Hořava gravity: Symmetries and Generalized Particle Dynamics

by

Dario Capasso

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Prof. A. Polychronakos

Date

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Prof. S. Greenbaum

Date

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The CITY UNIVERSITY of NEW YORK

Abstract

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Dario Capasso

Adviser: Prof. Alexios Polychronakos

In the search for a theory of Quantum Gravity a new proposal was recently made by P. Hořava. The main feature of this new proposed theory is that it is power-counting renormalizable by construction, and could prove to be truly renormalizable, although more work is needed in this direction.

The renormalizability of the theory is a central issue. Indeed, General Relativity does not have this property, implying that to construct its quantum version we need to "complete" the theory in the UV. Hořava suggested a possible way to provide a UV completion of GR by giving up full spacetime reparametrization symmetry, which is one of the fundamental assumptions of GR, and adding appropriate higher order terms in the action.

In this Thesis we review Hořava's theory and analyze some of the issues related to the breaking of the spacetime structure.

Specifically, we derive the general static spherically symmetric solutions for Hořava's theory with a nonvanishing radial "shift" field g_{tr}. Such "hedgehog" configurations are not considered in GR, since g_{tr} can be mapped to zero with an appropriate reparametrization, but they are physically distinct solutions in Hořava gravity where the reparametrization is not allowed by the reduced symmetry. These new solutions exhibit specific properties from the particle dynamics point

of view and possess an extra gauge symmetry.

We also study the deformed kinematics of point particles allowed by the reduced reparametrization symmetry. The main result is that particles can have generalized dispersion relations that include higher even powers of the momentum. We analyze the implications of this and provide some examples that may be converted into possible experimental tests for the deviations of this new theory of gravity from standard GR.

Contents

Chapter 1

Hořava-Lifshitz Gravity

In this chapter the main ideas behind Hořava theory of gravity and its initial formulation will be introduced, showing how the relativistic property of spacetime can be considered as emergent in this context.

The aim of this introduction is to have a general view of the theory, while some structural and formal problems will be discussed in the following chapters, as well as the derived dynamical properties for point-like particles.

1.1 The Lifshitz Scalar

The model introduced by Hořava in [1] and subsequently adapted to a $(3+1)$-dimensional theory of gravity in [2] is based on a previous model for anisotropic scalars constructed by Lifshitz [3].

The Lifshitz model describes a scalar with an action anisotropic with respect to space and time. The isotropy is recovered only in the IR limit of the theory making the "relativistic invariance" an emergent property.

Let us start with a massless scalar theory for a D-dimensional Euclidean flat space:

$$W = \frac{1}{2}\int d^D x \partial_i \phi \partial^i \phi.$$

We want to construct a $(D+1)$-dimensional theory having the same configuration space of W and such that its ground state $\Psi_0[\phi]$ satisfies the property

$$e^{-W[\phi]} = \Psi_0^*[\phi]\Psi_0[\phi];$$

therefore $\Psi_0^*[\phi]\Psi_0[\phi]$ describes the density distribution in the partition function

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-W[\phi]} = \int \mathcal{D}\phi \Psi_0^*[\phi]\Psi_0[\phi].$$

Such a theory is given by the following action

$$S = \frac{1}{2}\int dt d^D x \left[(\dot{\phi})^2 - \frac{1}{4}(\partial_i \partial_i \phi)^2\right] \tag{1.1}$$

and is referred as **Lifshitz Scalar theory**.

The action (1.1) can be divided in a kinetic term

$$S_K = \frac{1}{2}\int dt d^D x (\dot{\phi})^2 \tag{1.2}$$

and in a potential term

$$S_V = \frac{1}{2}\int dt d^D x \left[-\frac{1}{4}(\partial_i \partial_i \phi)^2\right] \tag{1.3}$$

that are evidently anisotropic under a space and time rescaling. Space and time are treated in a different way in this model that, consequently, will not be relativistically invariant by construction.

Parametrizing the rescaling of spatial coordinates, the x's, and of the time coordinate t as

$$x \to bx \qquad t \to b^z t,$$

it is easy to see that, independently of D, $z = 2$. Hence the conformal dimensions of the variable and of the scalar field are

$$[x]_s = -1 \qquad [t]_s = -z \qquad [\phi]_s = \frac{D-z}{2}.$$

We then conclude that the Lifshitz scalar action describes a free-field with a fixed point where the dynamical critical exponent $z(= 2)$ can be defined in terms of the scaling properties of two point functions as follows:

$$\langle \phi(x,t)\phi(0,0)\rangle = \frac{1}{|x|^{2[\phi]_s}} f\left(\frac{x}{t^{1/z}}\right). \tag{1.4}$$

The Hamiltonian associated to (1.1) in the Schrödinger representation will be then given by

$$H = \frac{1}{2}\int d^D x \left[\left(-i\frac{\delta}{\delta\phi}\right)^2 + \frac{1}{4}(\partial_i\partial_i\phi)^2\right] =$$
$$= \frac{1}{2}\int d^D x \left[\left(-\frac{\delta}{\delta\phi} - \frac{1}{2}\partial_i\partial_i\phi\right)\left(\frac{\delta}{\delta\phi} - \frac{1}{2}\partial_i\partial_i\phi\right)\right].$$

The ground state can be obtained solving the condition:

$$\left(\frac{\delta}{\delta\phi} - \frac{1}{2}\partial_i\partial_i\phi\right)\Psi_0[\phi] = 0.$$

We can rewrite the last relation as follows

$$\frac{\delta}{\delta\phi}\Psi_0[\phi] = \frac{1}{2}\partial_i\partial_i\phi\Psi_0[\phi] \tag{1.5}$$

which is solved by

$$\Psi_0[\phi] = e^{-\frac{1}{4}\int d^D x \partial_i\phi\partial_i\phi} = e^{-\frac{1}{2}W},$$

as we wanted.

The above property is a consequence of the fact that the potential term can be rewritten as

$$S_V = -\frac{1}{8}\int dtd^Dx\,\left[E^2\right]$$

where we defined

$$E = \partial_i\partial^i\Phi.$$

Such potential then satisfies the **detailed balance condition**; namely it can be obtained as the square of the variation of an Euclidean action that, in our case, corresponds to W:

$$E = -\frac{\delta W}{\delta\phi}. \tag{1.6}$$

The critical exponent $z = 2$ is then due to the particular choice of the potential term. With an opportune choice of the potential term we can add also an isotropic phase with $z = 1$. Such a term is given by

$$-c^2\int dtd^Dx\partial_i\phi\partial_i\phi. \tag{1.7}$$

This term is a **relevant deformation**, that is, a deformation of the potential that is not obtained through a detailed balance condition and that influences only the low energy behavior of the theory. Indeed the potential term (1.3) contributes only in the UV. In this way the theory in the IR flows naturally to the fixed point with dynamical critical exponent $z = 1$ accidentally restoring Lorentz invariance. The constant velocity of light is then originated from the dimensionful coupling constant of the relevant deformation term.

The theory just described can be used as a model to construct an anisotropic gravitational theory as proposed by Hořava [1]. Such a theory will not have exactly the same properties as the Lifshitz-scalar, as we will see in the following section, but it will have some appealing properties that make the theory a possible candidate for a UV-completion of GR.

1.2 Hořava-Lifshitz gravity: Building the Theory

The main idea behind Hořava's choice to consider a theory with anisotropic rescaling properties is that different potential terms correspond to coupling constants with different conformal dimension. This means that there exists a particular class of potential terms for which the coupling constant is dimensionless or with negative conformal dimension producing a theory power counting renormalizable.

The Hořava-Lifshitz (HL) model introduced in [1, 2], describing a gravity in which time and space are not treated in the same way, is constructed on a space-time of the form $M = \mathbb{R} \times \Sigma$ where Σ is a space-like and, for simplicity, a compact D-dimensional surface while $\mathbb{R}$ parametrizes the time. The topological structure of the manifold M contains in addition a foliation structure $\mathcal{F}$ or, more precisely, a codimensional-one foliation (see [4, 5]):

A ***codimension-q foliation*** *$\mathcal{F}$ on a d-dimensional manifold M means that there exists an atlas (y^a, x^i) with $a = 1, ..., q$ and $i = 1, ...d - q$, such that the transition function acts as follows*

$$x^i \to \tilde{x}^i = \tilde{x}^i(x, y) \qquad y^a \to \tilde{y}^a = \tilde{y}^a(y),$$

that is, we consider only the subgroup of the diffeomorphisms that leave unchanged the foliation structure.

Therefore in the Hořava-Lifshitz theory we consider only the diffeomorfisms

$$x^i \to \tilde{x}^i = \tilde{x}^i(x, t) \qquad t \to \tilde{t} = \tilde{t}(t).$$

In the context of a metric theory with a space-like foliation the metric tensor $g_{\mu\nu}$ can be globally decomposed in terms of its **Arnowitt-Deser-Misner** (ADM) components:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i & N_j \\ N_i & h_{ij} \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix} \tag{1.8}$$

where $h_{ij}(x,t)$ is the metric on Σ and $N(x,t)$ and $N_i(x,t)$ are called, respectively, lapse and shift variables.

In principle such a metric makes sense only if there exists a space-time structure or at least if a particle would move along the geodesic associated to $g_{\alpha\beta}$ and, in general, this is not necessarily the case (ch. 4). In the Hořava-Lifshitz model, General Relativity should be recovered in the IR limit and hence also the spacetime must be seen as emerging from the theory in the low energy limit; therefore we expect that the fields of such a theory should be $h_{ij}(x,t), N(x,t), N_i(x,t)$ and that they should describe a metric only in the IR limit. This turns out to be a problematic point due to the survival of an extra unstable scalar mode in the theory. In general, and in particular in the UV limit, the only quantity that can be interpreted as a metric is h_{ij} on Σ.

We can now start constructing a model similar to the Lifshitz-scalar, taking into account that the Hilbert-Einstein action in terms of the ADM components of the metric is given by

$$S_{HE} = \frac{1}{16\pi Gc}\int \sqrt{g} d^4x [R - 2\Lambda_E] = \frac{1}{16\pi Gc}\int dt d^3x N\sqrt{h}[K_{ij}K^{ij} - K^2 + c^2\mathcal{R} - 2\Lambda_E] \tag{1.9}$$

where

$$K_{ij} = \frac{1}{2N}[\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i], \tag{1.10}$$

is the **extrinsic curvature**, ∇_i is the covariant derivative compatible with the metric h_{ij}, $\mathcal{R}$ is the curvature of Σ constructed from h_{ij} and Λ_E is the cosmological constant.

Similarly to the Lifshitz scalar we can construct the action for the gravitational field as

$$S_{HL} = S_K - S_V$$

with

$$S_K = \frac{2}{k^2}\int dtd^3x\sqrt{h}NK_{ij}G^{ijkl}K_{kl}$$

$$S_V = \frac{k^2}{2}\int dtd^3x\sqrt{h}NE^{ij}G_{ijkl}E^{kl}$$

where G^{ijkl} and its inverse G_{ijkl} are a generalization of the De Witt metric (in which $\lambda = 1$) and are defined by

$$G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}, \quad G_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) - \frac{\lambda}{D\lambda - 1}g_{ij}g^{kl}$$

$$G^{ijkl}G_{klmn} = \frac{1}{2}(\delta^i_m\delta^j_n + \delta^j_m\delta^i_n).$$

This choice for the kinetic term

$$S_K = \frac{2}{\kappa^2}\int dtd^3x\sqrt{h}N(K_{ij}K^{ij} - \lambda K^2)$$

is due to the fact that K_{ij} is, respect to the foliation preserving diffeomorphisms, a covariant generalization of the time derivative of the field h_{ij}; λ is introduced because, quantizing the theory, we should require a general coupling constant for each term. In GR $\lambda = 1$ and this value does not change in the quantization process because it is preserved by the whole diffeomorphism invariance, contrarily to HL gravity.

The choice of the potential term will fix the UV critical point and hence its conformal dimension is crucial to construct a power-counting renormalizable theory.

From the kinetic term we deduce the following relations between the conformal dimensions of the fields and the coupling constant κ:

$$[h_{ij}]_s + z = [N_i]_s + 1 \qquad [K]_s = [h_{ij}]_s + z - [N]_s$$

$$D + z + 2[\kappa]_s = \frac{[h_{ij}]_s}{2} + [N]_s + 2[K]_s + 2[h_{ij}]_s = \frac{9[h_{ij}]_s}{2} - [N]_s + 2z.$$

Being h_{ij} a metric we can postulate $[h_{ij}]_s = 0$ and $[N]_s = 0$ obtaining (2.24)

$$[N_i]_s = z - 1 \quad [\kappa]_s = \frac{z - D}{2}.$$

As consequence we have that the coupling constant κ is dimensionless in the case $z = D$.

Hence, to gain power-counting renormalizability, we need to construct a theory with a potential term such that the Lifshitz fixed point in the UV regime has $z = D$ (for $z > D$ becomes power-counting super-renormalizable) and that in the IR case flows to a theory with a fixed point with dynamical critical constant $z = 1$ reproducing the isotropy. Moreover the action in the low energy limit must reproduce the Hilbert-Einstein action, that is the potential term should reduce to the $3D$ Ricci scalar plus, eventually, the cosmological constant as in (1.9).

Following the construction of the Lifshitz scalar, the potential part can be constructed using the detailed balance condition, that is,

$$\sqrt{h}E^{ij} = \frac{\delta W}{\delta h_{ij}}.$$

Therefore the starting point is to construct W as the most general D-dimensional Euclidean Gravity invariant under spatial diffeomorphims. The choice of W depends essentially on the spatial dimension D. Let us consider the case $D = 3$ which corresponds to $z = 3$.

A possible candidate is the Chern-Simons character

$$W = \frac{1}{w^2}\int \mathrm{Tr}\left[{}^{(3)}\Gamma \wedge d^{(3)}\Gamma + \frac{2}{3}{}^{(3)}\Gamma \wedge^{(3)}\Gamma \wedge^{(3)}\Gamma\right]$$

where the Γ's are the Christoffel symbols relative to h_{ij}.

The theory at $z = 3$ contains λ, κ, w as dynamical coupling constants, but only w^2 controls the strength of the interactions: the IR limit correspond to $w^2 \to \infty$ and the UV limit to $w^2, \kappa^2 \to 0$ with

$$\gamma = \frac{\kappa^2}{w^2}$$

finite. Therefore in the UV limit we have a two-parameter family, in terms of λ and γ, of free-field fixed points.

Evaluating the variation we find

$$E^{ij} = \frac{2}{w^2}C^{ij}$$

where

$$C^{ij} = \varepsilon^{ikl}\nabla_k\left({}^{(3)}R^j_l - \frac{1}{4}{}^{(3)}\delta^j_l\right)$$

is the Cotton tensor[1] on Σ. Using the traceless and the symmetry properties we obtain the following potential term

$$S_V = \frac{\kappa^2}{2w^4}\int dtd^3x\sqrt{h}NC^{ij}C_{ij}.$$

[1]The Cotton tensor has conformal weight $-5/2$, that is,

$$h \to e^{2\Omega}g \quad \Rightarrow \quad C \to e^{-5\Omega}C$$

and the following properties

$$C^{ij} = C^{ji} \qquad C^{ij}h_{ij} = 0 \qquad \nabla_i C^{ij} = 0.$$

The conformal dimension of the cotton tensor is $[C]_s = 3$ giving $[w]_s = 0$. Therefore the added term has a conformal dimension of $6 = 2D$.

Such a theory in the IR limit should reduce just to the kinetic term and hence we must consider the presence of other terms in W relevant in the IR limit to make HL-gravity a UV extension of GR. This will be described in sec. 1.3. Moreover the correct IR limit can be recovered only if the dynamical coupling constant λ flows to 1 in the IR limit.

A running λ may become 1 in the $z = 1$ fixed point, but other issues prevent the Hořava-Lifshitz theory from recovering Einstein gravity, such as the persistence of strong coupling and an extra scalar mode [6, 7], which will not be analyzed here.

1.2.1 Gravity and Detailed Balance Condition

Although the detailed balance in the context of Hořava gravity is just a prescription to generate potentials, it does imply some properties, although they do not have a direct correspondence with the one of the Lifshitz scalar (sec. 1.1).

To see the main consequence of the detailed balance condition, let us consider a Wick rotation $t \to i\tau$ of the above Hořava-Lifshitz action

$$S = i \int d\tau d^3x \sqrt{h} N \left(\frac{1}{\kappa} K_{ij} - \frac{\kappa}{2w^2} C_{ij} \right) G^{ijkl} \left(\frac{1}{\kappa} K_{kl} - \frac{\kappa}{2w^2} C_{kl} \right)$$

where we used the traceless property of the Cotton tensor, $C^{ij} h_{ij} = 0$, and the fact that the mixed term $K_{ij} G^{ijkl} C_{kl}$ is a total derivative:

$$\frac{1}{w^2} \int d\tau d^3x \sqrt{h} N K_{ij} G^{ijkl} C_{kl} = \frac{1}{2w^2} \int d\tau d^3x \sqrt{h} (\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i) C^{ij} =$$

$$= \int d\tau d^3x [\dot{h}_{ij} \frac{\delta W}{\delta h_{ij}} + \frac{1}{w^2} \sqrt{h} \nabla_i (N_j C^{ij})] = \int d\tau d^3x [\dot{\mathcal{L}} + \frac{1}{w^2} \partial_i (\sqrt{h} N_j C^{ij})]$$

where we used the symmetry and the transverse invariance, $\nabla_i C^{ij} = 0$, of the

Cotton tensor and where $\mathcal{L}$ is the Lagrangian corresponding to the action W.

Introducing an auxiliary field B_{ij} we can rewrite the action as follows

$$S = i \int d\tau d^3x \sqrt{h} N \left\{ B^{ij} \left(\frac{1}{\kappa} K_{ij} - \frac{\kappa}{2w^2} C_{ij} \right) - B^{ij} G_{ijkl} B^{kl} \right\}.$$

The coefficient of the term linear in B^{ij} is the gradient flow equation for h_{ij}

$$\dot{h}_{ij} = \frac{\kappa}{\sqrt{h}} \frac{\delta W}{\delta h_{ij}} + \nabla_i N_j + \nabla_j N_i;$$

this is the consequence of the detailed balance condition.

For the Lifshitz scalar the action W on the D-dimensional space was a solution of the Schrödinger equation of the $D+1$-dimensional theory, in particular it corresponded to the ground state. It is not possible to construct a similar solution of the Schrödinger equation associated to the Hořava-Lifshitz gravity in canonical quantization because the state

$$\psi_0[h_{ij}] = e^{-\frac{1}{2w^2} \in \mathrm{Tr}(\Gamma \wedge \Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma)}$$

is unphysical. This means only that the ground state of the theory has not such a simple form.

1.3 Relevant Deformations

We concentrated on terms in the action with the highest possible engineering dimension. These terms determine the behavior at the fixed point ($z = 3$) such terms are called **marginal**.

Adding terms involving lower dimensional operators, called **relevant** terms

$$S = \underbrace{\int dt d^D x \sqrt{h} \sum_{[O_J]=2D} \lambda_J O_J}_{marginal} + \underbrace{\int dt d^D x \sqrt{h} \sum_{[O_A]<2D} \lambda_A O_A}_{relevant}$$

still preserves the power counting renormalizability

Such terms are quite arbitrary, the only request being the compatibility with the symmetries of the theory, as the foliation preserving diffeomorphisms. Such terms, by general arguments from effective field theory, will be generated by quantum effects and will dominate in the long distance behavior (IR); therefore the most simple way to determine the presence of such terms is to request a certain IR behavior that, in our case, corresponds in the request that the theory flows naturally to the relativistic scale invariance $z = 1$ reducing to the Hilbert-Einstein action.

One way to add relevant terms is to modify directly the spatial action W. In this case the detailed balance condition is preserved. The other possibility is to add relevant terms, not obtainable from the detailed balance procedure and that do not modify the IR limit, directly in the action; in this way the UV limit is still described by a potential term derived from a detailed balance condition. In this case we will say that the detailed balance condition is **softly broken**.

1.3.1 Relevant Deformation Satisfying the Detailed Balance Condition

To reproduce the Hilbert-Einstein action, respecting the detailed balance condition, we can consider the following relevant deformed action to obtain the potential part of the S_{HL} action:

$$W = \frac{1}{w^2} \int w_3(\Gamma) + \mu \int d^3 x \sqrt{h} ({}^{(3)}R - 2\Lambda_W).$$

By construction we have:

$$[\mu]_s = 1 \qquad [\Lambda_W]_s = 2$$

and the full **Hořava-Lifshitz action**, in order of descending dimension, becomes

$$\begin{aligned} S &= \int dtd^3x\sqrt{h}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij}-\lambda K^2)-\frac{\kappa^2}{2w^4}C_{ij}C^{ij}+\frac{\kappa^2\mu}{2w^2}\varepsilon^{ijk}R_{il}\nabla_j R^l_k \right.\\ & \left. -\frac{\kappa^2\mu^2}{8}R_{ij}R^{ij}+\frac{\kappa^2\mu^2}{8(1-3\lambda)}\left(\frac{1-4\lambda}{4}R^2+\Lambda_W R-3\Lambda_W^2\right)\right\}. \end{aligned} \tag{1.11}$$

In the IR limit, $w \to \infty$ and the terms quadratic in the curvature go to zero so the HL-action reduces to

$$S = \int dtd^3x\sqrt{h}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij}-\lambda K^2)+\frac{\kappa^2\mu^2}{8(1-3\lambda)}\Lambda_W\left(R-3\Lambda_W\right)\right\}$$

that, compared with the Hilbert-Einstein action

$$S_E = \frac{1}{16\pi Gc}\int\sqrt{g}d^4x[R-2\Lambda_E] = \frac{1}{16\pi Gc}\int\sqrt{h}d^4xN[K_{ij}K^{ij}-K^2+c^2\mathcal{R}-2\Lambda_E],$$

gives the emergent velocity of light

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}} \qquad [c]_s = (z-1), \tag{1.12}$$

the emergent Newton constant

$$G_N = \frac{\kappa^2}{32\pi c} \qquad [G_N]_s = -2 \tag{1.13}$$

and the cosmological constant

$$\Lambda = \frac{3}{2}\Lambda_W. \tag{1.14}$$

The theory then describes a space-time with a cosmological constant. In particular, looking at (1.12), it is evident that $\Lambda \geq 0$ if $\lambda \leq 1/3$ and $\Lambda < 0$ if $\lambda > 1/3$. With an analytic continuation of the theory

$$\mu \to i\mu \qquad w^2 \to -iw^2$$

we can reverse the situation, that is, $\Lambda \leq 0$ if $\lambda \leq 1/3$ and $\Lambda > 0$ if $\lambda > 1/3$.

Moreover from the IR limit of the Hořava-Lifshitz action it is also evident that it does not correspond to the Hilbert-Einstein action for $\lambda \neq 1$. The quantity λ becomes a running constant once we quantize the theory[2], therefore it is necessary to study the RG equation to verify that $\lambda = 1$ in the $z = 1$ fixed point.

1.4 Graviton Propagator

In Einstein's theory of Gravity the main problem in employing usual quantum field theory techniques is that the coupling constant G_N has dimension -2 in mass units yielding a theory that is not power counting renormalizable; moreover the graviton propagator is

$$\frac{1}{w^2 - k^2}$$

and, at increasing loop orders, requires counterterms of ever-increasing degree in the curvature. The theory can still be treated as an effective field theory but it needs an UV completion.

In the Hořava-Lifshitz model, as consequence of the non-relativistic value of the dynamical critical exponent z, the dimension of the coupling constants that determines the UV behavior makes the theory power counting renormalizable.

[2]In GR the condition $\lambda = 1$ is preserved by the full covariance.

Schematically the graviton propagator will take the form [2]

$$\frac{1}{w^2 - c^2k^2 + ... - G(k^2)^z}$$

where in general there will be also other powers of k^2 between 1 and z in the denominator, but they are important in our discussion.

In the IR limit we can expand the propagator as

$$\frac{1}{w^2 - c^2k^2 - G(k^2)^z} = \frac{1}{w^2 - c^2k^2} + \frac{1}{w^2 - c^2k^2}G(k^2)^z\frac{1}{w^2 - c^2k^2} + ...$$

reproducing the expected graviton propagator. Moreover no higher time derivatives are generated as usually happens for theories with higher powers in the curvature.

In the UV limit, instead, we can expand the propagator as follows

$$\frac{1}{w^2 - c^2k^2 - G(k^2)^z} = \frac{1}{w^2 - G(k^2)^z} + \frac{1}{w^2 - G(k^2)^z}c^2k^2\frac{1}{w^2 - G(k^2)^z} + ...$$

where the short distance behavior is improved by a suitable choice of the scale factor z that changes the critical dimension at which the theory is power countable renormalizable. This then shows that HL-gravity may have a better UV behavior.

1.5 The Scalar Field Theory

The next step is to introduce a scalar field in the theory.

Because of the reduced symmetry we can add new non-relativistic terms to the action, the implications of which will be analyzed in chapter 4 where the kinematics of point-like particles, derived as optical limit of this scalar field, is analyzed.

To introduce a scalar field we can follow the same steps as for the Lifshitz scalar or for the HL-action, that is, we start from a kinetic term, that is, a "time derivative" of the field and then we add the most general potential term.

Because we want to recover the Klein-Gordon action in the IR limit let us start looking to the Klein Gordon action for a free massless scalar theory in terms of the ADM components of the metric:

$$S = \int d^4x\sqrt{-g}\left[\frac{\sqrt{h}}{N}\left(\partial_t\phi - \frac{N^i}{c}\partial_i\phi\right)^2 + h^{ij}\partial_i\phi\partial_j\phi\right].$$

Here we are using the usual notation, the ∂_i's indicate derivatives with respect to the coordinates on the space-like surface.

Then it is clear that, in order to recover the relativistic theory in the IR limit, we have to consider the following kinetic term

$$\frac{1}{N^2}(\partial_t\phi - \frac{N^i}{c}\partial_i\phi)^2$$

while the second term, after an integration by part, rewritten as $\phi\Delta\phi$ - where $\Delta = h^{ij}\nabla_i\nabla_j$ -, will be included, together with the eventual mass term, in the potential part, containing only space derivatives.

In [8] the authors suggest the following general action for a scalar field theory

$$S = \frac{1}{2}\int dtd^dx\sqrt{h}N\left[\frac{1}{N^2}(\partial_t\phi - N^i\partial_i\phi)^2 - \sum_{J\geq 2} O_J \star \phi^J\right] \tag{1.15}$$

where

$$O_J = \sum_{n=0}^{n_J}(-1)^n\frac{\lambda_{J,n}}{M^{2n+\frac{d-1}{2}J-d-1}}\Delta^n$$

and the $\star$ product contains all the possible combinations in the application of $\Delta = h^{ij}\nabla_i\nabla_j$ to ϕ; e.g.

$$\Delta^2 \star \phi^3 = c_1(\Delta\phi)^2\phi + c_2\phi^2\Delta^2\phi$$

with c_1, c_2 constants. The $\lambda_{J,n}$ are energy dimensionless coupling constants, while M has the dimension of the energy. From the kinetic term we get

$$[\phi]_s = \frac{d-z}{2}.$$

Therefore in the case $z = d$ the scalar field is dimensionless then every power of ϕ is allowed in the interaction term.

The scaling dimension of the coupling constant will be

$$[\lambda_{J,n}]_s = z + d + \frac{z-d}{2}J - 2n.$$

Then, to have a power counting renormalizable theory, we must impose the condition $[\lambda_{J,n}]_s \leq 0$ that implies

$$n_J = \max\left\{\frac{z+d}{2} + \frac{z-d}{4}J\right\}.$$

If $z < d$ then $J < 2\frac{z+d}{d-z}$. If $z \geq d$ then J is unbounded and hence we can have interaction terms of any power in the field; in particular if $z = d$ there is no dependence from J and $n_J = z = d$.

We are interested in the evaluation of the free field theory so we will consider only the effective mass term, that is, only the case $J = 2$:

$$S_M = \frac{1}{2}\int d^4x\sqrt{h}N\left\{\frac{1}{N^2}(\partial_t\phi - N^i\partial_i\phi)^2 - \sum_{n=0}^{z}(-1)^n\frac{\lambda_{2,n}}{M^{2(n-1)}}\Delta^n \star \phi^2\right\}. \quad (1.16)$$

The case studied in [8] contains an analysis of the $J = 2$ case restricted to a Minkowskian metric, that is, $N = 1$, $N^\alpha = 0$ and $h_{ij} = \delta_{ij}$.

To study the transformation properties of a matter field we generalize the action (1.16) to the following diffeomorphism-invariant action

$$\begin{aligned} S_M &= \frac{1}{2}\int d^4x\sqrt{-g}\left\{-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* - \lambda_{2,0}M^2\phi\phi^* + \left(\lambda_{2,1}-\frac{1}{2}\right)(\phi\Delta\phi^* + \phi^*\Delta\phi)\right. \\ &\quad \left. -\sum_{n=2}^{z}\sum_{k=0}^{n}(-1)^n\frac{\lambda_{2,n,k}}{M^{2(n-1)}}\Delta^{n-k}\phi\Delta^k\phi^*\right\} \end{aligned} \tag{1.17}$$

where $\lambda_{2,n,k} = \lambda_{2,n,n-k}$ to make the action real. The action (1.17) it is a generalization of the Klein-Gordon action where the terms involving the various λ's - excluding the term $\lambda_{2,0}M^2\phi\phi^*$ that corresponds to the usual mass term in the Klein-Gordon equation - must be considered as small corrections.

Note that the action (1.17) reduces to the original action (1.16) for a Minkowski space-time in ADM coordinates, which corresponds to the case studied in [8].

In section 4.1 we will find the optical limit approximation of the scalar theory (1.17) obtaining the general equation of motion of massive and massless particles. In particular we will study the kinematics in a flat space-time in section 4.2 and the dynamics in a static spherical symmetric space-time in section 4.3.

Chapter 2

Variations on Hořava-Lifshitz Theory

The original action that goes under the name of Hořava-Lifshitz action was first introduced by P. Hořava in [2] mostly as described in Chapter 1. The expression is the following (1.11):

$$\begin{aligned} S &= \int dt d^3x \sqrt{h} N \left\{ \frac{2}{k^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{k^2}{2w^4} C_{ij}C^{ij} + \frac{k^2\mu}{2w^2} \varepsilon^{ijk} R_{il} \nabla_j R^l_k \right. \\ & \left. - \frac{k^2\mu^2}{8} R_{ij}R^{ij} + \frac{k^2\mu^2}{8(1-3\lambda)} \left(\frac{1-4\lambda}{4} R^2 + \Lambda_W R - 3\Lambda_W^2 \right) \right\} . \end{aligned} \tag{2.1}$$

The initial theory has been modified in several versions for different reasons: zero-Cosmological constant limit, generalizations without the use of the detailed balance condition, a "healthy extension", a covariant generalization, etc..

Other than these modifications, some of them suggested by Hořava himself in [2], Hořava distinguished between two possible branches of theories: the projectable and the non-projectable theory.

This main distinction has to do with the way in which we want to constrain the lapse function N: if N is a space-time dependent function the theory is called

non-projectable, if it is a function of t only will give rise to **projectable** theories.

