. However, it has been used extensively with estimated mean and variance and still is more discriminating than the Chi-Square test. As to the sample size, Kolmogorov- Smirnov test is valid for any sample size, whereas Chi-Square test is generally valid only in an asymptotic sense.

Covariance and Dependent Arrivals

The covariance of D_n and $D_{n+\tau}$ is used to measure the degree of dependence between D_n and $D_{n+\tau}$. τ is the number of lagged intervals. D_n is the interdeparture time between the $n+1^{th}$ and the n^{th} customers, and $D_{n+\tau}$ is the interdeparture interval lagged by τ number of customers.

In the class of M/G/1/N queueing systems, King (1971) showed that the corr(D_n , $D_{n+\tau}$) decreases monotonically as τ increases. And in G/M/1 queue, Daley (1968) also showed that Cov(D_n , $D_{n+\tau}$) converges to zero monotonically as τ increases. These findings imply that the strength of dependent relationship of interdeparture times separated by τ intervals decreases as τ increases, and the interdeparture time of a customer will be independent of the interdeparture time of the customers who already left the system for a long length of time. Therefore, the effect of interdeparture times lagged by a long interval on the waiting time of the next stage will be insignificant.

In this study, we only consider the lag 1 covariance because it indicates the manner of dependence of consecutive departing customers, and may have significant effect on the waiting time of the next stage in a tandem queue.

If we consider the consecutive interdeparture times as a stationary time series, the covariance between D_n and $D_{n+\tau}$ is

$$\mathrm{Cov}(D_n, D_{n+\tau}) = E\left[(D_n - \mu)(D_{n+\tau} - \mu) \right]$$

This covariance signifies the dependency or independency relationship between D_n and $D_{n+\tau}$, and is used to measure the linear relationship between D_n and $D_{n+\tau}$. However, its actual magnitude does not provide the degree of relationship. The correlation coefficient provides better measure of the linear relationship of D_n and $D_{n+\tau}$ since it removes the individual variability of each D_n and $D_{n+\tau}$ by dividing the covariance by the product of the standard deviations. That is,

$$r_\tau = \frac{E[(D_n - \mu)(D_{n+\tau} - \mu)]}{\sigma(D_n) \; \sigma(D_{n+\tau})}$$

The significance of r_τ is then tested statistically by simple t-test. The composite t-test is also employed to predict the sign of correlation so as to accept whether the correlation is significantly positive or negative.

Waiting Time in GI/M/1

In the GI/M/1 queue, the expected waiting time is defined as

$$E(W) = \frac{\nu}{\mu(1-\nu)}$$

Where,

μ is the average service rate

ν is the unique root in (0, 1) of the equation

$$\nu = L[\mu(1-\nu)]$$

and

L(s) is the Laplace transform of the interarrival time distribution. Or the Laplace transform of the interdeparture time distribution of the preceeding stage.

For the Gamma distribution

$$L(s) = \left[\frac{1/\beta}{s + 1/\beta}\right]^{\alpha}$$

Where, α is the shape parameter and

β is the scale parameter

Therefore,

$$L[\mu(1-\nu)] = \left[\frac{1/\beta}{\mu(1-\nu)+1/\beta}\right]^{\alpha} = \nu$$

or,

$$\nu[\mu-\mu\nu+1/\beta]^{\alpha} = 1/\beta^{\alpha}$$

For $\alpha=1$, $\nu= 1/\beta\mu$ is a unique feasible root.

If we let $1/\beta\mu = \rho$

$$E(W) = \frac{\rho}{\mu(1-\rho)} \quad ,$$

It is the expected waiting time of M/M/1 with independent arrivals.

For $\alpha= 2$, we obtain the cubic equation:

$$\mu^2\nu^3-2\mu(\mu+\beta)\nu^2+(\mu-\beta)^2\nu-\beta^2=0$$

If we let $1/\beta\mu = \rho$, the roots of this equation can be found as

$\nu = 1$, (This is always a root)

$\nu = (1/2)(1+\sqrt{4\rho + 1}) + \rho$

(This is greater than 1 since $1+ 4\rho > 1$)

or $\nu = (1/2)(1-\sqrt{4\rho + 1}) + \rho$,

(This is a unique feasible root because $0<\nu<1$)

For the unique feasible root, $\nu = (1/2)(1-\sqrt{4\rho + 1}) + \rho$, it can be shown that $0 < \nu < \rho$.
Because for $0 < \rho < 1$,

$\nu < \rho$, since $1-\sqrt{4\rho + 1} < 0$, and
$\nu > 0$, since $\sqrt{4\rho + 1} < 2\rho + 1$.

The expected waiting time for $\alpha=2$ (i.e., $E_2/M/1$) then becomes

$$E(W) = \frac{(1/2)(1-\sqrt{1+4\rho}) + \rho}{\mu[(1/2)(1+\sqrt{1+4\rho}) - \rho]}$$

This value of E(W) for the $E_2/M/1$ queue is lower than that for the M/M/1 queue because its numerator, (1/2)(1-

$\sqrt{1+4\rho}$) + ρ , is less than ρ , and its denominator is greater than $\mu(1-\rho)$. This phenomenon can also be found in Hiller and Yu's (1981) simulation results. In that study, the queue length of the E_K/ M/1 queueing system decreases as K increases.

For other values of α, which are not restricted to be integers, we use Newton's approximation method to estimate ν. E(W) of the next stage of the tandem queue can then be obtained.

CHAPTER IV

TESTING RESULTS AND COVARIANCE EFFECT

Testing Results on the Output Process

To test the hypotheses formulated in this study, five independent runs are made on 3000 transactions for a wide range of the $E_j/E_K/1$ queueing systems. That is, j=2, 3, 4, 5, 10, 20, 30, k= 1, 5, 10, 20, 30, and ρ= .1, .3, .5, .7, .8, .9. The interdeparture times of the last 1000 transactions are taken as samples for testing goodness of fit between the Gamma distribution and the observed data.

The observed data are first arranged in an ordered manner, and the histogram of the samples is then constructed. The number of intervals is specified as 30 for the sample size of 1000. And the size of interval is found by dividing the range of sample values by the number of intervals. The shape of distribution can then be observed from the histogram constructed. For all the cases considered, their shapes of distribution are observed to be approximately Gamma with different shape and scale. Some have display shapes that are approximately normal. These cases were those with shape parameters approximately five or more.

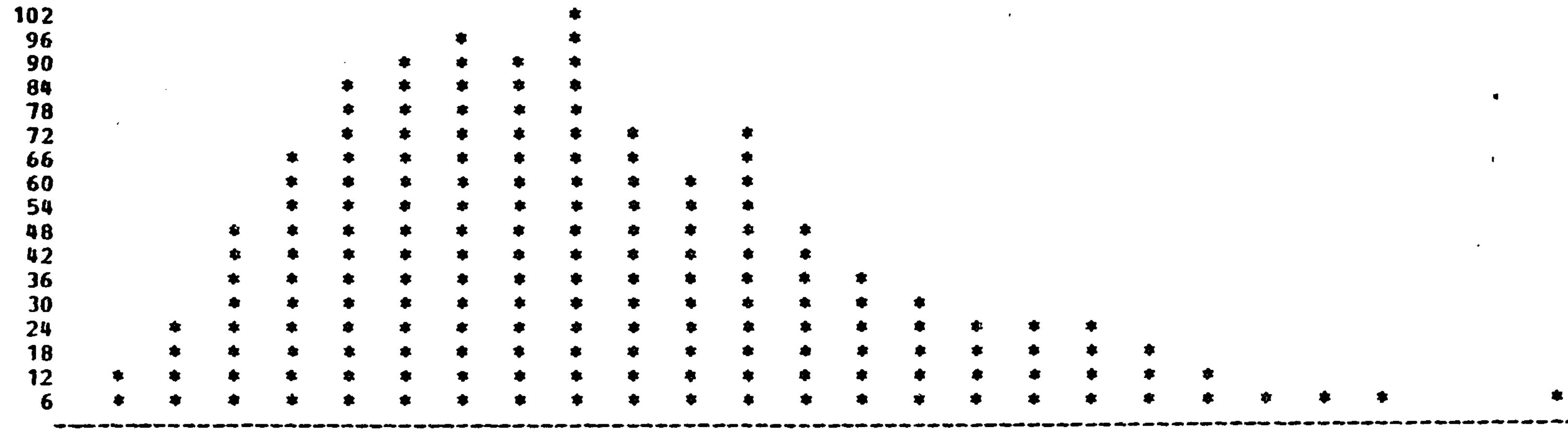
To illustrate, the histogram for the system $(E_{10}/E_{10}/1, \rho=.8)$ is shown in Figure 4. In that, the sample values are the interdeparture times of consecutive transactions which range from 0.258 to 2.792, with mean= .98983, and variance= .14886. This histogram only provides a visual synopsis of the observed data. It does not provide any statistical information on the parameters of the distribution. The parameters of the approximated Gamma distribution are then estimated using the method of moments:

$\alpha = \mu^2/\sigma^2 = 6.58188$, the shape parameter

$\beta = \sigma^2/\mu = .15039$, the scale parameter

Therefore, the mean of the approximated Gamma distribution is $\alpha\beta = .9898$, which is approximately equal to 1. And the mean interarrival time of the queueing system under study has been specified as 1 in the simulation runs. From this phenomenon, we conclude that the steady-state has been reached.

To state the output distribution statistically, the hypothesis statement is formulated as



NUMBER	10	23	48	65	80	87	92	90	101	70	56	67	45	34	29	21	22	21	16	7	4	5	4	0	0	1
CLASS	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

START 0.258 STOP 2.792 SIZE OF INTERVAL 0.0845
CALCULATED MEAN= 0.98983 CALCULATED VARIANCE= 0.14886

****

HYPOTHESIS STATEMENT

NULL HYPOTHESIS= POPULATION IS GAMMA WITH TRUE MEAN= 0.99
ALTERNATE HYPOTHESIS= POPULATION IS NOT GAMMA WITH TRUE MEAN = 0.99

****

Figure 4: Histogram Presentation of Interdeparture Times of 1000 Samples for the $E_{10}/E_{10}/1$ queueing system with $\rho=.8$

Null Hypothesis

H_0: Population is Gamma with true mean =.99

i.e., with shape parameter = 6.58188

scale parameter = .15039

Alternative Hypothesis

H_1: population is not Gamma with true mean = .99

The estimated parameters of the approximated Gamma distribution and their goodness of fit test statistics for all cases considered were obtained and are summarized in Appendix B. The Chi-square test for the previous example (E_{10} /E_{10}/1, ρ=.8). is shown in Figure 5. The number of intervals were based on two rules usually used in literature. They are, each interval contains equal probability of observations and the expected number of observations in each interval is not too small. Therefore, the number of intervals is first specified as 20, (i.e., each range will contain 5% of the 1000 observations). The ranges are then found for each interval based on the values of the samples. The expected number of observations

CHI - SQUARE TEST

CELLS	RANGE FROM (INCLUSIVE)	RANGE TO (EXCLUSIVE)	OBSERVED	EXPECTED	$(OBS-EXP)^2/EXP$
1	0.25854	0.53638	100.00000	98.45792	0.02415
2	0.53931	0.59961	50.00000	47.99210	0.08401
3	0.60034	0.65112	50.00000	45.68826	0.40691
4	0.65112	0.70630	50.00000	54.20451	0.32613
5	0.70679	0.75366	50.00000	49.68773	0.00196
6	0.75464	0.79688	50.00000	46.96005	0.19679
7	0.79883	0.84399	50.00000	51.99939	0.07688
8	0.84473	0.89624	50.00000	57.53922	0.98784
9	0.89746	0.93823	50.00000	45.39078	0.46804
10	0.93848	0.98389	50.00000	47.85858	0.09582
11	0.98438	1.02124	50.00000	37.61839	4.07524
12	1.02222	1.08057	50.00000	56.34743	0.71503
13	1.08179	1.13989	50.00000	51.60332	0.04982
14	1.13989	1.21289	50.00000	56.51384	0.75079
15	1.21533	1.28516	50.00000	48.24066	0.06416
16	1.28809	1.39575	50.00000	59.69762	1.57533
17	1.39819	1.53174	50.00000	52.98418	0.16807
18	1.53198	1.71704	50.00000	44.69788	0.62895
19	1.72095	2.79150	50.00000	46.51813	0.26062

CHI - SQUARE = 10.95655

DEGREES OF FREEDOM= 16

PARAMETERS OF THE GAMMA DISTRIBUTION

ALPHA= 6.581878662 BETA= 0.150387466

Figure 5: Chi-square Test on the Interdeparture Times of 1000 Samples for the $E_{10}/E_{10}/1$ Queueing System with ρ=.8

for each interval is then calculated using the Gamma distribution. To avoid too small group size in one interval, the minimum number of expected observations is specified as 10. If it is lower than 10, it is then combined into the next group. As can be seen in the example (Figure 5), the first group contains 100 numbers of observed data and 98.45792 numbers of expected data. The other groups contain equal number of observations.

The Chi-square statistic is calculated as

$$\sum \frac{(Obs - Exp)^2}{Exp} = 10.95655$$

with degrees of freedom = 16

This is quite satisfactory when compared with the critical value of X^2 distribution at 95 % confidence level (it is 26.3). The Chi-square statistic is sensitive to the number of intervals specified, or the sample size. It is applicable appropriately only in an asymtotical sense. The Chi-square statistics of large group size grouping is more satisfactory than that of small group size grouping. However, the Kolmogorov-Smirnov test statistic is much more stable for various group size.

In the Kolmogorov-Smirnov test, the observed cumulative distribution is compared with the expected cumulative distribution at each sample point and the maximum of the deviations is the test statistic. As the group size increases from one (i.e. no grouping), the discreminating power is decreasing, and the test statistic increases. In the cases under study, the test statistic increases as the group size increases. However, when group size is as large as 40, the statistic is 0.03630 (as shown in Figure 6). It is still quite satisfactory as compared with the critical value of Kolmogorov-Smirnov statistic with 25 degrees of freedom at 95% confidence level (it is 0.264). When the group size decreases to 25, the test statistics decreases to 0.02365 with 40 degrees of freedom, and its critical statistic is 0.21.

Based on the Chi-square test statistics and Kolmogorov-Smirnov test statistics obtained for all cases considered in this study (as shown in Appendix B), the null hypothesis formulated will be accepted. That is, the interdeparture times of the consecutive departing transactions from the $E_j / E_k / 1$ queueing systems are approximately Gamma distributed.

KOLMOGOROV - SMIRNOV TEST

CELLS	RANGE FROM	TO	OBSERVED	OBSERVED FREQUENCY	CUMULATVE OBSERVED FREQUENCY	THEORETICAL FREQUENCY	CUMULATIVE THEORETICAL FREQUENCY	KOLMOGOROV - SMIRNOV STATISTIC
1	0.25854	0.44531	40.000	0.040	0.04000	0.00370	0.00370	0.03630
2	0.44751	0.50977	40.000	0.040	0.08000	0.07755	0.08125	0.00125
3	0.50977	0.56226	40.000	0.040	0.12000	0.03568	0.11693	0.00307
4	0.56274	0.61084	40.000	0.040	0.16000	0.03902	0.15595	0.00405
5	0.61304	0.65112	40.000	0.040	0.20000	0.03619	0.19214	0.00786
6	0.65112	0.69604	40.000	0.040	0.24000	0.04378	0.23592	0.00408
7	0.69629	0.73218	40.000	0.040	0.28000	0.03728	0.27320	0.00680
8	0.73242	0.77612	40.000	0.040	0.32000	0.04709	0.32030	0.00030
9	0.77612	0.80688	40.000	0.040	0.36000	0.03369	0.35399	0.00601
10	0.80737	0.84399	40.000	0.040	0.40000	0.04100	0.39499	0.00501
11	0.84473	0.88232	40.000	0.040	0.44000	0.04229	0.43728	0.00272
12	0.88257	0.92334	40.000	0.040	0.48000	0.04467	0.48195	0.00195
13	0.92358	0.95581	40.000	0.040	0.52000	0.03461	0.51656	0.00344
14	0.95752	0.99194	40.000	0.040	0.56000	0.03745	0.55401	0.00599
15	0.99243	1.02124	40.000	0.040	0.60000	0.02938	0.58340	0.01660
16	1.02222	1.06494	40.000	0.040	0.64000	0.04194	0.62534	0.01466
17	1.06641	1.12329	40.000	0.040	0.68000	0.05207	0.67741	0.00259
18	1.12524	1.18091	40.000	0.040	0.72000	0.04666	0.72407	0.00407
19	1.18555	1.22607	40.000	0.040	0.76000	0.03316	0.75723	0.00277
20	1.22827	1.28516	40.000	0.040	0.80000	0.03887	0.79610	0.00390
21	1.28809	1.36401	40.000	0.040	0.84000	0.04422	0.84032	0.00033
22	1.36694	1.47266	40.000	0.040	0.88000	0.04793	0.88825	0.00825
23	1.47437	1.61475	40.000	0.040	0.92000	0.04384	0.93209	0.01210
24	1.61621	1.77563	40.000	0.040	0.96000	0.03070	0.96280	0.00280
25	1.77783	2.79150	40.000	0.040	1.00000	0.03677	0.99956	0.00044

NUMBER OF OBSERVATIONS= 1000

THE KOLMOGOROV - SMIRNOV STATISTIC = 0.03630

DEGREES OF FREEDOM= 25

Figure 6: Kolmogorov-Smirnov Test on the Interdeparture Times of 1000 Samples for the $E_{10}/E_{10}/1$ Queueing System with $\rho=.8$

Relationship Between Input and Output Parameters:

Since the hypothesis on the approximated Gamma distribution of the output process have been accepted statistically, we further proceeded to observe some relationships between the estimated output parameters and the input and transformation parameters. We obtained the following results for the estimated parameters (α and β) of the fitted Gamma distribution.

