70-1109

ROITBERG, Yael, 1942-
SOME RESULTS ON PARAFREE GROUPS.

The City University of New York, Ph.D., 1969
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

SOME RESULTS ON PARAFREE GROUPS

by

YAEL ROITBERG

A dissertation submitted to the
Graduate Faculty in Mathematics in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy,
The City University of New York.

1969

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

May 22, 1969
date

Gilbert Baumslag
Chairman of Examining Committee
Professor Gilbert Baumslag

date

Eldon Dyer
Executive Officer
Professor Eldon Dyer

Professor Louis Auslander

Professor Harvey M. Hyman

Supervisory Committee

Preliminaries

We refer the reader to H. Neumann [11] for the definition of varieties, product varieties, metabelian groups, nilpotent groups, residually nilpotent groups, lower central series of groups, relatively free groups, basic commutators and left normed basic commutators. (The definition of basic commutators in H. Neumann is given for finitely many generators only. When infinitely many generators are involved, only obvious modifications are required, see e.g. [13]).

Next we give certain definitions which were introduced by G. Baumslag and can be found in his papers [4] and [5].

Let G be any group and let $\gamma_n G$ be the n-th term of the lower central series. Then the sequence

$$G/\gamma_2 G, \; G/\gamma_3 G, \; \ldots$$

is called the lower central sequence of G. Let H be another group. Then H and G have the same lower central sequence if there are isomorphisms θ_k from $G/\gamma_k G$ onto $H/\gamma_k H$ such that θ_k induces θ_{k-1} on $G/\gamma_{k-1} G$ to $H/\gamma_{k-1} H$ for $k = 2,3,\ldots$.

A group P is termed parafree in a variety $\underline{\underline{V}}$ if

(i) $P \in \underline{\underline{V}}$,

(ii) P is residually nilpotent, and

(iii) P has the same lower central sequence as a free group in $\underline{\underline{V}}$.

A subset X of a parafree group P is called a parabasis for P if X freely generates P modulo $\gamma_2 P$. It follows that if X is a parabasis of P, then X freely generates P modulo $\gamma_n P$ for $n \geq 2$.

We now list the notation which will be used throughout this paper.

Notation

x^y	the transform $y^{-1}xy$ of x by y, $x,y \in G$.
$x^{y_1+\dots+y_n}$	$x^{y_1} x^{y_2} \dots x^{y_n}$ ($x,y_1,\dots,y_n$ in the group G).
$[x_1,x_2]$	the commutator $x_1^{-1}x_2^{-1}x_1x_2$ of x_1 and x_2.
$[x_1,x_2,\dots,x_n]$	the left normed commutator $[[x_1,x_2,\dots,x_{n-1}],x_n]$.
$[x_1,k_2x_2,\dots,k_nx_n]$	the left normed commutator with k_i repetitions of x_i, $i=2,\dots,n$, and $k_i \geq 0$. If $k_i = 0$ the element x_i does not appear at all.
$H \leq G$	H is a subgroup of G.
gp (X)	the subgroup of G generated by X, where X is a subset of G.
$gp_G(X)$	the normal subgroup G generated by X.
$[A,B]$	the group generated by the commutators $[a,b]$, $a \in A$, $b \in B$.
$\gamma_n G$	the nth term of the lower central series of a group G.
$\lvert S \rvert$	the cardinality of a set S.
$S \setminus T$	the set theoretical difference between S and T, where T is a subset of S.
$\{s_1,\dots,\hat{s}_k,\dots,s_n\}$	the set $\{s_1,\dots,s_n\} \setminus \{s_k\}$.
$\mathbb{Z}$	the set of integers.
$m \mid n$	m divides n, $m,n \in \mathbb{Z}$.
$m \nmid n$	m does not divide n, $m,n \in \mathbb{Z}$.
$\underline{\underline{A}}$	the variety of all abelian groups.
$\underline{\underline{A}}_n$	variety of all abelian groups of exp. dividing n ($n \geq 2$).

Introduction

The purpose of this thesis is to provide analogs, for parafree groups in certain varieties, of a number of theorems concerning free groups in these varieites. These latter theorems in turn, are generalizations of classical theorems about absolutely free groups, i.e. free groups in the variety of all groups. We shall not be specifically concerned here, however, with absolutely parafree groups, although some of our results do hold in this case. In fact, we shall be dealing mainly with certain types of product varieties.

The paper is divided into three chapters, as follows.

In Chapter I, we consider a well-known theorem of O. Schreier [12], asserting that a nontrivial normal subgroup of an absolutely free group of finite rank is finitely generated if and only if it is of finite index. G. Baumslag [3] considers possible generalizations of this theorem for certain relatively free groups and obtains the following result:

Let $\underline{\underline{U}}$ and $\underline{\underline{V}}$ be varieties of groups and let F be a noncyclic free group in $\underline{\underline{U}}\,\underline{\underline{V}}$. Let V(F) denote the (unique) minimal subgroup of F such that F/V(F) lies in $\underline{\underline{V}}$. Suppose that N is a nontrivial normal subgroup of F such that F/V(F)N is infinite. Then N is not finitely generated.

Note that in the case where $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are both the varieties of all groups, Baumslag's theorem reduces to the "only if" part of Schreier's theorem. Our contribution is to prove a form of Baumslag's theorem, replacing the free group F by a parafree group P. However, we must make a drastic restriction on the varieties $\underline{\underline{U}}$ and $\underline{\underline{V}}$. More precisely we shall prove:

Let P be a parafree group of rank ≥ 2 in the variety $\underline{\underline{A}}^2$ of all metabelian groups. Let N be a normal subgroup of P such that $|P/NP'|$ is infinite. Then, either $N = 1$ or N is not finitely generated.

Notice that the free groups in $\underline{\underline{A}}^2$ are actually parafree, being residually nilpotent, and therefore our result can be viewed as a partial generalization of Baumslag's theorem.

In Chapter II, we shall prove the following theorem:

If P is a parafree group in any variety $\underline{\underline{V}}$ containing the variety of all metabelian groups, and if x and y are any two elements of P which are independent modulo $\gamma_2 P$, then the commutator $[x,y]$ is not a proper power.

This result generalizes the analogous result for free groups, provided $\underline{\underline{V}}$ is either the variety of all groups (G. Baumslag [2]; W. Magnus, A. Karrass and D. Solitar [8]) or the variety of all metabelian groups (G. Baumslag, B. H. Neumann, H. Neumann and P. M. Neumann [6]). If $\underline{\underline{U}}$ is a variety which lies strictly between the variety of all groups and the variety of all metabelian groups, then a free group in $\underline{\underline{U}}$ is not necessarily a parafree group, and therefore our theorem does not apply directly to free groups in such varieties. Nevertheless, the result is still true for free groups F in $\underline{\underline{U}}$, $\underline{\underline{U}} \supset \underline{\underline{A}}^2$. Indeed, we need only observe that F/F'' is free in $\underline{\underline{A}}^2$.

In Chapter III, we establish the following result:

A parafree group of infinite rank in the product variety $\underline{\underline{A}}\underline{\underline{A}}_2$ has a trivial center.

The analogous result for free groups is known to be true for a much larger class of varieties. (See, e.g., M. Auslander and R. C. Lyndon [1].) Unfortunately, at the moment, our methods do not permit us to extend the result to parafree groups in other varieties, indeed not even to parafree groups in $\underline{\underline{A}}\underline{\underline{A}}_p$, with p an odd prime. The reason for our inability to establish more general theorems of this type, even for $\underline{\underline{A}}\underline{\underline{A}}_3$, is that combinatorial difficulties arise. More precisely, the problem of the triviality of the center translates into a problem concerning the augmentation ideal of the integral group ring of the free group in the variety $\underline{\underline{A}}_3$. This latter problem is totally intractable.

The reader can find a more detailed outline of the results contained in this paper in the introductory sections of the various chapters.

At this point, I would like to express my deep appreciation to my thesis advisor, Professor G. Baumslag, without whose patient guidance and encouragement this thesis would not have materialized.

Table of Contents

Chapter I

§1. The purpose of this chapter is to prove a generalization of a theorem of G. Baumslag [3]. Our result can be stated as follows:

Theorem I. Let P be a parafree group of rank ≥ 2 in the variety $\underline{\underline{A}}^2$ of all metabelian groups, let N be a normal subgroup of P such that $|P/N\cdot P'|$ is infinite. Then either $N = 1$ or N is not finitely generated.

The main tool used in the proof of the theorem is a formula expressing an arbitrary element of P as an infinite product in an appropriate sense. (This formula will also be used in subsequent chapters.) Aside from several special cases, the proof of the theorem proceeds by contradiction, first reducing to the case when P is parafree of rank 2. The assumption that N is finitely generated, together with the other assumptions, leads to (several) systems of (k+1) linear inhomogeneous real equations with k unknowns (where k+1 = number of generators of N). These systems must, on the one hand, always be solvable, but, on the other hand, we can arrange matters so that some of these systems are not solvable. This involves making certain estimates on the terms of the system.

The rest of Chapter I is arranged as follows. In §2 we prove our infinite product formula. Some technical lemmas concerning commutator formulas are derived in §3 and §4, and in §5 we introduce certain real functions which are used in making our estimates. Then, in §6, we carry out our reduction to the rank 2 case, and, finally, we complete the proof in §7.

§2. Let g be an arbitrary element of a group G. We write

$$g = \prod_{i=1}^{\infty} g_i \quad \text{iff} \quad \Big(\prod_{i=1}^{n} g_i\Big)^{-1} g \in \gamma_n G \,, \qquad \text{for all } n.$$

Now suppose that G is residually nilpotent and let $g \in G$. Then we may choose elements $g_1, g_2, \ldots$ of G such that $g_i \in \gamma_i G$ and such that

$$g \equiv g_1 \, g_2 \, \ldots \, g_n \quad \text{modulo} \quad \gamma_{n+1} G \, .$$

It follows immediately then that

$$g = \prod_{i=1}^{\infty} g_i \, .$$

If one imposes certain extra conditions on the group G, then every element of G can be written uniquely as a particular infinite product. This is the essence of the following discussion.

Let G be a residually nilpotent group. For every $n \geq 1$, suppose

$$S_n = \{ g_{n,\alpha(n)} \mid \alpha(n) \in \Lambda_n \}$$

where $g_{n,\alpha(n)} \in \gamma_n G$ for all $\alpha(n) \in \Lambda_n$. Now well-order S_n in any way. Suppose further that every element g_n of $\gamma_n G$ can be expressed uniquely modulo $\gamma_{n+1} G$ in the following way:

$$g_n \equiv \prod_{i=1}^{k} g_{n,\alpha_i(n)}^{\rho_{n,\alpha_i(n)}} \quad \text{modulo } \gamma_{n+1}G$$

where $g_{n,\alpha_1(n)} < g_{n,\alpha_2(n)} < \cdots < g_{n,\alpha_k(n)}$ and $\rho_{n,\alpha_i(n)}$ belongs to a fixed subset of integers depending on n.

Clearly, every element g of G can be written as follows:

$$g = \prod_{n=1}^{\infty} \prod_{i=1}^{k(n)} g_{n,\alpha_i(n)}^{\rho_{n,\alpha_i(n)}} ,$$

where k(n) is a finite non-negative integer, with the understanding that $\prod_{i=1}^{0} g_{n,\alpha_i(n)} = 1$, and

$$g_{n,\alpha_1(n)} < g_{n,\alpha_2(n)} < \cdots < g_{n,\alpha_{k(n)}(n)} .$$

Suppose

$$g = \prod_{n=1}^{\infty} \prod_{i=1}^{k(n)} g_{n,\alpha_i(n)}^{\rho_{n,\alpha_i(n)}} = \prod_{n=1}^{\infty} \prod_{j=1}^{\ell(n)} g_{n,\beta_j(n)}^{\rho_{n,\beta_j(n)}}$$

where

$$0 \leq k(n) < \infty, \qquad 0 \leq \ell(n) < \infty ,$$

$$g_{n,\alpha_1(n)} < \cdots < g_{n,\alpha_k(n)}(n)$$

and

$$g_{n,\beta_1(n)} < \cdots < g_{n,\beta_{\ell(n)}}(n) .$$

Then for all $n \geq 1$, $k(n) = \ell(n)$ and $\alpha_i(n) = \beta_i(n)$ for all $n \geq 1$ and $1 \leq i \leq k(n)$.

Let P be a parafree group in the variety of all metabelian groups, with x_λ, $\lambda \in \Lambda$ as a parabasis. For every $n \geq 1$, let S_n be the set of all left normed basis commutators in x_λ, $\lambda \in \Lambda$ of weight n. (Of course, we assume that the sets S_n are well-ordered.) Using a theorem of W. Magnus (see e.g. [11]) on the independence of the left normed basic commutators of weight ≥ 2 in a free metabelian group, it follows immediately from the preceding discussion that every element of P has a unique form as an infinite product of powers left normed basic commutators in the x_λ's, arranged in an increasing order. This representation of elements of a parafree group in $\underline{\underline{A}}^2$ will be used in Chapter II as well as in Chapter I.

§3. Throughout this section we shall mainly deal with metabelian groups. The first four lemmas are known, and hence their proofs will be omitted.

Lemma 3.1. Let G be any group, and let x and y be any two elements of G. Then

(i) $[x,y^{-1}] = ([x,y]^{-1})^{y^{-1}}$

(ii) $[x^{-1},y] = ([x,y]^{-1})^{x^{-1}}$

(iii) $[x^{-1},y^{-1}] = [x,y]^{(xy)^{-1}}$.

Lemma 3.2. (P. Hall) Let G be any group, and let x, y and z be arbitrary elements of G. Then

$$[x,y^{-1},z]^y\,[y,z^{-1},x]^z\,[z,x^{-1},y]^x = 1.$$

Lemma 3.3. Let G be any group, and let x, y, and z be arbitrary elements of G. Then

(i) $[xy, z] = [x,z]^y\,[y,z]$

(ii) $[x,yz] = [x,z]\,[x,y]^z$

(iii) $[x,y]^z = [x,y]\,[x,y,z]$.

Moreover, if G is a metabelian group, then

(iv) whenever $x \in \gamma_2 G$, $x^{yz} = x^{zy}$

(v) $[x^r, y^s] = [x,y]^{(1+x+\ldots+x^{r-1})(1+y+\ldots\ y^{s-1})}$

where r and s are positive integers.

Lemma 3.4. Let G be any metabelian group, and let x and y be any two elements of G. Then

$$[y,\ kx,\ \ell y,\ mx] = [y,\ (k+m)x,\ \ell y]$$

where $k \geq 1$, $\ell \geq 0$ and $m \geq 0$.

Proof: The proof can be found in H. Neumann [11; p. 96].

Lemma 3.5. Let G be any residually nilpotent metabelian group. Let g be an element of $\gamma_2 G$, and let h be any element of G. If we write $g = \prod_{i=2}^{\infty} g_i$ where $g_i \in \gamma_i G$ $(i \geq 2)$, then

(i) $$[g,h] = \prod_{i=2}^{\infty} [g_i, h]$$

(ii) $$g^h = \prod_{i=2}^{\infty} g_i^h$$

Proof: Using Lemma 3.3 and the fact that $\gamma_2 G$ is abelian we get

(1) $$[g,h] \equiv \prod_{i=2}^{n} [g_i, h] \text{ modulo } \gamma_{n+2} G \qquad (n \geq 2) .$$

Hence,

$$\left(\prod_{i=2}^{n} [g_i,h]\right)^{-1} [g,h] \in \gamma_{n+2}G \qquad (n \geq 2).$$

Thus, (i) is proved. The proof of (ii) is similar, and hence is omitted.

Before stating the next lemmas, we shall introduce certain notation.

Let G be any group, and let x be an arbitrary element of G. If k and ℓ are any non-negative integers, let $\alpha(k,\ell,x)$ denote the following element of $\mathbb{Z}(G)$, the group ring of G over the integers:

$$\alpha(0,i,x) = x^i \quad \text{for all} \quad i \geq 0$$

and immediately,

$$\alpha(k,\ell,x) = \sum_{i=k-1}^{\ell-1} \alpha(k-1,i,x) \quad \text{for all } k > 0 \text{ and } \ell \geq 0 ,$$

where our convention is that $\sum_{i=k-1}^{\ell-1} \alpha(k-1,i,x) = 0$ if $\ell < k$. Thus, $\alpha(k,\ell,x) = 0$ whenever $\ell < k$. Moreover, one should observe that $\alpha(k,k,x) = 1$ for all $k \geq 0$.