In this chapter we will explain how the projectable case comes out naturally and why it's physically odd, at least once we relate the theory to GR, and why we preferred to choose a non-projectable theory for our studies. Moreover we will introduce some of the interesting reformulation of Hořava gravity working out in deeper details the one we used in the rest of the chapters. We will also introduce the covariant formulation for the original Hořava gravity [9, 10] achieved introducing a Stuckelberg field, and the canonical formulation of the theory.

2.1 Symmetries and Projectability

On a geometrical ground the main difference between GR and HL gravity is in the symmetries of space-time. General Relativity is covariant under a generic change of coordinates while, because of the anisotropic rescaling properties, Hořava gravity is invariant only under a foliation preserving diffeomorphism

$$\tilde{x}^i = \phi^i(x^i, t) \qquad \tilde{t} = \psi(t) \tag{2.2}$$

that preserves the distinction between "space" and "time". Then it comes natural to use the ADM decomposition of the metric (1.8) to study the fields of the theory.

Having in mind that we would like to recover GR in the infrared limit, we need to require that the lapse function N, the shift vector N_i and the spatial metric h_{ij} transform under a foliation preserving diffeomorphism (2.2) in the same way as their combined field, the metric, transforms

$$g_{\mu\nu}(x,t) \to g_{\alpha\beta}(\tilde{x},\tilde{t})\partial_\mu \Phi^\alpha \partial_\nu \Phi^\beta$$

and to consider its non-relativistic limit ($c \to \infty$). The result is then:

$$\begin{aligned}
\delta h_{ij} &= \partial_i \psi^k h_{jk} + \partial_j \psi^k h_{ik} + \psi^k \partial_k h_{ij} + \phi \dot{h}_{ij} \\
\delta N_i &= N_k \partial_i \psi^k + \psi^k \partial_k N_i + \dot{\psi}^k h_{ki} + \dot{\phi} N_i + \phi \dot{N}_i \\
\delta N &= \psi^k \partial_k N + \dot{\phi} N + \phi \dot{N}.
\end{aligned} \tag{2.3}$$

As consequence of the invariance under a foliation preserving transformation we obtain the following conservation laws for the ADM components of the energy-momentum tensor:

$$\dot{h}_{ij}\tau^{ij} - \frac{N}{\sqrt{h}}\frac{d}{dt}(\sqrt{h}\rho) - \frac{N_i}{\sqrt{h}}\frac{d}{dt}(\sqrt{h}v^i) = 0 \tag{2.4}$$

$$2\nabla^i \tau_{ij} - \rho \nabla_j N + \frac{1}{\sqrt{h}}\frac{d}{dt}(\sqrt{h}v_j) + N_j \nabla_i v^i + 2v^i \nabla_{[i} N_{j]} = 0. \tag{2.5}$$

As we will see later the lapse function is related to the normalization of the time-direction orthogonal to the spatial hypersurface Σ, therefore it comes natural to consider the case in which such a field is just time dependent considering that this choice is preserved by the time reparametrization $t \to \psi(t)$.

In what will follow in chapter 3 and 4 we will not consider the projectable theory because it is overly constrained and does not reproduce the Schwarzschild spherically symmetric solution.

2.1.1 The Projectable Theories

The shift vector N_i and the lapse function N can be easily interpreted as the gauge fields associated, respectively, to the time-dependent space reparametrization and to the time reparametrization. Therefore it comes natural to consider only time-dependent lapse functions as fields. A theory with such a restriction is called **projectable**.

Because of the restricted time symmetry due to the physical foliation, the symmetry group has one less generator with respect to GR; the projectable case then represents a way to match this constriction reducing the degree of freedom of N [1].

In terms of the canonical variables (sec. 2.5) we have that the momenta of N and N_i are identically zero (respectively $\mathcal{H}$ and $\mathcal{H}_i$) giving the two constraints

$$\mathcal{H}_i = 0$$

and, in the projectable case,

$$\int d^D x \mathcal{H} = 0;$$

only in the case in which N has a full space-time dependence this constraints corresponds to the GR's energy constraint $\mathcal{H} = 0$. In the projectable case then the energy constraint becomes a global constraint while in GR it is local. This makes clear that the projectable case will not recover GR in a naive IR limit. A possible way out was described in [11].

Then in the projectable case we have that the constraints form a Lie algebra [1]

$$[\int d^D x \mathcal{H}(x), \int d^D y \mathcal{H}(y)] = 0 \qquad [\int d^d \zeta^i(x) \mathcal{H}_i(x), \int d^D y \mathcal{H}(y)] = 0.$$

From this analysis it is possible to show [12] that the expected number of degrees of freedom of the theory are [1]:

$$\sharp(DoF) = \frac{\sharp(\text{field components}) - 2 \times \sharp(\text{first-class constraints})}{2} = \frac{(D+1)(D-2)}{2} + 1,$$

[1]Note that, using the gauge freedom in rescaling time, we can always set the lapse function to 1.

that is one more than GR. The corresponding scalar degree of freedom is called **khronon**[2].

An unappealing feature is the strong coupling of the extra scalar mode at low energy [6, 7, 13] which might make the theory unstable in the present formulation; moreover the unstable scalar mode does not decouple in the IR limit, therefore GR is not really recovered. Let us analyze this point.

As suggested in [13] the projectable condition can be forced introducing in the action the term

$$\int d^3x dt \sqrt{h} N \frac{M^4}{4}\left(\frac{1}{N^2}-1\right)^2$$

that constraints N^2 to be equal to 1 in the limit $M \to \infty$. Neglecting the terms with higher spatial derivatives the Hořava action can be rewritten as (the covariant formulation will be introduced in sec 2.4)

$$\frac{1}{2}\int d^4x\sqrt{-g}\Bigg[-M_P^2 R + \frac{M^4}{2}(\partial_\mu\chi\partial^\mu\chi - 1)^2 + \\ -(\lambda-1)\frac{M_P^2}{\partial_\alpha\chi\partial^\alpha\chi}\left(\Box\chi - \frac{\partial^\mu\chi\partial^\nu\chi}{\partial_\alpha\chi\partial^\alpha\chi}\nabla_\mu\nabla_\nu\chi\right)\Bigg]$$

where χ is the khronon. Considering the following perturbation with respect to flat space-time

$$N = 1+\phi, \qquad N_i = \frac{\partial_i}{\sqrt{\Delta}}B, \qquad h_{ij} = \delta_{ij} - 2\left(\delta_{ij} - \frac{\partial_i\partial_j}{\Delta}\right)\psi - 2\frac{\partial_i\partial_j}{\Delta}E \qquad \chi = t+\theta,$$

at the quadratic level we have

$$S_I^{(2)} = \int d^4x \left[\frac{M_P^2}{2}\left(-2\dot\psi^2 - 2\psi\Delta\psi + 4\phi\Delta\psi + 4\psi\sqrt{\Delta}\dot B + 4\psi\ddot E\right) + \right. \\ \left. + M^4(\dot\theta - \phi)^2 - (\lambda-1)\frac{M_P^2}{2}\left(\sqrt{\Delta}B + \dot E + 2\dot\psi + \Delta\theta\right)^2\right], \tag{2.6}$$

[2] Later the khronon will be simply called foliation.

where the first line is just the linearized Hilbert-Einstein action. Evaluating the equation of motion for $\dot{E}$ and $\sqrt{\Delta}B$ and substituting them back into the action we have

$$\mathcal{L}^{(2)} = \frac{M_P^2}{2}\left\{\frac{2(3\lambda-1)}{\lambda-1}\dot{\psi}^2 - 2\psi\Delta\psi\right\}$$

that produces the following dispersion relation

$$\omega^2 = -\frac{\lambda-1}{(3\lambda-1)}p^2.$$

Asking the positivity of the kinetic term, that is, imposing

$$\frac{(3\lambda-1)}{\lambda-1} > 0,$$

we obtain, from the dispersion relation, that ϕ must be tachyonic, at least at low energies where the higher powers of p^2 do not contribute. Considering also the term p^4 it is possible to show that, being such a term a positive contribution to the dispersion relation [13], the instability does not develop at least in the Universe life-time if

$$|\lambda - 1| \lesssim 10^{-61}. \tag{2.7}$$

Under this condition we can neglect the term $\dot{\psi}^2$ with respect to $\psi\Delta\psi$ in (2.6).

To study the khronon we are going to choose the gauge

$$B = 0, \qquad 2\psi + E = 0$$

that reduces the action to

$$S_I^{(2)} = \int d^4x \left[M_P^2\left(-\psi\Delta\psi + 2\phi\Delta\psi\right) + M^4(\dot{\theta}-\phi)^2 - (\lambda-1)\frac{M_P^2}{2}\left(\Delta\theta\right)^2\right].$$

The equation of motion for ψ then is $\psi = \phi$; substituted back in the action yields

$$S_I^{(2)} = \int d^4x \left[M_P^2 \phi \Delta \phi + M^4 (\dot{\theta} - \phi)^2 - (\lambda - 1) \frac{M_P^2}{2} (\Delta\theta)^2 \right].$$

The equation of motion for ϕ becomes

$$\phi = \frac{M^4 \dot{\theta}}{M_P^2 + M^4}$$

that substituted back in the action gives

$$S_I^{(2)} = \int d^4x \left[\frac{M_P^2 M^4}{M_P^2 + M^4} \dot{\theta} \Delta \dot{\theta} - (\lambda - 1) \frac{M_P^2}{2} (\Delta\theta)^2 \right].$$

The leading interaction term is given by the cubic interaction with the smallest number of time derivatives and, following the treatment used for the quadratic part, it will correspond to

$$-\int d^4x \frac{M_P^2 M^4}{M_P^2 \Delta + M^4} \Delta \dot{\theta} (\partial_i \theta)^2.$$

Therefore, in the projectable limit $M^4 \to \infty$, the action reduces to

$$\int d^4x \left[M_P^2 \dot{\theta} \Delta \dot{\theta} - (\lambda - 1) \frac{M_P^2}{2} (\Delta\theta)^2 - M_P^2 \Delta \dot{\theta} (\partial_i \theta)^2 \right]$$

and can be set into canonical form with the rescaling

$$t = \frac{t'}{\sqrt{(\lambda - 1)}} \qquad \theta = \frac{\theta'}{M_P (\lambda - 1)^{1/4}}$$

from which we get

$$\int d^4x \left[\dot{\theta}' \Delta \dot{\theta}' - \frac{1}{2} (\Delta\theta')^2 - \frac{1}{M_P (\lambda - 1)^{3/4}} \Delta \dot{\theta}' (\partial_i \theta')^2 \right];$$

from this expression we ca read out the cutoff

$$\Lambda_p = M_P(\lambda - 1)^{3/4}$$

that under the constraint (2.7) goes to zero, showing that in the IR limit the khronon is strongly coupled.

Such a problem is also present in the non-projectable case and there are proposals to solve it [13, 14].

2.1.2 The non-Projectable Theories

The non-projectable case, on the other hand, is more tractable although it still suffers from a strong coupling problem. This is particularly evident if we consider also the presence of a matter term S_m [6].

Let us consider the action

$$S = S_{GR} + S_{UV} + S_m$$

where

$$\begin{aligned} S_{GR} &= \frac{1}{16\pi G c} \int dt d^3x \sqrt{h} N [K_{ij}K^{ij} - K^2 - c^2(\mathcal{R} - 3\Lambda)] \\ S_{UV} &= \int dt d^3x \sqrt{h} N \left[\frac{\kappa}{2}(1-\lambda)K^2 - \frac{\kappa^2}{8} E^{ij} G_{ijkl} E^{kl} \right]. \end{aligned}$$

Expanding around a background metric $(\overline{N} + n, \overline{N}_i + n_i, \overline{h}_{ij} + \gamma_{ij})$, the matter term at the linear level becomes

$$S_m \simeq -\int dt dx^3 \sqrt{\overline{h}\overline{N}}[n\rho + n_i v^i + \gamma_{ij}\tau^{ij}]$$

while

$$S_{UV} \simeq \int dtd^3x\sqrt{hN}\left[\frac{\kappa}{2}(1-\lambda)\frac{1}{4\overline{N}^2}\left(\dot{\gamma}-2\nabla^i n_i\right)^2-\frac{\kappa^2}{8}\left(\frac{1}{\omega^2}\epsilon^{kl(i}\nabla_k\psi_l^{j)}-\frac{\mu}{2}\psi^{ij}\right)\overline{G}_{ijkl}e^{kl}\right]$$

where

$$\psi^{ij}=-\frac{1}{2}\nabla^2(\gamma^{ij}-\gamma\overline{g}^{ij})+\nabla^{(i}\nabla_k\gamma^{j)k}-\frac{1}{2}\nabla^i\nabla^j\gamma+\overline{g}^{ij}\nabla_k\nabla_l\gamma^{kl}+\frac{\Lambda}{2}\gamma^{ij}.$$

The transformation rules (2.3)

$$N \rightarrow N+\zeta^k\nabla_k N+\dot{f}N+f\dot{N} \tag{2.8}$$

$$N_i \rightarrow N_i+\nabla_i(\zeta^j N_j)-2\zeta^j\nabla_{[i}N_{j]}+\dot{\zeta}^j h_{ij}+\dot{f}N_i+f\dot{N}_i \tag{2.9}$$

$$h_{ij} \rightarrow h_{ij}+2\nabla_{(i}\zeta_{j)}+f\dot{h}_{ij} \tag{2.10}$$

describe the symmetries of Hořava gravity. We want to use the Stuckelberg formalism, therefore we consider an extension of this symmetry treating f as a space-time dependent coefficient to generate the Stuckelberg field. In this case the transformation rules, considering an expansion around a time-independent 3-dimensional metric with $N_i = 0$, that corresponds to a particular choice of gauge, become

$$n \rightarrow n+\zeta^k\nabla_k\overline{N}+\dot{f}\overline{N}+f\dot{\overline{N}} \tag{2.11}$$

$$n_i \rightarrow n_i+\dot{\zeta}^j\overline{h}_{ij}-\overline{N}^2c^2\nabla_i f \tag{2.12}$$

$$\gamma_{ij} \rightarrow \gamma_{ij}+2\nabla_{(i}\zeta_{j)}. \tag{2.13}$$

Because the time-dependent space diffeomorphim corresponding to the field ζ^k is a symmetry also of the generally covariant theory (2.4.3), it does not produce any

Stuckelberg field, therefore we can just redefine the fields as follows

$$n \rightarrow n + \dot{\phi}\overline{N} + \phi\dot{\overline{N}} \tag{2.14}$$

$$n_i \rightarrow n_i - \overline{N}^2 c^2 \nabla_i \phi \tag{2.15}$$

where $\phi(x,t)$ is the Stuckelberg field. The action then reduces to

$$-\int dt dx^3 \sqrt{h}\overline{N}c^2 \Big[\frac{\kappa^2}{2}(1-\lambda)\left(\frac{1}{\overline{N}^2}\nabla^i(\overline{N}^2\nabla_i\phi)(\dot{\gamma} - 2\nabla^i n_i) + \frac{c^2}{\overline{N}^2}(\nabla^i(\overline{N}^2\nabla_i\phi))^2\right) + \\ -\frac{\phi}{\overline{N}}\nabla_i(\overline{N}^2 v^i)\Big]$$

where we considered that S_{GR} does not give any contribution because it is invariant under a general diffeomorphism, that only the first term of S_{UV} might contribute in the IR limit and that

$$\frac{\overline{N}}{\sqrt{h}}\frac{d}{dt}(\sqrt{h}\rho) = 0$$

as consequence of the energy-momentum conservation laws (2.4), (2.5).

The corresponding equation of motion is

$$\kappa^2(1-\lambda)\nabla_i\left[\overline{N}^2\nabla^i\left(\frac{1}{2\overline{N}}(\dot{\gamma} - 2\nabla^i n_i) + \frac{c^2}{\overline{N}^2}\nabla^j(\overline{N}^2\nabla_j\phi)\right)\right] = \nabla_i(\overline{N}^2 v^i).$$

Unlike GR where $\nabla_\mu T^{\mu\nu} = 0$ is satisfied, the right hand-side of the above equation is not zero implying that the limit $\lambda \rightarrow 1$ is not continuous; therefore the theory is strongly coupled in the IR limit: the Stuckelberg field does not decouple and hence Hořava gravity does not recover GR in the IR limit.

We are going to consider anyway a non-projectable theory for our discussions in the following chapters because it recovers GR solutions for $\lambda = 1$.

2.2 To Balance or not to Balance

Initially Hořava introduced the detailed balance condition to reduce the number of parameters in the theory giving a recipe to produce possible potential terms [2]. More general expressions were considered in [15, 16, 17].

In particular Kiritsis and Kofinas in [15] studied more general solutions considering the Hořava-Lifshitz action with generic (independent) coupling constants

$$S = \int dt d^3x \sqrt{h} N \Big[\alpha (K_{ij}K^{ij} - \lambda K^2) + \beta C_{ij}C^{ij} + \gamma \eta^{ijk} \mathcal{R}_{il} \nabla_j \mathcal{R}^l{}_k + \\ + \zeta \mathcal{R}_{ij}\mathcal{R}^{ij} + \eta \mathcal{R}^2 + \xi \mathcal{R} + \sigma \Big] , \tag{2.16}$$

in the detailed balance case the constants correspond to

$$\alpha_{DB} = \frac{2}{\kappa^2} \qquad \beta_{DB} = -\frac{\kappa^2}{2w^4} \qquad \gamma_{DB} = \frac{\kappa^2 \mu}{2w^2} \qquad \zeta_{DB} = -\frac{\kappa^2\mu^2}{8}$$

$$\eta_{DB} = \frac{\kappa^2\mu^2}{8(1-3\lambda)}\frac{1-4\lambda}{4} \qquad \xi_{DB} = \frac{\kappa^2\mu^2}{8(1-3\lambda)}\Lambda_W \qquad \sigma_{DB} = -3\frac{\kappa^2\mu^2}{8(1-3\lambda)}\Lambda_W^2 .$$

Such an action is not the most general allowed by the foliation preserving symmetry, it is indeed possible to consider terms with six spatial derivatives arriving to cubic terms in the spatial curvature. In [15] such terms are not considered to study the deformations with respect to the original HL gravity with (softly) broken detailed balance.

The emerging constants for this action become

$$c = \sqrt{\frac{\xi}{\alpha}} \qquad G = \frac{1}{16\pi}\frac{1}{\sqrt{\alpha\xi}} \qquad \Lambda = -\frac{\sigma}{2\xi} .$$

In most of the cases the theory acquires specific properties or, at least, results to be easier in the case the detailed balance is employed (e.g. the case $3\zeta + 8\eta$ with $\zeta\eta \neq 0$ in [15] corresponds to the detailed balance with $\lambda = 1$). Whether the detailed balance is important or just a guiding principle to generate easier potentials is not clear until now.

2.3 The Kehagias-Sfetsos Action

Although, as described in the previous section, there is no true guiding principle to construct the potential term there is GR as constraint on the IR behavior of the theory. With the aim to recover GR and hence the spacetime isotropy, Hořava introduces, through the detailed balance condition, a linear term in the curvature and a constant term in the action (1.11). With this construction we need necessarily the presence of a cosmological constant to recover GR as it is evident from (1.12).

In his second germinal paper [2], Hořava suggests also possible modifications, in terms of soft violations, of his action. The authors of [18] developed Hořava's suggestion studying the solution of the so called Kehagias-Sfetsos action:

$$\begin{aligned} S &= \int dtd^3x\sqrt{h}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij}-\lambda K^2)-\frac{\kappa^2}{2w^4}C_{ij}C^{ij}+\frac{\kappa^2\mu}{2w^2}\epsilon^{ijk}\mathcal{R}_{il}\nabla_j\mathcal{R}^l{}_k+\right.\\ &\left.-\frac{\kappa^2\mu^2}{8}\mathcal{R}_{ij}\mathcal{R}^{ij}+\frac{\kappa^2\mu^2}{8(1-3\lambda)}\left(\frac{1-4\lambda}{4}\mathcal{R}^2+\Lambda_W\mathcal{R}-3\Lambda_W^2\right)+\mu^4\mathcal{R}\right\}. \end{aligned} \tag{2.17}$$

The soft breaking of the detailed balance is achieved introducing the term $\mu^4\mathcal{R}$ by hand in the action; therefore the cosmological constant Λ_W is no longer necessary. With this action is then possible to study asymptotically flat solutions under the well-behaved limit

$$\Lambda_W \to 0;$$

the theory therefore admits a Minkowski vacuum. This is the theory that has a central role in our work.

For this action we obtain the following emergent velocity of light, the emergent

Newton constant and the cosmological constant:

$$c = \frac{\kappa\mu}{4}\sqrt{8\mu^2 + \frac{\kappa^2\Lambda_W}{1-3\lambda}}, \qquad G_N = \frac{\kappa^2}{32\pi c}, \qquad \Lambda = \frac{3\kappa^2\Lambda_W^2}{16(1-3\lambda)\mu^2 + 2\kappa^2\Lambda_W};$$

in the limit $\Lambda_W \to 0$ they reduce to

$$c = \frac{\sqrt{2}\kappa\mu^2}{2}, \qquad G_N = \frac{\kappa^2}{32\pi c}, \qquad \Lambda = 0.$$

Several aspects of the Kehagias-Sfetsos action were analyzed in the literature: cosmological solutions [19, 20, 21], possible tests [22, 23, 24, 25, 26], fundamental aspects of the theory [6, 7, 8, 27, 28, 29, 30, 31, 32, 33, 34, 35], black hole solutions [15, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46], the specific case of $\lambda = 1/3$ [47] and possible extensions of the theory [48, 49].

We are mostly interested in the spherical solutions relative to the KS-action. The static spherically symmetric solution with zero-shift vector was derived first in [18]. Such a solution is not the most general one, as it happens in GR, because of the reduced coordinate invariance symmetry, as it is observed in sec. 3.1.

2.4 The Covariant Action

The first formulation of Hořava gravity [1, 2] was based on an explicit distinction between space and time where, essentially, the only object with a "geometric interpretation" was the space-like surface Σ endowed with a metric h_{ij}. The Lapse function and the shift vector were interpreted as two external fields.

A covariant approach, with its own geometrical interpretation, would make easier the study of differences between GR and Hořava gravity.

Let us start reviewing the ADM formalism, then we can introduce the covariant scalar-tensor theory.

2.4.1 The ADM formulation of GR

The Arnowitt-Deser-Misner (ADM) formalism was introduced to obtain a Hamiltonian formulation for GR. The main ingredient is the existence of a global space-like foliation for the space-time. Such a foliation will allow a definition of a time direction, giving the basis for a Hamiltonian formulation.

Let us then assume that such a global space-like foliation exists. The time t then identifies the specific space-like hypersurface Σ_t (that from now on we will just write as Σ) in the foliation. From a covariant point of view what is geometrically relevant is the time-like direction field t^α identified with the vector field tangent to the time-like *time flow* generated by points of Σ during the time evolution, and the space-like hypersurfaces Σ. The time direction t^α can be decomposed as

$$t^\alpha = N^\alpha + N n^\alpha \qquad (N^\alpha n_\alpha = 0) \tag{2.18}$$

where N^α is called **shift vector** and is defined as the projection of t^α on Σ while N, called **Lapse function**, is the component of the projection of t^α along the time-like direction n^α orthonormal to Σ. With this structure the metric $g_{\alpha\beta}$ can be easily decomposed into [10]

$$g_{\alpha\beta} = h_{\alpha\beta} - n_\alpha n_\beta \tag{2.19}$$

where $h_{\alpha\beta}$ corresponds to the embedded metric of the space-like surface Σ.

It is evident then that, raising one index with the metric $g_{\alpha\beta}$, we can interpret ${h_\alpha}^\beta$ and $-n_\alpha n^\beta$, respectively, as the projector on Σ and its orthogonal projector (${\delta_\alpha}^\beta = {h_\alpha}^\beta - n_\alpha n^\beta$).

Considering that we are going to interpret $h_{\alpha\beta}$ as the "metric on Σ" we need to define a covariant derivative D_α with respect to this metric and compatible with it. Such a derivative D_α is just the projection of the covariant derivative on the space-like surface Σ; i.e

$$D_\alpha T^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m} = h_\alpha{}^\beta h_{\mu_1}{}^{\lambda_1}\ldots h_{\mu_m}{}^{\lambda_m} h_{\rho_1}{}^{\nu_1}\ldots h_{\rho_n}{}^{\nu_n}\nabla_\beta T^{\rho_1\ldots\rho_n}_{\lambda_1\ldots\lambda_m}. \tag{2.20}$$

Using the orthogonality of n_α with respect to the space-like hypersurface Σ, that is, $h_\beta{}^\alpha n_\alpha = 0$, it is simple to show that $D_\alpha h^{\mu\nu} = 0$.

The foliation is completely determined by a scalar function χ, indeed setting χ equal to a constant k (later such a constant will be identified with the time t after a choice of coordinates) we identify a hypersurface Σ; the continuous set of hypersurfaces then corresponds to the foliation. Hence the hortogonal vector to the hypersurfaces is directly identified with

$$n_\alpha = -N\partial_\alpha\chi \qquad \text{where} \quad n_\alpha n^\alpha = -1, \quad N = \frac{1}{\sqrt{-\partial_\alpha\chi\partial^\alpha\chi}}. \tag{2.21}$$

Note that $n_\alpha n^\alpha = -1$ implies

$$n^\gamma\nabla_\alpha n_\gamma = n_\gamma\nabla_\alpha n^\gamma = \nabla_\alpha(n^\gamma n_\gamma) - n^\gamma\nabla_\alpha n_\gamma = -n^\gamma\nabla_\alpha n_\gamma,$$

that is,

$$n^\gamma\nabla_\alpha n_\gamma = 0. \tag{2.22}$$

The defining condition $n_\alpha = -N\partial_\alpha\chi$ verifies the Frobenius integrability condition[3]

$$\mathcal{F}_{\mu\nu} = D_{[\mu}n_{\nu]} = 0, \tag{2.23}$$

which means that a zero vorticity condition for n_α is satisfied; this is a necessary condition to have a foliation structure.

[3]The Frobenius theorem proves that the algebra of fields on Σ must be close for Σ to be a manifold.

From the definition

$$D_\mu n_\nu = h_\mu{}^\alpha h_\nu{}^\beta \nabla_\alpha n_\beta = h_\mu{}^\alpha \nabla_\alpha n_\nu = \nabla_\mu n_\nu + n_\mu n^\alpha \nabla_\alpha n_\nu$$

we can rewrite the integrability condition as

$$0 = D_{[\mu} n_{\nu]} = \nabla_{[\mu} n_{\nu]} + n^\alpha \nabla_\alpha n_{[\nu} n_{\mu]}.$$

Having that $\chi = k$ identifies a particular hypersurface in the foliation along the time direction t^α, we may interpret k as a time parameter, its monotonic increase being the time evolution between one hypersurface Σ to another. The easiest choice is to set $t = k$. For the moment t is still just a parameter and will not be interpreted as a coordinate because in this case we would break the diffeomorphism invariance. From this choice we have that

$$n_\alpha = -N\nabla_\alpha \chi = -N\nabla_\alpha t$$

and, recalling that (2.19)

$$t^\alpha = N^\alpha + N n^\alpha,$$

we deduce

$$-N = t^\alpha n_\alpha = -N t^\alpha \nabla_\alpha t$$

that implies

$$t^\alpha \nabla_\alpha t = 1, \qquad N^\alpha \nabla_\alpha t = 0;$$

therefore the choice to parametrize the hypersurfaces with the time parameter t gives also as return a natural parameter for the curves describing the time flow evolution. Multiplying the expression $n_\alpha = -N\nabla_\alpha t$ by n_α, we find an other relation to construct the lapse function:

$$N = (n^\alpha \nabla_\alpha t)^{-1}. \tag{2.24}$$

Using this structure it is possible to decompose the Riemannian and the Ricci tensor as well as the Ricci scalar with respect to n_α and Σ. In the decomposition two kind of curvatures can be found: the **intrinsic curvature**, the curvature of Σ, and the **extrinsic curvature**.

The extrinsic curvature is defined as the Lie derivative of the spatial metric along the orthogonal direction to Σ:

$$K_{\alpha\beta} = \frac{1}{2}\mathcal{L}_n h_{\alpha\beta}.$$

From the definition we have

$$K_{\mu\nu} = \frac{1}{2}(n^\gamma \nabla_\gamma h_{\mu\nu} + h_{\gamma\nu}\nabla_\mu n^\gamma + h_{\mu\gamma}\nabla_\nu n^\gamma) = \frac{1}{2}(n^\gamma \nabla_\gamma (n_\mu n_\nu) + \nabla_\mu n_\nu + \nabla_\nu n_\mu) =$$

$$= n^\gamma \nabla_\gamma n_{(\mu} n_{\nu)} + \nabla_{(\mu} n_{\nu)} = \nabla_\gamma n_{(\mu} h_{\nu)}{}^\gamma = h^\rho_{(\mu} h_{\nu)}{}^\gamma \nabla_\gamma n_\rho = D_{(\mu} n_{\nu)}$$

where we used (2.22). From the integrability condition (2.23) we then have

$$K_{\mu\nu} = D_\mu n_\nu.$$

Note the extrinsic curvature $K_{\alpha\beta}$ is orthogonal to n^α by construction

$$0 = \frac{1}{2}\mathcal{L}_n(h_{\alpha\beta}n^\beta) = K_{\alpha\beta}n^\beta + h_{\alpha\beta}\mathcal{L}_n n^\beta = K_{\alpha\beta}n^\beta.$$

The previous expression of the extrinsic curvature does not involve explicitly the fields N, N_α and $h_{\alpha\beta}$ but an explicit expression can be obtained

$$K_{\alpha\beta} = \frac{1}{2}\mathcal{L}_n h_{\alpha\beta} = \frac{1}{2}(n^\gamma \nabla_\gamma h_{\alpha\beta} + h_{\alpha\gamma}\nabla_\beta n^\gamma + h_{\gamma\beta}\nabla_\alpha n^\gamma) =$$

$$= \frac{1}{2N}(Nn^\gamma \nabla_\gamma h_{\alpha\beta} + Nh_{\alpha\gamma}\nabla_\beta n^\gamma + Nh_{\gamma\beta}\nabla_\alpha n^\gamma) =$$

$$= \frac{1}{2N}[Nn^\gamma \nabla_\gamma h_{\alpha\beta} + h_{\alpha\gamma}\nabla_\beta (Nn^\gamma) + h_{\gamma\beta}\nabla_\alpha (Nn^\gamma)]$$

where we used $n^\gamma h_{\gamma\beta} = 0$; from the decomposition $Nn^\gamma = t^\gamma - N^\gamma$ we deduce

$$K_{\alpha\beta} = \frac{1}{2N}[\mathcal{L}_t h_{\alpha\beta} - \mathcal{L}_N h_{\alpha\beta}] = \frac{1}{2N}[\dot{h}_{\alpha\beta} - D_\alpha N_\beta - D_\beta N_\alpha]$$

where $\mathcal{L}_t h_{\alpha\beta} = \partial_t h_{\alpha\beta} \equiv \dot{h}_{\alpha\beta}$ by definition and $\mathcal{L}_N h_{\alpha\beta} = D_\alpha N_\beta + D_\beta N_\alpha$ is obtained writing the Lie derivative in terms of the covariant derivative D_α.