1. For the $M/E_K/1$ queues, when system utilization is low, α reacts rather insensitively to an increase of k for all $k > 1$. When system utilization is high, α increases monotonically as K increase for all $K > 1$. These relationships are shown in Figure 7 and 8.

2. For the $E_j/M/1$ queues, when system utilization is extremely light, α increases sharply as j increases for all $j > 1$. And when system utilization is high, α remains insensitive to the change of j values for all $j > 1$. These relationships are shown in Figure 9 and 10.

3. For the E_j/E_K /1 queues, when $j = K$, α increases monotonically as $j=k$ increases (as shown in Figure 11). For

any j or k, α is also a function of ρ. That is, α decreases as system utilization increases up to a certain level, and then turns to increase (as shown in Figure 12). From these phenomena, we found that the output of the $E_j/E_k/1$ queues ($j=k$) is neither E_j nor E_k except when the system utilization is extremely light.

4. In general, for the $E_j/E_k/1$ queues, α is a function of ρ. α decreases as ρ increases when $j >> k$, and α increases as ρ increases when $j << k$. For the case with k approximately equal to j, α will first decrease as ρ increases, and then increase as ρ increases beyond a certain value of ρ. Therefore, we may expect that for the $E_j/E_k/1$ queues,

when $j \neq k$

$\alpha \to j$ when $\rho \to 0$, and

$\alpha \to k$ when $\rho \to 1$

for both $j << k$, and $j >> k$ (as can be seen from Figure 13 & 14).

Thus far, we will expect that as $\rho \to 0$, the output distribution tends to be the same as input distribution due to extremely light utilization of the system. And , as $\rho \to$