Next, we shall denote by $m(s,t)$, where $s \geq 0$, $t \geq 0$, the following non-negative integer:

$$m(0,t) = 1 \quad \text{for all} \quad t \geq 0$$

and inductively,

$$m(s,t) = \sum_{i=s-1}^{t-1} m(s-1,i) \quad \text{for all } s > 0 \text{ and } t \geq 0$$

and again our convention is that whenever $t < s$,

$\sum_{i=s-1}^{t-1} m(s-1,i) = 0$. Thus $m(s,t) = 0$ whenever $t < s$. Furthermore, it can easily be checked that $m(s,s) = 1$ for all $s \geq 0$.

Now we are ready to proceed with the lemmas.

Lemma 3.6. Let G be any metabelian group. If $x_1,x_2,\ldots,x_n$, and z are arbitrary elements of G such that $n \geq 2$, then

$$[x_1,x_2,\ldots,x_n]^{\alpha(k,\ell,z)} = [x_1,x_2,\ldots,x_n]^{m(k,\ell)}[x_1,\ldots x_n,z]^{\alpha(k+1,\ell,2)}$$

where $k \geq 0$ and $\ell \geq 0$.

Proof: The proof is by induction on k. If $k = 0$, we must show that

$$(1)\quad [x_1,x_2,\ldots x_n]^{\alpha(0,\ell,z)} = [x_1,x_2,\ldots x_n]^{m(0,\ell)} [x_1,x_2,\ldots,x_n,z]^{\alpha(1,\ell,z)}$$

By definition we get

$$(2)\qquad \alpha(0,\ell,z) = z^{\ell}$$

and

$$(3)\qquad \alpha(1,\ell,z) = \sum_{i=0}^{\ell-1} \alpha(0,i,z) .$$

We observe that if $\ell = 0$ equation (1) holds trivially. Thus, assume that $\ell > 0$. Using Lemma 3.3 and equations (2) and (3) we get

$$\begin{aligned}[x_1,x_2,\ldots,x_n]^{z^{\ell}} &= [x_1,x_2,\ldots,x_n][x_1,x_2,\ldots,x_n,z^{\ell}] \\ &= [x_1,x_2,\ldots,x_n][x_1,x_2,\ldots,x_n,z]^{1+z+\ldots+z^{\ell-1}} \\ &= [x_1,x_2,\ldots,x_n][x_1,x_2,\ldots,x_n,z]^{\alpha(1,\ell,z)}\end{aligned}$$

Thus, equation (1) holds. Next, we shall assume that the result holds for some fixed $k > 0$, and we shall prove that the following identity holds:

$$(4)\quad [x_1,x_2,\ldots,x_n]^{\alpha(k+1,\ell,z)}$$

$$= [x_1,x_2,\ldots,x_n]^{m(k+1,\ell)}[x_1,x_2,\ldots,x_n,z]^{\alpha(k+2,\ell,z)} .$$

Using the induction hypothesis and the fact that γ_2G is abelian, we get

$$(5)\quad [x_1,x_2,\ldots,x_n]^{\alpha(k+1,\ell,z)} = [x_1,x_2,\ldots,x_n]^{\sum_{i=k}^{\ell-1}\alpha(k,i,z)}$$

$$= [x_1,x_2,\ldots,x_n]^{\sum_{i=k}^{\ell-1}m(k,i)}$$

$$\cdot\, [x_1,x_2,\ldots,x_n,z]^{\sum_{i=k}^{\ell-1}\alpha(k+1,i,z)} .$$

By definition we know that

$$(6)\qquad \sum_{i=k}^{\ell-1} m(k,i) = m(k+1,\ell)$$

and

$$(7)\qquad \sum_{i=k}^{\ell-1}\alpha(k+1,i,z) = \sum_{i=k+1}^{\ell-1}\alpha(k+1,i,z) = \alpha(k+2,\ell,z) .$$

Now, equation (4) follows immediately from equations (5), (6) and (7), and thus the proof of the lemma is complete.

Lemma 3.7. Let G be any metabelian group. If $x_1,x_2,\ldots,x_n$ and z are arbitrary elements of G such tnat $n \geq 2$, then

$$[x_1,x_2,\ldots,x_n]^{z^\ell} = \prod_{i=0}^{\ell} [x_1,x_2,\ldots,x_n,iz]^{m(i,\ell)}$$

where $\ell \geq 0$.

Proof: The proof is by induction on ℓ. Suppose $\ell = 0$. Then

$$[x_1,x_2,\ldots,x_n]^{z^0} = [x_1,x_2,\ldots,x_n]^{m(0,0)}$$

because $m(0,0) = 1$. Next, we shall assume that

$$(1)\quad [x_1,x_2,\ldots,x_n]^{z^j} = \prod_{i=0}^{j} [x_1,x_2,\ldots,x_n,iz]^{m(i,j)} \text{ for } j \leq \ell$$

where ℓ is a fixed positive integer, and we must show that

$$(2)\quad [x_1,x_2,\ldots,x_n]^{z^{\ell+1}} = \prod_{i=0}^{\ell+1} [x_1,x_2,\ldots,x_n,iz]^{m(i,\ell+1)} .$$

We first observe that by definition $z^\ell = \alpha(0,\ell,z)$. Thus, using Lemma 3.6 we get

$$\begin{aligned}(3)\quad [x_1,x_2,\ldots,x_n]^{z^{\ell+1}} &= [x_1,x_2,\ldots,x_n]^{\alpha(0,\ell+1,z)}\\ &= [x_1,x_2,\ldots,x_n]^{m(0,\ell+1)}[x_1,x_2,\ldots,x_n,z]^{\alpha(1,\ell+1,z)}\\ &= [x_1,x_2,\ldots,x_n]^{m(0,\ell+1)}[x_1,x_2,\ldots,x_n,z]^{\sum_{j=0}^{\ell} \alpha(0,j,z)} .\end{aligned}$$

Using the induction hypothesis, equation (3) and the fact that $\gamma_2 G$ is abelian, it follows that

$$(4)\quad [x_1,x_2,\dots,x_n]^{z^{\ell+1}} = [x_1,x_2,\dots,x_n]^{m(0,\ell+)} \prod_{j=0}^{\ell} \prod_{i=0}^{j} [x_1,x_2,\dots,x_n,(1+i)z]^{m(i,j)}$$

$$= [x_1,x_2,\dots,x_n]^{m(0,\ell+1)} \prod_{i=0}^{\ell} [x_1,x_2,\dots x_n,(i+1)z]^{\sum_{j=i}^{\ell} m(i,j)} .$$

By the definition, we get

$$(5)\qquad m(i+1,\ell+1) = \sum_{j=i}^{\ell} m(i,j) .$$

Hence it follows immediately from equations (4) and (5) that

$$(6)\quad [x_1,x_2,\dots,x_n]^{z^{\ell+1}} = [x_1,x_2,\dots,x_n]^{n(0,\ell+1)} \prod_{i=0}^{\ell} [x_1,x_2,\dots,x_n,(1+i)z]^{m(i+1,\ell+1)}$$

$$= \prod_{i=0}^{\ell+1} [x_1,x_2,\dots,x_n,iz]^{m(i,\ell+1)}$$

and the lemma is proved.

<u>Lemma 3.8</u>. Let G be any metabelian group. If x and y are any elements of G, then

$$[y,mx,ny]^{x^{\ell}} = \prod_{i=0}^{\ell} [y,(m+i)x,ny]^{m(i,\ell)}$$

where $m > 0$, $n \geq 0$ and $\ell \geq 0$.

Proof: The proof of this lemma follows easily from Lemmas 3.4 and 3.7.

§4. In this section, we shall deal with parafree groups of rank 2 in the variety of all metabelian groups. Using the ideas developed in Section 2, it is easy to see that if a and b are a parabasis of P, then every element p of P can be written uniquely (modulo permutation of the commutators) as follows:

$$p = a^{k_{0,0}} b^{k_{0,1}} \prod_{\substack{m \geq 1 \\ m \geq 0}} [b,ma,nb]^{k_{m,n}} ;$$

here $k_{0,0}$, $k_{0,1}$ and $k_{m,n}$ (for $m \geq 1$, $n \geq 0$) are any integers. This representation of elements of p will be used repeatedly in this section.

The following lemma is essentially well-known [7].

Lemma 4.1. Let P be a parafree group of rank 2 in $\underline{\underline{A}}^2$. If a and b form a parabasis of P, then

$$[b,ma,nb]^{b^{-1}} = \prod_{i=0}^{\infty} [b,ma,(n+i)b]^{(-1)^i}$$

where $m > 0$ and $n \geq 0$.

Proof: We shall first show that for an arbitrary positive integer k the following identity holds:

$$(1) \quad [b,ma,nb]^{b^{-1}} = \prod_{i=0}^{k-1} [b,ma,(n+i)b]^{(-1)^i} \cdot ([b,ma,(n+k)b]^{(-1)^k})^{b^{-1}} .$$

The proof of equation (1) is by induction. Suppose $k = 1$. Then we must show that

$$(2) \quad [b,ma,nb]^{b^{-1}} = [b,ma,nb]([b,ma,(n+1)b]^{-1})^{b^{-1}}.$$

It follows from Lemmas 3.1 and 3.3 that

$$(3) \quad [b,ma,nb]^{b^{-1}} = [b,ma,nb]\ [b,ma,nb,b^{-1}]$$

$$= [b,ma,nb]\ ([b,ma,(n+1)b]^{-1})^{b^{-1}}.$$

Next, suppose equation (1) holds for an arbitrary fixed $k > 1$. Then we shall show that

$$(4) \quad [b,ma,nb]^{b^{-1}} = \prod_{i=0}^{k} [b,ma,(n+i)b]^{(-1)^i} \cdot ([b,ma,(n+k+1)b]^{(-1)^{k+1}})^{b^{-1}}.$$

Using the induction hypothesis, Lemmas 3.1 and 3.3 and the fact that $\gamma_2 P$ is abelian we get

$$(5) \quad [b,ma,nb]^{b^{-1}}$$

$$= \prod_{i=0}^{k-1} [b,ma,(n+i)b]^{(-1)^i} \cdot ([b,ma,(n+k)b]^{(-1)^k})^{b^{-1}}$$

$$= \prod_{i=0}^{k} [b,ma,(n+i)b]^{(-1)^i}\ [b,ma,(n+k)b,b^{-1}]^{(-1)^k}$$

$$= \prod_{i=0}^{k} [b,ma,(n+i)b]^{(-1)^i}\ \{([b,ma,(n+k+1)b]^{-1})^{b^{-1}}\}^{(-1)^k}$$

$$= \prod_{i=0}^{k} [b,ma,(n+i)b]^{(-1)^i}\ ([b,ma,(n+k+1)b]^{(-1)^{k+1}})^{b^{-1}}.$$

Thus, equation (4) holds, and hence equation (1) is true for all $k \geq 0$. Since the commutator $([b,ma,(n+k)b]^{(-1)^k})^{b^{-1}}$ belongs to $\gamma_{m+n+1+k}P$, it follows immediately from the definition of an infinite product that

$$[b,ma,nb]^{b^{-1}} = \prod_{i=0}^{\infty} [b,ma,(n+i)b]^{(-1)^i} ,$$

and thus the lemma is proved.

Corollary. Let P be a parafree group of rank 2 in $\underline{\underline{A}}^2$. If a and b form a parabasis of P, then

$$[b,ma,nb,b^{-1}] = \prod_{i=1}^{\infty} [b,ma,(n+i)b]^{(-1)^i} .$$

Proof: The proof follows immediately from Lemmas 3.3 and 4.1.

Lemma 4.2. Let P be a parafree group of rank 2 in $\underline{\underline{A}}^2$, with a and b as a parabasis. Then

$$[b,ma,nb,kb^{-1}] = \prod_{i=k}^{\infty} [b,ma,(n+i)b]^{\mu_i}$$

where $\mu_i \in \mathbb{Z}$, $m > 0$, $n \geq 0$ and $k \geq 1$.

Proof: The proof is by induction on k. If k = 1, then the statement of Lemma 4.2 is reduced to the statement of the corollary of Lemma 4.1. Thus, suppose the lemma holds for an arbitrary fixed k > 1. Then we shall show that

$$[b,ma,nb,(k+1)b^{-1}] = \prod_{i=k+1}^{\infty} [b,ma,(n+i)b]^{\mu_i} .$$

where $\mu_i \in \mathbb{Z}$, $m > 0$, $n \geq 0$ and $k > 1$.
Using the induction hypothesis, Lemma 3.5 and the corollary of Lemma 4.1, we get

$$\begin{aligned}[b,ma,nb,(k+1)b^{-1}] &= [b,ma,nb,kb^{-1},b^{-1}] \\ &= [\prod_{i=k}^{\infty} [b,ma,(n+i)b]^{\mu_i},b^{-1}] \\ &= \prod_{i=k}^{\infty} [b,ma,(n+i)b,b^{-1}]^{\mu_i} \\ &= \prod_{i=k}^{\infty} (\prod_{j=1}^{\infty} [b,ma,(n+i+j)b]^{(-1)^j})^{\mu_i} \\ &= \prod_{\ell=k+1}^{\infty} [b,ma,(n+\ell)b]^{\rho_\ell} ,\end{aligned}$$

where $\rho_\ell = \sum_{\substack{i+j=\ell \\ j\geq 1 \\ i\geq k}} (-1)^j \mu_i$.

Clearly $\rho_\ell \in \mathbb{Z}$, and hence the lemma is proved.

Lemma 4.3. Let P be a parafree group of rank 2 in $\underline{\underline{A}}^2$, with a and b as a parabasis. Let p be an arbitrary element of $\gamma_2 P$. If we write

$$(1) \qquad p = \prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{k_{m,n}}$$

and

$$(2) \qquad p^{b^s} = \prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{\ell_{m,n}} ,$$

where $k_{m,n}$, $\ell_{m,n}$ and s are any integers, then

$$k_{m,0} = \ell_{m,0} \quad \text{for all} \quad m .$$

Proof: We shall first suppose that $s \geq 0$. Using Lemmas 3.5 and 3.7 and equation (1) we get

$$(3)\quad p^{b^s} = \Big(\prod_{\substack{m>0\\ n\geq 0}} [b,ma,nb]^{k_{m,n}}\Big)^{b^s}$$

$$= \prod_{\substack{m>0\\ n\geq 0}} \big([b,ma,nb]^{b^s}\big)^{k_{m,n}}$$

$$= \prod_{\substack{m>0\\ n\geq 0}} \Big(\prod_{i=0}^{s} [b,ma,(n+i)b]^{m(i,s)}\Big)^{k_{m,n}}$$

$$= \prod_{\substack{m>0\\ n\geq 0}} \prod_{i=0}^{s} [b,ma,(n+i)b]^{k_{m,n}\cdot m(i,s)} .$$

Now, we observe that the infinite product, $\prod_{\substack{m>0\\ n\geq 0}} \prod_{i=0}^{s} [b,ma,(n+i)b]^{k_{m,n}\cdot m(i,s)}$ contains a commutator ending with an a if and only if $n = 0$. Furthermore, if $n = 0$, the commutator $[b,ma]$ appears to the power $k_{m,0}\, m(0,s) = k_{m,0}$, for all $m \geq 1$. Using the above fact, Lemma 2.1 and equation (2), it is clear that $k_{m,0} = \ell_{m,0}$ for all $m \geq 1$.