The Riemannian curvature $\mathcal{R}^\delta{}_{\gamma\alpha\beta}$ on Σ can be defined as the commutator of spatial covariant derivatives acting on a 1-form on Σ:

$$[D_\alpha, D_\beta]w_\gamma = -\mathcal{R}^\delta{}_{\gamma\alpha\beta} w_\delta.$$

Following the definition (2.20) we have

$$D_\alpha D_\beta w_\gamma = h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\delta \nabla_\mu [h_\nu{}^\rho h_\delta{}^\sigma \nabla_\rho w_\sigma] =$$

$$= h_\alpha{}^\mu h_\beta{}^\rho h_\gamma{}^\sigma \nabla_\mu \nabla_\rho w_\sigma + h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\sigma \nabla_\mu [h_\nu{}^\rho] \nabla_\rho w_\sigma + h_\alpha{}^\mu h_\beta{}^\rho h_\gamma{}^\delta \nabla_\mu [h_\delta{}^\sigma] \nabla_\rho w_\sigma$$

$$= h_\alpha{}^\mu h_\beta{}^\rho h_\gamma{}^\sigma \nabla_\mu \nabla_\rho w_\sigma + h_\gamma{}^\sigma K_{\alpha\beta} n^\rho \nabla_\rho w_\sigma + h_\beta{}^\rho K_{\alpha\gamma} n^\sigma \nabla_\rho w_\sigma$$

that implies

$$\mathcal{R}^\delta{}_{\gamma\alpha\beta} = \left[h_\alpha{}^\mu h_\beta{}^\rho h_\gamma{}^\sigma R^\lambda{}_{\sigma\mu\rho} + h_{[\beta}{}^\rho K_{\alpha]\gamma} \nabla_\rho n^\lambda\right] h_\lambda{}^\delta = h_\lambda{}^\delta h_\alpha{}^\mu h_\beta{}^\rho h_\gamma{}^\sigma R^\lambda{}_{\sigma\mu\rho} + K_{[\beta}{}^\delta K_{\alpha]\gamma},$$

where we used $n^\alpha w_\alpha = 0$, w_α being defined on Σ.

The Ricci tensor will be

$$\mathcal{R}_{\gamma\beta} = h_\lambda{}^\mu h_\beta{}^\rho h_\gamma{}^\sigma R^\lambda{}_{\sigma\mu\rho} + K_\beta{}^\alpha K_{\alpha\gamma} - K_\alpha{}^\alpha K_{\beta\gamma},$$

while the Ricci scalar is

$$\mathcal{R} = h_\lambda{}^\mu h^{\rho\sigma} R^\lambda{}_{\sigma\mu\rho} + K_\beta{}^\alpha K_\alpha{}^\beta - K_\alpha{}^\alpha K_\beta{}^\beta$$

$$= R + 2n^\lambda n^\mu R_{\lambda\mu} + K_\beta{}^\alpha K_\alpha{}^\beta - K_\alpha{}^\alpha K_\beta{}^\beta = 2n^\lambda n^\mu G_{\lambda\mu} + K_\beta{}^\alpha K_\alpha{}^\beta - K_\alpha{}^\alpha K_\beta{}^\beta.$$

On the other hand

$$n^\alpha n^\beta R_{\alpha\beta} = n^\beta R^\gamma{}_{\alpha\gamma\beta} n^\alpha = n^\beta [\nabla_\gamma, \nabla_\beta] n^\gamma = n^\beta \nabla_\gamma \nabla_\beta n^\gamma - n^\beta \nabla_\beta \nabla_\gamma n^\gamma =$$

$$= -\nabla_\gamma n^\beta \nabla_\beta n^\gamma + \nabla_\gamma [n^\beta \nabla_\beta n^\gamma] + \nabla_\beta n^\beta \nabla_\gamma n^\gamma - \nabla_\beta [n^\beta \nabla_\gamma n^\gamma]$$

$$= -K_\gamma{}^\beta K_\beta{}^\gamma + K_\beta{}^\beta K_\gamma{}^\gamma + \nabla_\gamma [n^\beta \nabla_\beta n^\gamma] - \nabla_\beta [n^\beta \nabla_\gamma n^\gamma]$$

therefore we have the following expression for the Ricci scalar:

$$R = \mathcal{R} + K_\beta{}^\alpha K_\alpha{}^\beta - K_\alpha{}^\alpha K_\beta{}^\beta - 2\nabla_\gamma [n^\beta \nabla_\beta n^\gamma] + 2\nabla_\beta [n^\beta \nabla_\gamma n^\gamma].$$

(By construction also the intrinsic curvature is orthogonal to n^α.)

From the expression of the Ricci tensor we can directly write the GR action in the ADM decomposition

$$S_{HE} = \frac{1}{16\pi G} \int \sqrt{g} d^4 x R = \frac{1}{16\pi G} \int dt d^3 x N \sqrt{h} [K_{\alpha\beta} K^{\alpha\beta} - K^2 + \mathcal{R}]$$

in which we neglected the pure divergence terms.

2.4.2 Coordinate Expressions

The particular choice of coordinates that allows us to recover the expressions used in the original formulation of Hořava gravity is the gauge choice

$$x^0 = t.$$

In this particular choice of coordinates the time direction t^α reduces to the vector $(1,0,0,0)$ and

$$n_\alpha = -N\nabla_\alpha\chi = -N\partial_\alpha t = (-N,0,0,0).$$

From the above expression and the orthogonality relation $N^\alpha n_\alpha = 0$ we readily deduce

$$N^\alpha = (0, N^i)$$

which give us a simple way to find the expression for n^α

$$n^\alpha = \frac{1}{N}(t^\alpha - N^\alpha) = \left(\frac{1}{N}, -\frac{N^i}{N}\right).$$

Therefore from the relation (2.20)

$$\delta_\alpha{}^\beta = h_\alpha{}^\beta - n_\alpha n^\beta$$

we have

$$-n_\alpha n^\beta = \begin{pmatrix} 1 & -N^i \\ 0 & 0 \end{pmatrix}$$

and hence the expression for the projector on Σ becomes

$$h_\alpha{}^\beta = \begin{pmatrix} 0 & N^i \\ 0 & \delta^i_j \end{pmatrix}. \tag{2.25}$$

Note that the index β in the projector above will take only spatial values because the 0-components are zero; this allow us to substitute $h_\alpha{}^\beta \to h_\alpha{}^i$. This implies that every tensor defined only on Σ, that is, such that each index contracted with n^α gives zero, can always be written in terms of its associated spatial components

$$T_{\alpha_1 \dots \alpha_n} = h_{\alpha_1}^{\ \ \beta_1} \dots h_{\alpha_1}^{\ \ \beta_1} T_{\beta_1 \dots \beta_n} = h_{\alpha_1}^{\ \ i_1} \dots h_{\alpha_1}^{\ \ i_1} T_{i_1 \dots i_n}.$$

In particular we have

$$h_{\alpha\beta} = h_\alpha{}^i h_\beta{}^j h_{ij}.$$

The next step is to construct the tensor $h^{\alpha\beta}$. It can be easily obtained using the fact that $h^{\alpha\beta}$ is orthogonal to n_β

$$h^{\alpha\beta} n_\beta = (g^{\alpha\beta} + n^\alpha n^\beta) n_\beta = g^{\alpha\beta} n_\beta - n^\alpha = 0$$

where

$$0 = h^{\alpha\beta} n_\beta = -N h^{\alpha 0};$$

hence the only non-zero components are the spatial components h^{ij}:

$$h^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & h^{ij} \end{pmatrix}.$$

From here we are finally able to write the inverse metric $g^{\alpha\beta}$

$$g^{\alpha\beta} = h^{\alpha\beta} - n^\alpha n_\beta = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}$$

which inverted gives

$$g_{\alpha\beta} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & h_{ij} \end{pmatrix},$$

where we defined h_{ij} as the inverse metric[4] of h^{ij} ($h_{ij}h^{jk} = \delta_i^k$) and $N_i = h_{ij}N^j$; from the above expression we can then read

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta = \begin{pmatrix} N_i N^i & N_j \\ N_i & h_{ij} \end{pmatrix}$$

and evaluate

$$N_\alpha = h_{\alpha\beta}N^\beta = (N_k N^k, N_i).$$

The coordinate expression of the extrinsic curvature can be obtained from the definition

$$K_{\alpha\beta} = \frac{1}{2}\mathcal{L}_{\mathbf{n}} h_{\alpha\beta} = \frac{1}{2N}(\mathcal{L}_{\mathbf{t}} h_{\alpha\beta} - \mathcal{L}_{\mathbf{N}} h_{\alpha\beta})$$

where

$$\mathcal{L}_{\mathbf{t}} h_{ab} = \partial_t h_{ab} \qquad \mathcal{L}_{\mathbf{N}} h_{ab} = N^i \partial_i h_{ab} + h_{ib}\partial_a N^i + h_{ai}\partial_b N^i,$$

that is,

$$K_{\alpha\beta} = \frac{1}{2N}\left[\begin{pmatrix} \partial_t(N^k N_k) & \partial_t N_j \\ \partial_t N_i & \partial_t h_{ij} \end{pmatrix} - \begin{pmatrix} N^k \partial_k (N^l N_l) & N^k \partial_k N_j \\ N^k \partial_k N_i & N^k \partial_k h_{ij} \end{pmatrix} + \right.$$

$$\left. - \begin{pmatrix} \partial_t N^k N_k & \partial_t N^k h_{kj} \\ N_k \partial_i N^k & \partial_i N^k h_{kj} \end{pmatrix} - \begin{pmatrix} \partial_t N^k N_k & N_k \partial_j N^k \\ \partial_t N^k h_{ki} & \partial_j N^k h_{ki} \end{pmatrix}\right] =$$

[4] The spatial tensor h^{ij} is a metric; indeed it is symmetric by construction and is non-degenerate as consequence of the non-degeneracy of $g^{\alpha\beta}$.

$$= \frac{1}{2N}\begin{pmatrix} N^kN^l\partial_t(h_{kl}) - N^k\partial_k(N^lN_l) & N^k\partial_t h_{kj} - N^k\partial_k N_j - N_k\partial_j N^k \\ N^k\partial_t h_{ki} - N^k\partial_k N_i - N_k\partial_i N^k & \partial_t h_{ij} - N^k\partial_k h_{ij} - \partial_i N^k h_{kj} - \partial_j N^k h_{ki} \end{pmatrix}$$

$$= \begin{pmatrix} N^kN^lK_{kl} & N^kK_{kj} \\ N^kK_{ki} & K_{ij} \end{pmatrix} = h_\alpha{}^i h_\beta{}^j K_{ij}$$

where

$$K_{ij} \equiv \frac{1}{2N}\Big(\partial_t h_{ij} - N^k\partial_k h_{ij} - \partial_i N^k h_{kj} - \partial_j N^k h_{ki}\Big) = \frac{1}{2N}\left(\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_i\right)$$

and ∇_i is the covariant derivative with respect to the metric h_{ij} and corresponds with the coordinate expressions of D_α.

Now let us consider the ingredients we need for the covariantization of HL gravity.

The trace $K_\alpha{}^\alpha = K_{\alpha\beta}h^{\beta\alpha}$ is given by

$$K_{\alpha\beta}h^{\alpha\beta} = h_\alpha{}^i h_\beta{}^j K_{ij} h^{\alpha\beta} = K_{ij}h^{ij} = \frac{1}{2N}\Big(h^{ij}\partial_t h_{ij} - h^{ij}N^k\partial_k h_{ij} - 2\partial_i N^i\Big)$$

or, equivalently, by

$$K_{\alpha\beta}h^{\alpha\beta} = \frac{1}{2N}\left(h^{ij}\partial_t h_{ij} - 2\nabla_i N^i\right) = h^{ij}K_{ij}.$$

The product $K_{\alpha\beta}K^{\alpha\beta}$ is given by

$$K_{\alpha\beta}K^{\alpha\beta} = h_\alpha{}^i h_\beta{}^j K_{ij}K^{\alpha\beta} = K_{ij}K^{ij}.$$

Note that

$$K_{\alpha\beta}K^{\alpha\beta} - \lambda K^2 = K_{ij}K^{ij} - \lambda (K^i{}_i)^2$$

reproduces exactly the kinetic term in the Hořava-Lifshitz action. The same can be said for the intrinsic curvature and for the Cotton tensor appearing in the potential term of Hořava gravity, noticing that

$$\mathcal{R}_{\alpha\beta} = h_\alpha{}^i h_\beta{}^j \mathcal{R}_{ij} \qquad \mathcal{R} = \mathcal{R}_{\alpha\beta} g^{\alpha\beta} = \mathcal{R}_{ij} h^{ij}$$

and that

$$C_{\alpha\beta} = h_\alpha{}^i h_\beta{}^j C_{ij}$$

where the covariant expression of the Cotton tensor is given by

$$C^{\mu\nu} = \eta^{\mu\alpha\beta} D_\alpha \left[\mathcal{R}^\nu{}_\beta - \frac{1}{4}\mathcal{R}\delta^\nu{}_\beta \right]$$

with $\eta^{\mu\alpha\beta} \equiv \eta^{\mu\alpha\beta\delta} n_\delta$ the 3-dimensional volume form.

2.4.3 A Scalar-Tensor Theory

The reduced symmetries in Hořava gravity, due to a particular choice of foliation, can be interpreted in terms of a new degree of freedom, with respect to GR, that is responsible for the breaking of the symmetry. Such a degree of freedom corresponds with the foliation χ which is the only ingredient we need to construct the remaining geometrical objects we need for the ADM decomposition. The field χ then will be the scalar degree of freedom together with the metric $g_{\alpha\beta}$. Such a field enters in the action thorough the vector n_α:

$$n_\alpha = -\frac{\partial_\alpha \chi}{\sqrt{-\partial_\gamma \chi \partial^\gamma \chi}};$$

Considering $g_{\alpha\beta}$ and χ as the two independent degrees of freedom of the theory, n_α results automatically normalized without the use of a Lagrangian multiplier [9].

Here we are going to consider only the IR limit of the theory, this being the only part we need to discuss about regarding the covariantization (the UV terms, once rewritten in a covariant form, can be easily integrated in the following discussion):

$$S_{(IR)} = \int \sqrt{-g} d^4x (K_{\alpha\beta}K^{\alpha\beta} - \lambda K^2 + \mathcal{R});$$

the above relation can be rewritten as

$$S = \int \sqrt{-g} d^4x [(1-\lambda)K^2 + R].$$

From this expression it is clear that only the term K^2 contains the scalar field χ, being the only term involving the normal vector n_α.

The term K^2 contains also the field $g_{\alpha\beta}$ therefore the variation of this term with respect to $g_{\alpha\beta}$ will be like an energy-momentum tensor for the scalar field χ.

The variation of the action $\int \sqrt{-g} d^4x K^2$ gives

$$\delta \int \sqrt{-g} d^4x K^2 = \int \sqrt{-g} d^4x \left\{ \frac{1}{2} K^2 g^{\alpha\beta} \delta g_{\alpha\beta} + \right.$$

$$\left. +2K \left[-\nabla^\alpha n^\beta \delta g_{\alpha\beta} - (n^\alpha g^{\delta\beta} + n^\beta g^{\alpha\delta} - n^\delta g^{\alpha\beta}) \frac{\nabla_\delta \delta g_{\alpha\beta}}{2} + g^{\alpha\beta} \nabla_\alpha \delta_n n_\beta \right] \right\} =$$

$$= \int \sqrt{-g} d^4x \left\{ \frac{1}{2} K^2 g^{\alpha\beta} \delta g_{\alpha\beta} - 2K \nabla^\alpha n^\beta \delta g_{\alpha\beta} + \right.$$

$$\left. +(2n^{(\alpha}\nabla^{\beta)}K - g^{\alpha\beta} n^\delta \nabla_\delta K) \delta g_{\alpha\beta} + K(2\nabla^{(\beta}n^{\alpha)} - K g^{\alpha\beta}) \delta g_{\alpha\beta} - 2\nabla^\beta K \delta_n n_\beta \right\} =$$

$$= \int \sqrt{-g} d^4x \left\{ \left(-\frac{1}{2} K^2 g^{\alpha\beta} + 2n^{(\alpha}\nabla^{\beta)}K - g^{\alpha\beta} n^\delta \nabla_\delta K \right) \delta g_{\alpha\beta} - 2\nabla^\beta K \delta_n n_\beta \right\}.$$

Noting that

$$\delta n_\alpha = -\frac{h_\alpha{}^\beta \partial_\beta \delta\phi}{\sqrt{-\partial_\gamma \phi \partial^\gamma \phi}} - \frac{1}{2} n_\alpha n^\mu n^\nu \delta g_{\mu\nu}$$

we finally have

$$= \int \sqrt{-g} d^4x \left\{ \left(-\frac{1}{2} K^2 g^{\alpha\beta} + 2n^{(\alpha} \nabla^{\beta)} K - (g^{\alpha\beta} - n^\alpha n^\beta) n^\delta \nabla_\delta K \right) \delta g_{\alpha\beta} \right.$$
$$\left. + 2\nabla_\alpha K \frac{h^{\alpha\beta} \partial_\beta \delta\phi}{\sqrt{-\partial_\gamma \phi \partial^\gamma \phi}} \right\}.$$

Therefore

$$\delta S = \delta \int \sqrt{-g} d^4x [(1-\lambda) K^2 + R] =$$

$$= \int \sqrt{-g} d^4x \left\{ (1-\lambda) \left(-\frac{1}{2} K^2 g^{\alpha\beta} + 2n^{(\alpha} \nabla^{\beta)} K - (g^{\alpha\beta} - n^\alpha n^\beta) n^\delta \nabla_\delta K \right) \delta g_{\alpha\beta} + \right.$$
$$\left. -G^{\alpha\beta} \delta g_{\alpha\beta} + 2(1-\lambda) \nabla_\alpha K \frac{h^{\alpha\beta} \partial_\beta \delta\phi}{\sqrt{-\partial_\gamma \phi \partial^\gamma \phi}} \right\}$$

where we used te fact that the $(3+1)$-Ricci tensor R does not depend on the foliation.

The equations of motion then are

$$(1-\lambda) \left(-\frac{1}{2} K^2 g^{\alpha\beta} + 2n^{(\alpha} \nabla^{\beta)} K - (g^{\alpha\beta} - n^\alpha n^\beta) n^\delta \nabla_\delta K \right) - G^{\alpha\beta} = 0 \tag{2.26}$$

$$-2(1-\lambda) \nabla_\beta \left(\frac{h^{\beta\alpha} \nabla_\alpha K}{\sqrt{-\partial_\gamma \phi \partial^\gamma \phi}} \right) = 0. \tag{2.27}$$

Multiplying the first equation by ${h_\alpha}^\mu {h_\beta}^\nu$, ${h_\alpha}^\mu n_\beta$ and $n_\alpha n_\beta$ we have, respectively

$$(1-\lambda) \left(-\frac{1}{2} K^2 h^{\mu\nu} - h^{\mu\nu} n^\delta \nabla_\delta K \right) - G^{\alpha\beta} {h_\alpha}^\mu {h_\beta}^\nu = 0$$

$$(1-\lambda) \left(-{h_\alpha}^\mu \nabla^\alpha K \right) - G^{\alpha\beta} {h_\alpha}^\mu n_\beta = 0$$

$$(1-\lambda) \frac{1}{2} K^2 - G^{\alpha\beta} n_\alpha n_\beta = 0;$$

using the Gauss-Codacci relation

$$D_\alpha K^\alpha{}_\beta - D_\beta K = R_{\mu\nu} n^\nu h^\mu{}_\beta,$$

and

$$G^{\mu\nu} n_\mu n_\nu = \frac{1}{2}(\mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu}),$$

the equations become

$$(1-\lambda)\left(-\frac{1}{2}K^2 h^{\mu\nu} - h^{\mu\nu} n^\delta \nabla_\delta K\right) - G^{\alpha\beta} h_\alpha{}^\mu h_\beta{}^\nu = 0$$

$$D_\alpha (K^{\alpha\mu} - \lambda h^{\alpha\mu} K) = 0$$

$$\mathcal{R} - K_{\mu\nu}K^{\mu\nu} + \lambda K^2 = 0.$$

For $\lambda = 1$ the equations of motion reproduce the equations of General Relativity [10] but, as we already discussed in sec. 2.1.2, the limit $\lambda \to 1$ does not behave well implying that the scalar field does not decouple in the IR regime.

Let us now analyze the connections between the equations of motion in the covariant version and in the ADM context. The full expression of the covariant generalization of the KS action (2.17) is given by[9]

$$\begin{aligned} S_{GKS} =& \int d^4x \sqrt{-g} \left\{ \frac{2}{\kappa^2}(K_{\alpha\beta}K^{\alpha\beta} - \lambda K^2) - \frac{\kappa^2}{2\omega^4} C_{\alpha\beta}C^{\alpha\beta} + \frac{\kappa^2 \mu}{2\omega^2} C_{\alpha\beta}\mathcal{R}^{\alpha\beta} + \right. \\ & \left. - \frac{\kappa^2\mu^2}{8}\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta} + \frac{\kappa^2\mu^2}{8(1-3\lambda)}\left(\frac{1-4\lambda}{4}\mathcal{R}^2 + \Lambda_W \mathcal{R} - 3\Lambda_W\right) + \eta^4 \mathcal{R} \right\} \end{aligned} \tag{2.28}$$

In the ADM formalism we choose the gauge $\phi = t$ then

$$\delta g_{\mu\nu} = \begin{pmatrix} -2N\delta N + 2N^k \delta N_k - N^k N^l \delta h_{kl} & \delta N_j \\ \delta N_i & \delta h_{ij} \end{pmatrix} \qquad \delta n_\mu = (-\delta N, 0, 0, 0).$$

Setting

$$EM_g^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \qquad EM_n^{\alpha} = \frac{\delta\mathcal{L}}{\delta n_{\alpha}}$$

we have

$$\delta\mathcal{L} = EM_n^{\alpha}\delta n_{\alpha} + EM_g^{\mu\nu}\delta g_{\mu\nu} =$$

$$-EM_n^0\delta N + EM_g^{00}(-2N\delta N + 2N^k\delta N_k - N^kN^l\delta h_{kl}) + 2EM_g^{0i}\delta N_i + EM_g^{ij}\delta h_{ij}$$

$$= -(EM_n^0 + 2NEM_g^{00})\delta N + 2(EM_g^{00}N^i + EM_g^{0i})\delta N_i + (EM_g^{ij} - EM_g^{00}N^iN^j)\delta h_{ij}.$$

The relative equation of motion can be obtained projecting the equations of motion for $g_{\alpha\beta}$ and n_{α}.

The equation of motion $EM_g^{\mu\nu} = 0$ multiplied by $n_{\mu}h_{\nu}{}^{i}$ in the gauge $\phi = t$ becomes

$$0 = EM_g^{\mu\nu}n_{\mu}h_{\nu}{}^{i} = -N(EM_g^{0i} + EM_g^{00}N^i)$$

multiplied by $h_{\mu}{}^{i}h_{\nu}{}^{j}$ becomes

$$\begin{aligned} 0 &= EM_g^{\mu\nu}h_{\mu}{}^{i}h_{\nu}{}^{j} = EM_g^{ij} + EM_g^{00}N^iN^j + EM_g^{0j}N^i + EM_g^{i0}N^j \\ &= EM_g^{ij} + (EM_g^{00}N^j + EM_g^{0j})N^i + (EM_g^{i0} + EM_g^{00}N^i)N^j - EM_g^{00}N^iN^j \\ &= EM_g^{ij} - EM_g^{00}N^iN^j \end{aligned}$$

where we used the previous result. The last component is obtained multiplying the equation of motion $EM_g^{\mu\nu} = 0$ for the last projector $n_{\mu}n_{\nu}$ that gives

$$0 = EM_g^{\mu\nu}n_{\mu}n_{\nu} = N^2EM_g^{00}$$

together with the equation of motion for n_{α}.

2.5 The Hamiltonian Formalism

Now that we have a covariant expression it is easier to look for a Hamiltonian formulation of HL gravity. We will again focus only on the IR limit - with a general λ - for the same reasons. The Hamiltonian formulation will follow the same steps as in GR, the only difference being in the modified de Witt metric $G_{\alpha\beta\mu\nu}$ that contains λ.

The Lagrangian describing the HL theory in the IR limit is

$$\mathcal{L} = N\sqrt{h}(K_{\alpha\beta}G^{\alpha\beta\mu\nu}K_{\mu\nu} + \mathcal{R}),$$

where the degree of freedom are N, N_α and $h_{\alpha\beta}$; it easy to show that the only non-zero canonical momentum is the one associated to $h_{\alpha\beta}$

$$\pi^{\alpha\beta} = \frac{\delta\mathcal{L}}{\delta\dot{h}_{\alpha\beta}} = \sqrt{h}K_{\alpha\beta}G^{\alpha\beta\mu\nu} = \sqrt{h}(K^{\mu\nu} - \lambda K h^{\mu\nu}).$$

The Hamiltonian then takes the form

$$H = \int d^3x[\pi^{\alpha\beta}\dot{h}_{\alpha\beta} - \mathcal{L}] = \int d^3x \left[\frac{N}{\sqrt{h}}\pi^{\alpha\beta}G_{\alpha\beta\mu\nu}\pi^{\mu\nu} - N\sqrt{h}\mathcal{R} + 2\pi^{\alpha\beta}D_\alpha N_\beta\right].$$

The variation of the Hamiltonian with respect to N gives the energy constraint

$$\mathcal{H} = \frac{1}{\sqrt{h}}\pi^{\alpha\beta}G_{\alpha\beta\mu\nu}\pi^{\mu\nu} - \sqrt{h}\mathcal{R} = 0$$

that in the Lagrangian formalisms corresponds to the energy constraint

$$0 = \sqrt{h}(K_{\mu\nu}K^{\mu\nu} - \lambda K^2) - \sqrt{h}\mathcal{R} = -\sqrt{h}[(\lambda - 1)K^2 + G_{\alpha\beta}n^\alpha n^\beta];$$

the variation with respect to N_β gives the momentum constraint

$$\mathcal{H}^\beta = -2\sqrt{h}D_\alpha \frac{\pi^{\alpha\beta}}{\sqrt{h}} = 0$$

that in the Lagrangian formalism corresponds to

$$0 = D_\alpha(K^{\alpha\beta} - \lambda K h^{\alpha\beta}).$$

The equation of motion relative to the metric $h_{\alpha\beta}$ are:

$$\dot{h}_{\alpha\beta} = \frac{\delta H}{\delta \pi^{\alpha\beta}} = 2\frac{N}{\sqrt{h}}\pi^{\mu\nu}G_{\mu\nu\alpha\beta} + 2D_{(\alpha}N_{\beta)}$$

$$\dot{\pi}^{\alpha\beta} = -\frac{\delta H}{\delta h_{\alpha\beta}} = -N\sqrt{h}\left(\mathcal{R}^{\beta\delta} - \frac{1}{2}\mathcal{R}h^{\beta\delta}\right) + \frac{1}{2}\frac{N}{\sqrt{h}}h^{\beta\delta}\left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{2}\pi^2\right) +$$

$$-2\frac{N}{\sqrt{h}}\left(\pi^{\beta\nu}\pi_\nu{}^\delta - \frac{1}{2}\pi\pi^{\beta\delta}\right) - 2D_\alpha\pi^{\alpha(\beta}N^{\delta)} + D_\alpha(N^\alpha\pi^{\beta\delta}) - \pi^{\alpha(\beta}D_\alpha N^{\delta)}$$

where the first equation is just the definition of the extrinsic curvature.

The Poisson brackets structure is given by

$$\{N(x), Q_N(y)\} = \delta^3(x-y) \tag{2.29}$$

$$\{N_i(x), Q_N^j(y)\} = \delta_i^j\delta^3(x-y) \tag{2.30}$$

$$\{h_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta^3(x-y) \tag{2.31}$$

while the others are all zero.

Chapter 3

Spherically Symmetric Solutions

In this chapter we study and derive spherically symmetric solutions for the non-projectable Kehagias-Sfetsos action (2.17) with general λ and nonzero shift variables [50, 51]. We call these "hedgehog" solutions, in analogy with the field theoretic soliton configurations of the same name, as they possess radially-pointing "hair" due to the shift field. In the process we uncover conserved quantities for the system, and a special "deformed" gauge invariance for the case $\lambda = 1$. The conformal value $\lambda = \frac{1}{3}$ will also turn out to have special properties. In this context we study the presence of horizons, singularities and the related Hawking temperature.

In the IR limit the infinite set of solutions, found for $\lambda = 1$, corresponds to the invariance of General Relativity under a spacetime reparametrization. In general, not being a coordinate transformation, the symmetry in the action responsible for the infinite set of solutions does not have a clear physical interpretation. Indeed it is broken by the relativistic matter term in the action. In this chapter we also study the behavior of the solutions for generic values of the gauge parameter $g^2(r)$.

3.1 The Spherically Symmetric Ansatz

Since the beginning of studies on Hořava gravity the spherical solution played one of the main roles, being the easier solution - excluding the flat configuration - and because it allows for solar system tests. Between the formulations of Hořava gravity (chapter 2) we focused on the KS-action (2.17) mainly because it allows for asymptotically flat solutions. As already observed in sec. 1.2, there are two relevant parameters: ω that describes the strength of the fields and λ. The first corresponds to the deformation with respect to the IR limit, that is "GR", while the second strictly depends on the behavior of the renormalization. In the attempt to study general solutions in terms of these two parameters we rescaled

$$\mu^2 \to (3\lambda - 1)\mu^2$$

allowing us to recover a nontrivial conformal limit when $\lambda = \frac{1}{3}$. We will also denote the total coefficient of the linear Ricci scalar $\mathcal{R}$ (which receives contributions both from Λ_W in the action and from the added extra terms) as $\omega\kappa^2\mu^2/8$. Finally, we will use the freedom to rescale time and N_i (which amounts to a choice of time units) in order to make the coefficient of the kinetic term equal to the coefficient of the Ricci scalar. This will ensure that in the IR limit the speed of light comes out equal to 1. With these choices, the action (2.17) becomes

$$\begin{aligned} S &= \frac{\kappa^2\mu^2}{8}\int dtd^3x\sqrt{h}N\Big\{\omega(K_{ij}K^{ij} - \lambda K^2) + \frac{4}{\mu w^2}\sqrt{3\lambda-1}\epsilon^{ijk}\mathcal{R}_{il}\nabla_j\mathcal{R}^l{}_k \\ &\quad - \frac{4}{\mu^2 w^4}C_{ij}C^{ij} - (3\lambda-1)\mathcal{R}_{ij}\mathcal{R}^{ij} + \frac{4\lambda-1}{4}\mathcal{R}^2 + 3\Lambda_W^2 + \omega\mathcal{R}\Big\}. \end{aligned} \tag{3.1}$$

(Note that our ω corresponds to $\omega - \Lambda_W$ in [19].) The standard Einstein gravity is recovered in the limit $\omega \to \infty$ and for $\lambda \to 1$; the cosmological constant Λ in this limit is identified as

$$\Lambda = \frac{3\Lambda_W^2}{2\omega}.$$

We shall keep λ arbitrary, as there may be measurable deviations from its general relativistic value ($\lambda = 1$).