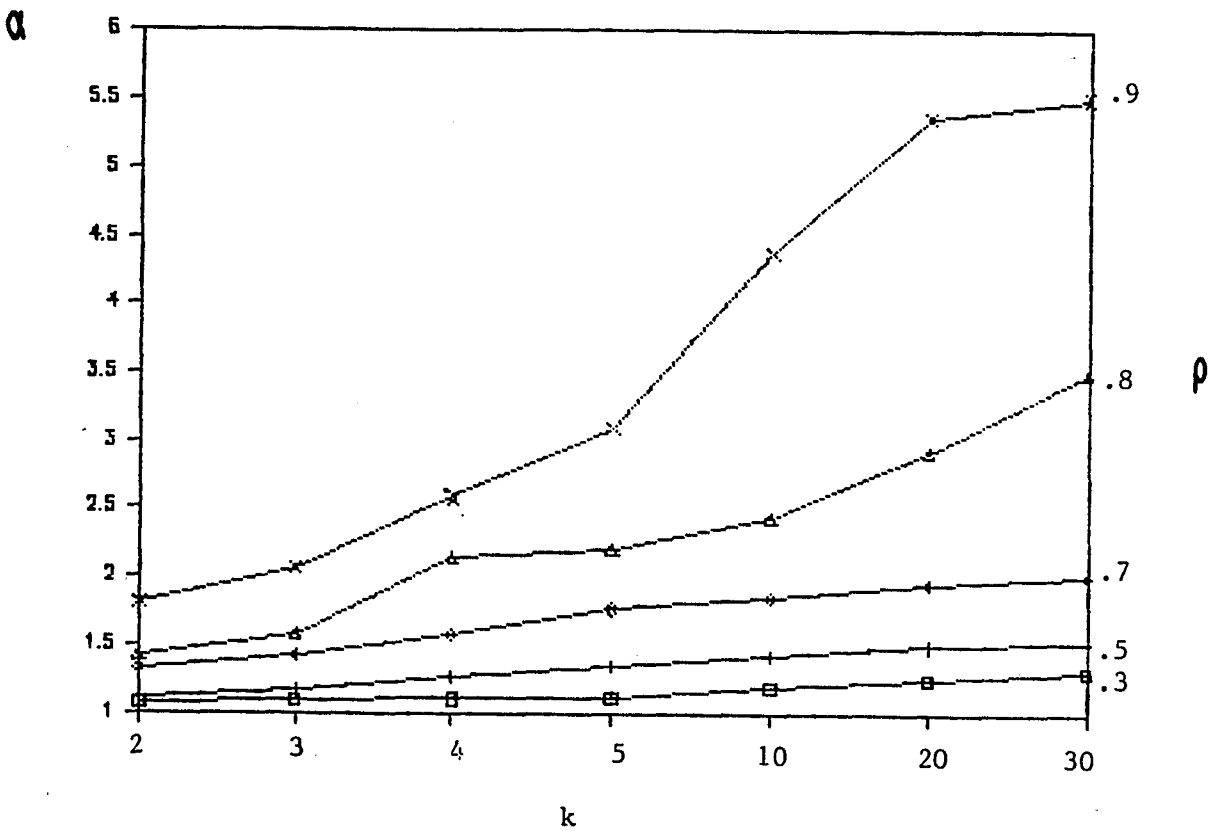


Figure 7: α as a Function of k for j=1 in $E_j/E_k/1$ Queues

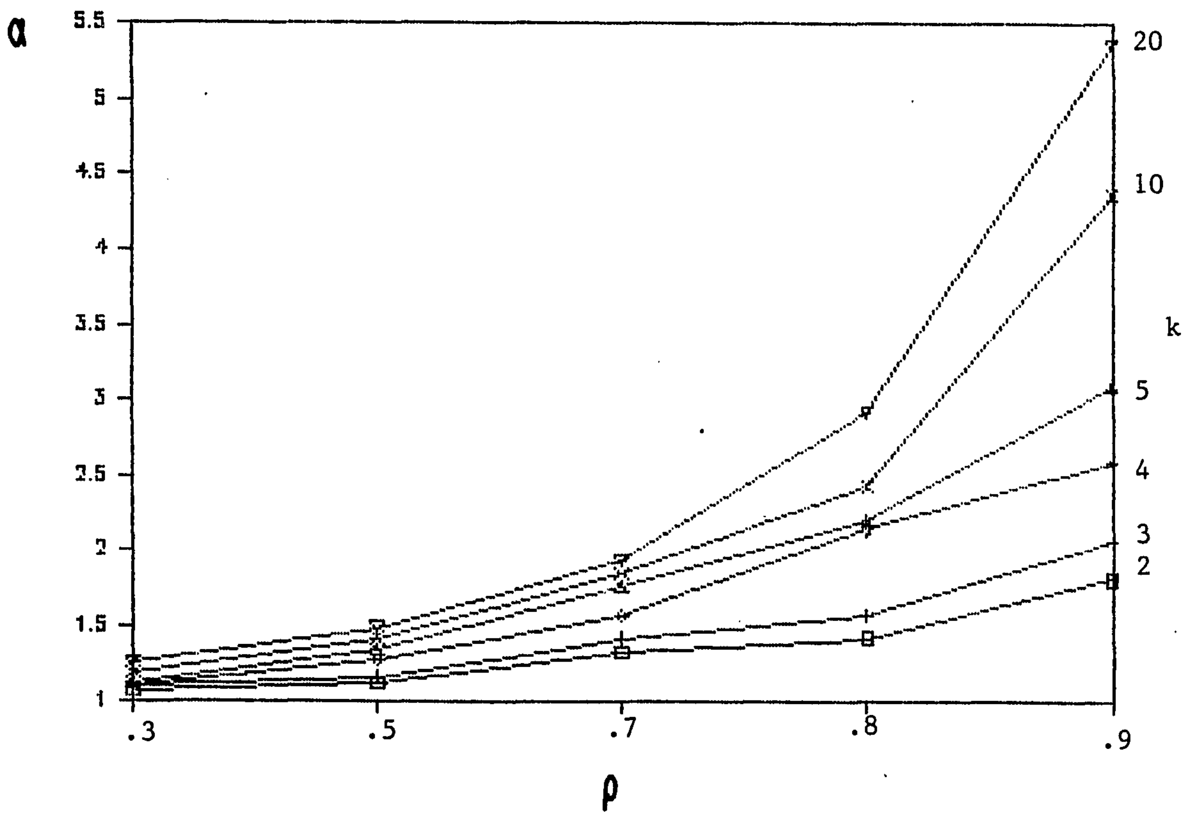


Figure 8: α as a Function of ρ for j=1 in $E_j/E_k/1$ Queues

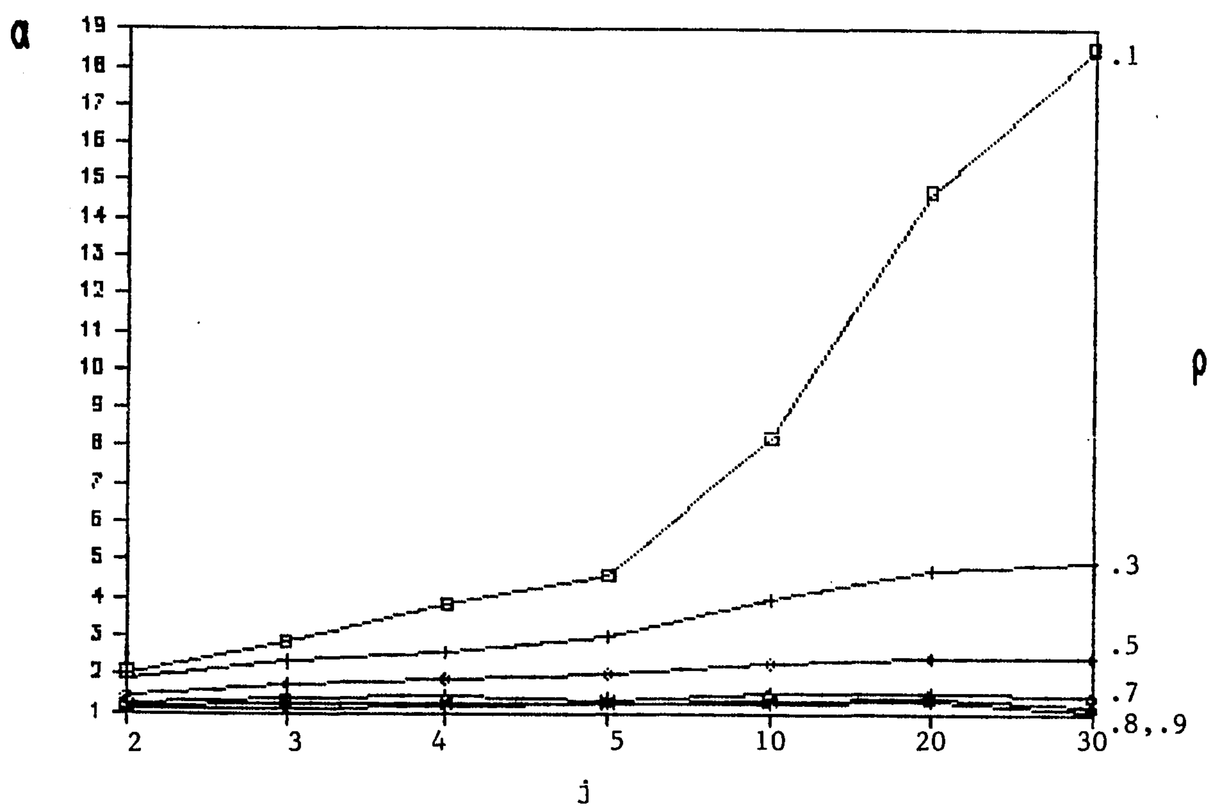


Figure 9: α as a Function of j for k=1 in $E_j/E_K/1$ Queues

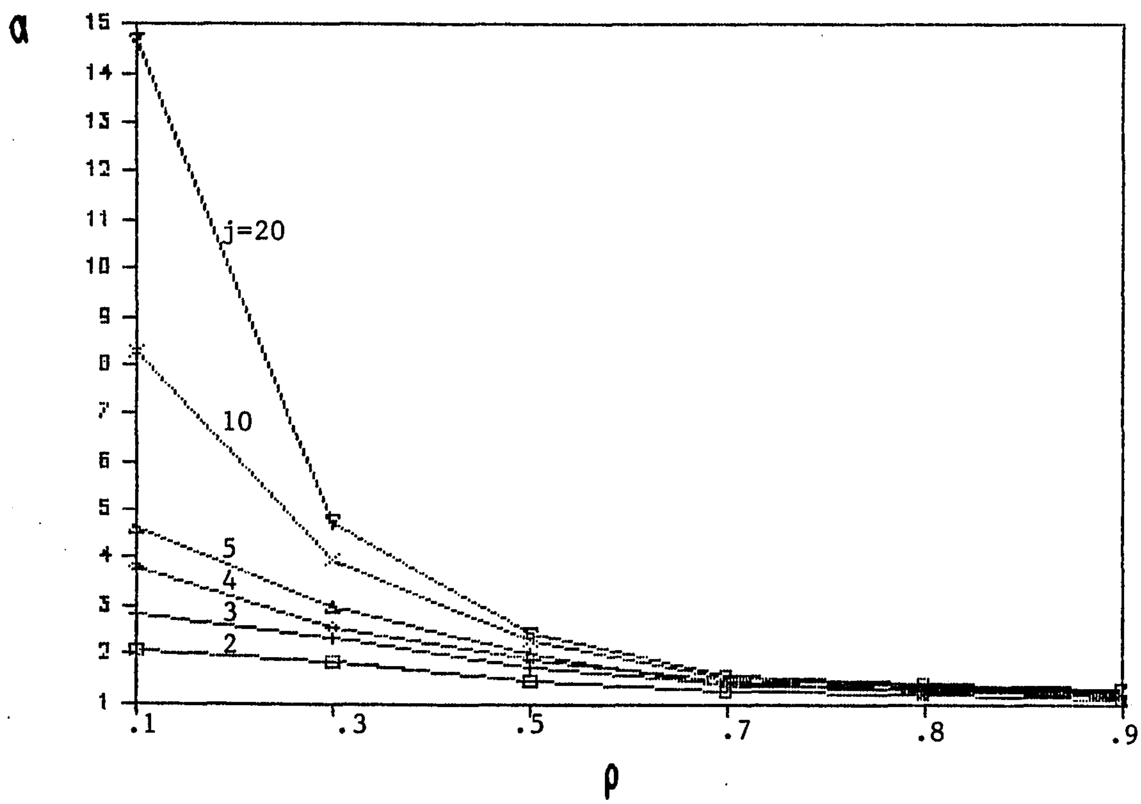


Figure 10: α as a Function of ρ for k=1 in $E_j/E_k/1$ Queues

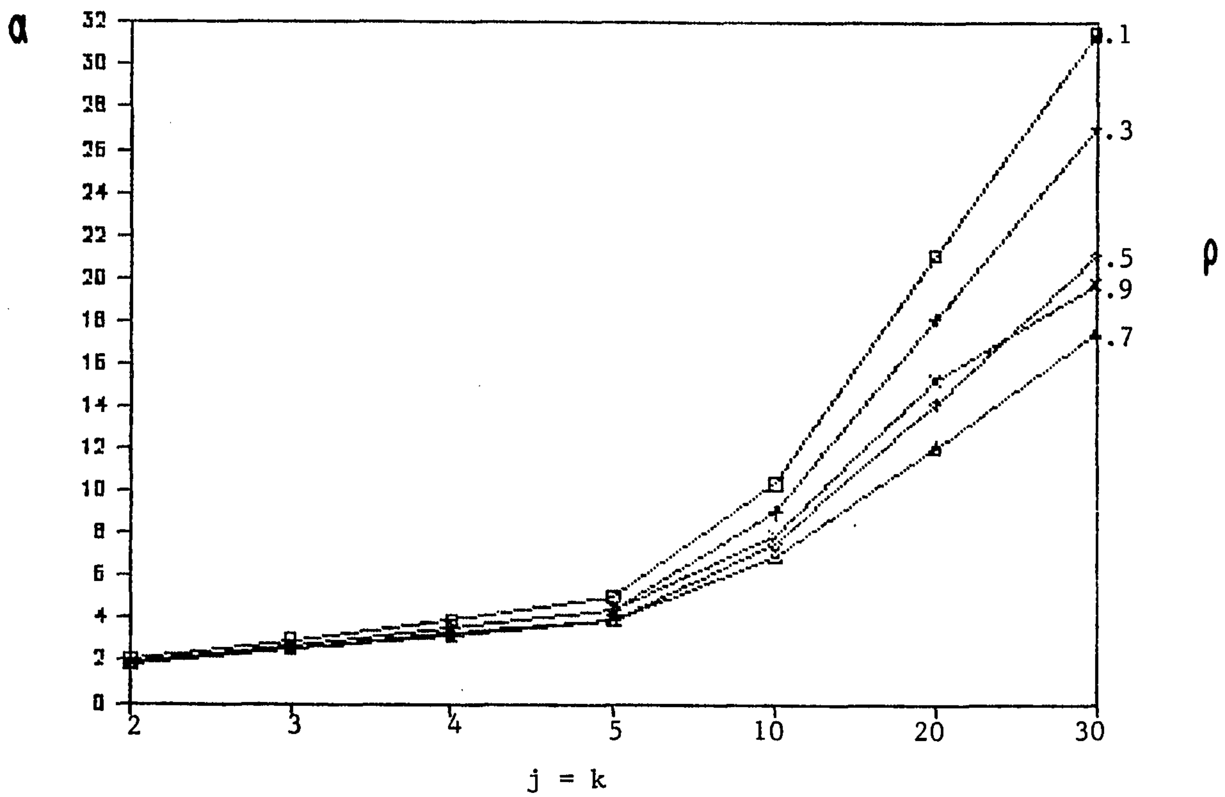


Figure 11: α as a Function of j and k for j=k in $E_j/E_k/1$ Queues

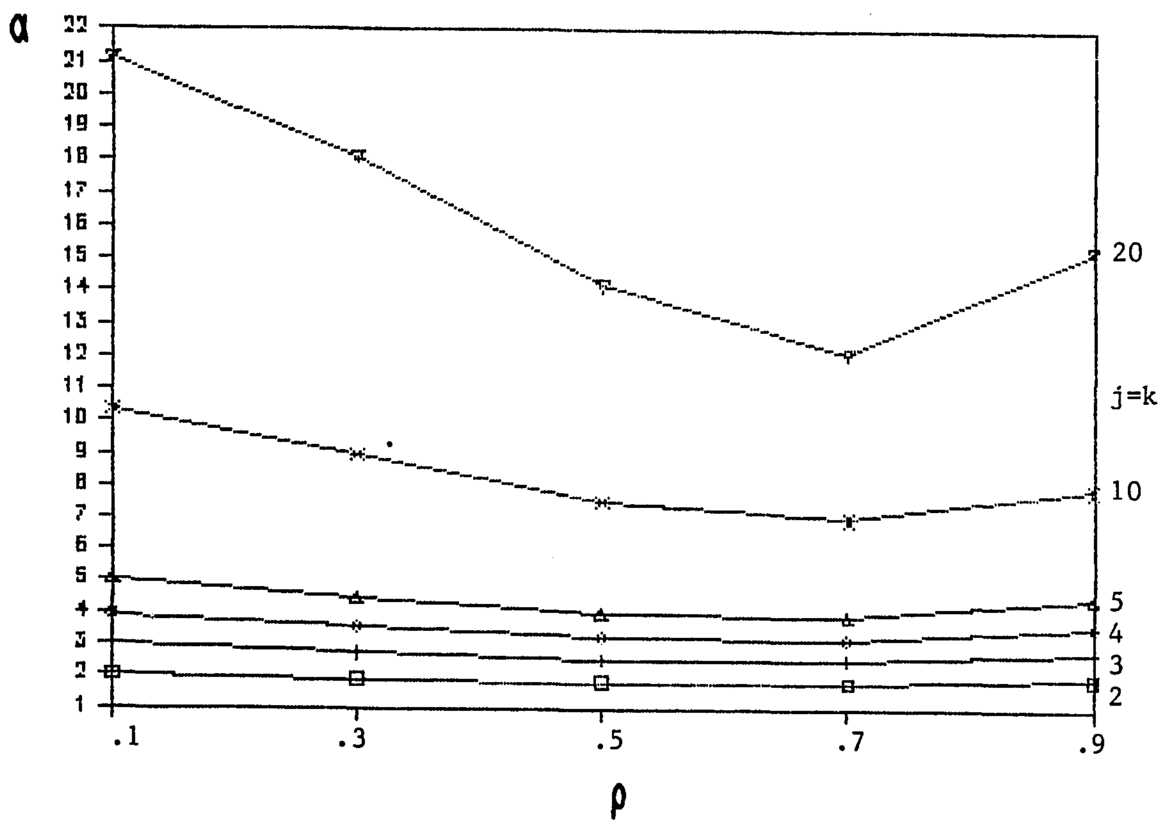


Figure 12: α as a Function of ρ for j=k in $E_j/E_k/1$ Queues

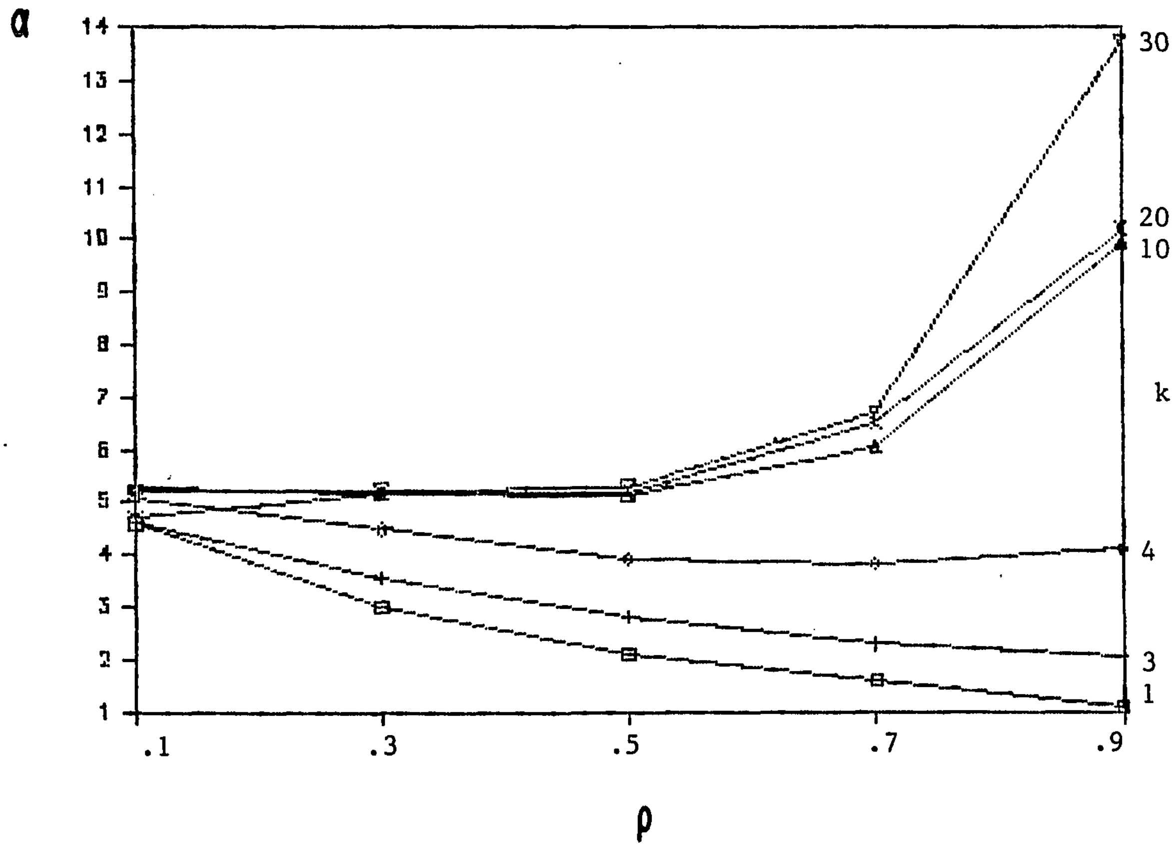


Figure 13: α as a Function of ρ for j=5 in $E_j/E_k/1$ Queues

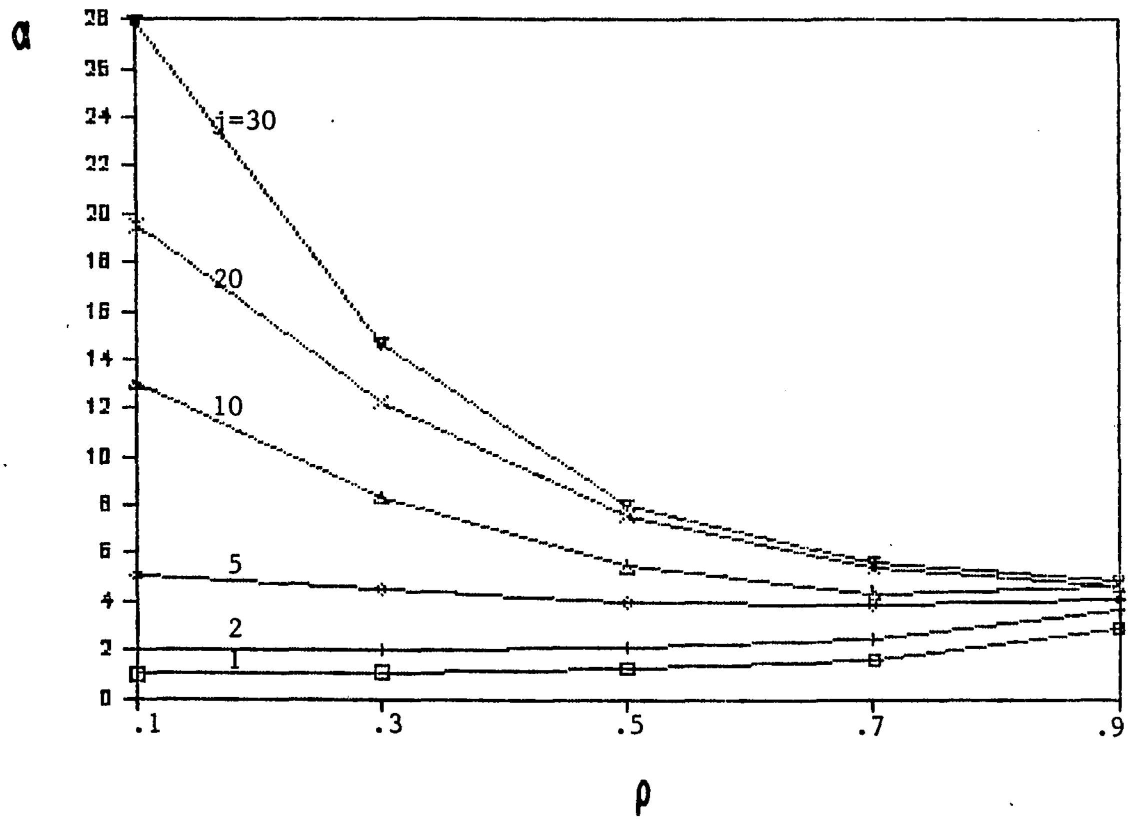


Figure 14: α as a Function of ρ for k=5 in $E_j/E_k/1$ Queues

1, the output distribution tends to be the same as processing time distribution due to extremely heavy utilization of the system.

The statistical relationship of these phenomena can be fitted into a structured power function using multiple regression approximation technique.

For the $E_j/E_K/1$ queues,

Let $\alpha = c1\rho^{c2} j^{c3} K^{c4}$

α is the estimated shape parameter of the fitted interdeparture time distribution.

ρ is the system utilization factor.

Where, c1, c2, c3, and c4 are estimated based on the parameters of all cases of the $E_j/E_K/1$ queues considered in this study. That is, j= 2, 3, 4, 5, 10, 20, 30, K= 1, 5, 10, 20, 30, and ρ= .1, .3, .5, .7, .8, .9 of the $E_j/E_K/1$ queueing systems. Based on the previously observed phenomena, these cases are then decomposed into three groups of cases. These groups are: j much smaller than K (j<<K), j much larger

than k (j>>k), and j approximately equal to k (j $\cong$ k). The coefficients of the power function are then estimated for each group. The results are summaried in Table 1.

As shown in Table 1. The respective R^2 of these approximations are rather significant. The F-statistics are also quite satisfactory when compared with the critical values of the F distribution at 95% confidence level with respective degrees of freedom. The T-statistics of the estimated coeffidcients are also very satisfactory when compared with its critical value at 95% confidence level.

There are some functional relationships that can be examined from the characteristics of these coefficients. Some previously observed relationships between α and ρ , j, or k are now strongly supported by the empirical evidence.

Considering for all cases, the power function was:

$$\alpha = .74674\rho^{-.08358} j^{.55597} k^{.38074}$$

These coefficients agree with some of our previous findings. These were overall, as ρ increases, α will decrease with very low degree of effect. And j plays a

Table 1: Power Approximation on $\alpha = c1\rho^{c2} j^{c3} k^{c4}$ for the $E_j/E_k/1$ queues

	c1	c2	c3	c4	R^2	F-stat. (d. f.)
All Cases	.74674	-.08358	.55597	.38074	.733	227.13(3, 248)
(t-stat)	-3.73869	-2.29829	20.90830	16.0136		
j << k	1.50343	.34550	.73022	.17235	.787	120.88(3, 98)
t-stat.	3.35712	9.64420	14.52690	3.98530		
j >> k	.72352	-.67584	.25616	.63791	.929	300.18(3, 68)
t-stat.	-3.40434	-16.55900	6.22310	15.94920		
j ≅ k	.92633	-.09799	.58711	.31535	.970	811.44(3, 74)
t-stat.	-1.63951	-4.70133	13.58310	7.77585		

more important role than k in determining α value.

For j << k,

$$\alpha = 1.50343\rho^{.345501} j^{.730216} k^{.172354}$$

That is, α will increase as ρ increases, and as $\rho \to 1$, α is more likely determined by j than by k.

For j >> k,

$$\alpha = .72352\rho^{-.675837} j^{.256160} k^{.63791}$$

That is, α will decrease as ρ increases, and as $\rho \to 1$, α is more likely determined by k than by j.

For $j \cong k$,

$$\alpha = .926325\rho^{-.097987} j^{.587105} k^{.315348}$$

That is, α is more likely determined by j or k, than by ρ. However, α is neither equal to j nor equal to k. Therefore, the output process of $E_j/E_k/1$ is not E_j or E_k.

In general, k plays a major role only when j is much greater than k and only as $\rho \to 1$. For other situations, j becomes a more critical role than k. And ρ is significant only when j is much smaller than k, or when j is much greater than k.

To show the validity of the fitted power functions, the α values for k=1, ρ=.3 and j=2, 3, 4, 5, 10, 20, 30 are calculated using the fitted power function

$$\alpha = .72352\rho^{-.675837} j^{.256160} k^{.63791}, \quad j >> k$$

These α values are then plotted and compared with the actual observations (as shown in Figure 15).

Since α is the shape parameter and β is the scale parameter in the fitted Gamma distribution, the mean of the Gamma distribution is therefore equal to $\alpha\beta$. In the $E_j/E_k/1$ queueing system under study, the mean interdeparture time is expected to be approximately equal to the mean interarrival time based on the input-output equivalence property of a steady-state queueing system. The numerical evidence on the steady-state property is obtained when the β function is also approximated in the following same power function.

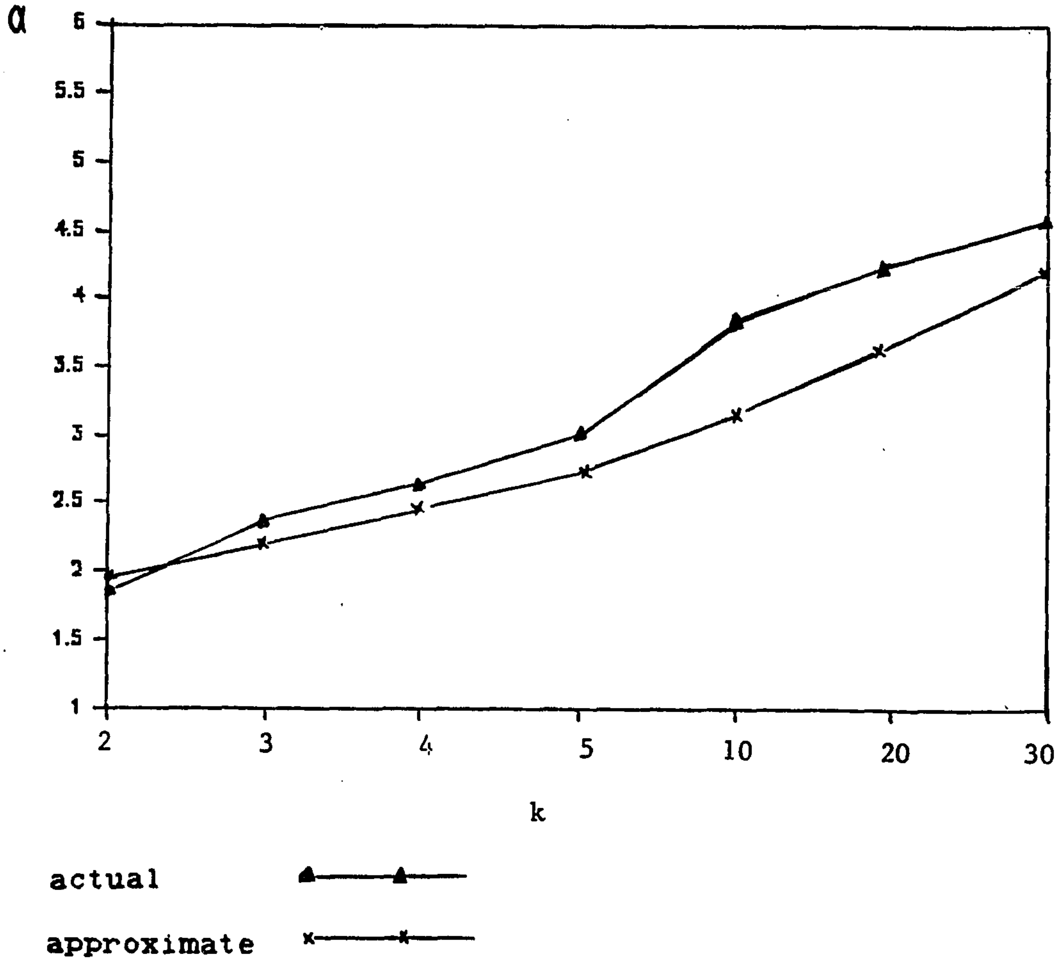


Figure 15. The actual vs. the approximate α values for k=1 in $E_j/E_k/1$ queues with ρ=.3

For the $E_j/E_K/1$ queues, let

$$\beta = e1\rho^{e2} j^{e3} K^{e4}$$

β is the estimated scale parameter of the fitted Gamma distribution

ρ is the system utilization factor.

Again, the coefficients e1 , e2 , e3 , and e4 are estimated based on the parameters of all cases of the E_j/E_K /1 queues under study. These cases are then decomposed into three groups: $j << k$, $j >> k$, and $j \cong k$. And the coefficients are estimated for each group separately. The results are summarized in Table 2.

For all cases,

$$\beta = 1.34474\rho^{.086144} j^{-.559362} K^{-.378276}$$

That is, overall, the role of ρ in β function is quite

Table 2: Power Approximation on $\beta = e1\rho^{e2} j^{e3} k^{e4}$ for the $E_j/E_k/1$ queues

	e1	e2	e3	e4	R^2	F-stat. (d. f.)
All Cases	1.34474	.08614	-.55936	-.37828	.737	231.32 (3, 248)
t-stat.	3.82020	2.38639	-21.19190	-16.02810		
j << k	.66259	-.33881	-.74430	-.16275	.802	132.50 (3, 98)
t-stat.	-3.53706	-9.87123	-15.45490	-3.92780		
j >> k	1.38016	.67576	-.25654	-.63804	.930	301.68 (3, 68)
t-stat.	3.39677	16.59310	-6.24594	-15.98730		
j ≅ k	1.07933	.09759	-.59684	-.30628	.970	820.98 (3, 74)
t-stat.	1.64452	4.70837	-13.88470	-7.59396		

minimal, and β is more likely determined by k than by j.

