

Asymptotics for the parabolic, hyperbolic, and elliptic
Eisenstein series through hyperbolic and elliptic
degeneration

by

Daniel J. Garbin

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Jay Jorgenson

Date

Chair of Examining Committee

Józef Dodziuk

Date

Executive Officer

Jay Jorgenson

Gautam Chinta

Józef Dodziuk

Cormac O'Sullivan

Supervisory committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Advisor: Professor Jay Jorgenson

Let Γ be a Fuchsian group of the first kind acting on the hyperbolic upper half plane \mathbb{H} , and let $M = \Gamma \backslash \mathbb{H}$ be the associated (connected) finite volume hyperbolic Riemann surface. We will allow the presence of both parabolic and elliptic elements as part of the group Γ , so that the surface has cusps (coming from parabolic elements) and conical points (coming from elliptic elements). To each primitive Γ -inconjugate parabolic element there is an associated parabolic Eisenstein series which is more commonly referred to in the literature as the non-holomorphic Eisenstein series. If γ is a primitive Γ -inconjugate hyperbolic element, then following the work due to Kudla and Millson, there is an associated hyperbolic Eisenstein series. More recently, Jorgenson and Kramer have introduced an elliptic Eisenstein series associated to a primitive Γ -inconjugate elliptic element γ of the discontinuous group Γ .

In this note, we look at the behavior of these Eisenstein series on families of hyperbolic Riemann surfaces of finite volume. In particular, there are two types of families that we study. The first family is obtained by hyperbolic degeneration which is a process that involves pinching primitive simple closed geodesic. The second family is obtained by elliptic degeneration, a process

in which the order of ramification becomes unbounded, namely the order of elliptic fixed points associated to the conical points of the surface runs off to infinity. The main results are as follows. The Eisenstein series that are not associated to degenerating elements will converge to their correspondents in the limiting surface. For the Eisenstein series that are associated to degenerating elements the situation is as follows. In the case of hyperbolic degeneration, the hyperbolic Eisenstein series associated to a pinched geodesic will converge (up to a multiplicative factor) to the parabolic Eisenstein series associated to the newly developed cusp(s) in the limit surface. In the case of elliptic degeneration, a strikingly similar result occurs since the elliptic Eisenstein series associated to a degenerating conical point converges (up to a multiplicative factor) to the parabolic Eisenstein series associated to the newly developed cusp in the limit surface. The striking similarity lays in the fact that the above multiplicative factors involve the parameters defining the two type of degeneration, namely the length of the pinched geodesic in the case of hyperbolic degeneration and the angle of the pinched cone in the case of elliptic degeneration.

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1 Preliminaries

1.1 Riemann surfaces

The material in this section is rather classical in the mathematics literature. It can be found in several books on the subject of Riemann surfaces (see [FK 92] or [Bea 83] for instance).

In essence, a Riemann surface M is a 1-(complex) dimensional connected topological manifold which is equipped with a complex analytic structure. The uniformization theorem tells us that there are three possibilities (up to biholomorphism) for the universal cover of a Riemann surface, namely the complex plane \mathbb{C} , the sphere $\mathbb{C} \cup \{\infty\}$, and the upper half plane \mathbb{H} .

In this paper we will focus on the hyperbolic Riemann surfaces, namely those whose universal cover is the upper half plane. In other words, we realize these surfaces as the quotient $\Gamma \backslash \mathbb{H}$ where Γ denotes a discrete group of automorphisms of the upper half plane. Thus, the study of such surfaces amounts to understanding the so called uniformizing group Γ .

The group $\mathrm{PSL}(2, \mathbb{R})$ is the group of (orientation preserving) automorphisms of the upper half plane. It consists of the collection of mappings of the form $z \mapsto (az + b)/(cz + d)$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The uniformizing group Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. The quantity $|a + d|$ (also referred to as trace) allows us to classify the types of elements of $\mathrm{PSL}(2, \mathbb{R})$. We call an automorphism parabolic, hyperbolic, or elliptic if its trace is greater than 2, equal to 2, or less than 2, respectively. The presence of elliptic elements in the uniformizing group Γ results in a surface with conical points referred to as orbifolds. We will allow the presence of such elliptic elements. In an abuse

of language, we will still use the term Riemann surface for such orbifolds. However, all these surfaces whether compact or non-compact will have finite volume.

To see that the quotient $\Gamma \backslash \mathbb{H}$ can be equipped with a complex analytic structure we reason as follows. First we need two definitions. If the discontinuous group Γ contains a parabolic element γ , then γ has a fixed point say $P \in \mathbb{R} \cup \{\infty\}$. We call P a cusp for the group Γ . Now suppose that $\gamma \in \Gamma$ is an elliptic transformation. Then γ has two fixed points one in the upper-half plane the other in the lower-half plane. Denote by E the fixed point of γ contained in the upper-half plane. Then E is called an elliptic point or conical point of Γ . Now consider the projection map $\pi : \mathbb{H} \mapsto \Gamma \backslash \mathbb{H}$ given by z goes to Γz , that is the orbit of the z under the action of the group Γ . Denote by Γ_z the stabilizer of z under the action of Γ , namely

$$\Gamma_z = \{\gamma \in \Gamma \mid \gamma(z) = z\}.$$

Since the group Γ is discontinuous, for each $z \in \mathbb{H}$ there exists a neighborhood U of z such that

$$\Gamma_z = \{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \emptyset\}.$$

Let us look at the inclusion map $\Gamma_z \backslash U \hookrightarrow \Gamma_z \backslash \mathbb{H}$. If $z \in \mathbb{H}$ does not correspond to a cusp or elliptic point, then its stabilizer consists only of the identity element of Γ . Then $\pi : U \mapsto \Gamma_z \backslash U$ is a homeomorphism and we can take $(\Gamma_z \backslash U, \pi^{-1})$ as local chart about z .

Now suppose that z is some cusp P of Γ . Choose a scaling matrix $\sigma \in$

$\mathrm{PSL}(2, \mathbb{R})$ such that $\sigma(P) = \infty$. Then $\sigma\Gamma_P\sigma^{-1}$ stabilizes ∞ and is an infinite cyclic group generated by $\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}$, where w denotes the width of the cusp. Define the homeomorphism φ from $\Gamma_P \backslash U$ to some open set of the \mathbb{C} given by $\varphi(\pi(z)) = \exp(2\pi i\sigma(z)/w)$. Then $(\Gamma_P \backslash U, \varphi)$ represents a local chart about the cusp P .

If z is some elliptic point E of Γ , choose a bi-holomorphism $\sigma : \mathbb{H} \mapsto \mathbb{D}$ with $\sigma(E) = 0$. Then $\sigma\Gamma_E\sigma^{-1}$ stabilizes $0 \in \mathbb{D}$ and is a finite cyclic group (of rotations) of order n generated by $w \mapsto \exp(2\pi i/n)w$. Define the homeomorphism φ from $\Gamma_E \backslash U$ to some open set of the \mathbb{C} given by $\varphi(\pi(z)) = \sigma(z)^n$. Then $(\Gamma_E \backslash U, \varphi)$ represents a local chart about the elliptic point E .

1.2 The upper-half plane

Denote by \mathbb{C} the complex plane. Then one can define the upper half plane as $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ equipped with the Riemannian metric given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

With respect to the coordinates the volume form and Laplace operator are given by

$$d\mu(z) = \frac{dx dy}{y^2} \quad \text{and} \quad \Delta = -y^2 (\partial_{xx} + \partial_{yy}).$$

Under the change of coordinates $x = e^r \cos \theta$ and $y = e^r \sin \theta$, the hyperbolic

metric becomes

$$ds^2 = \frac{dr^2 + d\theta^2}{\sin^2 \theta}$$

with corresponding volume form and Laplace operator

$$d\mu(z) = \frac{drd\theta}{\sin^2 \theta} \quad \text{and} \quad \Delta = -\sin^2 \theta (\partial_{rr} + \partial_{\theta\theta}).$$

It will be useful for us to introduce the hyperbolic polar coordinates (ρ, θ) about $i \in \mathbb{H}$, where $\rho(z)$ denotes the hyperbolic distance between i and z , and $\theta(z)$ is the angle between the positive y -axis and the tangent (Euclidean) line at the point i to the unique geodesic from i to z . In this coordinates, the hyperbolic metric is given by

$$ds^2 = d\rho^2 + \sinh^2(\rho)d\theta^2$$

whereas the volume form and Laplace operator by

$$d\mu(z) = \frac{d\rho d\theta}{\sinh(\rho)} \quad \text{and} \quad \Delta = -\partial_{\rho\rho} - \frac{1}{\tanh(\rho)}\partial_{\rho} - \frac{1}{\sinh^2(\rho)}\partial_{\theta\theta}.$$

1.3 Eisenstein series

We end this chapter with a brief description of Eisenstein series. We will take a look at three types of Eisenstein series, namely parabolic (associated to cusps), hyperbolic (associated to primitive closed geodesics), and elliptic (associated to conical points). We decide to keep this section concise. Nevertheless, precise statements will be made in Section 3.1.

By now, the study of parabolic Eisenstein series associated to a cusp P on a non-compact, finite volume hyperbolic Riemann surface $M = \Gamma \backslash \mathbb{H}$ is a classical aspect of mathematics (see, for example, [He 83], [Iw 02] or [Ku 73]). To recall, for any $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, we define the parabolic Eisenstein series $E_{\text{par};M,P}(z, s)$ by

$$E_{\text{par};M,P}(z, s) = w^{-s} \sum_{\eta \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \eta z)^s$$

where Γ_∞ denotes the stabilizer of the cusp at ∞ . In our definition, we assume without loss of generality that the cusp $P = \infty$, in other words the uniformizing group Γ is reduced at ∞ . It is standard in the mathematical literature to normalize cusps so that the width w is equal to one. We will work slightly more generally and, as a result, include the multiplicative factor of w^{-s} .

Our next task is to define the hyperbolic Eisenstein series. In [KM 79], the authors introduced a vector-valued type of Eisenstein series associated to primitive closed geodesics. We will use a scalar-valued analog of this. Suppose that $\gamma \in \Gamma$ is a primitive hyperbolic element which is associated to a primitive closed geodesic on M . Again, we will make the assumption that γ is represented by a diagonal matrix and fixes 0 and ∞ . Using the (r, θ) coordinates coming from Section 1.2, we formally define the hyperbolic Eisenstein series $E_{\text{hyp};M,\gamma}(z, s)$ by

$$E_{\text{hyp};M,\gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} (\sin \theta(\eta z))^s$$

where Γ_γ denotes the stabilizer of γ . The series above converges absolutely and locally uniformly for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$.

In [JK 03], the authors introduced an elliptic Eisenstein series associated

to an elliptic element $\gamma \in \Gamma$ with fixed point E on a finite volume hyperbolic Riemann surface. For any point $z \in \mathbb{H}$ which is not Γ -equivalent to the associated elliptic fixed point E and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, we define the elliptic Eisenstein series $E_{\text{ell},M,\gamma}(z, s)$ by

$$E_{\text{ell},M,\gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \sinh(\rho(\eta z))^{-s}.$$

In the above series, we are using the (ρ, θ) coordinates as described in Section 1.2. Furthermore, we assume that the uniformizing group Γ is reduced at i in the sense that the element γ fixes i . As before, Γ_γ denotes the stabilizer of the elliptic representative γ .

2 Degeneration of Riemann surfaces

The main result of this chapter deals with the uniform convergence of the hyperbolic metrics away from the newly developed nodes. After reviewing some facts about moduli space of genus g , we will introduce in Section 2.2 the notion of hyperbolic degeneration of Riemann surfaces of genus g . In short, degeneration occurs by pinching some primitive simple closed geodesics on the surface. The limiting surface will develop new nodes or cusps. In other words, through hyperbolic degeneration one moves inside the moduli space of genus g towards its boundary. Section 2.3 concludes the chapter by review the concept of elliptic degeneration. In this setting, we start with a hyperbolic Riemann surface with conical points (really this is an orbifold) where degeneration occurs by pinching conical points or differently said, letting the order of ramification go to infinity.

2.1 Moduli spaces

In this section we will discuss the Teichmüller space of genus g and the corresponding moduli space. Our presentation here will closely follow the ideas starting on Section 1.2 of [IT 92]. We will take the approach from [IT 92] by describing first the genus 1 case and then generalize to higher genus.

Our discussion here will focus on the closed Riemann surfaces, by which we mean Riemann surfaces that are homeomorphic with a sphere having g handles attached to it. Let us start by looking at the moduli space of tori. A torus is a Riemann surface which can be realized as the quotient $\Gamma \backslash \mathbb{C}$, where \mathbb{C} denotes the complex plane and Γ is lattice group on \mathbb{C} . The lattice

is spanned by two points (vectors) in \mathbb{C} . It is common to normalize these two points so that one of them is 1 and the other is $\tau \in \mathbb{H}$. Thus one can identify the lattice group Γ as the direct sum $\mathbb{Z} \oplus \mathbb{Z}\tau$. We will write Γ_τ for Γ in order to emphasize the relation with the second generator τ of the lattice group. Now let us look at two tori $R_1 = \Gamma_{\tau_1} \backslash \mathbb{C}$ and $R_2 = \Gamma_{\tau_2} \backslash \mathbb{C}$. It is not hard to show that the tori are biholomorphic if and only if there exists a fractional linear transformation $\gamma \in \text{PSL}(2, \mathbb{Z})$ with $\gamma(\tau_1) = \tau_2$. This in turn allows us to introduce biholomorphic equivalence classes of tori. The collection of equivalence classes of tori is called the moduli space of tori \mathcal{M}_1 , and this moduli space of tori can be identified with the quotient of the upper half plane \mathbb{H} by the modular group $\text{PSL}(2, \mathbb{Z})$, namely

$$\mathcal{M}_1 \cong \text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}.$$

Let us discuss the genus 1 case. One way to introduce the notion of Teichmüller space and moduli space is via the fundamental group. Consider a torus R_τ with generators 1 and τ . The torus can be constructed as follows. Consider the fundamental parallelogram in \mathbb{C} with vertices at 0, 1, τ and $\tau + 1$. If we identify the segment $[0, 1]$ with segment $[\tau, \tau + 1]$ and the segment $[0, \tau]$ with the segment $[1, \tau + 1]$ we obtain the torus R_τ . The segment $[0, 1]$ in \mathbb{C} determines a simple closed curve on the torus R_τ . The same can be said about the segment $[0, \tau]$. Call the first simple closed curve A and the second B . Choose $p = [0]$ as a base point on R_τ . The homotopy classes $[A]$ and $[B]$ determine a system of generators for the fundamental group $\pi_1(R_\tau, p)$. Furthermore, the lattice Γ_τ is isomorphic to the fundamental group $\pi_1(R_\tau, p)$

under the following correspondence

$$\begin{aligned} 1 &\mapsto [A] \\ \tau &\mapsto [B]. \end{aligned}$$

Consider now a biholomorphism between two tori $f : R_{\tau_1} \mapsto R_{\tau_2}$. This induces an isomorphism between the fundamental groups $f_* : \pi_1(R_{\tau_1}, p_1) \mapsto \pi_1(R_{\tau_2}, p_2)$ by

$$\begin{aligned} f_*([A_1]) &= [f(A_1)] = [A_2] \\ f_*([B_1]) &= [f(B_1)] = [B_2], \end{aligned}$$

corresponding to the isomorphism between the lattice groups Γ_{τ_1} and Γ_{τ_2} .

Consider a torus R . Take a set of generators $\Sigma_p = \{[A], [B]\}$ (also referred to as marking on R) for the fundamental group $\pi_1(R, p)$. Define an equivalence relation as follows. Two markings $\Sigma_p = \{[A_1], [B_1]\}$ and $\Sigma_q = \{[A_2], [B_2]\}$ are equivalent if there exist a continuous curve $C_{p,q}$ from the p to q and this curve induces an isomorphism between $\pi_1(R, p)$ and $\pi_1(R, q)$ by sending $[A_1]$ to $[A_2]$ and $[B_1]$ to $[B_2]$. More generally, this isomorphism sends a homotopy class $[C] \in \pi_1(R, p)$ to the homotopy class $[C_{p,q}^{-1} \cdot C \cdot C_{p,q}] \in \pi_1(R, q)$. The next step is to consider two pairs (R, Σ_p) and (S, Σ_q) . The two pairs are equivalent if there exist a biholomorphism $f : R \mapsto S$ and the marking $f_*(\Sigma_p)$ is equivalent to the marking Σ_q . The equivalence class $[(R, \Sigma_p)]$ is called a marked torus. The Teichmüller space of genus 1 which we denote by \mathcal{T}_1 consists of the collection of marked tori. Given a marking Σ_p we can identify it with the set of homology classes determined by the generators 1 and τ of the corresponding torus and

write $\Sigma(\tau)$ for Σ_p . Suppose that $[R, \Sigma(\tau_1)]$ and $[S, \Sigma(\tau_2)]$ are two elements in \mathcal{T}_1 . It is not difficult to show that the two elements are equal if and only if $\tau_1 = \tau_2$. But then this implies that the Teichmüller space of genus 1 can be identified with the upper half plane.

