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CAPITAL INVESTMENT IN A DUOPOLY AS A DIFFERENTIAL GAME

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City University of New York, 1987

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CAPITAL INVESTMENT IN A DUOPOLY AS A DIFFERENTIAL GAME

by

DIEP NGUYEN

A dissertation submitted to the Graduate Faculty
in Economics in partial fulfillment of the
requirements for the degree of Doctor of
Philosophy, The City University of New York.

1987

This manuscript has been read and accepted for the Graduate Faculty in Economics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

May 27, 1987
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Abstract

CAPITAL INVESTMENT IN A DUOPOLY AS A DIFFERENTIAL GAME

by

Diep Nguyen

Adviser: Professor Ronald W. Anderson

Capital investment strategy in production and in research and development in a two-firm industry formulated as a differential game is examined in this dissertation. In the first part of our work we study firms' capital investment rates under certainty in an infinite horizon, continuous time model with discounting. We find that while the follower firm continuously invests as quickly as possible, the leader's optimal strategy depends on the firms' initial capitals. It may invest continuously, or start by not investing and then invest when its rival attains a certain size, or may start by investing but interrupt its growth for a time. We characterize analytically the locus of the switch points (from growth to no-growth or vice versa).

The second part analyzes firms' R & D expenditures under uncertainty of a technological breakthrough. We first review some classical static and dynamic models in the literature, in which firms are assumed identical. This assumption is relaxed in our dynamic, finite horizon two-firm model. We derive a system of simple equations characterizing the equilibrium and highlight the R & D competition effect. We also discuss properties of the equilibrium from the numerical solution of these equations.

ACKNOWLEDGEMENTS

I am grateful to:

Professor Ronald Anderson who guided me in the field of Industrial Organization, made very helpful comments and corrections, and above all, taught me to think as an economist,

Professor Michael Grossman who by his efforts of broadening the scope of the Department of Economics gave me a chance of getting exposed to new fields of research,

Professor Salih Neftci who encouraged me to apply modern engineering methodology in my research.

DEDICATION

To my mother who gave me life and spirit,

my wife Rosemarie who fully supported and stood by me during the difficult school years,

my son Daniel, a competitor for fame,

and my brothers and sisters on whom I can always count.

TABLE OF CONTENTS

PART I

CAPTIAL INVESTMENT IN PRODUCTION

1. INTRODUCTION.

In the past ten years economists have intensively studied investment strategy and the deterrent effect of irreversible capital commitment on market entry by competitors. Spence [14], Dixit [4], Eaton and Lipsey [5], and Fudenberg and Tirole [6] and [7], in particular, have used repetitive or continuous-time investment games to model investment competition. This first part extends and formalizes the findings of two of those papers regarding steady state outcome and the investment path.

Spence, in his continuous-time duopoly model with discounting and bounded investment rates for both firms, showed diagramatically (a) that the final equilibrium outcome is on the follower firm's steady state reaction curve; (b) that the follower firm's optimal strategy is to invest as quickly as possible until the end; and (c) that the leader firm's optimal strategy is to invest as quickly as possible until its capital stock reaches a level not exceeding the Stackelberg point on the follower firm's reaction curve.

Fudenberg and Tirole adapted Spence's model for the non-discounting case, showing that Spence's solution constitutes a set of perfect equilibrium strategy pairs and that the final outcome is the same as Spence's. They went on to propose an early-stop equilibrium for the non-discounting case, and to extend this to the discounting case. Fudenberg and Tirole did not examine the optimal path for the discounting case, but they conjectured that it typically has two switch points.

We reformulate Spence's infinite-horizon continuous-time two-firm model with discounting as a non-cooperative non-zero sum differential game, in order to examine rigorously the statements about steady state equilibrium and optimal path which Spence as well as Fudenberg and Tirole supported only informally.

The objective of each firm is to maximize the net present value of cash-flow; the firm's strategies are their investment rates; and the state of the game is the set of the firm's net current capital stocks. To derive the equilibrium we will use the optimal control method, obtaining a set of differential equations of the state and costate variables. The boundary conditions are the initial states and the transversality conditions on the terminal surface which will be derived as a natural result of the game.

Section 2 introduces our notation and approach through the formulation of a single firm's investment policy. Sections 3 and 4 present strategies in a duopoly and extend Spence's and Fudenberg and Tirole's propositions. Section 5 reconsiders Fudenberg and Tirole's early-stop Pareto dominance equilibrium.

Appendix C gives an example of the model and extends it to the non-discounting case.

2. INVESTMENT POLICY OF A SINGLE FIRM.

In this section, the optimal control technique is used to derive a capital investment policy of a single firm assuming there is no strategic interaction between the firm and the rest of the industry.

The firm's capital stock is the state variable $x(t)$. We assume that there is no depreciation in capital stock:

$$\dot{x}(t) = u(t)$$

where $u(t)$ is the firm's investment rate. We also assume that investment is irreversible, and that the firm's investment rate is bounded by a constant : $u(t) \in [0,\lambda]$

The initial condition is $x(0) = \xi$.

The firm wants to maximize the net present value of cash flow:

$$U = \int_0^\infty e^{-rt}[\pi(x(t))-u(t)]\,dt$$

where the discount rate r is assumed to be constant and the profit function $\pi(x)$ defined as total revenue minus operating cost is assumed to be twice continuously differentiable with negative second derivative.

The control variable u maximizes the Hamiltonian

$$H = e^{-rt}.\ \pi(x) + (p - e^{-rt})\,u$$

The costate variable p verifies $\dot{p} = -\dfrac{\partial H}{\partial x}$.

The control is bang-bang: $u = \lambda$ if $p - e^{-rt} > 0$ and $u = 0$ if $p - e^{-rt} < 0$. We can see that if the firm starts by investing, it will invest at maximum rate and as long as $p > e^{-rt}$. It can be shown that

Proposition 1. To maximize the present value of cash flow, the firm will invest as quickly as possible and as long as the marginal profitability of capital $\pi_x(x)$ is greater than the discount rate. If the initial marginal profitability of capital is less than the discount rate, the firm will not invest.

Proof: See Appendix A.

3. INVESTMENT GAME IN A DUOPOLY - THE MODEL.

We assume there are two firms interacting in the market. Other firms' investment policy and general market conditions such as demand and input price are time invariant and known with certainty.

The state variable is the capital stocks $(x(t), y(t))$ of firms X and Y at time t. We assume that there is no capital stock depreciation, that investment is irreversible, and that investment rates $u(t)$ and $v(t)$ of firms X and Y are bounded by the constants λ and μ respectively.

$$\begin{aligned} 0 &\leq \dot{x} = u(t) \leq \lambda \\ 0 &\leq \dot{y} = v(t) \leq \mu \end{aligned} \tag{3.1}$$

The initial conditions are non-negative

$$\begin{aligned} x(0) &= \xi \geq 0 \\ y(0) &= \eta \geq 0 \end{aligned} \tag{3.2}$$

The objective of the firms is to maximize the net present value of cash flow by means of the control variables $u(t)$

and v(t) for firms X and Y respectively

$$(3.3)\qquad \begin{aligned} U &= \int_0^\infty e^{-rt}\,[\ \pi(x(t),y(t)) - u(t)]dt \\ V &= \int_0^\infty e^{rt}\ [\ \phi(x(t),y(t)) - v(t)]dt \end{aligned}$$

where r is the constant positive discount rate and $\pi(x,y)$, $\phi(x,y)$ the instantaneous profit functions depending solely on the state (x,y) which in turn is a function of time t. $\pi(x,y)$ is assumed to be C^2-differentiable with $\pi_{xx} < 0$, $\pi_y < 0$ and $\pi_{xy} < 0$. The last condition means that the reaction curve in the (x,y) space is downward sloping. The profit function $\phi(x,y)$ has properties similar to those of $\pi(x,y)$.

At every instant, each firm knows the state (x,y), knows how the other firm is going to proceed, and selects the investment rate to maximize the net present value of the return cash flow.