The most general static spherically symmetric ansatz involves a spherically symmetric 3-dimensional metric in terms of a radial coordinate r with metric $f^{-1}(r)$ and spherical angles θ, ϕ, a lapse function $N(r)$ depending only on r and a "hedgehog" configuration for the shift vector N_i of the form $N_r = N_r(r)$, $N_\theta = N_\phi = 0$. In this parametrization the metric is

$$ds^2 = -(N^2 - N_r^2 f)dt^2 + 2N_r dtdr + \frac{1}{f}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi). \tag{3.2}$$

In the general relativistic case the term $dtdr$ can be eliminated by an appropriate redefinition $t \to t + F(r)$. In the present case, however, such a transformation is not an invariance of the action, and the variable N_r remains a relevant degree of freedom (app. C); this point was also stressed in [15].

For nonvanishing N_i the kinetic term for h_{ij} (involving the extrinsic curvature) is nonvanishing and must be included in the action. Using the expressions derived in appendix B for the extrinsic and intrinsic curvatures for the spherical metric (3.2), the action (3.1) after integration over the angular part - omitting the trivial integration over t - becomes

$$S = 4\pi\frac{\kappa^2\mu^2}{8}\int dr\,[L_K - L_V] \tag{3.3}$$

where

$$L_V = \frac{N}{\sqrt{f}}\left[(2\lambda-1)\frac{(f-1)^2}{r^2} - 2\lambda\frac{f-1}{r}f' + \frac{\lambda-1}{2}f'^2 - 2\omega(1-f-rf') - 3\Lambda_W^2 r^2\right]$$
$$L_K = \omega\frac{\sqrt{f}}{N}\left[(1-\lambda)\frac{r^2}{f}\left(fN_r' + \frac{f'N_r}{2}\right)^2 + 2(1-2\lambda)fN_r^2 - 4\lambda r\left(fN_r' + \frac{f'N_r}{2}\right)N_r\right]$$

in which prime denotes differentiation with respect to r.

To facilitate the treatment of the problem and identify its essential mathematical structure, we define new fields:

$$p = \frac{1+\omega r^2 - f}{\sqrt{\omega^2 - \Lambda_W^2}\, r^2}, \quad q = \sqrt{\frac{2\omega f}{\omega^2 - \Lambda_W^2}}\, r^2 N_r, \quad M = \frac{N r^3}{\sqrt{f}} \tag{3.4}$$

assuming $\omega > |\Lambda_W|$ (other signs of ω and $\omega^2 - \Lambda_W^2$ can be treated by analytically continuing $p \to ip$ and/or $q \to iq$). We further define a new logarithmic radial coordinate

$$s = \ln r. \tag{3.5}$$

In terms of the new variables and coordinate, the action becomes

$$\begin{aligned} S &= 2\pi\kappa^2\mu^2(\omega^2 - \Lambda_W^2)\int ds\ \mathcal{L} \\ \mathcal{L} &= M\left(\frac{\lambda-1}{2}\dot{p}^2 - 2p\dot{p} - 3p^2 + 3\right) + \frac{1}{M}\left(\frac{\lambda-1}{2}\dot{q}^2 + 2q\dot{q} - 3q^2\right) \end{aligned} \tag{3.6}$$

where overdot denotes derivative with respect to s. Note that the case of unbroken detailed balance (studied in [36]) corresponds to $\omega = -\Lambda_W$, which, upon rescaling of the p and q variables, corresponds to eliminating the term $3M$ from the action (3.6).

For the classical theory the overall coefficient in the action is irrelevant and will be omitted from now on.

In the above form, some features are immediately obvious: the explicit appearance of the radial variable has dropped. Further, the only relevant parameter is λ, all other parameters (such as ω and Λ_W) having been absorbed in field redefinitions. Also note that the fields p (spatial metric) and q (shift vector) enter in the action in a remarkably similar way.

The equations of motion are

$$\begin{aligned}
\frac{\lambda-1}{2}\dot{p}^2 - 2p\dot{p} - 3p^2 + 3 &= \frac{1}{M^2}\left(\frac{\lambda-1}{2}\dot{q}^2 + 2q\dot{q} - 3q^2\right) && (3.7)\\
-M(2\dot{p}+6p) &= \frac{d}{ds}\{M[(\lambda-1)\dot{p} - 2p]\} && (3.8)\\
\frac{1}{M}(2\dot{q}-6q) &= \frac{d}{ds}\left\{\frac{1}{M}[(\lambda-1)\dot{q}+2q]\right\}. && (3.9)
\end{aligned}$$

Upon elimination of M using its (algebraic) equation of motion the above reduce to two coupled second-order differential equations for p and q. The general solution will contain 4 integration constants. The equations of motion, however, are invariant under a simultaneous rescaling of N and N_r, or

$$M \to cM\ , \qquad q \to cq \tag{3.10}$$

for any constant c, corresponding to a rescaling of time in the metric. This can be used to set their scale (usually by requiring $N \to 1$ as $r \to \infty$) thus eliminating one integration constant. The solutions will therefore contain 3 relevant constants, corresponding to the mass of the black hole plus two additional "hair" parameters.

The above equations are invariant under independent changes of sign for M, p and q, so the solution manifold will exhibit this symmetry. The flip $M \to -M$ is inconsequential, since only N^2 appears in the spacetime structure. The flip $q \to -q$ is essentially time reversal and corresponds to inverting the hedgehog direction $N_r \to -N_r$, while the flip $p \to -p$ corresponds to changing the radial metric as $f \to 2 + \omega r^2 - f$.

In addition to the above, the action (3.6) has two radial "invariants", that is, two first integrals of the equations of motion. The first one is obvious: the action is invariant under shifts $s \to s + \epsilon$, that is, under the infinitesimal variations

$$\delta M = \dot{M}, \qquad \delta p = \dot{p}, \qquad \delta q = \dot{q},$$

since $\mathcal{L}$ does not depend explicitly on the parameter s. Under such a transformation the Lagrangian changes by a total derivative,

$$\delta \mathcal{L} = \dot{\mathcal{L}}$$

and so the conserved quantity, analogous to energy for the radial coordinate s, is:

$$\begin{aligned} E &= \frac{\partial \mathcal{L}}{\partial \dot{M}} \delta M + \frac{\partial \mathcal{L}}{\partial \dot{p}} \delta p + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q - \mathcal{L} \\ &= M \left(\frac{\lambda - 1}{2} \dot{p}^2 + 3p^2 - 3 \right) + \frac{1}{M} \left(\frac{\lambda - 1}{2} \dot{q}^2 + 3q^2 \right) \end{aligned} \tag{3.11}$$

E is essentially the mass parameter, reducing to $E = 12m$ in the case of an ordinary (de Sitter) black hole.

The other invariance is more nontrivial. The fact that p and q enter the action in a similar form suggests a possible new invariance under a variation involving just these two fields. Indeed, it can be checked that the variation

$$\delta p = \frac{1}{M}[(\lambda - 1)\dot{q} + 2q], \qquad \delta q = M[(\lambda - 1)\dot{p} - 2p] \tag{3.12}$$

will make $\delta \mathcal{L} = \dot{K}$ a total derivative. Therefore the conserved quantity is

$$\begin{aligned} G &= \left(\frac{\partial \mathcal{L}}{\partial \dot{p}} \delta p + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q - K \right) \\ &= 2(\lambda - 1) \left(\frac{\lambda - 1}{2} \dot{p}\dot{q} + q\dot{p} - p\dot{q} + 3pq \right) \end{aligned} \tag{3.13}$$

This is one of the nontrivial "hair" parameters of the black hole.

The above two constants of motion allow in principle for the reduction of the system into one ordinary differential equation. Indeed, E, G, and the equation of motion for M (3.7) are algebraic expressions in M, $\dot{p}$ and $\dot{q}$, and therefore can be used to express M, $\dot{p}$ and $\dot{q}$ in terms of p and q:

$$\dot{p} = P(p,q), \qquad \dot{q} = Q(p,q).$$

Considering p as the new independent variable, q can be obtained by solving the equation

$$\frac{dq}{dp} = \frac{Q(p,q)}{P(p,q)}$$

after which M and the variable s can be determined.

Due to the rather complicated form of $P(p,q)$ and $Q(p,q)$, the above procedure is quite involved. There are, however, special values of λ ($= 1, 1/3$) with interesting features for which the problem can be readily solved, and we expose them in the next sections. Further, a more explicit solution for general λ can be found in the "bald" configuration $N_r = 0$ and will be analyzed in section 3.1.3.

3.1.1 The case $\lambda = 1$

The value $\lambda = 1$ is special, as it is required for recovering general relativity (together with $\omega \to \infty$). The equations of motion (3.7,3.8,3.9) for $\lambda = 1$ become first-order and simplify dramatically:

$$\begin{aligned} -2p\dot{p} - 3p^2 + 3 &= \frac{1}{M^2}\left(2q\dot{q} - 3q^2\right) && (3.14)\\ (\dot{M} - 3M)\,p &= 0 && (3.15)\\ (\dot{M} - 3M)\frac{q}{M} &= 0. && (3.16) \end{aligned}$$

In particular the last two equations become essentially identical and imply

$$\dot{M} = 3M \qquad \Rightarrow \qquad M = c\,e^{3s} = c\,r^3.$$

(The other solution, $p = q = 0$, implies trivially $M = 0$.) Using the time scale invariance (3.10) to set $c = 1$, we obtain

$$N = \sqrt{f} \tag{3.17}$$

as in the standard general relativistic case. The remaining equation can then be written as

$$\frac{d}{ds}[r^3(1-p^2)] = \frac{d}{ds}\left(\frac{q^2}{r^3}\right) \tag{3.18}$$

which determines p in terms of q or vice-versa:

$$p^2 = 1 + \frac{k}{r^3} - \frac{q^2}{r^6}. \tag{3.19}$$

It is evident that the case $\lambda = 1$ has an infinity of solutions. The corresponding solutions for the metric function $f(r)$ and the shift variable $N_r(r)$ in terms of an arbitrary function $g(r)$ read

$$f = 1 + \omega r^2 \pm \sqrt{(\omega^2 - \Lambda_W^2)r^4 + 4\omega m r - 2\omega g^2(r)}, \tag{3.20}$$

$$N_r = \frac{g(r)}{r\sqrt{f}}. \tag{3.21}$$

The expressions for f obtained in [18, 19] are recovered for $g(r) = 0$, once we choose the negative sign for p and identify our ω with their $\omega - \Lambda_W$.

From (3.20) it is evident that

$$\frac{\omega^2 - \Lambda_W^2}{2\omega} r^4 + 2mr \geq g^2(r) \tag{3.22}$$

must be verified; in particular this means that $g(r=0)=0$.

The expression 3.18 suggests that the theory for $\lambda = 1$ has a gauge invariance. A further indication for this is that the integral of motion G (3.13) is identically zero for $\lambda = 1$. Indeed, the $\lambda = 1$ action is invariant under the variation

$$\delta(q^2) = -M^2\delta(p^2), \qquad \delta M = 0 \tag{3.23}$$

with $\delta(p^2)$ an arbitrary function of r. Clearly the symmetry transformation (3.12) is a special case of the above gauge transformation, justifying the vanishing of its charge.

The above symmetry (3.23) reduces to the usual reparametrization invariance under $t \to t + F(r)$ in the IR limit $\omega \to \infty$, as can be checked by using the expressions (3.20,3.21). For finite ω, however, it corresponds to a "deformed" transformation.

The gauge invariance of the solutions for $\lambda = 1$ corresponds to a "deformed" coordinate transformation, not previously observed because the usually chosen condition $g_{rt} = 0$ fixes the gauge. This invariance does not survive for $\lambda \neq 1$. A specific gauge can thus be fixed by continuity as $\lambda \to 1$ (we must take into account that in the theory λ is a running constant), for example by matching the value of

$$\lim_{\lambda \to 1} \frac{G}{\lambda - 1},$$

which remains finite and nonzero, or, alternatively, by coupling the theory to matter which will not present this gauge invariance.

One must also consider that, although it is hoped (and required) that λ goes to 1 in the IR limit of the theory, this has not been proved yet.

3.1.2 The case $\lambda = 1/3$

As observed in [2] the value $\lambda = \frac{1}{3}$ corresponds to the action being invariant under an anisotropic conformal symmetry. That this value is special also manifests in the fact that the action in this case becomes a sum of perfect squares:

$$\mathcal{L} = -\frac{M}{3}(\dot{p}+3p)^2 + 3M - \frac{1}{3M}(\dot{q}-3q)^2 = -\frac{\bar{M}}{3}\dot{\bar{p}}^2 + 3\bar{M}r^6 - \frac{1}{3\bar{M}}\dot{\bar{p}}^2$$

where we redefined

$$\bar{p} = r^3 p, \quad \bar{q} = \frac{q}{r^3}, \quad \bar{M} = \frac{M}{r^6}.$$

The equations of motion (3.7,3.8,3.9) simplify accordingly

$$\frac{\dot{\bar{q}}^2}{\bar{M}^2} + 9r^6 = \dot{\bar{p}}^2 \tag{3.24}$$

$$\frac{d}{ds}(\bar{M}\dot{\bar{p}}) = 0 \tag{3.25}$$

$$\frac{d}{ds}\left(\frac{\dot{\bar{q}}}{\bar{M}}\right) = 0. \tag{3.26}$$

The above equations integrate readily giving

$$\dot{\bar{p}} = \pm 3\sqrt{A^2 + r^6}, \qquad \dot{\bar{q}} = 3A\bar{M} = \frac{3AB}{\sqrt{A^2+r^6}}$$

with A and B integration constants. From these, the fields p, q and M are obtained as

$$p = \pm\frac{3}{r^3}\int \frac{dr}{r}\sqrt{A^2+r^6} = \pm\frac{1}{r^3}\left(\sqrt{A^2+r^6} + \frac{A}{2}\ln\frac{\sqrt{A^2+r^6}-A}{\sqrt{A^2+r^6}+A}\right) + \frac{K_1}{r^3}$$

$$q = 3ABr^3\int \frac{dr}{r\sqrt{A^2+r^6}} = \frac{Br^3}{2}\ln\frac{\sqrt{A^2+r^6}-A}{\sqrt{A^2+r^6}+A} + K_2 r^3$$

$$M = \frac{Br^6}{\sqrt{A^2+r^6}}$$

with K_1, K_2 new integration constants. Moreover fixing the scale of M by choosing $B = 1$, the corresponding solutions for f, N and N_r are

$$f = 1 + \omega r^2 \pm \frac{\sqrt{(\omega^2 - \Lambda_W^2)}}{r}\left(\sqrt{A^2 + r^6} + \frac{A}{2}\ln\frac{\sqrt{A^2 + r^6} - A}{\sqrt{A^2 + r^6} + A}\right) + \frac{K_1}{r} \tag{3.27}$$

$$N_r = r\sqrt{\frac{\omega^2 - \Lambda_W^2}{2\omega f}}\left(\frac{1}{2}\ln\frac{\sqrt{A^2 + r^6} - A}{\sqrt{A^2 + r^6} + A} + K_2\right) \qquad N = \frac{r^3\sqrt{f}}{\sqrt{A^2 + r^6}}. \tag{3.28}$$

The condition $N_r = 0$ corresponds to choosing $A, K_2 = 0$. In this case the above expressions reduce to:

$$f = 1 + \omega r^2 \pm \sqrt{(\omega^2 - \Lambda_W^2)}r^2 + \frac{K_1}{r} \qquad N = \sqrt{f}.$$

3.1.3 $N_r = 0$ solutions

For $q = 0$ the equation of motion (3.9) is satisfied and we can determine p from (3.7):

$$\frac{\lambda - 1}{2}\dot{p}^2 - 2p\dot{p} - 3(p^2 - 1) = 0. \tag{3.29}$$

Solving for $\dot{p}$ we have

$$\dot{p} = \frac{2p - \epsilon\sqrt{4p^2 + 6(\lambda - 1)(p^2 - 1)}}{\lambda - 1}$$

where $\epsilon = \pm 1$. Note that only the case $\epsilon = +1$ has a finite limit for $\lambda \to 1$. This equation is trivially separable, giving

$$\frac{dr}{r} = \frac{\lambda - 1}{2}\frac{dp}{p - \epsilon\sqrt{\frac{3\lambda - 1}{2}p^2 - \frac{3}{2}(\lambda - 1)}}$$

and upon doing the integral we obtain the implicit expression

$$\ln[Cr] = -\frac{1}{6}\left\{\ln\left[\frac{\sqrt{ap^2+b}-\epsilon p}{\sqrt{ap^2+b}+\epsilon p}\right] + \ln[b+(a-1)p^2] + 2\epsilon\sqrt{a}\ln[ap+\sqrt{a}\sqrt{b+ap^2}]\right\} \tag{3.30}$$

where

$$a = \frac{3\lambda - 1}{2} \qquad b = -\frac{3}{2}(\lambda - 1)$$

and C is an integration constant.

Although such an expression is not explicit, it becomes explicit by considering p as the independent variable and expressing r in terms of p in the spacetime structure. Moreover, it allows for a qualitative investigation of the behavior of the solution under a variation of λ. It also reproduces the explicit solutions found earlier in the limit $\lambda \to 1$ and $\lambda \to 1/3$.

3.2 Horizons and Hawking Temperature

In this section we will start studying the physical implications of the solutions found before. To study the coupling with matter we will restrict our analysis only to relativistic matter coupled to the full metric $g_{\mu\nu}$ in the standard way. This excludes matter with modified dispersion relation described in [31, 32, 33, 34] (we will introduce this topic in chapter 4). The notion of horizon could be meaningless for non-relativistic matter since, in principle, particles could travel at any velocity breaking the causality structure of General Relativity, therefore this kind of matter will not be considered in this context.

The horizon can be defined as the surface on which photons are static. Then in the static spherically symmetric case a photon on the horizon has four-momentum

$$p^\mu = (p^t, 0, 0, 0)$$

proportional to the static Killing vector ∂_t. The horizon will be the surface at $r = r_h$ defined by the condition

$$0 = p^\mu p_\mu = -(N^2 - N_r^2 f)(p^t)^2. \tag{3.31}$$

Since matter couples to the $g_{\mu\nu}$ constructed from N, N_r, f in a covariant way, we can perform an ordinary reparametrization transformation to the above $g_{\mu\nu}$ (this is not an invariance of the gravitational part of the HL-action, but it is an invariance of particle dynamics) choosing a diagonal coordinate system in which the new shift field vanishes, we have for the new (tilded) variables

$$\tilde{N}^2 = N^2 - fN_r^2, \qquad \tilde{f} = f(1 - N_r^2), \qquad \tilde{N}_r = 0.$$

(Note, further, that if $N = \sqrt{f}$ then we also have $\tilde{N} = \sqrt{\tilde{f}}$.)
We see that the horizon condition (3.31) is now the usual statement $\tilde{N}^2 = \tilde{g}_{00} = 0$.

For $\lambda = 1$ the horizon condition translates into

$$0 = p^\mu p_\mu = -\left(f - \frac{g^2(r_h)}{r_h^2}\right)(p^t)^2,$$

that is,

$$f(r_h) = \frac{g^2(r_h)}{r_h^2} \tag{3.32}$$

where we used (3.17,3.21). Obviously the above condition reduces to $f = 0$ if $g(r)$ is identically equal to zero. In the case $\lambda = 1/3$ we have a similar situation:

$$0 = -(N^2 - N_r^2 f)(p^t)^2 = -\left[\frac{fr^6}{A^2 + r^6} - r^2\frac{\omega^2 - \Lambda_W^2}{2\omega}\left(\frac{1}{2}\ln\frac{\sqrt{A^2 + r^6} - A}{\sqrt{A^2 + r^6} + A} + K_2\right)^2\right](p^t)^2,$$

that is,

$$f = \frac{\omega^2 - \Lambda_W^2}{2\omega} \frac{A^2 + r^6}{r^4} \left(\frac{1}{2} \ln \frac{\sqrt{A^2 + r^6} - A}{\sqrt{A^2 + r^6} + A} + K_2 \right)^2 \tag{3.33}$$

that reproduces the condition $f = 0$ in the case of no hair, $A = 0$ and $K_2 = 0$.

To evaluate the Hawking temperature we need to look at space-times that are asymptotically flat, so from now on in this section we will consider

$$\Lambda_W = 0$$

and choose the negative sign in the expressions (3.20,3.27) for f. Expanding the expression (3.20) for f for $\omega r^2 \gg 1$, we find for the metric functions in the diagonal form of the metric in the case $\lambda = 1$

$$\tilde{N}^2 = \tilde{f} \simeq 1 - \frac{2m}{r} + \frac{1}{2\omega} \left(\frac{2m}{r^2} - \frac{g^2}{r^3} \right)^2 ,$$

therefore to recover asymptotically flat space-time we need also to impose the following condition

$$\lim_{r \to \infty} \frac{g^2}{r^3} = 0.$$

This condition, together with the constraint (3.22), which can be cast in the form

$$\frac{g^2}{r^2} - 1 \leq \frac{\omega}{2} r^2 + \frac{2m}{r} - 1, \tag{3.34}$$

still leaves an infinity of possible gauges.

Moreover, the horizon condition (3.32), using relation (3.20) with the minus sign, reads as:

$$\left(\frac{g^2(r_h)}{r_h^2} - 1 \right)^2 = 2\omega r_h (2m - r_h). \tag{3.35}$$

One immediate corollary is that

$$r_h \leq 2m$$

implying that the radius of the horizon must be less the the GR value.

From the above we see that the function g^2/r^2 is constrained to be within the envelope defined in (3.34), while a horizon will form for it satisfies (3.35). We have different possible situations, depending on the value of the parameter $m^2\omega$ and the form of $g^2(r)$: we could have no horizons at all, one, or several. In any case, the Hawking temperature, if a horizon exists, will be evaluated for the biggest possible value of the radius for the horizon. For $\omega r_h^2 \gg 1$ the value of the external horizon acquires the correction

$$r_h = 2m - \frac{1}{4\omega m^2}\left(\frac{g^2(2m)}{4m^2} - 1\right)^2 .$$

In the case $\lambda = 1/3$ the horizon radius, if it exists, can be determined once we know all the constants in the expression (3.33).

Since we are dealing with black hole solutions with hair, we cannot directly adopt the definition of Hawking temperature used in [19, 41, 42, 52].

The Hawking temperature T_h is proportional to the surface gravity κ ($T_h = \kappa/2\pi$) on the horizon of the black hole (see, e.g., [10]) which corresponds to the force at $r = \infty$ necessary to keep a particle from falling into the horizon. The surface gravity is related to the Killing vector $\boldsymbol{\chi} = \partial_t$, asymptotically normalized ($\lim_{r\to\infty} \chi^\alpha \chi_\alpha = -1$), as follows

$$\left(\nabla^\alpha(\chi_\beta\chi^\beta)\right)_{r_h} = -\left(g^{\alpha r}\partial_r(N^2 - N_r^2 f)\right)_{r_h} = -2\kappa\chi^\alpha(r_h);$$

then κ reduces to

$$\kappa = \frac{1}{2N_r(r_h)} \left(\partial_r(N^2 - N_r^2 f)\right)_{r_h}.$$

For $\lambda = 1$ the Hawking temperature is then given by

$$T_h = \frac{1}{4\pi}\partial_r \left(f - \frac{g^2}{r^2}\right)_{r_h} = \frac{1}{4\pi}\partial_r \tilde{f}(r_h)$$

which corresponds to the expression used in [19, 41, 42, 52] in the case without hair ($g = 0$) and has the correct relativistic limit for $\omega \to \infty$. For $\lambda = 1/3$ it reduces to

$$T_h = \frac{1}{4\pi}\frac{\sqrt{A^2 + r_h^6}}{r_h^3} \left(\partial_r(N^2 - N_r^2 f)\right)_{r_h}.$$

Note that the Hawking temperature for $\lambda = 1/3$ for the no-hair case $(A, K_2 = 0)$

$$f_{1/3} = 1 + \omega\frac{K_1}{r} \qquad N_{1/3} = \sqrt{f_{1/3}},$$

reduces to an expression similar to the case $\lambda = 1$

$$T_h = \frac{1}{4\pi}\partial_r f = -\frac{1}{2\pi\omega K_1}.$$

3.3 Spherically Symmetric Solutions for $\lambda = 1$

The infinite set of solutions (3.17,3.20,3.21) for the case $\lambda = 1$

$$f = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega M r - 2\omega g^2 r^2} \qquad N^2 = f \qquad N_r = \pm\sqrt{\frac{g^2}{f}} \tag{3.36}$$

is not physically reasonable. Indeed, although the solutions (3.36) are the result of an hidden symmetry of the pure gravity theory (3.6) (from here we will consider the case of a zero cosmological constant)

$$
\begin{aligned}
S &= \frac{\kappa^2\mu^2}{8}\int dtd^3x\sqrt{h}N\Big\{\omega(K_{ij}K^{ij}-\lambda K^2)+\omega\mathcal{R}-\frac{4}{\mu^2w^4}C_{ij}C^{ij}+ \\
&+\frac{4}{\mu w^2}\sqrt{3\lambda-1}\epsilon^{ijk}\mathcal{R}_{il}\nabla_j\mathcal{R}^l{}_k-(3\lambda-1)\mathcal{R}_{ij}\mathcal{R}^{ij}+\frac{4\lambda-1}{4}\mathcal{R}^2\Big\},
\end{aligned}
$$

they corresponds to physically different space-times for the simple reason that such a symmetry is not an invariance of a relativistic mass term.

Although in principle we could construct a matter term that is invariant under such a gauge symmetry, the physical relativistic matter breaks the gauge explicitly: therefore every value of the function g^2 corresponds to a different space-time background.

The reduced symmetries of HL gravity make unclear the meaning of the symmetry (3.46) from a physical point of view. Such a gauge invariance may be just an accident for the present formulation of the theory, but if it is not an accident it can be used, if generalized to a generic space-time background, to fix λ to the value of 1 in the quantization process.

In this and in the next sections of this chapter we will be studying the behavior of the solutions (3.36) for different g^2.

The choice $\lambda = 1$ is dictated not only by the fact that it is the expected value in the IR limit, but also by the fact that the gauge invariance found in [50] and described in sec. 3.1.1 could be used to fix $\lambda = 1$ from the beginning once the theory is quantized (there are also different works in which it is shown how, leaving λ general, it is possible to achieve a correct IR limit introducing second order constraints [53]). A promising advance in this direction, although it is not clear yet if there is any relation with our gauge symmetry, was made by Hořava and Melby-Thompson in [11]. In [54, 55] spherically symmetric solutions relative to the theory described in [11] are discussed.

In section 3.4 we will study the constraints to which $g(r)$ is subject to have a well defined metric, while in sections 3.5 and 3.6 we will review some physical aspects of the problem, i.e. the possible measurement of $g(r)$ from astrophysical data and the behavior of spacetime behind the horizon.

To discuss the topic we are going to use redefined coordinates for which the metric becomes diagonal, exactly as we have already done in sec. 3.2. Let us start with a brief review.

As shown in the appendix C, in HL gravity, unlike GR, the non-diagonal metric

$$ds^2 = -(N^2 - N_r^2 f)dt^2 + 2N_r dr dt + \frac{dr^2}{f} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad (3.37)$$

where we assume f and g^2 to be analytic functions, and the diagonal one

$$ds^2 = -N^{*2} dt^{*2} + \frac{1}{f^*} dr^{*2} + r^{*2} d\theta^2 + r^{*2} \sin^2\theta d\phi^2 \qquad (3.38)$$

are not equivalent because we cannot perform the relevant coordinate transformation:

$$\begin{gathered} dt = dt^* + \tfrac{N_r}{N^2 - N_r^2 f} dr \\ \text{with} \qquad N^{*2} = N^2 - N_r^2 f \qquad f^* = \tfrac{f(N^2 - N_r^2 f)}{N^2}. \end{gathered} \qquad (3.39)$$

Hence the most general spherically symmetric ansatz in HL gravity is (3.2).

It is evident that these solutions are well defined only for

$$f > 0 \qquad \text{and} \qquad g^2 \leq \frac{\omega}{2} r^2 + \frac{2M}{r};$$

in the following, and in particular in section 3.4, we will find other constraints on g^2 to have a well behaved metric.

Although we cannot consider the two form of the metric, (3.2) and (3.38), to be physically equivalent in this context, we can still perform the coordinate transformation (3.39) to obtain intermediate expressions and then go back to the non-diagonal coordinates to study the physical results. In all the cases we consider here the relations used are relativistic because we consider the standard relativistic coupling with matter, therefore, in the diagonal coordinates, the expressions for the equations of motion for a test particle, the bending of light or the relation for the position of the horizon are exactly like the one obtained in the GR context, only in the wrong coordinate system. This means that we can just use the relativistic relation and perform the change of coordinates to go to non-diagonal coordinates to obtain the result we seek.

In our case the coordinate transformation for the diagonalization becomes

$$dt = dt^* + \sqrt{\frac{g^2}{f}}\frac{1}{(f - g^2)}dr; \tag{3.40}$$

which is defined only for

$$f > 0 \qquad \text{and} \qquad f \neq g^2 \quad (f^* \neq 0). \tag{3.41}$$

The first condition, $f > 0$, implies a constraint on g^2

$$g^2 \geq -1 + \frac{2M}{r} - \frac{1}{2\omega r^2}, \tag{3.42}$$

while the second condition is related to the position r_h of the horizon, which is obtained by solving the condition

$$f^*(r_h) = 0 \qquad \Rightarrow \qquad f(r_h) = g^2(r_h) \tag{3.43}$$

and choosing the outer solution. In particular, from the relations above, we have shown in sec. 3.2 that

$$0 \leq (g^2(r_h) - 1)^2 = 2\omega r_h(2M - r_h), \tag{3.44}$$

which corresponds to say that no horizon beyond the Schwarzschild radius is possible ($r_h \leq 2M$).

Under the conditions (3.41) the coefficients in the diagonal metric become:

$$N^{*2} = f^* = f - g^2 = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega M r - 2\omega g^2 r^2} - g^2. \tag{3.45}$$

The implications of conditions (3.41) are that the change of coordinates is allowed only in the region outside the horizon, exactly as in GR. With an opportune change of coordinates in GR we can extend the spacetime inside the horizon, but it is not clear how to proceed in HL gravity, where the foliation has a geometrical meaning, not the metric. We will briefly investigate this problem in section 3.6.

3.4 Constraints on g^2

Let us forget for a moment about the problems related to the extension of the metric inside the horizon and concentrate on analyzing the infinite set of all possible solutions to the equations of motion, looking for the constraints which the generic function g must obey to reproduce a well behaved metric.

The solutions in [50] for $\lambda = 1$ (sec. 3.1.1) exhibit the following invariance:

$$\frac{\delta_g N^2}{N^2} = \frac{\delta_g f}{f} \qquad \delta_g N_r^2 = \frac{1 + \omega r^2 - f - \omega r^2 N_r^2 f / N^2}{\omega r^2 f} \frac{N^2 \delta_g f}{f}. \tag{3.46}$$

This invariance is not related to a coordinate transformation (although it recovers the usual GR coordinate invariance in the limit $\omega r^2 \to \infty$) and therefore does not admit yet a simple physical interpretation. Moreover, as will be evident from section 3.5, we need to treat each possible value of g as the source of a different solution. In this context it makes sense to discard some of the solutions based on the study of the constraints that g^2 must fulfill.