Next, suppose $s < 0$, and put $t = -s$ and $c = b^{-1}$. Then, as in equation (3) we get

$$(4)\qquad p^{b^s} = p^{(b^{-1})^{-s}}$$

$$= p^{c^t}$$

$$= \prod_{\substack{m>0\\ n\geq 0}} \prod_{i=0}^{t} [b,ma,nb,ic]^{k_{m,n}\cdot m(i,t)}$$

$$= \prod_{\substack{m>0\\ n\geq 0}} \prod_{i=0}^{t} [b,ma,nb,ib^{-1}]^{k_{m,n}\cdot m(i,t)} .$$

Using Lemma 4.2 and equation (4) we get

$$(5)\quad p^{b^s} = \prod_{\substack{m>0\\ n\geq 0}} \prod_{i=0}^{t} \Big(\prod_{j=i}^{\infty} [b,ma,(n+j)b]^{\mu_{j,i}}\Big)^{k_{m,n}\cdot m(i,t)},$$

where $\mu_{j,i}$ $(j \geq i)$ are integers such that $\mu_{0,0} = 1$ and $\mu_{j,0} = 0$ whenever $j > 0$. Thus, we observe that the infinite product on the right hand side of equation (5) contains a commutator ending with an a if and only if $n = 0$. Furthermore, if $n = 0$, then the commutator $[b,ma]$ appears to the power $k_{m,0}\cdot m(0,t) = k_{m,0}$, for all $m \geq 1$. Thus, as in the previous case, it follows that $k_{m,0} = \ell_{m,0}$ for all $m \geq 1$. Hence the lemma is proved.

Lemma 4.4. Let P be a parafree group of rank 2, with a and b as a parabasis. If we write

$$p = b^s \prod_{\substack{m>0\\ n\geq 0}} [b,ma,nb]^{k_{m,n}},$$

then

$$p^{a^t} = b^s[b,a]^{st+k_{1,0}} \cdot \prod \text{commutators of higher weight},$$

where $s \geq 0$ and $t \geq 0$.

Proof: We first note that

$$p \equiv b^s[b,a]^{k_{1,0}} \quad \text{modulo} \quad \gamma_3 P .$$

Thus, using Lemma 3.3 we get

$$\begin{aligned} p^{a^t} &\equiv b^s[b^s,a^t]\,([b,a]^{k_{1,0}})^{a^t} \\ &\equiv b^s[b,a]^{st}\,[b,a]^{k_{1,0}} \equiv \end{aligned}$$

$$\equiv \ b^s[b,a]^{st+k_{1,0}} \text{ modul } \gamma_3 P \ .$$

Hence the lemma is proved.

§5. <u>Lemma 5.1</u>. If we put

$$f_n(x) = \frac{\{x - (n-1)\}^n}{n!}$$

where $x \in R$ and n is an integer ≥ 0, then

$$m(n,t) \geq f_n(t) \quad \text{for all} \quad t \geq n \ .$$

<u>Proof</u>: The proof is by induction on n. Suppose $n = 0$. Then we must show that

$$(1) \qquad m(0,t) \geq f_0(t) \quad \text{for all} \quad t \geq 0 \ .$$

But, by the definitions, we get

$$(2) \qquad m(0,t) = 1 \qquad \text{for all} \quad t \geq 0$$

and

$$(3) \qquad f_0(t) = \frac{(t+1)^0}{0!} = 1 \quad \text{for all } t \geq 0 \ .$$

Now, equation (1) follows immediately from equations (2) and (3). Next, we shall assume that the lemma is true for an arbitrary fixed n, and we shall show that

$$(4) \qquad m(n+1,t) \geq f_{n+1}(t) \quad \text{for all } t \geq n+1,$$

We first observe that $f_n(x)$ is an increasing function for all $x \geq (n-1)$, and thus the sum, $\sum_{i=n}^{t-1} f_n(i)$ $(t > n)$, is an upper approximation of the integral $\int_{n-1}^{t-1} f_n(x)\,dx$.

But

(5)
$$\int_{n-1}^{t-1} f_n(x)\,dx = \int_{n-1}^{t-1} \frac{\{x-(n-1)\}^n}{n!}\,dx = \left.\frac{\{x-(n-1)\}^{n+1}}{(n+1)!}\right]_{n-1}^{t-1} = \frac{(t-n)^{n+1}}{(n+1)!}$$

Using the induction hypothesis we know that

(6)
$$f_n(i) \leq m(n,i) \quad \text{for} \quad i \geq n .$$

Thus, using the upper approximation together with equations (5) and (6), we get

(7)
$$\frac{(t-n)^{n+1}}{(n+1)!} \leq \sum_{i=n}^{t-1} f_n(i) \leq \sum_{i=n}^{t-1} m(n,i) = m(n+1,t)$$

for all $t \geq n+1$. But, by definition,

(8)
$$f_{n+1}(t) = \frac{(t-n)^{n+1}}{(n+1)!} .$$

Now, the lemma follows immediately from equations (7) and (8).

Lemma 5.2. If we put

$$g_0(x) = 1$$

and

$$g_n(x) = \frac{x^n}{n!} ,$$

where $x \in R$ and n is an integer > 0, then

$$g_n(t) \geq m(n,t) \quad \text{for all} \quad t \geq n.$$

Proof: The proof of Lemma 5.2 is similar to the proof of Lemma 5.1 and hence is omitted.

§6. Lemma 6.1. Let P be a parafree group in the variety $\underline{\underline{A}}^2$. If N is a normal subgroup of P such that $N < \gamma_2 P$ and $|P/NP'|$ is infinite, then either $N = 1$ or N is not finitely generated.

Proof: Suppose $N \neq 1$ and N is finitely generated. Since $\gamma_2 P$ is a torsion free abelian group, N is free abelian of finite rank. Now, let a be any element of N different from the identity. As the center of P is 1, it follows that there is an element b of P such that $[a,b] \neq 1$. Using a theorem of G. Baumslag [3] it follows that

$$H = \text{gp}\ (a^{b^i} \mid i = 0,1,2,\ldots)$$

is free abelian of infinite rank. But $H \leq N$, and hence we arrive at a contradiction. Thus, the lemma is proved.

Lemma 6.2. Suppose Theorem I is true whenever P is a parafree group of rank 2. Then Theorem I is true for an arbitrary, non-cyclic, parafree group P.

Proof: We shall first assume that P/NP' is periodic. Then, since $|P/NP'|$ is infinite, it follows that P is of infinite rank. Let η be the natural homomorphism of P onto P/P'. Then, clearly, it is enough to show that $N\eta$ is not finitely generated. Now, suppose $N\eta$ is finitely generated. Then, $P\eta/N\eta$ is not periodic, because $P\eta$ is a free abelian group of infinite rank. Thus we have arrived at a contradiction.

Next, we shall assume that P/NP' is not periodic. Using Lemma 6.1, it is enough to consider the case when N is not a subgroup of $\gamma_2 P$. Now, suppose N is finitely generated. Then using the basis theorem for free abelian groups, there exists a parabasis, X, of P and a finite subset, $\{x_1, \ldots x_n\}$, of X such that the group $N\gamma_2 P$ is generated by the element $x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}$ modulo $\gamma_2 P$, where $\alpha_i \in \mathbb{Z}$ $(i = 1,2,\ldots n)$. Furthermore, since P/NP' is not periodic and N is not a subgroup of $\gamma_2 P$, it is easy to see that X and its subset $\{x_1, \ldots x_n\}$ can be chosen so that $\alpha_1 = 0$, $\alpha_2 \neq 0$ and $x_2^{\alpha_2} \notin \gamma_2 P$. Now, put

$$K = gp_P\,(X \setminus \{x_1, x_2\}) \tag{1}$$

and

$$J = \bigcap_n \gamma_n P \cdot K \,. \tag{2}$$

Using a theorem of G. Baumslag [3], we know that P/J is a parafree group of rank 2. Let μ be the natural homomorphism of P onto P/J. Then, it is left to show that

$$N\eta \neq 1 \tag{3}$$

and

$$(4) \qquad |P\eta / N\eta (P\eta)'| \text{ is infinite.}$$

We shall first show that equation (3) holds. Since $x_2^{\alpha_2} \notin \gamma_2 P$ there exists an element, n, of N such tnat $n \neq 1$ and $n = x_2^{\alpha_2} \cdot P'$ where $P' \in \gamma_2 P$. Now, $P/K \,/\, \gamma_2(P/K)$ is free abelian of rank 2, freely generated by $x_1 K$ and $x_2 K$. So let ρ be the homomorphism of $P/K \,/\, \gamma_2(P/K)$ into $P/\gamma_2 P$ defined by

$$x_i K \cdot (\gamma_2(P/K))\rho = x_i \gamma_2 P \qquad (i = 1,2).$$

Then, clearly, $n \cdot K(\gamma_2 \; P/K)\rho \neq 1$ and thus $n \notin \gamma_2 P \cdot K \subset J$. Hence $N\eta \neq 1$.

Next, we shall prove equation (4). Let μ be the homomorphism of $P/J \,/\, \gamma_2(P/J)$ into $P/\gamma_2 P$ defined by

$$x_i J(\gamma_2 \; P/J)\mu = x_i \; \gamma_2 P \qquad (i = 1,2).$$

Now clearly

$$(5) \qquad \{(N\eta) \cdot (\gamma_2 \; P/J)\}\mu = \{\mathrm{gp}\; (x_2^{\alpha_2} J) \cdot (\gamma_2 \; P/J)\}\mu$$
$$= \mathrm{gp}\; (x_2^{\alpha_2} \; \gamma_2 P) .$$

Suppose $x_1^k \cdot J \in (N\eta)\cdot(P\eta)'$ for some $k \neq 0$. Then using equation (5) we get

$$x_1^k \; \gamma_2 P \in \mathrm{gp}\; (x_2^{\alpha_2} \; \gamma_2 P) .$$

But x_1 and x_2 are independent modulo $\gamma_2 P$. Hence $x_1^k \eta \notin (N\eta)\cdot(P\eta)'$ for all $k \neq 0$, and thus equation (4) is true.

§7. Lemma 7.1. Let P be a parafree group of rank 2 in the variety $\underline{\underline{A}}^2$. If N is a nontrivial normal subgroup of P such that $|P/N\,\gamma_2P|$ is infinite, then N is not finitely generated.

Proof: It follows from Lemma 6.1 that we can assume that N is not a subgroup of γ_2P. Thus $NP'/P' \neq 1$. Using the basis theorem for free abelian groups, it follows that we can choose a parabasis, $\{a,b\}$, of P such that a^t and b^s, where s and t are integers, generates $N\,\gamma_2P$ modulo γ_2P. Since $NP'/P' \neq 1$ and $|NP'/P'|$ is infinite, without loss of generality we shall assume that $s \neq 0$ and $t = 0$. Furthermore, we shall assume that $s > 0$. Suppose N is finitely generated, i.e. suppose

$$(1) \qquad N = gp\,(h_1,\ldots,h_k)\ .$$

Clearly $h_i = b^{sn_i} h_i'$, where $n_i \in \mathbb{Z}$ and $h_i' \in \gamma_2P$ for all $1 \leq i \leq k$. Since $b^s \cdot P'$ is an element of NP'/P', there is an element p of N such that $p = b^s p'$, where $p' \in \gamma_2P$. Put

$$(2) \qquad g_i = (p)^{-n_i}\, h_i \quad \text{for} \quad 1 \leq i \leq k\ .$$

Clearly, $g_i \in \gamma_2P$ for $1 \leq i \leq k$. Using equations (1) and (2) we get

$$(3) \qquad N = gp\,(h_1,\ldots,h_k) = gp\,(p,g_1,\ldots,g_k)$$
$$= gp\,(p)\cdot gp(g_1^{b^{st}}, \ldots, g_k^{b^{st}} \mid t \in \mathbb{Z}).$$

Now, the proof of the theorem is broken into two cases, depending on whether $k = 0$ or $k \neq 0$.

<u>Case 1</u>. Suppose $k = 0$. Then $N = \mathrm{gp}\,(b^s p')$. Since N is a normal subgroup of P, the element p^a belongs to N. Thus we must have

$$(4) \qquad p^a = p^{\ell} \quad \text{where} \quad \ell \in \mathbb{Z}\,.$$

We also know that $p = b^s p'$ can be written as follows:

$$(5) \qquad p = b^s \prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{k_{m,n}}\,,$$

where $k_{m,n} \in \mathbb{Z}$. Using equation (5) and Lemma 4.4, we get

$$(6) \qquad p^a \equiv b^s [b,a]^{s+k_{1,0}} \text{ modulo } \gamma_3 P\,.$$

Moreover, $p^{\ell} \equiv b^{\ell s}$ modulo $\gamma_2 P$. Hence $\ell = 1$. Thus, by equation (4) $p^a \equiv p$ modulo $\gamma_3 P$. But this implies that $s = 0$, and thus leads to a contradiction.

<u>Case 2</u>. Suppose $k \neq 0$. Then

$$(7) \qquad N = \mathrm{gp}\,(p) \cdot \mathrm{gp}(g_1^{b^{st}}, \ldots, g_k^{b^{st}} \mid t \in \mathbb{Z})$$

where $g_i \in \gamma_2 P$, $1 \leq i \leq k$. Put

$$(8) \qquad g_i = \prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{\ell^{(i)}_{m,n}} \quad \text{for} \quad 1 \leq i \leq k\,,$$

where $\ell^{(i)}_{m,n} \in \mathbb{Z}$, and

$$(9) \qquad g_i^{b^{st}} = \prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{\ell^{(i,t)}_{m,n}}$$

where $\ell_{m,n}^{(i,t)} \in \mathbb{Z}$. It follows immediately from Lemma 4.3 and equations (8) and (9) that

$$\ell_{m,0}^{(i)} = \ell_{m,0}^{(i,t)}$$

for all $m > 0$, $1 \leq i \leq k$ and $t \in \mathbb{Z}$. Next, we shall show that for at least one i, $1 \leq i \leq k$, $\ell_{1,0}^{i} \neq 0$. Since $p^a \in N$ we can write

$$p^a = p^{\ell} \prod_{i=1}^{k} \prod_{j=1}^{r_i} g_i^{b^{s \cdot t_{i,j}}} \tag{10}$$

where $t_{i,j}$, r_i and ℓ are integers. Using equations (6) and (10), it follows that $\ell = 1$. Moreover, using equations (9) and (10), we get

$$p^a = b^s[b,a]^{k_{1,0}+\sum_{i=1}^{k} r_i \ell_{1,0}^{i}} \cdot \prod \text{commutators of higher weight.} \tag{11}$$

On the other hand, using Lemma 4.4, we get

$$p^a = b^s[b,a]^{s+k_{1,0}} \cdot \prod \text{commutators of higher weight.} \tag{12}$$

Hence, the following identity holds

$$s = \sum_{i=1}^{k} r_i \ \ell_{1,0}^{i} \ . \tag{13}$$

But $s \neq 0$, and so there is an i, $1 \leq i \leq k$ such that $\ell_{1,0}^{i} \neq 0$. Without loss of generality we may assume that $\ell_{1,0}^{1} \neq 0$. Next, we shall write the element $g_1^{a^u}$ $(u > 0)$ in two ways. We shall then derive, for each u, a system of linear inhomogeneous equations by comparing the two different representations of $g_1^{a^u}$. On the one hand, using Lemma 3.5, we get

$$(14) \qquad g_1^{a^u} = \Big(\prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{\ell^1_{m,n}} \Big)^{a^u}$$

$$= \prod_{\substack{m>0 \\ n\geq 0}} \big([b,ma,nb]^{a^u} \big)^{\ell^1_{m,n}} .$$

Using equation (14) and Lemma 3.8, we get, for every u > 0,

$$(15) \quad g_1^{a^u} = \prod_{\substack{m>0 \\ n\geq 0}} \Big(\prod_{i=0}^{u} [b,(m+i)a,nb]^{m(i,u)} \Big)^{\ell^1_{m,n}}$$

$$= \prod_{j=1}^{k+1} [b,ja]^{\sum_{i=1}^{j} \ell^1_{i,0}\, m(j-i,u)}$$

$$\cdot \prod_{\substack{m>0 \\ n\geq 0 \\ (m+i,n)\in\{(1,0,\ldots(k+1,0)\}}} \Big(\prod_{i=0}^{u} [b,(m+i)a,nb]^{m(i,u)} \Big)^{\ell^1_{m,n}} .$$