The above formulation defines a two-person non-cooperative differential game. The strategy set of the game is the set of all control functions (u(t),v(t)) satisfying (3.1) which also define a path of the game in the (x,y)-plane. The players' payoffs as functions of (u(t),v(t)) and defined by (3.3) are called the payoff of the game. $(u^*(t),v^*(t))$ is the Nash equilibrium strategy if for all t, $U(u^*,v^*) \geq U(u,v^*)$ and $V(u^*,v^*) \geq V(u^*,v)$ for all (u,v) of the strategy set. (u^*,v^*) is also called the optimal strategy or optimal control. The related path (defined by $\overset{\circ}{x} = u^*$ and $\overset{\circ}{y} = v^*$) is called the optimal path. $(U(u^*,v^*),V(u^*,v^*))$ is called the value of the game. Since the time horizon is infinite, the value of the game is solely a function of the initial state.

We have to find (a) whether the game defined above has a Nash equilibrium strategy, (b) if the optimal path exists and is unique, given an initial (ξ, η), and (c) whether the equilibrium strategy is perfect in Selten sense.

As the game is defined, there is no restriction on the end of the game. Although the time horizon is infinite, it will be shown that in some finite time both firms stop investing and the path reaches a terminal state.

The optimal paths will be specified in a simple way by means of the Industry Growth Path (IGP), introduced by Spence [7] as the locus of the points (x,y) when both firms invest as quickly as possible, given the initial state (ξ, η). In our model the IGP starting from (ξ, η) is a straight line in the (x,y) plan.

The Necessary Conditions.

The Hamiltonians for a 2-person differential game take the form (see for instance Intriligator [10])

$$H_1 = e^{-rt} \pi(x,y) + (p-e^{-rt})u + qv$$

$$H_2 = e^{-rt} \phi(x,y) + \sigma u + (s-e^{-rt})v$$

The optimal controls u^*, v^* maximizing H_1 and H_2 respectively verify the necessary conditions

$$(3.4) \quad \begin{aligned} u^* &= \lambda \text{ when } p-e^{-rt} > 0 \\ u^* &= 0 \text{ when } p-e^{-rt} < 0 \\ v^* &= \mu \text{ when } s-e^{-rt} > 0 \end{aligned}$$

$$v^* = 0 \text{ when } s - e^{-rt} < 0$$

The adjoint equations are

$$(3.5) \quad \begin{array}{ll} \mathring{p} = -e^{-rt} \; \pi_x(x^*,y^*) & \mathring{q} = -e^{rt} \; \pi_y(x^*,y^*) \\ \dot{\sigma} = -e^{-rt} \; \phi_x(x^*,y^*) & \dot{s} = -e^{rt} \; \phi_y(x^*,y^*) \end{array}$$

From (3.4) we see that each firm's capital stock is a non-decreasing, piecewise linear in time. To find the optimal path starting from state (ξ , η) and verifying the six differential equations (3.1) and (3.5), four additional boundary conditions are needed. They are provided by the terminal conditions defined below.

Terminal Time and Terminal State.

The game ends when both firms stop investing. The existence of the terminal time t_f and terminal state (x_f, y_f) can be shown in the same way as in the case of single firm. Each firm's capital stock is a non-decreasing piecewise linear function of time. The competitor's increase in capital deteriorates each firm's current profit and marginal profitability. If we assume that both profit functions $\pi(x,y)$ and $\phi(x,y)$ are bounded then a finite terminal state and a finite terminal time exist. However, neither the terminal state and terminal time are specified yet. This will be done in the next section.

4. INVESTMENT GAME IN A DUOPOLY - THE OPTIMAL PATH.

In the following the superscript * designating the value on the optimal path will be omitted. Introducing

$$(4.1)\qquad \begin{aligned} \delta(t) &= p(t) - e^{-rt} \\ \varepsilon(t) &= s(t) - e^{-rt} \end{aligned}$$

it follows from (3.4) that firm X will invest at rate λ if $\delta(t)$ is positive and firm Y will invest at rate μ if $\varepsilon(t)$ is positive.

From (3.5) the time derivatives of $\delta(t)$ and $\varepsilon(t)$ become

$$(4.2)\qquad \begin{aligned} \dot{\delta}(t) &= e^{-rt}\ (r - \pi_x(x,y)) \\ \mathring{\varepsilon}(t) &= e^{-rt}\ (r - \phi_y(x,y)) \end{aligned}$$

Let R_1 and R_2 be the steady-state reaction curves defined by[(1)] R_1: $\pi_x(x,y) - r = 0$ and R_2: $\phi_y(x,y) - r = 0$, and N their intersection. Also let Ω be the set of $(x,y;\ x > 0, y > 0)$ to the right of the IGP through N and including the border line. Within Ω firm X has an initial advantage in the sense that if both firms invest at their maximal rate firm X will reach its static Nash capital stock before firm Y.

In the sequel, only initial points belonging to Ω will be considered. Points where Y has an initial advantage,

---

(1) the terminal condition being (t_f, x_f, y_f), firm X's payoff can be written as

$$U = \int_0^{t_f} e^{-rt}\ \{\pi(x,y) - rx\}\ dt + \frac{1}{r}\ e^{-rt_f}\ \{\pi(x_f,y_f) - rx_f\}$$

The net present value of the steady-state return is

$$U_{ss} = \frac{1}{r} e^{-rt_f}\ \{\pi(x_f,y_f)\ - rx_f\ \}$$

The steady-state reaction curve of firm X is defined by

$$\frac{\partial U_{ss}}{\partial x_f} = 0 \quad \text{or} \quad \pi_x(x,y) - r = 0.$$

i.e., not belonging to Ω can be treated similarly. We also assume that in Ω the steady state reaction curve R_1 is below R_2 (see Fig. 4.1)

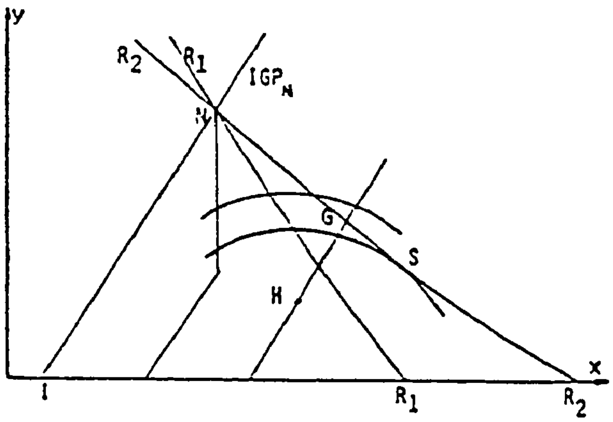

Fig. 4.1

Free Terminal State.

The game has a priori no specification on the terminal state. The transversality conditions are (see appendix B)

$$\text{(4.3)} \qquad \begin{aligned} \delta(t) &= e^{-rt}\ (\ \pi_x(x_f,y_f)\ -\ r)\ /\ r \\ \varepsilon(t) &= e^{-rt}\ (\ \phi_y(x_f,y_f)\ -\ r)\ /\ r \end{aligned}$$

It follows that firm X would stop investing on R_1 and firm Y on R_2. Consider then IGP_N, the IGP through N. For all points on IGP_N below N, both $\mathring{\delta}(t)$ and $\mathring{\varepsilon}(t)$ are negative or $\delta(t)$ and $\varepsilon(t)$ decreasing which means they are positive since according to (4.3), $\delta(t_N) = \varepsilon(t_N) = 0$.

We have established:

Proposition 2. For every initial state on the IGP_N below N, both firms will invest as quickly as possible until they reach the Nash-Counot equilibrium point N where both firms stop investing. The IGP_N up to N is an optimal path.

NR_2 As Derived Terminal Surface. For any point in the region INR_2 except on NR_2, since $\phi_y - r > 0$, from (4.3), $\varepsilon(t)$ is positive and firm Y's strategy is to invest as quickly as possible all the way until R_2 and to stop investing once at R_2. If the optimal path reaches the segment IN of the IGP_N then, as we have seen above, the state will move along the IGP_N until N. If the optimal path does not reach the segment IN it will have to intersect R_2. At the point of intersection of the path and R_2, firm Y would stop investing. Firm X's interest is to have the terminal point as "close" as possible to R_1 once it is already above R_1: firm X stops investing as soon as firm Y does. Thus, given firm Y's strategy, the terminal state is no longer free: the terminal surface becomes NR_2.