For j << k,

$$\beta = .662594\rho^{-.338809} j^{-.744298} k^{-.162745}$$

That is, as ρ increases, β will decrease, and as ρ approaches 1, β is more likely determined by k than by j.

For j >> k,

$$\beta = 1.380162\rho^{.675757} j^{-.256540} k^{-.638040}$$

That is, as ρ increases, β will increase, and as ρ approaches 1, β is more likely determined by j than by k.

For j $\cong$ k

$$\beta = 1.07933\rho^{.0975941} j^{-.596837} k^{-.306278}$$

That is, the role of ρ is quite minimal, and β is more likely determined by j or k.

These results can be considered as complementary to the results found for α, since $\alpha\beta = 1$. As α increases β will decrease, therefore, the relationship between β and ρ, j, or k obtained from the approximated β function is the complement of those for the approximated α function. The R^2 statistics, F- statistics and t- statistics for the functions are quite close to those obtained for the function. The coefficients e1 , e2, e3, and e4 are complementary of c1 , c2, c3, and c4 in the following approximated relationships: (as can be seen from Table 3).

In that,

$$c1 \cdot e1 \cong 1$$

$$c2 + e2 \cong 0$$

$$c3 + e3 \cong 0$$

$$c4 + e4 \cong 0$$

Here, $ci + ei \cong 0$ $(i \neq 1)$ are as expected since α & β are estimated by method of moments and $\alpha\beta$ is the mean of the approximated Gamma distribution.

Table 3: Relationship of c_i and e_i of Power Approximation on the α and β function

	c1 + e1	c2 + e2	c3 + e3	c4 + e4
All Cases	1.00417	.00256	-.00339	.00246
j << k	.99616	.00669	-.01408	.00960
j >> k	.99857	-.00008	-.00038	-.00013
j ≅ k	.99982	-.00040	-.00973	.00907

Therefore,

$$\alpha\beta = c1\rho^{c2} j^{c3} K^{c4} \; e1\rho^{e2} j^{e3} K^{e4}$$

$$= c1e1\rho^{c2+e2} j^{c3+e3} K^{c4+e4}$$

$$\cong c1e1\rho^{0} j^{0} K^{0}$$

$$\cong c1 \cdot e1$$

From this result, we know that the mean of the approximated Gamma distribution is approximately equal to the constant $c1 \cdot e1$ which is approximately equal to 1 in these cases. Since the mean interarrival times of the E_j/E_K /1 queues have been set to be 1, $\alpha\beta \cong 1$ is as expected. Therefore, the input-output equivalence of a steady-state queueing system has been obtained for the class of queueing system (E_j/E_K /1) under study. These relationships of the estimated coefficients also indicate the validity of the queueing models under study.

Based on the testing results, we conclude that the output process of the $E_j/E_K/1$ queues approximately follows the Gamma distribution. And its shape parameter and scale parameter are well approximated by a power function of the input and processing time parameters.

Interdeparture Time Correlation Structure

For the $E_j/E_K/1$ queueing systems considered, the interrelationship between the consecutive departing transactions is under examination by means of covariance analysis. The covariance of consecutive departure intervals may indicate the interdependence between the departing transactions and signify some degree of effect on the waiting time measurement in the following stations when queues are arranged in series.

Again, for the cases considered, five independent runs are made for 3000 transactions. The Lag-1 covariance and autocorrelation are calculated by taking a common mean

i.e.,

$$Cov(D_n, D_{n+1}) = E[(D_n - \mu)(D_{n+1} - \mu)]$$

and are also calculated by using their individual mean i.e.,

$$Cov(D_n, D_{n+1}) = E\{[D_n - \mu(D_n)][D_{n+1} - \mu(D_{n+1})]\}$$

The fact that there was no significant difference between the covariances obtained indicates that a reasonable steady state had been reached. The correlation coefficients are also calculated to obtain a better measure of the linear relationship between D_n and D_{n+1}. The covariances and correlations obtained are summarized in Appendix C.

Based on the results summarized in Appendix C, some observations can be made about the relationship between the covariance or correlation and the input and transformation parameters. These are:

1. For the $M/E_k/1$ $(k \neq 1)$ queues, correlations are positive and first increase as ρ increases up to a certain level and then decrease (as shown in Figure 16). It can be seen that the correlation is also a function of k. The correlation increases as k increases and approaches zero as k moves much lower.

2. For the $E_j/M/1$ $(j \neq 1)$ queues, correlations are negative and decrease as ρ increases up to a certain level and then increase (as shown in Figure 17). The correlation is also a function of j. As j decreases, the correlation decreases and approaches to zero as j moves much lower.

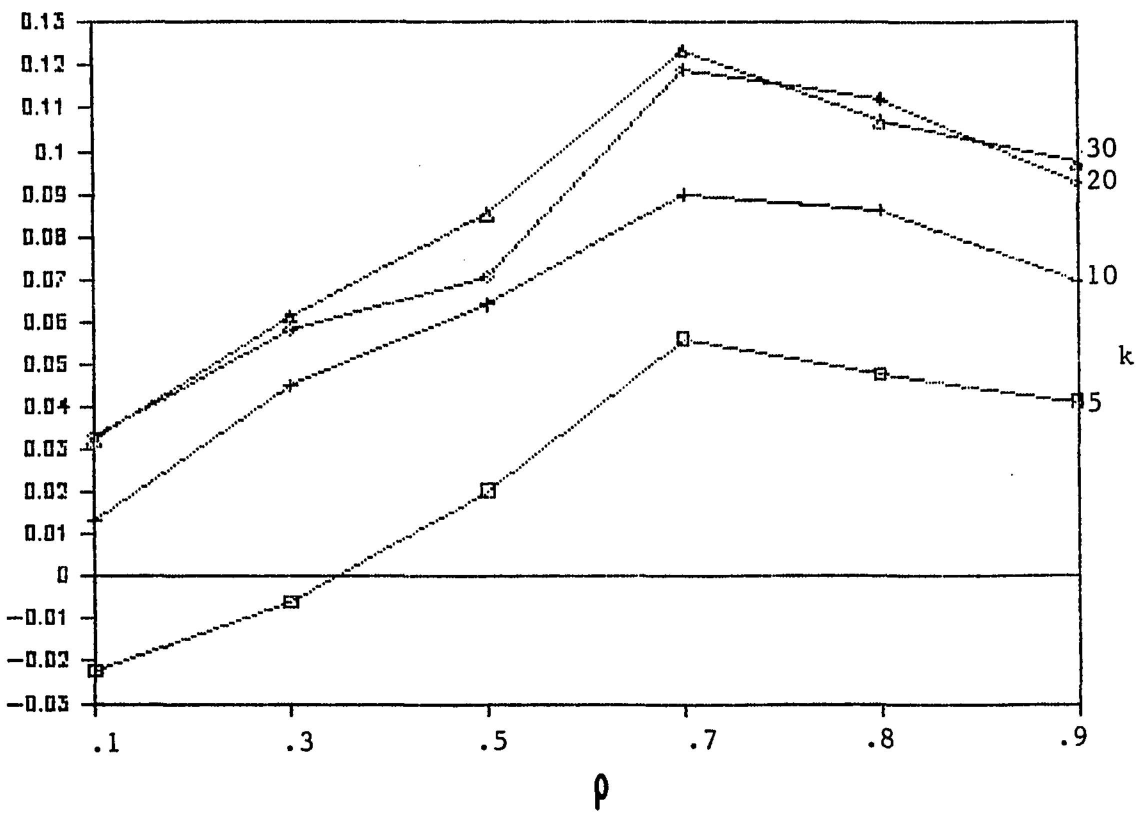


Figure 16: Correlation as a Function of ρ for j=1 in $E_j/E_K/1$ Queues

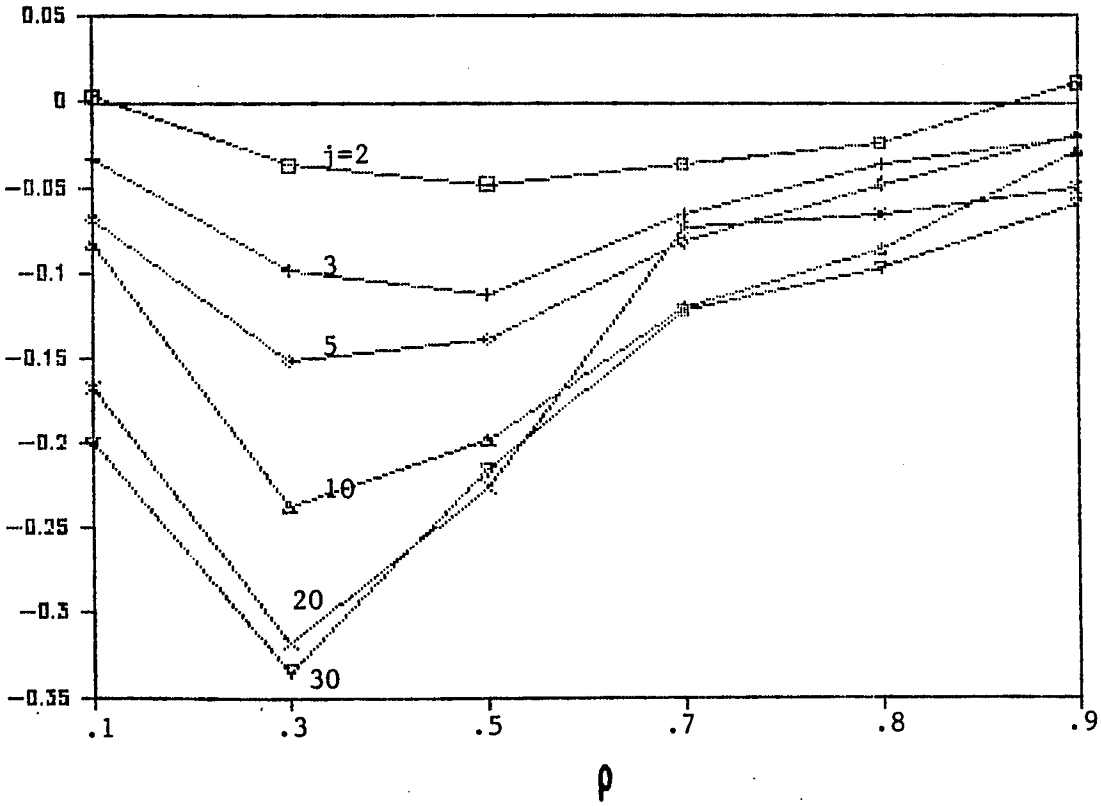


Figure 17: Correlation as a Function of ρ for k=1 in $E_j/E_K/1$ Queues

3. For the $E_j/E_K/1$ queues, when $j = k \neq 1$, correlations are negative and their magnitude increases as $j = k$ increases. As ρ increases the magnitude of correlations first increases then decreases as ρ moves higher. The maximum magnitude is reached around a moderate system utilization level (i.e. approximately .5 for $j = k$ is lower than 5, approximately .7 for $j = k$ is 5 or higher). When ρ is extremely low, correlation is not significant and is independent of j or k (as shown in Figure 18).

4. In general, for the $E_j/E_K/1$ ($j \neq 1$, and $k \neq 1$) queues, correlation or covariance is a function of ρ, j, and k. When $j \cong k$, or j is greater than k, correlation is negative and its magnitude increases as ρ increases up to a certain level then turns to decrease. On the other hand when j is much smaller than k, correlation is positive and its magnitude increases as ρ increases up to a particular level then turn to decrease (as shown in Figure 19).

Based on these phenomena, we may expect that for the $E_j/E_K/1$ queues,

as ρ -> 0, correlation -> 0

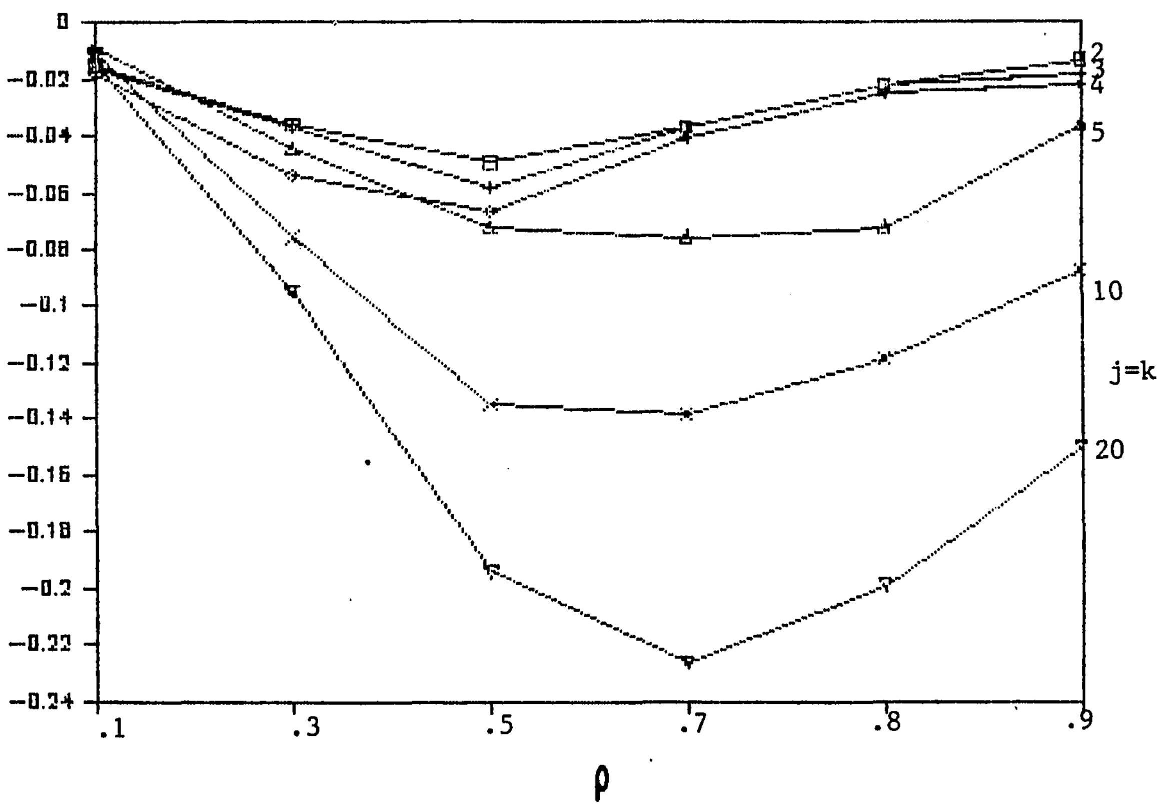


Figure 18: Correlation as a Function of ρ for j=k in $E_j/E_k/1$ Queues

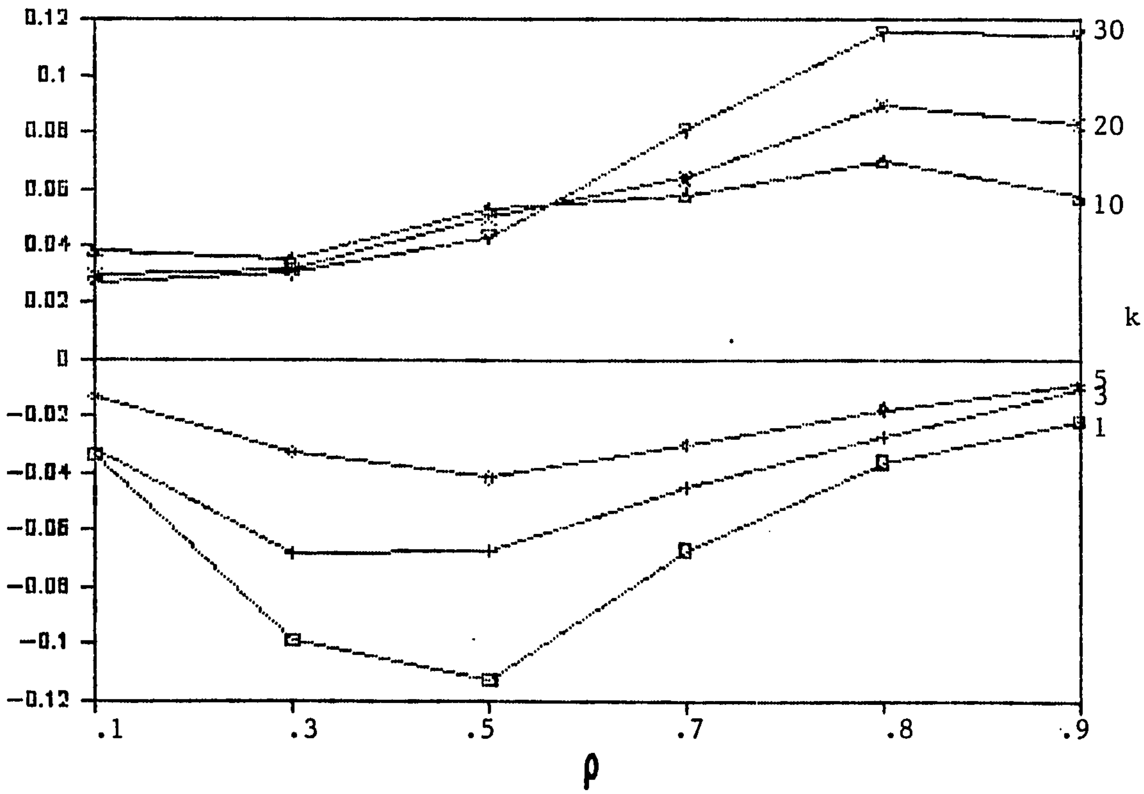


Figure 19: Correlation as a Function of ρ for j=3 in $E_j/E_k/1$ Queues

This indicates that the consecutive departure intervals are independent of each other. That is, as the system utilization is extremely low, there will be almost no congestion at the service station and the departure process is approximately the joint distribution of the arrival distribution and the service time distribution. The departure process is therefore statistically independent.

as $\rho \rightarrow 1$, correlation $\rightarrow 0$

This again indicates an independent departure process. That is, when the system utilization is extremely high, there will be almost no idle time occurred in the service facility and the interdeparture times will correspond approximately to the independent service times.