A second method to construct the Teichmüller space of genus 1 is via orientation-preserving diffeomorphisms. Let R be a torus with a marking $\Sigma = \{[A_1], [B_1]\}$. Consider a pair (S, f) of a torus S and an orientation-preserving diffeomorphism $f : R \mapsto S$. Then $f_*(\Sigma) = \{f_*([A_1]), f_*([B_1])\}$ is a marking on S . Furthermore, two such pairs (S_1, f_1) and (S_2, f_2) are equivalent if and only if the map $f_2 \circ f_1^{-1} : S_1 \mapsto S_2$ is homotopic to a biholomorphism $f : S_1 \mapsto S_2$ (see Theorem 1.3 on page 12 of [IT 92]). The Teichmüller space of R which is denoted by $\mathcal{T}(R)$ consists of the collection of equivalence classes $[S, f]$.

To see how to identify the spaces \mathcal{T}_1 and $\mathcal{T}(R)$ we reason as follows. Let $[S, \Sigma_p]$ be a marked torus. Then we can find an orientation-preserving diffeomorphism $f : R \mapsto S$ with $[S, \Sigma_p] = [S, f_*(\Sigma)]$. Choose two points $\tau, \tau_1 \in \mathbb{H}$ such that $[R, \Sigma] = [R_\tau, \Sigma(\tau)]$ and $[S, \Sigma_p] = [R_{\tau_1}, \Sigma(\tau_1)]$. Then the linear map (which maps 1 to 1 and τ_1 to τ) given by

$$\tilde{f}_\tau(z) = \frac{(\tau - \bar{\tau}_1)z - (\tau - \tau_1)\bar{z}}{\tau_1 - \bar{\tau}_1}, \text{ with } z \in \mathbb{C}$$

gives rise to an orientation-preserving diffeomorphism $f_\tau : R_{\tau_1} \mapsto R_\tau$ such that

$f_\tau([z]) = [\tilde{f}_\tau(z)]$ as in the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{f}_\tau} & \mathbb{C} \\ \pi \downarrow & & \downarrow \pi \\ R_{\tau_1} & \xrightarrow{f_\tau} & R_\tau \end{array}$$

where π denotes the universal covering map. The correspondence that sends $[R_\tau, \Sigma(\tau)]$ to $[R_\tau, f_\tau]$ identifies the \mathcal{T}_1 with $\mathcal{T}(R)$.

Now the group $\mathrm{PSL}(2, \mathbb{Z})$ acts on the Teichmüller space $\mathcal{T}(R)$. On the one hand, for $\gamma \in \mathrm{PSL}(2, \mathbb{Z})$ with $\gamma(z) = (az + b)/(cz + d)$, the action is defined by sending an element $[R_\tau, f_\tau]$ to $[R_{\gamma(\tau)}, f_{\gamma(\tau)}]$. On the other hand, for a point $\tau \in \mathbb{H}$, the automorphism $\tilde{h}_\tau : \mathbb{C} \mapsto \mathbb{C}$ given by $\tilde{h}_\tau(z) = (c\tau + d)z$ induces the biholomorphism “downstairs” $h_\tau : R_{\gamma(\tau)} \mapsto R_\tau$ by $h_\tau([z]) = [\tilde{h}_\tau(z)]$. Letting $R_{\tau_0} = R$, we see that $h_\tau \circ f_{\gamma(\tau)} = f_\tau \circ h_{\tau_0} \circ f_{\gamma(\tau_0)}$ (see the diagram below), and this in turn yields $[R_{\gamma(\tau)}, f_{\gamma(\tau)}] = [R_\tau, f_\tau \circ h_{\tau_0} \circ f_{\gamma(\tau_0)}]$ in the Teichmüller space $\mathcal{T}(R)$.

$$\begin{array}{ccc} R_{\tau_0} & \xleftarrow{h_{\tau_0}} & R_{\gamma(\tau_0)} \\ \cong \uparrow & \nearrow f_{\gamma(\tau_0)} & \\ R & \xrightarrow{f_\tau} & R_\tau \\ & \searrow f_{\gamma(\tau)} & \uparrow h_\tau \\ & & R_{\gamma(\tau)} \end{array}$$

Putting these two together, we see that $[R_\tau, f_\tau] = [R_\tau, f_\tau \circ h_{\tau_0} \circ f_{\gamma(\tau_0)}]$ and this defines an action on $\mathcal{T}(R)$. Namely for $\omega_\gamma = (h_{\tau_0} \circ f_{\gamma(\tau_0)})^{-1}$, which is a diffeomorphism of R , we have $[\omega_\gamma]_*[R_\tau, f_\tau] = [R_\tau, f_\tau \circ \omega_\gamma^{-1}]$. Since $[\omega_\gamma]_*$ is a biholomorphism on $\mathcal{T}(R)$ and corresponds to $\gamma \in \mathrm{PSL}(2, \mathbb{Z})$, we can identify the moduli space \mathcal{M}_1 of tori by the quotient $\mathrm{PSL}(2, \mathbb{Z}) \backslash \mathcal{T}(R)$. This concludes the discussion in the genus 1 case.

The genus 1 construction generalizes to higher genus g . In the latter case, the fundamental group $\pi_1(R, p)$ has $2g$ generators, so that a marking consists of the set $\Sigma_p = \{[A_i], [B_i]\}_{i=1}^g$. As before, we have that two pairs (R, Σ_p) and (S, Σ_q) are equivalent if and only if there exist a biholomorphism $h : R \mapsto S$ such that the marking $h_*(\Sigma_p)$ on S is equivalent to the marking Σ_q on S . We call such an equivalence class a marked closed Riemann surface of genus g . The Teichmüller space of genus g which is denoted by \mathcal{T}_g consists of the collection of marked closed Riemann surfaces of genus g .

As before, we can construct the Teichmüller space $\mathcal{T}(R)$ of a closed Riemann surface of genus g by using the method of orientation-preserving diffeomorphisms. Consider a pair (S, f) of a closed Riemann surface of genus g and an orientation-preserving diffeomorphism $f : R \mapsto S$. Two such pairs (S_1, f_1) and (S_2, f_2) are equivalent if and only if the map $f_2 \circ f_1^{-1} : S_1 \mapsto S_2$ is homotopic to a biholomorphism $f : S_1 \mapsto S_2$. The space $\mathcal{T}(R)$ consists of the collection of equivalence classes $[S, f]$. Furthermore, the Teichmüller space \mathcal{T}_g of genus g can be identified with $\mathcal{T}(R)$ (see Theorem 1.4 of [IT 92]).

Finally, let us introduce the moduli space of genus g . Denote by $\text{Mod}(R)$ the mapping class group, namely the collection of all homotopy classes of diffeomorphisms $\omega : R \mapsto R$. We can introduce an action of $\text{Mod}(R)$ on $\mathcal{T}(R)$ as follows. For $[\omega] \in \text{Mod}(R)$ let $[\omega]_*[S, f] = [S, f \circ \omega^{-1}]$. The moduli space \mathcal{M}_g of genus g can be identified as the quotient $\text{Mod}(R) \backslash \mathcal{T}(R)$.

2.2 Hyperbolic degeneration

In this section we present the concept of (hyperbolically) degenerating Riemann surfaces. The discussion in this section is taken from [Fa 73] (see also

[IT 92]) and is repeated here for the convenience of the reader.

Let M_1 and M_2 be two compact Riemann surfaces of genus g_1 and g_2 respectively. Each surface has one distinguished (removed) point denoted by $P_i \in M_i$ ($i = 1, 2$). Let $z_1 : U_1 \mapsto \mathbb{D}$ and $z_2 : U_2 \mapsto \mathbb{D}$ be local coordinates in the neighborhoods U_1 and U_2 of the two distinguished points. Here \mathbb{D} represents the (open) unit disc in \mathbb{C} . Put

$$W_i^t = \{x_i \in M_i \mid x_i \in M_i \setminus U_i \text{ or } x_i \in U_i \text{ with } |z_i(x_i)| > |t|\}$$

and

$$C_t = \{(X, Y) \in \mathbb{D} \times \mathbb{D} \mid X \cdot Y = t\}.$$

The above surface C_t denotes a “complex hyperbola” with parameter $t \in \mathbb{C}$ of small absolute value. We then can sew together W_1^t and W_2^t with the collar C_t , obtaining a Riemann surface of genus $g = g_1 + g_2$ as follows:

$$x_1 \in W_1^t \cap U_1 \text{ is identified with } \left(z_1(x_1), \frac{t}{z_1(x_1)} \right) \in C_t$$

and

$$x_2 \in W_2^t \cap U_2 \text{ is identified with } \left(\frac{t}{z_2(x_2)}, z_2(x_2) \right) \in C_t.$$

Then we can set $x = (X+Y)/2$ and $y = (X-Y)/2$ as holomorphic coordinates on C_t , so that the space $M_t = W_1^t \cup W_2^t \cup C_t$ is a compact Riemann surface of genus $g = g_1 + g_2$. For $t = 0$ we obtain a Riemann surface with two components

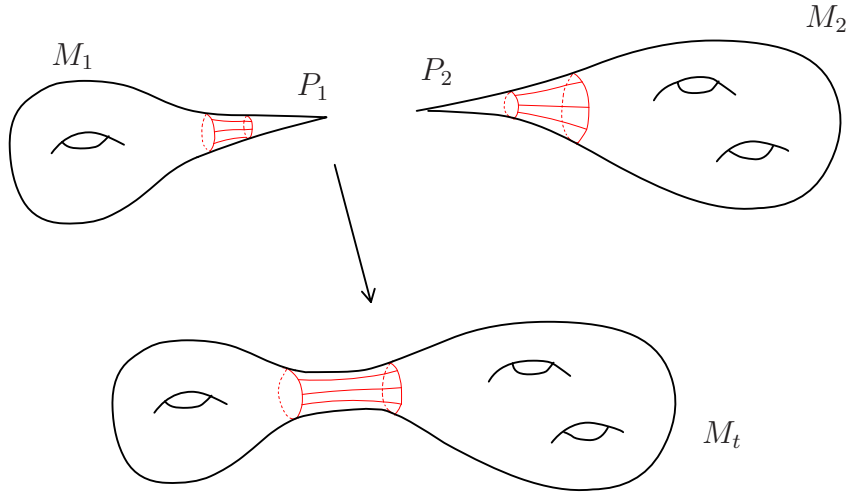


Figure 1: Hyperbolic degeneration with two components

and one node, coming from identifying to two removed points $P_1 \in M_1$ and $P_2 \in M_2$ (see Fig. 1). In this way, we have constructed a degenerating family $\{M_t \mid |t| < 1\}$ of Riemann surfaces of genus $g = g_1 + g_2$.

We will now describe another case of hyperbolic degeneration. Start with a compact Riemann surface M of genus $g (\geq 2)$, having two distinct removed points P_1 and P_2 . Choose local coordinates about P_i , namely $z_i : U_i \mapsto \mathbb{D}$ where U_1 and U_2 are disjoint neighborhoods of P_1 and P_2 respectively. For any $t \in \mathbb{D} \setminus \{0\}$, let

$$W_t = \{x \in M \mid x \in M \setminus (U_1 \cup U_2) \text{ or } x \in U_i \text{ with } |z_i(x)| > |t|\}.$$

With C_t defined as above, we can form a Riemann surface of genus $g + 1$ by sewing together W_t and U_i as follows:

$$x \in W_t \cap U_1 \text{ is identified with } \left(z_1(x), \frac{t}{z_1(x)} \right) \in C_t$$

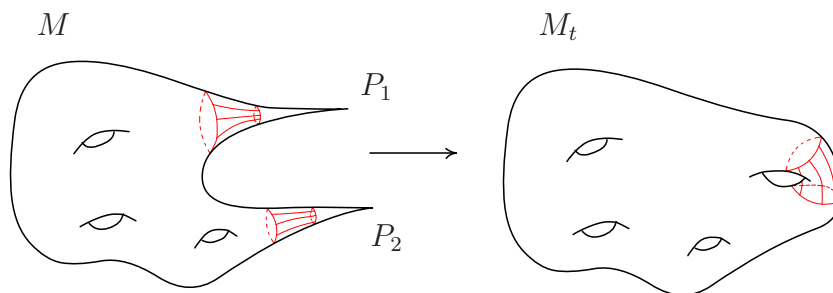


Figure 2: Hyperbolic degeneration with one component

and

$$x \in W_t \cap U_2 \text{ is identified with } \left(\frac{t}{z_2(x)}, z_2(x) \right) \in C_t.$$

As before, we can set $x = (X + Y)/2$ and $y = (X - Y)/2$ as holomorphic coordinates on C_t . When $t = 0$, we obtain a surface M_0 with one component and one node. Once again, we call $\{M_t \mid |t| < 1\}$ a hyperbolically degenerating family (see Fig. 2).

Now suppose that M_t is a hyperbolically degenerating family of compact Riemann surfaces. We will often use the phrase “bounded away from the developing node (cusps)” to refer to points that are contained in W_t . Another way to say this is that such points are outside the collar C_{t_0} , and hence outside any collar C_t for $|t| \leq |t_0|$. The above construction allows us to identify points on M_t for sufficiently small $|t|$. Furthermore, we can use the above construction to study the behavior of functions defined on M_t for values bounded away from the developing node. Another point here is that the above degeneration reduces to the case of one developing node. It is clear though that one can repeat the degeneration for as many nodes as wanted.

One can repeat the above two arguments to form a hyperbolically degenerating family of non-compact Riemann surfaces of finite volume. First choose

distinguished points lying away from the developing nodes. After removing these distinguished points we obtain cusps on the family M_t . In the non-compact case, bounded away from developing node must also include bounded away from the removed points.

In the above constructions, we parametrized the degenerating family by t , the parameter defining the collar C_t . Now we can equip the collar C_t with the hyperbolic metric. In section 2 of [Wo 87], the author models the collar as an annular region and expresses the hyperbolic metric in terms of the z coordinate in \mathbb{C} . Then one can look at the short geodesic in the homology class of $H_1(C_t, \mathbb{Z})$ (which is isomorphic to \mathbb{Z}). From [Wo 87] we have the following result about $[\gamma]$ – the homology class generating $H_1(C_t, \mathbb{Z})$. As $|t|$ goes to zero, one has

$$\frac{\ell_\gamma}{2\pi} = \frac{2\pi}{-\log |t|^2} + o((\log |t|)^{-2}),$$

where ℓ_γ denotes the length of the geodesic path in $[\gamma]$. As such, one can use the lengths $\ell = (\ell_1, \dots, \ell_n)$ to parametrize the hyperbolically degenerating family M_ℓ .

Let us summarize the hyperbolic degeneration in the language of hyperbolic geometry. Let M_ℓ be a degenerating family of connected, hyperbolic Riemann surfaces with p degenerating geodesics. Denote by ℓ the p -tuple corresponding to the lengths of the degenerating geodesics. To say that ℓ approaches zero means that the length of each degenerating geodesic is approaching zero. Although each M_ℓ is connected when $\ell > 0$, the limit surface M_0 need not be connected and, indeed, the number of cusps on M_0 is equal to the number of cusps on M_ℓ plus $2p$.

For $\ell > 0$, let \mathcal{C}_ℓ denote the hyperbolic infinite cylinder with simple, closed geodesic of length ℓ . One can describe \mathcal{C}_ℓ as

$$\mathcal{C}_\ell = \{(\rho, \theta) \mid \rho \in \mathbb{R}, \theta \in [0, 1)\}$$

equipped with the Riemannian metric

$$ds^s = d\rho^2 + \ell^2 \cosh^2(\rho) d\theta^2$$

and consequent volume form

$$d\mu = \ell \cosh(\rho) d\rho d\theta.$$

It easily follows that the length of the meridian at ρ equals to $\ell \cosh(\rho)$. Consequently, the shortest meridian is at $\rho = 0$ and the length of the meridian increases without bound as ρ approaches positive or negative infinity.