As already observed the first constraint on g^2 comes from the expression (3.45), for which we must have

$$g^2 \leq \frac{\omega}{2} r^2 + \frac{2M}{r}. \tag{3.47}$$

This, in particular, implies

$$g^2 r^2|_{r=0} = 0. \tag{3.48}$$

The constraint (3.42) is always compatible with the upper bound for g^2 since

$$-1 + \frac{2M}{r} - \frac{1}{2\omega r^2} \leq \frac{\omega}{2} r^2 + \frac{2M}{r}$$

and is relevant only when the l.h.s is positive (remember that $g^2 > 0$):

$$-1 + \frac{2M}{r} - \frac{1}{2\omega r^2} > 0 \qquad \Rightarrow \qquad M - \sqrt{M^2 - \frac{1}{2\omega}} < r < M + \sqrt{M^2 - \frac{1}{2\omega}},$$

that is, between the internal and the external horizon radius for the KS metric ($N_r = 0$). This in particular means that only when $f^* \leq 0$ the condition $f > 0$ may become relevant. Indeed, for a generic value of g^2, f satisfies

$$1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega M r} \leq f < 1 + \omega r^2.$$

The limits correspond, respectively, to $g^2 = 0$ and $g^2 = \frac{\omega}{2} r^2 + \frac{2M}{r}$.

Also if we consider only a constrained subclass of possible g^2, it is still an infinite set. We can then study the range in which f^* may vary for different g^2's.

For $g = 0$ we have the well known KS solution

$$f_0^* = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega M r};$$

while for $g^2 = \frac{\omega}{2}r^2 + \frac{2M}{r}$ we have

$$f^*_{max} = 1 - \frac{2M}{r} + \frac{\omega}{2}r^2,$$

that is, a de Sitter-like solution. The inequality

$$f^*_{max} \geq 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega M r - 2\omega g^2 r^2} - g^2$$

is true for

$$-\frac{3}{2}\omega r^2 + \frac{2M}{r} \leq g^2$$

which is always verified for

$$-\frac{3}{2}\omega r^2 + \frac{2M}{r} \leq 0,$$

that is,

$$r \geq \left(\frac{4M}{3\omega}\right)^{1/3}.$$

Then, restricting to the case $r \gg \left(\frac{4M}{3\omega}\right)^{1/3}$, which for $\omega \gg 1$ is much less than the classical horizon $2M$, we can consider f^*_{max} to be the upper bound for f^*.

Minimizing f^* with respect to g we find two values

$$g = 0 \qquad \text{and} \qquad g = \sqrt{\frac{2M}{r}}.$$

For $g^2 = \frac{2M}{r}$ we have

$$f^*_{min} = 1 - \frac{2M}{r}. \tag{3.49}$$

Therefore the Schwarzschild solution is the lower bound solution for the metric (fig.3.1).

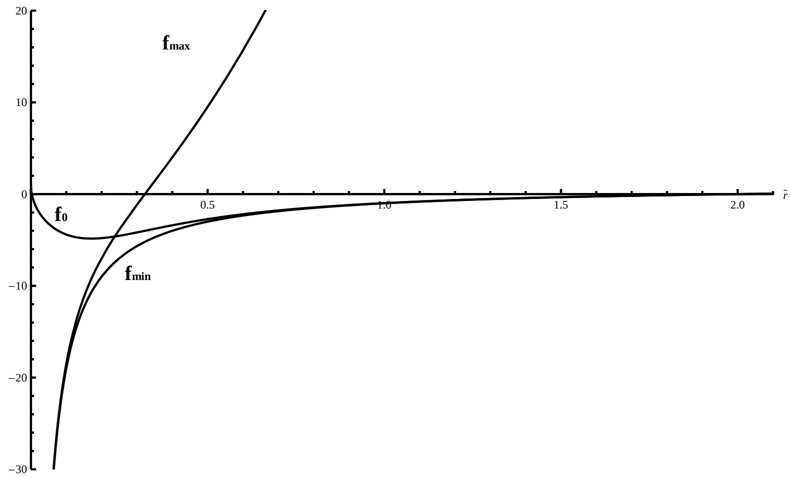

Figure 3.1: In this figure we plotted f^*_{min}, f^*_{max} and f^*_0 with $\tilde{r} = r/M$ and $\tilde{\omega} = \omega M^2 = 100$ (for bigger values of $\tilde{\omega}$ the plot is qualitatively the same).

3.4.1 Asymptotic Behavior

In this section we want to find what are the restrictions to the asymptotic behavior of g^2 to have a flat asymptotic space-time. Let us then consider $f^*(r)$ for $r \to \infty$. Writing

$$f^* = 1 + \omega r^2 \left[1 - \sqrt{1 + \frac{4M}{\omega r^3} - \frac{2g^2}{\omega r^2}} \right] - g^2$$

it is easy two see that we can encounter three cases for $r \to \infty$: the terms $\frac{2g^2}{\omega r^2}$ is negligible with respect to $\frac{4M}{\omega r^3}$, it is comparable with $\frac{4M}{\omega r^3}$ or $\frac{4M}{\omega r^3}$ is negligible with respect to $\frac{2g^2}{\omega r^2}$. In the first two cases we can directly obtain that the asymptotic limit is flat space-time; in the third case we can neglect the term $\frac{4M}{\omega r^3}$ and, considering that asymptotically $2g^2 \leq \omega r^2$, we can expand f^* in terms of g^2 as follows:

$$f^* = 1 + \sum_{n=2} \frac{(2n-3)!!}{n!} \frac{g^{2n}}{\omega^{n-1} r^{2n-2}}.$$

The above relation implies that g^2, asymptotically, must grow less fast than r to have a Minkowski flat asymptotic space-time.

For $g^2 = Cr$, with C constant, the function f^* asymptotically goes as

$$1 + \sum_{n=2} \frac{(2n-3)!!}{n!} \frac{C^n}{\omega^{n-1}} r^{2-n} \simeq 1 + \frac{C^2}{2\omega};$$

in this case the asymptotic metric interval becomes

$$ds^2 = -(1 + \frac{C^2}{2\omega})dt^{*2} + \frac{dr^2}{1 + \frac{C^2}{2\omega}} + r^2 d\Omega \tag{3.50}$$

that, after a rescaling of time and the radial coordinate ($\sqrt{1 + \frac{C^2}{2\omega}} t^* \to t^*$ and $r \to \sqrt{1 + \frac{C^2}{2\omega}} r$), gives

$$ds^2 = -dt^{*2} + dr^2 + (1 + \frac{C^2}{2\omega})r^2 d\Omega$$

showing a conical singularity.

Such a space is asymptotically flat in the sense that there exists a coordinate transformation (3.40) for which the diagonal metric is Minkowsky-like. Going back to the time coordinate determined by the foliation, the metric (3.50) asymptotically becomes

$$ds^2 = -(1 + \frac{C^2}{2\omega})dt^2 + 2drdt + dr^2 + r^2 d\Omega.$$

To have a true flat asymptotic space-time in the physical coordinates frame we need to impose the condition $\lim_{r\to\infty} N_r = 0$ that simply implies $\lim_{r\to\infty} g(r) = 0$, being $\lim_{r\to\infty} f(r)$ finite by construction. Under this condition we directly obtain a Minkowskian asymptotic behavior.

3.5 The Bending of Light Measure

In this section we are going to show explicitly that the dynamics of a relativistic particle is different for different values of the gauge parameter. As consequence of this statement it will be possible to measure deformations with respect to the Schwarzschild behavior (the analysis with respect the parameter ω will be considered for the case $N_r = 0$ in the section 4.3) measuring carefully the kinematics of free falling objects. To study how, let us consider as example the bending of light. Although this method will not produce good results because of the lack of precision in this kind of measure, it is one of the few easy options available.

It is easy to show that in both coordinate frames, the physical and the diagonal one, the Killing vectors that correspond, respectively, to the energy E and to the angular momentum L take the same form

$$\boldsymbol{\xi}_E = \partial_t = \partial_{t^*} \qquad \boldsymbol{\xi}_L = \partial_\phi .$$

In particular the energy is given by

$$E = -p_0^* = N^{*2} p^{*0} = (N^2 - N_r^2 f)\left[p^0 - \frac{N_r p^r}{N^2 - N_r^2 f}\right] = (N^2 - N_r^2 f)p^0 - N_r p^r \equiv -p_0 .$$

Therefore for a particle of mass m the dispersion relation yields

$$-\frac{\varepsilon^2}{N^{*2}} + \frac{\dot{r}^2}{f^*} + \frac{l^2}{r^2} + 1 = 0,$$

where

$$\varepsilon = E/m \qquad l = L/m \qquad k = \begin{cases} 1 & \text{massive particle} \\ 0 & \text{massless particle} \end{cases} .$$

The equations of motion are

$$\varepsilon = (N^2 - N_r^2 f)\dot{t} - N_r \dot{r} \tag{3.51}$$

$$l = r^2 \dot{\phi} \tag{3.52}$$

$$\dot{r}^2 = \varepsilon^2 - V_{eff} \tag{3.53}$$

where we defined the effective potential as

$$V_{eff} = f^* \left(\frac{l^2}{r^2} + 1 \right).$$

The equations of motion are explicitly not invariant under the gauge transformation (3.46), so metrics with different g's represent physically different backgrounds. Therefore the only way we have to fix g^2 is to study the trajectory of a test particle and reconstruct from it the function g^2: the measurement of the bending of light as a function of the radial coordinate r, that is, $\frac{d\phi}{dr}$, and the impact parameter can be used to determine the metric.

For a light ray the radial equation of motion is

$$\dot{r}^2 = E^2 - \frac{L^2}{r^2} f^*$$

and the impact parameter is defined as usual as

$$b = \frac{L}{E} = \left. \sqrt{\frac{r^2}{f^*}} \right|_{r=R_0}$$

where R_0 is the closest distance to the star in the trajectory. Using the equation of motion for ϕ, as in GR, we obtain

$$\frac{d\phi}{dr} = \frac{1}{r^2} \left[\frac{1}{b^2} - \frac{f^*}{r^2} \right]^{-1/2}.$$

Knowing the impact parameter b and the function $\frac{d\phi}{dr}$ the we can in principle obtain completely g^2.

A nonzero value of g^2 can be also observed measuring the total bending angle, although it will not be possible to reconstruct the whole function.

The deflection angle is given by

$$\delta\phi = 2\int_{R_0}^{\infty}\frac{dr}{r^2}\left[\frac{1}{b^2}-\frac{f^*}{r^2}\right]^{-1/2} - \pi.$$

Therefore, considering the functions g_1^2 and g_2^2, we have that

$$\delta\phi_2 \geq \delta\phi_1$$

is true if the relative radial functions f_1^* and f_2^* satisfy the following requirements:

$$f_1^*(r) \geq f_2^*(r) \qquad \text{for } r \geq R_0 \qquad \text{and} \tag{3.54}$$

$$\frac{f_1^*(r)}{r^2}, \frac{f_2^*(r)}{r^2} \qquad \text{are monotonically decreasing.} \tag{3.55}$$

The above requirements are not the most general but are easy enough to deduce what happens for some of the metrics considered here. First of all we know that if we choose the metric (3.49), that is, $f_2^* = f_{min}^* = 1 - \frac{2M}{r}$, the condition (3.54) will be verified for any other possible f^*. The condition (3.55) is also verified for f_{min}^*. So we can conclude that for every g^2 such that the relative f^* satisfies condition (3.55) the deflection angle is smaller than what we expect from GR.

The case $f^* = 1$ with $g^2 = 0$ and $M = 0$ corresponds to a flat Minkowski space-time giving rise to a zero deflection angle. This means that for an f^* satisfying condition (3.55) and such that $f^* > 1$ the deflection angle is negative, the force being repulsive.

Let us consider as example the KS metric. Using the expansion for the KS metric for $\omega r^2 \gg 1$, $f^* = 1 - \frac{2M}{r} + \frac{2M^2}{\omega r^4}$ the deflection angle can be approximated as

$$\begin{aligned}
\delta\phi &\simeq 2\int_{R_0}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b_{min}^2} - \frac{f_{min}^*}{r^2} + \frac{2M^2}{\omega}\left(\frac{1}{R_0^6} - \frac{1}{r^6}\right)\right]^{-1/2} - \pi \\
&\simeq 2\int_{R_0}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b_{min}^2} - \frac{f_{min}^*}{r^2}\right]^{-1/2}\left[1 - \frac{M^2}{\omega}\frac{1/R_0^6 - 1/r^6}{\frac{1}{b_{min}^2} - \frac{f_{min}^*}{r^2}}\right] - \pi \\
&= \delta\phi_{GR} - 2\int_{R_0}^{\infty} \frac{dr}{r^2}\frac{M^2}{\omega}\frac{1/R_0^6 - 1/r^6}{\left[\frac{1}{b_{min}^2} - \frac{f_{min}^*}{r^2}\right]^{3/2}}.
\end{aligned}$$

Defining

$$\omega\delta\phi_\omega \equiv 2\int_{R_0}^{\infty} \frac{dr}{r^2} M^2 \frac{1/R_0^6 - 1/r^6}{\left[\frac{1}{b_{min}^2} - \frac{f_{min}^*}{r^2}\right]^{3/2}}$$

we ca rewrite the above expression as

$$\delta\phi \simeq \delta\phi_{GR} - \frac{\omega\delta\phi_\omega}{\omega},$$

that will give the following estimated value for ω:

$$\omega = \frac{\omega\delta\phi_\omega}{\delta\phi_{GR}}\frac{1}{1-\gamma},$$

where we defined $\gamma \equiv \delta\phi/\delta\phi_{GR}$. In 1970 the National Radio Astronomy Observatory [56, pag. 1105] reports the value $\gamma = 0.90 \pm 0.05$ for the deflection angle of a ray barely touching the surface of the Sun. Being $\delta\phi_{GR} = 1.75''$ and $\omega\delta\phi_\omega = 1.13 \cdot 10^{-23}$, we obtain the following estimation for the minimal value of ω:

$$\omega_{\text{min}} = 6.5 \cdot 10^{-23}(1.0 \pm 0.5).$$

A small value was expected, being already observed in [23, 26]. Such a small value is mainly consequence of the lack of precision in the measurement. The other values reported in [56, pag. 1105] will reproduce a negative estimation for ω, then corresponding to no constraint at all. A possible explanation for a deflection angle bigger than the GR value is in considering a metric with a g^2 not satisfying the conditions (3.54, 3.55). At the moment the choice of such value of g^2 is quite arbitrary.

3.6 The Singularity

Let us now consider what happens inside the horizon.

As already observed no Kruskal extensions make sense in the HL-gravity context because of the reduced symmetry due to the foliation preservation. This means that the concept of singularity must be reviewed.

The eventual external horizon satisfies the condition $f^*(r_h) = 0$, so we need first of all to check if there are solutions to the condition $f^* > 0$. Such a case satisfies conditions (3.41), therefore we can just analyze the diagonal metric to study its properties. Using relation (3.44) we can just restrict to the case $r \leq 2M$, f^* being positive for $r > 2M$.

At $r = 0$ we have that

$$f^*(0) = 1 - g^2(0)$$

where we used condition (3.48). Moreover the condition $f^* > 0$ yields

$$1 + \omega r^2 - g^2 > \sqrt{\omega^2 r^4 + 4\omega M r - 2\omega g^2 r^2}.$$

Let us consider a g^2 such that $1 + \omega r^2 > g^2$ (otherwise there is a range for which $f^* < 0$ and hence we have a horizon), then

$$g^4 - 2g^2 + 1 + 2\omega r^2 - 4\omega M r > 0$$

which implies

$$g^2 < 1 - \sqrt{2\omega r^2 \left(\frac{2M}{r} - 1\right)} \qquad g^2 > 1 + \sqrt{2\omega r^2 \left(\frac{2M}{r} - 1\right)}.$$

The first condition corresponds to a positive g^2 for $r < M - \sqrt{M^2 - 1/2\omega}$ and $r > M + \sqrt{M^2 - 1/2\omega}$ if $M > \sqrt{\frac{1}{2\omega}}$, while for $M \leq \sqrt{\frac{1}{2\omega}}$ any g^2 such that $0 < g^2 < 1 - \sqrt{2\omega r^2 \left(\frac{2M}{r} - 1\right)}$ satisfies all the requirements. The second case, instead, does not satisfy the condition

$$1 + \omega r^2 > 1 + \sqrt{2\omega r^2 \left(\frac{2M}{r} - 1\right)}$$

for any $r < 2M$ so it must be excluded if we want g^2 to be a continuous function (only in the case $M = 0$[1] we can consider such a case).

Therefore we cannot have vacuum solutions with no horizon other than in the case (we are considering only the expression of g^2 for $r \leq 2M$ because for $r > 2M$ we can consider any analytic continuation)

$$g^2 < 1 - \sqrt{2\omega r\,(2M - r)} \qquad \text{with} \qquad M < \sqrt{\frac{1}{2\omega}};$$

note that in this case there is no singularity at $r = 0$, $f^*(0) = 1 - g^2(0) > 0$, only a possible pinch. In particular this means that there are no naked singularities, if we exclude the pinch.

Unlike GR, in HL gravity the 4D metric $g_{\mu\nu}$ is not physically important: the foliation structure is geometrically and physically relevant. In HL gravity the foliation is determined by a scalar function χ, as described in section 2.4. It is evident then that, asking for a well behaving foliation, we need to have a surface

[1] For $M = 0$, whatever is the source for N_r, there are g^2 for which $f^* > 0$ but the condition $1 + \omega r^2 \geq g^2$ does not allow to have singularities, being $g^2(0) < 1$.

Σ with a well defined orthogonal vector n^α. In the gauge $t = \phi$

$$n_\alpha = -N\partial_\alpha \chi = (-N, 0, 0, 0)$$

hence the foliation is well defined if $N \neq 0$. In our case $N = \pm\sqrt{f}$, thus a foliation is well defined only if $f > 0$. Let us call r_f the outer radius satisfying the condition $f(r_f) = 0$. Surprisingly we have

$$f^*(r_f) = f(r_f) - g^2(r_f) = -g^2(r_f) \leq 0,$$

that is,

$$r_f \leq r_h. \tag{3.56}$$

This means that the foliation may also be well defined also behind the horizon. For the KS metric the condition $g^2(r_f) = 0$ implies $r_f = r_h$, but for any metric such that $g^2(r_f) \neq 0$ it is always possible to define a foliation also behind the horizon. We already discussed in section 3.4 the implications of condition $f > 0$ (3.42) and they simply implies that if an r_f exists is between two horizons, being $f^*(r_f) \leq 0$.

As example we have that the metric relative to f^*_{min} has a well defined foliation, being $N^2 = f = 1$.

In general it is not clear what happens for $r \leq r_f$ because the foliation structure breaks down, introducing a different kind of singularity. To explore what happens to a particle travelling toward r_h and then toward r_f let us consider a photon of energy E following a radial trajectory. Unlike GR there are no constraints from the fact that a particle is space-like, null-like or time-like because in this context the $4D$-metric has no direct physical meaning (there is not a clear causality structure). Here we will base our discussion on the geometrical properties of space-time in terms of its physical foliation and we will consider the equations of motion for a particle to be the same everywhere, inside or outside the horizons:

$$\text{ingoing particle} \quad \dot r = -E,\ \dot t^* = \tfrac{E}{N^{*2}} \quad \Rightarrow \quad \dot t = -\left[\frac{1}{N^{*2}} - \frac{N_r}{f^*}\right]\dot r \tag{3.57}$$

$$\text{outgoing particle} \quad \dot r = +E,\ \dot t^* = \tfrac{E}{N^{*2}} \quad \Rightarrow \quad \dot t = +\left[\frac{1}{N^{*2}} + \frac{N_r}{f^*}\right]\dot r. \tag{3.58}$$

Because we are considering asymptotically flat spherically symmetric space-times, $\lim_{r\to\infty} g^2 = 0$ and hence $\lim_{r\to\infty} f^* = \lim_{r\to\infty} f = 1$. This implies that outside the outer horizon f^* and f are both positive and that the outer horizon $r_h^{(0)}$ is the first zero of f^*. The consequence of this statement is that outside the black hole $N_r^2 < 1$, while in general we have

$$\begin{aligned} f^* > 0 \quad &\Rightarrow \quad N_r^2 < 1 \\ f^* < 0 \quad &\Rightarrow \quad N_r^2 > 1. \end{aligned}$$

We will assume that ${f^*}'(r_h^{(i)}) \neq 0$ and $f'(r_f) \neq 0$.

On the horizon, if $r_f \neq r_h^{(i)}$, $N_r^2(r_h^{(i)}) = 1$ and we can approximate N_r near $r_h^{(i)}$ as follows

$$\begin{aligned} N_r > 0: \qquad & N_r \simeq +1 - \tfrac{1}{2}\tfrac{{f^*}'(r_h^{(i)})}{f(r_h^{(i)})}(r - r_h^{(i)}) \\ N_r < 0: \qquad & N_r \simeq -1 + \tfrac{1}{2}\tfrac{{f^*}'(r_h^{(i)})}{f(r_h^{(i)})}(r - r_h^{(i)}). \end{aligned}$$

If instead $r_f = r_h^{(i)}$ for a given i, then g^2 near $r_h^{(i)}$ goes like $g^2 \simeq \frac{D^{2n}(g^2)(r_h^{(i)})}{(2n)!}(r - r_h^{(i)})^{2n}$, where $[D^{2n}g^2](r_h^{(i)})$ is the first - even, being $g^2 > 0$, - non zero derivative of g^2 in $r_h^{(i)}$. Therefore, near $r_h^{(i)}$, N_r goes like

$$N_r \simeq \pm\sqrt{\frac{[D^{2n}g^2](r_h^{(i)})}{f'(r_h^{(i)})}\frac{(r - r_h^{(i)})^{2n-1}}{(2n)!}};$$

in particular we have $N_r(r_h^{(i)}) = 0$. In the above relation we used the fact that

${f^*}'(r_h^{(i)}) \neq 0$, which implies $f'(r_h^{(i)}) \neq 0$, being ${g^2}'(r_h^{(i)}) = 0$. The case $g^2(r) = 0$ is then included in the case $g^2(r_h^{(i)}) = 0$ considering that all the following derivatives of g^2 are all zero.

The last case to consider is what happens in r_f for $r_f \neq r_h^{(i)}$ for any i. Assuming that $f'(r_f) \neq 0$ we have that $f'(r_f) > 0$ because of the asymptotic flatness. Moreover $f^*(r_f) = -g^2(r_f) < 0$, otherwise we fall in the above case for $g^2(r_h^{(i)}) = 0$. Then N_r near r_f is given by

$$N_r \simeq \pm\sqrt{\frac{g^2(r_f)}{f'(r_f)(r - r_f)}}$$

and it is singular in r_f. This behavior is expected considering that in r_f the time direction become tangential and that N^{-1} is singular in r_f.

Let's start considering a photon near[2] the outer horizon $r_h^{(0)}$, for which ${f^*}'(r_h^{(0)}) > 0$, and crossing it from outside 3.2:

$$\begin{array}{lll}
N_r(r_h^{(0)}) > 0 : & \Delta t \simeq -\int_{r_h+\delta}^{r} \frac{dr}{2f(r_h^{(0)})} = \frac{r_h^{(0)}+\delta-r}{2f(r_h^{(0)})} & \\
N_r(r_h^{(0)}) < 0 : & \Delta t \simeq -\int_{r_h+\delta}^{r} \frac{2dr}{{f^*}'(r_h^{(0)})(r-r_h^{(0)})} = \frac{2}{{f^*}'(r_h^{(0)})} \ln \left| \frac{\delta}{r-r_h^{(0)}} \right. & \\
N_r(r_h^{(0)}) = 0 : & \Delta t \simeq -\int_{r_h+\delta}^{r} \frac{dr}{{f^*}'(r_h^{(0)})(r-r_h^{(0)})} = \frac{1}{{f^*}'(r_h^{(0)})} \ln \left. \frac{\delta}{r-r_h^{(0)}} \right| . &
\end{array}$$

For $N_r(r_h^{(0)}) > 0$ we can extend the integral to $r \leq r_h^{(0)}$ (in this case $f^* < 0$) obtaining a finite positive value ($f(r_h^{(0)}) > 0$ by construction), obviously inside the limits for which our approximation is still valid. In the remaining two cases the coordinate time interval goes to $+\infty$ for $r \to r_h^{(0)}$. This means that for $N_r(r_h^{(0)}) \leq 0$ the black hole behaves just like a Schwarzschild black hole, while for $N_r(r_h^{(0)}) > 0$ a particle can cross the horizon in a finite coordinate time and if we consider the limit of integrations to be from $r < r_h^{(0)}$ to $r_h^{(0)} - \delta$ (in this case $f^* < 0$) the interval

[2]We shall consider only the time intervals around the points of interest because we want to show only if they are finite or no, positive or no.

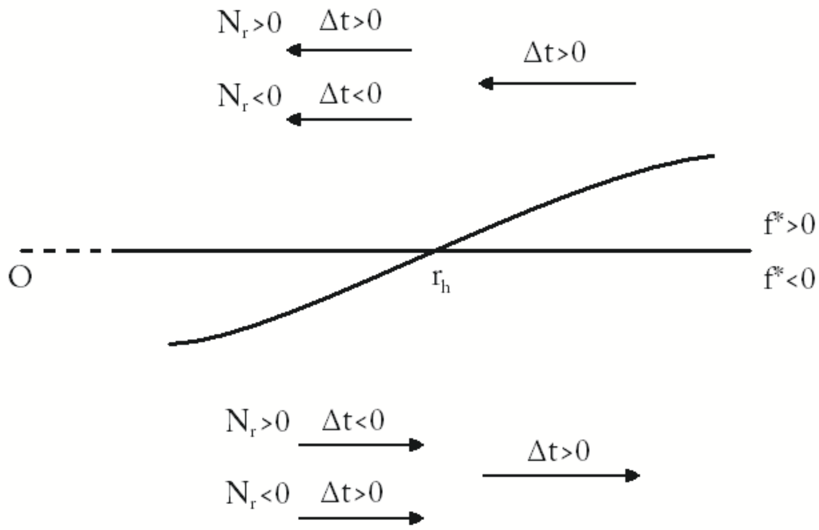

Figure 3.2: A particle crossing inward or outward a horizon with $f^{*\prime}(r_h)^{(i)} > 0$.

of time becomes negative and is divergent for $r \to r_h^{(0)}$. This last statement can be physically interpreted saying that for $N_r(r_h^{(0)}) \leq 0$ particles behind the horizon $(r < r_h^{(0)})$ can travel only outward. Will see that this is indeed possible once we shall look to the motion of outgoing particles.

There may exists an other horizon $r_h^{(1)}$ just behind $r_h^{(0)}$ but, in this case, $f^{*\prime}(r_h^{(1)}) < 0$. In a similar way we can show that we obtain the same results as before.

If there exist other horizons then we go back considering one of the to above cases.

Following the same steps we find that the situation for an outgoing particle for $f^{*'}(r_h^{(0)}) > 0$ (fig. 3.3) is reversed ($N_r > 0 \Leftrightarrow N_r < 0$), giving

$$\begin{aligned}
N_r(r_h^{(0)}) > 0: &\qquad \Delta t \simeq \int_r^{r_h+\delta} \frac{2dr}{f^{*'}(r_h^{(0)})(r-r_h^{(0)})} = \frac{2}{f^{*'}(r_h^{(0)})} \ln\left|\frac{\delta}{r-r_h^{(0)}}\right| \\
N_r(r_h^{(0)}) < 0: &\qquad \Delta t \simeq \int_r^{r_h+\delta} \frac{dr}{2f(r_h^{(0)})} = \frac{r_h^{(0)}+\delta-r}{2f(r_h^{(0)})} \\
N_r(r_h^{(0)}) = 0: &\qquad \Delta t \simeq \int_r^{r_h+\delta} \frac{dr}{f^{*'}(r_h^{(0)})(r-r_h^{(0)})} = \frac{1}{f^{*'}(r_h^{(0)})} \ln\left|\frac{\delta}{r-r_h^{(0)}}\right|.
\end{aligned}$$

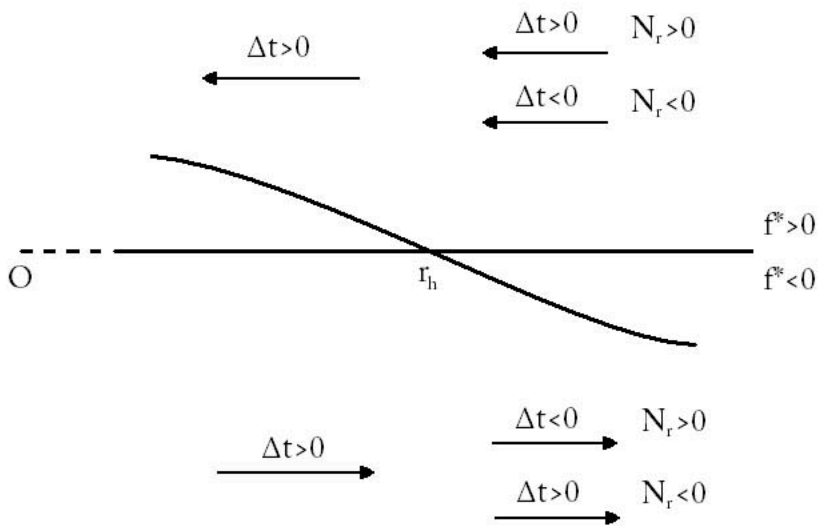

Figure 3.3: A particle crossing inward or outward a horizon with $f^{*'}(r_h)^{(i)} < 0$.

Therefore for $N_r(r_h^{(0)}) \geq 0$ the coordinate becomes infinite for $r \to r_h^{(0)}$ and in particular it is negative if the particle travels toward $r_h^{(0)}$ from inside. As before the case $f^{*'} < 0$ gives similar results.

Therefore we can deduce that while $N_r(r_h^{(i)}) > 0$ a photon can travel toward the center of the black hole in a finite coordinate time while if $N_r(r_h^{(i)}) < 0$ and the photon is an a region in which $f^*(r) > 0$, the photon take an infinite coordinate time to reach the horizon toward which is traveling. Moreover if $N_r(r_h^{(i)}) < 0$

and the photon is a region in which $f^* < 0$ the photon can travel only toward outside. If the photon is traveling toward outside the situation is completely reversed: particles can come out in a finite coordinate time for $N_r(r_h^{(i)}) < 0$ and need an infinite coordinate time to move away from an horizon if $f^*(r_h^{(i)}) > 0$ and $N_r(r_h^{(i)}) > 0$.

If there is an $r_h^{(i')} = r_f$ then $N_r(r_h^{(i')}) = 0$ and this horizon behaves just like in GR, that is, no photon can go away from the horizon surface in a finite coordinate time and no photon reaches the horizon in a finite coordinate time. In this last case the foliation structure also breaks down in this point so we will not worry about what happens inside the horizon.