The above observed phenomena do not seem to fit into the power function as previously used to obtain the α and β parameters. In fitting the following power functions, let

$$\text{Covariance} = c_1 \rho^{c_2} j^{c_3} K^{c_4}$$

$$\text{Correlation} = e_1 \rho^{e_2} j^{e_3} K^{e_4}$$

The covariance and correlation observations were then divided respectively into two groups. One group contained all positive values, the other group contained all negative values. The R^2 statistics and F-statistics for both the negative and positive covariances and correlations turn out to be rather insignificant. This is intuitively true since the concave or convex distribution of the data do not fit into the power function.

The functional relationships of these phenomena are then fitted into a quadratic function using multiple regression approximation technique.

Let

$$Cov = c1 + c2\rho^2 + c3j^2 + c4k^2 + c5\rho j + c6\rho k + c7j + c8k + c9\rho$$

and

$$Corr = e1 + e2\rho^2 + e3j^2 + e4k^2 + e5\rho j + e6\rho k + e7j + e8k + e9\rho$$

Where, ci and ei (i= 1, 2, . . . , 9) are estimated based on the parameters of all cases of the E_j /E_k /1 queues. The results obtained are summarized in Table 4. In

Table 4: Approximation on the Covariance and Correlation Function for $E_j/E_K/1$ Queues

Covariance			Correlation		
	Coefficients	T-statistics		Coefficients	T-statistics
c1	-.01026	-2.29303	e1	.02168	1.38066
c2	.07091	5.20515	e2	.39735	8.31024
c3	.00008	6.89848	e3	.00028	6.46554
c4	-.00011	-10.16220	e4	-.00012	-3.36083
c5	-.00157	-5.05560	e5	-.00442	-4.04473
c6	.00087	3.05925	e6	.00032	.32055
c7	-.06703	-4.55023	e7	-.32496	-6.28515
c8	-.00280	-6.63327	e8	-.01325	-8.94726
c9	.00409	11.14820	e9	.00684	5.30570
R^2=.71816			R^2=.77056		
F-statistic= 77.3988(8, 243)			F-statistic= 102.012(8, 243)		

that, the R^2 statistics on the covariance function is 0.7182 and is 0.7706 on the correlation function. The F-statistics on the covariance is 77.3988 and is 102.012 on the correlation function with degrees of freedom equal 8 and 243. These statistics are quite satisfactory when compared with the critical value at 5% significance level (it is 2.93).

Based on the results shown in Table 4, ρ plays the most significant role in these approximated functions. Overall, as ρ increases, covariance and correlation will increase. However, the relationship between j/k and covariance/correlation is negative, as j or k increases, covariance or correlation will decrease.

So far, the first hypothesis in this study has been tested numerically. Based on these numerical evidence, the first hypothesis is accepted. That is, the output process of $E_j / E_k / 1$ queueing system follows approximately Gamma distribution, and the estimated poarameters of the Gamma distribution are approximately in a power function of the parameters of the interarrival time and service time distributions. The covariance of interdeparture time is approximately in a quadratic function of the parameters of the interarrival time and service time distributions.

Hypothesis Testing For Significance of Correlation:

The covariances obtained for all cases were found to be nonzero. This implies that the interdeparture times between the consecutive transactions are <u>not</u> independent. The significance level of the non-zero covariance and its degree of effect in a tandem queue are therefore questioned. To resolve these questions, the significance level of the covariance should be tested statistically, and its effect in a tandem queue should be investigated by comparing the results of independency assumptions with that of dependency considerations on the arrival process of the second station.

To test the significance of the correlation, the t-statistic is computed. The hypothesis of zero correlation between consecutive departing transactions is formulated as follows:

Null Hypothesis: $H_0 : q = 0$
(i.e., the consecutive departure intervals are independent of each other.)

Alternative Hypothesis: $H_1 : q \neq 0$

(i.e., the consecutive departure intervals are dependent of each other)

Where q is the true Lag-1 autocorrelation of the consecutive interdeparture_times of the departing transactions.

Then,

$t = (r - q)/S_r$

has t distribution with N-2 degrees of freedom.

Where,

r: the sample lag-1 autocorrelation between the consecutive departing transactions

S_r: $\sqrt{(1-r^2)/(N-2)}$ is the standard error of the sample autocorrelation.

The decision rule is:

At the α% significance level,

Accept H_0 if $t < t_{1/2\alpha;N-2}$,

Accept H_1, otherwise

To reject the null hypothesis, the absolute value of correlation must be greater than 0.0619 for sample size equals 1000. Among the cases under study, the absolute value of the correlation is found to be larger than the critical statistic value for some but not all cases. These include cases with j is much smaller than k, j is much larger than k, or j and k are both large. _However, when the system utilization is extremely low or extremely high, the absolute value of correlation is smaller than the critical statistic value except when both j and k are large. The correlation which is larger than .0619 is considered significantly different from zero at 5% significance level. This indicates the high significance of consecutive transaction interrelationship. Therefore, the consecutive departing transactions will be statistically independent when the system utilization is extremely low or extremely high when j and k are not large.

Based on these results, we expect to find the effect of this interrelationship on the following stations in a tandem queue. And the sign of the correlations is concerned when the nature of the effect is under examination. For the cases under study, the sign of the correlations is found to be positive for those cases where j is much smaller than k; and negative for those cases where j is close to k or j is larger than k.

To test whether the correlation is significantly positive or negative, the alternative hypothesis can be formulated as a one-sided hypothesis test.

That is,

H_1': $q > 0$ (i.e., the true autocorrelation is positive)

or,

H_1'': $q < 0$ (i.e., the true autocorrelation is negative)

The decision rules become:

At the α% significance level,

Accept H_0 if $t < t_{\alpha; N-2}$

Accept H_1', otherwise (i.e., the correlation is significantly positive)

or,

Accept H_0 if $t > t\alpha; N-2$

Accept H_1'', otherwise (i.e., the correlation is significantly negative)

To accept the alternative hypothesis of $q > 0$, the value of correlation must be greater than 0.052, and be smaller than -0.052 for accepting $q < 0$. For the cases studied, the null hypothesis is rejected (i.e., accept H_1') when j is much smaller than k with moderate or high system utilization. That is, the sign of correlation is predicted to be positive. And the null hypothesis is also rejected (i.e., accept H_1'') when j is much larger than k or when both j and k are large with moderate or high system utilization. This indicates that correlation is significantly negative under those circumstances..

Covariance Effect in Tandem Queues:

The results of t-test in the previous section demonstrated the interdependence of consecutive departing transactions of the $E_j/E_K/1$ queueing systems. The effect of this dependence on the performance measurement of the next station in a tandem queue is next investigated. In order to find out the degree of the covariance effect, the waiting time in the next station is found by decomposition as well as by compression techniques. The discrepancy between these two sets of waiting times is then computed and is compared to the characteristics of covariance found previously. The purpose is to find the relationship between the underestimate/overestimate of waiting time and both the positive/negative covariance and the magnitude of the covariance values.

Decomposition in tandem queues treats each station separately and ignores the dependence between the departure and the arrival of an intermediate station. In measuring the average total time a transaction stays in the entire system, it will just sum up the average time spent in each independent station as the total time. This treatment may result in an underestimate or overestimate of the actual time spent in the system. The performance measurement of the entire system will be inaccurate due to a failure to consider the interaction between consecutive stages. In the input-output control technique, the knowledge of the actual system performance is rather critical.

In the tandem queue (E_j /E_K/1 -->./M/1), the output of the E_j /E_K/1 queue forms the input of the second station. Therefore, the mean interarrival time of the second station is just the mean interdeparture time of the first station. Logically, the dependent departure process of the first station will result in a dependent arrival process at the second station. The dependent departure process has been strongly supported by the non-zero covariance found for the E_j/E_K /1 queue in previous section. In order to find the non-zero covariance effect on the waiting time in the second station, the interaction between the consecutive stations will be first ignored. The results are then compared with those of dependency considerations.

Here, the second station is treated as a GI/M/1 queue. That is, the interarrival process is a sequence of independent and identically distributed random variables. By using the previously obtained Gamma distribution of the output process of the E_j /E_K /1 queues, the arrival process of the GI/M/1 queue can be approximated as Gamma distribution with parameter α and β. Therefore, in the

GI/M/1 queue,

$$Ew = \frac{\nu}{\mu(1-\nu)}$$

Where ν is an unique root of

$$\nu[(\mu - \mu\nu) - 1/\beta]^{\alpha} = 1/\beta^{\alpha}$$ (as shown in Chapter III).

Where α and β are as previously found, and μ is the service rate in the second station.

By substituting α and β of each case into this equation, with given value of μ, ν can be obtained by Newton's approximation method since α is not exactly equal to 1 or 2 for which an exact solution can be obtained (as shown in Chapter III). The expected waiting times are thus obtained for the GI/M/1 queues.

The approximate waiting time can be found as a function of α and β for a specified μ. It decreases as ρ increases for $j << k$, and increases as ρ increases for $j >> k$. As μ decreases from 2 to 1.25, the expected approximate waiting time increases. That is, as the system utilization is higher, the expected waiting time will be longer.

Compression technique however considers the interrelationship between the consecutive stations in the system flow. With the aid of computer techniques, the expected actual system performance can be obtained. However, sometimes the large number of stations in a system for which the exact measurement can be obtained is limited by the computer capacity and cost concerned. The decomposition technique is usually employed to approximate the system performance. The knowledge of % error from the independence treatment may therefore be used to adjust the decomposition results and reduce the discrepancy in the estimation of system performance so as to achieve a better input-output control of the entire system.

With compression, the E_j/E_K /1 -->./M/1 tandem queue is treated as a whole system flow. As the transaction departs from the first station, it will join the queues in front of the second station if the service facility in the second station is busy; otherwise enters the service facility directly. The waiting time in the second queue of each transaction are collected and the average waiting time is calculated. This is considered as the actual waiting time in the second station.

The 95% confidence interval of the actual waiting time is constructed for each case. The approximate waiting time found is then compared with the confidence interval. If it falls outside the confidence interval, the approximate waiting time is said to be significantly different from the actual waiting time at 5% significance level.

For those cases with $j \cong k$, the approximate waiting times are not significantly different from their actual waiting times except when both j and k are large. For those cases with j much smaller than k or with j much larger than k, the approximate waiting times fall outside the confidence intervals only when system utilization is extremely low or extremely high. Based on these confidence intervals, the % error is considered to be significant if it is 6% or higher. The % errors are computed for cases with j is much smaller than k, j is much larger than k, and j is approximately equal to k. These results are summarized in Table 5, 6 and 7.

For the cases with j much smaller than k, the % errors are negative. That is, the expected waiting time in the second station will be underestimated if interdependence between the consecutive arrivals is ignored. On the other hand, for the cases where j much larger than k, the % errors are positive. That is, the expected waiting time in the second station will be

Table 5: Waiting Time in the Second Stage of $E_j/E_k/1$ -> ./M/1 Queues (j << k)

			μ = 2			μ = 1.25		
ρ	j	k	EW	WT	% ERROR	EW	WT	% ERROR
.3	2	20	.3021	.3227	-6.35	2.2748	2.5133	-9.49
.5	2	20	.2800	.3073	-8.88	2.1564	2.3831	-9.51
.7	2	20	.2357	.2575	-8.47	1.7987	2.0015	-10.13
.9	2	20	.1835	.2097	-12.49	1.5582	1.7542	-11.17
.3	2	30	.3018	.3318	-9.04	2.1281	2.3282	-8.59
.5	2	30	.2797	.3081	-9.22	2.0648	2.2686	-8.98
.7	2	30	.2437	.2805	-13.12	1.8334	2.1307	-13.95
.9	2	30	.1943	.2314	-16.03	1.3710	1.6180	-15.27
.3	3	20	.2469	.2898	-14.80	1.8417	2.2198	-17.03
.5	3	20	.2374	.2646	-10.28	1.7941	2.1364	-16.02
.7	3	20	.2099	.2414	-13.05	1.6503	2.0075	-17.79
.9	3	20	.1717	.2005	-14.36	1.3820	1.7455	-20.82
.3	3	30	.2537	.2721	-6.76	1.9471	2.2360	-12.92
.5	3	30	.2420	.2850	-15.09	1.8720	2.2040	-15.06
.7	3	30	.2178	.2399	-9.21	1.7636	1.9714	-10.54
.9	3	30	.1682	.1842	-8.69	1.3512	1.6940	-20.24
.3	4	20	.2074	.2259	-8.19	1.7690	2.1785	-18.80
.5	4	20	.2036	.2176	-6.43	1.7425	2.2934	-24.02
.7	4	20	.1883	.1982	-4.99	1.7184	1.9385	-11.35
.9	4	20	.1637	.1775	-7.77	1.5037	1.6686	-9.88
.3	4	30	.2256	.2344	-3.75	1.8705	2.3230	-19.48
.5	4	30	.2191	.2542	-13.81	1.8028	2.1871	-17.57
.7	4	30	.1974	.2200	-10.27	1.6800	1.8978	-11.48
.9	4	30	.1617	.1868	-13.44	1.3419	1.6290	-17.62

EW: The approximated waiting time by decomposition
WT: The actual waiting time by compression
ρ: System utilization
μ: Service rate in the second station

TABLE 6: Waiting Time in the Second Stage of $E_j/E_k/1$ -> ./M/1 Queues (j >> k)

			μ = 2			μ = 1.25		
ρ	j	k	EW	WT	% ERROR	EW	WT	% ERROR
.3	10	1	.2179	.2050	6.29	1.7731	1.4764	20.10
.5	10	1	.2873	.2414	19.01	2.0839	1.8332	13.68
.7	10	1	.3679	.3335	10.31	2.4362	2.1489	13.37
.9	10	1	.4522	.4340	4.19	2.9473	2.4613	19.75
.3	10	5	.1759	.1694	3.84	1.5697	1.3195	18.96
.5	10	5	.1918	.1608	19.28	1.6581	1.3597	21.95
.7	10	5	.2037	.1829	11.37	1.7241	1.4437	19.42
.9	10	5	.2039	.1924	5.98	1.6660	1.5535	7.24
.3	20	1	.2014	.1892	6.45	1.7954	1.5020	19.53
.5	20	1	.2831	.2528	11.99	2.1843	1.8174	20.19
.7	20	1	.3570	.3030	17.82	2.3892	1.9257	24.07
.9	20	1	.4381	.4127	6.15	2.8360	2.1279	33.28
.3	20	5	.1559	.1467	6.27	1.5472	1.3814	12.00
.5	20	5	.1743	.1526	14.22	1.6304	1.4071	15.87
.7	20	5	.1971	.1812	8.77	1.7074	1.5371	11.08
.9	20	5	.1996	.1915	4.23	1.7034	1.5950	6.80
.3	30	1	.2024	.1846	9.64	1.7362	1.5177	14.40
.5	30	1	.2808	.2414	16.32	2.0953	1.7913	16.97
.7	30	1	.3835	.3486	10.01	2.6863	2.1972	22.26
.9	30	1	.4892	.4662	4.93	3.4009	2.5774	31.95
.3	30	5	.1536	.1351	13.69	1.4841	1.2660	17.23
.5	30	5	.1743	.1556	12.02	1.6464	1.4708	11.94
.7	30	5	.1954	.1595	22.51	1.7365	1.5609	11.25
.9	30	5	.2042	.1937	5.42	1.6828	1.4647	14.89

EW: The approximated waiting time by decomposition
WT: The actual waiting time by compression
ρ: System utilization
μ: Service rate in the second station

TABLE 7: Waiting Time in the Second Stage of $E_j/E_k/1$ -> ./M/1 Queues ($j \cong k$)

			μ = 2			μ = 1.25		
ρ	j	k	EW	WT	% ERROR	EW	WT	% ERROR
.3	2	2	.3196	.3038	5.20	2.1548	2.1489	0.27
.5	2	2	.3245	.3128	3.74	2.2271	2.1827	2.03
.7	2	2	.3221	.3157	2.03	2.1902	2.1048	4.06
.9	2	2	.3157	.3111	1.48	2.1834	2.1363	2.20
.3	3	3	.2588	.2542	1.81	1.9199	1.8358	4.58
.5	3	3	.2679	.2613	2.53	1.9942	1.8825	5.93
.7	3	3	.2684	.2748	-2.33	2.0420	1.9413	5.19
.9	3	3	.2595	.2689	-3.49	1.9470	1.9673	-1.03
.3	4	4	.2282	.2315	-1.43	1.8358	1.8560	-1.09
.5	4	4	.2382	.2335	2.01	1.8886	1.8564	1.73
.7	4	4	.2413	.2437	-0.98	1.9017	1.8059	5.30
.9	4	4	.2249	.2392	-5.98	1.8475	1.9091	-3.23
.3	4	5	.2226	.2138	4.12	1.7235	1.6853	2.27
.5	4	5	.2300	.2352	-2.21	1.7564	1.7550	0.80
.7	4	5	.2271	.2216	2.48	1.7207	1.6477	4.43
.9	4	5	.2138	.2108	1.42	1.6903	1.6414	2.98
.3	5	5	.2108	.2033	3.69	1.8065	1.8397	-1.80
.5	5	5	.2203	.2097	5.05	1.8708	1.7969	4.11
.7	5	5	.2221	.2189	1.46	1.8461	1.8071	2.16
.9	5	5	.2102	.2200	-4.45	1.7466	1.7359	0.62
.3	10	10	.1704	.1665	2.34	1.5424	1.4232	4.69
.5	10	10	.1788	.1737	2.94	1.5932	1.6123	-1.18
.7	10	10	.1822	.1819	0.16	1.6082	1.7027	-5.55
.9	10	10	.1742	.1678	3.81	1.6041	1.6919	-5.19

EW: The approximated waiting time by decomposition
WT: The actual waiting time by compression
ρ: System utilization
μ: Service rate in the second station

overestimated if the interdependence between the consecutive arrivals is ignored. For the cases where j approximately equal to k, the approximate results can be reasonably used since the % errors are statistically insignificant except when both j and k are large.