A convenient fundamental domain for \mathcal{C}_ℓ in \mathbb{H} is

$$\{r \exp(i\theta) \mid 1 \leq r < \exp(\ell), 0 < \theta < \pi\}, \quad (2.2.1)$$

with hyperbolic metric induced from \mathbb{H} and uniformizing group $\{\exp(k\ell) \mid k \in \mathbb{Z}\}$ which acts on \mathbb{H} by multiplication. For any $\varepsilon > 0$, let $\mathcal{C}_{\ell, \varepsilon}$ denote the symmetric submanifold of \mathcal{C}_ℓ about the geodesic defined by γ with total volume equal to ε . Using the expression of the metric in (r, θ) coordinates as in Section 1.2, we have that

$$\varepsilon = \int_{-a}^a \int_1^\ell \frac{dr d\theta}{\sin^2 \theta} = 2\ell \cot(a).$$

A model for $\mathcal{C}_{\ell,\varepsilon}$ in (2.2.1) is obtained by adding the restriction

$$\cot^{-1}(\varepsilon/(2\ell)) < \alpha < \pi - \cot^{-1}(\varepsilon/(2\ell)).$$

In order to compute the length of each boundary component of $\mathcal{C}_{\ell,\varepsilon}$ we integrate the line element as the r -coordinate varies and $\theta = \cot^{-1}(\varepsilon/2\ell)$ stays constant, giving

$$\text{length of } \partial\mathcal{C}_{\ell,\varepsilon} = \int_1^\ell \frac{dr}{\sin \theta} = \frac{\ell}{\sin(\cot^{-1}(\varepsilon/2\ell))} = \sqrt{\ell^2 + \varepsilon^2/4}.$$

For $\varepsilon_1 > \varepsilon_0$, the distance between the boundary of $\mathcal{C}_{\ell,\varepsilon_1}$ and that of $\mathcal{C}_{\ell,\varepsilon_0}$ can be computed as follows

$$\begin{aligned} d_{\mathbb{H}}(\partial\mathcal{C}_{\ell,\varepsilon_0}, \partial\mathcal{C}_{\ell,\varepsilon_1}) &= \int_{\cot^{-1}(\varepsilon_1/(2\ell))}^{\cot^{-1}(\varepsilon_0/(2\ell))} \frac{d\theta}{\sin \theta} \\ &= -\log(\csc \theta + \cot \theta) \Big|_{\cot^{-1}(\varepsilon_1/(2\ell))}^{\cot^{-1}(\varepsilon_0/(2\ell))} \\ &= \log \left((\varepsilon_1 + \sqrt{(\varepsilon_1^2 + 4\ell^2)}) / (\varepsilon_0 + \sqrt{(\varepsilon_0^2 + 4\ell^2)}) \right). \end{aligned} \quad (2.2.2)$$

From [Ra 79] we have that for any $0 < \varepsilon < 1/2$, the surface $\mathcal{C}_{\ell,\varepsilon}$ embeds isometrically into M_ℓ . The surface M_0 contains $2p$ embedded copies of $\mathcal{C}_{0,\varepsilon}$ which is the limit of $\mathcal{C}_{\ell,\varepsilon} \subset M_\ell$. One can model $\mathcal{C}_{0,\varepsilon}$ as two copies of a symmetric neighborhood of the origin in the punctured unit disc with its complete hyperbolic metric. Throughout this article we make use of the following fundamental result which we cite without proof from [Ab 77], stated as Theorem 8, page 37.

Proposition 2.2.1. *With notation as above, the hyperbolic metrics on the*

degenerating family M_ℓ convergence to the hyperbolic metric on M_0 . Furthermore, the convergence is uniform on compact subsets of M_0 bounded away from the developing cusps.

We refer the reader to [Ab 77] for a complete proof of Proposition 2.2.1.

2.3 Elliptic degeneration

In this section we describe how to construct a family of elliptically degenerating hyperbolic Riemann surfaces of finite volume. For further details, we refer to [Ab 80], [Ber 74], [Fa 73], as well as the more recent papers [Ju 95] and [Ju 98].

Let q be a positive integer, and write C_q for the infinite hyperbolic cone of order q . One can describe C_q as

$$C_q = \{(\rho, \theta) \mid \rho > 0, \theta \in [0, 2\pi)\} \quad (2.3.1)$$

with Riemannian metric given by

$$ds^2 = d\rho^2 + q^{-2} \sinh^2(\rho) d\theta^2 \quad (2.3.2)$$

and volume form

$$d\mu = q^{-1} \sinh(\rho) d\rho d\theta. \quad (2.3.3)$$

A fundamental domain for C_q in the hyperbolic unit disc model is provided by a sector with vertex at the origin and with angle $2\pi/q$. In coordinates, we

write

$$\{\alpha \exp(i\phi) \mid 0 \leq \alpha < 1, 0 \leq \phi < 2\pi/q\}.$$

The hyperbolic metric on C_q is that which is induced by the disc on this fundamental domain. The uniformizing group which corresponds to this fundamental domain consists of the numbers $\exp(2\pi in/q)$ for $n = 1, 2, \dots, q$ acting by multiplication.

Denote by $C_{q,\varepsilon}$ the submanifold of C_q of volume ε . Integrating the volume form in (2.3.1) above, we have that

$$\varepsilon = \int_0^a \int_0^{2\pi} q^{-1} \sinh \rho d\theta d\rho = \frac{2\pi}{q} \int_0^a \sinh \rho d\rho = \frac{2\pi}{q} (\cosh a - 1).$$

Then $C_{q,\varepsilon}$ can be obtained by restricting the first coordinate of (ρ, θ) to $0 \leq \rho < \cosh^{-1}(1 + \varepsilon q/2\pi)$.

A fundamental domain for $C_{q,\varepsilon}$ in the unit disc model is the subset of that given above with the added restriction that $\alpha < (\varepsilon q/(4\pi + \varepsilon q))^{1/2}$. It easily follows that the volume of $C_{q,\varepsilon}$ equals ε , and the length of its boundary is $(4\pi\varepsilon/q + \varepsilon^2)^{1/2}$.

For $\varepsilon_1 < \varepsilon_2$, we can compute the the length between the boundaries of the two nested cones C_{q,ε_1} and C_{q,ε_2} as follows

$$\begin{aligned} d_{\mathbb{H}}(\partial C_{q,\varepsilon_1}, \partial C_{q,\varepsilon_2}) &= \int_{\cosh^{-1}(1+\varepsilon_1 q/2\pi)}^{\cosh^{-1}(1+\varepsilon_2 q/2\pi)} d\rho \\ &= \cosh^{-1}(1 + \varepsilon_2 q/2\pi) - \cosh^{-1}(1 + \varepsilon_1 q/2\pi) \\ &= \log \left(\frac{\varepsilon_2 q + 2\pi + \sqrt{\varepsilon_2 q(4\pi + \varepsilon_2 q)}}{\varepsilon_1 q + 2\pi + \sqrt{\varepsilon_1 q(4\pi + \varepsilon_1 q)}} \right), \end{aligned} \quad (2.3.4)$$

where we used the fact that $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$.

C_∞ will denote an infinite cusp. A fundamental domain for C_∞ in the upper half plane is given by the set $\{x + iy \mid y > 0, 0 < x < 1\}$. C_∞ is obtained from this fundamental domain by identifying the boundary points iy with $1 + iy$. The uniformizing group that corresponds to this fundamental domain consists of \mathbb{Z} acting by addition.

$C_{\infty,\varepsilon}$ will denote the submanifold of C_∞ obtained by restricting the y coordinate of the fundamental domain given above to $y > 1/\varepsilon$. Easy computations show that $\text{Vol}(C_{\infty,\varepsilon}) = \varepsilon$, and the length of the boundary of $C_{\infty,\varepsilon}$ is also ε .

We now introduce some notation that will be convenient for studying surfaces with more than one distinguished, elliptic point. Let $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ be a vector of the orders of these elliptic fixed points. In this case we define $C_\omega = \cup_{k=1}^m C_{\omega_k}$. $C_{\omega,\varepsilon}$ is similarly defined as a union over the components of ω . We say that the vector ω approaches infinity if and only if each of its components approach infinity. It will be convenient to define

$$|\omega| = \min_{1 \leq k \leq m} \omega_k$$

We say that the Riemannian manifold C_ω (with m connected components) converges to m copies of the limit Riemannian manifold C_∞ as $\omega \rightarrow \infty$. Similarly, $C_{\omega,\varepsilon}$ converges to m copies of $C_{\infty,\varepsilon}$. We shall write these limits as $m \times C_\infty$ and $m \times C_{\infty,\varepsilon}$.

A family of finite volume hyperbolic surfaces M_ω parametrized by the m -vector ω will be called an elliptically degenerating surface if it has the following properties (see Fig. 3):

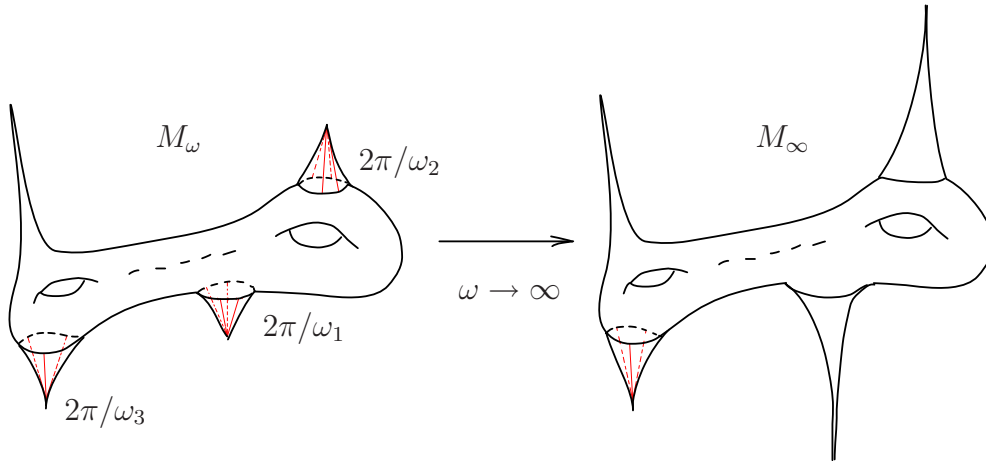


Figure 3: Elliptic degeneration of ω_1 and ω_2

1. For any $\varepsilon < 1/2$, the surface $C_{\omega,\varepsilon}$ (with m components) embeds isometrically into M_ω .
2. As $\omega \rightarrow \infty$, M_ω converges to a complete, hyperbolic surface M_∞ in the following sense. M_∞ contains m embedded copies of $C_{\infty,\varepsilon}$ which is the limit of $C_{\omega,\varepsilon} \subset M_\omega$. And, the geometry of $M_\omega \setminus C_{\omega,\varepsilon}$ converges to the geometry of $M_\infty \setminus (m \times C_{\infty,\varepsilon})$. In the last expression, $m \times C_{\infty,\varepsilon}$ refers to the “new” cusps of M_∞ , that is, the cusps which developed through degeneration. In particular, for every ω , it is possible to identify points x_ω and y_ω on $M_\omega \setminus C_{\omega,\varepsilon}$ such that $\lim_{\omega \rightarrow \infty} d_\omega(x_\omega, y_\omega) = d_\infty(x_\infty, y_\infty)$. Henceforth, we shall suppress the ω dependence of points which are identified during degeneration and simply write x and y . The volume forms induced by the converging metrics also converge uniformly on $M_\omega \setminus C_{\omega,\varepsilon}$, and all such measures are absolutely continuous with respect to each other. In general, the hyperbolic volume form occurring in an integral will be denoted by μ with an appropriate subscript when needed (for example, μ_ω). Length measure will be denoted by ρ .

The description of the degeneration of M_ω to the limit surface M_∞ also applies to the degeneration of C_ω and $C_{\omega,\delta}$ (with $\varepsilon < \delta$) to their limit surfaces, $m \times C_\infty$ and $m \times C_{\infty,\delta}$ respectively.

Let us describe how one can convert a cusp to a cone. The discussion follows [Ju 98] (see also section 2 of [Ju 95]), which for reasons of transparency we decide to include here. First consider the infinite cylinder $\mathbb{S}^1 \times (0, \infty)$ equipped with some smooth element of arc length. Denote by $\ell(t)$ the length of the circle $\mathbb{S}^1 \times \{t\}$ and by $d(s, t)$ the signed distance between the two circles $\mathbb{S}^1 \times \{t\}$ and $\mathbb{S}^1 \times \{s\}$. Then we can define the cone angle as the rate of change of the ℓ with respect to d , namely

$$\alpha = \lim_{t \rightarrow \infty} \left[\lim_{s \rightarrow t} \frac{\ell(s) - \ell(t)}{d(s, t)} \right]$$

provided that the limit exists. In such case, we call α the angle of the cone with apex at ∞ .

To model a cusp, one can chose the metric $m_0 = y^{-2}(dx^2 + dy^2)$. First note that the metric has constant negative curvature equal to -1 . It also easily follows that the length of the circle $\mathbb{S}^1 \times \{y\}$ is given by $\ell(y) = 1/y$ whereas the distance between two circles $\mathbb{S}^1 \times \{y_1\}$ and $\mathbb{S}^1 \times \{y_2\}$ is $d(y_1, y_2) = \log(y_2/y_1)$. Using the above notion of angle, one can compute the angle of the cone as follows

$$\alpha = \lim_{y \rightarrow \infty} \left[\lim_{y_1 \rightarrow y} \frac{\frac{1}{y_1} - \frac{1}{y}}{\log y_1 - \log y} \right] = \lim_{y \rightarrow \infty} \frac{-\frac{1}{y^2}}{\frac{1}{y}} = 0.$$

Hence, $(\mathbb{S}^1 \times (0, \infty), m_0)$ describes a hyperbolic cone with angle 0 and apex at ∞ , i.e. a hyperbolic cusp.

To model a cone with angle α and apex at ∞ , we consider $(\mathbb{S}^1 \times (0, \infty), m_\alpha)$ where

$$m_\alpha = \rho_\alpha(y)^2(dx^2 + dy^2) \text{ with } \rho_\alpha(y) = \frac{\alpha}{\sinh(\alpha y)}.$$

Recall that the curvature in local coordinates (x, y) is given by

$$\kappa(x, y) = \frac{\Delta \log \rho(x, y)}{\rho(x, y)^2},$$

where the function ρ represents the density of the metric and Δ represents the Euclidean Laplace operator $\Delta = -(\partial_{xx} + \partial_{yy})$. First we observe that $\Delta \log \rho_\alpha(y) = -\rho_\alpha(y)^2$. This in turn shows the metric m_α has constant curvature equal to -1 . Furthermore, let us evaluate the length of the circle $\mathbb{S}^1 \times \{y\}$. Namely, we have

$$\ell_\alpha(y) = \int_0^1 \frac{\alpha dx}{\sinh(\alpha y)} = \frac{\alpha}{\sinh(\alpha y)}.$$

As per the length between to meridians $\mathbb{S}^1 \times \{y_1\}$ and $\mathbb{S}^1 \times \{y_2\}$, it follows that

$$d_\alpha(y_1, y_2) = \int_{y_1}^{y_2} \frac{\alpha dy}{\sinh(\alpha y)} = \log \left| \frac{\operatorname{csch}(\alpha y_1) + \operatorname{coth}(\alpha y_1)}{\operatorname{csch}(\alpha y_2) + \operatorname{coth}(\alpha y_2)} \right|.$$

Then we can evaluate the angle of the cone as follows

$$\begin{aligned} & \lim_{y \rightarrow \infty} \left[\lim_{y_1 \rightarrow y} \frac{\alpha / \sinh(\alpha y_1) - \alpha / \sinh(\alpha y)}{\log(\operatorname{csch}(\alpha y_1) + \operatorname{coth}(\alpha y_1)) - \log(\operatorname{csch}(\alpha y) + \operatorname{coth}(\alpha y))} \right] \\ &= \lim_{y \rightarrow \infty} \frac{\alpha \cosh(\alpha y)}{\sinh(\alpha y)} = \alpha. \end{aligned}$$

This in turn makes $(\mathbb{S}^1 \times (0, \infty), m_\alpha)$ a hyperbolic cone of angle α with apex

at ∞ . Note that as α approaches 0, one obtains the metric $y^{-2}(dx^2 + dy^2)$ which is just the metric m_0 .

We note that all the computations involving the m_α model of the cone agree with the computations performed at the beginning of the section where we used different coordinates. To see this, one makes the change of variables

$$\begin{cases} \theta &= 2\pi x \\ \rho &= 2 \tanh^{-1}(e^{-\alpha y}), \quad \text{with } \alpha = \frac{2\pi}{q} \end{cases}$$

which then immediately gives the metric expressed in hyperbolic polar coordinates as in (2.3.1).

Now suppose that M is a hyperbolic Riemann surface with conical ends. For simplicity, let us assume that M has exactly one such end $E \subset M$. Consider a hyperbolic metric g on M such that (E, g) is a cone as described above. The main theorem in [Ju 98] states the following.

Let M be a surface with one end E . Let $[g]$ be a pointwise conformal class of hyperbolic metrics on M . For each $\alpha \in [0, 2\pi(1 - \chi(M))]$, let g_α be the unique hyperbolic metric on M such that (E, g_α) is a cone of angle α . Define the conformal factor w_α by $g_\alpha = e^{w_\alpha} g_0$. Then the map that sends α to the smooth factor w_α is real analytic.

In other words, a hyperbolic surface with cusps belongs to a unique conformal deformation of hyperbolic surfaces with cones and this deformation is real-analytically parametrized by the angles of the cones. For us, it give the following proposition.

Proposition 2.3.1. *Let M_∞ be an hyperbolic surface of finite volume with p cusps. For $0 < m \leq p$, there exists a degenerating family of surfaces M_ω with*

$p - m$ cusps indexed by the m -tuple ω such that

$$\lim_{\omega \rightarrow \infty} M_\omega = M_\infty$$

Proof. See Theorem 1 of [Ju 98].