With NR_2 as terminal surface, the transversality conditions become (see appendix B, relation (B.6))

$$(4.4)\qquad \begin{aligned} r(p-e^{-rt})(u\phi_{xy}+v\phi_{yy}) &= e^{-rt}v((\pi_x-r)\phi_{yy}-\pi_y\phi_{xy}) \\ r(s-e^{-rt})(u\phi_{xy}+v\phi_{yy}) &= -e^{-rt}\,u\phi_x\phi_{yy} \end{aligned}$$

with u equal to either λ or 0 and $v = \mu$. From (4.4) we conclude that (a) $\varepsilon(t)$ is non-negative and (b) $\delta(t)$ is positive if

$$(4.5)\qquad -\frac{\pi_x - r}{\pi_y} \geq -\frac{\phi_{xy}}{\phi_{yy}}$$

The equality in (4.5) defines the Stackelberg point where a curve of the family $\pi(x,y) - rx = \text{constant}$ is tangent to the steady-state reaction curve R_2. Notice that the curves $\pi(x,y) - rx = k$ are the isostatic profit curves having their peaks on the reaction curve R_1.

The sign of $\delta(t)$ shows that on the way to the terminal surface NR_2 and in the neighborhood of R_2, firm X is investing if the terminal point is on the region NS and is not investing if the terminal point is on the region SR_2. Firm Y is continuously investing at maximum rate from any initial state below NR_2 since $\varepsilon(t)$ is positive below NR_2.

Switch Lines. (see Fig. 4.2)

Let G (x_G, y_G) be a terminal point on the manifold NS and H (x_H, y_H) an initial point on the IGP through G, i.e., $t_H = 0$. We have then

$$(4.6) \qquad t_G = (x_G - x_H)/\lambda \quad = (y_G - y_H)/\mu$$

$\delta(t_G)$ is positive ($\delta(t_G) = 0$ when G at S) and $\mathring{\delta}(t_G) > 0$. Firm X is investing in the neighborhood of G. H is a switch point from non-investing to investing if $\delta(t_H) = 0$. Point H, if it exists, has to be in the region NR_1R_2 between the two reaction curves. Considering H as the initial point, we have

$$(4.7) \qquad \delta(t_G) = \int_0^{t_G} e^{-rt} \left(r - \pi_x(x_H+\lambda t, y_H+\mu t)\right) dt$$

where

$$(4.8) \qquad \delta(t_G) = \frac{e^{-rt}}{a} \mu \frac{\pi(\pi_x - r)\phi_{yy} - \pi_y\phi_{xy}}{\lambda\phi_{xy} + \mu\phi_{yy}} \Bigg|_{\substack{x=x_G \\ y=y_G}}$$

(4.8) is derived from (4.4). These three relations (4.6-7-8) define H as function of G. They are the equations of the switch line SF with F on R_1. On the switch line SF, firm X switches from not investing to investing. The portion of the optimal path arriving on H is vertical.

Moving backward from H on the vertical portion of the optimal path, since H is in the region NR_1R_2, $\dot{\delta}(t_H) > 0$ and $\delta(t)$ is decreasing (as t decreases) as long as the current state is in the region NR_1R_2. From the intersection of the optimal path with R_1, $\delta(t)$ increases as t decreases. The vertical segment of the path ends at J where $\delta(t_J)$ becomes zero. J is defined from H by

$$\int_{t_J}^{t_H} e^{-rt}\,(r - \pi_x(x_J, y_J+\mu t))\,dt = 0 \tag{4.9}$$

Notice that by virtue of (4.9), J is below NR_1 and when H is at F, J is also at F. Equation (4.9) defines the switch line FT, locus of points J where firm X switches from investing to non-investing.

Let G now be on the portion SR_2 of the terminal surface. Since $\delta(t_G)$ is negative, the optimal arrives on G vertically. Point H no longer exists but point J may exist. J is defined from G by

$$\int_{t_J}^{t_G} e^{-rt}\,(r - \pi_x(x_J, y_J+\mu t))\,dt = 0 \;\Bigg|_{\substack{x_J\geq 0\\ y_J\geq 0}} \tag{4.10}$$

When G is at S, J is at T. The switch curve FT is continuous but should have a kink at T.

To summarize, let us consider for reason of symmetry the set Ω of state variables (x,y) in the first quadrant on

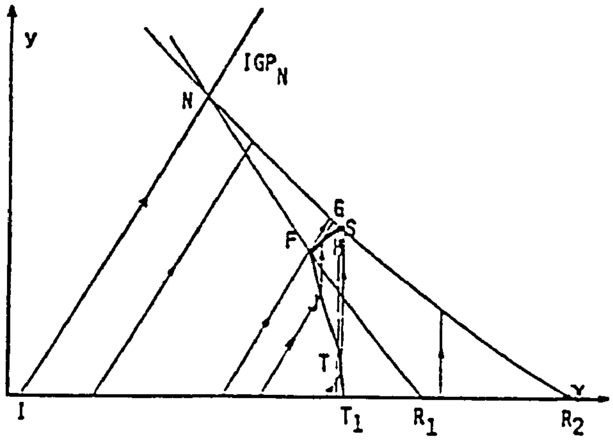

Fig. 4.2

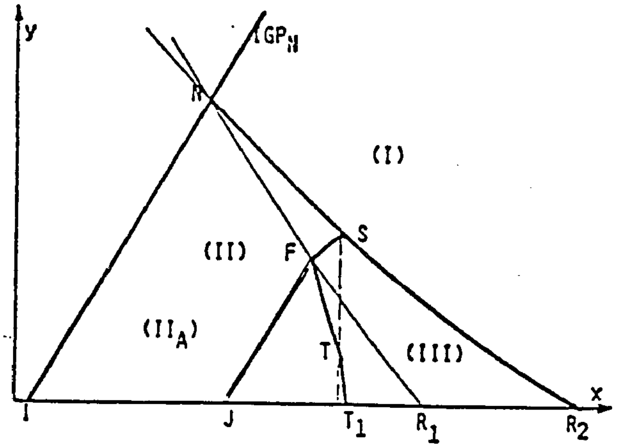

Fig. 4.3

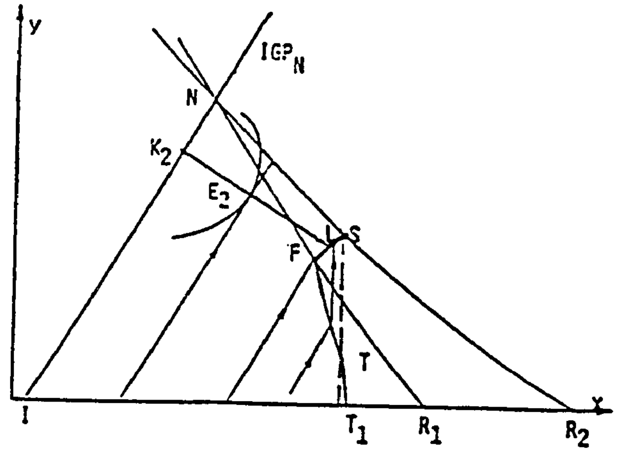

Fig. 5.1

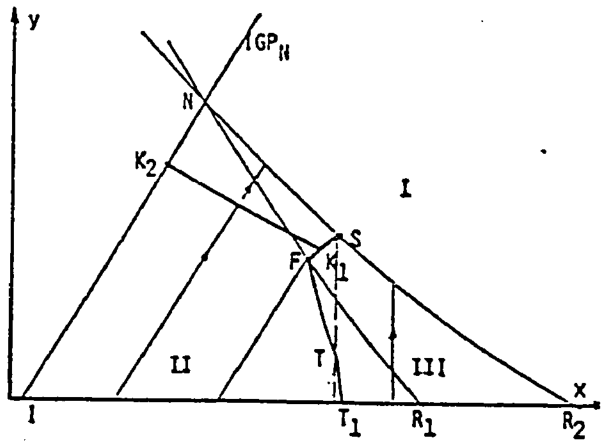

Fig. 5.2

and below the IGP_N. We define a firm's simple strategy as a partition of Ω into two subsets, one for investing at maximum rate and one for not investing. The strategy pair is a pair of simple strategies used by two firms, that is a partition of Ω into four subsets. We have then (see figures 4.2 and 4.3.)