As these phenomena correspond to the characteristics of the lag-1 covariances found previously, we may state that the expected waiting time will be underestimated when the lag-1 covariance is positive and will be overestimated when the lag-1 covariance is negative.

The magnitude of the % errors seems not quite consistent with the magnitude of the lag-1 correlations. There are some cases with significant lag-1 correlation while their % errors are small; and there are also some cases with large % errors while their lag-1 correlations are not significant.

Based on these results, the second hypothesis in this study can therefore be only partially accepted. That is, the covariance of the output process has a significant effect on the waiting time of the next stage in tandem queues. The positive covariance results in an underestimate on the actual waiting time when independence is assumed, and the negative covariance results in an overestimate. However, the size of the estimation error does not correspond proportionally to the magnitude of covariance.

We therefore suspect there might be some factors, other than the lag-1 covariance effect, that effect the % errors for the expected waiting time measurement. The higher order correlations of departure intervals (e.g., lag-2 or lag-3) might be statistically significant. We also suspect that the serial correlation of departure intervals separated by one or more intervals might not be of the same sign and their values might not decrease monotonically. The fluctuation of higher order correlations might lead to the inconsistent results in the investigation of lag-1 covariance effect. Therefore, higher order lags deserve further study.

CHAPTER V

CONCLUSIONS AND SUGGESTIONS FOR FUTURE STUDY

This study has investigated extensively the E_j /E_K /1 queueing system in terms of the departure process and the covariance/correlation structure for consecutive departure intervals. The lag-1 covariance effect on the performance measurement of the second station in the tandem queues (E_j /E_K /1 -> · /M/1) has also been investigated. The Erlang phases in the E_j/E_K /1 queues are considered to cover a wide range of the distribution (from single phase to 30 phases) so as to represent a more general queueing system.

The output process of the E_j /E_K/1 queues (known to be a nonrenewal process when j = 1 and k = 1) was statistically analyzed. It was found to follow approximately the Gamma distribution. The goodness-of-fit of the Gamma distribution has been tested through use of the Chi-square and Kolmogorov- Smirnov tests. The test statistics are quite satisfactory when compared with the critical values of the Chi-square and Kolmogorov-Smirnov statistics at a certain significance level.

The estimated parameters of the output process have been related through a power function to the input and the transformation parameters with strong empirical evidence. The approximate power function of the shape parameters and that of the scale parameters are complementary to each other due to the fact that the mean interdeparture time is approximately equal to the mean interarrival time in a steady-state queueing system.

From the observations made on the estimated parameters and on their fitted power functions, we found that as the system utilization approaches 1, the output parameter tends to be the same as the processing time parameter due to extremely high system utilization; and as system utilization approaches 0, the output parameter tends to be the same as the input parameter due to extremely light system utilization.

The lag-1 covariances and correlations of the consecutive interdeparture times have been computed and fitted into a quadratic function of the input and processing time parameters with very significant statistical support.

The significance of the correlations has been tested statistically through a simple t-test. The degree of significance of the correlation is found mainly determined by the system utilization and the number of Erlang phases.

The signs of the lag-1 covariances are found to be positive for some cases and negative for other cases. The positive or negative sign depends on the Erlang phases of the input distribution and the processing time distribution. The positive and negative sign of the correlation have also been tested by composite t-test. It is found that when j is much smaller than k, the correlation is significantly positive; and the correlation is significantly negative when j is much greater than k or when both j and k are large.

The lag-1 correlation between interdeparture times is insignificantly different from zero when system utilization is extremely low or extremely high. As system utilization approaches 0, there is almost no congestion at the service station and the departure process is just approximately the joint distribution of the arrival distribution and the service time distribution. The departure process is therefore statistically independent. As system utilization approaches 1, there is almost no idle time in the service station and the interdeparture times are approximately equal to the service times which are independent. We also found the relationship between the Erlang phases and the significance level of the lag-1 correlations. The lag-1 correlation is significantly different from zero only for those cases where j is much smaller than k, or j is much larger than k, or when both j and k are large.

The significant lag-1 correlations found demonstrated the interdependence of the consecutive departure intervals. The lag-1 covariance effect on the measurement of the expected waiting time in the second station of a tandem queue has also been examined. The positive lag-1 covariance results in an underestimate on the waiting time measurement; negative lag-1 covariance results in an overestimate. However, the magnitude of the waiting time measurement errors found seems not quite consistent with the magnitude of the lag-1 correlations.

The measurement error is said to be significant if it is 6% or more based on the confidence interval of the actual waiting time. The significant measurement discrepancy on the waiting time might be attributed to the presence of the non-zero lag-1 covariance. However, the lag-1 covariance seems not be the only factor that induces these measurement errors. The higher order correlations are therefore suspected to be also statistically significant and have some degree of effect on the expected waiting time measurement. The higher order correlations might even be more statistically significant than the lag-1 correlation for some cases of the E_j / E_k /1 queues. The effect of some lagged departure intervals covariances (e.g., lag-2, lag-3,...) and their combined effect therefore deserve further study.

In this study, we only considered two-station tandem queues with infinite waiting spaces. Usually, an operating system consists of more than two work stations with finite intermediate waiting space or capacity. When an operating system is extended to consist of more than two work stations (e.g., E_j /E_k/1 -> . /E_1/1 ->. . . ->. /E_m /1), it becomes more difficult to predict the actual input from other work stations in the intermediate station. The output process and its correlation structure of the intermediate stations are expected to be different from that of the first station due to the dependent arrivals in the intermediate stations.

When the intermediate waiting spaces are limited to be finite (e.g., E_j /E_k/1 -> . /E_1/1/N -> ... -> . /E_m/1), the output process and its correlation structure are also expected to be different from that of the unlimited-waiting-room systems since the arrival may be suppressed by the overflow of the available capacity. A study on the output process and its correlation structure of the intermediate stations in a multistage system and/or with finite waiting rooms and their covariance effect on the performance measurement is very significant.

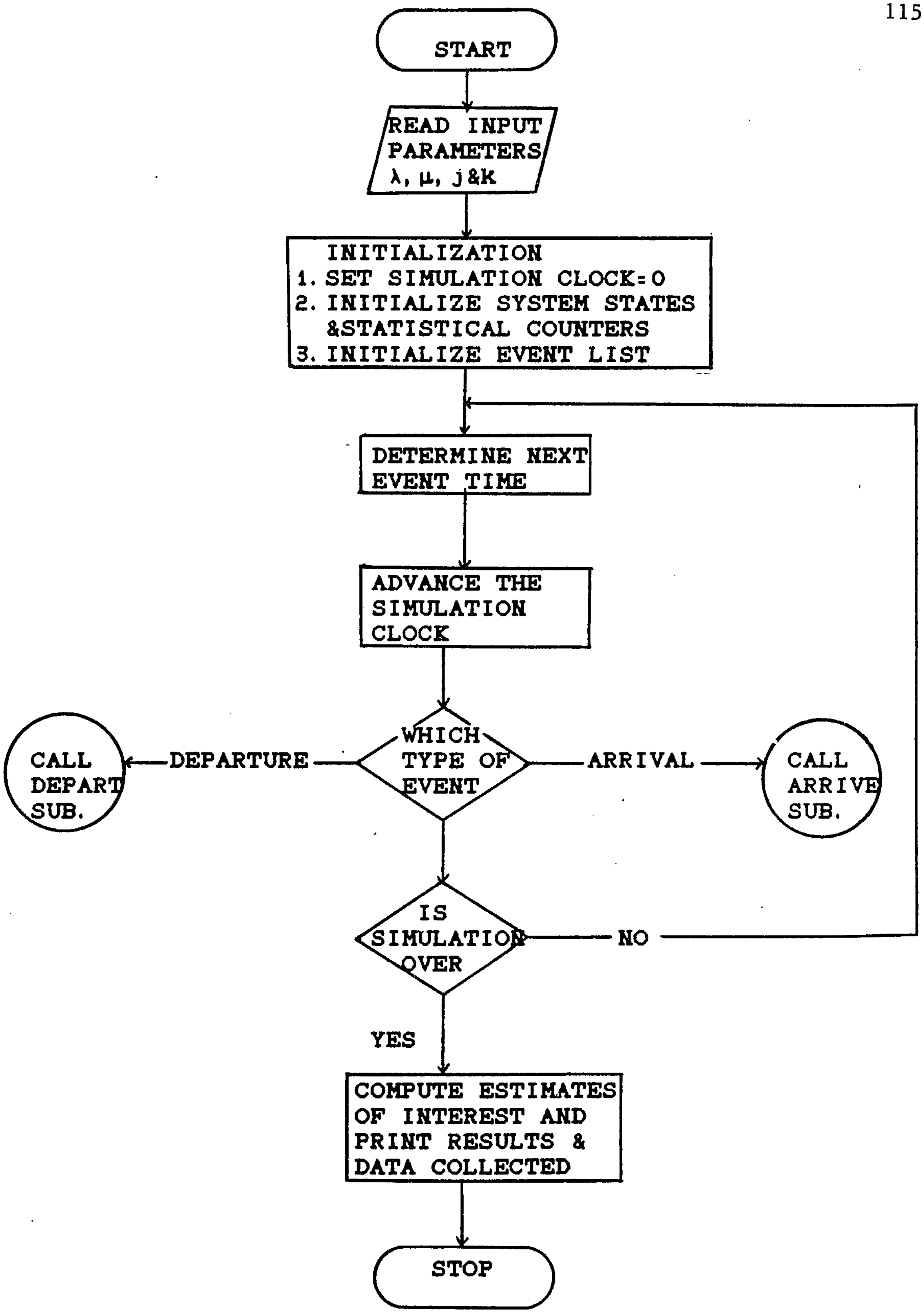
The results of this study can be useful in the design and control of a multistage production system. The knowledge of the significance level of the performance measurement error may simplify the control process of such system. If the measurement error is statistically insignificant, the approximate results can be reasonably used. The findings of this study on the class of queueing system (E_j /E_k /1 ->. /M/1) therefore suggest that the decomposition technique can be used to obtain the system performance measurement when j and k are small, or when system utilization is extremely low or extremely high. That is, the decomposition technique developed in this study based on the estimated parameters of the approximate departure process can be used for some particular cases. For some groups of cases in which the measurement error is significant, the independence treatment of the decomposition procedure may result in large errors.

The knowledge of the underestimate or overestimate of performance measurement at the intermediate stations in the production flow can help to improve performance of the production line and meet the desired throughput rate. Consider that if an underestimate of waiting time occurs, the actual output rate will be smaller than the expected output rate. And if an overestimate occurs, the actual output rate will be larger than the expected output rate. In a multistage production system, these imbalances between successive production stages may cause an unexpected underutilization or overflow of the intermediate stages. And as the number of stages increases, the discrepancy between the actual output rate and expected output rate may increase to the point where the system is out of control. To stabilize the underutilization or overflow of the intermediate stations, the production planner would exercise some control techniques based on the results obtained in this kind of study, i.e., consider changing some of the parameters affecting the outputs.

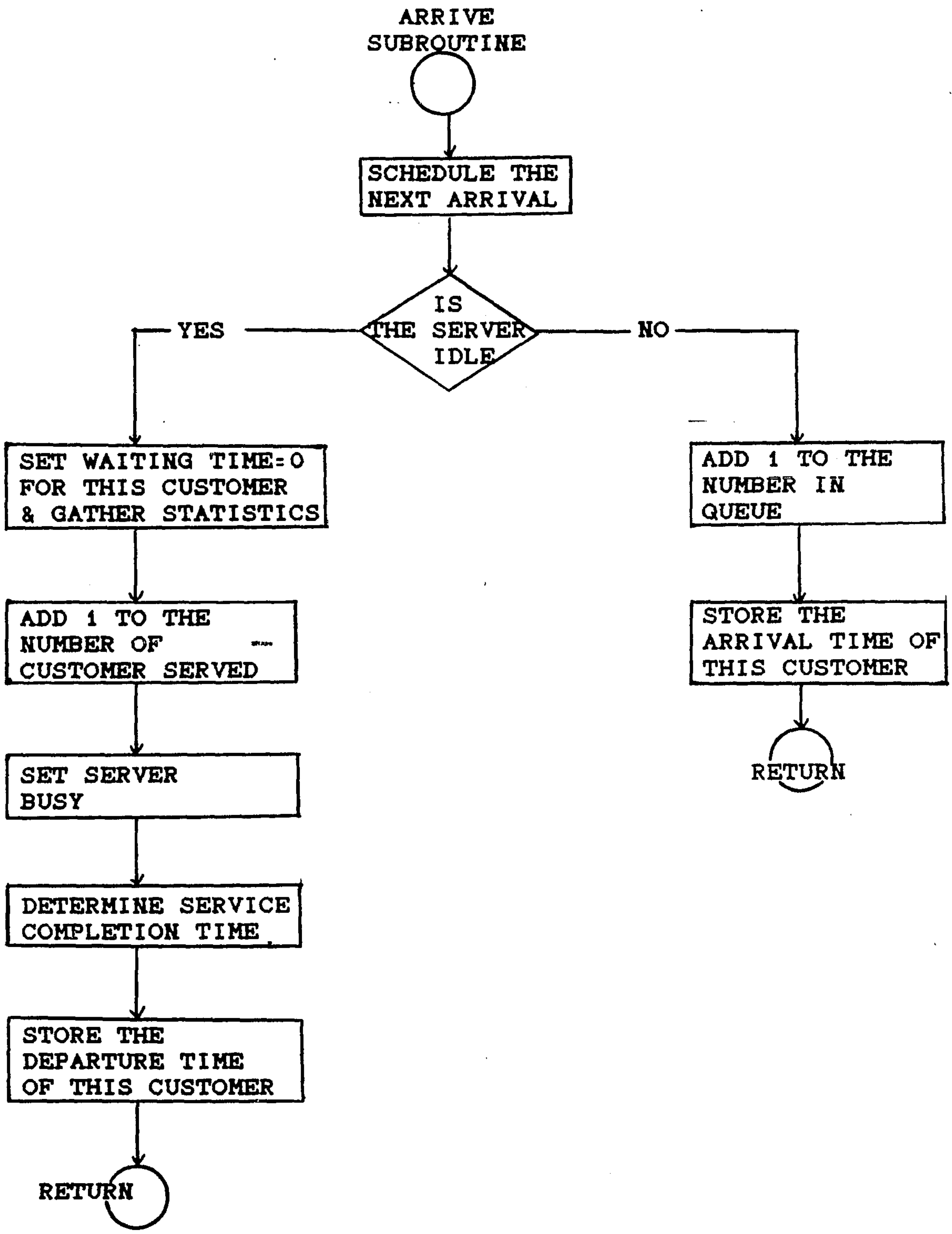
How to integrate the results of this kind of study into the production system design and control is a very significant problem and deserves further study.