□

3 Eisenstein series through degeneration

We begin this chapter by recalling some facts about the Eisenstein series introduced in Section 1.3 as well as express these Eisenstein series as Stieltjes integrals with some appropriate measure, which is basically a counting function. Aside from this, we also give bounds for these counting functions. The main results of the chapter are then presented in Sections 3.2 and 3.3. Namely, we show the convergence of the parabolic, hyperbolic, and elliptic Eisenstein series through both hyperbolic and elliptic degeneration to their corresponding Eisenstein series on the limiting surface. In obtaining these results, one uses the integral representation, the bounds for counting functions, and the convergence of metrics as described in Sections 2.2 and 2.3.

3.1 Integral representation

We start this section by making precise definitions of the three Eisenstein series introduced in Section 1.3. Then we will represent these series as Stieltjes integrals against geodesic counting functions.

Let us start with the parabolic Eisenstein series. For a reference on this one can look at [Ku 73] for instance. Suppose that γ is a primitive parabolic element of the uniformizing group Γ , namely γ corresponds to one of the Γ -inequivalent cusp representatives $\kappa_1, \dots, \kappa_h$. Let Γ_γ be the stabilizer in Γ of one the cusps P , that is $\Gamma_\gamma = \{\sigma \in \Gamma \mid \sigma P = P\}$. Then, there exist an element $\sigma \in \text{PSL}(2, \mathbb{R})$ (referred to as the scaling matrix for the cusp P) such

that $\sigma_\infty = P$ and the group $\sigma^{-1}\Gamma_\gamma\sigma$ is generated by the matrix

$$\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix},$$

where w is the width of the cusp P which is the fixed point of γ . Then for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ one can define the parabolic Eisenstein series $E_{\text{par};M,P}(z, s)$ associated to the cusp P by

$$E_{\text{par};M,P}(z, s) = w^{-s} \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \text{Im}(\sigma^{-1}\eta z)^s.$$

The parabolic Eisenstein series as defined above converges absolutely and locally uniformly for any $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$ (see [He 83], [Iw 02], or [Ku 73]).

Proposition 3.1.1. *For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, the parabolic Eisenstein series $E_{\text{par};M,P}(z, s)$ is an automorphic form with respect to the group Γ , namely it is invariant under Γ and is an eigenfunction of the Laplace operator, that is*

$$(\Delta_z - s(1-s))E_{\text{par};M,P}(z, s) = 0.$$

Proof. Let σ be any element of the group Γ . Then we can write

$$E_{\text{par};M,P}(\sigma z, s) = w^{-s} \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \text{Im}(\sigma\eta z)^s = w^{-s} \sum_{\eta' \in \Gamma_\gamma \backslash \Gamma} \text{Im}(\eta' z)^s = E_{\text{par};M,P}(z, s),$$

whenever the series converges. Over the domain of convergence of the series, we can interchange the sum with the Laplace operator. In the (x, y) coordinates,

the Laplace operator has the following expression

$$\Delta_z = -y^2(\partial_{xx} + \partial_{yy}).$$

With this we can write

$$\begin{aligned} \Delta_z(y^s) &= (-y^2)\partial_{yy}(y^s) = (-y^2)s(s-1)y^{s-2} \\ &= s(1-s)y^s, \end{aligned}$$

which proves the differential equation. \square

In what is to follow, we will assume that the group Γ is reduced at ∞ , that is ∞ is a cusp and its stabilizer Γ_∞ is infinite cyclic generated by $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. If P is a cusp which not equivalent to ∞ , we can use a scaling matrix $\sigma \in \text{PSL}(2, \mathbb{R})$ with $\sigma\infty = P$ and then the group $\sigma^{-1}\Gamma\sigma$ is reduced at ∞ (see [Ku 73]). With this assumption, our definition of the parabolic Eisenstein series is consistent with the one mentioned in Section 1.3, namely

$$E_{\text{par};M,P}(z, s) = w^{-s} \sum_{\eta \in \Gamma_\infty \backslash \Gamma} \text{Im}(\eta z)^s. \quad (3.1.1)$$

Our next task is to define the geodesic counting functions. Choose and fix any point $z \in M$, which we lift to a point $z \in \mathbb{H}$. Since Γ is acts discontinuously, it follows that the set

$$\{\eta z \mid \eta \in \Gamma\} \cap \text{neighborhood of } \infty$$

has finitely many elements. Then we can choose $y_0 \in \mathbb{R}$ sufficiently large so

that $y_0 > \text{Im}(\eta z)$ for all $\eta \in \Gamma$. Let \mathcal{L}_{y_0} be the horocycle in \mathbb{H} defined by $\{\text{Im}(z) = y_0\}$. This is basically the boundary of a neighborhood of the cusp at ∞ . For any point $z \in M$, which we lift to a point $z \in \mathbb{H}$, let $d_{\mathbb{H}}(z, \mathcal{L}_{y_0})$ denote the geodesic distance from z to \mathcal{L}_{y_0} . With all this, we define the parabolic counting function associated to γ and y_0 to be

$$N_{\text{par};M,P}(T; z, y_0) = \text{card} \{ \eta \in \Gamma_{\infty} \setminus \Gamma \mid d_{\mathbb{H}}(\eta z, \mathcal{L}_{y_0}) < T \}.$$

Observe that when defining the parabolic counting function, we needed to use the length from z to a horocycle \mathcal{L}_{y_0} since the cusp is at infinite distance.

An important part of our analysis is played by the following bound for the counting function $N_{\text{par};M,P}(u; z, y_0)$.

Lemma 3.1.2. *For any point $z \in M$ with injectivity radius r , and any $u \geq T_0 > r$, we have*

$$N_{\text{par};M,P}(u; z, y_0) \leq N_{\text{par};M,P}(T; z, y_0) + \frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})}.$$

Proof. The arguments from [Lu 93] apply to yield the following bound. However, to make the reading more self-contained we present a proof here. So fix some point $z \in M$. The injectivity radius at z is given by

$$r = \frac{1}{2} \min\{d_{\mathbb{H}}(z, \eta z) \mid \eta \in \Gamma - \{\text{id}\}\}.$$

Now for two distinct representatives η_1 and η_2 of $\Gamma_{\infty} \setminus \Gamma$ we have that the hyperbolic balls of radius r about $\eta_1 z$ and $\eta_2 z$ are disjoint. But then, for $u \geq T_0 > r$ we can estimate the number of points ηz which are trapped inside

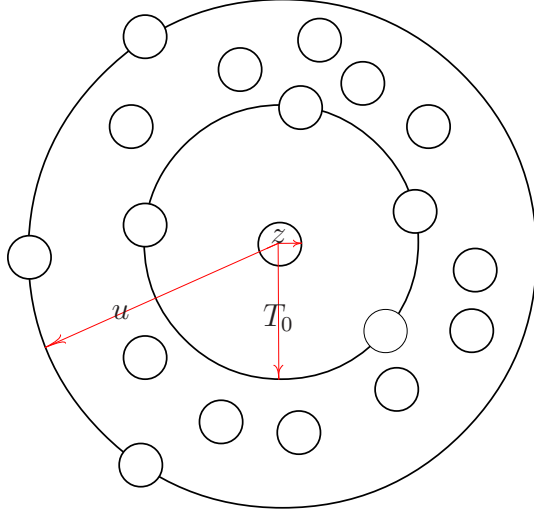


Figure 4: Hyperbolic balls of radius r trapped inside $B_z(u+r) \setminus B_z(T_0-r)$

the hyperbolic balls of radii T_0 and u about z , by comparing the aggregate volume of such hyperbolic balls (see Fig. 4). Namely, for representatives $\eta \in \Gamma_\infty \setminus \Gamma$ for which $T_0 \leq d_{\mathbb{H}}(z, \eta z) \leq u$, we have that

$$\sum_{\eta \in \Gamma_\infty \setminus \Gamma} \text{vol}(B_{\eta z}(r)) \leq \text{vol}(B_z(u+r)) - \text{vol}(B_z(T_0-r)),$$

where the “corrections” $+r$ and $-r$ in the right hand side above are justified by the fact that for some coset representatives it might happen that $d_{\mathbb{H}}(z, \eta z) = T_0$ or $d_{\mathbb{H}}(z, \eta z) = u$.

Recalling that the volume of a hyperbolic ball of radius r is equal to $4\pi \sinh^2(r/2)$, we can further write

$$\begin{aligned} 4\pi \sinh^2\left(\frac{r}{2}\right) \cdot \left(N_{\text{par};M,P}(u; z, y_0) - N_{\text{par};M,P}(T; z, y_0)\right) \\ \leq 4\pi \sinh^2\left(\frac{u+r}{2}\right) - 4\pi \sinh^2\left(\frac{T_0-r}{2}\right). \end{aligned}$$

The result of the lemma now easily follows from the above inequality. \square

Now we are ready to give an integral representation for the parabolic Eisenstein series. For any point $z \in \mathbb{H}$ and $y_0 \in \mathbb{R}$ with $\text{Im}(z) < y_0$, we have that

$$d_{\mathbb{H}}(z, \mathcal{L}_{y_0}) = \int_{\text{Im}(z)}^{y_0} \frac{dy}{y} = \log \left(\frac{y_0}{\text{Im}(z)} \right),$$

so then

$$\text{Im}(z)^s = y_0^s \exp(-s \cdot d_{\mathbb{H}}(z, \mathcal{L}_{y_0})).$$

With this observation, we have the following result.

Proposition 3.1.3. *The parabolic Eisenstein series $E_{\text{par};M,P}(z, s)$ has the following integral representation*

$$E_{\text{par};M,P}(z, s) = (y_0/w)^s \int_0^\infty e^{-su} dN_{\text{par};M,P}(u; z, y_0),$$

for all $z \in \mathbb{H}$ and all $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

Note that the above definition is independent of the choice of y_0 . To see this, let us choose $y_1 \in \mathbb{R}$ sufficiently large such that for any lift $z \in \mathbb{H}$ of a point $z \in M$, we have $y_1 > \text{Im}(z)$. Without loss of generality we may assume that $y_1 \geq y_0$. Then we can write,

$$\begin{aligned} \text{Im}(z)^s &= y_1^s \exp(-s \cdot d_{\mathbb{H}}(z, \mathcal{L}_{y_1})) \\ &= y_1^s \exp(-s \cdot (d_{\mathbb{H}}(z, \mathcal{L}_{y_0}) + d_{\mathbb{H}}(\mathcal{L}_{y_0}, \mathcal{L}_{y_1}))) \\ &= y_0^s \exp(-s \cdot d_{\mathbb{H}}(z, \mathcal{L}_{y_0})), \end{aligned}$$

since $d_{\mathbb{H}}(\mathcal{L}_{y_0}, \mathcal{L}_{y_1}) = \log(y_1/y_0)$.

As we will see, one can use the bound in Lemma 3.1.2 and Proposition 3.1.3 to prove the well-known result that the series defined in (3.1.1) converges uniformly and absolutely for $\operatorname{Re}(s) > 1$.

Now that we are done with the integral representation for the parabolic Eisenstein series, we proceed with the integral representation of the hyperbolic Eisenstein series. Let γ be a primitive hyperbolic element of Γ . Then we can find a matrix $\sigma \in \operatorname{PSL}(2, \mathbb{R})$ such that

$$\sigma^{-1}\gamma\sigma = \gamma_\ell = \begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix},$$

where ℓ denotes the length of the simple closed geodesic L_γ on M in the homotopy class of γ . We then lift the geodesic L_γ to the geodesic \mathcal{L}_0 lying on $\{z \in \mathbb{H} \mid \operatorname{Re}(z) = 0\}$ in the upper half plane. Using the (r, θ) coordinates introduced in Section 1.2, we define hyperbolic Eisenstein series $E_{\text{hyp}; M, \gamma}(z, s)$ associated to γ by

$$E_{\text{hyp}; M, \gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \sin(\theta(\sigma^{-1}\eta z))^s.$$

As in the parabolic case, we can use the notion of reduce at 0 and ∞ , so that there is no need for the scaling matrix σ . With this in mind, our definition of the hyperbolic Eisenstein series is consistent with definition given in Section 1.3, that is

$$E_{\text{hyp}; M, \gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \sin(\theta(\eta z))^s. \quad (3.1.2)$$

Using the hyperbolic metric in the (r, θ) coordinates as given in Section 1.2, we have that

$$\begin{aligned} d_{\mathbb{H}}(z, \mathcal{L}_0) &= \int_{\theta(z)}^{\pi/2} \frac{d\theta}{\sin \theta} \\ &= -\log(\csc \theta + \cot \theta) \Big|_{\theta(z)}^{\pi/2} \\ &= \log(\csc \theta(z) + \cot \theta(z)), \end{aligned}$$

where we assume that $\theta(z) \leq \pi/2$, for otherwise the limits of integration must be swapped. This allows us to write

$$\exp(d_{\mathbb{H}}(z, \mathcal{L}_0)) = \frac{1 + \cos \theta(z)}{\sin \theta(z)}$$

and

$$\exp(-d_{\mathbb{H}}(z, \mathcal{L}_0)) = \frac{\sin \theta(z)}{1 + \cos \theta(z)},$$

from which we get that

$$\sin(\theta(z)) \cdot \cosh(d_{\mathbb{H}}(z, \mathcal{L}_0)) = 1.$$

This in turn, allows us to express (3.1.2) as

$$E_{\text{hyp}; M, \gamma}(z, s) = \sum_{\eta \in \Gamma_{\gamma} \setminus \Gamma} (\cosh(d_{\mathbb{H}}(\eta z, \mathcal{L}_0)))^{-s}.$$

The series given in (3.1.2) converges absolutely and locally uniformly for $z \in \mathbb{H}$

and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ (see [Fa 07], [GJM 08], [vP], or [Ri 07]).

Proposition 3.1.4. *For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the hyperbolic Eisenstein series $E_{\text{hyp};M,\gamma}(z, s)$ is invariant under Γ in the z variable, but it is not an eigenfunction of the Laplace operator since it satisfies the following differential equation*

$$(\Delta_z - s(1 - s))E_{\text{hyp};M,\gamma}(z, s) = s^2 E_{\text{hyp};M,\gamma}(z, s + 2).$$

Proof. Let σ be an element of Γ . Then as we have seen in the parabolic case (cf. Proposition 3.1.1), we have that $E_{\text{hyp};M,\gamma}(\sigma z, s) = E_{\text{hyp};M,\gamma}(z, s)$ whenever the series converges. Over the domain of convergence, we can interchange the sum and the Laplace operator. Recall that the Laplacian in the (r, θ) coordinates is given by

$$\Delta = -(\sin \theta)^2 (\partial_{rr} + \partial_{\theta\theta}).$$

Working with the series as given by (3.1.2), we have that

$$\begin{aligned} \Delta(\sin(\theta)^s) &= -\sin(\theta)^2 \partial_{\theta\theta}(\sin(\theta)^s) = -\sin(\theta)^2 \partial_{\theta}(s \sin(\theta)^{s-1} \cos \theta) \\ &= -\sin(\theta)^2 [s(s-1) \sin(\theta)^{s-2} \cos(\theta)^2 - s \sin(\theta)^{s-1} \sin \theta] \\ &= s(1-s) \sin(\theta)^s \cos(\theta)^2 + s \sin(\theta)^{s+2} \\ &= s(1-s) \sin(\theta)^s - s(1-s) \sin(\theta)^{s+2} + s \sin(\theta)^{s+2} \\ &= s(1-s) \sin(\theta)^s + s^2 \sin(\theta)^{s+2}, \end{aligned}$$

which completes the proof. □

We now proceed to define the counting functions in the hyperbolic case.

For any point $z \in M$, which we lift to a point $z \in \mathbb{H}$, let $d_{\mathbb{H}}(z, \mathcal{L}_0)$ denote the geodesic distance from z to \mathcal{L}_0 . With all this, we define the hyperbolic counting function as

$$N_{\text{hyp};M,\gamma}(T; z) = \text{card} \{ \eta \in \Gamma_\gamma \backslash \Gamma \mid d_{\mathbb{H}}(\eta z, \mathcal{L}_0) < T \}.$$

Equivalently, one can count the number of geodesic paths from $z \in M$ to the closed geodesic on M in the homotopy class determined by γ .

As in the parabolic case, we can establish the following bound for the hyperbolic counting function.

Lemma 3.1.5. *For any point $z \in M$ with injectivity radius r , and any $u \geq T_0 > r$, we have*

$$N_{\text{hyp};M,\gamma}(u; z) \leq N_{\text{hyp};M,\gamma}(T_0; z) + \frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})}.$$

Proof. The result follows by mimicking the arguments employed in Lemma 3.1.2 above. \square

We can now express the hyperbolic Eisenstein series given by (3.1.2) as a Stieltjes integral, namely we have the following result.

Proposition 3.1.6. *The hyperbolic Eisenstein series $E_{\text{hyp};M,\gamma}(z, s)$ has the following integral representation*

$$E_{\text{hyp};M,\gamma}(z, s) = \int_0^\infty (\cosh u)^{-s} dN_{\text{hyp};M,\gamma}(u; z).$$

for all $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

By combining the bound in Lemma 3.1.5 and Proposition 3.1.6, we can show that the series defining the hyperbolic Eisenstein series (3.1.2) converges uniformly and absolutely for $\operatorname{Re}(s) > 1$.