The last case to consider is when a photon travels toward $r_f \neq r_h^{(i)}$. We already pointed out that $f^*(r_f) \leq 0$ (we already studied the case in which the equality is true so will exclude it from the following analysis) therefore we already know that for $N_r > 0$ we need to consider only photons moving toward r_f and for $N_r < 0$ only photons moving away from r_f:

$$\begin{array}{ll} N_r(r_f) > 0: & \Delta t \simeq -\int_{r_f+\delta}^{r} \frac{dr}{f^*(r_f)}\left[1-\sqrt{\frac{g^2(r_f)}{f'(r_f)(r-r_f)}}\right] \xrightarrow{r\,\to\,r_f} \frac{1}{f^*(r_f)}\left[\delta - 2\sqrt{\frac{g^2(r_f)\delta}{f'(r_f)}}\right] \\ N_r(r_f) < 0: & \Delta t \simeq \int_{r}^{r_f+\delta} \frac{dr}{f^*(r_f)}\left[1-\sqrt{\frac{g^2(r_f)}{f'(r_f)(r-r_f)}}\right] \xrightarrow{r\,\to\,r_f} \frac{1}{f^*(r_f)}\left[\delta - 2\sqrt{\frac{g^2(r_f)\delta}{f'(r_f)}}\right]. \end{array}$$

The above results are both finite and positive in our approximation ($\delta < 4\frac{g^2(r_f)}{f'(r_f)}$). This means that for $N_r(r_f) > 0$ the photon will hit in a finite coordinate time the singularity r_f while for $N_r(r_f) < 0$ photons can come out of the singularity in a finite coordinate time.

Again behind r_f it is not clear if it is possible to extend space-time.

Going back to the case in which $r_f = r_h^{(i)}$ for a given i, like in GR, we obtain that it is necessary a finite proper time to reach the horizon. The KS metric is an example:

the contribution to the proper time around ($\delta \ll r_h^{(0)}$) at the turning point $r_h^{(0)}$

for a radially falling (time-like) particle with energy εm

$$\Delta\tau = -\int_{r_h^{(0)}+\delta}^{r_h^{(0)}} \frac{dr}{\sqrt{\varepsilon^2 - f^*}} \simeq -\int_{r_h^{(0)}+\delta}^{r_h^{(0)}} \frac{dr}{\sqrt{f^{*\prime}(r_h^{(0)})(r_h^{(0)}+\delta-r)}} = 2\sqrt{\frac{\delta}{f^{*\prime}(r_h^{(0)})}}$$

is finite, being $f^{*\prime}(r_h^{(0)}) > 0$. In general if $f^{*\prime}(r_h^{(0)}) = 0$, then the integral is divergent. In particular for an energy $1-(2\omega M^2)^{1/3} \leq \epsilon < 1$, between the two horizons, the motion is periodic with a finite proper time period.

Being the proper time finite we can imagine that something like a Kruskal extension is possible. In GR the Kruscal extension shows that r_h is not a singular point but the procedure works because of the general covariance that allows us to consider the same solution in a non-singular coordinate frame system. Here we cannot perform any change of coordinates mixing space and time, so a Kruskal-like extension does not exist. On the contrary it is still possible that a particular interaction term for matter allows only well defined foliations.

An other point to consider in introducing an extension is the behavior of the singularity in $r = 0$.

Supposing that we are in the conditions for which a particle will hit the center of the system, what happens after the particle hits $r = 0$ is unclear because for $M \neq 0$ the slope of the KS metric goes like

$$f_0^{*\prime}(r) = 2\omega\left(r - \frac{\omega r^3 + M}{\sqrt{\omega^2 r^4 + 4\omega M r}}\right) \qquad \Rightarrow \qquad \lim_{r\to 0} f_0^{*\prime}(r) = -\infty$$

showing the presence of a singularity, a pinch (the Ricci scalar near $r = 0$ goes like $\mathcal{R} \simeq -\frac{6\sqrt{\omega M}}{r^{3/2}}$).

To have a smooth behavior at $r = 0$, that is, to have a space-time that looks locally flat at $r = 0$ letting the particle go through, we need $f^*(0)$ to be finite and $f^{*\prime}(0) = 0$. The first condition implies that $g^2(0)$ is finite while the second reduces to

$$\left[2\omega r - \frac{2\omega^2 r^3 + 2\omega M - 2\omega g^2 r - \omega (g^2)' r^2}{\sqrt{\omega^2 r^4 + 4\omega M r - 2\omega g^2 r^2}} - (g^2)'\right]_{r=0} = 0.$$

For $r \simeq 0$, $f^{*'}$ reduces to

$$2\omega r - \frac{2\omega M - \omega (g^2)' r^2}{\sqrt{4\omega M r}} - (g^2)' \simeq 0,$$

that is,

$$(g^2)' \simeq -\sqrt{\frac{\omega M}{r}} \qquad \Rightarrow \qquad g^2 \simeq -2\sqrt{\omega M r}$$

showing that we cannot have a smooth behavior at the origin for $M \neq 0$, then the presence of a point-mass still correspond to a singularity in space-time.

If we consider the case $M = 0$ with $N_r \neq 0$, then

$$\left[2\omega r - \frac{2\omega^2 r^3 - 2\omega g^2 r - \omega (g^2)' r^2}{\sqrt{\omega^2 r^4 - 2\omega g^2 r^2}} - (g^2)'\right]_{r=0} = 0.$$

For $r \simeq 0$, $f^{*'}$ reduces to ($g^2 < \omega r^2/2$ for $r \simeq 0$)

$$2\omega r - \frac{2\omega^2 r^3 - 2\omega g^2 r - \omega (g^2)' r^2}{\omega r^2} - (g^2)' \simeq 0,$$

that is, $\left.\frac{g^2}{r}\right|_{r=0} = 0$. This property means that are possible locally non-flat vacuum solutions with $M = 0$ and $N_r \neq 0$ and smooth in $r = 0$. In this case there must be some other source, other than M, responsible for an $N_r \neq 0$. This possibility will then depend strictly on the particular coupling with matter.

Here we do not consider any model for the collapse so we do not worry if it is possible to have trapped particle between two horizons during the collapse but we simply analyze how long it takes to move toward to or away from a horizon. As simple consequence we have that if a black hole has a radial shift vector toward outside then massless particles can travel in a finite coordinate time toward inside, while if the shift vector is directed inwardly massless particles can came out in a finite coordinate time.

With this definition we do not need to ask for any extension of the metric behind such a point because the geometric structure, the foliation, is not well defined; moreover, unlike GR,we are not supposed to consider the problem in a different set of coordinates because it would be unphysical. It is still possible that a particular coupling with matter or perhaps also the standard one would imply, once the collapse is studied, that the foliation is always well defined under certain physical conditions.

Chapter 4

Particle Kinematics in Hořava-Lifshitz Gravity

The anisotropy of Hořava gravity has as consequence on the particle dynamics that the Lorentz group is not anymore fundamental: a boost is not anymore a "physical" coordinate transformation corresponding to a change of foliation. In this chapter we are going to introduce a generalized particle dynamics to study the deviation from the standard relativistic behavior.

In [29, 30, 33, 34] several formulations were analyzed to introduce particles in the theory, mainly using as only prescription the reduced symmetry. In this chapter we will follow [32] and we will introduce the particle action as the optical limit of the scalar field action (1.16) described in section 1.5.

The action (1.17) is a deformation of the Klein-Gordon action in the sense that it introduces new interacting terms small in the IR behavior. Such a deformation has nothing to do with the deformation of the Hilbert-Einstein theory used to construct the Hořava-Lifshitz gravity, but it is allowed by the breaking of the Lorentz symmetry of space-time. This means that a scalar may still be described by the usual Klein-Gordon action without any consequence for the whole theory.

A deformed dynamics not invariant under the full Lorentz group may support, as we will see, non-standard kinematics as superluminal or massive luminal particles. These deviations from the standard behavior could be used for experimental tests to check whether or not the Lorentz invariance is broken. The results of such experiments may suggest the existence of new parameters, other than the mass, describing a particle, related to the λ's in (1.17), or may give, at least, an upper limit for them.

4.1 The Optical Limit

As guiding principle to construct the action describing the motion of a particle we can use the dynamical symmetries allowed by the gravitational theory in consideration. A different approach, that has built in it these symmetries, is to consider the action for a generalized scalar field, obtaining the dynamics of a particle in the same way we obtain the geometrical optics from the electromagnetic wave equation.

To obtain the **ray optical structure**[1] (See [57] for a review) which describes the optical limit behavior, we write the equation of motion for the scalar field ϕ obtained from the action (1.17) and express the scalar field as

$$\phi = Se^{i\psi} \tag{4.1}$$

to find the eikonal equation. In deriving the ray (optical) limit of the scalar matter field we will make the usual assumption that the wavelength of the ray λ_{w} is small compared to relevant gravitational length scales in the problem, and thus we will neglect all terms that are small compared to the inverse of the wavelength (the momentum of the particle). This includes derivatives of the wavefront S, because

[1]The ray optical structure H is the **Super Hamiltonian** of our system. This implies that the dispersion relation will be given by the condition $H = 0$.

we are considering almost planar wavefronts, as well as spatial derivatives of the four-momentum - defined in terms of the field ψ as $p_\mu \equiv \partial_\mu \psi$ - considering that it changes slowly ($\partial_\alpha p_\beta \ll p^2 \sim \frac{1}{\lambda_w^2}$). We will also neglect terms proportional to the derivatives of the metric (in the presence of terms proportional to the inverse wavelength) because we consider that the scale of variation of space-time quantities is large compared to the wavelength (the curvature is small compared to the square of the inverse wavelength). The details of this approximation are given in the Appendix D. Our approximation breaks down for high energy particles (which would gravitate) or for strong gravitational fields, therefore we are going to exclude these conditions.

The equation of motion obtained from the action (1.17), using the results in the appendix D, becomes the eikonal equation for the particle associated to the scalar field.

Note that the term $\nabla_\mu \nabla_\nu \phi$, with ϕ given by (4.1), can be expanded as

$$\nabla_\mu \nabla_\nu \phi = \nabla_\mu \partial_\nu \phi = \nabla_\mu (\partial_\nu S e^{i\psi} + i\phi \partial_\nu \psi)$$

and that, considering the approximation that the wave front is locally planar, $\partial_\nu S = 0$, it becomes

$$\simeq \nabla_\mu (i\phi \partial_\nu \psi) = i(\partial_\mu S e^{i\psi} \partial_\nu \psi + i\phi \partial_\mu \psi \partial_\nu \psi + i\phi \partial_\mu \partial_\nu \psi) \simeq -\phi \partial_\mu \psi \partial_\nu \psi$$

where in the last step we have used the fact that the four-momentum changes slowly. Thus we have the following expansions:

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi \simeq -\phi g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi$$

$$\Delta \phi = h^{\mu\nu} \nabla_\mu \nabla_\nu \phi \simeq -\phi h^{\mu\nu} \partial_\mu \psi \partial_\nu \psi$$

$$\Delta^n\phi \simeq \Delta^{n-1}(-\phi h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi) \simeq -\Delta^{n-1}(\phi)h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \simeq (-1)^n(h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi)^n\phi.$$

Hence the eikonal equation is given by

$$g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z}\tilde{\lambda}_{2,n}(h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi)^n = 0 \tag{4.2}$$

where the new constants $\tilde{\lambda}_{2,n} = \lambda_{2,n}M^{2(1-n)}$ are combinations of the old λ's appearing in the equation of motion for ϕ. The terms containing the $\tilde{\lambda}_{2,n}$'s will be considered small and treated as corrections to the usual eikonal equation derived from the unmodified Klein-Gordon action[2]. In contrast, $\tilde{\lambda}_{2,0}$ can be simply interpreted as the square of the mass of the particle[3]. From the eikonal equation we can deduce the ray optical structure H for the theory replacing $\partial_\mu\psi$ with the momenta p_μ; therefore the Hamiltonian is given by

$$H = \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}}\left\{g^{\mu\nu}p_\mu p_\nu + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z}\tilde{\lambda}_{2,n}(h^{\mu\nu}p_\mu p_\nu)^n\right\} = 0 \tag{4.3}$$

from which we deduce the following equations of motion with respect to an affine parameter τ analogous to proper time:

$$\dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha} = -\frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}}\left\{\partial_\alpha g^{\mu\nu}p_\mu p_\nu + \sum_{n=1}^{z}\tilde{\lambda}_{2,n}n(h^{\rho\lambda}p_\rho p_\lambda)^{n-1}\partial_\alpha h^{\mu\nu}p_\mu p_\nu\right\} \tag{4.4}$$

$$\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}}\left\{g^{\alpha\nu}p_\nu + \sum_{n=1}^{z}\tilde{\lambda}_{2,n}n(h^{\mu\nu}p_\mu p_\nu)^{n-1}h^{\alpha\nu}p_\nu\right\} \tag{4.5}$$

(as usual, overdot stands for τ differentiantion). For the massless case we find the

[2]From now on, when we speak generally of $\tilde{\lambda}_{2,n}$'s or simply of $\tilde{\lambda}$'s, we will always refer to the $\tilde{\lambda}_{2,n}$'s with $n \geq 1$.

[3]In this context the velocity of light c, the emerging velocity of light in the IR limit, is just a conversion constant and will eventually be set to 1.

ray optical structure to be

$$H = \frac{1}{2}\left\{ g^{\mu\nu}p_\mu p_\nu + \sum_{n=1}^{z}\tilde{\lambda}_{2,n}(h^{\mu\nu}p_\mu p_\nu)^n \right\} = 0 \tag{4.6}$$

that yields the following equations of motion

$$\dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha} = -\frac{1}{2}\left\{ \partial_\alpha g^{\mu\nu}p_\mu p_\nu + \sum_{n=1}^{z}\tilde{\lambda}_{2,n} n (h^{\rho\lambda}p_\rho p_\lambda)^{n-1}\partial_\alpha h^{\mu\nu}p_\mu p_\nu \right\} \tag{4.7}$$

$$\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha} = \left\{ g^{\alpha\nu}p_\nu + \sum_{n=1}^{z}\tilde{\lambda}_{2,n} n (h^{\mu\nu}p_\mu p_\nu)^{n-1} h^{\alpha\nu}p_\nu \right\}. \tag{4.8}$$

In the upcoming sections we will analyze the motion of a particle for a Minkowski space-time and for a static spherically symmetric metric. In the next section, instead, we will study the corrections to the geodesic equation due to the deformed kinematics.

4.1.1 Corrections to the Geodesic Equation

The deformed optical structure tells us essentially that the free-falling motion of a particle will only be approximately a geodesic of the metric $g_{\mu\nu}$. Here we will evaluate the first correction in the $\tilde{\lambda}$'s to the geodesic equation. To write down the exact equation of motion we need to invert $\dot{x}^\alpha$, finding p_μ as a function of it. Here, because corrections higher than first order in the $\tilde{\lambda}$'s are negligible for low energies, we will retain only terms first order in the $\tilde{\lambda}$'s.

Differentiating (4.5) with respect to the parameter τ we have:

$$\begin{aligned}\ddot{x}^\alpha = &\frac{1}{\sqrt{\tilde{\lambda}_{2,0}}}\left[\sqrt{\tilde{\lambda}_{2,0}} g_{\nu\gamma}\dot{x}^\gamma \partial_\beta g^{\alpha\nu}\dot{x}^\beta - \partial_\beta g^{\alpha\nu}\dot{x}^\beta \sum_{n=1}^{z}\tilde{\lambda}_{2,n} n (h^{\mu\nu}p_\mu p_\nu)^{n-1} h_\nu{}^\gamma p_\gamma \right] + \\ &+\frac{g^{\alpha\nu}\dot{p}_\nu}{\sqrt{\tilde{\lambda}_{2,0}}} + \sum_{n=1}^{z}\frac{\tilde{\lambda}_{2,n} n}{\sqrt{\tilde{\lambda}_{2,0}}}\Big[(n-1)(h^{\mu\nu}p_\mu p_\nu)^{n-2}(\dot{h}^{\mu\nu}p_\mu p_\nu + 2h^{\mu\nu}\dot{p}_\mu p_\nu)h^{\alpha\nu}p_\nu + \\ &+(h^{\mu\nu}p_\mu p_\nu)^{n-1}(\dot{h}^{\alpha\nu}p_\nu + h^{\alpha\nu}\dot{p}_\nu)\Big].\end{aligned}$$

The momentum p_α, using the equation of motion (4.5), can be expressed as

$$p_\alpha = \sqrt{\tilde{\lambda}_{2,0}} g_{\alpha\beta}\dot{x}^\beta - \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (h^{\mu\nu} p_\mu p_\nu)^{n-1} h_\alpha{}^\nu p_\nu.$$

Moreover, using the expression above, we have the following approximation:

$$h^{\mu\nu} p_\mu p_\nu \simeq \tilde{\lambda}_{2,0} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + O(\tilde{\lambda}).$$

Substituting the two expressions above for p_α and $h^{\mu\nu}p_\mu p_\nu$ in (4.4) we have

$$\dot{p}_\alpha = -\frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}} \partial_\alpha g^{\mu\nu} p_\mu p_\nu - \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}} \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (h^{\rho\lambda} p_\rho p_\lambda)^{n-1} \partial_\alpha h^{\mu\nu} p_\mu p_\nu \simeq$$

$$\frac{\sqrt{\tilde{\lambda}_{2,0}}}{2} \left\{ \partial_\alpha g_{\beta\delta} \dot{x}^\beta \dot{x}^\delta - [2\partial_\alpha g_{\mu\beta} h^{\mu\gamma} g_{\delta\gamma} + \partial_\alpha h^{\mu\nu} g_{\mu\delta} g_{\nu\beta}] \left[\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (\tilde{\lambda}_{2,0} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{n-1} \right] \dot{x}^\delta \dot{x}^\beta \right\}$$

Finally, putting everything together we have

$$\dot{x}^\beta \nabla_\beta \dot{x}^\alpha \simeq \left[-\partial_\beta g^{\alpha\nu} \dot{x}^\beta \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (\tilde{\lambda}_{2,0} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{n-1} h_\nu{}^\gamma g_{\gamma\delta} \dot{x}^\delta \right] +$$

$$+ \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n \left[\left(-g^{\alpha\nu} \partial_\nu g_{\mu\beta} h^{\mu\gamma} g_{\delta\gamma} - \frac{1}{2} g^{\alpha\nu} \partial_\nu h^{\mu\gamma} g_{\mu\delta} g_{\gamma\beta} \right) (\tilde{\lambda}_{2,0} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{n-1} \dot{x}^\delta \dot{x}^\beta + \right.$$

$$+ (n-1)(\tilde{\lambda}_{2,0} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{n-2} (\tilde{\lambda}_{2,0} \dot{h}^{\beta\gamma} g_{\beta\delta} g_{\gamma\rho} \dot{x}^\rho \dot{x}^\delta + \tilde{\lambda}_{2,0} h^{\beta\gamma} \partial_\beta g_{\delta\rho} \dot{x}^\delta \dot{x}^\rho g_{\gamma\sigma} \dot{x}^\sigma) h^{\alpha\nu} g_{\nu\phi} \dot{x}^\phi +$$

$$\left. + (\tilde{\lambda}_{2,0} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{n-1} (\dot{h}^{\alpha\nu} g_{\nu\delta} \dot{x}^\delta + \frac{1}{2} h^{\alpha\nu} \partial_\nu g_{\gamma\delta} \dot{x}^\gamma \dot{x}^\delta) \right] + O(\tilde{\lambda}^2).$$

As expected, the geodesic equation, relative to the metric $g_{\mu\nu}$, is recovered as the zero-order approximation in the $\tilde{\lambda}$'s.

In this deformed kinematics the equation of motion depends on the values of the $\tilde{\lambda}$'s and hence each particle, having different $\tilde{\lambda}$'s, will follow a slightly different trajectory; this makes possible to verify experimentally if the kinematics is deformed or not. Moreover an interesting feature is that the motion will depend also on the mass of the particle.

Similarly for the massless case $\tilde{\lambda}_{2,0} = 0$ we obtain

$$\dot{x}^\beta \nabla_\beta \dot{x}^\alpha \simeq -\partial_\beta g^{\alpha\nu} \dot{x}^\beta \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{n-1} h_\nu{}^\gamma g_{\gamma\delta}\dot{x}^\delta +$$

$$+ \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n \left[\left(-g^{\alpha\nu}\partial_\nu g_{\mu\beta} h^{\mu\gamma} g_{\delta\gamma} - \frac{1}{2} g^{\alpha\nu}\partial_\nu h^{\mu\gamma} g_{\mu\delta} g_{\gamma\beta} \right) (h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{n-1} \dot{x}^\delta \dot{x}^\beta + \right.$$

$$+ (n-1)(h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{n-2} (\dot{h}^{\beta\gamma} g_{\beta\delta} g_{\gamma\rho} \dot{x}^\rho \dot{x}^\delta + h^{\beta\gamma}\partial_\beta g_{\delta\rho} \dot{x}^\delta \dot{x}^\rho g_{\gamma\sigma}\dot{x}^\sigma) h^{\alpha\nu} g_{\nu\phi}\dot{x}^\phi +$$

$$\left. + (h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{n-1} (\dot{h}^{\alpha\nu} g_{\nu\delta}\dot{x}^\delta + \frac{1}{2} h^{\alpha\nu}\partial_\nu g_{\gamma\delta}\dot{x}^\gamma \dot{x}^\delta) \right] + O(\tilde{\lambda}^2).$$

We can give to the above modified geodesics a geometrical interpretation in terms of the standard metric by interpreting the right hand side of the geodesic equation as an extra velocity-dependent gravitational force due to the coupling of the matter field to the vector field n_α. Such a coupling implies that the force depends on the foliation, as expected from the fact that the foliation has a physical meaning in Hořava-Lifshitz gravity. Here we will not analyze this issue because we are interested in studying the motion of test particles and hence we keep fixed the background.

4.2 The Minkowski Case

We will study the behavior of particles in a curved - static and spherically symmetric - gravitational field in section 4.3. For the moment we will concentrate on the flat case to generalize Special Relativity.

The full Hořava-Lifshitz gravity is invariant under local rotations and translations but this symmetry is not enough to construct the kinematics without having any prescription on how to add velocities; it is so because there is no allowed transformation mixing space and time.

We shall study the modification in the kinematics considering the action of the Poincaré group, mainly for two reasons: the first is that Special Relativity and General Relativity are well verified theories and are based on the Lorentz group; the second is related to the first but it is more practical: we used low-energy light, which appears to have the same property light has in Special Relativity, as test particles to define operationally "inertial" frame systems.

As noted in [18] the action (2.28) admits as solution the Minkowski vacuum with $g = \{-1,1,1,1\}$ and $n_\alpha = (-1,0,0,0)$, that is, with $h = \{0,1,1,1\}$. This solution corresponds to a particular choice of coordinates. In general, we also have to consider how to pass from one coordinate frame system to another. In this section we will call the frame system in which the metric is given by $g = \{-1,1,1,1\}$ and $h = \{0,1,1,1\}$ the "*preferred frame system*" and we will introduce the notion of "*inertial frame systems*" which will have the same Minkowskian metric $g = \{-1,1,1,1\}$ but a different $h^{\mu\nu}$. The group of coordinate transformations which leave invariant the Minkowski metric is the Poincaré group, but the ray optical structure is invariant only under rotations and translations[4]. In this context, although the metric remains invariant, we move from the preferred frame system in which $n_\alpha = (-1,0,0,0)$ to others in which this vector, and consequently $h^{\mu\nu}$, will be different. The dynamics of particles then will be different in these frames allowing, in principle, to distinguish between any inertial frame system and the preferred one. Therefore, in such a theory the preferred frame system plays the role of an absolute frame system.

[4]The linear transformations of coordinates that leave unchanged the metric $g = \{-1,1,1,1\}$ and the vector $n_\alpha = (-1,0,0,0)$ are spatial rotations and spacial and time translations:

$$\mathbf{x}' = R\mathbf{x} + \mathbf{x}_0 \quad t' = t + t_0.$$

4.2.1 Massive Particles in the "Preferred Frame System"

Let us start considering the equation of motion in the preferred frame system ($h = \{0, 1, 1, 1\}$); we will choose for simplicity and w.l.o.g.[5] that the particle is moving along the x direction with $p_y = p_z = 0$. Then, the equations (4.4,4.5) reduce to

$$\dot{t} = -\frac{p_t}{\sqrt{\tilde{\lambda}_{2,0}}} \quad \dot{x} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ 1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)} \right\} p_x \quad \dot{y} = 0 \qquad \dot{z} = 0 \qquad (4.9)$$

and

$$\dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha} = 0. \qquad (4.10)$$

Note that in this case the conditions $p_x = 0$ and $\dot{x} = 0$ are consistent; that is, when a particle is at rest the linear momentum is zero, which is not the case in other frame systems.

Integrating with respect to the parameter τ, we obtain the trajectories

$$\begin{aligned} t(\tau) &= -\frac{p_t}{\sqrt{\tilde{\lambda}_{2,0}}}\tau + t_0, \\ x(\tau) &= \left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)} \right] \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} p_x \tau + x_0, \\ y(\tau) &= y_0 \qquad \text{and} \qquad z(\tau) \;=\; z_0 \end{aligned} \qquad (4.11)$$

with the dispersion relation following from (4.3)

$$-p_t^2 + p_x^2 + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} p_x^{2n} = 0. \qquad (4.12)$$

The momentum p_x^2 can be bounded from above or not, depending on the values

[5] The action is invariant under the spacial rotation group.

of the $\tilde{\lambda}_{2,n}$'s as is evident from the consistency condition[6]

$$p_t^2 = p_x^2 + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} p_x^{2n} \geq 0. \tag{4.13}$$

If the preferred frame is the rest frame system of a particle we have $p_x = 0$ and, from the dispersion relation, $p_t = -\sqrt{\tilde{\lambda}_{2,0}}$. In this case τ can be interpreted as the proper time for this particle, but in general τ will be just an affine parameter.

To study how the motion changes moving from one "inertial frame system" to another we need to understand the kinematics of particles moving with constant velocity in the preferred frame system, that is, to solve for p_x as a function of the constant velocity $v = \frac{\dot{x}}{\dot{t}}$. From the equation of motion (4.9) we have:

$$v_x = \frac{dx}{dt} = -\left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right] \frac{p_x}{p_t} \quad \rightarrow \quad p_t = -\left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right] \frac{p_x}{v}. \tag{4.14}$$

Inserting the last relation in the dispersion relation (4.12), without any approximation, we have

$$-p_x^2 \frac{1}{\gamma^2 v^2} + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} \left(1 - \frac{2n}{v^2}\right) p_x^{2n} - \left(\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right)^2 \frac{p_x^2}{v^2} = 0$$

which, approximated to the first order in the $\tilde{\lambda}_{2,n}$'s and for $v \ll 1$ ($v \ll c$), becomes

$$p_x^2 - \tilde{\lambda}_{2,0} \gamma^2 v^2 - \sum_{n=1}^{z} \tilde{\lambda}_{2,n} \gamma^2 (v^2 - 2n) p_x^{2n} \simeq 0 \qquad \left(\gamma^2 \equiv \frac{1}{1 - v^2}\right).$$

[6] From the dispersion relation we have

$$-p_t^2 + p_x^2 = -\tilde{\lambda}_{2,0} - \sum_{n=1}^{z} \tilde{\lambda}_{2,n} p_x^{2n}$$

therefore there is no condition on the four momentum to be space-, time- or light-like.

Without any approximation we have

$$v^2 = \left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right]^2 \frac{p_x^2}{\tilde{\lambda}_{2,0} + p_x^2 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} p_x^{2n}}. \tag{4.15}$$

Note that the behavior of v^2 depends on the sign of the $\tilde{\lambda}_{2,n}$'s and, although it must be always positive under the condition (4.13), it may not be a monotonic increasing function of p_x. To study the behavior of v^2 as a function of p_x^2 we need to know the positive roots and the stationary points of the polynomial $\tilde{\lambda}_{2,0}+p_x^2+\sum_{n=1}^{z} \tilde{\lambda}_{2,n} p_x^{2n}$. Concerning the roots, we can simply see that if the polynomial has at least one positive root then v^2 goes to $+\infty$ for a finite momentum p_x but in general not monotonically; if the polynomial has no positive roots then v^2 will go, in general not monotonically, to $+\infty$ for $p_x \to \infty$ if $\tilde{\lambda}_{2,n} \neq 0$ for at least one $n > 1$; if $\tilde{\lambda}_{2,n} = 0$ for $n > 1$ then the particle will always have a finite velocity. The non-monotonic behavior means that the velocity can be 0 even for non-vanishing p_x, as in the case in which the first derivative of the polynomial, that is, $1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}$, has positive roots.

In the extreme case $p_x \to \infty$ the validity of our model breaks down because the energy of the particle becomes strong enough to gravitate and modify the gravitational background. (We will study the transition from subluminal to superluminal speed with an example in section 4.2.5.)

In the zero order expansion in the $\tilde{\lambda}_{2,n}$'s, (4.15) reduces to

$$v^2 = \frac{p_x^2}{\tilde{\lambda}_{2,0} + p_x^2}$$

which corresponds to the relativistic relation

$$p_x^2 = \tilde{\lambda}_{2,0} \gamma^2 v^2$$

reproducing the expected behavior (with the usual definition of the relativistic factor γ). Now we can use the Newton algorithm to find the roots of a polynomial using as starting point the value $\tilde\lambda_{2,0}\gamma^2 v^2$, the other terms of the polynomial being small corrections. After the first step[7] we have

$$p_x^2 \simeq \tilde\lambda_{2,0}\gamma^2 v^2 - \frac{-\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^n\gamma^2[v^2-2n]}{1-\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^{n-1}\gamma^2 n[v^2-2n]} \simeq$$

$$\tilde\lambda_{2,0}\gamma^2 v^2 + \sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^n\gamma^2[v^2-2n].$$

Therefore, p_x and p_t of a particle moving at a constant velocity v in the preferred frame system, at the first order in the $\tilde\lambda$'s, are

$$p_x \simeq \left[1+\frac{1}{2}\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^{n-1}\gamma^2[v^2-2n]\right]\sqrt{\tilde\lambda_{2,0}}\,\gamma v \tag{4.16}$$

$$p_t \simeq -\left[1-\frac{1}{2\tilde\lambda_{2,0}}\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^{n}[2n-1]\right]\sqrt{\tilde\lambda_{2,0}}\,\gamma \tag{4.17}$$

where to evaluate p_t we used the relation (4.14). The first order equations of motion then become

$$t(\tau) \simeq \left[1-\frac{1}{2\tilde\lambda_{2,0}}\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^{n}[2n-1]\right]\gamma\tau + t_0 \tag{4.18}$$

$$x(\tau) \simeq \left[1-\frac{1}{2\tilde\lambda_{2,0}}\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^{n}[2n-1]\right]\gamma v\tau + x_0. \tag{4.19}$$

In section 4.2.3 we will obtain the equations of motion in a generic inertial frame system. Before that, however, we need to construct operationally a notion

[7]

$$(p^2)_0 = \tilde\lambda_{2,0}\gamma^2 v^2 \quad y_0 = -\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^n\gamma^2[v^2-2n]$$

$$m = \left.\frac{\partial y}{\partial p^2}\right|_{(p^2)_0} = 1-\sum_{n=1}^{z}\tilde\lambda_{2,n}(\tilde\lambda_{2,0}\gamma^2 v^2)^{n-1}\gamma^2 n[v^2-2n]$$

of "inertial frame"; this will be done in the next section.