Proposition 3. The following strategy pair constitutes a perfect Nash equilibrium solution of the differential game:

.in subset I, neither firm invests

.in subset II, both firms invest at maximum rate

.in subset III, firm Y (the follower) invests at maximum rate and firm X does not invest.

The switch curve SF is given by equations (4.6), (4.7) and (4.8). The switch curve FT is given by equations (4.6) and either (4.9) or (4.10).

The strategy pair for initial states (ξ, η) above the IGP_N can be obtained in a similar way with R_1 as terminal surface.

The switch lines are illustrated in appendix C by an example in which the two firms are assumed to have identical production functions depending solely on capital. In general the switch lines can only be obtained numerically even in the case of constant returns to scale where the expressions (4.7 and (4.8) of $\delta(t)$ can be integrated. It is further shown for the special case in appendix C that, at the limit when the discount rate goes to zero, the switch lines still exist (SF is above but very close to IGP_S); this means that for every initial point in the region SFT, the optimal path starts out vertical, then becomes an IGP (rather than beginning as a branch of an IGP and then becomming a vertical segment through S).

Clearly, the strategy pair defined above is unique for a given initial point. It is a state dependent (closed-loop) Nash equilibrium. Thus the set of strategy pairs defined in proposition 3 above is a set of perfect state-space equilibrium (see Fudenberg & Tirole [7], page 11).

Proposition 3 sheds light on the papers of Spence and Fudenberg and Tirole. We have found in a rigorous examination of Spence's model (i.e., with discounting) that his rule that a firm invests as quickly as possible and then stops investing permanently is true only for certain initial conditions, i.e., points in region II_a of fig. 4.3 (INSFJ). Fudenberg and Tirole, based on their geometric analysis of the model without discounting, conjectured as much, stating that typically the optimal path has two switch points. Our analysis shows that the optimal path can have zero, one, or two switch points. The number of switch points of the optimal path starting from a given initial state of capital stocks can be easily derived from our proposition 3.

Finally, our analysis sheds light on Fudenberg and Tirole's conjecture that for a given level of initial capital for one firm there is a wide range of initial capital stocks of its rival such that the equilibrium path terminates in the steady state Stackelberg point (i.e., point S in Fig. 4.3). In the discounting case, this above conjecture generally is not correct, namely, for a given firm 1 stock, there is at most one value of initial stock for firm 2 such that the resulting equilibrium terminates in S. In the limiting game where the discount rate goes to zero, the conjecture is only correct if point T (of the switch line FT on the vertical of S) exists, and the switch line TT_1 exists and is vertical.

5. EARLY STOP SURFACE.

In the two previous sections, the terminal surface R_2 is derived from the firm leader's anticipation that the follower firm stops investing only on its steady state reaction curve R_2. Let us reexamine the firms' payoffs

$$U = \frac{1}{r} e^{-rT} \pi(x_T, y_T) + \int_0^T e^{-rt}(\pi(x(t), y(t)) - u(t))\, dt$$

$$V = \frac{1}{r} e^{-rT} \phi(x_T, y_T) + \int_0^T e^{-rt}(\phi(x(t), y(t)) - v(t))\, dt$$

T being the terminal time considered as a parameter. Assume that the initial state (ξ, η) is in the region below NFT so that both firms invest initially and as quickly as possible. Along the IGP through (ξ, η), $x = \xi + \lambda t$, $y = \eta + \mu t$.

On the IGP, firm X prefers that both firms stop investing at E_1 where

$$\frac{dU}{dT} = \frac{e^{-rT}}{r}(\lambda(\pi_x - r) + \mu\pi_y) = 0$$

and similarly, firm Y prefers the stop point E_2 where (see fig. 5.1)

$$\frac{dV}{dT} = \frac{e^{-rT}}{r}(\lambda\phi_x + \mu(\phi_y - r)) = 0$$

E_1 and E_2 are the contact points of the IGP with the isostatic profit curves $\pi(x,y) - ax$ = constant and $\phi(x,y) - ay$ = constant. Assume that E_2 is above E_1 and both points are below the reaction curve R_2. Firm Y certainly would not like to stop at E_1 since it can improve its payoff by moving up the IGP to E_2. Would firm X prefer to stop at E_2? Firm X's outcome at E_2 is better than its outcome in a subgame starting from E_2 and using the strategy pair defined in section 4 above. In other words, firm's outcome at E_2 strictly dominates all other outcomes of subgames starting from E_2. As mentioned in Fudenberg and Tirole [4], both firms have incentive to propose, through coordination, E_2 as early stop point. The locus of E_2 is defined by

$$(5.1) \qquad \lambda\pi_x + \mu(\phi_y - r) = 0$$

It starts from K_1 on SF and ends at K_2 on the IGP_N. (see fig. 5.2)

Now let us consider a point L on SK_1 where firm X restarts investing. Both firms' outcomes at L strictly dominate all other outcomes of subgames starting from L. L is an early stop point and SK_1 is the early stop surface extended from K_1K_2.

Analytically, the terminal surface being defined by equation (5.1), the transversality conditions (see appendix B, relation (B.7)) will lead to following conclusions:

. $\varepsilon(t) = s - e^{rt}$ is always positive if $\phi_y - r$ positive or point (x,y) below R_2.

. $\delta(t) = p - e^{-rt}$ positive if $-\dfrac{\pi_x - r}{\pi_y} \geq -\dfrac{\lambda\phi_{xx} + \mu\phi_{xy}}{\lambda\phi_{xy} + \mu\phi_{yy}}$

The equality defines the point where the isostatic profit curve $\pi - ax$ = constant is tangent to the curve K_1K_2. The contact point is K_1 since from K_1 to K_2, $\delta(t)$ is positive and $\delta(t) = 0$ at K_1. By the same token, point L on SK_1 is the corner solution to the problem of maximizing U on the vertical portion of the optimal path. The difference between the portions SK_1 and K_1K_2 is that on SK_1 there is no need for coordination since firm X is not investing.

The early stop surface is incorporated in the solution to the game as follows: (see fig. 5.2)

Proposition 4. The strategy pair defined in the above section remains unchanged except that both firms stop investing on the manifold SK_1K_2. Then every point on SK_1K_2 is a Nash equilibrium point and the modified strategy pair constitute a perfect equilibrium of the differential game. This strategy pair Pareto dominates the previous perfect Nash equilibrium defined in the section 4 above.

6. Conclusion.

By optimal control methodology we have established equations of the switch lines which can be determined numerically. These lines in turn will enable us to define graphically the family of optimal paths in the capital investment game.

In a non-cooperative no-coordination game, the optimal path ends on the steady-state reaction curve R_2 of the follower firm which continuously invests as quickly as possible until R_2. The firm with competitive edge does not always invest but whenever it does, the maximum rate is used. The leader firm may invest continuously, or start by not investing and then invest when the follower firm's capital is large enough, or may start by investing and interrupt its growth for a time. Depending on the firms' initial capitals, the optimal path can have no, one, or two switch points. The terminal state related to an optimal path having two switch points is above (and close to) the Stackelberg point S.

As the discount rate becomes smaller and smaller, the switch line SF gets closer and closer to the IGP_S. The resulting optimal paths in the non-disounting case can have up to two switch points, in contrast to Fudenberg and Tirole's proposed investment paths which seemingly are not optimal in every instance. Similarly, Fudenberg and Tirole's early stop line should be partially replaced by a portion of the switch line SF.