We end this section with the integral representation of the elliptic Eisenstein series. Let γ be a primitive elliptic element of Γ . Then there is an element $\sigma \in \operatorname{PSL}(2, \mathbb{R})$ such that

$$\sigma^{-1}\gamma\sigma = \gamma_q = \begin{pmatrix} \cos(2\pi/q) & \sin(2\pi/q) \\ -\sin(2\pi/q) & \cos(2\pi/q) \end{pmatrix},$$

where q denotes the order of the elliptic fixed point E associated to γ . The quotient $\alpha_q = 2\pi/q$ denotes the cone angle. Using the hyperbolic polar coordinates (ρ, θ) (coming from Section 1.2), we define the elliptic Eisenstein series $E_{\text{ell};M,\gamma}(z, s)$ associated to the elliptic element γ by

$$E_{\text{ell};M,\gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \sinh(\rho(\sigma^{-1}\eta z))^{-s}.$$

As in the previous two cases, we can use the notion of reduce at i , so that there is no need for the scaling matrix σ . With this in mind, our definition of the elliptic Eisenstein series is consistent with definition given in Section 1.3, that is

$$E_{\text{ell};M,\gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \sinh(\rho(\eta z))^{-s}. \quad (3.1.3)$$

The series converges absolutely and locally uniformly for all $z \in \mathbb{H}$ which are not Γ -equivalent to the associated elliptic point and all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ (see [JK 03] or [vP]).

Proposition 3.1.7. *For all $z \in \mathbb{H}$ which are not Γ -equivalent to the associated elliptic point and all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the elliptic Eisenstein series $E_{\text{ell};M,\gamma}(z, s)$ is invariant under Γ in the z variable, but fails to be an eigenfunction of the Laplace operator, since it satisfies the following differential equation*

$$(\Delta_z - s(1 - s))E_{\text{ell};M,\gamma}(z, s) = -s^2 E_{\text{ell};M,\gamma}(z, s + 2).$$

Proof. That the elliptic Eisenstein series is Γ -invariant in the z variable follows from the definition given in (3.1.3). Recall the expression in (ρ, θ) coordinates of the Laplace operator, namely

$$\Delta = -\partial_{\rho\rho} - \frac{1}{\tanh \rho} \partial_{\rho} - \frac{1}{\sinh(\rho)^2} \partial_{\theta\theta}.$$

Then we can write

$$\begin{aligned} \Delta(\sinh(\rho)^{-s}) &= -\partial_{\rho\rho}(\sinh(\rho)^{-s}) - \frac{1}{\tanh \rho} \partial_{\rho}(\sinh(\rho)^{-s}) \\ &= -\partial_{\rho}(-s \sinh(\rho)^{-s-1} \cosh(\rho)) \\ &\quad - \frac{\cosh(\rho)}{\sinh(\rho)} (-s \sinh(\rho)^{-s-1} \cosh(\rho)) \\ &= s(-s - 1) \sinh(\rho)^{-s-2} \cosh(\rho)^2 + s \sinh(\rho)^{-s} \\ &\quad + s \sinh(\rho)^{-s-2} \cosh(\rho)^2 \\ &= -s^2 \sinh(\rho)^{-s-2} \cosh(\rho)^2 + s \sinh(\rho)^{-s} \\ &= -s^2 \sinh(\rho)^{-s-2} (1 + \sinh(\rho)^2) + s \sinh(\rho)^{-s} \\ &= s(1 - s) \sinh(\rho)^{-s} - s^2 \sinh(\rho)^{-s-2}, \end{aligned}$$

which proves the differential equation. \square

We define the elliptic counting function

$$N_{\text{ell};M,\gamma}(T; z) = \text{card} \{ \eta \in \Gamma_\gamma \setminus \Gamma \mid d_{\mathbb{H}}(\eta z, i) < T \},$$

which counts the number of geodesic paths from $z \in M$ to the elliptic fixed point E of length less than T .

Similar to Lemmas 3.1.2 and 3.1.5 above, we obtain the following bound involving the elliptic counting function.

Lemma 3.1.8. *For any point $z \in M$ with injectivity radius r , and any $u \geq T_0 > r$, we have*

$$N_{\text{ell};M,\gamma}(u; z) \leq N_{\text{ell};M,\gamma}(T_0; z) + \frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})}.$$

Proof. This follows by the same methods used in Lemma 3.1.2 above. \square

We can now express the elliptic Eisenstein series (3.1.3) as a Stieltjes integral, namely we have the following result.

Proposition 3.1.9. *The elliptic Eisenstein series $E_{\text{ell};M,\gamma}(z, s)$ has the following integral representation*

$$E_{\text{ell};M,\gamma}(z, s) = \int_{u_0}^{\infty} (\sinh u)^{-s} dN_{\text{ell};M,\gamma}(u; z)$$

for all $z \in \mathbb{H}$ which are not Γ -equivalent to the associated conical point and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. The lower limit of integration u_0 depends on the variable z in the sense that $u_0 = u_0(z) = \min\{d_{\mathbb{H}}(\eta z, i) \mid \eta \in \Gamma_\gamma \setminus \Gamma\}$.

Note the $u_0 = u_0(z) > 0$ since z is not Γ -equivalent to the associated elliptic fixed point. By combining the bound in Lemma 3.1.8 and Proposition 3.1.9, we can show that the series defining the hyperbolic Eisenstein series (3.1.3) converges uniformly and absolutely for $\operatorname{Re}(s) > 1$.

3.2 Eisenstein series through hyperbolic degeneration

In this section, we study the limiting behavior of the three Eisenstein series on a hyperbolically degenerating family of finite volume hyperbolic Riemann surfaces. In particular, we prove the following result. If $\gamma \in \Gamma$ corresponds to a degenerating hyperbolic element, then a multiple of the associated hyperbolic Eisenstein series converges to parabolic Eisenstein series on the limit surface. Namely, we have the following theorem which is based on joint work with J. Jorgenson and M. Munn (see [GJM 08]). This convergence is also studied in [Fa 07]. However, the methods employed here rely on the uniform convergence of counting functions and are quite different from the methods used in [Fa 07].

Theorem 3.2.1. *Let M_ℓ be a hyperbolically degenerating family of hyperbolic Riemann surfaces of finite volume, with limit surface M_0 .*

- (i) *Let $E_{\text{hyp};M_\ell,\gamma}(z, s)$ be the hyperbolic Eisenstein series on M_ℓ associated to the hyperbolic element γ . If γ corresponds to a non-degenerating hyperbolic element, then*

$$\lim_{\ell \rightarrow 0} E_{\text{hyp};M_\ell,\gamma}(z, s) = E_{\text{hyp};M_0,\gamma}(z, s).$$

- (ii) *Let $E_{\text{par};M_\ell,P}(z, s)$ be the parabolic Eisenstein series on M_ℓ associated to*

the cusp P . Then

$$\lim_{\ell_\gamma \rightarrow 0} E_{\text{par}; M_\ell, P}(z, s) = E_{\text{par}; M_0, P}(z, s).$$

(iii) Let $E_{\text{ell}; M_\ell, E}(z, s)$ be the elliptic Eisenstein series on M_ℓ associated to the conical point E . Then

$$\lim_{\ell_\gamma \rightarrow 0} E_{\text{ell}; M_\ell, E}(z, s) = E_{\text{ell}; M_0, E}(z, s).$$

(iv) Let $E_{\text{hyp}; M_\ell, \gamma}(z, s)$ be the hyperbolic Eisenstein series on M_ℓ associated to the hyperbolic element γ , whose geodesic has length ℓ_γ . If γ corresponds to a degenerating hyperbolic element which results in the new cusp P , then

$$\lim_{\ell_\gamma \rightarrow 0} \ell_\gamma^{-s} E_{\text{hyp}; M_\ell, \gamma}(z, s) = E_{\text{par}; M_0, P}(z, s).$$

In all instances, the convergence is uniform on compact subsets of M_0 bounded away from the developing cusps, and in half-planes of the form $\text{Re}(s) \geq 1 + \delta$ for any $\delta > 0$.

In order to keep the statement of part (iv) manageable, we employed a slight abuse of notation, which we wish to clarify. If M_ℓ has a single pinching geodesic which is separating, then the limit surface M_0 has two components, which we denote by $M_{0;1}$ and $M_{0;2}$, each with a single newly formed cusp, denoted by P_1 and P_2 . In this case, the right-hand-side of (iv) depends on the location of the point z : If $z \in M_{0;1}$ (resp. $z \in M_{0;2}$), then the right-hand-side of (iv) signifies the function $E_{\text{par}; M_{0;1}, P_1}(z, s)$ (resp. $E_{\text{par}; M_{0;2}, P_2}(z, s)$). If M_ℓ

has a single pinching geodesic which is non-separating, then the limit surface M_0 has one component with two newly formed cusps, denoted by P_1 and P_2 . In this case, the right-hand-side of (iv) signifies the function $E_{\text{par};M_0,P_1}(z, s) + E_{\text{par};M_0,P_2}(z, s)$. To consider the general case when M_ℓ has several pinching geodesics, then one simply iterates the results from Theorem 3.2.1 by pinching one geodesic at a time.

In the first part of the section we will look at the behavior of the counting functions mentioned in Section 3.1. Then we will use the convergence of counting function and the integral representation of the Eisenstein series to prove the aforementioned result. For simplicity, we will assume that M_ℓ has a single family of degenerating geodesics; the more general situation is easily obtained from the arguments presented here with only a slight modification of notation.

Lemma 3.2.2. *With notation as above, we have the following limits:*

(a) *If γ does not correspond to a degenerating hyperbolic element, then*

$$\lim_{\ell \rightarrow 0} N_{\text{hyp};M_\ell,\gamma}(T; z) = N_{\text{hyp};M_0,\gamma}(T; z);$$

(b) *For any cusp P , we have*

$$\lim_{\ell \rightarrow 0} N_{\text{par};M_\ell,P}(T; z, y_0) = N_{\text{par};M_0,P}(T; z, y_0).$$

(c) For any elliptic element γ , we have

$$\lim_{\ell \rightarrow 0} N_{\text{ell}; M_\ell, \gamma}(T; z) = N_{\text{ell}; M_0, \gamma}(T; z).$$

In all instances, the convergence is uniform on compact subsets of M_0 bounded away from the developing cusps.

Proof. The first step is to choose ε_1 sufficiently small so that the point z is bounded away from the degenerating collar, that is $z \in M_\ell \setminus \mathcal{C}_{\ell, \varepsilon_1}$. The second step is to choose $\varepsilon_0 < \varepsilon_1$ so that the distance from the boundary of $\mathcal{C}_{\ell, \varepsilon_0}$ to the boundary of $\mathcal{C}_{\ell, \varepsilon_1}$ is greater than T . Namely, one can choose ε_0 such that the following inequality holds

$$\frac{\varepsilon_0 + \sqrt{\varepsilon_0 + 4\ell^2}}{\varepsilon_1 + \sqrt{\varepsilon_1 + 4\ell^2}} < e^{-T},$$

which comes from formula (2.2.2) in Section 2.2. But then any geodesic path from z to the non-pinching geodesic γ with length bounded by T will necessarily avoid entering the inner collar $\mathcal{C}_{\ell, \varepsilon_0}$, for otherwise it will gain a length of at least T . Said differently, such a geodesic will be contained entirely in the $M_\ell \setminus \mathcal{C}_{\ell, \varepsilon_0}$. From Proposition 2.2.1, we know that the family of hyperbolic metrics converge uniformly away on $M_\ell \setminus \mathcal{C}_{\ell, \varepsilon_0}$, which proves part (a).

The convergence statements asserted in (b) and (c) follow by a similar argument. □

Lemma 3.2.3. *With notation as above, let*

$$g(y_0, \ell) = \log \left(\frac{w}{y_0 \ell} + \sqrt{\left(\frac{w}{y_0 \ell} \right)^2 + 1} \right)$$

and let γ correspond to a degenerating hyperbolic element.

(a) Assume $\varepsilon > 0$ is sufficiently small so that $\mathcal{C}_{\ell,\varepsilon}$ is embedded in M_ℓ , and, for $z \in M_\ell$, let $\mathcal{C}_{\ell,\varepsilon}^z$ denote the half of $\mathcal{C}_{\ell,\varepsilon}$ closest to z . Define

$$N_{\text{hyp};M_\ell,\partial\mathcal{C}_{\ell,\varepsilon}^z}(T; z) = \text{card}\{\eta \in \Gamma_\gamma \setminus \Gamma \mid d_{\mathbb{H}}(\eta z, \partial\mathcal{C}_{\ell,\varepsilon}^z) < T\}. \quad (3.2.1)$$

Then, for any $T > 0$, we have

$$N_{\text{hyp};M_\ell,\gamma}(T + g(y_0, \ell); z) = N_{\text{hyp};M_\ell,\partial\mathcal{C}_{\ell,\varepsilon}^z}(T; z).$$

(b) For any fixed $T > 0$, we have that

$$\lim_{\ell \rightarrow 0} N_{\text{hyp};M_\ell,\gamma}(T + g(y_0, \ell); z) = N_{\text{par};M_0,P}(T; z, y_0).$$

Proof. For fixed ℓ , one can identify M_ℓ with a fundamental domain in \mathbb{H} , requiring that the lift of the pinching geodesic γ lies along the line $\text{Re}(z) = 0$. With this choice of fundamental domain, it is easy to see that the boundary $\partial\mathcal{C}_{\ell,\varepsilon}^z$ of $\mathcal{C}_{\ell,\varepsilon}^z$ lies along a path for which $\theta(z)$ is constant, namely $\theta(z) = \cot^{-1}(\varepsilon/2\ell)$. The curve $\partial\mathcal{C}_{\ell,\varepsilon}^z$ is orthogonal to the geodesics which transverse the sub-cylinder $\mathcal{C}_{\ell,\varepsilon}^z$, so $\partial\mathcal{C}_{\ell,\varepsilon}^z$ converges to a path on M_0 which is perpendicular to the geodesics which transverse a neighborhood of the cusp, meaning $\partial\mathcal{C}_{\ell,\varepsilon}$ converges to a horocyclic path \mathcal{L}_{y_0} in a neighborhood of the new cusp; see Fig. 5. The area of $\mathcal{C}_{\ell,\varepsilon}^z$ equals the area on M_0 above \mathcal{L}_{y_0} by the choice of $\varepsilon = \varepsilon(\ell)$. By direct computation, we have

$$\text{Area}(\text{region on } M_0 \text{ above } \mathcal{L}_{y_0}) = \int_{y_0}^{\infty} \int_0^{\omega} \frac{dx dy}{y^2} = \frac{\omega}{y_0},$$

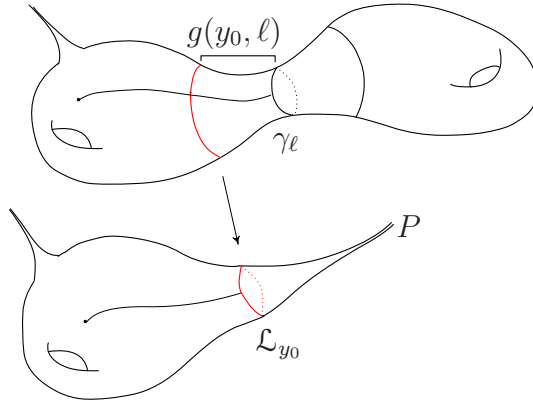


Figure 5: The Riemann surfaces M_ℓ and M_0

giving the relation $\varepsilon = 2w/y_0$. We define $g(y_0, \ell)$ to be the distance from $\partial\mathcal{C}_{\ell, \varepsilon}^z$ to the geodesic in $\mathcal{C}_{\ell, \varepsilon}^z$. Using the computations from Section 2.2, we then have that

$$\begin{aligned}
 g(y_0, \ell) &= \int_{\cot^{-1}(\varepsilon/2\ell)}^{\pi/2} \frac{d\theta}{\sin \theta} \\
 &= -\log(\csc \theta + \cot \theta) \Big|_{\cot^{-1}(\varepsilon/2\ell)}^{\pi/2} \\
 &= \log \left(\frac{\varepsilon}{2\ell} + \sqrt{1 + \left(\frac{\varepsilon}{2\ell}\right)^2} \right).
 \end{aligned}$$

Recalling that $\varepsilon = 2w/y_0$, allows to write

$$g(y_0, \ell) = \log \left(\frac{w}{y_0\ell} + \sqrt{1 + \left(\frac{w}{y_0\ell}\right)^2} \right).$$

Furthermore, consider a coset representative $\eta \in \Gamma_\gamma \backslash \Gamma$. For an appropriate such representative, we may assume that ηz lies in the fundamental domain for Γ_ℓ from Section 2.2, meaning $0 \leq \log |\eta z| < \ell$. It follows that the geodesic path from ηz to the $\{\operatorname{Re}(z) = 0\} \cap \mathbb{H}$ lies along the path $\rho = \text{constant}$, which

then is seen to be orthogonal to each path $\theta = \text{constant}$. Using the triangle inequality (which in this case reduces to triangle equality), we have that

$$d_{\mathbb{H}}(\eta z, \gamma) = d_{\mathbb{H}}(\eta z, \partial\mathcal{C}_{\ell, \varepsilon}^z) + d_{\mathbb{H}}(\partial\mathcal{C}_{\ell, \varepsilon}^z, \gamma) = d_{\mathbb{H}}(\eta z, \partial\mathcal{C}_{\ell, \varepsilon}^z) + g(y_0, \ell). \quad (3.2.2)$$

From (3.2.2) it immediately follows that

$$d_{\mathbb{H}}(\eta z, \gamma) < T + g(y_0, \ell) \iff d_{\mathbb{H}}(\eta z, \partial\mathcal{C}_{\ell, \varepsilon}^z) < T,$$

which finishes the proof of part (a).