APPENDIX A1

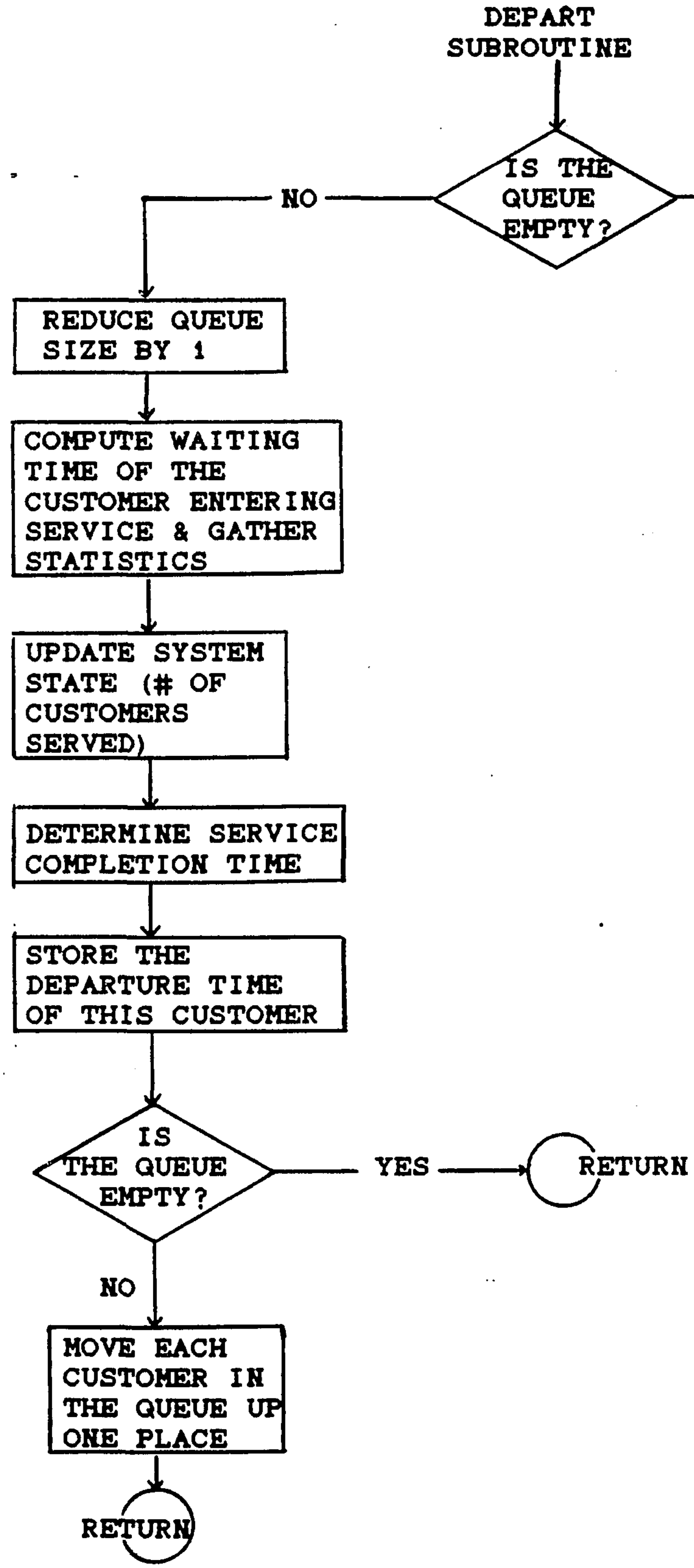
FLOW CHART FOR THE $E_j/E_K/1$ QUEUEING SYSTEM



Flow Chart for the $E_j/E_k/1$ Queueing System
(Event Triggered Variable Time Increment)



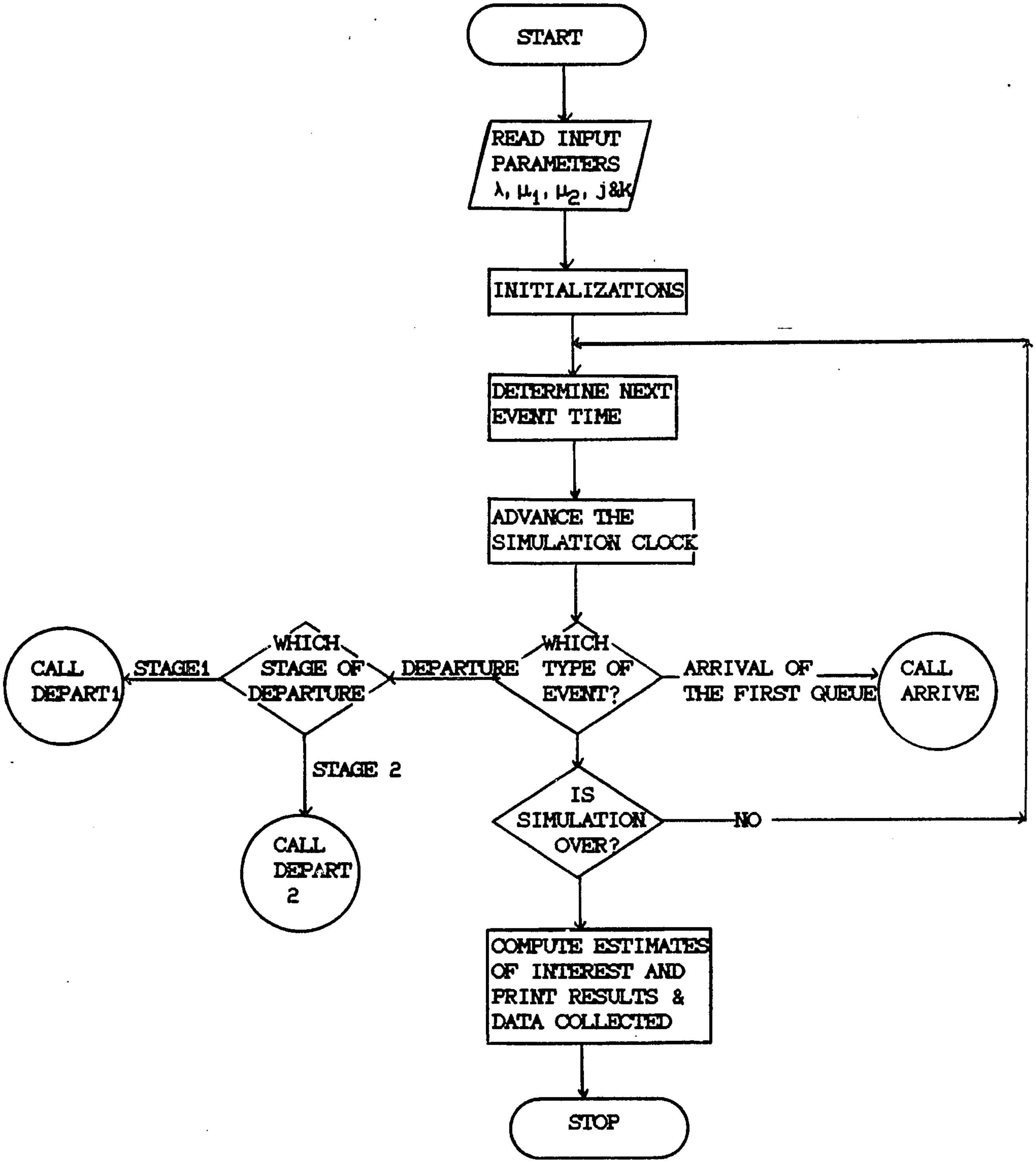
ARRIVE Subroutine of the $E_j/E_K/1$ queueing system



Depart Subroutine of the $E_j/E_K/1$ queueing system

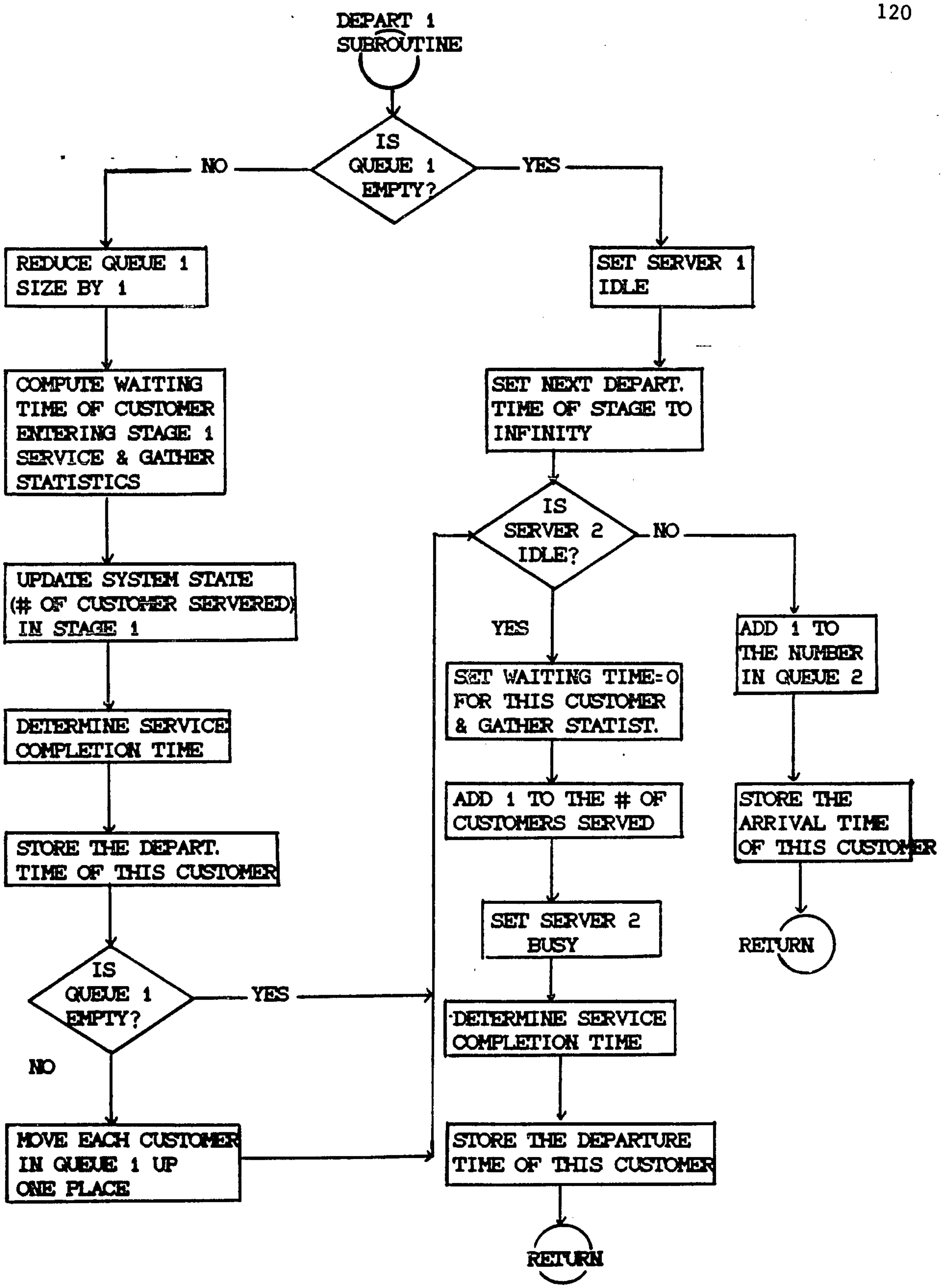
APPENDIX A2

FLOW CHART FOR THE $E_j/E_k/1$ -> ./M/1 QUEUEING SYSTEM



NOTE: DEPART 2 Subroutine is the same as DEPART Subroutine in the $E_j/E_K/1$

Flow Chart for the $E_j/E_K/1$ ->. /M/1 Queueing System



DEPART 1 Subroutine of the $E_j/E_K/1 \rightarrow ./M/1$ Queueing System

APPENDIX B

CHI-SQUARE AND KOLMOGOROV-SMIRNOV TEST STATISTICS ON THE ESTIMATED PARAMETERS (α AND β) OF THE FITTED GAMMA DISTRIBUTION

Degrees of Freedom: 25 (Kolmogorov-Smirnov Test)

Degrees of Freedom: 16 (Chi-square Test)

Critical Values of Chi-square Test:

α= .05 : 26.30

α= .01 : 32.00

Critical Values of Kolmogorov-Smirnov Test:

α= .05 : 0.26

α= .01 : 0.32

ρ	j	k	α	β	X^2	K&S
0.1	2	1	1.9841	.4981	14.722	.0436
0.3	2	1	1.7432	.5697	7.043	.0706
0.5	2	1	1.4655	.6750	13.458	.0733
0.7	2	1	1.2524	.7891	13.431	.0650
0.8	2	1	1.1382	.8657	11.326	.0407
0.9	2	1	1.0810	.9162	9.320	.0457
0.1	2	2	2.0117	.5012	7.371	.0397
0.3	2	2	1.8992	.5286	8.406	.0379
0.5	2	2	1.8165	.5499	6.076	.0366
0.7	2	2	1.7942	.5605	7.176	.0388
0.8	2	2	1.8262	.5473	7.679	.0384
0.9	2	2	1.9164	.5274	7.058	.0354
0.1	2	5	2.0541	.4806	5.813	.0341
0.3	2	5	2.0262	.4689	12.665	.0351
0.5	2	5	2.1121	.4654	9.615	.0607
0.7	2	5	2.4500	.4019	5.264	.0514
0.8	2	5	2.9414	.3339	10.588	.0464
0.9	2	5	3.6873	.2662	5.463	.0582
0.1	2	10	2.0702	.4890	1.479	.0317
0.3	2	10	2.1171	.4787	7.796	.0303
0.5	2	10	2.3423	.4352	11.021	.0613
0.7	2	10	3.0518	.3336	16.415	.0871
0.8	2	10	3.8480	.2643	17.864	.1138
0.9	2	10	5.4767	.1824	7.378	.0535
0.1	2	20	2.0438	.4868	12.911	.0327
0.3	2	20	2.1127	.4714	8.756	.0395
0.5	2	20	2.4318	.4096	8.596	.1074
0.7	2	20	3.3115	.3054	16.407	.1368
0.8	2	20	4.2096	.2467	18.243	.0172
0.9	2	20	6.4052	.1576	9.151	.1331
0.1	2	30	2.1812	.4585	14.114	.0318
0.3	2	30	2.2873	.4384	10.599	.0392
0.5	2	30	2.6265	.3804	17.495	.0995
0.7	2	30	3.5594	.2768	14.593	.1437
0.8	2	30	4.9963	.1994	11.708	.1427
0.9	2	30	10.3132	.1004	11.414	.1276

ρ	j	k	α	β	X^2	K&S
0.1	3	1	2.8582	.3495	8.181	.0399
0.3	3	1	2.2758	.4390	3.965	.0396
0.5	3	1	1.7249	.5774	13.500	.0521
0.7	3	1	1.3673	.7285	7.922	.0487
0.8	3	1	1.2611	.7919	16.401	.0399
0.9	3	1	1.1302	.8885	6.317	.0456
0.1	3	3	2.9876	.3383	2.236	.0376
0.3	3	3	2.7310	.3700	1.833	.0361
0.5	3	3	2.5504	.3951	3.737	.0347
0.7	3	3	2.5508	.3899	5.339	.0351
0.8	3	3	2.6049	.3817	8.836	.0361
0.9	3	3	2.7443	.3633	5.491	.0379
0.1	3	5	2.9851	.3320	4.705	.0383
0.3	3	5	2.8680	.3453	9.609	.0366
0.5	3	5	2.8436	.3486	5.139	.0371
0.7	3	5	3.1135	.3188	3.423	.0353
0.8	3	5	3.4473	.2891	6.936	.0437
0.9	3	5	3.9447	.2551	11.608	.0346
0.1	3	10	3.0032	.3358	9.704	.0386
0.3	3	10	3.0088	.3354	9.356	.0279
0.5	3	10	3.1668	.3186	7.993	.0336
0.7	3	10	3.8756	.2606	12.974	.0599
0.8	3	10	4.4467	.2280	12.350	.0588
0.9	3	10	6.4156	.1561	15.438	.0539
0.1	3	20	3.0552	.3280	5.469	.0354
0.3	3	20	3.0928	.3232	6.507	.0221
0.5	3	20	3.4203	.2921	9.397	.0643
0.7	3	20	4.3385	.2310	11.708	.0906
0.8	3	20	5.5060	.1828	15.286	.1009
0.9	3	20	7.4910	.1377	14.088	.1097
0.1	3	30	3.1396	.3184	7.364	.0351
0.3	3	30	3.1490	.3173	5.664	.0301
0.5	3	30	3.4175	.2932	14.743	.0564
0.7	3	30	4.3374	.2309	18.422	.1243
0.8	3	30	5.7623	.1733	10.445	.1440
0.9	3	30	9.0633	.1123	11.044	.1059

ρ	j	k	α	β	X^2	K&S
0.1	4	1	3.8164	.2633	6.514	.0400
0.3	4	1	2.6683	.3764	9.687	.0400
0.5	4	1	1.8912	.5305	9.563	.0460
0.7	4	1	1.4701	.6854	7.651	.0451
0.8	4	1	1.2753	.7907	10.511	.0398
0.9	4	1	1.1377	.8738	6.914	.0400
0.1	4	4	3.9505	.2524	3.379	.0399
0.3	4	4	3.5622	.2799	5.219	.0387
0.5	4	4	3.2704	.3047	5.276	.0382
0.7	4	4	3.1630	.3147	0.848	.0377
0.8	4	4	3.3748	.2926	12.931	.0397
0.9	4	4	3.5570	.2818	10.445	.0391
0.1	4	5	4.0165	.2476	9.547	.0346
0.3	4	5	3.6541	.2722	6.542	.0362
0.5	4	5	3.4014	.2930	6.730	.0425
0.7	4	5	3.5054	.2850	1.649	.0402
0.8	4	5	3.8494	.2613	12.498	.0388
0.9	4	5	4.2759	.2323	11.043	.0370
0.1	4	10	3.9833	.2522	8.309	.0394
0.3	4	10	3.8420	.2615	9.624	.0389
0.5	4	10	3.8457	.2613	8.446	.0336
0.7	4	10	4.3691	.2292	7.447	.0375
0.8	4	10	5.0962	.1976	9.247	.0437
0.9	4	10	6.3794	.1587	11.850	.0458
0.1	4	20	3.8825	.2593	6.941	.0374
0.3	4	20	3.8156	.2639	4.026	.0352
0.5	4	20	3.9919	.2524	7.167	.0244
0.7	4	20	4.8165	.2080	5.203	.0673
0.8	4	20	5.9918	.1667	3.585	.0693
0.9	4	20	8.9705	.1122	10.941	.0929
0.1	4	30	3.9980	.2557	11.217	.0394
0.3	4	30	3.9956	.2489	8.729	.0268
0.5	4	30	4.2782	.2333	15.618	.0534
0.7	4	30	5.4849	.1825	10.526	.0946
0.8	4	30	7.1083	.1417	11.362	.0903
0.9	4	30	10.4416	.0981	12.849	.0838

ρ	j	k	α	β	X^2	K&S
0.1	5	1	4.6186	.2158	7.726	.0400
0.3	5	1	3.0126	.3307	3.688	.0399
0.5	5	1	2.0166	.4942	18.749	.0477
0.7	5	1	1.4629	.6798	6.610	.0509
0.8	5	1	1.2740	.7781	9.274	.0397
0.9	5	1	1.2059	.8343	6.757	.0393
0.1	5	5	5.0701	.1951	1.876	.0399
0.3	5	5	4.4992	.2197	6.658	.0391
0.5	5	5	3.9961	.2474	10.817	.0392
0.7	5	5	3.8955	.2540	11.090	.0431
0.8	5	5	4.0543	.2440	12.285	.0424
0.9	5	5	4.4457	.2249	7.556	.0398
0.1	5	10	5.2593	.1924	9.058	.0339
0.3	5	10	4.9269	.2059	4.159	.0375
0.5	5	10	4.7717	.2133	4.992	.0305
0.7	5	10	5.1827	.1971	6.064	.0455
0.8	5	10	5.8835	.1739	7.767	.0410
0.9	5	10	7.4101	.1349	5.960	.0359
0.1	5	20	5.3946	.1845	5.461	.0384
0.3	5	20	5.1780	.1924	11.286	.0383
0.5	5	20	5.1815	.1923	3.966	.0409
0.7	5	20	6.0499	.1649	6.976	.0699
0.8	5	20	7.1419	.1398	8.211	.0722
0.9	5	20	9.9478	.1023	12.849	.0491
0.1	5	30	5.2221	.1864	4.423	.0398
0.3	5	30	5.1360	.1894	9.901	.0329
0.5	5	30	5.3268	.1826	9.720	.0509
0.7	5	30	6.6999	.1453	6.075	.1022
0.8	5	30	8.5375	.1161	9.895	.1165
0.9	5	30	13.7474	.0723	8.865	.0768