PART II

RESEARCH AND DEVELOPMENT EXPENDITURES

7. INTRODUCTION.

In this second part we examine a firm's R & D expenditures in a patent race. Several sources of uncertainty may affect a firm's investment decision: uncertainty in the invention process, uncertainty in the value of the patent, etc. We limit our study to the uncertainty of the date of invention. Since there is only one winner among the firms engaging in the same R & D project, we assume the value of the patent is sufficient to at least offset the risks of failure.

Clearly, the first question is why competition affects a firm's R & D activity. If the product market is competitive, firms undertaking R & D expenditures buy the possibility of acquiring the monopoly power and associated quasi-rents or increasing the post-innovation market share with low production cost. In a different environment, a firm may, with patent protection, have the monopoly at the present time. But other firms engage in R & D and compete for the market in the future. The monopolist, threatened by potential competitors, has to pursue the R & D activity lest its market share be eroded.

How does the competition affect a firm's R & D activity? The relationship between the structure of an industry and the technological progress has been extensively studied in the literature. Both theoretical and empirical works came to the conclusion that there is a positive correlation between industrial R & D expenditures and the degree of concentration of the industry. Up to the early 70's however, much of the studies analyzed the causal association between the degree of concentration and innovative activity as well as the behavior of an individual firm that views industrial structure parametrically. Later, the economists recognized that, as the expected profit of the firms pursuing R & D activity in the industry is positive, in the absence of barrier to entry, additional firms are expected to enter the R & D competition. Thus the relationship between the degree of concentration and R & D activity is not a causal one. The two factors are endogeneous and ought to be determined simultaneously at market equilibrium. In other words, there is generally some degree of concentration intermediate between pure monopoly and perfect competition at which a firm's pursuit of R & D is most vigorous.

Many models were established to formulate the quantitative relationship at equilibrium between the degree of industry concentration in one hand and the R & D expenditures and the speed of research on the other hand: Dagusta and Stiglitz [2 & 3], Loury [12], Lee & Wilde [11], Reinganum [13], Telser [15], Grossman & Shapiro [8], to quote only a few. They have many common characteristics:

(1) The R & D competition is formulated as a single stage game where each firm's objective is to maximize the expected present value of its net return and each firm's strategy is defined by the rate of its R & D expenditure.

(2) Firms are independent and identical.

(3) The time of invention is a random variable with hazard rate as a function of the firm's R & D expenditures.(1)

(4) The first firm that makes the invention gets the reward

(5) The value of the reward does not change with the winner.

These models also address the following issues:

(a) What is the effect of industrial structure on the expected industry introduction time for the innovation?

(b) How is the equilibrium level of R & D expenditures compared to the socially optimal level?

(c) Can the R & D activity in an industry be regulated to make the long run equilibrium in R & D socially optimum?

In section 8, we study the models of Dagusta & Stiglitz [3], Loury [12], Lee & Wilde [11], and Reinganum [13]. The first three models are static in the sense that the firms'

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(1) In Grossman & Shapiro's model, the progress is a random variable. In Reinganum's model, the hazard rate is constant.

commitments are assumed binding, i.e., taken as known at the initial moment. They can represent short life R & D programs. Reinganum's model is dynamic in the sense that investment rates are time varying. The R & D competition is modeled as a differential game between two identical firms. In section 9 and 10, Reinganum's model is reformulated. The assumption of symmetry is relaxed: firms' cost functions and rewards can be different and one firm may have a headstart. In addition, the hazard rate of each firm is assumed to be a function of the firm's current stock of knowledge. We use the optimal control technique to find and isolate the effect of competition on R & D expenditures equilibrium strategy. We will not address the multistage race, a topic in industrial organization being investigated by many economists at the present time.

A short note on the hazard rate is helpful to make our paper easier to follow.

The Hazard Rate. Let x be firm X's capital stock in R & D at time t, that can be firm X's accumulated R & D expenditures or knowledge, depending on the model. Let F(x) be the probability that firm X succeeds in making the discovery with stock x or less. It follows that the density distribution function of success is dF(x), defined as the probability that the discovery occurs when the stock level is between x and x+dx. It implies that the discovery can only occur if there is an increase in stock level x, i.e. by spending on R & D activity. Given that no discovery is made with stock level of x or less, the conditional probability of success when the level x increases by dx is

$$h(x)\,dx = \frac{dF(x)}{1 - F(x)} \tag{7.1}$$

h(x) is the hazard rate. It must satisfy

$$h(x) \geq 0$$

and

$$\int_{x_o}^{\infty} h(x)\,dx = \infty$$

The second condition means that $F(\infty) = 1$. h(x) is always assumed to be twice continuously differentiable. In Reinganum's model, h(x) is a constant. In all other models including ours, h(x) is assumed to be strictly increasing, i.e., $h'(x) > 0$.

The cumulative distribution function F(x) can be defined from (7.1) with $F(x_0) = 0$ and $F(\infty) = 1$ as

$$F(x) = 1 - \exp(-\int_{x_o}^{x} h(s)\,ds)\ .$$

The condition $F(x_o) = 0$ implies that at time $t = 0$, when the firm starts its R & D activity, even with a non-zero stock x, surely it has not yet made the invention. The condition $F(\infty) = 1$ means that the firm will make the discovery at some finite time if it invests continuously.

Some models assume $h(0) = 0$, $\lim_{x \to \infty} h'(x) = 0$,

and $h''(x) \gtrless 0$ as $x \lessgtr \bar{x}$. If $\bar{x} > 0$,

there is an initial range of increasing returns to scale in R & D technology. In this case there is a x* such that $h'(x) = h(x)/x$. (see fig. 7.1)

In our numerical example, we use the hazard rate derived from the well known Gamma distribution function

$$h(x) = \frac{c^2 x}{1 + cx} \quad \text{where } c \text{ is a constant.}$$

In our example, h(x) is strictly increasing in x, h(0) = 0, and h(∞) = c. We say that for x large enough, the "variation" of h(x) is small. This property will be used when we will examine the transversality condition. (see fig. 7.2)

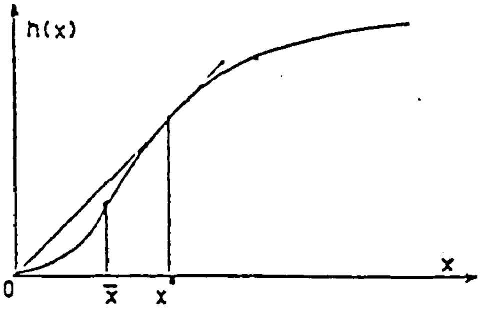


fig. 7.1

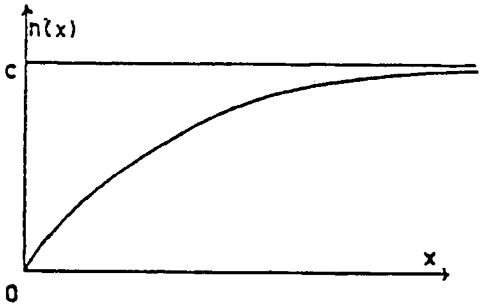


fig. 7.2

8. CLASSICAL MODELS.

First we review three static models of R & D expenditures formulated by Dagusta & Stiglitz, Loury, and Lee & Wilde, then we study Reinganum's dynamic model. The three static models have the same assumptions mentioned in the previous section. We use the following notation: i designates the individual firm; in industry equilibrium, since all firms are identical, subscript i is no longer required.