Part (b) follows from combining part (a) with the convergence of the hyperbolic metric on M_ℓ away from the developing cusps to the hyperbolic metric on M_0 , as stated in Proposition 2.2.1. \square

Remark 3.2.4. As mentioned after the statement of Theorem 3.2.1, there are two cases one needs to consider in part (b) of Lemma 3.2.3. In the first case the pinching geodesic separates, that is the limiting surface M_0 is disconnected. If γ is separating, then the statement of part (b) can be taken as is. In the second case, the pinching geodesic γ is non-separating. In such a case, one needs to take into account that geodesic lengths from z to γ enter the cylinder about the pinching geodesic from the two different sides. The proof of part (b) easily extends to show that in the non-separating case the right-hand-side is actually the sum of two parabolic counting functions corresponding to the two newly formed cusps. With this noted, we choose to use the statement of part (b) in slight abuse of notation in both the separating and non-separating cases in order to avoid additional notation.

Remark 3.2.5. Note that, the convergence of the counting functions in Lemma

3.2.2 is an immediate consequence of the convergence of the hyperbolic metrics away from the developing cusps, as stated in Proposition 2.2.1. In Lemma 3.2.3, we have the added feature that the hyperbolic counting function involves the distances from the orbits of a point z to the geodesic corresponding to the hyperbolic element, but the parabolic counting function involves distances to a chosen horocycle. The distances to the geodesic associated to a degenerating hyperbolic element are growing without bound; however, Lemma 3.2.3 can be viewed as establishing a type of “regularized convergence”. Namely, the function $g(y_0, \ell)$ depends solely on y_0 and ℓ , and no other aspect of the family M_ℓ . This allows us to “regularize” the counting function associated to a degenerating hyperbolic representative γ by defining it as

$$N_{\text{hyp}; M_\ell, \gamma}(T; z) = \text{card} \{ \eta \in \Gamma_\gamma \backslash \Gamma \mid d_{\mathbb{H}}(\eta z, \mathcal{L}_0) < T + g(y_0, \ell) \}.$$

With this regularization, Lemma 3.2.3 states that the counting function $N_{\text{hyp}; M_\ell, \gamma}$ associated to a degenerating hyperbolic element converges to the counting function $N_{\text{par}; M_\ell, P}$ associated to the newly formed cusp.

A key component in our analysis is an integral inequality for Stieltjes integrals, which we quote from [JK 01] and, for the sake of completeness, we state here.

Lemma 3.2.6. *Let F be a real valued, non-negative, smooth, decreasing function defined for $u > 0$ and let g_1, g_2 be real valued, non decreasing functions defined for $u \geq a > 0$ and satisfying $g_1(u) \leq g_2(u)$ for $u \geq a$. Then, the*

following inequality of Stieltjes integrals

$$\int_a^\infty F(u) dg_1(u) + F(a) g_1(a) \leq \int_a^\infty F(u) dg_2(u) + F(a) g_2(a)$$

holds, provided both integrals exist.

We now proceed to prove Theorem 3.2.1. In brief, our proof uses the convergence of the counting functions for fixed T (Lemma 3.2.2 and Lemma 3.2.3), the uniform bounds for the counting functions (section 3.1) and the Stieltjes integral inequality (see Lemma 3.2.6). We present the arguments in the setting of a single degenerating hyperbolic element γ whose associated geodesic has length ℓ ; in order to consider the general situation where there are a number of degenerating geodesics, one simply needs notational changes.

Proof of the Theorem 3.2.1 part (i). The proof will proceed as follows. First, we split the Stieltjes integral defining the hyperbolic Eisenstein series into two integrals, namely for some $T_0 > 0$ we have an integral over $[0, T_0]$ and one over the interval $[T_0, \infty)$. For the latter, we show that for an appropriate value T_0 , the absolute value of this integral is arbitrarily small. Then we continue by showing that the former integral converges through degeneration.

So let $T_0 > 0$ and write

$$\begin{aligned} E_{\text{hyp}; M_\ell, \gamma}(z, s) &= \int_0^{T_0} (\cosh u)^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z) \\ &\quad + \int_{T_0}^\infty (\cosh u)^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z). \end{aligned} \quad (3.2.3)$$

Choose any $\delta > 0$ and restrict $s \in \mathbb{C}$ to the half-plane $\text{Re}(s) \geq 1 + \delta$ for some

fixed $\delta > 0$. Trivially, we have that

$$\left| \int_{T_0}^{\infty} (\cosh u)^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z) \right| \leq \int_{T_0}^{\infty} (\cosh u)^{-(1+\delta)} dN_{\text{hyp}; M_\ell, \gamma}(u; z).$$

We now wish to show that the integral in the right hand side above can be made arbitrarily small. Namely, given any $\varepsilon > 0$, we want to produce a value $T_0 = T_0(\varepsilon, \delta, r)$, where r is the injectivity radius at z , such that for each $\ell \geq 0$, we have

$$\int_{T_0}^{\infty} (\cosh u)^{-(1+\delta)} dN_{\text{hyp}; M_\ell, \gamma}(u; z) < \varepsilon. \quad (3.2.4)$$

The bound (3.2.4) above follows via similar arguments as in the proof of Lemma 1.4 from [JoLu 97a], which we repeat here. In the notation of Lemma 3.2.6, let $F(u) = (\cosh u)^{-(1+\delta)}$, which evidently is real-valued, non-negative, smooth, and decreasing. For $u \geq T_0$, we set

$$\begin{aligned} g_1(u) &= N_{\text{hyp}; M_\ell, \gamma}(u; z) \\ g_2(u) &= N_{\text{hyp}; M_\ell, \gamma}(T_0; z) + \frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})}. \end{aligned}$$

Using the Lemma 3.1.5 above, we have that $g_1(u) \leq g_2(u)$. In addition to this, both g_1 and g_2 are real-valued and non-decreasing for $u \geq T_0 > r$. Then the hypothesis of the Stieltjes integral inequality are met (c.f. Lemma 3.2.6),

giving the bound

$$\begin{aligned}
& \int_{T_0}^{\infty} (\cosh u)^{-(1+\delta)} dN_{\text{hyp}; M_\ell, \gamma}(u; z) \\
& \leq \int_{T_0}^{\infty} (\cosh u)^{-(1+\delta)} dg_2(u) \\
& \quad + (\cosh T_0)^{-(1+\delta)} \left\{ \frac{\sinh^2(\frac{T_0+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})} \right\}.
\end{aligned}$$

We have that

$$\begin{aligned}
dg_2(u) &= \frac{d}{du} \left(\frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})} \right) \\
&= \frac{1}{\sinh^2(\frac{r}{2})} \frac{d}{du} \left(\sinh^2 \left(\frac{u+r}{2} \right) \right) \\
&= \frac{1}{\sinh^2(\frac{r}{2})} \cdot 2 \sinh \left(\frac{u+r}{2} \right) \cosh \left(\frac{u+r}{2} \right) \frac{1}{2} du \\
&= \frac{\sinh(u+r)}{2 \sinh^2(\frac{r}{2})} du,
\end{aligned}$$

as well as

$$\sinh^2 \left(\frac{T_0+r}{2} \right) - \sinh^2 \left(\frac{T_0-r}{2} \right) = \sinh r \sinh T_0.$$

The trivial inequalities $\sinh u \leq e^u/2$ and $\cosh u \geq e^u/2$, allows for further

bound, namely

$$\begin{aligned}
\int_{T_0}^{\infty} (\cosh u)^{-(1+\delta)} dg_1(u) &\leq \frac{1}{2 \sinh^2(\frac{r}{2})} \int_{T_0}^{\infty} (\cosh u)^{-(1+\delta)} \sinh(u+r) du \\
&\quad + (\cosh T_0)^{-(1+\delta)} \frac{\sinh r \sinh T_0}{\sinh^2(\frac{r}{2})} \\
&\leq \frac{2^\delta e^r}{\sinh^2(\frac{r}{2})} \int_{T_0}^{\infty} e^{-\delta \cdot u} du \\
&\quad + (\cosh T_0)^{-(1+\delta)} \frac{\sinh r \sinh T_0}{\sinh^2(\frac{r}{2})} \\
&\leq e^{-\delta \cdot T_0} \left(\frac{2^\delta e^r}{\sinh^2(\frac{r}{2})} \left(\frac{1}{\delta} + 1 \right) \right), \tag{3.2.5}
\end{aligned}$$

which clearly can be made smaller than any $\varepsilon > 0$, namely, by taking

$$T_0 \geq \frac{1}{\delta} \left(-\log \varepsilon + \log \left(\frac{2^\delta e^r}{\sinh^2(\frac{r}{2})} \left(\frac{1}{\delta} + 1 \right) \right) \right). \tag{3.2.6}$$

Therefore, we have proved the bound asserted in (3.2.4).

In addition to the requirement on T_0 as in (3.2.6), let us assume that T_0 is a point of continuity of $N_{\text{hyp}; M_0, \gamma}(T; z)$. By this, we mean that there is no geodesic path from z to γ on M_0 with length equal to T_0 . Then, with T_0 chosen, there is an integer N and a positive number ℓ_0 sufficiently small such that for $\ell < \ell_0$, we have $N = N_{\text{hyp}; M_\ell, \gamma}(T_0; z) = N_{\text{hyp}; M_0, \gamma}(T_0; z)$. Let $\{d_{k, M_\ell}\} \subset [0, T_0]$ be the set of lengths on M_ℓ such that for any $\eta > 0$ we have

$$N_{\text{hyp}; M_\ell, \gamma}(d_{k, M_\ell} - \eta; z) < N_{\text{hyp}; M_\ell, \gamma}(d_{k, M_\ell} + \eta; z).$$

For simplicity, we count the elements in the set $\{d_{k, M_\ell}\}$ with multiplicities so

that we have

$$\int_0^{T_0} (\cosh u)^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z) = \sum_{k=1}^N (\cosh d_{k, M_\ell})^{-s}.$$

With this, we can write

$$\begin{aligned} \int_0^{T_0} (\cosh u)^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z) &- \int_0^{T_0} (\cosh u)^{-s} dN_{\text{hyp}; M_0, \gamma}(u; z) \\ &= \sum_{k=1}^N [(\cosh d_{k, M_\ell})^{-s} - (\cosh d_{k, M_0})^{-s}]. \end{aligned}$$

Observe now that the function $(\cosh u)^{-s}$ is uniformly continuous and absolutely continuous on $[0, T_0]$. By Lemma 3.2.3, which we apply for all $T < T_0$, there is an ℓ'_0 such that for $\ell < \ell'_0$ we have

$$|d_{k, M_\ell} - d_{k, M_0}| < \frac{\delta}{N} \text{ for all } k,$$

so then, $\sum_{k=1}^N |d_{k, M_\ell} - d_{k, M_0}| < \delta$. By the absolute continuity of $(\cosh u)^{-s}$ on $[0, T_0]$ we arrive at the bound

$$\begin{aligned} &\left| \sum_{k=1}^N [(\cosh d_{k, M_\ell})^{-s} - (\cosh d_{k, M_0})^{-s}] \right| \\ &\leq \sum_{k=1}^N |(\cosh d_{k, M_\ell})^{-s} - (\cosh d_{k, M_0})^{-s}| < \varepsilon. \end{aligned} \quad (3.2.7)$$

Combining all these results, we can write

$$\begin{aligned}
& |E_{\text{hyp};M_\ell,\gamma}(z, s) - E_{\text{hyp};M_0,\gamma}(z, s)| \leq \\
& \left| \int_0^{T_0} (\cosh u)^{-s} dN_{\text{hyp};M_\ell,\gamma}(u; z) - \int_0^{T_0} (\cosh u)^{-s} dN_{\text{hyp};M_0,\gamma}(u; z) \right| \\
& + \left| \int_{T_0}^{\infty} (\cosh u)^{-s} dN_{\text{hyp};M_\ell,\gamma}(u; z) \right| + \left| \int_{T_0}^{\infty} (\cosh u)^{-s} dN_{\text{hyp};M_0,\gamma}(u; z) \right|. \quad (3.2.8)
\end{aligned}$$

The second and third terms on the right hand side are arbitrarily small by taking T_0 as in (3.2.6), and the first term on the right hand side is arbitrarily small by (3.2.7). This completes the proof of part (i) of the Theorem 3.2.1. \square

Proof of the Theorem 3.2.1 part (ii). The proof of part (ii) follows the pattern of the proof of part (i) with slight modifications which we sketch below. The function $(\cosh u)^{-s}$ is replaced by e^{-su} and for $T_0 > 0$ we write

$$\begin{aligned}
E_{\text{par};M_\ell,P}(z, s) &= \left(\frac{y_0}{w}\right)^s \int_0^{T_0} e^{-su} dN_{\text{par};M_\ell,P}(u; z, y_0) \\
&+ \left(\frac{y_0}{w}\right)^s \int_{T_0}^{\infty} e^{-su} dN_{\text{par};M_\ell,P}(u; z, y_0).
\end{aligned}$$

The absolute value of the second integral above can be made arbitrarily small for an appropriate choice of T_0 . To show this, we use the Stieltjes integral inequality coming from Lemma 3.2.6 applied to the function $F(u) = e^{-(1+\delta)u}$ (where $\delta > 0$ and $\text{Re}(s) \geq 1 + \delta$) together with the function $g_1(u)$ and $g_2(u)$ which we set in the manner of the Lemma 3.1.2 above. Using this bound, we can easily derive the fact that for any $\varepsilon > 0$ we have

$$\int_{T_0}^{\infty} e^{-(1+\delta)u} dN_{\text{par};M_\ell,P}(u; z, y_0) < \varepsilon,$$

provided that we choose

$$T_0 > \frac{1}{\delta} \left(-\log \varepsilon + \log \left(\frac{e^r}{\sinh^2(r/2)} \right) \right).$$

With T_0 chosen as above, for the integral over the interval $[0, T_0]$ we use the absolute continuity of the function $f(u) = e^{-su}$ on this interval together with the added requirement on T_0 that it is also a point of continuity on $N_{\text{par}; M_0, P}(T; z, y_0)$. This allows us to show that the absolute value of the difference between the integral over $[0, T_0]$ associated to the parabolic Eisenstein series on M_ℓ and the integral over $[0, T_0]$ associated to the parabolic Eisenstein series on M_0 can be made arbitrarily small. The convergence as stated in Theorem 3.2.1 part (ii) then follows. \square

Proof of the Theorem 3.2.1 part (iii). The proof of part (iii) follows by similar methods used in part (i) and (ii) above (also see the proof of Theorem 3.3.1 part (i) below). \square

Proof of the Theorem 3.2.1 part (iv). We proceed as in the proof of parts (i) through (iii) above with some slight changes. First, we split the integral that defines the hyperbolic Eisenstein series into two integrals, namely

$$\begin{aligned} E_{\text{hyp}; M_\ell, \gamma}(z, s) &= \int_0^{T_0 + g(y_0, \ell)} (\cosh u)^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z) \\ &\quad + \int_{T_0 + g(y_0, \ell)}^\infty (\cosh u)^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z), \end{aligned} \quad (3.2.9)$$

where $g(y_0, \ell)$ is given in Lemma 3.2.3. Then multiply both sides of (3.2.9) by $2^{-s} e^{sg(y_0, \ell)}$ and let ℓ approach zero. For the integral over $[T_0 + g(y_0, \ell), \infty)$, we

first use part (a) of Lemma 3.2.3 to write

$$\begin{aligned} & \int_{T_0+g(y_0,\ell)}^{\infty} (\cosh u)^{-s} dN_{\text{hyp};M_\ell,\gamma}(u; z) \\ &= \int_{T_0}^{\infty} (\cosh(u + g(y_0, \ell)))^{-s} dN_{\text{hyp};M_\ell,\partial\mathcal{C}_{\ell,\varepsilon}}(u; z). \end{aligned}$$

The geometric argument which produced Lemmas 3.1.5 and 3.1.2 can be easily adapted to show that

$$N_{\text{hyp};M_\ell,\partial\mathcal{C}_{\ell,\varepsilon}}(u; z) \leq N_{\text{hyp};M_\ell,\partial\mathcal{C}_{\ell,\varepsilon}}(T_0; z) + \frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})},$$

for $u > T_0 > r$ where, as before, r is the injectivity radius at z . Following the computations in (3.2.5), we arrive at the estimate

$$\left| 2^{-s} e^{sg(y_0,\ell)} \int_{T_0+g(y_0,\ell)}^{\infty} (\cosh u)^{-s} dN_{\text{hyp};M_\ell,\gamma}(u; z) \right| \leq e^{-\delta \cdot T_0} \frac{e^r}{\sinh^2(\frac{r}{2})} \left(\frac{1}{\delta} + 1 \right), \quad (3.2.10)$$

where we have written $\text{Re}(s) = 1 + \delta$. By choosing

$$T_0 \geq \frac{1}{\delta} \left(-\log \varepsilon + \log \left(\frac{e^r}{\sinh^2(\frac{r}{2})} \left(\frac{1}{\delta} + 1 \right) \right) \right)$$

we have that the upper bound in (3.2.10) is less than ε .