On the other hand, since N is normal, we get

$$(16) \qquad g_1^{a^u} = P^{\ell_u} \prod_{i=1}^{k} \prod_{j=1}^{r_{i,u}} g_i^{st_{i,j}}$$

where $u > 0$, $\ell_u \in \mathbb{Z}$, $r_{i,u} \in \mathbb{Z}$ and $t_{i,j} \in \mathbb{Z}$. Since $g_1 \in \gamma_2 P$, it follows that $\ell_u = 0$ for all $u > 0$. Thus, using equations (9) and (16) and Lemma 4.3, we get

$$(17)\quad g_1^{a^u} = \prod_{i=1}^{k}\ \prod_{j=1}^{r_{i,u}}\ \prod_{\substack{m>0\\ n\geq 0}} [b,ma,nb]^{\ell_{m,n}^{i,t_{i,j}}}$$

$$= \prod_{j=1}^{k+1} \left([b,ja]^{\sum_{i=1}^{k} \ell_{j,0}^{i} r_{i,u}}\right) \prod_{i=1}^{k}\ \prod_{j=1}^{r_{i,u}}\ \prod_{\substack{m>0\\ n\geq 0\\ (m,n)\notin\{(1,0),\ldots,(k+1,0)\}}} [b,ma,nb]^{\ell_{m,n}^{i,t_{i,j}}}$$

Using equations (15) and (17) and Lemma 2.1, we get, for each $u > 0$, the following $k+1$ equations:

$$(18)\quad \sum_{i=1}^{k} \ell_{j,0}^{i}\, r_{i,u} = \sum_{i=1}^{j} \ell_{i,0}^{1}\, m(j-i,u) \quad \text{for } j = 1,2,\ldots k+1.$$

Now, consider a real vector space of dimension k, generated by $r_{i,u}$, $i = 1,2,\ldots,k$. Then the $k+1$ vectors $\sum_{i=0}^{k} \ell_{j,0}^{i}\, r_{i,u}$ $(j = 1,\ldots,k+1)$ must be linearly dependent. Hence, there exist non-zero real numbers $n_1,\ldots,n_{k+1}$ such that

$$(19)\qquad \sum_{j=1}^{k+1} n_j \sum_{i=1}^{k} \ell_{j,0}^{i}\, r_{i,u} = 0\ .$$

Using equations (18) and (19), it follows that

$$(20)\qquad \sum_{j=1}^{k+1} n_j \sum_{i=1}^{j} \ell_{i,0}^{1}\, m(j-i,u) = 0\ .$$

Next, we shall show that if u is sufficiently large, equation (20) does not hold. It follows from Lemmas 5.1 and 5.2 that whenever $u > 0$

$$(21)\qquad \frac{u^{j-i}}{(j-i)!} \geq m(j-i,u) \geq \frac{\{u-(j-i-1)\}^{j-i}}{(j-i)!}\ .$$

We first note that for every $u > 0$ we get

$$(22)\quad \left|n_{k+1} \sum_{i=2}^{k+1} \ell^{1}_{i,0}\, m(k+1-i,u) + \sum_{j=1}^{k} n_j \sum_{i=1}^{j} \ell^{1}_{i,0}\, m(j-i,u)\right|$$

$$\leq |n_{k+1}| \sum_{i=2}^{k+1} |\ell^{1}_{i,0}|\, m(k+1-i,u)$$

$$+ \sum_{j=1}^{k} |n_j| \sum_{i=1}^{j} |\ell^{1}_{i,0}|\, m(j-i,u)$$

$$\leq |n_{k+1}| \sum_{i=2}^{k+1} |\ell^{1}_{i,0}| \frac{u^{k+1-i}}{(k+1-i)!} +$$

$$+ \sum_{j=1}^{k} |n_j| \sum_{i=1}^{j} |\ell^{1}_{i,0}| \frac{u^{j-i}}{(k+1-i)!}$$

$$\leq |n_{k+1}| \sum_{i=2}^{k+1} |\ell^{1}_{i,0}| u^{k+1-i} + \sum_{j=1}^{k} |n_j| \sum_{i=1}^{j} |\ell^{1}_{i,0}| u^{j-i}.$$

Using equation (22), it follows immediately that there exist positive integers K_1 and N_1 such that whenever $u > N_1$ we get

$$(23)\quad \left|n_{k+1} \sum_{i=2}^{k+1} \ell^{1}_{i,0}\, m(k+1-i,u) + \sum_{j=1}^{k} n_j \sum_{i=1}^{j} \ell^{1}_{i,0} m(j-i,u)\right| \leq K_1 u^{k-1}.$$

Moreover, we note that

$$(24)\qquad \left|n_{k+1}\ \ell^{1}_{1,0}\, m(k,u)\right| \;\geq\; \left|n_{k+1}\ \ell^{1}_{1,0}\right| \frac{(u-k)^k}{k!} \quad .$$

Put $|n_{k+1}\ \ell^{1}_{1,0}| = K_2$. Then $K_2 \neq 0$ because $n_{k+1} \neq 0$ and $\ell^{1}_{1,0} \neq 0$. Next, we observe that there exists a positive integer, N_2, such that $N_2 > N_1$ and such that whenever $u > N_2$ we get

$$(25)\qquad K_2 \frac{(u-k)^k}{k!} \;>\; K_1 u^{k-1} \quad .$$

Thus it follows from equations (23), (24) and (25) that whenever $u > N_2$ we get

$$(26) \quad \left| \sum_{j=1}^{k+1} n_j \sum_{i=1}^{j} \ell^1_{i,0}\, m(j-i,u) \right|$$

$$\geq \left| n_{k+1}\, \ell^1_{1,0}\, m(k,u) \right|$$

$$- \left| n_{k+1} \sum_{i=2}^{k+1} \ell^1_{i,0} m(k+1-i,u) + \sum_{j=1}^{k} n_j \sum_{i=1}^{j} \ell^1_{i,0} m(j-i,u) \right|$$

$$\geq K_2 \frac{(u-k)^k}{k!} - K_1 u^{k-1} > 0 .$$

Hence, equation (20) does not hold for every $u > N_2$. Thus Lemma 6.1, and hence Theorem I, is proved.

CHAPTER II

§1. In this chapter, we shall generalize a theorem of G. Baumslag, B. H. Neumann, H. Neumann and P. M. Neumann [6] concerning free groups in the variety of all metabelian groups to parafree groups in any variety larger than the variety of all metabelian groups. Precisely, we shall prove

Theorem II. Let P be a parafree group in any variety $\underline{\underline{V}}$ containing the variety $\underline{\underline{A}}^2$ of all metabelian groups. If x and y are any two elements of P which are independent modulo $\gamma_2 P$, then there is no element $z \in P$ such that $[x,y] = z^m$, $m > 1$.

To prove the theorem, we begin by reducing to the case where $\underline{\underline{V}} = \underline{\underline{A}}^2$ and P is of rank 2. We then examine the commutator $[x,y]$, expressing it in terms of basic commutators, using again our infinite product formula. The expression thus obtained for $[x,y]$ shows that $[x,y]$ cannot be a proper power.

§2. Lemma 2.1. Suppose Theorem II is true whenever P is a parafree group of rank 2 in $\underline{\underline{A}}^2$. Then Theorem II is true for an arbitrary, non-cyclic, parafree group P in $\underline{\underline{A}}^2$.

Proof. The proof is by contradiction. Suppose there exist elements x, y and z of P such that x and y are independent modulo $\gamma_2 P$ and $[x,y] = z^m$ for some $m > 1$. Now, since x and y are independent modulo $\gamma_2 P$, there exists a parabasis X of P such that x and y belong to X. Next, let

(1) $$K = gp_P\,(X \setminus \{x,y\})$$

and

(2) $$J = \bigcap_n (\gamma_n P \cdot K)\ .$$

Using a theorem of G. Baumslag, we know that P/J is a parafree group of rank 2. Now, if η denotes the natural homomorphism of P onto P/J, then, clearly $x\eta$ and $y\eta$ are independent modulo $\gamma_2(P/J)$ and $[x\eta, y\eta] = (z\eta)^m$, where $m > 1$. This leads to a contradiction, and hence the lemma is proved.

Lemma 2.2. Suppose Theorem II is true whenever P is a parafree group of rank 2 in $\underline{\underline{A}}^2$. Then Theorem II is true for an arbitrary, noncyclic, parafree group P in $\underline{\underline{V}}$, where $\underline{\underline{V}}$ is any variety which contains $\underline{\underline{A}}^2$.

Proof. Let P be any noncyclic parafree group in $\underline{\underline{V}}$. As in Lemma 2.1, suppose there exist elements x, y and z of P such that x and y are independent modulo $\gamma_2 P$ and $[x,y] = z^m$ for some $m > 1$. Now let

(1) $$J = \bigcap_n (\gamma_n P \cdot P'')\ .$$

Then it is easy to see that P/J is a parafree group in $\underline{\underline{A}}^2$. Next we notice that if η denotes the natural homomorphism of P onto P/J, then $x\eta$ and $y\eta$ are independent modulo $\gamma_2(P/J)$ and $[x\eta, y\eta] = (z\eta)^m$, $m > 1$. Thus, using Lemma 2.1 we arrive at a contradiction, and hence the lemma is proved.

Before proving Theorem II, we shall prove a few technical lemmas.

Lemma 2.3. Let G be any metabelian group. If x and y are arbitrary elements of G, then

$$[y,x]^{y^s x^t} = \prod_{j=0}^{s} \Big(\prod_{i=0}^{t} [y,(1+i)x,jy]^{m(i,t)} \Big)^{m(j,s)}$$

where $s \geq 0$ and $t \geq 0$.

Proof. Using Lemma 3.3 of Chapter I we get

$$(1) \quad [y,x]^{y^s x^t} = [y,x]\,[y,x,y^s x^t]$$

$$= [y,x]\,[y,x,x^t]\,[y,x,y^s]^{x^t} .$$

Using equation (1) and Lemmas 3.6 and 3.7 of Chapter I we get

$$(2) \quad [y,x]^{y^s x^t} = [y,x] \prod_{i=1}^{t} [y,(1+i)x]^{m(i,t)} \Big(\prod_{j=1}^{s} [y,x,jy]^{m(j,s)}\Big)^{x^t}$$

$$= \prod_{i=0}^{t} [y,(1+i)x]^{m(i,t)} \prod_{j=1}^{s} \big([y,x,jy]^{x^t}\big)^{m(j,s)}$$

$$= \prod_{i=0}^{t} [y,(1+i)x]^{m(i,t)} \prod_{j=1}^{s} \Big(\prod_{i=0}^{t} [y,(1+i)x,jy]^{m(i,t)} \Big)^{m(j,s)}$$

$$= \prod_{j=0}^{s} \Big(\prod_{i=0}^{t} [y,(1+i)x,jy]^{m(i,t)} \Big)^{m(j,s)} .$$

Thus, the lemma is proved.

Lemma 2.4. Let G be any metabelian group. If x and y are arbitrary elements of G, then

$$[y^k,x^\ell] = \prod_{\substack{0\leq s\leq k-1\\ 0\leq t\leq \ell-1}} \prod_{j=0}^{s} \Big(\prod_{i=0}^{t} [y,(1+i)x,jy]^{m(i,t)}\Big)^{m(j,s)}$$

where $k \geq 0$ and $\ell \geq 0$.

Proof. Using Lemma 2.3 of Chapter I and Lemma 2.3 of Chapter II we get

$$\begin{aligned}[y^k,x^\ell] &= [y,x]^{(1+y+\dots+y^{k-1})(1+x+\dots x^{\ell-1})}\\ &= \prod_{\substack{0\leq s\leq k-1\\ 0\leq t\leq \ell-1}} [y,x]^{y^s x^t}\\ &= \prod_{\substack{0\leq s\leq k-1\\ 0\leq t\leq \ell-1}} \prod_{j=0}^{s} \Big(\prod_{i=0}^{t} [y,(1+i)x,jy]^{m(i,t)}\Big)^{m(j,s)}\end{aligned}$$

Hence, the lemma is proved.

Lemma 2.5. Let G be any metabelian group. If x and y are elements of G such that

$$x = g_1^\ell \prod_{i=2}^{\infty} g_i \tag{1}$$

and

$$y = h_1^k \prod_{i=2}^{\infty} h_i \tag{2}$$

where $g_i \in \gamma_i G$ and $h_i \in \gamma_i G$, $i \geq 1$, and $\ell, k > 0$, then

$$[x,y] = [g_1^\ell, h_1^k] \prod_{i=2}^{\infty} h_i^{-g_1^\ell} \prod_{i=2}^{\infty} g_i^{h_1^k} \prod_{i=2}^{\infty} h_i \prod_{i=2}^{\infty} g_i^{-1}$$

Proof. Put

$$g' = \prod_{i=2}^{\infty} g_i \tag{1}$$

and

$$(2) \qquad h' = \prod_{i=2}^{\infty} h_i .$$

Clearly g' and h' belong to $\gamma_2 G$. Now, using Lemma 3.3 of Chapter I and the fact that $\gamma_2 G$ is abelian we get

$$\begin{aligned}(3) \qquad [x,y] &= [g_1^{\ell} g', h_1^{k} h'] \\ &= [g_1^{\ell}, h_1^{k}]\,[g_1^{\ell}, h']\,[g', h_1^{k}] \\ &= [g_1^{\ell}, h_1^{k}]\,(h'^{-1})^{g_1^{\ell}}\, h'\, g'^{-1}\, g'^{h_1^{k}} .\end{aligned}$$

Using Lemma 3.5 of Chapter I we know that

$$(4) \qquad g'^{-1} = \prod_{i=2}^{\infty} g_i^{-1}$$

and

$$(5) \qquad h'^{-1} = \prod_{i=2}^{\infty} h_i^{-1} .$$

Thus, using equations (3), (4) and (5) and Lemma 3.5 of Chapter I we get

$$\begin{aligned}(6) \quad [x,y] &= [g_1^{\ell}, h_1^{k}] \prod_{i=2}^{\infty} h_i^{-g_1^{\ell}} \prod_{i=2}^{\infty} h_i \prod_{i=2}^{\infty} g_i^{-1} \prod_{i=2}^{\infty} g_i^{h_1^{k}} \\ &= [g_1^{\ell}, h_1^{k}] \prod_{i=2}^{\infty} h_i^{-g_1^{\ell}} \prod_{i=2}^{\infty} g_i^{h_1^{k}} \prod_{i=2}^{\infty} h_i \prod_{i=2}^{\infty} g_i^{-1} .\end{aligned}$$

Thus, the lemma is proved.

Now we are ready to proceed with the proof of Theorem II. Using Lemmas 2.1 and 2.2 of this chapter, the proof of Theorem II will follow immediately from the next lemma.

Lemma 2.6. Let P be a parafree group in the variety $\underline{\underline{A}}^2$ of rank 2. If x and y are any two elements of P which are independent modulo γ_2P, then there is no element z of P such that $[x,y] = z^m$, $m > 1$.

Proof. The proof is by contradiction. Suppose there exist elements x, y and z of P such that x and y are independent modulo γ_2P and $[x,y] = z^m$ for some $m > 1$. Without loss of generality, we may assume that m is a prime. Now, clearly, z must lie in γ_2P. Furthermore, since x and y are independent modulo γ_2P, there exists a parabasis, consisting of elements a and b of P, such that

(1) $$y \equiv a^{\ell} \text{ modulo } \gamma_2P ,$$

(2) $$x \equiv b^{k} \text{ modulo } \gamma_2P ,$$

where $k > 0$ and $\ell > 0$. Thus it follows from the discussion in Section 2 of Chapter I that x and y can be written in the following way:

(3) $$y = a^{\ell} \prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{\ell_{m,n}} ,$$

and

(4) $$x = b^{k} \prod_{\substack{m>0 \\ n\geq 0}} [b,ma,nb]^{k_{m,n}} .$$

Using Lemma 2.5 of Chapter II we get

$$(5)\quad [x,y] = [b^k,a^\ell]\prod_{\substack{m>0\\ n\geq 0}}([b,ma,nb]^{b^k})^{-\ell_{m,n}}\cdot\prod_{\substack{m>0\\ n\geq 0}}([b,ma,nb]^{a^\ell})^{k_{m,n}}$$

$$\cdot\prod_{\substack{m>0\\ n\geq 0}}[b,ma,nb]^{\ell_{m,n}}\prod_{\substack{m>0\\ n\geq 0}}[b,ma,nb]^{-k_{m,n}}.$$

Now, it follows from equation (5) and Lemmas 3.6 and 3.7 of Chapter I and 2.4 of Chapter II that

$$(6)\quad [x,y] = \prod_{\substack{0\leq s\leq k-1\\ 0\leq t\leq \ell-1}}\prod_{j=0}^{s}\Big(\prod_{i=0}^{t}[b,(1+i)a,jb]^{m(i,t)}\Big)^{m(j,s)}$$

$$\cdot\prod_{\substack{m>0\\ n\geq 0}}\Big(\prod_{i=0}^{k}[b,ma,(n+i)b]^{m(i,k)}\Big)^{-\ell_{m,n}}$$

$$\cdot\prod_{\substack{m>0\\ n\geq 0}}\Big(\prod_{i=0}^{\ell}[b,(m+i)a,nb]^{m(i,\ell)}\Big)^{k_{m,n}}$$

$$\cdot\prod_{\substack{m>0\\ n\geq 0}}[b,ma,nb]^{\ell_{m,n}}\prod_{\substack{m>0\\ n\geq 0}}[b,ma,nb]^{-k_{m,n}}.$$

Since we are assuming that $[x,y] = z^m$ where m is a positive prime, the exponent of each commutator [b,ma,nb] on the left side of equation (6) must divide m. However, we shall show that this is not the case.