4.2.2 Massless Particles

The equations of motion for a massless particle ($\tilde{\lambda}_{2,0} = 0$) moving along the x-axis in the preferred frame system, as derived from the Hamiltonian (4.6) are

$$\dot{t} = -p_t \qquad \dot{x} = \left\{ 1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (p_x)^{2(n-1)} \right\} p_x \qquad \dot{y} = 0 \qquad \dot{z} = 0.$$

Following the same procedure as in the massive case, in the massless case we find

$$v_x = \frac{dx}{dt} = -\left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)} \right] \frac{p_x}{p_t} \quad \rightarrow \quad p_t = -\left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)} \right] \frac{p_x}{v}. \tag{4.20}$$

Inserting the last relation into the dispersion relation we have

$$v^2 = \frac{\left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)} \right]^2}{1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} p_x^{2(n-1)}} \simeq \left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} (2n-1) p_x^{2(n-1)} \right].$$

Note that low energy massless particles have a constant velocity of $\sqrt{1 + \tilde{\lambda}_{2,1}}$ and that they behave as in special relativity, that is,

$$-p_t = |p_x|$$

since in this range no $\tilde{\lambda}$'s, other than $\tilde{\lambda}_{2,1}$, appear in the relations. We can use this property to define "inertial" frame systems as the frames in which light move with a constant velocity. We will single out one kind of massless particles, namely photons, and use their speed $c \equiv \sqrt{1 + \tilde{\lambda}_{2,1}^{light}}$ as a conversion factor between space and time. The constant speed of light c can then be set equal to 1 with a rescaling of time units, and this is what we will assume in what will follow (this rescaling corresponds to set $\tilde{\lambda}_{2,1}^{light} = 0$).

To define an "inertial" frame system we start with a frame in which a photon with low energy travels at a constant speed equal to 1. Such a definition of "inertial" frame leads us to consider only frame systems obtained by applying the usual Lorentz transformations[8]. Indeed, only such transformations preserve the relation

$$\dot{t}^2 - \dot{x}^2 = p_t^2 - p_x^2 = 0$$

in the low energy regime. This, in particular, means that in such frame systems, lengths and time intervals change as in special relativity. Further, in a generic "non-inertial" frame the metric will not be Minkowskian any more and hence the equations of motion will be different from

$$\dot{x} = p_x \qquad \dot{t} = -p_t. \tag{4.21}$$

As an example, let us consider a boost with velocity u along the x-direction with respect an "inertial frame". Then the equations of motion change as follows:

$$\dot{x}' = \gamma(\dot{x} - u\dot{t}) = \gamma(p_x - up_t) \qquad \dot{t}' = \gamma(\dot{t} - u\dot{x}) = \gamma(p_t - up_x).$$

Considering that in the new frame system, by definition of "inertial frame", we must have the same kind of equations of motion as (4.21), we deduce that

$$p_x' = \gamma(p_x - up_t) \qquad p_t' = \gamma(p_t - up_x).$$

Therefore, as expected, the use of light to define an inertial frame implies that we need to use the usual Lorentz transformations to go from one inertial frame to an other.

[8]We consider as a more appropriate definition of an "inertial frame system" the unique coordinates and units obtained using Lorentz transformations, once we fixed the units in one of the "inertial" frame systems.

4.2.3 Particle Motion in a Generic "Inertial" Frame System

Consider a particle in the origin of the rest frame O' that coincide with the origin O of a moving frame at $t_0 = t'_{0'} = 0$; moreover let us assume that these frames have parallel spatial axes and that the moving frame is moving with a velocity $-u$ along the x-direction. Consider for the moment O' to be the preferred frame ($h'^{\alpha\beta} = \{0, 1, 1, 1\}$). Then the optical structure takes the following form in O'

$$H = \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}} \left\{ \eta'^{\mu\nu} p'_\mu p'_\nu + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} (h'^{\mu\nu} p'_\mu p'_\nu)^n \right\}.$$

Using the fact that the new coordinates are related to the primed ones by the boost

$$x = \gamma(x' + ut') \qquad t = \gamma(t' + ux')$$

and that the quantities appearing in H are vectors and tensors and hence transform with the matrix $\frac{\partial x'^\alpha}{\partial x^\beta}$, we have that, in O, the metric becomes

$$h^{\alpha\beta} = \begin{pmatrix} \gamma^2 u^2 & \gamma^2 u & 0 & 0 \\ \gamma^2 u & \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \eta^{\alpha\beta} = \eta'^{\alpha\beta} \qquad n_\alpha = (-\gamma, \gamma u, 0, 0) \tag{4.22}$$

and H in the new frame system can be written as

$$H = \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}} \left\{ -p_t^2 + p_x^2 + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} [\gamma^2 (up_t + p_x)^2]^n \right\}$$

yielding to the following equations of motion:

$$\dot{t} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ -p_t + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n [\gamma(up_t + p_x)]^{2n-1} \gamma u \right\}$$

$$\dot{x} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ p_x + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n [\gamma(up_t + p_x)]^{2n-1} \gamma \right\} \quad \dot{y} = 0 \quad \dot{z} = 0$$

with all the momenta constant and related to the primed momenta, being covectors, by the same transformation rules

$$p_t = \gamma(p'_t - up'_x) = -\gamma\sqrt{\tilde{\lambda}_{2,0}} \quad p_x = \gamma(p'_x - up'_t) = \gamma\sqrt{\tilde{\lambda}_{2,0}}u.$$

This means that $up_t + p_x = p'_x/\gamma = 0$ and hence the kinematics is described by the equation of motion

$$\dot{t} = -\frac{p_t}{\sqrt{\tilde{\lambda}_{2,0}}} \quad \dot{x} = \frac{p_x}{\sqrt{\tilde{\lambda}_{2,0}}} \quad \dot{y} = 0 \quad \dot{z} = 0$$

with the usual dispersion relation

$$-p_t^2 + p_x^2 = 0.$$

Now let us consider the same situation with an O', the rest frame of the particle, which is not the preferred frame but it is itself moving with a constant velocity $-v^{(r)}$ with respect to the absolute frame system. Using the fact that in O' the metric takes the same form (4.22) with $v^{(r)}$ instead of u, labeling the four-momentum in O' with ${}^{(r)}$, the equations of motion in O' are given by

$$\begin{aligned} \dot{t}' &= \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ -p_t^{(r)} + \textstyle\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n [\gamma^{(r)}(v^{(r)} p_t^{(r)} + p_x^{(r)})]^{2n-1} \gamma^{(r)} v^{(r)} \right\} \\ \dot{x}' &= \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ p_x^{(r)} + \textstyle\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n [\gamma^{(r)}(v^{(r)} p_t^{(r)} + p_x^{(r)})]^{2n-1} \gamma^{(r)} \right\} \quad \dot{y}' = 0 \quad \dot{z}' = 0 \end{aligned} \tag{4.23}$$

with the dispersion relation

$$-{p^{(r)}}^2_t + {p^{(r)}}^2_x + \tilde{\lambda}_{2,0} + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} [{\gamma^{(r)}}^2 (v^{(r)} p_t^{(r)} + p_x^{(r)})^2]^n = 0.$$

Since O' is the particle's rest frame, we have the condition $\dot{x}' = 0$; from this, using the Newton method with starting point $p_x^{(r)} = 0$, we obtain

$$p_x^{(r)} \simeq -\frac{\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n({\gamma^{(r)}}^2 {v^{(r)}}^2 {p^{(r)}_t}^2)^{n-1} {\gamma^{(r)}}^2 v^{(r)} p_t^{(r)}}{1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n(2n-1)({\gamma^{(r)}}^2 {v^{(r)}}^2 {p^{(r)}_t}^2)^{n-1} {\gamma^{(r)}}^2} \simeq$$

$$-\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n({\gamma^{(r)}}^2 {v^{(r)}}^2 {p^{(r)}_t}^2)^{n-1} {\gamma^{(r)}}^2 v^{(r)} p_t^{(r)}.$$

We can plug this approximate result into the dispersion relation to find p_t using the Newton method with starting point ${p_t^{(r)}}^2 = \tilde{\lambda}_{2,0}$ obtaining

$$\begin{aligned} p_t^{(r)} &\simeq -\sqrt{\tilde{\lambda}_{2,0}} \left[1 + \tfrac{1}{2\tilde{\lambda}_{2,0}} \sum_{n=1}^{z} \tilde{\lambda}_{2,n} ({\gamma^{(r)}}^2 {v^{(r)}}^2 \tilde{\lambda}_{2,0})^n \right] \\ p_x^{(r)} &\simeq -\sqrt{\tilde{\lambda}_{2,0}} \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n ({\gamma^{(r)}}^2 {v^{(r)}}^2 \tilde{\lambda}_{2,0})^{n-1} {\gamma^{(r)}}^2 v^{(r)}. \end{aligned} \tag{4.24}$$

It is evident, then, that only particles at rest in the preferred absolute frame have Lorentz-like dispersion relations and that, in general, the linear momentum is not zero, even if the particle is at rest. Moreover the dispersion relation depends from the velocity $v^{(r)}$, the velocity of the particle with respect to the preferred frame system.

Boosting from a non-preferred "inertial" rest frame to another will then make the kinematics, through the momenta, $\tilde{\lambda}$-dependent making possible experimental verification regarding the existence of such parameters. For example, in section 4.2.3 we show how the results of a scattering experiment turn out to be frame dependent, while in section 4.2.5 we show that a particle, under certain conditions, can reach superluminal velocities by applying a finite constant force.

To evaluate the equation of motion of a particle in a generic "inertial" frame system we start with the previous result; therefore we will consider as given the values of $p_x^{(r)}$ and $p_t^{(r)}$ in the particle rest frame. Noting that in the rest frame we have the condition $\dot{x}^{(r)} = 0$ we simply have that

$$p_x^{(r)} = -\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n [\gamma^{(r)}(v^{(r)} p_t^{(r)} + p_x^{(r)})]^{2n-1} \gamma^{(r)} = -\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n [h^{\mu\nu} p_\mu p_\nu]^{n-1/2} \gamma^{(r)} \tag{4.25}$$

where $v^{(r)}$ is the relative velocity between the "inertial" rest frame of the particle and the preferred frame. Thus the equation of motion in a generic "inertial" frame system for a particle moving with a velocity u, without any approximation, are

$$\begin{aligned} \dot{t} &= \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ -p_t - \frac{p_x^{(r)}}{\gamma^{(r)}} \gamma^{(T)} u^{(T)} \right\} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ -p_t - p_x^{(r)} \gamma_u (v^{(r)} + u) \right\} \\ \dot{x} &= \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ p_x - \frac{p_x^{(r)}}{\gamma^{(r)}} \gamma^{(T)} \right\} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left\{ p_x - p_x^{(r)} \gamma_u (1 + u v^{(r)}) \right\} \quad \dot{y} = 0 \quad \dot{z} = 0 \end{aligned} \tag{4.26}$$

where

$$u^{(T)} = \frac{v^{(r)} + u}{1 + v^{(r)} u} \quad \gamma^{(T)} = \gamma^{(r)} \gamma_u (1 + u v^{(r)}).$$

In the massless case we can follow the same procedure arriving to the following equations of motion:

$$\begin{aligned} \dot{t} &= \left\{ -p_t - \frac{p_x^{(r)}}{\gamma^{(r)}} \gamma^{(T)} u^{(T)} \right\} = \left\{ -p_t - p_x^{(r)} \gamma_u (v^{(r)} + u) \right\} \\ \dot{x} &= \left\{ p_x - \frac{p_x^{(r)}}{\gamma^{(r)}} \gamma^{(T)} \right\} = \left\{ p_x - p_x^{(r)} \gamma_u (1 + u v^{(r)}) \right\} \quad \dot{y} = 0 \quad \dot{z} = 0. \end{aligned} \tag{4.27}$$

The expressions for $p_x^{(r)}$ and $p_t^{(r)}$, however will be different. As in the massive case from the condition $\dot{x}' = 0$ we obtain

$$p_x^{(r)} \simeq -\frac{\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n ({\gamma^{(r)}}^2 {v^{(r)}}^2 {p^{(r)}}_t^2)^{n-1} {\gamma^{(r)}}^2 v^{(r)} p_t^{(r)}}{1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (2n-1) ({\gamma^{(r)}}^2 {v^{(r)}}^2 {p^{(r)}}_t^2)^{n-1} {\gamma^{(r)}}^2} \simeq$$

$$-\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n ({\gamma^{(r)}}^2 {v^{(r)}}^2 {p^{(r)}}_t^2)^{n-1} {\gamma^{(r)}}^2 v^{(r)} p_t^{(r)}.$$

which, substituted in the dispersion relation

$$-{p^{(r)}}_t^2 + {p^{(r)}}_x^2 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} [{\gamma^{(r)}}^2 (v^{(r)} p_t^{(r)} + p_x^{(r)})^2]^n = 0,$$

at the first order in the $\tilde{\lambda}$'s, yields

$$-{p^{(r)}}_t^2 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} [\gamma^{(r)} v^{(r)} p_t^{(r)}]^{2n} = 0.$$

The first solution, $p_t^{(r)} = 0$, must be rejected because it implies also $p_x^{(r)} = 0$, that is, the particle has vanishing energy and momentum. This corresponds to the familiar result, valid for $\tilde{\lambda} = 0$, that no photons at rest exist. The physical solution for a massless particle at rest, to first order in $\tilde{\lambda}$'s, would correspond to the roots of the polynomial

$$\sum_{n=1}^{z} \tilde{\lambda}_{2,n} [\gamma^{(r)} v^{(r)} p_t^{(r)}]^{2(n-1)} = 1.$$

In this case we see that the generic solution would involve large momenta, of order $p_t^{(r)} \sim \tilde{\lambda}_{2,2}^{-1/2}$ or similar, in which case the first-order approximation breaks down as we cannot neglect higher order terms. The existence and magnitude of solutions is hard to estimate, especially since we do not have any information about massless particles at rest, nor any indication of their existence.

The equations of motion (4.26) and (4.27) are the same equations we obtain by boosting the rest frame equation of motion. For example there is no difference, with respect to the prediction of Special Relativity, in the dilation of time

$$\frac{\Delta t}{\Delta t^{(r)}} = \frac{\dot{t}}{\dot{t}^{(r)}} = \frac{-p_t - p_x^{(r)} \gamma_u (v^{(r)} + u)}{-p_t^{(r)} - p_x^{(r)} v^{(r)}} = \frac{-\gamma_u (p_t^{(r)} - u p_x^{(r)}) - p_x^{(r)} \gamma_u (v^{(r)} + u)}{-p_t^{(r)} - p_x^{(r)} v^{(r)}} = \gamma_u$$

or in the contraction of lengths. Therefore it looks like if nothing is really changed.

But still, the kinematics of a particle, as is evident from equations (4.26), strictly depends on the relative velocity between the particle and the preferred frame. This will be evident once we study the dynamics in the next section. Then we will complete the study of the particle motion in section 4.2.5 analyzing the general case of luminal and superluminal particles.

4.2.4 Scattering

Suppose we have two identical particles[9] in an "inertial" frame O, with respect to which the preferred frame system P is moving with velocity u; the particles P_1 and P_2 are moving, respectively, with a velocity $-v$ and a velocity v symmetrically toward the origin O. After they collide, a unique particle is created. We want to find the dependence of this scattering on the particular "inertial" frame system.

The dynamics is described by the super Hamiltonian $H = H_1 + H_2$ before the collision and, after, by H_T [the formal structure of the Hamiltonians does not change with respect to (4.3)]. The total conserved time component of the momentum is

$$(p_t)_1 + (p_t)_2 = \gamma_v \left\{ [(p_t^{(r)})_1 + (p_t^{(r)})_2] + v[(p_x^{(r)})_1 - (p_x^{(r)})_2] \right\} = (p_t)_T$$

and the conserved spatial component is

$$(p_x)_1 + (p_x)_2 = \gamma_v \left\{ [(p_x^{(r)})_1 + (p_x^{(r)})_2] + v[(p_t^{(r)})_1 - (p_t^{(r)})_2] \right\} = (p_x)_T.$$

Now, using the approximate expressions in (4.24), we can evaluate the two four-momenta in the rest frame as functions of the relative velocity with respect to the

[9] Here with identical particles we mean particles with the same λ's and the same mass.

preferred frame:

$$
\begin{array}{lcl}
(p_t^{(r)})_1 & \simeq & -\sqrt{\tilde{\lambda}_{2,0}}\left[1+\frac{1}{2\tilde{\lambda}_{2,0}}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}[\gamma_v^2\gamma_u^2(u+v)^2\tilde{\lambda}_{2,0}]^n\right] \\
(p_x^{(r)})_1 & \simeq & -\sqrt{\tilde{\lambda}_{2,0}}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n[\gamma_v^2\gamma_u^2(u+v)^2\tilde{\lambda}_{2,0}]^{n-1}\gamma_v^2\gamma_u^2(1+uv)(-u-v) \\
(p_t^{(r)})_2 & \simeq & -\sqrt{\tilde{\lambda}_{2,0}}\left[1+\frac{1}{2\tilde{\lambda}_{2,0}}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}[\gamma_v^2\gamma_u^2(v-u)^2\tilde{\lambda}_{2,0}]^n\right] \\
(p_x^{(r)})_2 & \simeq & -\sqrt{\tilde{\lambda}_{2,0}}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n[\gamma_v^2\gamma_u^2(v-u)^2\tilde{\lambda}_{2,0}]^{n-1}\gamma_v^2\gamma_u^2(1-uv)(v-u)
\end{array}
\tag{4.28}
$$

where we used the relativistic sum of the velocities

$$
v_1^{(P)}=\frac{-v-u}{1+uv}\qquad v_2^{(P)}=\frac{v-u}{1-uv}.
$$

The approximate $(p_t)_T$ is then given by

$$
\begin{aligned}
(p_t)_T=\gamma_v&\left\{[(p_t^{(r)})_1+(p_t^{(r)})_2]+v[(p_x^{(r)})_1-(p_x^{(r)})_2]\right\}\simeq \\
&-3\sqrt{\tilde{\lambda}_{2,0}}\gamma_v\left\{\left[1+\frac{1}{2\tilde{\lambda}_{2,0}}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}[\gamma_v^2\gamma_u^2\tilde{\lambda}_{2,0}]^n\sum_{k=0}^{n}\binom{2n}{2k}u^{2(n-k)}v^{2k}\right]+\right. \\
&-v\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n[\gamma_v^2\gamma_u^2\tilde{\lambda}_{2,0}]^{n-1}\gamma_v^2\gamma_u^2\left[\sum_{k=1}^{n}\binom{2n-1}{2k-1}u^{2(n-k)}v^{2k-1}\right. \\
&\left.\left.\qquad\qquad+uv\sum_{k=0}^{n-1}\binom{2n-1}{2k}u^{2(n-k)-1}v^{2k}\right]\right\}= \\
=-2\sqrt{\tilde{\lambda}_{2,0}}\gamma_v&\left\{1+\sum_{n=1}^{z}\frac{\tilde{\lambda}_{2,n}}{2\tilde{\lambda}_{2,0}}[\gamma_v^2\gamma_u^2\tilde{\lambda}_{2,0}]^n\sum_{k=0}^{n}\frac{(2n)!u^{2(n-k)}v^{2k}}{(2k)!(2n-2k)!}\left(1-2nv^2-2\frac{k}{\gamma_v^2}\right)\right\}
\end{aligned}
$$

while the approximate linear momentum is

$$
(p_x)_T=\gamma_v\left\{[(p_x^{(r)})_1+(p_x^{(r)})_2]+v[(p_t^{(r)})_1-(p_t^{(r)})_2]\right\}\simeq
$$

$$
\frac{\gamma_v}{\sqrt{\tilde{\lambda}_{2,0}}}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}[\gamma_v^2\gamma_u^2\tilde{\lambda}_{2,0}]^n\sum_{k=1}^{2n-1}\frac{(2n)!u^{2(n-k)+1}v^{2k-1}}{(2k-1)!(2n-2k+1)!}\left(\frac{(2k-1)}{v\gamma_v^2}+(2n-1)v\right).
$$

If we know the final velocity and the $\tilde{\lambda}$'s of the new particle, then we can relate the old $\tilde{\lambda}$'s to the new ones. In general we can expect two possible results: the final particle P_T is at rest in O or is moving in O. In the case in which the new particle P_T is found to be at rest then, noting that the expected four-momentum of the final particle is (4.24)

$$\begin{aligned} (p_t^{(r)})_T &\simeq -\sqrt{\tilde{\lambda}_{2,0}^T}\left[1+\frac{1}{2\tilde{\lambda}_{2,0}^T}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}^T[\gamma_u^2u^2\tilde{\lambda}_{2,0}^T]^n\right] \\ (p_x^{(r)})_T &\simeq -\sqrt{\tilde{\lambda}_{2,0}^T}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}^T n[\gamma_u^2u^2\tilde{\lambda}_{2,0}^T]^{n-1}\gamma_u^2(-u), \end{aligned} \tag{4.29}$$

$-u$ being the relative velocity of the particle with respect to the preferred frame, we deduce that, although the mass $2\sqrt{\tilde{\lambda}_{2,0}}\gamma_v$ is the same as predicted by special relativity, the other λ's come out to be dependent on u as well on v. On the other hand, in the case in which the particle P_T has a non zero velocity in O, we deduce that such a velocity must be first order in the $\tilde{\lambda}$'s; this is so because in the zero order in the $\tilde{\lambda}$'s we expect a particle at rest. Then the velocity of P_T and its set of $\tilde{\lambda}$'s will depend on the velocities respect to O, the masses, and the $\tilde{\lambda}$'s of the two scattered particles as well as on the relative velocity of O with respect to the preferred frame.

Therefore, in both cases we deduce that the physics is different in different "inertial" frame systems; that is, the same scattering in two different "inertial" frame systems either produces two different kinds of particle because the set of λ's depends on the relative velocity with the preferred frame system, or produces the same kind of particle (same λ's) but with different kinematics (momenta), depending on the different initial frames.

4.2.5 Luminal and Superluminal Particles

The equations of motion (4.26) and (4.27) describe, respectively, massive and massless subluminal particles, being subluminal in every "inertial" frame. The motion is determined once we know the four-momentum of the particle in its rest frame, which can be approximately evaluated knowing all the λ's. Also luminal ($|u| = 1$)[10] and superluminal particles ($|u| > 1$), respectively, are luminal and superluminal in every "inertial" frame, but for such particles there does not exist an "inertial" rest frame. Therefore, the simplest choice is to write the equations of motion for a generic "inertial" frame in terms of the particle four-momentum in the preferred frame.

Consider, then, the case of a luminal particle in the preferred frame (all other cases can be obtained by appropriate Lorentz transformations). We cannot define the four-momentum in the rest frame for luminal particles, therefore we need to use the equation of motion (4.9) and the equivalent in the massless case. The condition $u = \frac{\dot{x}}{\dot{t}} = 1$, using the equations (4.9), translates into

$$\left\{1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right\}^2 p_x^2 = p_t^2$$

which, using the dispersion relation (4.12), gives

$$\left[\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right]^2 p_x^2 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n}(2n-1) p_x^{2n} = \tilde{\lambda}_{2,0}.$$

The relation above allows us to find p_x and then p_t in the preferred frame. In the massless case it reduces to

$$\left[\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right]^2 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n}(2n-1) p_x^{2(n-1)} = 0$$

where we excluded the solution $p_x = 0$ corresponding to the case of no motion. Knowing p_x and then p_t in the preferred frame we can construct the kinematics in any other "inertial" frame system. Note that in this case we cannot use the

[10] With luminal particles we mean particles moving with the same velocity as low energy photons, that is, $c = \sqrt{1 + \tilde{\lambda}_{2,1}^{light}} \equiv 1$.

Newton method as in the other cases because we cannot choose an opportune starting point not knowing any expected behavior of the four-momentum nor if the $\tilde{\lambda}$'s are small. Therefore in this case it is necessary to know the $\tilde{\lambda}$'s.

For superluminal particles we can proceed as in the luminal case by considering the four-momentum only in the preferred frame because there does not exist any Minkowskian frame in which a superluminal particle is at rest. Then, using the equations of motion (4.9), the first condition is

$$\left\{1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right\}^2 p_x^2 = (v^{(P)})^2 p_t^2$$

where $v^{(P)}$ is the velocity of the superluminal particle in the preferred frame. The velocity in any other "inertial" frame system is obtained with the usual relativistic addition of velocities rule. Using the dispersion relation we then have

$$[1 - (v^{(P)})^2] p_t^2 + \left[\sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)}\right]^2 p_x^2 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} (2n-1) p_x^{2n} = \tilde{\lambda}_{2,0}.$$

We can again find the four-momentum in the preferred frame, this time as a function of $v^{(P)}$, allowing us to know the kinematics in any "inertial" frame system. In this case also it is necessary to know the λ's because we do not have any known behavior of such particles to use as starting point in the Newton method.

We have seen that the deformed kinematics considered here allows in general the existense of superluminal particles and massive luminal particles, depending on the values of the the $\tilde{\lambda}$'s. Indeed, from (4.15) we have that the velocity can be bounded or unbounded depending on the values of the $\tilde{\lambda}$'s. Moreover, if we have a particle with $\tilde{\lambda}$'s such that the velocity is unbounded, in principle we can accelerate such a particle from rest to superluminal velocities with a constant force.

Let us consider for example the case of a charged particle at rest in the preferred frame. Turning on a constant electric field directed along the x direction the particle starts to move. This can be achieved with the minimal substitution $p_\mu \to p_\mu + eA_\mu$ in the super Hamiltonian, where $A_\mu = (-Ex, 0, 0, 0)$. The equations of motion for $\dot{x}$ and $\dot{t}$ are still the same, while the equations for $\dot{p}_\mu$ turn out to be

$$\dot{p}_x = \frac{eEp_t}{\sqrt{\tilde{\lambda}_{2,0}}} \qquad \dot{p}_t = \frac{eE}{\sqrt{\tilde{\lambda}_{2,0}}} p_x \left(1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n p_x^{2(n-1)} \right).$$

If the particle has only a positive $\tilde{\lambda}_{2,1}$ then the mechanics is unchanged with the only difference that the new maximum velocity is $c' = \sqrt{1 + \tilde{\lambda}_{2,1}} (> c)$. The velocity of the particle is given by

$$v = c' \tanh \frac{eEc'}{\sqrt{\tilde{\lambda}_{2,0}}} \tau$$

then making possible to reach the velocity of light in a finite proper time; the associated energy is given by

$$E = -p_t = \frac{\sqrt{\tilde{\lambda}_{2,0}}}{\sqrt{1 - \left(\frac{v}{c'}\right)^2}} \simeq \frac{\sqrt{\tilde{\lambda}_{2,0}}}{\sqrt{\tilde{\lambda}_{2,1}}}.$$

This result shows that particles with small masses can, in principle, reach superluminal velocities within energies for which the particles will not modify the background gravity.

Considering that $\tilde{\lambda}_{2,1} \neq 0$ is just a shifting of the velocity of light let us now consider the case in which only $\tilde{\lambda}_{2,n}$ with $n > 1$ is not zero. For $\tilde{\lambda}_{2,n}$ small we have the following momenta to the first order in $\tilde{\lambda}_{2,n}$:

$$p_x = -\frac{\sqrt{\tilde{\lambda}_{2,0}}}{c'} \sinh \frac{eEc'}{\sqrt{\tilde{\lambda}_{2,0}}} \tau + \tilde{\lambda}_{2,n} g(\tau) \qquad p_t = -\sqrt{\tilde{\lambda}_{2,0}} \cosh \frac{eEc'}{\sqrt{\tilde{\lambda}_{2,0}}} \tau - \tilde{\lambda}_{2,n} \frac{\sqrt{\tilde{\lambda}_{2,0}}}{eE} \dot{g}$$

where $g(\tau)$ is the first order correction.

In general the value of p_x for which the particle reaches a luminal velocity depends on the magnitude of the $\tilde{\lambda}$'s. If p_x is such that the associated energy p_t becomes high enough to generate a relevant gravitational field our analysis is not valid any more. In principle, using relation (4.15), the condition to reach a luminal velocity to the first order in the $\tilde{\lambda}$'s is

$$\tilde{\lambda}_{2,0} \simeq \sum_{n=1}^{z} \tilde{\lambda}_{2,n}(2n-1)p_x^{2n}$$

in which case p_x^2 becomes of the order of $\sqrt[n]{\tilde{\lambda}_{2,0}/\tilde{\lambda}_{2,n}}$ and therefore our approximation breaks down; if instead we consider particles with sizeable $\tilde{\lambda}$'s we could have a solution of the condition $v^2 = 1$ (relation (4.15) assumes no approximations) with not too high p_x^2, that is, with energies small enough to not generate an appreciable gravitational field. This conclusion turns out to be true also if the polynomial $\tilde{\lambda}_{2,0} + p_x^2 + \sum \tilde{\lambda}_{2,n} p_x^{2n}$ has a positive root.

The above analysis was made starting from the equations of motion. These equations are actually exact for a Minkowski background and hence can be considered for generic values of the $\tilde{\lambda}$'s. On a general curved spacetime one must be more careful; indeed, as it is evident from the evaluation of the optical limit in the appendix D, we must also impose the condition $\tilde{\lambda}_{2,n} < \lambda_{\mathrm{w}}^{2(n-1)}$, which is a good approximation if we consider usual relativistic matter whose momentum $p \sim \hbar/\lambda_{\mathrm{w}}$ is not too large.

4.2.6 Non-covariant vs Covariant Hořava Theory

In this section we want to point out the differences between the non-covariant form of the modified Hořava-Lifshitz action (2.17) and its covariant generalization (2.28). The main difference is that the action (2.17) is written in terms of the fields h_{ij}, N, N_i while its covariant generalization is in terms of $g_{\alpha\beta}$ and n_α; this implies that, as described in section 2.4, the covariant action reduces to (2.17) only if it is written in a frame in which the metric $g_{\alpha\beta}$ can be decomposed in terms of h_{ij}, N , N_i.

In the case of the Minkowsky metric such frames are obtained by rotations and translations of the preferred frame system. Indeed, the metric (4.22), obtained by boosting the preferred frame, is not writable in terms of ADM components. Therefore, the action (2.28) is a generalization of the non-covariant action (2.17) because it includes solutions which are absent in the non-covariant form.

Let $g_{\alpha\beta}$ and n_α be a solution of the Kehagias-Sfetsos action (2.17); then $g_{\alpha\beta}$ and n_α can be decomposed in terms of its ADM components and it is a solution also for covariant action (2.28). If we consider a generic change of coordinates $x^\mu(x^\alpha)$, then

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}$$

is another solution for (2.28) but not for (2.17). If the covariant action (2.28) is equivalent to the Kehagias-Sfetsos action (2.17), then, after the change of coordinates $x^\mu(x^\alpha)$, the new form assumed by (2.17) can be rewritten in terms $g'_{\alpha\beta}$ and will have the same form as (2.28).