ρ	j	k	α	β	X^2	K&S
0.1	10	1	8.2731	.1206	10.468	.0400
0.3	10	1	3.9901	.2517	6.863	.0508
0.5	10	1	2.2857	.4405	9.711	.0413
0.7	10	1	1.5344	.6579	10.570	.0395
0.8	10	1	1.3419	.7466	5.938	.0399
0.9	10	1	1.1576	.8639	2.637	.0377
0.1	10	5	9.7472	.1033	5.608	.0398
0.3	10	5	7.6475	.1317	5.788	.0399
0.5	10	5	5.6519	.1780	9.380	.0416
0.7	10	5	4.6739	.2149	8.236	.0363
0.8	10	5	4.5493	.2213	9.977	.0384
0.9	10	5	4.7413	.2129	10.338	.0283
0.1	10	10	10.3558	.0962	2.137	.0381
0.3	10	10	8.9817	.1108	7.081	.0398
0.5	10	10	7.5014	.1326	5.285	.0397
0.7	10	10	6.9769	.1426	9.746	.0375
0.8	10	10	7.2004	.1385	10.956	.0363
0.9	10	10	7.8809	.1267	5.567	.0349
0.1	10	20	10.0730	.0995	5.057	.0346
0.3	10	20	9.3579	.1071	9.030	.0363
0.5	10	20	8.5628	.1171	5.901	.0378
0.7	10	20	8.6741	.1154	8.721	.0262
0.8	10	20	9.6392	.1041	9.964	.0399
0.9	10	20	11.7013	.0864	9.731	.0435
0.1	10	30	10.5568	.0952	3.244	.0337
0.3	10	30	10.0154	.1003	5.393	.0293
0.5	10	30	9.3959	.1069	3.278	.0339
0.7	10	30	9.8020	.1026	5.420	.0546
0.8	10	30	11.2955	.0895	6.189	.0406
0.9	10	30	15.4691	.0660	9.872	.0543

ρ	j	k	α	β	X^2	K&S
0.1	20	1	14.7080	.0676	6.807	.0236
0.3	20	1	4.7603	.2089	5.830	.0400
0.5	20	1	2.4331	.4084	12.250	.0413
0.7	20	1	1.5444	.6453	2.906	.0400
0.8	20	1	1.2983	.7666	11.898	.0395
0.9	20	1	1.1816	.8510	5.337	.0394
0.1	20	5	19.4944	.0510	6.378	.0258
0.3	20	5	12.2546	.0812	12.275	.0499
0.5	20	5	7.5653	.1317	8.218	.0399
0.7	20	5	5.6000	.1781	6.226	.0399
0.8	20	5	5.2392	.1902	3.475	.0412
0.9	20	5	5.2287	.1913	6.760	.0371
0.1	20	10	19.6878	.0505	8.701	.0337
0.3	20	10	14.8181	.0672	4.092	.0594
0.5	20	10	10.4244	.0955	10.220	.0384
0.7	20	10	8.4683	.1173	4.067	.0364
0.8	20	10	8.1500	.1218	5.711	.0363
0.9	20	10	8.4862	.1175	5.973	.0258
0.1	20	20	21.1140	.0472	2.362	.0377
0.3	20	20	18.0982	.0550	6.985	.0771
0.5	20	20	14.1529	.0704	14.843	.0354
0.7	20	20	12.0742	.0824	8.260	.0398
0.8	20	20	12.5443	.0794	10.098	.0363
0.9	20	20	15.2555	.0656	5.555	.0351
0.1	20	30	22.0150	.0453	13.400	.0398
0.3	20	30	19.8607	.0503	5.920	.0376
0.5	20	30	16.7014	.0597	4.121	.0386
0.7	20	30	15.1684	.0654	4.697	.0351
0.8	20	30	16.2117	.0613	5.423	.0603
0.9	20	30	19.3909	.0513	4.430	.0479

ρ	j	k	α	β	X^2	K&S
0.1	30	1	18.4937	.0540	7.307	.0569
0.3	30	1	4.9641	.2012	4.879	.0495
0.5	30	1	2.4291	.4123	8.369	.0801
0.7	30	1	1.4267	.6989	15.618	.0514
0.8	30	1	1.1882	.8340	9.138	.0380
0.9	30	1	1.0467	.9395	8.127	.0379
0.1	30	5	27.9601	.0356	10.307	.0391
0.3	30	5	14.6545	.0682	4.925	.0399
0.5	30	5	7.9906	.1243	7.164	.0536
0.7	30	5	5.5905	.1774	6.766	.0398
0.8	30	5	4.9776	.1993	8.927	.0387
0.9	30	5	4.7803	.2086	9.078	.0404
0.1	30	10	28.2716	.0353	3.602	.0366
0.3	30	10	19.8229	.0503	5.254	.0380
0.5	30	10	12.9370	.0768	5.979	.0387
0.7	30	10	9.6038	.1033	9.009	.0395
0.8	30	10	9.1719	.1082	9.078	.0404
0.9	30	10	9.4573	.1055	16.170	.0544
0.1	30	20	30.8628	.0325	8.367	.0337
0.3	30	20	23.9782	.0417	12.080	.0553
0.5	30	20	17.2694	.0578	10.770	.0631
0.7	30	20	13.7869	.0724	9.370	.0445
0.8	30	20	13.5261	.0739	6.325	.0325
0.9	30	20	15.0667	.0663	6.190	.0324
0.1	30	30	31.4950	.0319	6.216	.0574
0.3	30	30	27.0927	.0372	5.707	.0515
0.5	30	30	21.1084	.0477	6.378	.0750
0.7	30	30	17.4545	.0574	6.046	.0434
0.8	30	30	17.3318	.0576	9.314	.0710
0.9	30	30	19.7574	.0503	6.216	.0462

APPENDIX C

LAG-1 COVARIANCES AND CORRELATIONS OF THE CONSECUTIVE INTERDEPARTURE TIMES OF THE $E_j/E_k/1$ QUEUEING SYSTEM

ρ	j	k	COV	CORR
0.1	2	1	-0.00145	-0.00329
0.3	2	1	-0.02256	-0.03948
0.5	2	1	-0.03049	-0.04510
0.7	2	1	-0.02584	-0.03250
0.8	2	1	-0.02368	-0.02797
0.9	2	1	0.00143	0.00146
0.1	2	2	-0.00705	-0.01552
0.3	2	2	-0.01815	-0.03589
0.5	2	2	-0.02629	-0.04948
0.7	2	2	-0.02642	-0.04721
0.8	2	2	-0.01220	-0.02189
0.9	2	2	-0.00885	-0.01326
0.1	2	5	-0.01419	-0.02825
0.3	2	5	-0.01242	-0.02480
0.5	2	5	0.00297	0.00811
0.7	2	5	0.00814	0.02108
0.8	2	5	0.00813	0.02414
0.9	2	5	-0.00262	-0.01005
0.1	2	10	-0.00206	-0.00452
0.3	2	10	0.00184	0.00282
0.5	2	10	0.01041	0.02152
0.7	2	10	0.01444	0.03870
0.8	2	10	0.01354	0.04489
0.9	2	10	0.01113	0.05241
0.1	2	20	-0.00054	-0.00319
0.3	2	20	0.00606	0.01113
0.5	2	20	0.02424	0.06040
0.7	2	20	0.02346	0.07995
0.8	2	20	0.02289	0.09808
0.9	2	20	0.01768	0.13165
0.1	2	30	0.00198	0.00537
0.3	2	30	0.00917	0.02239
0.5	2	30	0.02873	0.07648
0.7	2	30	0.03683	0.13893
0.8	2	30	0.03884	0.19520
0.9	2	30	0.01930	0.13528

ρ	j	k	COV	CORR
0.1	3	1	-0.00837	-0.02389
0.3	3	1	-0.03902	-0.08733
0.5	3	1	-0.06319	-0.10842
0.7	3	1	-0.05135	-0.06842
0.8	3	1	-0.04188	-0.04965
0.9	3	1	-0.02036	-0.02341
0.1	3	3	-0.00619	-0.01683
0.3	3	3	-0.01593	-0.04151
0.5	3	3	-0.01514	-0.03818
0.7	3	3	-0.01438	-0.03693
0.8	3	3	-0.00941	-0.02238
0.9	3	3	-0.00329	-0.00793
0.1	3	5	-0.00843	-0.02279
0.3	3	5	-0.01410	-0.03747
0.5	3	5	-0.01514	-0.04118
0.7	3	5	-0.01035	-0.03089
0.8	3	5	-0.00836	-0.02832
0.9	3	5	-0.00604	-0.02320
0.1	3	10	0.00663	0.02097
0.3	3	10	0.00620	0.01978
0.5	3	10	0.00891	0.02907
0.7	3	10	0.00981	0.03910
0.8	3	10	0.01085	0.04889
0.9	3	10	0.00623	0.04059
0.1	3	20	0.00095	0.00085
0.3	3	20	0.00561	0.01347
0.5	3	20	0.01602	0.04893
0.7	3	20	0.02134	0.08587
0.8	3	20	0.02101	0.10677
0.9	3	20	0.01628	0.11739
0.1	3	30	0.01038	0.03219
0.3	3	30	0.01104	0.03486
0.5	3	30	0.01415	0.04735
0.7	3	30	0.01775	0.07753
0.8	3	30	0.01609	0.08937
0.9	3	30	0.00901	0.09199

ρ	j	k	COV	CORR
0.1	4	1	-0.01420	-0.05057
0.3	4	1	-0.04920	-0.12969
0.5	4	1	-0.05983	-0.11361
0.7	4	1	-0.04089	-0.06049
0.8	4	1	-0.03766	-0.04651
0.9	4	1	-0.01281	-0.01465
0.1	4	4	-0.00357	-0.01349
0.3	4	4	-0.01584	-0.05461
0.5	4	4	-0.02420	-0.07677
0.7	4	4	-0.01780	-0.05447
0.8	4	4	-0.01286	-0.03993
0.9	4	4	-0.00900	-0.02993
0.1	4	5	0.00266	0.01256
0.3	4	5	-0.00886	-0.02932
0.5	4	5	-0.01605	-0.05115
0.7	4	5	-0.01615	-0.05124
0.8	4	5	-0.00790	-0.02587
0.9	4	5	-0.00390	-0.01074
0.1	4	10	-0.00101	-0.00519
0.3	4	10	-0.00377	-0.01560
0.5	4	10	-0.00146	-0.00662
0.7	4	10	0.00397	0.01698
0.8	4	10	0.00403	0.02329
0.9	4	10	0.00209	0.01611
0.1	4	20	-0.00772	-0.02785
0.3	4	20	-0.00992	-0.03575
0.5	4	20	-0.00655	-0.02428
0.7	4	20	0.00127	0.00670
0.8	4	20	0.00246	0.01371
0.9	4	20	0.00651	0.05485
0.1	4	30	0.00161	0.00558
0.3	4	30	0.00093	0.00289
0.5	4	30	0.00413	0.01683
0.7	4	30	0.01089	0.06075
0.8	4	30	0.01267	0.09106
0.9	4	30	0.00749	0.08051

ρ	j	k	COV	CORR
0.1	5	1	-0.01519	-0.06808
0.3	5	1	-0.05877	-0.17177
0.5	5	1	-0.08146	-0.16015
0.7	5	1	-0.06549	-0.09383
0.8	5	1	-0.05037	-0.06370
0.9	5	1	-0.00733	-0.00893
0.1	5	5	0.00173	0.00893
0.3	5	5	-0.00984	-0.04454
0.5	5	5	-0.01782	-0.07246
0.7	5	5	-0.01953	-0.07578
0.8	5	5	-0.02072	-0.08197
0.9	5	5	-0.00830	-0.03563
0.1	5	10	0.00007	-0.00100
0.3	5	10	-0.00586	-0.02770
0.5	5	10	-0.00723	-0.03274
0.7	5	10	-0.00311	-0.01301
0.8	5	10	0.00039	0.00757
0.9	5	10	0.00038	0.00306
0.1	5	20	-0.00268	-0.01417
0.3	5	20	-0.00608	-0.03037
0.5	5	20	-0.00657	-0.03326
0.7	5	20	-0.00043	-0.00410
0.8	5	20	0.00209	0.01300
0.9	5	20	0.00001	0.00418
0.1	5	30	-0.00627	-0.03411
0.3	5	30	-0.00711	-0.03805
0.5	5	30	-0.00346	-0.02003
0.7	5	30	0.00634	0.04126
0.8	5	30	0.00997	0.07987
0.9	5	30	0.00750	0.09197

ρ	j	k	COV	CORR
0.1	10	1	-0.01088	-0.09024
0.3	10	1	-0.06126	-0.24562
0.5	10	1	-0.08879	-0.20305
0.7	10	1	-0.08276	-0.12468
0.8	10	1	-0.05912	-0.07482
0.9	10	1	-0.02528	-0.03111
0.1	10	5	-0.00166	-0.01562
0.3	10	5	-0.01518	-0.11212
0.5	10	5	-0.03213	-0.17873
0.7	10	5	-0.03384	-0.16043
0.8	10	5	-0.03128	-0.14486
0.9	10	5	-0.02175	-0.10117
0.1	10	10	-0.00361	-0.03561
0.3	10	10	-0.01170	-0.10104
0.5	10	10	-0.02128	-0.15366
0.7	10	10	-0.02086	-0.14016
0.8	10	10	-0.01428	-0.09837
0.9	10	10	-0.00868	-0.06708
0.1	10	20	0.00176	0.01435
0.3	10	20	-0.00049	-0.00677
0.5	10	20	-0.00335	-0.02980
0.7	10	20	-0.00145	-0.01402
0.8	10	20	0.00175	0.01397
0.9	10	20	0.00085	0.00799
0.1	10	30	-0.00396	-0.04171
0.3	10	30	-0.00690	-0.06972
0.5	10	30	-0.00914	-0.08752
0.7	10	30	-0.00523	-0.05316
0.8	10	30	-0.00159	-0.01723
0.9	10	30	0.00093	0.01597

ρ	j	k	COV	CORR
0.1	20	1	-0.01135	-0.16619
0.3	20	1	-0.06781	-0.32998
0.5	20	1	-0.09747	-0.24442
0.7	20	1	-0.05554	-0.08945
0.8	20	1	-0.05759	-0.07707
0.9	20	1	-0.03972	-0.04742
0.1	20	5	-0.00306	-0.06036
0.3	20	5	-0.01948	-0.23516
0.5	20	5	-0.04187	-0.30927
0.7	20	5	-0.04277	-0.23677
0.8	20	5	-0.03219	-0.16515
0.9	20	5	-0.00873	-0.04830
0.1	20	10	-0.00222	-0.04496
0.3	20	10	-0.01132	-0.16491
0.5	20	10	-0.02433	-0.24650
0.7	20	10	-0.02974	-0.24014
0.8	20	10	-0.02568	-0.20481
0.9	20	10	-0.01671	-0.13980
0.1	20	20	0.00110	0.01863
0.3	20	20	-0.00266	-0.04963
0.5	20	20	-0.00981	-0.13886
0.7	20	20	-0.01453	-0.17605
0.8	20	20	-0.01170	-0.14735
0.9	20	20	-0.00515	-0.07825
0.1	20	30	-0.00094	-0.02752
0.3	20	30	-0.00365	-0.07681
0.5	20	30	-0.00852	-0.14261
0.7	20	30	-0.01033	-0.15447
0.8	20	30	-0.00814	-0.13103
0.9	20	30	-0.00428	-0.08448

p	j	k	COV	CORR
0.1	30	1	-0.01230	-0.21642
0.3	30	1	-0.06972	-0.33260
0.5	30	1	-0.08171	-0.19294
0.7	30	1	-0.06981	-0.09936
0.8	30	1	-0.06750	-0.08073
0.9	30	1	-0.05703	-0.05872
0.1	30	5	-0.00320	-0.08728
0.3	30	5	-0.02147	-0.30053
0.5	30	5	-0.04719	-0.36438
0.7	30	5	-0.05566	-0.29566
0.8	30	5	-0.04765	-0.22819
0.9	30	5	-0.03253	-0.14267
0.1	30	10	-0.00127	-0.03535
0.3	30	10	-0.00807	-0.15966
0.5	30	10	-0.01848	-0.24176
0.7	30	10	-0.02169	-0.20897
0.8	30	10	-0.01541	-0.13818
0.9	30	10	-0.00760	-0.07154
0.1	30	20	-0.00017	-0.00549
0.3	30	20	-0.00465	-0.10940
0.5	30	20	-0.01274	-0.21655
0.7	30	20	-0.01639	-0.22352
0.8	30	20	-0.01425	-0.18542
0.9	30	20	-0.01017	-0.13606
0.1	30	30	-0.00073	-0.02193
0.3	30	30	-0.00312	-0.08743
0.5	30	30	-0.00807	-0.17925
0.7	30	30	-0.01216	-0.22053
0.8	30	30	-0.01087	-0.19480
0.9	30	30	-0.00630	-0.13027

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