The base model is Dagusta & Stiglitz's one. Let n be the endogeneous number of independent, identical firms competing in R & D. At time t = 0, firm i chooses to invest x_i. The time t_i of its invention is a random variable with hazard rate $h(x_i)$. The probability that firm i succeeds during the interval [t + dt) is

$$\begin{aligned} & h(x_i)\cdot \exp(-h(x_i)t)\cdot \exp(-\sum_{j\neq i} h(x_j)t) \\ = \; & h(x_i)\cdot \exp(-\sum_j h(x_j)t) \ . \end{aligned}$$

Since all n independent firms are identical, it follows that the industry probability of discovery during [t,t+dt) is $nh(x) \cdot \exp(-nh(x)t)$ and the expected date of discovery is $1/nh(x)$. This model postulates that the hazard rate h(x) has an initial range of increasing returns, i.e., there exists an x* verifying

$$(8.1) \qquad h'(x) = \frac{h(x)}{x}$$

x* is called efficient level of investment for the reason stated below. With this postulate, we examine a firm's R & D activity in three environments:

<u>1. Socially Managed Economy</u>. Let V_s be the value of social benefits of the invention at t. n_s and x_s are chosen to maximize the overall expected return

$$\begin{aligned} (8.2) \qquad & nV_s \int_0^\infty h(x)\cdot\exp(-nh(x)t - rt)\,dt - nx \\ = \; & n\left(\frac{V_s\, h(x)}{nh(x) + r} - x\right) \end{aligned}$$

Assuming that n is a continuous variable, which can be justified if the efficient size x* of each unit is small, n_s

and x_s have to verify respectively

$$V_s\, h(x)\, r = x\, (nh(x) + r)^2 \tag{8.3a}$$

$$V_s\, h'(x)\, r = (nh(x) + r)^2 \tag{8.3b}$$

These equations imply (8.1) or $x_s = x^*$: firm's level of investment in R & D is efficient and independent of the number of firms.

2. <u>Monopoly</u>. Assume that the monopoly firm operates n independent, identical laboratories. Let V_m be the value of the reward at the date of discovery. It can be shown that $V_m < V_s$. The monopolist chooses n_m and x_m to maximize (8.2). We see that each unit also operates at the efficient level x^* and since $V_m < V_s$, from (8.3a), we have $n_m < n_s$. Thus, the monopolist delays innovation.

3. <u>Fully Competitive Market</u>. V_e being the present value of the reward at the patent date, each firm chooses the R & D expenditure x_i to maximize the expected present value of the return:

$$V_e \int_0^\infty h(x_i) \cdot \exp(-\sum_j h(x_j)t - rt)\, dt - x_i \tag{8.4}$$

$$= V_e \frac{h(x_i)}{\sum_j h(x_j) + r} - x_i$$

Dagusta & Stiglitz assume that the number of firms is large enough so that the choice of x_i has no effect on $\sum_j h(x_j)$. Then, at Cournot-Nash equilibrium, x_e verifies

$$V_e\, h'(x) = nh(x) + r \tag{8.5}$$

With free entry, the number n_e of firms has to be such that the expected present value of each firm is zero. Then n_e verifies

$$(8.6) \qquad V_e\, h(x) = (nh(x) + r)x$$

From (8.5&6), we see that each firm operates at the efficient level. With free entry, V_e is close to V_s. But since the profit in the socially managed economy is positive, from (8.2) and (8.6), we derive that $n_e > n_s$. It follows that in a perfect competition market, the number of firms is excessive and the aggregate R & D expenditure is larger than what is socially optimum.

G. Loury does not assume a large number of firms in his model. Each firm's R & D expenditure x_i maximizes

$$\frac{V_e \cdot h(x_i)}{a + h + r} - x_i$$

where $a = \sum_{j \neq i} h(x_j)$. At equilibrium, we have

$$(8.7) \qquad V_e h'(x)\,((n-1)h(x) + r) = (nh(x) + r)^2$$

since a becomes (n-1) h(x). n is defined by firms' zero expected profit at long run equilibrium: $V_e h(x) = x(nh(x)+r)$.

Loury shows that $\frac{dx}{dn} < 0$, i.e., as the number of firms increases, the equilibrium investment in R & D decreases.

Moreover, from (8.7), we have

$$\frac{h'(x)}{h(x)/x} = \frac{nh(x) + r}{(n-1)h(x) + r} > 1$$

or $x_e < x^*$: when there are increasing returns to scale in the R & D technology, firms operate with "excess capacity" in the sense that they do not exploit all of the scale economies in the innovation technology.

Assuming that $V_e = V_s$, Loury also shows that (a) given a fixed market structure n, in the industry equilibrium each firm invests more in R & D than what is socially optimal, (b) there exists a finite patent life or an entry tax (possibly negative) in the presence of which the long run industry equilibrium is socially optimal.

While Dagusta & Stiglitz and Loury papers focus on the role of fixed costs, Lee & Wilde emphasize the importance of variable costs. They assume that each firm commits at t = 0 a fixed cost F and a flow cost x until the invention is made by one of the firms. Again let $a = \sum_{j \neq i} h(x_j)$. a can be interpreted as the degree of rivalry. The expected benefit of investing in R & D is Vh/(a+h+r) and the expected costs are x/(a+h+r) + F. The following results are obtained when profit is maximized:

(a) As long as firms earn non negative profit, $\frac{dx}{da} \geq 0$, i.e., competition stimulates R & D activity.

(b) $\frac{dx}{dn} > 0$: as the number of firms increases, the equilibrium investment in R & D increases. This result is different from the one obtained by Loury.

(c) An increase of number of firms in the industry results in earlier time of invention and a decrease in expected profits.

(d) Let (n_e, x_e) be any competitive equilibrium such that the firms' expected profit is positive. The monopolist operating n_e identical projects independently will set $x_m < x_e$.

(e) Let (n_z, x_z) be the competitive equilibrium such that the firms' expected profit is zero. The monopolist will set $n_m < nz$. Moreover $xm < xz$.

The three static models we just reviewed are constructed to analyze the industry R & D competitive equilibrium defined by two endogeneous variables which are the number of identical firms n (or units in case of monopoly), and each firm's initial commitment in R & D expenditures x. These models represent short life programs of innovation.

In Reinganum's dynamic model, two identical firms compete in R & D for a technological breakthrough. Each firm i, invests on its own project to acquire additional experience resulting in an increase in its stock of knowledge x_i. The rate of acquisition of knowledge u_i is a function of time, so is x_i. Because of the uncertainty regarding the invention date, firms have to make the contingency plan for investing in R & D over a planning period T. If the invention occurs during the interval (O,T), the winner will collect the reward of constant present value P (discounted to t=0), independent of the invention date. The R & D program is stopped by both firms after the invention is made by the winner and they forfeit their investment to date. If both firms fail to succeed by T, their projects are abandoned at T with no residual value. The cost of obtaining the rate of acquisition of knowledge u_i is assumed to be $C(ui) = .5\ u_i^2$. The model is formulated as a non-cooperative differential game in bounded time horizon. The strategy of each firm i, defined by u_i, is chosen to maximize firm's expected payoff in present value.

Reinganum assumes two identical firms and uses a constant hazard rate h: the probability of making the invention with stock of knowledge less or equal xi is $1 - \exp(-hx_i)$. It follows that firm i's probability of discovery when its stock of knowledge is between x and x+dx is $he^{-hx}\,dx = he^{-hx}\,u\,dt$. Firm i's expected payoff is

$$(8.8)\quad J^1(u_1,u_2) = \int_0^T (P.\exp(-h(x_1 + x_2)).hu_1 - \exp(-rt-h(x_1+x_2)).u_1^2/2)\,dt$$

The first term of the integrand represents the instantaneous expected reward if firm i makes the invention during (t,t+dt), the second term, the instantaneous expected cost if no discovery is made by t. The expression for the second player's payoff is analogous.

Reinganum uses the dynamic programming approach to obtain the system of Hamilton-Jacobi equations for the value function. The closed form of solution of these particular H-J equations can be obtained and provides following results:

(1) Let $m(t) = Ph^2\,(e^{rt} - e^{rT})\,/\,r$

In the non-cooperative game, the Nash equilibrium strategy pair is (superscript e for competitive Nash equilibrium):

$$u_1^e(t) = u_2^e(t) = 2Phe^{rt}/(3 - \exp(m(t))$$

and the value of the game is

$$J_1^e(u_1,u_2) = J_2^e(u_1,u_2) = P(1 - 2/m(0)).$$

(2) $\dfrac{\partial u_1^e}{\partial P} > 0$ and $\dfrac{\partial u_1^e}{\partial T} < 0$:

The Nash equilibrium rate of acquisition of knowledge increases with the discount patent value and decreases with the planning horizon.