Define a positive integer γ as follows: If $m\nmid m(1,\ell)$, put $\gamma = 1$. If $m\mid m(1,\ell)$, let γ be the least positive integer such that $m\nmid m(\gamma,\ell)$. Since $m(\ell,\ell) = 1$, it follows that γ exists and $1\leq\gamma\leq\ell$.

Similarly, define a positive integer δ as follows: If $m\nmid m(1,k)$, put $\delta = 0$. If $m\mid m(1,k)$, let $\delta+1$ be the least positive integer such that $m \nmid m(\delta+1,k)$. Again, since $m(k,k) = 1$, it follows that δ exists and $0 \leq \delta \leq k-1$.

Now, we shall show that the exponent ε of the commutator $[b,\gamma a,\delta b]$ on the left hand side of equation (6) does not divide m. Using equation (6) and the definition of $m(s,t)$ we get

$$(7)\quad \varepsilon = \sum_{\substack{0\leq t\leq k-1\\ 0\leq s\leq \ell-1}} m(\gamma-1,t)m(\delta,s) - \sum_{i=0}^{\delta} \ell_{\gamma,\delta-i}m(i,k) + \sum_{i=0}^{\gamma-1} k_{\gamma-i,\delta}m(i,\ell) + \ell_{\gamma,\delta} - k_{\gamma,\delta}$$

$$= \sum_{t=0}^{k-1} m(\gamma-1,t) \sum_{s=0}^{\ell-1} m(\delta,s) - \sum_{i=1}^{\delta} \ell_{\gamma,\delta-i}m(i,k) + \sum_{i=1}^{\gamma-1} k_{\gamma-i,\delta}\, m(i,\ell)$$

$$= m(\gamma,k)\; m(\delta+1,\ell) - \sum_{i=1}^{\delta} \ell_{\gamma,\delta-i}m(i,k) + \sum_{i=1}^{\gamma-1} k_{\gamma-i,\delta}m(i,\ell).$$

Now by the choice of γ and δ we know that m does not divide $m(\gamma,k)m(\delta+1,\ell)$ but m does divide every other term on the left side of equation (7). Thus m does not divide ε and the lemma is proved.

Chapter III

§1. It has been shown for several varieties $\underline{\underline{V}}$ that the free group F of rank n in $\underline{\underline{V}}$ has a trivial center. It seems natural to inquire whether, in such a variety $\underline{\underline{V}}$, any parafree group of rank n also has a trivial center.

Here we attempt to prove this for the variety $\underline{\underline{AA}}_p$, p a prime, but we succeed only in the case $p = 2$, $n = \infty$. (It is known [1] that for any prime p, the free group of rank n in $\underline{\underline{AA}}_p$ has a trivial center if and only if $n = \infty$.) Even this seemingly innocent result is rather difficult to prove, involving a considerable amount of combinatorial technique. For $p > 2$, the combinatorial work becomes even much more formidable and, so far, seems to be out of reach, at least using the methods developed here.

A key fact used in the proof of our result is an analog, in the variety $\underline{\underline{AA}}_2$, of a theorem of W. Magnus (see e.g. [11]) which asserts that the left normed basic commutators of weight ≥ 2 of a free group in the variety $\underline{\underline{A}}^2$ are independent.

The organization of the rest of the chapter is as follows. In §2, we prove a number of general results about the left normed basic commutators of a free group in $\underline{\underline{AA}}_m$ (m an arbitrary integer ≥ 2). In particular, we prove two special cases of our first main theorem (the analog of Magnus' theorem), namely those cases in which commutators of weights one and two are involved. In fact, we obtain a unique way of writing elements of F modulo γ_2F and elements of γ_2F modulo γ_3F. From §3 on, we specialize to the case m = 2. In §3, we obtain further technical results about the left normed basic commutators in $\underline{\underline{A}}\underline{\underline{A}}_2$. We also derive some results concerning the augmentation ideal of the integral groupring of a free group in the variety $\underline{\underline{A}}_2$ which are used later on. In §4, we obtain some specific relations between the left normed basic commutators of the free group in $\underline{\underline{A}}\underline{\underline{A}}_2$ and in §5, using the relations of §4, we arrive at a subset of the set of left normed basic commutators, which allows us to prove the first main theorem. Finally, in §6, we prove our second main theorem on the centers of parafree groups of infinite rank in $\underline{\underline{A}}\underline{\underline{A}}_2$.

§2. Let F be a free group in the product variety AA_n, $n > 1$, freely generated by a well-ordered set $\{x_\lambda, \lambda \in \Lambda\}$. Under these conditions we have the following four lemmas.

Lemma 2.1. F/γ_2F is a free abelian group, and F is freely generated modulo γ_2F by the x_λ, $\lambda \in \Lambda$.

Lemma 2.2. If $c = [x_{\lambda_1}, \ldots, x_{\lambda_k}]$ is an arbitrary commutator in F of weight k, then

(i) $c^{n^2} \in \gamma_3F$ if $k = 2$,

and

(ii) $c^n \in \gamma_{k+1}F$ if $k > 2$.

Lemma 2.3. $\gamma_k F/\gamma_{k+1} F$ is an abelian group and $\gamma_k F$ is generated modulo $\gamma_{k+1} F$ by the left normed basic commutators in x_λ, $\lambda \in \Lambda$, of weight k. Moreover, if $f = \prod_{i=1}^{m} c_i^{\ell_i}$, $\ell_i \neq 0$ ($i = 1, \dots m$), where the c_i's are distinct left normed basic commutators in the x_λ's of weight 2, and if n^2 does not divide ℓ_i for some $1 \leq i \leq m$, then $f \notin \gamma_3 F$.

Lemma 2.4. Well order the set of all left normed basic commutators in the x_λ's of weight 2. Then every element in $\gamma_2 F$ can be written uniquely, modulo $\gamma_3 F$, as a product, $c_1^{k_1} \dots c_m^{k_m}$, of left normed basic commutators of weight 2 such that $c_1 < \dots < c_m$ and $1 \leq k_i < n^2$ for $i = 1, \dots, m$.

Proof of Lemma 2.1. Since F is free in the variety $\mathbf{A}\mathbf{A}_n$, $F/\gamma_2 F \cong G/\gamma_2 G$ where G is an absolutely free group of rank $|\Lambda|$. Hence the lemma is obvious.

Before proceeding with the proofs of the next lemmas, observe that if $\mathfrak{F}$ denotes the free metabelian group of rank $|\Lambda|$, then

$$(A) \qquad F \cong \mathfrak{F}/[\mathfrak{F}', \mathfrak{F}^n][\mathfrak{F}^n, \mathfrak{F}^n] .$$

Using equation (A) we get

$$(B) \quad \gamma_k F/\gamma_{k+1} F \cong \gamma_k \mathfrak{F}\cdot[\mathfrak{F}', \mathfrak{F}^n][\mathfrak{F}^n, \mathfrak{F}^n]/\gamma_{k+1}\mathfrak{F}\cdot[\mathfrak{F}', \mathfrak{F}^n][\mathfrak{F}^n, \mathfrak{F}^n].$$

Using equation (A) we shall identify F with

$\mathcal{F}/[\mathcal{F}', \mathcal{F}^n][\mathcal{F}^n, \mathcal{F}^n]$ and hope no confusion will arise. Thus, if we let $\{y_\lambda\}$, $\lambda \in \Lambda$ denote the free generators of $\mathcal{F}$ we shall put $x_\lambda = y_\lambda[\mathcal{F}', \mathcal{F}^n][\mathcal{F}^n, \mathcal{F}^n]$.

<u>Proof of Lemma 2.2</u>. Using the above remark we get

(1) $$c = [y_{\lambda_1}, \dots y_{\lambda_k}][\mathcal{F}', \mathcal{F}^n][\mathcal{F}^n, \mathcal{F}^n] .$$

First, we shall show that if $k = 2$, $c^{n^2} \in \gamma_{k+1}F$. It follows from equations (B) and (1) that it is enough to show that

(2) $$[y_{\lambda_1}, y_{\lambda_2}]^{n^2} \in \gamma_3\mathcal{F} \cdot [\mathcal{F}', \mathcal{F}^n][\mathcal{F}^n, \mathcal{F}^n] .$$

By Lemma 3.3 of Chapter I we get

(3) $$[y^n_{\lambda_1}, y^n_{\lambda_2}] \equiv [y_{\lambda_1}, y_{\lambda_2}]^{n^2} \text{ modulo } \gamma_3\mathcal{F} .$$

Clearly $[y^n_{\lambda_1}, y^n_{\lambda_2}] \in [\mathcal{F}^n, \mathcal{F}^n]$. Thus it follows from equation (3) that (2) holds.

Next we shall show that if $k > 2$, $c^n \in \gamma_{k+1}F$. As before, it follows from equations (B) and (1) that it is enough to show that

(4) $$[y_{\lambda_1}, \dots, y_{\lambda_k}] \in \gamma_{k+1}[\mathcal{F}', \mathcal{F}^n][\mathcal{F}^n, \mathcal{F}^n] .$$

Using Lemma 3.3 of Chapter I we get

(5) $$[y_{\lambda_1}, y_{\lambda_2}, \dots, y^n_{\lambda_k}] \equiv [y_{\lambda_1}, y_{\lambda_2}, \dots, y_{\lambda_k}]^n \text{ modulo } \gamma_{k+1}\mathcal{F}.$$

Furthermore, since $k > 2$, $[y_{\lambda_1}, y_{\lambda_2}, \dots, y^n_{\lambda_k}] \in [\mathcal{F}', \mathcal{F}^n]$.

Hence it follows from equation (5) that (4) holds, and thus the proof of the lemma is complete.

<u>Proof of Lemma 2.4</u>. To prove this lemma we shall make use of the following theorem due to W. Magnus, see e.g. [11]. In every free group G in the variety of all metabelian groups, the left normed basic commutators of weight ≥ 2 freely generate a free abelian subgroup of the derived group. Moreover, the left normed basic commutators of weight k generate $\gamma_k G$ modulo $\gamma_{k+1} G$.

It follows immediately from equation (B) and Magnus' theorem that $\gamma_k F$ is generated, modulo $\gamma_{k+1} F$, by the left normed basic commutators in the x_λ's of weight k. Suppose $f \in \gamma_3 F$. Then we shall show that n^2 divides ℓ_i $(i = 1,\ldots,m)$. Let d_i $(i = 1,\ldots,m)$ be the left normed basic commutator in the y_λ's corresponding to the commutator c_i in the x_λ's, by changing the x_λ's into y_λ's, and put

$$h = \prod_{i=1}^{m} d_i^{\ell_i} . \qquad (1)$$

Clearly, the d_i's are distinct commutators in the y_λ's. Using equation (B) we have

$$f \in \gamma_3 F \text{ if and only if } h \in \gamma_3 \mathfrak{F} \cdot [\mathfrak{F}^n, \mathfrak{F}^n].$$

Now suppose $h \in \gamma_3 \mathfrak{F} \cdot [\mathfrak{F}^n, \mathfrak{F}^n]$. Then we can write

$$h \equiv \prod_{i=1}^{r} [f_i^n, g_i^n] \text{ modulo } \gamma_3 \mathfrak{F} , \qquad (2)$$

where $f_i, g_i \in \mathcal{F}$ for $i = 1,\dots,r$. Using Magnus' theorem we can write

$$(3) \qquad [f_i, g_i] \equiv \prod_{j=1}^{m_i} d_{i,j}^{\ell_{i,j}} \quad \text{modulo } \gamma_3 \mathcal{F}$$

where $1 \leq i \leq r$ and the $d_{i,j}$ are left normed basic commutators in the y_λ's of weight 2. Using Lemma 3.3 of Chapter I and equations (2) and (3) we get

$$(4) \qquad h \equiv \prod_{i=1}^{r} [f_i, g_i]^{n^2}$$
$$\equiv \prod_{i=1}^{r} \Big(\prod_{j=1}^{m_i} d_{i,j}^{\ell_{i,j}} \Big)^{n^2} \quad \text{modulo } \gamma_3 \mathcal{F}.$$

Thus, it follows from Magnu's theorem and equations (1) and (4) that $n^2 \mid \ell_i$, $1 \leq i \leq r$, and the lemma is proved.

Proof of Lemma 2.4. Using Lemmas 2.2 and 2.3 it follows that every element of $\gamma_2 F$ can be written, modulo $\gamma_3 F$, as a product, $c_1^{k_1} \dots c_m^{k_m}$, of left normed basic commutators of weight 2 such that $c_1 < \dots < c_m$ and $1 \leq k_i < n^2$ for $i = 1,\dots,m$. Thus it is left to prove the uniqueness. Suppose

$$(1) \qquad c_1^{k_1} \dots c_m^{k_m} \equiv d_1^{\ell_1} \dots d_n^{\ell_n} \quad \text{modulo } \gamma_3 F,$$

where $c_1 < \dots < c_m$, $d_1 < \dots < d_n$, $1 \leq k_i < n^2$ $(i = 1,\dots,m)$ and $1 \leq \ell_j < n^2$ $(j = 1,\dots,n)$. Put

$$(2) \qquad f = c_1^{k_1} \dots c_m^{k_m} d_n^{-\ell_n} \dots d_1^{-\ell_1}.$$

It follows from (1) that $f \in \gamma_3 F$. Since F is a metabelian group it follows from Lemma 2.3 and equation (1) that $m = n$ and $c_i = d_i$ for $i = 1,\ldots,m$. Hence

$$f = \prod_{i=1}^{m} c_i^{k_i - \ell_i} . \tag{3}$$

Since $1 \leq k_i, \ell_i < n^2$ for $i = 1,\ldots,m$, it follows from Lemma 2.3 and equation (3) that $k_i = \ell_i$ for $i = 1,\ldots,m$. Hence the lemma is proved.

§3. From now on, we consider a free group F in the variety $\underline{\underline{A}}\underline{\underline{A}}_2$, freely generated by x_λ, $\lambda \in \Lambda$ (where $|\Lambda|$ is finite or infinite). By definition, we have $F \cong \mathcal{F}/\underline{\underline{A}}\underline{\underline{A}}_2\mathcal{F}$, where $\mathcal{F}$ is an absolutely free group.

Let $A_2 = \mathcal{F}/\mathcal{F}' \cdot \mathcal{F}^2$ denote the free group of rank $|\Lambda|$ in the variety $\underline{\underline{A}}_2$, freely generated by y_λ, $\lambda \in \Lambda$, and let M be a free $\mathbb{Z}A_2$-module (where $\mathbb{Z}A_2$ is the group ring of A_2 over the integers) of rank $|\Lambda|$, freely generated by t_λ, $\lambda \in \Lambda$. According to an embedding theorem due to W. Magnus [9], the set of matrices

$$S = \left\{ \begin{pmatrix} a & 0 \\ m & 1 \end{pmatrix} \;\middle|\; a \in A_2,\ m \in M \right\}$$

forms a multiplicative group, and the mapping ϕ of F into S given by

$$x_\lambda \phi = \begin{pmatrix} y_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix}$$

is a monomorphism. Using this fact we shall consider F as embedded in S and put

$$x_\lambda = \begin{pmatrix} y_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix}$$

This latter equality will be used repeatedly in what follows.