For the first integral in (3.2.9), we begin by writing

$$\begin{aligned} & \int_0^{T_0+g(y_0,\ell)} (\cosh u)^{-s} dN_{\text{hyp};M_\ell,\gamma}(u; z) \\ &= \int_0^{T_0} (\cosh(u + g(y_0, \ell)))^{-s} dN_{\text{hyp};M_\ell,\partial\mathcal{C}_{\ell,\varepsilon}}(u; z). \end{aligned}$$

Also, we observe the following result. For fixed $x > 0$ and $s \in \mathbb{C}$ with $\text{Re}(s) >$

0, we have

$$\lim_{r \rightarrow \infty} 2^{-s} e^{rs} (\cosh(x+r))^{-s} = e^{-sx}. \quad (3.2.11)$$

Furthermore, the limit (3.2.11) is uniform for all $x > 0$ and $\operatorname{Re}(s) \geq 1 + \delta$.

Let $f(s, \ell) = 2^{-s} y_0^s e^{sg(y_0, \ell)}$. Then, by Lemma 3.2.3 and the argument yielding (3.2.7), we have, for any T_0 as in (3.2.6), the limit

$$\begin{aligned} \lim_{\ell \rightarrow 0} f(s, \ell) \int_0^{T_0 + g(y_0, \ell)} (\cosh(u))^{-s} dN_{\text{hyp}; M_\ell, \gamma}(u; z) \\ = y_0^s \int_0^{T_0} e^{-su} dN_{\text{par}; M_0, P}(u; z, y_0). \end{aligned} \quad (3.2.12)$$

We now use (3.2.10) and (3.2.12) and the triangle inequality, as in (3.2.8), in order to prove

$$\lim_{\ell \rightarrow 0} f(s, \ell) E_{\text{hyp}; M_\ell, \gamma}(z, s) = w^s E_{\text{par}; M_0, P}(z, s). \quad (3.2.13)$$

To complete the proof of part (iv), it remains to evaluate $f(s, \ell)$. As shown in the proof of Lemma 3.2.3, we have

$$g(y_0, \ell) = \log \left(\frac{w}{y_0 \ell} + \sqrt{\left(\frac{w}{y_0 \ell} \right)^2 + 1} \right),$$

from which we immediately derive the relation

$$\begin{aligned} f(s, \ell) &= 2^{-s} y_0^s \left(\frac{w}{y_0 \ell} + \sqrt{\left(\frac{w}{y_0 \ell} \right)^2 + 1} \right)^s \\ &= (w/\ell)^s + o((w/\ell)^s) \quad \text{as } \ell \rightarrow 0. \end{aligned} \quad (3.2.14)$$

Substituting (3.2.14) into (3.2.13), then multiplying both sides by w^{-s} , completes the proof of part (iv) of our Theorem 3.2.1. \square

Remark 3.2.7. In the setting of part (iv) of Theorem 3.2.1, consider the differential equation satisfied by $E_{\text{hyp};M_\ell,\gamma}(z, s)$ which, after multiplying by ℓ^{-s} , is the identity

$$\begin{aligned} \Delta(\ell^{-s} E_{\text{hyp};M,\gamma}(z, s)) &= s(1-s)(\ell^{-s} E_{\text{hyp};M,\gamma}(z, s)) \\ &\quad + (s\ell)^2(\ell^{-s-2} E_{\text{hyp};M,\gamma}(s+2, z)). \end{aligned} \quad (3.2.15)$$

By part (iv) of Theorem 3.2.1, we have that

$$\begin{aligned} \lim_{\ell \rightarrow 0} \left(s(1-s)(\ell^{-s} E_{\text{hyp};M,\gamma}(z, s)) + s^2(\ell^{-s-2} E_{\text{hyp};M,\gamma}(s+2, z)) \cdot \ell^2 \right) \\ = s(1-s) E_{\text{par};M_0,P}(z, s), \end{aligned}$$

for all $\text{Re}(s) > 1$ and z bounded away from the developing cusps. The point here is that the second term on the right hand side of (3.2.15) vanishes through degeneration. Heuristically, this shows that in the setting of part (iv), the differential equation for the hyperbolic Eisenstein series limits to the differential equation for the parabolic Eisenstein series.

3.3 Eisenstein series through elliptic degeneration

In this section, we study the limiting behavior of the three Eisenstein series on an elliptically degenerating family of finite volume hyperbolic Riemann surfaces. The main result of the section has a similar flavor to the results from Section 3.2. We prove the following result. If $\gamma \in \Gamma$ corresponds to a degenerating elliptic element, then a multiple of the associated elliptic Eisenstein series converges to parabolic Eisenstein series on the limit surface. Namely,

we have the following theorem based on joint work with A. von Pippich (see [GvP]).

Theorem 3.3.1. *Let M_ω be an elliptically degenerating family of hyperbolic Riemann surfaces of finite volume, with limit surface M_∞ .*

(i) *If γ_ω is a degenerating elliptic element, and γ_q is a non-degenerating elliptic element, then*

$$\lim_{\omega \rightarrow \infty} E_{\text{ell}; M_\omega, \gamma_q}(z, s) = E_{\text{ell}; M_\infty, \gamma_q}(z, s).$$

(ii) *If γ_ω is a degenerating elliptic element, and P is a cusp (not among the newly developed ones), then*

$$\lim_{\omega \rightarrow \infty} E_{\text{par}; M_\omega, P}(z, s) = E_{\text{par}; M_\infty, P}(z, s).$$

(iii) *If γ_ω is a degenerating elliptic element, and γ is a hyperbolic element, then*

$$\lim_{\omega \rightarrow \infty} E_{\text{hyp}; M_\omega, \gamma}(z, s) = E_{\text{hyp}; M_\infty, \gamma}(z, s).$$

(iv) *If γ_ω corresponds to a degenerating elliptic element of order ω whose associated hyperbolic cone has angle $\alpha_\omega = 2\pi/\omega$, and P is the newly developed cusp on the limiting surface M_∞ , then*

$$\lim_{\alpha_\omega \rightarrow 0} (\alpha_\omega)^{-s} E_{\text{ell}; M_\omega, \gamma_\omega}(z, s) = E_{\text{par}; M_\infty, P}(z, s).$$

In all instances, the convergence is uniform on compact subsets of M_∞ bounded

away from the developing cusp and elliptic cones, and in half-planes of the form $\operatorname{Re}(s) \geq 1 + \delta$ for any $\delta > 0$.

As in the Section 3.2, we will establish first the convergence of the counting functions. Then we will use this together with the integral representation of the Eisenstein series and bounds on counting functions to prove Theorem 3.3.1. As with the hyperbolic case, we will assume that M_ω has a single degenerating elliptic element; the more general case can be obtained by repeating the argument for each elliptic element we want to degenerate.

In this section we will make use of Proposition 2.3.1, namely that one can construct a degenerating sequence which converges to M_∞ whose metrics converge uniformly away from the developing cusp.

Lemma 3.3.2. *Let M_ω be an elliptically degenerating family of finite volume hyperbolic Riemann surfaces which converges the limit surface M_∞ . Let γ_ω correspond to the degenerating elliptic element.*

(a) *If γ_q does not correspond to the degenerating elliptic element, then*

$$\lim_{\omega \rightarrow \infty} N_{\text{ell}, M_\omega, \gamma_q}(T; z) = N_{\text{ell}, M_\infty, \gamma_q}(T; z).$$

(b) *If γ corresponds to a primitive hyperbolic element, then*

$$\lim_{\omega \rightarrow \infty} N_{\text{hyp}, M_\omega, \gamma}(T; z) = N_{\text{hyp}, M_\infty, \gamma}(T; z).$$

(c) *If P is any cusp, then*

$$\lim_{\ell \rightarrow 0} N_{\text{par}, M_\omega, P}(T; z, y_0) = N_{\text{par}, M_\infty, P}(T; z, y_0).$$

In all instances, the convergence is uniform on compact subsets of M_∞ bounded away from the developing cusps.

Proof. We reason as in Lemma 3.2.2 above. Let ε_1 sufficiently small so that the point z lies in $M_\omega \setminus C_{\omega, \varepsilon_1}$. Choose $\varepsilon_0 < \varepsilon_1$ so that the distance from the boundary of $C_{\omega, \varepsilon_0}$ to the boundary of $C_{\omega, \varepsilon_1}$ is greater than T , namely one chooses ε_0 such that inequality

$$\frac{\varepsilon_0\omega + 2\pi + \sqrt{\varepsilon_0\omega(4\pi + \varepsilon_0\omega)}}{\varepsilon_1\omega + 2\pi + \sqrt{\varepsilon_1\omega(4\pi + \varepsilon_1\omega)}} < e^{-T}$$

holds. But then, any geodesic path from z to the non-degenerating elliptic element γ_q with length bounded by T necessarily lies entirely in the $M_\omega \setminus C_{\omega, \varepsilon_0}$. From Proposition 2.3.1 (see also Section 2.3), we know that the family of hyperbolic metrics converge uniformly away on $M_\omega \setminus C_{\omega, \varepsilon_0}$, which proves part (a).

Parts (b) and (c) follow by a similar argument. \square

Lemma 3.3.3. *Let M_ω be an elliptically degenerating family of finite volume hyperbolic Riemann surfaces which converges the limit surface M_∞ . Let γ_ω correspond to the degenerating elliptic conical point E and P be the corresponding newly developed cusp on M_∞ . Let*

$$g(y_0, \omega) = \log \left(1 + \frac{w\omega}{2\pi y_0} \left(1 + \sqrt{1 + 4\pi y_0 / (w\omega)} \right) \right).$$

(a) Choose $\varepsilon > 0$ such that the finite volume cone $C_{\omega, \varepsilon}$ embeds in M_ω . Define

$$N_{\text{ell}; M_\omega, \partial C_{\omega, \varepsilon}}(T; z) = \text{card}\{\eta \in \Gamma_{\gamma_\omega} \setminus \Gamma \mid d_{\mathbb{H}}(\eta z, \partial C_{\omega, \varepsilon}) < T\}.$$

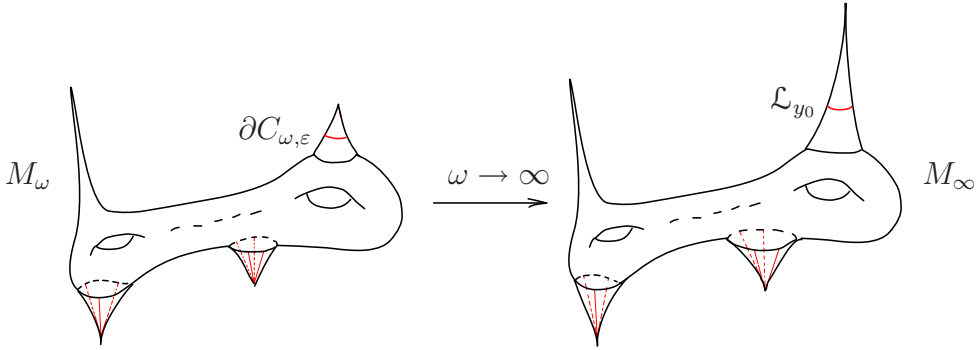


Figure 6: Convergence of $\partial C_{\omega, \epsilon}$ to \mathcal{L}_{y_0}

Then for any $T > 0$, we have

$$N_{\text{ell}; M_\omega, \gamma_\omega}(T + g(y_0, \omega); z) = N_{\text{ell}; M_\omega, \partial C_{\omega, \epsilon}}(T; z).$$

(b) For any fixed $T > 0$, we have

$$\lim_{\omega \rightarrow \infty} N_{\text{ell}; M_\omega, \gamma_\omega}(T + g(y_0, \omega); z) = N_{\text{par}; M_\omega, P}(T; z, y_0).$$

Proof. Start by fixing ω and identify M_ω with a fundamental domain such that the degenerating elliptic fixed point lies on the geodesic $|z| = 1$ with $y > 0$ in the upper half plane \mathbb{H} . Consequently, in hyperbolic polar coordinates (see Section 1.2) the boundary of the finite volume cone $C_{\omega, \epsilon}$ lies along the path where the ρ coordinate is constant, namely $\rho = \cosh^{-1}(1 + \epsilon\omega/2\pi)$. The boundary of this finite cone, $\partial C_{\omega, \epsilon}$, is perpendicular to the geodesics that traverse the cone and hit the elliptic fixed point. But then, this boundary must converge to a path on M_∞ which is perpendicular to the geodesics that traverse a neighborhood of the newly formed cusp. This means that $\partial C_{\omega, \epsilon}$ converges to the horocycle \mathcal{L}_{y_0} (see Fig. 6). On the one hand, the distance

between the degenerating elliptic point E and $\partial C_{\omega,\varepsilon}$ can be computed using (2.3.4) with $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon$, thus giving

$$d_{\mathbb{H}}(E, \partial C_{\omega,\varepsilon}) = \log \left(\frac{\varepsilon\omega + 2\pi + \sqrt{\varepsilon\omega(4\pi + \varepsilon\omega)}}{2\pi} \right). \quad (3.3.1)$$

On the other hand, the area of the finite cone $C_{\omega,\varepsilon}$ converges to the area of the neighborhood of the cusp above the horocycle \mathcal{L}_{y_0} , that is $C_{\infty,\varepsilon}$. The latter can be easily determined as

$$\varepsilon = \int_{y_0}^{\infty} \int_0^w \frac{dx dy}{y^2} = \frac{w}{y_0},$$

giving us the relation $\varepsilon = w/y_0$. This together with (3.3.1) allows us to write

$$g(y_0, \omega) = \log \left(1 + \frac{w\omega}{2\pi y_0} \left(1 + \sqrt{1 + 4\pi y_0/(w\omega)} \right) \right). \quad (3.3.2)$$

Pick a representative $\eta \in \Gamma_{\gamma_\omega} \setminus \Gamma$ such that ηz lies in the fundamental domain of Γ_{γ_ω} . Then, the geodesic that joins ηz to the degenerating conical point E is perpendicular to the boundary of the finite cone $C_{\omega,\varepsilon}$. Then we can write

$$d_{\mathbb{H}}(\eta z, E) = d_{\mathbb{H}}(\eta z, \partial C_{\omega,\varepsilon}) + d_{\mathbb{H}}(\partial C_{\omega,\varepsilon}, E) = d_{\mathbb{H}}(\eta z, \partial C_{\omega,\varepsilon}) + g(y_0, \omega).$$

From this last equality we can infer that $d_{\mathbb{H}}(\eta z, E) < T + g(y_0, \omega)$ if and only if $d_{\mathbb{H}}(\eta z, \partial C_{\omega,\varepsilon}) < T$. This finishes part (a).

Part (b) follows from part (a) and the convergence of metrics away from developing cusps, as in Proposition 2.3.1 (also see section 2.3). \square

Remark 3.3.4. Lemma 3.3.3 can be viewed as establishing a type of “regularized convergence”. Namely, if we defined the counting function associated to a degenerating elliptic element γ_ω as

$$N_{\text{ell};M_\omega,\gamma_\omega}(T; z) = \text{card}\{\eta \in \Gamma_{\gamma_\omega} \setminus \Gamma \mid d_{\mathbb{H}}(\eta z, \gamma_\omega) < T + g(y_0, \omega)\},$$

then Lemma 3.3.3 would state that the “regularized” counting function $N_{\text{ell};M_\omega,\gamma_\omega}$ converges to the counting function $N_{\text{par};M_\infty,P}$ associated to the newly developed cusp.

We are now ready to prove Theorem 3.3.1. Our proof uses the convergence of the counting functions for fixed T , the uniform bounds for the counting functions, and the Stieltjes integral inequality (see Lemma 3.2.6).

Proof of the Theorem 3.3.1 part (i). We show first the convergence of the elliptic Eisenstein series associated to an elliptic element other than the degenerating one. Recall that γ_ω is the degenerating elliptic element whereas γ_q is any non-degenerating elliptic.