(3) The Nash equilibrium rate of investment $(.5u_i^2)$ increases with t.

<u>Case of Socially Managed Economy</u>. The two identical firms being independent, the social planner chooses (u1,u2) to maximize

$$(8.9)\quad J(u) = \int_o^T \left(P_s.\exp(-h(x_1 + x_2)).(u_1 + u_2) - \exp(-rt-h(x_1+x_2)).(u_1^2 + u_2^2)/2\right) dt$$

where P_s is the discount social value of the discovery. The solution is

$$(8.9b)\quad u_1 = u_2 = P_s e^{rt} h/(1 - m(t)).$$

If $P_s = P$, it can be shown that $u^e(t) > u^s(t)$ for all $t < T$. They are equal at $t = T$. The result is that if $P_s = P$, then firms spend more in R & D than what is socially optimal. We arrive to the same result as in the static models.

The static and dynamic models reviewed in this section examine the competitive R&D expenditures in industry equilibrium of identical firms. While the static models consider the number of firms as an endogeneous variable, the dynamic model introduces the time dimension and takes the number of firms as given. We note that Reinganum is able to obtain the closed form of the equilibrium investment rate because the cost and the hazard rate have special forms and the reward is constant in present value. The assumption of two identical firms is another drawback of her model. Indeed, the investment path (x1,x2) is always the first diagonal of the (x1,x2) plan. For the game to be more meaningful economically, two firms should have different characteristics including initial stocks of knowledge, cost functions, hazard rates, and values of the reward. The general model with less restricted assumptions is developed in the following sections. Each firm's hazard rate is a function of the firm's stock of knowledge. The project time span is assumed to be bounded, as in Reinganum's model.

9. R & D INVESTMENT POLICY OF A SINGLE FIRM.

A firm X, without competition, engages in an R & D activity for a particular technological breakthrough. The firm spends its resources on the R & D project to acquire experience resulting in an increase in its stock of relevant knowledge x. By increasing x, the firm improves its chance of making the discovery. The time period t from the beginning of the project to the date of discovery is a random variable, depending on the stock level x to date. We assume that the budget for this particular project is planned for a finite period T. We also assume that the firm X starts the project with an initial stock of knowledge $x_0 = x(t=0)$ and there is no start-up cost. Let u(t) be the rate of increase in x, $u(t) = \overset{\circ}{x}(t)$, and r, the constant discount rate. The objective of the firm is to maximize the expected payoff in present value by means of the control variable u(t) which defines the R & D investment policy of the firm concerning this particular project.

Conditional on not having made the invention by t, X has to increase during the time interval dt its stock of knowledge x by $dx = u\,dt$ at cost of $C(u)\,dt$ in current value. Such cost can be for instance researchers' time, laboratory material and computer time. $C(u)$ is assumed to be independent of x, continuous, and strictly increasing, with Cuu > 0. We assume there is no upper limit for $u(t)$. The conditional probability of success during dt, when the level of stock x increases by dx, is (see 7.1)

$$\frac{dF(x)}{1 - F(x)} = h(x)\,dx = h(x)\,u\,dt$$

In case of success, the firm will get a reward of current value P. The firm's net present value of the expected return is

$$U = \int^{T} (1 - F(x))\cdot(Pe^{-rt}h(x)u - e^{-rt}\,C(u))\,dt$$

$F(x)$ is the cumulative distribution function and $h(x)$, the hazard rate. Noting that $F(x(0)) = 0$, we obtain by integrating U

$$(9.1)\quad U = PF(x(T))e^{-rT} + \int_{0}^{T} e^{-rt}\,(rFP - (1-F)C)\,dt$$

<u>The Equation of Motion</u>. We form the Hamiltonian

$$H = e^{-rt}\,rFP - e^{-rt}\,(1 - F)C + pu$$

with u maximizing H for every t:

$$(9.2)\qquad p = e^{-rt}\,(1 - F)\,C_u$$

The costate variable p verifies

(9.3) $$\overset{\circ}{p} = -\frac{\partial H}{\partial x} = e^{-rt}(rP + C)\,F_x$$

Let $z(T) = e^{-rT}P\,F(x(T))$, the transversality condition is

$$p^*(T) = -\frac{\partial z}{\partial x}(x(T)) = e^{-rT}PF_x(x(T))$$

Taking the time derivative of p in (9.2), then using (9.3) and the definition $h(x) = F_x(x)/(1 - F(x))$, we obtain the following final result.

Proposition 1. The optimal investment policy of a single firm verifies the equation of motion

(9.4) $$C_{uu}\,\overset{\circ}{u} = h(uC_u - C) + r(C_u - hP)$$

The change of variable (9.2) applied to the transversality condition gives

Proposition 2.

(9.5) $$C_u(u(T)) = P\,h(x(T))$$

The marginal cost with respect to u, the rate of acquisition of additional knowledge, at the terminal time T equals the current value of the reward weighted by the hazard rate at the level of know-how at T.

Examine the equation of motion (9.4)

$$C_{uu}\,\dot{u} = hu(C_u - C/u) + r(C_u - hP)$$

The coefficient of hu is the difference between the marginal cost and the average cost with respect to u. If $C(u{=}0) = 0$, this difference is always positive. At T, since $C_u - hP = 0$, u(T) is positive. For $t < T$, $\overset{\circ}{u}$ is positive if $Cu - hP$ is positive. $\overset{\circ}{u}$ can become negative if $C_u - hP$ is negative but this is not a sufficient condition. Thus we can state that for t less than and close to the terminal time T, the R & D investment rate u(t) is increasing with t and attains at T the value u*(T) defined by (9.5). We have established

Proposition 3. If $C(u=0) = 0$ there exists a continuous time domain with upper bound T such that the investment rate u(t) is increasing.

Case of Constant Present Value of Reward.

The expected payoff becomes

$$U = \int_o^T (1 - F)\ (Phu - e^{-rt}C)\ dt$$

or

$$U = PF(x(T)) - \int_o^T e^{-rt}\ (1 - F)C\ dt$$

The Hamiltonian $H = -e^{-rt}(1 - F)C + pu$ is maximum for all t: we must have $p = e^{-rt}\ (1 - F)\ C_u$

The costate variable p verifies $\overset{\circ}{p} = -e^{-rt}\ F_x\ C.$

From the two relation above, the following equation can be derived

$$C_{uu}\ \overset{\circ}{u} = h(uC_u - C) + rC_u \tag{9.6}$$

Under the assumption of C_u and Cuu positive and $C(u=0) = 0$, $\overset{\circ}{u}$ is always positive (Reinganum [13], proposition 4, page 33), i.e., the R & D investment rate is always increasing in case of constant present value of reward.

Special Case of Reinganum's Model. Always in the case of constant present value of reward, if C(u) = .5u2 and h(x) is a constant, h(x) = d, the equation of motion (9.6) becomes

$$u = .5du^2 + ru \tag{9.6b}$$

and the transversality condition, $u(T) = dPe^{rT}$

With the change of variable u = 1/z, the above differential equation becomes linear of first order. Its solution is

$$u(t) = \frac{dPe^{rt}}{1 - d^2P(e^{rt} - e^{rT})/\ 2r} \tag{9.7}$$

which is the solution (8.9b): if the monopolist operates two identical, independent units, each unit's investment rate is given by (9.7).

An Example of the Generalized Model of a Single Firm. Our model and its optimal R & D investment policy is illustrated by the following example. Let the success rate be $h(x) = \frac{c^2x}{1+cx}$ which is the hazard rate of the well known Gamma distribution function. h(x) is an increasing function of x, with h(0) = 0 and h(∞) = c. c is an input parameter. With the condition $F(x(t=0)) = F(x_0) = 0$, the cumulative distribution function becomes

$$F(x) = 1 - \frac{1 + cx}{1 + cx_0} \exp(c(x_0 - x))$$

The cost function in our example is quadratic in u: C(u) = .5qu2 where q is an input parameter.