<u>Lemma 3.1</u>. Let F be as above. Then

$$\text{(i)} \qquad \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}^{\begin{pmatrix} y_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ y_\lambda m & 1 \end{pmatrix}, \quad m \in M .$$

$$\text{(ii)} \quad [x_{\lambda_1}, \ldots x_{\lambda_n}] = \begin{pmatrix} 1 & 0 \\ (y_{\lambda_n}-1)\ldots(y_{\lambda_3}-1)\{(y_{\lambda_2}-1)t_{\lambda_1} + (1-y_{\lambda_1})t_{\lambda_2} & 1 \end{pmatrix}$$

where $n \geq 2$.

<u>Proof</u>. We first observe that

$$\text{(1)} \qquad \begin{pmatrix} y_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix}^{-1} = \begin{pmatrix} y_\lambda^{-1} & 0 \\ -y_\lambda^{-1}t_\lambda & 1 \end{pmatrix} .$$

Hence

$$\text{(2)} \quad \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}^{\begin{pmatrix} y_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix}} = \begin{pmatrix} y_\lambda^{-1} & 0 \\ -y_\lambda^{-1}t_\lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}\begin{pmatrix} y_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix}$$

$$= \begin{pmatrix} y_\lambda^{-1} & 0 \\ -y_\lambda^{-1}t_\lambda+m & 1 \end{pmatrix}\begin{pmatrix} y_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ y_\lambda m & 1 \end{pmatrix} .$$

Thus, we have proved part (i). The proof of part (ii) is by induction. Suppose n = 2. Then, using equation (1) we get

$$(3) \quad [x_{\lambda_1},x_{\lambda_2}] = \begin{pmatrix} y_{\lambda_1}^{-1} & 0 \\ -y_{\lambda_1}^{-1}t_{\lambda_1} & 1 \end{pmatrix} \begin{pmatrix} y_{\lambda_2}^{-1} & 0 \\ -y_{\lambda_2}^{-1}t_{\lambda_2} & 1 \end{pmatrix} \begin{pmatrix} y_{\lambda_1} & 0 \\ t_{\lambda_1} & 1 \end{pmatrix} \begin{pmatrix} y_{\lambda_2} & 0 \\ t_{\lambda_2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} y_{\lambda_1}^{-1}y_{\lambda_2}^{-1} & 0 \\ -y_{\lambda_2}^{-1}y_{\lambda_1}^{-1}t_{\lambda_1} - y_{\lambda_2}^{-1}t_{\lambda_2} & 1 \end{pmatrix} \begin{pmatrix} y_{\lambda_1}y_{\lambda_2} & 0 \\ y_{\lambda_2}t_{\lambda_1} + t_{\lambda_2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -t_{\lambda_1} - y_{\lambda_1}t_{\lambda_2} + y_{\lambda_2}t_{\lambda_1} + t_{\lambda_2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ (y_{\lambda_2}-1)t_{\lambda_1} + (1-y_{\lambda_1})t_{\lambda_2} & 1 \end{pmatrix} .$$

Now suppose that

$$(4) \quad [x_{\lambda_1},\ldots x_{\lambda_n}] = \begin{pmatrix} 1 & 0 \\ \begin{matrix} (y_{\lambda_n}-1)\ \ldots (y_{\lambda_3}-1)\{(y_{\lambda_2}-1)t_{\lambda_1} \\ + (1-y_{\lambda_1})t_{\lambda_2} \end{matrix} & 1 \end{pmatrix}$$

for an arbitrary fixed n > 2. Then we must show that

$$(5)\quad [x_{\lambda_1},\dots x_{\lambda_n},x_{\lambda_{n+1}}] = \begin{pmatrix} 1 & 0 \\ (y_{\lambda_{n+1}}-1)(y_{\lambda_n}-1)\dots(y_{\lambda_3}-1) \\ \{(y_{\lambda_2}-1)t_{\lambda_1}+(1-y_{\lambda_1})t_{\lambda_2}\} & 1 \end{pmatrix}$$

Put $m = (y_{\lambda_n}-1)\ \dots\ (y_{\lambda_3}-1)\{(y_{\lambda_2}-1)t_{\lambda_1}+(1-y_{\lambda_1})t_{\lambda_2}\}$. Then using the induction hypothesis and part (i) of the lemma we get

$$(6)\quad [x_{\lambda_1},\dots,x_{\lambda_m},x_{\lambda_{m+1}}] = [x_{\lambda_1},\dots,x_{\lambda_m}]^{-1}[x_{\lambda_1},\dots,x_{\lambda_m}]^{x_{\lambda_{m+1}}}$$

$$= \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}^{\begin{pmatrix} y_{\lambda_{m+1}} & 0 \\ t_{\lambda_{m+1}} & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_{\lambda_{m+1}}m & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ (y_{\lambda_{m+1}}-1)m & 1 \end{pmatrix}.$$

As

$$(y_{\lambda_{m+1}}-1)m = (y_{\lambda_{m+1}}-1)\ \dots\ (y_{\lambda_3}-1)\{(y_{\lambda_2}-1)t_{\lambda_1}+(1-y_{\lambda_1})t_{\lambda_2}\},$$

the lemma is proved.

Before stating the next lemma, recall that the augmentation ideal of the integral group ring, $\mathbb{Z}G$, of a group G, is the kernel of the ring homomorphism $\mathbb{Z}G \longrightarrow \mathbb{Z}$ given by $\sum_i n_i g_i \longrightarrow \sum_i n_i$.

Lemma 3.2. Let A_2 be as above. If y is an arbitrary element of A_2 different from 1, then in $\mathbb{Z}A_2$ we have

$$(y-1)^n = (-2)^{n-1}(y-1) \quad \text{for all } n \geq 1.$$

Proof. The proof is by induction on n. Suppose n = 1. Then clearly

$$(y-1) = (-2)^0(y-1) .$$

Thus suppose

$$(y-1)^n = (-2)^{n-1}(y-1) \tag{2}$$

when n is an arbitrary fixed integer greater than 1. Since $y^2 = 1$ we get

$$(y-1)^2 = (-2)^1(y-1) . \tag{3}$$

It follows from equations (2) and (3) that

$$\begin{aligned}(y-1)^{n+1} &= (-2)^{n-1}(y-1)(y-1) \\ &= (-2)^n (y-1) .\end{aligned} \tag{4}$$

Hence the lemma is proved.

Lemma 3.3. Let F and A_2 be as above and let $\mathfrak{a}$ denote the augmentation ideal of $\mathbb{Z}A_2$. For $n \geq 1$, let M_n denote a free $\mathfrak{a}^n$ module, freely generated by the elements t_λ. (Certainly, as sets, $M \supseteq M_1 \supseteq M_2 \ldots$.) Then

$$\gamma_k F \leq \text{gp}\left\{ \begin{pmatrix} 1 & 0 \\ m_{k-1} & 1 \end{pmatrix} \middle| \; m_{k-1} \in M_{k-1} \right\} \text{ where } k \geq 2.$$

Proof. The proof follows immediately from Magnus' embedding theorem, Lemma 3.3 of Chapter I and Lemma 3.1 of Chapter III.

Now assume that G is a group generated by elements g_ω, $\omega \in \Omega$ and let $\mathcal{G}$ be the augmentation ideal of $\mathbb{Z}G$. It is known that as α ring, $\mathcal{G}$ is generated by the set $\{(g_\omega - 1), \omega \in \Omega\}$. See, e.g. [10]. Moreover, if H is any group, and if ϕ is any group homomorphism from G into H, then ϕ induces a ring homomorphism from $\mathbb{Z}G$ into $\mathbb{Z}H$, and hence, also, a ring homomorphism from $\mathcal{G}$ into $\mathcal{H}$, where $\mathcal{H}$ is the augmentation ideal of $\mathbb{Z}H$.

Suppose now that G is a free group in a variety $\underline{\underline{V}}$, with the g_ω, $\omega \in \Omega$ as free generators. If a is an element of we can write

$$a = \sum_{i=1}^{k} \sum_{j=1}^{\ell_i} n_{i,j} f_{i,j}$$

where the $n_{i,j}$ are integers and the $f_{i,j}$ are of the form

$$f_{i,j} = (g_{i,1}-1)^{m_{i,1,j}} \dots (g_{i,s_i}-1)^{m_{i,s_i,j}}$$

with $m_{i,\ell,j} > 0$ and where for $i \neq h$ the set $\{g_{i,1},\dots,g_{i,s_i}\}$ is different from the set $\{x_{h,1},\dots,x_{h,s_n}\}$.

Lemma 3.4. If $a = 0$, then $\sum_{j=1}^{\ell_i} n_{i,j} f_{i,j} = 0$ for $i = 1,\dots,k$.

Proof. This lemma follows immediately from the fact that a group homomorphism induces a ring homomorphism between the corresponding augmentation ideals.

Lemma 3.5. Let $\mathfrak{a}$ be, as above, the augmentation ideal of $\mathbb{Z}A_2$ and let $a \in \mathfrak{a}$ be of the following form:

$$(1) \qquad a = (y_{\lambda_1}-1)(y_{\lambda_2}-1) \ldots (y_{\lambda_{m-1}}-1)(y_{\lambda_m}-1)^k$$

where $m \geq 1$, $k \geq 1$ and $y_{\lambda_i} \neq y_{\lambda_j}$ for $i \neq j$. Then $a \notin \mathfrak{a}^{m+k}$.

Proof. Suppose $a \in \mathfrak{a}^{m+k}$. Then using Lemma 3.4 and the definition of $\mathfrak{a}^{m+k}$ we can write

$$(2) \qquad a = \sum_{i=1}^{\ell} n_i(y_{\lambda_1}-1)^{k_{i,1}} \ldots (y_{\lambda_m}-1)^{k_{i,m}}$$

with the $k_{i,j} \geq 1$ and where

$$\sum_{j=1}^{m} k_{i,j} \geq m+k \quad (i = 1,\ldots, \ell).$$

Now for every $1 \leq i \leq \ell$ put

$$K_i = \sum_{j=1}^{m} k_{i,j} .$$

Using equations (1) and (2) and Lemma 3.2 we get on the one hand

$$(4) \qquad a = (-2)^{k-1}(y_{\lambda_1}-1) \ldots (y_{\lambda_m}-1)$$

and on the other hand

$$(5) \qquad a = \Big(\sum_{i=1}^{\ell} n_i(-2)^{K_i-m}\Big)(y_{\lambda_1}-1) \ldots (y_{\lambda_m}-1) .$$

Hence, since $y_{\lambda_i} \neq y_{\lambda_j}$ for $i \neq j$ we get

$$(6) \qquad (-2)^{k-1} = \sum_{i=1}^{\ell} n_i(-2)^{K_i-m} .$$

But $K_i - m \geq k$. This leads to a contradiction and the lemma is proved.

§4. As in §3, F will be a free group in the variety $\underline{\underline{A}}\underline{\underline{A}}_2$, generated by x_λ, $\lambda \in \Lambda$. Well order these generators in some fashion and let $x_1 < x_2 < \ldots < x_r$ be some finite subset of them. Under these conditions we have the following three lemmas.

<u>Lemma 4.1</u>. Let $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$ be non-negative integers such that $m_i = 0$ if and only if $n_i = 0$ and $\sum_{i=1}^{r} m_i = \sum_{i=1}^{r} n_i$, and let k be an integer with $1 \leq k \leq r$. If

$$y = [x_k, x_1, m_1 x_1, \ldots, m_r x_r] \text{ and } z = [x_k, x_1, n_1 x_1, \ldots n_r x_r],$$

then

$$y = z.$$

<u>Lemma 4.2</u>. (i) Let $r \geq 3$, $n \geq 2$, $1 < k < r$. If

$$y = [x_k, x_1, x_1, \ldots, x_k, \ldots, (n-1)x_r][x_k, x_1, x_1, \ldots, \hat{x}_k, \ldots nx_r]$$
$$\cdot [x_k, x_1, \hat{x}_1, \ldots, x_k, \ldots, nx_r][x_k, x_1, \hat{x}_1, \ldots, \hat{x}_k, \ldots (n+1)x_r]$$

and

$$z = [x_k, x_1, x_1, \ldots, \hat{x}_k, \ldots (n+1)x_r]^{-1}$$
$$\cdot [x_k, x_1, \hat{x}_1, \ldots, x_k, \ldots, (n+1)x_r]^{-1},$$

then

$$y = z.$$

(ii) Let $r \geq 3$, $n \geq 2$, $k = r$. If

$$y = [x_r,x_1,x_1,\ldots,(n-1)x_r][x_r,x_1,x_1,\ldots,nx_{r-1}]$$

$$\cdot[x_r,x_1,\hat{x}_1,\ldots,nx_r][x_r,x_1,\hat{x}_1,\ldots,(n+1)x_{r-1}]$$

and

$$z = [x_r,x_1,x_1,\ldots,(n+1)x_{r-1}]^{-1}[x_r,x_1,\hat{x}_1,\ldots(n+1)x_r]^{-1},$$

then

$$y = z.$$

Lemma 4.3. (i) Let $r \geq 4$, $n \geq 1$ and suppose $2 \leq \ell < k < r$. If

$$y = [x_k,x_1,x_1,\ldots,x_\ell,\ldots,\hat{x}_k,\ldots,nx_r]$$

$$\cdot[x_k,x_1,\hat{x}_1,\ldots,x_\ell,\ldots,\hat{x}_k,\ldots,(n+1)x_r]$$

$$\cdot[x_\ell,x_1,x_1,\ldots,\hat{x}_\ell,\ldots,x_k,\ldots\ nx_r]$$

$$\cdot[x_\ell,x_1,\hat{x}_1,\ldots,\hat{x}_\ell,\ldots,x_k,\ldots\ (n+1)x_r]$$

and

$$z = [x_k,x_1,\hat{x}_1,\ldots,x_\ell,\ldots,\hat{x}_k,\ldots,(n+2)x_r]^{-1}$$

$$\cdot[x_\ell,x_1,x_1,\ldots,\hat{x}_\ell,\ldots,x_k,\ldots,(n+1)x_r]^{-1},$$

then

$$y = z.$$

(ii) Let $r \geq 3$, $n \geq 1$ and suppose $2 \leq \ell < k = r$. If

$$y = [x_r, x_1, x_1, \dots, x_\ell, \dots, nx_{r-1}][x_r, x_1, \hat{x}_1 \dots x_\ell, \dots (n+1)x_{r-1}]$$

$$\cdot [x_\ell, x_1, x_1, \dots, \hat{x}_\ell, \dots\ nx_r][x_\ell, x_1, \hat{x}_1, \dots, \hat{x}_\ell, \dots (n+1)x_r]$$

and

$$z = [x_r, x_1, \hat{x}_1, \dots, x_\ell, \dots, (n+2)x_{r-1}]^{-1}$$

$$\cdot [x_\ell, x_1, x_1, \dots, \hat{x}_\ell, \dots,\ (n+1)x_r]^{-1} ,$$

then

$$y = z.$$

<u>Proof of Lemma 4.1</u>. Using Lemma 3.1 we get

$$(1)\quad y = \begin{pmatrix} 1 & 0 \\ (y_r-1)^{m_r} \dots (y\ -1)^{m_1}\{(y_1-1)t_k+(1-y_k)t_1\} & 1 \end{pmatrix}$$

and

$$(2)\quad z = \begin{pmatrix} 1 & 0 \\ (y_r-1)^{n_r} \dots\ (y_1-1)^{n_1}\{(y_1-1)t_k+(1-y_k)t_1\} & 1 \end{pmatrix}$$

It follows from equations (1) and (2) that in order to show that $y = z$ it is enough to show that

$$(3)\quad (y_r-1)^{m_r} \dots (y_1-1)^{m_1} = (y_r-1)^{n_r} \dots (y_1-1)^{n_1} .$$

Now put $N = \sum_{i=1}^{r} m_i = \sum_{i=1}^{r} n_i$. Using Lemma 3.2 we get

$$(4)\quad (y_r-1)^{m_r} \dots (y_1-1)^{m_1} = (-2)^{m_r-1}(y_r-1)\dots(-2)^{m_1-1}(y_1-1)$$

$$= (-2)^{N-r}(y_r-1) \dots (y_1-1)$$

and

$$(5) \quad (y_r-1)^{n_r}\dots(y_1-1)^{n_1} = (-2)^{n_r-1}(y_r-1)\dots(-2)^{n_1-1}(y_1-1)$$

$$= (-2)^{N-r}(y_r-1)\dots(y_1-1)$$

Hence (3) is true and the lemma is proved.