For any $T_0 > r$ (which is the injectivity radius at z), write

$$\begin{aligned} E_{\text{ell};M_\omega,\gamma_q}(z, s) &= \int_{u_0}^{T_0} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_q}(u; z) \\ &\quad + \int_{T_0}^{\infty} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_q}(u; z). \end{aligned} \quad (3.3.3)$$

Choose any $\delta > 0$. Let $s \in \mathbb{C}$ with $\text{Re}(s) \geq 1 + \delta$ for some fixed $\delta > 0$. Clearly,

we have that

$$\left| \int_{T_0}^{\infty} (\sinh u)^{-s} dN_{\text{ell}; M_{\omega}, \gamma_q}(u; z) \right| \leq \int_{T_0}^{\infty} (\sinh u)^{-(1+\delta)} dN_{\text{ell}; M_{\omega}, \gamma_q}(u; z).$$

We then show that the right hand side can be made arbitrarily small, namely given any $\varepsilon > 0$, there is a $T_0 = T_0(\varepsilon, \delta, r)$, where r is the injectivity radius at z , such that for each ω , we have

$$\int_{T_0}^{\infty} (\sinh u)^{-(1+\delta)} dN_{\text{ell}; M_{\omega}, \gamma_q}(u; z) < \varepsilon. \quad (3.3.4)$$

The verification of (3.3.4) is as follows. In the notation of Lemma 3.2.6, let $F(u) = (\sinh u)^{-(1+\delta)}$, which evidently is real-valued, non-negative (for $u > 0$), smooth, and decreasing. For $u > T_0$, we let

$$\begin{aligned} g_1(u) &= N_{\text{ell}; M_{\omega}, \gamma_q}(u; z) \\ g_2(u) &= N_{\text{ell}; M_{\omega}, \gamma_q}(T_0; z) + \frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})}. \end{aligned}$$

Using Lemma 3.1.8, we have that $g_1(u) \leq g_2(u)$, and both functions are real-valued and non-decreasing for $u \geq T_0 > r$. With all this, the conditions of Lemma 3.2.6 apply, so then we obtain the bound

$$\begin{aligned} & \int_{T_0}^{\infty} (\sinh u)^{-(1+\delta)} dN_{\text{ell}; M_{\omega}, \gamma_q}(u; z) \\ & \leq \int_{T_0}^{\infty} (\sinh u)^{-(1+\delta)} dg_2(u) \\ & \quad + (\sinh T_0)^{-(1+\delta)} \left\{ \frac{\sinh^2(\frac{T_0+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})} \right\}. \end{aligned}$$

As before, let us recall the identities

$$dg_2(u) = \frac{\sinh(u+r)}{2 \sinh^2(\frac{r}{2})} du$$

and

$$\sinh^2\left(\frac{T_0+r}{2}\right) - \sinh^2\left(\frac{T_0-r}{2}\right) = \sinh r \sinh T_0.$$

Using the trivial bounds $\sinh u \leq e^u/2$ and for $u \geq \log \sqrt{2}$ one has $\sinh u \geq e^u/4$, we then obtain the estimate

$$\begin{aligned} \int_{T_0}^{\infty} (\sinh u)^{-(1+\delta)} dg_1(u) &\leq \frac{1}{2 \sinh^2(\frac{r}{2})} \int_{T_0}^{\infty} (\sinh u)^{-(1+\delta)} \sinh(u+r) du \\ &\quad + (\sinh T_0)^{-(1+\delta)} \frac{\sinh r \sinh T_0}{\sinh^2(\frac{r}{2})} \\ &\leq \frac{4^\delta e^r}{\sinh^2(\frac{r}{2})} \int_{T_0}^{\infty} e^{-\delta \cdot u} du + \left(\frac{e^{T_0}}{4}\right)^{-\delta} \frac{e^r}{2 \sinh^2(\frac{r}{2})} \\ &\leq e^{-\delta T_0} \cdot \frac{4^\delta e^r}{\sinh^2(\frac{r}{2})} \left(\frac{1}{\delta} + 1\right) \end{aligned} \quad (3.3.5)$$

which clearly can be made smaller than any $\varepsilon > 0$, namely, by taking

$$T_0 \geq \frac{1}{\delta} \left(-\log \varepsilon + \log \left(\frac{4^\delta e^r}{\sinh^2(\frac{r}{2})} \left(\frac{1}{\delta} + 1 \right) \right) \right). \quad (3.3.6)$$

Therefore, we have proved the bound asserted in (3.3.4).

In addition to (3.3.6), assume that T_0 is a point of continuity of $N_{\text{ell}; M_\infty, \gamma_q}(T; z)$, meaning there is no geodesic path from z to the conical point associated to γ_q on M_∞ with length equal to T_0 . Then, with T_0 chosen, there is an integer N and an ω_0 sufficiently large such that for $\omega > \omega_0$, we have $N =$

$N_{\text{ell};M_\omega,\gamma_q}(T_0; z) = N_{\text{ell};M_\infty,\gamma_q}(T_0; z)$. Let $\{d_{k,M_\omega}\} \subset [u_0, T_0]$ be the set of lengths on M_ω such that for any $\eta > 0$ we have

$$N_{\text{ell};M_\omega,\gamma_q}(d_{k,M_\omega} - \eta; z) < N_{\text{ell};M_\omega,\gamma_q}(d_{k,M_\omega} + \eta; z).$$

Counting the elements in the set $\{d_{k,M_\omega}\}$ with multiplicities, gives that

$$\int_{u_0}^{T_0} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_q}(u; z) = \sum_{k=1}^N (\sinh d_{k,M_\omega})^{-s}.$$

With this, we can write

$$\begin{aligned} \int_{u_0}^{T_0} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_q}(u; z) - \int_{u_0}^{T_0} (\sinh u)^{-s} dN_{\text{ell};M_\infty,\gamma_q}(u; z) \\ = \sum_{k=1}^N [(\sinh d_{k,M_\omega})^{-s} - (\sinh d_{k,M_\infty})^{-s}]. \end{aligned}$$

Observe now that the function $(\sinh u)^{-s}$ is uniformly continuous and absolutely continuous on $[u_0, T_0]$. By Lemma 3.3.2, which we apply for all $T < T_0$, there is an ω'_0 such that for $\omega > \omega'_0$ we have

$$|d_{k,M_\omega} - d_{k,M_\infty}| < \frac{\delta}{N} \text{ for all } k,$$

so then, $\sum_{k=1}^N |d_{k,M_\omega} - d_{k,M_\infty}| < \delta$. By the absolute continuity of $(\sinh u)^{-s}$ on $[u_0, T_0]$ we arrive at the bound

$$\begin{aligned} \left| \sum_{k=1}^N [(\sinh d_{k,M_\omega})^{-s} - (\sinh d_{k,M_\infty})^{-s}] \right| \\ \leq \sum_{k=1}^N \left| (\sinh d_{k,M_\omega})^{-s} - (\sinh d_{k,M_\infty})^{-s} \right| < \varepsilon. \end{aligned} \quad (3.3.7)$$

To put all this together, let us write

$$\begin{aligned}
& \left| E_{\text{ell};M_\omega,\gamma_q}(z,s) - E_{\text{ell};M_\infty,\gamma_q}(z,s) \right| \\
& \leq \left| \int_{u_0}^{T_0} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_q}(u;z) - \int_{u_0}^{T_0} (\sinh u)^{-s} dN_{\text{ell};M_\infty,\gamma_q}(u;z) \right| \\
& + \left| \int_{T_0}^{\infty} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_q}(u;z) \right| \\
& + \left| \int_{T_0}^{\infty} (\sinh u)^{-s} dN_{\text{ell};M_\infty,\gamma_q}(u;z) \right|. \tag{3.3.8}
\end{aligned}$$

The second and third terms on the right hand side are arbitrarily small by taking T_0 as in (3.3.6), and the first term on the right hand side is arbitrarily small by (3.3.7). Thus we have shown that

$$\lim_{\omega \rightarrow \infty} E_{\text{ell};M_\omega,\gamma_q}(z,s) = E_{\text{ell};M_\infty,\gamma_q}(z,s).$$

In order to show the convergence of the parabolic Eisenstein series through elliptic degeneration we use similar arguments as above with a slight modification, namely one needs to use $F(u) = e^{-su}$ instead of $F(u) = (\sinh u)^{-s}$. For the convergence of the hyperbolic Eisenstein series through elliptic degeneration one needs to use $F(u) = (\cosh u)^{-s}$ instead. This completes the proof of parts (i), (ii), and (iii) of Theorem 3.3.1. \square

Proof of Theorem 3.3.1 part (iv). In this part we show the convergence (up to a multiplicative factor) of the elliptic Eisenstein series associated to the degenerating elliptic element γ_ω to the parabolic Eisenstein series associated to the newly formed cusp P .

To begin, we write

$$\begin{aligned}
E_{\text{ell};M_\omega,\gamma_\omega}(z, s) &= \int_{u_0}^{T_0+g(y_0,\omega)} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_\omega}(u; z) \\
&\quad + \int_{T_0+g(y_0,\omega)}^{\infty} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_\omega}(u; z), \tag{3.3.9}
\end{aligned}$$

where $g(y_0, \omega)$ is given in Lemma 3.3.3. We shall multiply both sides of (3.3.9) by $4^{-s} e^{sg(y_0, \omega)}$ and let ω approach infinity. For the integral over $[T_0 + g(y_0, \omega), \infty)$, we first use part (a) of Lemma 3.3.3 to write

$$\begin{aligned}
&\int_{T_0+g(y_0,\omega)}^{\infty} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_\omega}(u; z) \\
&= \int_{T_0}^{\infty} (\sinh(u + g(y_0, \omega)))^{-s} dN_{\text{ell};M_\omega,\partial C_{\omega,\varepsilon}}(u; z).
\end{aligned}$$

We can easily adapt the method of proof as in the Lemmas 3.1.2, 3.1.5, and 3.1.8 to derive the bound

$$N_{\text{ell};M_\omega,\partial C_{\omega,\varepsilon}}(u; z) \leq N_{\text{ell};M_\omega,\partial C_{\omega,\varepsilon}}(T_0; z) + \frac{\sinh^2(\frac{u+r}{2}) - \sinh^2(\frac{T_0-r}{2})}{\sinh^2(\frac{r}{2})},$$

for $u > T_0 > r$ where, as before, r is the injectivity radius of M_ω at z . Following the computations in (3.3.5), we arrive at the estimate

$$\begin{aligned}
&\left| 4^{-s} e^{sg(y_0,\omega)} \int_{T_0+g(y_0,\omega)}^{\infty} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_\omega}(u; z) \right| \\
&\leq e^{-\delta \cdot T_0} \left(\frac{e^r}{\sinh^2(r/2)} \left(\frac{1}{\delta} + 1 \right) \right), \tag{3.3.10}
\end{aligned}$$

where we have written $\operatorname{Re}(s) = 1 + \delta$. By choosing

$$T_0 \geq \frac{1}{\delta} \left(-\log \varepsilon + \log \left(\frac{e^r}{\sinh^2(r/2)} \left(\frac{1}{\delta} + 1 \right) \right) \right),$$

we have that the upper bound in (3.3.10) is less than ε .

For the first integral in (3.3.9), we begin by writing

$$\begin{aligned} & \int_{u_0}^{T_0+g(y_0,\omega)} (\sinh u)^{-s} dN_{\text{ell};M_\omega,\gamma_\omega}(u; z) \\ &= \int_{u_0}^{T_0} (\sinh(u + g(y_0, \omega)))^{-s} dN_{\text{ell};M_\omega,\partial C_{\omega,\varepsilon}}(u; z). \end{aligned}$$

Also, we observe the following elementary result: For fixed $x > 0$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, we have

$$\lim_{r \rightarrow \infty} 4^{-s} e^{rs} (\sinh(x+r))^{-s} = 2^{-s} e^{-sx}. \quad (3.3.11)$$

Furthermore, the limit (3.3.11) is uniform for all $x > 0$ and $\operatorname{Re}(s) \geq 1 + \delta$.

Let $f(s, \omega) = 4^{-s} y_0^s e^{sg(y_0,\omega)}$. Then, by Lemma 3.3.3 and the argument yielding (3.3.7), we have, for any T_0 as in (3.3.6), the limit

$$\begin{aligned} & \lim_{\omega \rightarrow \infty} f(s, \omega) \int_{u_0}^{T_0+g(y_0,\omega)} (\sinh(u))^{-s} dN_{\text{ell};M_\omega,\gamma_\omega}(u; z) \\ &= 2^{-s} y_0^s \int_0^{T_0} e^{-su} dN_{\text{par};M_\infty,P}(u; z, y_0). \end{aligned} \quad (3.3.12)$$

We now use (3.3.10) and (3.3.12) and the triangle inequality, as in (3.3.8), in order to prove

$$\lim_{\omega \rightarrow \infty} f(s, \omega) E_{\text{ell};M_\omega,\gamma_\omega}(z, s) = 2^{-s} w^s E_{\text{par};M_\infty,P}(z, s). \quad (3.3.13)$$

To complete the proof of part (iv), it remains to evaluate $f(s, \omega)$. As shown in the proof of Lemma 3.3.3, we have

$$g(y_0, \omega) = \log \left(1 + \frac{w\omega}{2\pi y_0} \left(1 + \sqrt{1 + 4\pi y_0 / (w\omega)} \right) \right).$$

from which we immediately derive the relation

$$f(s, \omega) \sim 4^{-s} y_0^s \left(\frac{w\omega}{\pi y_0} \right)^s = \left(\frac{w\omega}{4\pi} \right)^s \quad (3.3.14)$$

for sufficiently large values of ω . Substituting (3.3.14) into (3.3.13), then multiplying both sides by $2^s \omega^{-s}$, we get

$$\lim_{\omega \rightarrow \infty} \left(\frac{\omega}{2\pi} \right)^s E_{\text{ell}; M_\omega, \gamma_\omega}(z, s) = E_{\text{par}; M_\infty, P}(z, s).$$

Since the degenerating elliptic cone is parametrized by angle $\alpha_\omega = 2\pi/\omega$ which approaches zero as the order of the corresponding conical point ω runs off to infinity, we can rewrite the above limit as

$$\lim_{\alpha_\omega \rightarrow 0} \alpha_\omega^{-s} E_{\text{ell}; M_\omega, \gamma_\omega}(z, s) = E_{\text{par}; M_\infty, P}(z, s).$$

This completes the proof. □

Remark 3.3.5. Consider the differential equation satisfied by $E_{\text{ell}; M_\omega, \gamma_\omega}(z, s)$ which, after multiplying by α^{-s} , gives

$$\begin{aligned} \Delta(\alpha^{-s} E_{\text{ell}; M_\omega, \gamma_\omega}(z, s)) &= s(1-s)(\alpha^{-s} E_{\text{ell}; M_\omega, \gamma_\omega}(z, s)) \\ &\quad - (s\alpha)^2 (\alpha^{-s-2} E_{\text{ell}; M_\omega, \gamma_\omega}(s+2, z)). \end{aligned} \quad (3.3.15)$$

By part (ii) of Theorem 3.3.1, we have that

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \left(s(1-s)(\alpha^{-s} E_{\text{ell}; M_\omega, \gamma_\omega}(z, s)) - s^2(\alpha^{-s-2} E_{\text{hyp}; M, \gamma}(s+2, z)) \cdot \alpha^2 \right) \\ = s(1-s) E_{\text{par}; M_\infty, P}(z, s), \end{aligned}$$

for all $\text{Re}(s) > 1$ and z bounded away from the developing cusps. The point here is that the second term on the right hand side of (3.3.15) vanishes through degeneration.

Remark 3.3.6. There is a quite striking similarity between Theorems 3.2.1 and 3.3.1, especially when it comes to parts (iv) of Theorem 3.2.1 and Theorem 3.3.1. In the former the multiplicative factor in front of the hyperbolic Eisenstein series associated to a degenerating hyperbolic element γ is a power of the length of the corresponding primitive geodesic, namely ℓ^{-s} . In the latter, one has that the multiplicative factor is a power of the angle of the degenerating cone, namely α^{-s} . Essentially, the ℓ 's are parameters which describe the hyperbolically degenerating family M_ℓ , whereas the α 's are parameters describing the elliptically degenerating family M_ω .

Remark 3.3.7. In [Wo 87], the author poses the following question: what is the behavior of spectral invariants for a degenerating family of metrics? In this direction, there has been quite some progress. For instance, in the setting of hyperbolically degenerating family of hyperbolic Riemann surfaces, the behavior of the spectral counting function is studied by [He 90] and [JZ 93] in the compact case. The compact as well as non-compact settings are further studied in [JoLu 97a], [JoLu 95b], [HJL 95], and [HJL 97].

Another spectral invariant is the Selberg zeta function. Its behavior in the

context of hyperbolic degeneration has been studied by [He 90] and [Wo 87] (the compact case) and by [JoLu 97b] in the compact and non-compact setting.

Other spectral invariants such as a regularized trace of the heat kernel, exceptional eigenvalues, the spectral zeta function, have been studied in [JoLu 97b]. Furthermore, in [DJ 98] the authors study the behavior of such invariants in the setting of degenerating hyperbolic 3-manifolds, whereas in [GJ] the authors look at such spectral invariants in the context of elliptic degeneration of hyperbolic Riemann surfaces with conical points (orbifolds). Eisenstein series turn out to be also invariants in both hyperbolic and elliptic degeneration of hyperbolic Riemann surfaces.

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