Our results are independent of the formulation chosen for the Kehagias-Sfetsos action (2.17) because the solutions we considered are solutions of both theories. Moreover the ray optical structure was constructed using these solutions as a background metric. In terms of the optical structure we have that the equation of motion $H[g_{\alpha\beta}(x), n_\alpha(x)] = 0$, under the coordinate transformation $x^\mu(x^\alpha)$, becomes

$$H'[g_{\alpha\beta}(x'), n_\alpha(x')] = 0 \quad \longrightarrow \quad H[g'_{\alpha\beta}(x'), n'_\alpha(x')] = 0$$

remaining unchanged, the action (2.28) being covariant. On the other hand, if we rewrite the same Hamiltonian in terms of its ADM components, that is,

$H[h(x), N^i(x),\ N(x)] = 0$, the same change of coordinates will change the equation of motion as

$$H'[h(x'), N^i(x'), N(x')] = 0.$$

Such equation is equivalent to $H'[g_{\alpha\beta}(x'), n_\alpha(x')] = 0$, being just its expansion in terms of the ADM components of $g_{\alpha\beta}$ but cannot be rewritten as

$$H[h'(x), {N'}^i(x), N'(x)] = 0$$

because the transformed metric $g'_{\alpha\beta}$ cannot be decomposed in ADM components.

Therefore, independently from the equivalence of the covariant and the non-covariant form of the modified Hořava action, the optical structure is more conveniently written in a covariant form because the derived equations of motion take a more compact form. The only difference is that, if the non-covariant and the covariant theory are not equivalent, then $g'_{\alpha\beta}$ cannot be interpreted as a metric tensor in the IR limit of Hořava gravity, as it cannot be decomposed in ADM components.

4.3 Spherical Symmetric Case

As the original Hořava-Lifshitz theory, the Kehagias-Sfetsos action (2.28) possesses a spherical solution [18] (here we are not going to consider the general case $N_r \neq 0$ and $\lambda \neq 1$ because we will focus on the deviations due to the presence of the $\tilde{\lambda}$'s). The solution is based on the following ansatz

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \qquad h^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \tag{4.30}$$

$$n^\alpha = (-\frac{1}{N}, 0, 0, 0)$$

and the functions N and f, for the case $\lambda = 1$, are found to be

$$N^2 = f = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}. \tag{4.31}$$

This solution in the IR limit ($\omega \to \infty$) reproduces the Schwarzschild solution.

The metric is a function only of the coordinates r and θ and is diagonal, therefore

$$\dot{p}_t = 0 \qquad \dot{p}_\phi = 0.$$

As in the relativistic case, we can define[11]

$$p_t = -E \qquad p_\phi = L.$$

Moreover

$$\dot{t} = \frac{1}{N^2} \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} E, \tag{4.32}$$

$$\dot{\theta} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}} \left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (h^{\mu\nu} p_\mu p_\nu)^{n-1} \right] \frac{1}{r^2} p_\theta,$$

$$\dot{p}_\theta = -\frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}} \left[1 + \sum_{n=1}^{z} \tilde{\lambda}_{2,n} n (h^{\rho\lambda} p_\rho p_\lambda)^{n-1} \right] \left(\frac{-2\cos\theta}{r^2 \sin^3\theta} \right) L^2;$$

the last two equations have as particular solution

$$\theta = \frac{\pi}{2} \qquad p_\theta = 0.$$

[11] Given a vector field K^α, the product $K_\alpha \dot{x}^\alpha$ is conserved along a path $x^\alpha(\tau)$ if K^α is a Killing vector and if the path is a geodesic:

$$\frac{d}{d\tau}(K_\alpha \dot{x}^\alpha) = \dot{x}^\beta \nabla_\beta (K_\alpha \dot{x}^\alpha) = \dot{x}^\beta \dot{x}^\alpha \nabla_\beta K_\alpha + \dot{x}^\beta K_\alpha \nabla_\beta \dot{x}^\alpha = 0.$$

Being $\dot{x}^\beta \nabla_\beta \dot{x}^\alpha \neq 0$, we deduce that we cannot use the notion of Killing vectors in this context.

Therefore, in what follows we will concentrate on the case of the equatorial motion. Moreover,

$$\dot{p}_r = \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}}\partial_r\left(\frac{1}{N^2}\right)E^2 - \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n(h^{\rho\lambda}p_\rho p_\lambda)^{n-1}\right]\partial_r h^{\mu\nu}p_\mu p_\nu \tag{4.33}$$

$$\dot{r} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n(h^{\mu\nu}p_\mu p_\nu)^{n-1}\right]f(r)p_r \tag{4.34}$$

$$\dot{\phi} = \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n(h^{\mu\nu}p_\mu p_\nu)^{n-1}\right]\frac{L}{r^2}. \tag{4.35}$$

From the dispersion relation, on the other hand, we have

$$0 = -\frac{E^2}{N^2} + \tilde{\lambda}_{2,0} + \left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(f(r)p_r^2+\frac{L^2}{r^2}\right)^{n-1}\right]\left[f(r)p_r^2+\frac{L^2}{r^2}\right].$$

Now, to find an approximate result, we use again Newton's method. Setting

$$x = f(r)p_r^2 + \frac{L^2}{r^2}$$

we choose as starting point

$$x_0 = \frac{E^2}{N^2} - \tilde{\lambda}_{2,0}$$

which corresponds to the zero order in the λ's. Then we have

$$y_0 = \sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^n \qquad m_0 = \left.\frac{dy}{dx}\right|_{x_0} = 1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}.$$

Therefore,

$$p_r^2 \simeq \frac{1}{f(r)}\left\{\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}-\frac{L^2}{r^2}-\frac{\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^n}{1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}}\right\} \simeq$$

$$\frac{1}{f(r)}\left\{\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}-\frac{L^2}{r^2}-\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^n\right\}=$$

$$\frac{1}{f(r)}\left[\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}-\frac{L^2}{r^2}\right]\left\{1-\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\frac{\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^n}{\left[\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}-\frac{L^2}{r^2}\right]}\right\}$$

which yields

$$p_r \simeq \frac{1}{\sqrt{f(r)}}\sqrt{\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}-\frac{L^2}{r^2}}\left\{1-\frac{1}{2}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\frac{\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^n}{\left[\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}-\frac{L^2}{r^2}\right]}\right\}. \tag{4.36}$$

Using the expressions (4.34), (4.35) and (4.36) we have the approximate orbit equation

$$\frac{\dot{r}}{\dot{\phi}}=\frac{dr}{d\phi}=\frac{r^2 f(r)p_r}{L}.$$

Orbits as well as other classical gravitational tests were studied in the context of the Kehagias-Sfetsos modification of Hořava-Lifshitz gravity [22, 23, 25] showing the presence of corrections with respect to the results of General Relativity. In these articles the kinematic is assumed to be described by the super Hamiltonian

$$H=\frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}}g_{\mu\nu}p^\mu p^\nu$$

while we generalize this assumption. Our results, being obtained as the optical limit of a scalar field theory, show that the kinematic is indeed described by that super Hamiltonian in the zero order in the $\tilde{\lambda}$'s, but we are going to consider $\tilde{\lambda}$-dependent corrections.

In the more general framework allowed by the Hořava-Lifshitz gravity, that is, for non-zero $\tilde{\lambda}$'s, we can consider two simple cases which can be used to verify the existence of such parameters: circular and radial orbits.

To have a circular orbit we must have $\dot{r}=0$ and therefore, from (4.34), $p_r=0$.

Using the equation (4.33) we have the condition

$$
\begin{aligned}
0 &= \partial_r\left(\frac{E^2}{N^2}\right) - \left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n\left(\frac{L^2}{r^2}\right)^{n-1}\right]\left(-2\frac{L^2}{r^3}\right) \\
&= \partial_r\left[\frac{E^2}{N^2}-\frac{L^2}{r^2}-\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{L^2}{r^2}\right)^{n}\right]
\end{aligned}
$$

which is satisfied if

$$\frac{E^2}{N^2}-\frac{L^2}{r^2}-\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{L^2}{r^2}\right)^{n} = K$$

where K is a constant. From the dispersion relation we simply deduce $K=\tilde{\lambda}_{2,0}$. From the previous expression then we obtain the energy associated with a circular motion

$$\frac{E^2}{N^2} = \tilde{\lambda}_{2,0}+\frac{L^2}{r^2}+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{L^2}{r^2}\right)^{n}.$$

From the expression (4.34) and the approximate relation for p_r^2 we obtain:

$$\dot{r}^2 \simeq \frac{f(r)}{\tilde{\lambda}_{2,0}}\left\{\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}-\frac{L^2}{r^2}-\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}\left[(1-2n)\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)+2n\frac{L^2}{r^2}\right]\right\}.$$

We can interpret the right hand side of this equation as a potential $V(r)$.

From the condition $\frac{dV}{dr}=0$ for circular orbits we have

$$-\frac{E^2}{N^3}N'+\frac{L^2}{r^3}+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{L^2}{r^2}\right)^{n-1}\left[-\frac{E^2}{N^3}N'(6n-2)-\frac{L^2}{r^3}2n\right]=0.$$

So it is evident that deviations for the radius of a circular orbits from the zero-order solution $\frac{E^2}{N^3}N'=\frac{L^2}{r^3}$ will increase for increasing angular momentum.

Another simple consequence of this deformed kinematics is that two particles with the same mass and different $\tilde{\lambda}$'s will fall radially toward the center with two different velocities.

From the equation (4.35) it is simple to see that a radial solution corresponds to the case $L = 0$. In this case we have that the equations of motion become

$$\begin{aligned}\dot{p}_r &= \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}}\partial_r\left(\frac{1}{N^2}\right)E^2 - \frac{1}{2\sqrt{\tilde{\lambda}_{2,0}}}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n[f(r)p_r^2]^{n-1}\right]\partial_r f(r)p_r^2 \\ \dot{r} &= \frac{1}{\sqrt{\tilde{\lambda}_{2,0}}}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n[f(r)p_r^2]^{n-1}\right]f(r)p_r\end{aligned}$$

and the dispersion relation reduces to

$$0 = -\frac{E^2}{N^2} + \tilde{\lambda}_{2,0} + \left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}[f(r)p_r^2]^{n-1}\right]f(r)p_r^2.$$

Using the approximate relation (4.36) for p_r we have

$$p_r \simeq \frac{1}{\sqrt{f(r)}}\sqrt{\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}}\left\{1-\frac{1}{2}\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}\right\}$$

from which we have

$$\begin{aligned}\dot{r} &\simeq \sqrt{\frac{fE^2}{\tilde{\lambda}_{2,0}N^2}-1}\left\{1-\sum_{n=1}^{z}\frac{\tilde{\lambda}_{2,n}}{2}\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}\right\}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}n\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}\right] \\ &\simeq \sqrt{\frac{f(r)}{\tilde{\lambda}_{2,0}}}\sqrt{\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(n-\frac{1}{2}\right)\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}\right].\end{aligned}$$

Using the equation (4.32) describing the coordinate time, we have

$$\frac{dr}{dt} \simeq \sqrt{f(r)}\frac{N^2}{E}\sqrt{\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}}\left[1+\sum_{n=1}^{z}\tilde{\lambda}_{2,n}\left(n-\frac{1}{2}\right)\left(\frac{E^2}{N^2}-\tilde{\lambda}_{2,0}\right)^{n-1}\right].$$

The $\tilde{\lambda}$ dependence shows again that the theory produces observable effects that can be used to measure the deformations with respect to the usual dynamics.

Chapter 5

Conclusions and Outlook

Ever since Special Relativity was established, we think of space and time as a unique spacetime structure. General Relativity further generalized flat Minkowsky spacetime to a generic Riemannian manifold on which Einstein's theory of gravity manifests in a completely covariant way, making the invariance under general coordinate transformations a fundamental symmetry of the theory.

Whether this fundamental covariant structure survives in the quantum theory of gravity is not a priori known. There are clues from string theory and other attempts to quantize gravity that at the Planck scale the physics is quite different and general covariance emerges only as an asymptotic long distance symmetry.

In his proposed theory of gravity Hořava chose to abandon full covariance at the level of a four dimensional field theory and to introduce a physically relevant foliation structure in space and time. This choice allowed him to construct a power-counting normalizable action, but it also implies that the dynamics described by this theory are in principle different than the one of GR. However, GR is a well tested and verified theory in the IR, and any deviation from the expected behavior must be consistent with experimental and observational bounds. It is, therefore, important to identify and analyze such deviations.

There are two kinds of deformations to look for: in the gravitational background itself and in the particle dynamics on it.

In this Thesis we analyzed static spherically symmetric backgrounds and showed that additional physically distinct solutions are allowed. For the case $\lambda = 1$ we identified an extra gauge symmetry. This symmetry is explicitly broken by a particle mass term, however, increasing the possible number of physically distinct solutions. We analyzed these solutions and showed that particle trajectories deviate from the expected ones, giving rise to possible experimental tests.

We also studied the space-time structure of these solutions. It turns out that the concepts of horizons and singularities must be reviewed in this context. We suggest that the breakdown of the foliation structure, which may happen before we reach the singular point in the center, represents the true singularity of the theory. Further study of these questions is warranted and could reveal interesting conceptual issues.

Regarding the particle dynamics we showed how it is possible to generalize the action describing a particle. The reduced symmetry of the theory allows for higher even powers of the momentum in the dispersion relation, modifying the geodesics. Under certain conditions such modifications are small, at least in the IR behavior. We suggested possible experiments in this context; all the observations we already have about the motion of particles in Special Relativity and in General Relativity suggest that the new parameters of the theory are very small, or simply that no known matter has such nontrivial dynamics.

New perspectives come from a recent proposal by Hořava for an extension of the theory [11]. This HL theory possesses an additional $U(1)$ gauge symmetry which aims to stabilize λ to 1 and to reduce the number of degrees of freedom by 1 to match the degrees of freedom of GR. This model is based on a projectable lapse function, but, with an opportune interpretation, it is possible to include a space dependence through the introduction of the $U(1)$ gauge potential in a generalized expression for the IR metric. It is possible that this extension of the theory is related, or can be related, to the gauge symmetry we find in the case $\lambda = 1$. This could resolve the issue raised in [58] regarding the stabilization of λ and lead to a viable quantum theory of gravity.

Appendices

Appendix A

Spherically Symmetric Christoffel Symbols

For a general static spherically symmetric metric (3.2) the only non-zero Christoffel symbols are

$$\Gamma^t_{tt} = -\frac{1}{2}g^{tr}\partial_r g_{tt}; \qquad \Gamma^t_{tr} = \frac{1}{2}g^{tt}\partial_r g_{tt}; \qquad \Gamma^t_{rr} = g^{tt}\partial_r g_{tr} + \frac{1}{2}g^{tr}\partial_r g_{rr};$$

$$\Gamma^t_{\theta\theta} = -\frac{1}{2}g^{tr}\partial_r g_{\theta\theta} = -r\frac{N_r f}{N^2}; \qquad \Gamma^t_{\phi\phi} = -\frac{1}{2}g^{tr}\partial_r g_{\phi\phi} = \Gamma^t_{\theta\theta}\sin^2\theta;$$

$$\Gamma^r_{tt} = -\frac{1}{2}g^{rr}\partial_r g_{tt}; \qquad \Gamma^r_{tr} = \frac{1}{2}g^{rt}\partial_r g_{tt}; \qquad \Gamma^r_{rr} = g^{rt}\partial_r g_{rt} + \frac{1}{2}g^{rr}\partial_r g_{rr};$$

$$\Gamma^r_{\theta\theta} = -\frac{1}{2}g^{rr}\partial_r g_{\theta\theta} = -r\left(f - \frac{N_r^2 f^2}{N^2}\right); \qquad \Gamma^r_{\phi\phi} = -\frac{1}{2}g^{rr}\partial_r g_{\phi\phi} = \Gamma^r_{\theta\theta}\sin^2\theta;$$

$$\Gamma^\theta_{\theta r} = \frac{1}{2}g^{\theta\theta}\partial_r g_{\theta\theta} = \frac{1}{r}; \qquad \Gamma^\theta_{\phi\phi} = -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\phi\phi} = -\sin\theta\cos\theta;$$

$$\Gamma^\phi_{\phi r} = \frac{1}{2}g^{\phi\phi}\partial_r g_{\phi\phi} = \Gamma^\theta_{\theta r}; \qquad \Gamma^\phi_{\phi\theta} = \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} = -\frac{1}{\sin^2\theta}\Gamma^\theta_{\phi\phi}.$$

Appendix B

The Action for the Spherically Symmetric case

The most general static spherically symmetric ansatz for a metric is

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_r^2 f & N_r & 0 & 0 \\ N_r & \frac{1}{f} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

for which

$$h_{\mu\nu} = \begin{pmatrix} N_r^2 f & N_r & 0 & 0 \\ N_r & \frac{1}{f} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \quad N_\alpha = (fN_r^2, N_r, 0, 0) \quad n_\alpha = (-N, 0, 0, 0)$$

where $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the metric on the space-like surface Σ orthogonal to the direction n_α, N_α the shift vector and N the lapse function.

The kinetic term in the action (2.17) is constructed from the extrinsic curvature defined as

$$K_{\alpha\beta} \equiv \frac{1}{2}\mathcal{L}_n h_{\alpha\beta} = \frac{1}{2N}[\partial_t h_{\alpha\beta} - D_\alpha N_\beta - D_\beta N_\alpha].$$

In our case the metric is static then

$$K_{\alpha\beta} = -\frac{1}{2N}[D_\alpha N_\beta + D_\beta N_\alpha].$$

In particular the spacial components of the extrinsic curvature are

$$K_{rr} = -\frac{1}{N}\left(N_r' + \frac{1}{2}\frac{f'}{f}N_r\right) \quad K_{\theta\theta} = -\frac{1}{N}fN_r r \quad K_{\phi\phi} = -\frac{1}{N}fN_r r\sin^2\theta$$

and, using the relation $K_{\alpha\beta} = h_\alpha{}^i h_\beta{}^j K_{ij}$, the remaining non-zero components are

$$K_{tt} = f^2 N_r^2 K_{rr} \quad K_{tr} = fN_r K_{rr}.$$

Raising one index with $g^{\mu\nu}$ we find

$$K_r{}^r = K_{rt}g^{tr} + K_{rr}g^{rr} = -\frac{1}{N}\left(N_r' + \frac{1}{2}\frac{f'}{f}N_r\right)\left(\frac{N_r^2 f^2}{N^2} + f - \frac{N_r^2 f}{N^2}\right) = -\frac{1}{N}\left(fN_r' + \frac{1}{2}f'N_r\right)$$

$$K_\theta{}^\theta = K_\phi{}^\phi = -\frac{1}{N}\frac{fN_r}{r} \quad K_t{}^t = 0 \quad K_r{}^t = K_{rt}g^{tt} + K_{rr}g^{rt} = K_{rr}\left(-\frac{fN_r}{N^2} + \frac{fN_r}{N^2}\right) = 0$$

$$K_t{}^r = K_{tt}g^{tr} + K_{tr}g^{rr} = K_{rr}\left(\frac{N_r^3 f^3}{N^2} + f^2 N_r - \frac{N_r^3 f^3}{N^2}\right) = K_{tt}g^{tr} + K_{tr}g^{rr} = K_r{}^r fN_r$$

Moreover

$$K^{rr} = g^{rt}K_t{}^r + g^{rr}K_r{}^r = -\frac{1}{N}\left(fN_r' + \frac{1}{2}f'N_r\right)f \qquad K^{tr} = g^{tt}K_t{}^r + g^{tr}K_r{}^r = 0$$

$$K^{tt} = g^{tt}K_t{}^t + g^{tr}K_r{}^t = 0 \qquad K^{\theta\theta} = -\frac{1}{N}\frac{fN_r}{r^3} \qquad K^{\phi\phi} = -\frac{1}{N}\frac{fN_r}{r^3\sin^2\theta}.$$

Therefore the kinetic term of the action (2.17) is given by

$$\begin{aligned} K_{ij}G^{ijkl}K^{kl} &= \frac{1}{N^2}\left(fN_r' + \frac{1}{2}f'N_r\right)^2 + \frac{2}{N^2}\frac{f^2N_r^2}{r^2} - \lambda\left[\frac{1}{N}\left(fN_r' + \frac{1}{2}f'N_r\right) + \frac{2}{N}\frac{fN_r}{r}\right]^2 = \\ &= \frac{1-\lambda}{N^2}\left(fN_r' + \frac{1}{2}f'N_r\right)^2 + \frac{2(1-\lambda)}{N^2}\frac{f^2N_r^2}{r^2} - \frac{4\lambda}{N^2}\left(fN_r' + \frac{1}{2}f'N_r\right)\frac{fN_r}{r} \end{aligned} \tag{B.1}$$

In the potential term the intrinsic curvature $\mathcal{R}_{\alpha\beta\gamma\delta}$ in different contractions is related to the $3+1$-dimensional curvature as follows

$$\mathcal{R}_{\alpha\beta\gamma\delta} = h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\rho h_\delta{}^\lambda R^{(4)}_{\mu\nu\rho\lambda} - 2K_{\beta[\delta}K_{\gamma]\alpha}.$$

In our case the intrinsic Ricci tensor for Σ has the following spatial non-zero components:

$$\mathcal{R}_{rr} = -\frac{1}{r}\frac{f'}{f} \quad \mathcal{R}_{\theta\theta} = -\frac{1}{2}f'r - (f-1) \quad \mathcal{R}_{\phi\phi} = \mathcal{R}_{\theta\theta}\sin^2\theta \tag{B.2}$$

giving

$$\begin{aligned} \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} &= \mathcal{R}_{\mu\nu}\mathcal{R}_{\alpha\beta}h^{\mu\alpha}h^{\nu\beta} = \mathcal{R}_{ij}\mathcal{R}_{kl}h^{ik}h^{jl} = \sum_{i=1}^{3}(\mathcal{R}_{ii}h^{ii})^2 = (\mathcal{R}_{rr}f)^2 + \frac{2}{r^4}(\mathcal{R}_{\theta\theta})^2 = \\ &= \frac{1}{r^2}f'^2 + \frac{2}{r^4}\left(-\frac{1}{2}f'r - (f-1)\right)^2 = \frac{3}{2}\frac{1}{r^2}f'^2 + \frac{2}{r^4}(f-1)^2 + \frac{2}{r^3}(f-1)f' \end{aligned} \tag{B.3}$$

and

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} = h^{ij}\mathcal{R}_{ij} = -\frac{1}{r}f' + \frac{2}{r^2}\left(-\frac{1}{2}f'r - (f-1)\right) = -\frac{2}{r^2}[f'r + (f-1)]. \tag{B.4}$$

The Cotton tensor, because of the spherical symmetry, is null:

$$C_{ij} = 0. \tag{B.5}$$

Therefore the potential and the kinetic term in the action, respectively, are:

$$\begin{aligned}
L_V &= \frac{N}{\sqrt{f}}\left[(2\lambda-1)\frac{(f-1)^2}{r^2} - 2\lambda\frac{f-1}{r}f' + \frac{\lambda-1}{2}f'^2 - 2\omega(1-f-rf') - 3\Lambda_W^2 r^2\right] \\
L_K &= \omega\frac{\sqrt{f}}{N}\left[(1-\lambda)\frac{r^2}{f}\left(fN_r' + \frac{1}{2}f'N_r\right)^2 + 2(1-2\lambda)fN_r^2 - 4\lambda r\left(fN_r' + \frac{1}{2}f'N_r\right)N_r\right].
\end{aligned}$$

Appendix C

Metric Diagonalization in the Spherically Symmetric Case

As pointed out in section 3.1 the most generic spherically symmetric metric is with a nonzero shift vector and is given by (3.2)

$$ds^2 = -(N^2 - N_r^2 f)dt^2 + 2N_r dr dt + \frac{dr^2}{f} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

In General Relativity we can always perform the following change of coordinates

$$dt = dt^* + F(r)dr^* \qquad r = r^* \tag{C.1}$$

obtaining

$$ds^2 = -(N^2 - N_r^2 f)dt^{*2} + 2[N_r - (N^2 - N_r^2 f)F]dr^* dt^* +$$

$$+ \left[\frac{1}{f} - (N^2 - N_r^2 f)F^2 + 2N_r F\right] dr^{*2} + r^{*2} d\theta^2 + r^{*2} \sin^2\theta d\phi^2.$$

Choosing

$$F = \frac{N_r}{N^2 - N_r^2 f} \tag{C.2}$$

and defining

$$N^{*2} = N^2 - N_r^2 f \qquad f^* = \frac{f(N^2 - N_r^2 f)}{N^2}$$

we have that the metric takes the usual diagonal form (3.38):

$$ds^2 = -N^{*2} dt^{*2} + \frac{1}{f^*} dr^{*2} + r^{*2} d\theta^2 + r^{*2} \sin^2\theta d\phi^2.$$

Moreover note that if $f = N^2$ we also have

$$\frac{f^*}{N^{*2}} = \frac{f}{N^2} = 1.$$

Unlike GR, in HL gravity we cannot perform the change of coordinates (C.1) because such a transformation does not preserve the foliation $M = \mathbb{R} \times \Sigma$ of spacetime.

Appendix D

The Optical Limit

In the action (1.17) the deviation from the Klein-Gordon action is described by the term

$$-\sum_{n=1}^{z}\sum_{k=0}^{n}(-1)^n \frac{\lambda_{2,n,k}}{M^{2(n-1)}}\Delta^{n-k}\phi\Delta^k\phi^*$$

where we set $\lambda_{2,1,0} \equiv \lambda_{2,1,1} = \lambda_{2,1} - \frac{1}{2}$. Such a term can be rewritten as:

$$\Delta^{n-k}\phi\Delta^k\phi^* = \Delta^{n-k}\phi(h^{\mu\nu}\nabla_\mu\nabla_\nu)\Delta^{k-1}\phi^* =$$

$$= \nabla_\mu\left[\Delta^{n-k}\phi h^{\mu\nu}\nabla_\nu\Delta^{k-1}\phi^*\right] - \nabla_\mu\left[\Delta^{n-k}\phi h^{\mu\nu}\right]\nabla_\nu\Delta^{k-1}\phi^* =$$

$$= \nabla_\mu\left[\Delta^{n-k}\phi h^{\mu\nu}\nabla_\nu\Delta^{k-1}\phi^*\right] - \nabla_\nu\left[\nabla_\mu\left(\Delta^{n-k}\phi h^{\mu\nu}\right)\Delta^{k-1}\phi^*\right] + \nabla_\nu\nabla_\mu\left[\Delta^{n-k}\phi h^{\mu\nu}\right]\Delta^{k-1}\phi^*.$$

Defining the operator $\overrightarrow{\Delta}$ such that its action is given by $\overrightarrow{\Delta}\phi = \nabla_\alpha\nabla_\beta(h^{\alpha\beta}\phi)$ and considering that all the total covariant derivatives are boundary terms in the action, we conclude that

$$\Delta^{n-k}\phi\Delta^k\phi^* = \overrightarrow{\Delta}\Delta^{n-k}\phi\Delta^{k-1}\phi^* + \text{total derivatives}.$$

Therefore, varying with respect to ϕ^*, we have

$$\delta_{\phi^*}\left[\sum_{k=0}^{n}\lambda_{2,n,k}\Delta^{n-k}\phi\Delta^k\phi^*\right]=\sum_{k=0}^{n}\lambda_{2,n,k}\overrightarrow{\Delta}^k\Delta^{n-k}\phi\delta\phi^*+\text{total derivativs.}$$

Let us now restrict the above relation to the case of interest in this paper: the optical limit. In this approximation we consider the wavelength of the field $\lambda_{\rm w}$ to be much smaller than the characteristic length L of the square root of the components of the Riemann tensor. We can keep track of the various orders of magnitude by expanding the scalar field as follows

$$\phi=Se^{i\psi}=(a_0+\epsilon a_1+...)e^{\frac{i}{\epsilon}\theta}$$

where $\epsilon=\lambda_{\rm w}/L$ is a small dimensionless parameter [56]. (Note that this is analogous to the WKB approximation in quantum mechanics.) Then, keeping only the highest order terms, we have

$$\Delta\phi=h^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi=h^{\alpha\beta}\nabla_\alpha\left[\partial_\beta S\phi+\frac{i}{\epsilon}\phi\partial_\beta\theta\right]\simeq\frac{i}{\epsilon}h^{\alpha\beta}\nabla_\alpha\left[\phi\partial_\beta\theta\right]=$$

$$\simeq\frac{i}{\epsilon}h^{\alpha\beta}\left[\frac{i}{\epsilon}\phi\partial_\alpha\theta\partial_\beta\theta+\phi\nabla_\alpha\partial_\beta\theta\right]\simeq-\frac{1}{\epsilon^2}h^{\alpha\beta}\partial_\alpha\theta\partial_\beta\theta\phi$$

and it is easy to show that in general

$$\Delta^k\phi\simeq\frac{(-1)^k}{\epsilon^{2k}}(h^{\alpha\beta}\partial_\alpha\theta\partial_\beta\theta)^k\phi=(-1)^k(h^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi)^k\phi. \tag{D.1}$$

In an entirely similar way we also deduce

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi\simeq-\frac{1}{\epsilon^2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi. \tag{D.2}$$

Further, noticing that in

$$\overrightarrow{\Delta}\Delta^k\phi = \nabla_\mu\nabla_\nu h^{\mu\nu}\Delta^k\phi + 2\nabla_\mu h^{\mu\nu}\nabla_\nu\Delta^k\phi + \Delta^{k+1}\phi$$

the first term on the right hand side is of order ε^{-2k}, the second of order ε^{-2k-1} and the third of order ε^{-2k-2}, we deduce that under the condition that the variation of the curvature is small with respect to the inverse of the wavelength of the particle we have that $\overrightarrow{\Delta}\Delta^k\phi \simeq \Delta^{k+1}\phi$. Up to total derivatives, this leads to

$$\delta_{\phi^*}\left[\sum_{k=0}^{n}\lambda_{2,n,k}\Delta^{n-k}\phi\Delta^k\phi^*\right] = \sum_{k=0}^{n}\lambda_{2,n,k}\overrightarrow{\Delta}^k\Delta^{n-k}\phi\delta\phi^* \simeq \lambda_{2,n}\Delta^n\phi\delta\phi^*.$$

where we defined $\lambda_{2,n} = \sum_{k=0}^{n}\lambda_{2,n,k}$.

Then the full equation of motion can be simplified, yielding

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \lambda_{2,0}M^2 + \lambda_{2,1}\Delta\phi - \frac{\lambda_{2,2}}{M^2}\Delta^2\phi + \frac{\lambda_{2,3}}{M^4}\Delta^3\phi + ... \simeq 0.$$

Then using (D.2) and (D.1) we obtain the eikonal equation

$$g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + \lambda_{2,0}M^2 + \lambda_{2,1}(h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi) + \frac{\lambda_{2,2}}{M^2}(h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi)^2 + ... \simeq 0. \quad \text{(D.3)}$$

In general, to ensure that the extra terms do not dominate the quadratic term in the optical approximation and that the λ corrections remain small, we need to assume that the order of the extra terms is no more than ϵ^{-2}, that is,

$$\lambda_{2,n} \lesssim (ML\epsilon)^{2(n-1)} = (\lambda_{\mathrm{w}}M)^{2(n-1)}.$$

This will be ensured if the wavelength of the particle λ_{w} is small but no smaller than the fundamental length scale $1/M$ defined by the mass scale of the theory, in which case the above relation imposes no essential restrictions on the $\lambda_{2,n}$'s.

For wavelengths much smaller than $1/M$, however, the higher order terms would eventually dominate and could not be treated as small corrections.

For the treatment of general curved backgrounds we assume that we are in the regime $M^{-1} < \lambda_{\rm w} \ll L$, in which the extra terms can be treated as perturbations. Notice, however, that in the flat Minkowski case we do not need any assumptions for $\lambda_{\rm w}$, as in that case the optical approximation becomes exact.

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