It will be convenient, for the proofs of Lemmas 4.2 and 4.3 to introduce the following notation:
If y is a 2 by 2 matrix, we denote by $\alpha_{21}y$ the lower left entry of y.

<u>Proof of Lemma 4.2</u>. The proofs of part (i) and (ii) are the same, hence we shall prove only part (i). Using Lemma 3.1 we get

$$(1) \quad y = \begin{pmatrix} 1 & 0 \\ \begin{array}{l}\{(y_r-1)^{n-1}\dots(y_1-1)+(y_r-1)^{n}\dots\widehat{(y_k-1)}\dots(y_1-1) \\ \;+(y_r-1)^{n}\dots(y_2-1)+(y_r-1)^{n+1}\dots\widehat{(y_k-1)}\dots(y_2-1)\} \\ \{(y_1-1)t_k+(1-y_k)t_1\}\end{array} & 1 \end{pmatrix}$$

and

$$(2) \quad z = \begin{pmatrix} 1 & 0 \\ \begin{array}{l} -\{(y_r-1)^{n+1}\dots\widehat{(y_k-1)}\dots(y_1-1)+(y_r-1)^{n+1}\dots(y_2-1)\} \\ \{(y_1-1)t_k+(1-y_k)t_1\}\end{array} & 1 \end{pmatrix}$$

It is enough to show that

$$(3) \qquad \alpha_{21}y = \alpha_{21}z\,.$$

Using Lemma 3.2 and the proof of Lemma 4.1 we get

$$(4)\ \alpha_{21}y = \{2(y_r-1)^n\dots(y_1-1) + 2(y_r-1)^{n+1}\dots(\widehat{y_k-1})\dots(y_1-1)\}t_k$$

$$+ \{-2(y_r-1)^n \dots (y_1-1) - 2(y_r-1)^{n+1} \dots (y_2-1)\}t_1$$

$$= \{(-1)^{n-1}\, 2^n(y_r-1)\dots(y_1-1)+(-1)^n\, 2^{n+1}(y_r-1)$$

$$\dots(\widehat{y_k-1}) \dots (y_1-1)\}t_k$$

$$+ \{(-2)^n(y_r-1) \dots (y_1-1) + (-2)^{n+1}(y_r-1)\dots(y_2-1)\}t_1$$

and

$$(5)\ \alpha_{21}z = \{-(y_r-1)^{n+2}\dots(\widehat{y_k-1})\dots(y_1-1)-(y_r-1)^{n+1}\dots(y_1-1)\}t_k$$

$$+ \{(y_r-1)^{n+1}\dots(y_1-1)+(y_r-1)^{n+2}\dots(y_2-1)\}t_1$$

$$= \{-(-2)^{n+1}(y_r-1)\dots(\widehat{y_k-1})\dots(y_1-1)-(-2)^n(y_r-1)\dots(y_1-1)\}t_k$$

$$+ \{(-2)^n(y_r-1)\dots(y_1-1) +(-2)^{n+1}(y_r-1)\dots(y_2-1)\}t_1.$$

It follows immediately from equations (4) and (5) that $\alpha_{21}\, y = \alpha_{21}\, z$ and thus the lemma is proved.

<u>Proof of Lemma 4.3</u>. Again, since the proofs of parts (i) and (ii) are identical we shall only prove part (i). Using Lemma 3.1 we get

$$(1)\ y = \begin{pmatrix} 1 & 0 \\ \begin{array}{l} \{(y_r-1)^n\dots(\widehat{y_k-1})\dots(y_1-1)+(y_r-1)^{n+1}\dots(\widehat{y_k-1})\dots(y_2-1)\} \\ \cdot\{(y_1-1)t_k + (1-y_k)t_1\} \end{array} & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 \\ \begin{array}{l} \{(y_r-1)^n\dots(\widehat{y_\ell-1})\dots(y_1-1)+(y_r-1)^{n+1}\dots(\widehat{y_\ell-1})\dots(y_2-1)\} \\ \cdot\{(y_1-1)t_\ell + (1-y_\ell)t_1\} \end{array} & 1 \end{pmatrix}$$

and

$$(2)\quad z = \begin{pmatrix} 1 & 0 \\ \begin{array}{l} -(y_r-1)^{n+2}\ldots(\widehat{y_k-1})\ldots(y_2-1) \\ \cdot\{(y_1-1)t_k+(1-y_k)t_1\} \\ +(y_r-1)^{n+1}\ldots(\widehat{y_\ell-1})\ldots(y_1-1) \\ \cdot\{(y_1-1)t_\ell+(1-y_\ell)t_1\} \end{array} & 1 \end{pmatrix}$$

It is enough to show that

$$(3)\qquad \alpha_{21}\, y = \alpha_{21}\, z\ .$$

Using Lemma 3.2 and the proof of Lemma 4.1 we get

$$\begin{aligned}(4)\quad \alpha_{21}y &= 2(y_r-1)^{n+1}\ldots(\widehat{y_k-1})\ldots(y_1-1)t_k \\ &\quad + 2(y_r-1)^{n+1}\ldots(\widehat{y_\ell-1})\ldots(y_1-1)t_\ell \\ &\quad - \{2(y_r-1)^{n}\ldots(y_1-1) + 2(y_r-1)^{n+1}\ldots(y_2-1)\}t_1, \\ &= 2(-2)^n(y_r-1)\ldots(\widehat{y_k-1})\ \ldots\ (y_1-1)t_k \\ &\quad + 2(-2)^n(y_r-1)\ \ldots\ (\widehat{y_\ell-1})\ \ldots\ (y_1-1)t_\ell \\ &\quad + \{(-2)^n(y_r-1)\ \ldots\ (y_1-1)+(-2)^{n+1}(y_r-1)\ldots(y_2-1)\}t_1\end{aligned}$$

and

$$\begin{aligned}(5)\quad \alpha_{21}z &= \{-(y_r-1)^{n+2}\ \ldots\ (\widehat{y_k-1})\ \ldots\ (y_1-1)\}t_k \\ &\quad + \{-(y_r-1)^{n+2}\ \ldots\ (\widehat{y_\ell-1})\ \ldots\ (y_1-1)\}t_\ell \\ &\quad + \{(y_r-1)^{n+2}\ \ldots\ (y_2-1) + (y_r-1)^{n+1}\ldots(y_1-1)\}t_1\end{aligned}$$

$$
\begin{aligned}
&= \{-(-2)^{n+1}(y_r-1)\ldots(\widehat{y_k-1})\ \ldots\ (y_1-1)\}t_k \\
&\quad + \{-(-2)^{n+1}(y_r-1)\ldots\ (\widehat{y_\ell-1})\ \ldots\ (y_1-1)\}t_\ell \\
&\quad + \{(-2)^{n+1}(y_r-1)\ \ldots(y_2-1)+(-2)^n(y_r-1)\ldots(y_1-1)\}\ t_1
\end{aligned}
$$

Thus it follows from equations (4) and (5) that the $\alpha_{21}\, y = \alpha_{21}\, z$ and so the lemma is proved.

§5. We come now to the proof of Theorem III.

Let Ω be a non-empty finite subset of the well-ordered set $\{x_\lambda\}$, $\lambda \in \Lambda$, of generators of F. Thus $\Omega = \{x_1, \ldots x_r\}$, where $r \geq 1$ and $x_1 < \ldots < x_r$. Let also t be an integer ≥ 3. Denote by $B(\Omega,t)$ the set of all left normed basic commutators of weight t, each one of which involves <u>all</u> of the generators coming from Ω and <u>none</u> of the other generators. Observe that $B(\Omega,t) = \emptyset$ if $t < r = |\Omega|$.

From Lemma 4.1, we obtain the following:

(A) Let $r = |\Omega| = 2$.

(i)

$$
B(\Omega,3) = \begin{cases} [x_2,x_1,x_1] \\ [x_2,x_1,x_1] \end{cases}
$$

(ii) For $t > 3$,

$$
B(\Omega,t) = \begin{cases} [x_2,x_1,(t-2)x_1] \\ [x_2,x_1,x_1,(t-3)x_2] \\ [x_2,x_1,(t-2)x_2] \end{cases}
$$

(B) Let $r = |\Omega| > 2$.

(i)

$$B(\Omega,r) = \begin{cases} u_{k,r} = [x_k, x_1, \hat{x}_1, \ldots, \hat{x}_k, \ldots, x_r] & \text{for } 2 \leq k < r \\ u_{r,r} = [x_r, x_1, \hat{x}_1, \ldots, x_{r-1}] & \end{cases}$$

(ii)

$$B(\Omega,r+1) = \begin{cases} u_{k,r+1} = [x_k, x_1, \hat{x}_1, \ldots, \hat{x}_k, \ldots 2x_r] & \text{for } 2 \leq k < r \\ u_{r,r+1} = [x_r, x_1, \hat{x}_1, \ldots, 2x_{r-1}] & \\ v_{k,r+1} = [x_k, x_1, x_1, \ldots, \hat{x}_k, \ldots, x_r] & \text{for } 2 \leq k < r \\ v_{r,r+1} = [x_r, x_1, x_1, \ldots, x_{r-1}] & \\ w_{k,r+1} = [x_k, x_1, \hat{x}_1, x_2, \ldots, x_r] & \text{for } 2 \leq k \leq r \end{cases}$$

(iii) For $t > r+1$,

$$B(\Omega,t) = \begin{cases} u_{k,t} = [x_k, x_1, \hat{x}_1, \ldots, \hat{x}_k, \ldots, (t+1-r)x_r], & 2 \leq k < r \\ u_{r,t} = [x_r, x_1, \hat{x}_1, \ldots, (t+1-r)x_{r-1}] & \\ v_{k,t} = [x_k, x_1, x_1, \ldots, \hat{x}_k, \ldots, (t-r)x_r] & \text{for } 2 \leq k < r \\ v_{r,t} = [x_r, x_1, x_1, \ldots, (t-r)x_{r-1}] & \\ w_{k,t} = [x_k, x_1, \hat{x}_1, \ldots, (t-r)x_r] & \text{for } 2 \leq k \leq r \\ z_{k,t} = [x_k, x_1, x_1, \ldots, (t-r-1)x_r] & \text{for } 2 \leq k \leq r \end{cases}$$

We next define a certain subset of $B(\Omega,t)$, $t > 2$, which we call $C(\Omega,t)$. It will turn out that for fixed t, the union $C(t)$ of the $C(\Omega,t)$, where the union ranges over all nonempty subsets Ω of the set $\{x_\lambda\}$, $\lambda \in \Lambda$, forms a set of independent generators of $\gamma_t F$ modulo $\gamma_{t+1}F$. (The notion of independence will be defined below.)

Definition.

(A) Let $r = |\Omega| = 2$. Put

$$C(\Omega,t) = B(\Omega,t).$$

(B) Let $r = |\Omega| > 2$.

(i) Put $C(\Omega,r) = B(\Omega,r)$.

(ii) If $t > r$, put

$$C(\Omega,t) = \begin{cases} u_{k,t}\,, & 2 \leq k \leq r \\ v_{2,t} & \\ w_{k,t}\,, & 2 \leq k \leq r \end{cases} .$$

Definition. Let $c_1,\ldots,c_n$ be an arbitrary finite number of distinct left normed basic commutators of F of weight t. Let $c = c_1^{p_1}\ldots c_n^{p_n}$, with $n \geq 1$ and $0 \leq p_i \leq 1$ $(i = 1,\ldots,n)$. We shall say that $c_1,\ldots,c_n$ are independent if whenever $c \in \gamma_{t+1}F$, then $p_i = 0$, $i = 1,\ldots,n$.

An arbitrary set of left normed basic commutators of weight t will be said to be independent if every finite subset is independent.

Lemma 5.1. The elements of $C(\Omega,t)$, $t \geq 3$, form an independent set.

Proof. First suppose $r = |\Omega| = 2$.

If $t = 3$, then we must show that the commutators $[x_2,x_1,x_1]$ and $[x_2,x_1,x_2]$ are independent. Thus, put

$$a = [x_2,x_1,x_1]^{\rho_1}[x_2,x_1,x_2]^{\rho_2}$$

where $0 \leq \rho_i \leq 1$, $i = 1,2$. Using Lemma 3.1 we get

$$(2)\quad a = \begin{pmatrix} 1 & 0 \\ \{\rho_1(y_1-1)+\rho_2(y_2-1)\} \cdot \{(y_1-1)t_2+(1-y_2)t_1\} & 1 \end{pmatrix} .$$

Suppose $a \in \gamma_4 F$. Then it follows from Lemma 3.3 that $\alpha_{21}a \in M_3$. Hence, in particular, we get

$$(3)\qquad \rho_1(y_1-1)^2 + \rho_2(y_2-1)(y_1-1) \in \mathfrak{a}^3.$$

It follows immediately from equation (3) and Lemmas 3.4 and 3.5 that $\rho_1 = \rho_2 = 0$. Hence the set $C(\Omega,3)$ is independent.

If $t > 3$, then we must show that the commutators $[x_2,x_1,(t-2)x_1]$, $[x_2,x_1,x_1,(t-3)x_2]$ and $[x_2,x_1,(t-2)x_2]$ are independent. Thus, put

$$(4)\qquad b = [x_2,x_1,(t-2)x_1]^{\rho_1}[x_2,x_1,x_1,(t-3)x_2]^{\rho_2} \cdot [x_2,x_1,(t-2)x_2]^{\rho_3}$$

where $0 \leq \rho_i \leq 1$ for $i = 1,2,3$. Using Lemma 3.1 we get

$$(5)\quad b = \begin{pmatrix} 1 & 0 \\ \{\rho_1(y_1-1)^{t-2}+\rho_2(y_2-1)^{t-3}(y_1-1)+\rho_3(y_2-1)^{t-2}\} \cdot \{(y_1-1)t_2 + (1-y_2)t_1\} & 1 \end{pmatrix}$$

Suppose $b \in \gamma_{t+1}F$. Then it follows from Lemma 3.3 that $\alpha_{21}a \in M_t$. Hence,

$$(6)\quad \rho_1(y_1-1)^{t-1} + \rho_2(y_2-1)^{t-3}(y_1-1)^2+\rho_3(y_2-1)^{t-2}(y_1-1) \in \mathfrak{a}^t$$

and

$$(7)\quad \rho_1(y_1-1)^{t-2}(1-y_2)-\rho_2(y_2-1)^{t-2}(y_1-1)-\rho_3(y_2-1)^{t-1} \in \mathcal{A}^t.$$

Then it follows immediately from Lemmas 3.4 and 3.5 that

$$\rho_1 = \rho_2 = \rho_3 = 0.$$

Next suppose $r = |\Omega| > 2$. We shall prove that the sets $C(\Omega,t)$, $t \geq r$, are independent. We first observe that it is enough to show that the set $C(\Omega,t)$, $t > r+1$, is independent. Put

$$(8)\qquad C = \Big(\prod_{k=2}^{r} u_{k,t}^{\rho_{k,t}}\Big)\cdot v_{2,t}^{\sigma}\cdot \prod_{k=2}^{r} w_{k,t}^{\tau_{k,t}}$$

where $0 \leq \rho_{k,t} \leq 1$, $0 \leq \tau_{k,t} \leq 1$ for $k = 2,\ldots,r$, and $0 \leq \sigma \leq 1$. Using Lemma 3.1 we get

$$(9)\quad C=\begin{pmatrix} 1 & 0 \\ \begin{array}{l}\{\sum_{k=2}^{r-1} \rho_{k,t}(y_r-1)^{t+1-r}\ldots(\widehat{y_k-1})\ldots(y_2-1)\\ \cdot\{(y_1-1)t_k+(1-y_k)t_1\}\end{array} & 1 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & 0 \\ \begin{array}{l}\rho_{r,t}(y_{r-1}-1)^{t+1-r}\ldots(y_2-1)\\ \cdot\{(y_1-1)t_r + (1-y_r)t_1\}\end{array} & 1 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & 0 \\ \begin{array}{l}\sigma(y_{r-1})^{t-r}\ldots(\widehat{y_2-1})(y_1-1)\\ \cdot\{(y_1-1)t_2 + (1-y_2)t_1\}\end{array} & 1 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & 0 \\ \sum_{k=2}^{r} \tau_{k,t}(y_r-1)^{t-r}\ldots(y_2-1) & \\ \cdot\{(y_1-1)t_k + (1-y_k)t_1 & 1 \end{pmatrix} .$$

Suppose $c \in \gamma_{t+1}F$. Then it follows from Lemma 3.3 that, for $2 < k < r$,

$$(10)\quad \rho_{k,t}(y_r-1)^{t+1-r}\ldots(\widehat{y_k-1})\ldots(y_1-1)+\tau_{k,t}(y_r-1)^{t-r}\ldots(y_1-1) \in \mathcal{a}^t,$$

for $k = r$,

$$(11)\quad \rho_{r,t}(y_{r-1}-1)^{t+1-r}\ldots(y_1-1) + \tau_{r,t}(y_r-1)^{t-r}\ldots(y_1-1) \in \mathcal{a}^t,$$

and for $k = 2$,

$$(12)\quad \rho_{2,t}(y_r-1)^{t+1-r}\ldots(\widehat{y_2-1})(y_1-1) + \sigma(y_r-1)^{t-r}\ldots(\widehat{y_2-1})(y_1-1)^2 + \tau_{2,t}(y_r-1)^{t-r}\ldots(y_1-1) \in \mathcal{a}^t.$$

Thus it follows from Lemmas 3.4 and 3.5 that

$$(13)\qquad \rho_{k,t} = \tau_{k,t} = \tau_{2,t} = 0 \quad \text{for } 2 < k \leq r.$$

Using equations (9) and (13) and Lemma 3.3 we get

$$(14)\quad \rho_{2,t}(y_r-1)^{t+1-r}\ldots(y_2-1) + \sigma(y_r-1)^{t-r}\ldots(y_1-1) \in \mathcal{a}^t.$$

Again, it follows from Lemmas 3.4 and 3.5 that

$$(15)\qquad \rho_{2,t} = \sigma = 0 .$$

Hence, by equations (12), (13) and (15) we get

$$(16)\qquad \rho_{2,t} = 0 ,$$

and so the lemma is proved.

Theorem III. The elements of $C(t)$, $t \geq 3$, form an independent set.

Proof. By definition of independence we must deal with only finitely many commutators at a time. Thus, without loss of generality we may assume that $|\Lambda|$ is finite. Now let G be any free group of finite rank in the variety $\underline{\underline{AA}}_2$, freely generated by a well-ordered finite set Γ. Then for $t \geq 3$, set

$$(1) \qquad D(\Gamma,t) = \bigcup_{\substack{\Omega \subset \Gamma \\ \Omega \neq \Phi}} C(\Omega,t)$$

Moreover, let $\Sigma = \{x_\lambda, \lambda \in \Lambda\}$ be the set of free generators of F. Then clearly

$$(2) \qquad D(\Sigma, t) = C(t) , \quad t \geq 3 .$$

Thus we shall prove that $D(\Sigma,t)$ is independent. The proof is by induction on the number of generators of Σ. If $|\Sigma| = 1$ or 2, we get $D(\Sigma,t) = C(\Sigma,t)$, and the result holds by Lemma 5.1. Thus, assume that $D(\Gamma,t)$ is independent whenever $|\Gamma| \leq s$, where s is an arbitrary fixed integer greater than two. If $\Sigma = \{x_1,\ldots,x_{s+1}\}$ where $x_1<\ldots<x_{s+1}$, we shall show that $D(\Sigma,t)$ is independent. Set

$$(3) \qquad \Sigma_k = \{x_1,\ldots,\hat{x}_k,\ldots,x_{s+1}\}$$

where $1 \leq k \leq s+1$, and let F_k be the free group in the variety $\underline{\underline{AA}}_2$, freely generated by Σ_k. By induction hypothesis the elements of $D(\Sigma_k, t)$ are independent, $1 \leq k \leq s+1$. Clearly

$$(4) \qquad D(\Sigma, t) = \left(\bigcup_{k=1}^{s+1} D(\Sigma_k, t) \right) \quad C(\Sigma, t)$$

It is obvious that the $D(\Sigma_k, t)$ are not pairwise disjoint. Thus, we shall define subsets of the $D(\Sigma_k, t)$ so that $D(\Sigma, t)$ will be written as a union of disjoint sets. Put

$$D_1 = D(\Sigma_1, t)$$

and, for $2 \leq k \leq s+1$, put

$$D_k = D(\Sigma_k, t) \setminus \bigcup_{i=1}^{k-1} D(\Sigma_i, t) .$$

Clearly

$$(5) \qquad D(\Sigma, t) = \bigcup_{k=1}^{s+1} D_k \cup C(\Sigma, t) .$$

It follows from the definition of D_k, $1 \leq k \leq s+1$, that D_1 contains all possible left normed basic commutators in the generators in $\Sigma_1 = \{x_2, \ldots, x_{s+1}\}$. Moreover, D_k, $2 \leq k \leq s+1$, contains all possible left normed basic commutators in the generators in $\Sigma_k = \{x_1, \ldots, \hat{x}_k, \ldots x_{s+1}\}$ which must involve the generators $\{x_1, \ldots, x_{k-1}\}$. Now we are ready to show that the set $D(\Sigma, t)$ is indpendent. Let a be a product of commutators of $D(\Sigma, t)$. Since the D_k's and $C(\Sigma, t)$ are pairwise disjoint, we can write

$$(6) \qquad a = \prod_{k=1}^{s+1} \prod_{i=1}^{r_k} d_i(k)^{\rho_i(k)} \prod_{i=1}^{r} c_i^{\sigma_i}$$

with $d_i(k) \in D_k$, $c_i \in C(\Sigma, t)$ and where $0 \leq \rho_i(k) \leq 1$ and

$0 \leq \sigma_i \leq 1$. Suppose $a \in \gamma_{t+1}F$. Then we must show that the $\rho_i(k)$'s and σ_i's are zero. To do this, let ϕ_k be the homomorphism of F onto F_k defined by:

$$x_i\,\phi_k = x_i\,, \quad 1 \leq i \leq s+1,\ i \neq k$$

and

$$x_k\,\phi_k = 1\ .$$

Now we shall show that $\rho_i(k) = 0$ for $1 \leq k \leq s+1$. The proof is by induction on k. If $k = 1$, then

$$a\phi_1 = \prod_{i=1}^{r_1} d_i(1)^{\rho_i(1)}\ . \tag{7}$$

But $a\phi_1 \in \gamma_{t+1}F_1$. Thus, using the induction hypothesis, $\rho_i(1) = 0$, $1 \leq i \leq r_1$. Suppose $\rho_i(k) = 0$ for all $k \leq n$, where n is a fixed integer $1 \leq n < s+1$. Then we shall show that $\rho_i(n+1) = 0$. But since $\rho_i(k) = 0$, $k \leq n$, we have

$$a = \prod_{k=n+1}^{s+1} \prod_{i=1}^{r_k} d_i(k)^{\rho_i(k)} \prod_{i=1}^{r} c_i^{\sigma_i}\ .$$

So, we get

$$a\phi_{n+1} = \prod_{i=1}^{r_{n+1}} d_i(n+1)^{\rho_i(n+1)}\ . \tag{8}$$

But, as before, $a\phi_{n+1} \in \gamma_{t+1}F_{n+1}$ and thus $\rho_i(n+1) = 0$. Hence

$$a = \prod_{i=1}^{r} c_i^{\sigma_i}\ . \tag{9}$$

Using Lemma 5.1 we know that the set $C(\sum,t)$ is independent, and so $\sigma_i = 0$, $1 \leq i \leq r$. Thus, Theorem III is proved.

Corollary. Well-order the set $C(t)$, $t \geq 3$. Then every element in γ_2F can be written uniquely, modulo $\gamma_{t+1}F$, as a product $c_1(t) \ldots c_\ell(t)$ with $c_i(t) \in C(t)$ $(i = 1, \ldots)$ and $c_1(t) < \ldots < c_\ell(t)$.

Proof. It follows immediately from Lemmas 2.2, part (ii), and 2.3 and the lemmas of Section 4 that every element of γ_2F can be written, modulo $\gamma_{t+1}F$, as such a product. The uniqueness follows immediately from Theorem III.

§6. Let P be a parafree group in the variety $\underline{\underline{AA}}_2$, with a well-ordered parabasis $\{x_\lambda\}$, $\lambda \in \Lambda$. Observe that the subgroup of P generated by the x_λ's is a free group in $\underline{\underline{AA}}_2$.

We recall that for $t \geq 3$

$$(A) \qquad C(t) = \bigcup_{\Omega} C(\Omega, t)$$

where the union ranges over all nonempty finite subsets Ω of the set $\{x_\lambda\}$, $\lambda \in \Lambda$. Now define

$$(B) \qquad C(1) = \{x_\lambda, \lambda \in \Lambda\}$$

and

(C) $C(2) = \{$all left normed basic commutators in the x_λ's of weight 2$\}$.

We shall denote by $c(t)$ an arbitrary element of $C(t)$.

Using Lemmas 2.1 and 2.4 and the Corollary to Theorem III, together with §2 of Chapter I, every element p of P can be written uniquely in the form

$$\text{(D)} \qquad p = \prod_{i=1}^{r_1} c_i(1)^{\rho_i} \prod_{i=1}^{r_2} c_i(2)^{\sigma_i} \prod_{t=3}^{\infty} \prod_{i=1}^{r_t} c_i(t)$$

with $\rho_i \in \mathbb{Z} - \{0\}$, $1 \leq \sigma_i \leq 3$, $0 \leq r_t < \infty$ and where, for each t,

$$c_1(t) < \ldots < c_{r_t}(t) .$$

(If some $r_t = 0$, then, by convention, the corresponding product is 1.)

Theorem IV. Let P be a parafree group of infinite rank in the product variety $\underline{\underline{A}}\underline{\underline{A}}_2$. Then P has a trivial center.

Proof. Let $\{x_\lambda, \lambda \in \Lambda\}$ be a well-ordered parabasis of P. Suppose p belongs to the center of P. Then using the above discussion, we can assume that p is represented as in equation (D). We must show that p = 1. The proof is broken into 3 cases.

Case 1. Suppose $r_1 \neq 0$. Without loss of generality assume that $c_1(1) = x_1$. Put

$$\text{(1)} \qquad b = [x_k, x_1^{\rho_1}]$$

where x_k is an arbitrary element of c(1) such that $x_k > x_1$. If we write b as a product of left normed basic commutators, then each commutator will involve only x_1 and x_k. Since $b \neq 1$, there exists an $n \geq 2$ such that $b \in \gamma_n P$ and $b \notin \gamma_{n+1} P$. Thus we can write

$$b \equiv \prod_{\alpha} c_{\alpha}(n) \quad \text{modulo } \gamma_{n+1}P$$

where $c_{\alpha}(n)$ are elements of $C(n)$ which involve only x_1 and x_k. Choose x_k so large such that x_k is greater than every generator which appears in $c_i(t)$, $1 \leq t \leq n$. (Here the $c_i(t)$'s are the commutators appearing in the expression for p given in equation (D).) Thus the representation of $x_k p$ will contain a commutator of weight n which involves x_1 and x_k alone and the representation of px_p will not. Hence, $r_1 = 0$.

Case 2. Suppose $r_1 = 0$ and $r_2 \neq 0$. Without loss of generality put $c_1(2) = [x_2, x_1]$ where $x_2 > x_1$. The proof of this case is divided into three subcases, depending on whether the exponent σ_1 of $c_1(2)$ is 1, 2 or 3. Let x_k be an arbitrary element of $C(1)$ such that $x_k > x_1$ and $x_k > x_2$. Put

$$c = ([x_2, x_1]^{\sigma_1})^{x_k}.$$

Using Lemma 3.3 of Chapter I and the unique representation of elements of P, we get

$$c = [x_2, x_1][x_2, x_1, x_k], \quad \text{if } \sigma_1 = 1 \tag{2}$$

and

$$c = [x_2, x_1]^3[x_2, x_1, x_k], \text{ if } \sigma_1 = 3 . \tag{3}$$

Thus if σ_1 is 1 or 3 choose x_k so large such that x_k is greater than every generator which appears in $c_i(t)$, $t \leq 3$.

(Here, again, the $c_i(t)$'s are the commutators appearing in the expression for p given in equation (D).) Thus the representation of px_k will contain the commutator $[x_2,x_1,x_k]$ and that of $x_k p$ will not. Suppose now $\sigma_1 = 2$. Then

$$(4) \qquad c = [x_2,x_1]^2[x_2,x_1,x_k]^2 .$$

However, equation (4) does not give the unique presentation of the element c in P. Thus consider

$$(5) \qquad [x_2,x_1,x_k^2] = [x_2,x_1,x_k]^2[x_2,x_1,x_k,x_k] .$$

Since $[x_2,x_1,x_k^2] = 1$ it follows from (4) and (5) that the unique representation of c is

$$(6) \qquad c = [x_2,x_1]^2[x_2,x_1,2x_k] .$$

Now choose x_k so large such that x_k is greater than every generator which appears in $c_i(t)$, $t \leq 4$. Clearly the commutators $[c_i(3),x_k]$ $(1 \leq i \leq r_3)$ and $[d,x_k]$ are distinct elements of $C(4)$. Hence px_k will contain the commutator $[x_2,x_1,2x_k]$ and $x_k p$ will not. Thus $r_2 = 0$, and it is left to prove the third case.

Case 3. Suppose $r_i = 0$ for $i > 2$. Then assume that n is the smallest integer such that $r_n \neq 0$. Then

$$p = \prod_{i=1}^{r_n} c_i(n) \prod_{i=1}^{r_{n+1}} c_i(n+1) \prod_{t=n+2}^{\infty} \prod_{i=1}^{r_t} c_i(t) .$$

Let x_k be an element of $C(1)$ such that x_k is greater than every generator which appears in $c_i(n)$ and $c_i(n+1)$. Clearly, it follows from the construction of the sets $C(t)$ that the commutators $[c_i(n), x_k]$ $(1 \leq i \leq r_n)$ and $c_i(n+1)$ $(1 \leq i \leq r_{n+1})$ are distinct elements of $C(n+1)$. Thus the unique representation of px_k will contain a commutator of weight $(n+1)$ which involves the generator x_k and that of $x_k p$ will not. This contradicts the fact that $r_n \neq 0$. Thus $r_n = 0$ for all n and $p = 1$.

Bibliography

[1] M. Auslander and R. C. Lyndon, Commutator subgroups of free groups, Amer. J. Math. 77 (1955), 929-931.

[2] G. Baumslag, Some aspects of groups with unique roots, Acta Math. 104 (1960), 217-303.

[3] __________ , Some theorems on the free groups of certain product varieties, J. Combinatorial Theory 2 (1967), 77-99.

[4] __________ , Groups with the same lower central sequence as a relatively free group. I. The groups, Trans. Amer. Math. Soc. 129 (1967), 308-321.

[5] __________ , Groups with the same lower central sequence as a relatively free group. II. Properties, to appear in Trans. Amer. Math. Soc.

[6] __________ , B. H. Neumann, H. Neumann and P. M. Neumann, On varieties generated by a finitely generated group, Math. Z. 86 (1964), 93-122.

[7] P. Hall, Some word problems, J. London Math. Soc. 33 (1958), 482-496.

[8] A. Karrass, W. Magnus and D. Solitar, Elements of finite order in groups with a single defining relation, Comm. Pure Appl. Math. 13 (1960), 57-66.

[9] W. Magnus, On a theorem of Marshall Hall, Ann. Math. 40 (1939), 764-768.

[10] A. I. Mal'cev, Generalized nilpotent algebras and their adjoint groups, Mat. Sb. 25 (1949), 171-208. English

translation: Amer. Math. Soc. Transl. 69 (1968), 1-21.

[11] H. Neumann, Varieties of groups, Springer-Verlag, New York, 1967.

[12] O. Schreier, Die Untergruppen der freien Gruppen, Abh. Math. Sem. Univ. Hamburg 5 (1928), 161-183.

[13] J.-P. Serre, Lie algebras and Lie groups, W. A. Benjamin, New York, 1965.

Autobiographical Statement

Name- Yael Roitberg

Date of Birth- April 14, 1942

Place of Birth-ISRAEL

Higher Education-

City College of New York, February 1961 - August 1965

B. S. - August 1965

City University of New York, September 1965-June 1969

Ph.D. - June 1969.