The equation of motion of the system becomes

$$\begin{aligned} \dot{x} &= u \\ \dot{u} &= h(x) * .5u^2 + r(u - h(x)\ P/q) \end{aligned} \tag{9.8}$$

The boundary conditions are

$$x(0) = x_0$$
$$u(T) = h(x(T)) \cdot P$$

The system can only be solved numerically. To find the solution satisfying the initial condition for one variable and the final condition for the other, the iterative technique of variation of extremals is used. We will start with two initial values of u, $u_1(0) = u0$ and $u2(0) = u0 + b$, b being a small constant. The integration of (9.8) with two different initial values $(x_0,u1)$ and $(x0,u2)$ will give two values of u(T) generally different from h(x(T)).P. The differences will be used to modify the initial value u_0. The process is repeated until the transversality condition is satisfied numerically.

A FORTRAN program in interactive mode was prepared (see appendix D) with six input parameters: discounting rate r, time period T, reward in current value P, coefficient c of the hazard rate, coefficient q of the cost function and finally, the initial level of stock of knowledge.

Figure (9.1) represents the investment rate u(t) and the cumulative distribution of success F(x(t)) for the following numerical values: r = .1, T = 5, P = 5, q = 1, c = .5 and x0 = 0, 1, 2, and 3 respectively. Figure (9.2) also represents u(t) and F(x(t)) for the same values of r, P, q, and c as above but x_0 unchanged (x0 = 0), and the life time T of the project varies (T = 2, 5, 8, 10 respectively).

In our example we see that

1. The growth rate $\mathring{u}$ is small ($\mathring{u}$ can be negative) at the beginning of the project and becomes much larger at the end.
2. The initial stock of knowledge has practically no effect on the rate except at the end of the life of the project. Toward the end of the planned period T, the higher is the initial stock, the larger the growth rate of u.
3. Also the higher the initial stock is, the higher the probability of making the breakthrough. This is quite predictable since h(x) is increasing with z.
4. The longer the life cycle T is, the smaller $\mathring{u}$ is at the beginning of the life cycle and the larger $\mathring{u}$ and u at the end of the planned period.

10. R & D INVESTMENT STRATEGY IN A DUOPOLY.

Basic Model and Assumptions. All the assumptions stated in the previous section will apply to this section unless specified otherwise. Our model consists of two firms X and Y of the same industry engaging in a specific R & D activity for the same particular technological breakthrough during a defined period of time T. The successful firm which makes the discovery before its rival does and before the end of T will collect a reward of constant current value. Both firms will then stop their particular R & D activity and forfeit their investment to date. If no breakthrough is made by T, the firms' projects are scrapped with no residual value. For each firm, the breakthrough is a stochastic event, the probability of which depends solely on the firm's stock of relevant knowledge.

$x(t)$ and $u(t) = \dot{x}(t)$ are firm X's stock of knowledge and its growth rate at t, and $h(x)$, firm X's hazard rate. Firm X's probability of success when its level of know-how is between x and x + dx is $F_x(x).dx = F_x.u.dt$ with $F(x)$ defined by

$$F(x) = 1 - \exp(- \int_{x_0}^{x} h(s)\, ds) .$$

Note that $F(x = x_0) = 0$ which means that at time $t = 0$, when firm X enters the patent race with initial stock of knowledge x_0, it did not make the discovery. Its rate of increase in knowledge $u(t)$ is obtained at cost $C(u)$ with C_u and C_{uu} positive and $C(u = 0) = 0$.

For firm Y, at time t, its stock of knowledge is $y(t)$, its growth rate is $v(t) = \dot{y}(t)$, its hazard rate is $k(y)$. Firm Y's cumulative distribution function of success $G(y)$ and cost function $D(v)$ have the same definition and properties as $F(x)$ and $C(u)$.

We assume there are no externalities in supply for the firms: the cost function $C(u)$ depends solely on u and is independent of x, y and v, and the cost function $D(v)$ independent of x, y and u. C(.) and D(.) do not have to be identical. C(.) and D(.) depend on the firm's size, the economy of scales, entrepreneurship of researchers, etc...

Let P be the constant current value of the reward for firm X when X is the winner and Q is the reward in current value for firm Y when Y is the winner. P can be different from Q. Indeed, for a finite patent life, the winner's profit derived from the innovated product depends on the firm's size and market share. At equal size, the monopolist of the present product market would collect a smaller reward if it makes the discovery than the challenger when the latter is the winner.

Finally we assume that at every instant, each firm is fully aware of the state of the competition expressed by the rival's accumulated stock of knowledge and its growth rate.

The objective of the firms is to maximize the expected present value of the net return by means of the control variables $u(t)$ and $v(t)$ for firm X and Y respectively.

We note that, since the cost functions $C(u)$ and $D(v)$ are strictly increasing, the correspondence between u and $C(u)$, and between v and $D(v)$ is one-to-one. Thus, instead of investigating the R & D rates of expenditures $C(u)$ and $D(v)$, we will focus our study on the rate of acquisition of additional knowledge u and v which are the control variables in our model. The characteristics of $C(u)$ and $D(v)$ can be easily derived from the characteristics of u and v. In the sequel, u and v are also called rates of investment.

<u>Formulation</u>. Assuming that by the time t no breakthrough is made by either firm. During the interval dt, firm X increases its stock of experience x by $dx = u(t).dt$ at cost $C(u)$ and has the conditional probability $h(x).dx = h(x).u.dt$ to make the discovery and get the reward of current value P. The expected pay-off for firm X during the interval $(t,t+dt)$ is defined as $(1-F)(1-G)[hP.u.dt - C.dt]$. We can now formulate our model as follows: Let $(x(t),y(t))$ be the state variable

at time t and (u(t), v(t)) the control variables of the firms X and Y respectively.

$$(10.1) \qquad \begin{array}{l} 0 < \overset{o}{x} = u(t) \\ 0 < \overset{o}{y} = v(t) \end{array} \qquad 0 \leq t \leq T$$

The initial conditions are

$$(10.2) \qquad \begin{array}{l} x(0) = x_0 \geq 0 \\ y(0) = y_0 \geq 0 \end{array}$$

The objective of the firms is to maximize the pay-offs U and V by means of the control variables u and v respectively

$$(10.3) \qquad \begin{array}{l} U = -U_0 + \int_0^T e^{-rt}(1-F)(1-G)(hPu-C)\, dt \\ V = -V_0 + \int_0^T e^{-rt}(1-F)(1-G)(kQv-D)\, dt \end{array}$$

where

$$(10.4) \qquad \begin{array}{l} h(x) = \dfrac{F_x(x)}{1 - F(x)} \\ k(y) = \dfrac{G_y(y)}{1 - G(y)} \end{array}$$

We assume that P and Q are large enough so that U and V in equilibrium are positive. Then U_0 and V_0 can be ignored in our study of the effect of the competition. U_0 and V_0 would have impact on entry deterrence, which is not considered here.

The above formulation defines a two-person non cooperative differential game. The strategy set of the game is the set of all controls (u,v) satisfying (3.1) which defines the path of the game in the (x,y)-plane. (u*,v*) is the Nash equilibrium strategy - or optimal strategy - if for all

$t \in [0,T]$, we have $U(u^*,v^*) \geq U(u,v^*)$ and $V(u^*,v^*) \geq V(u^*,v)$ for all (u,v) of the strategy set. (u^*,v^*) is also called optimal control.

The Necessary Conditions: The Equation of Motion. Using $dx = u.dt$ and $dy = vdt$, the expressions of U and V in (10.3) with $U_0 = V_0 = 0$ becomes

$$\begin{aligned} U &= PF(x(T))(1-G(y(T))e^{-rT} \\ &\quad + \int_0^T e^{-rt} (PFG_y v + (1-G)(rPF-(1-F)C))\, dt \\ V &= QG(y(T))(1-F(x(T))e^{-rT} \\ &\quad + \int_0^T e^{-rt} (QGF_x u + (1-F)(rQG-(1-G)D))\, dt \end{aligned}$$

We form the Hamiltonians

$$\begin{aligned} H_1 &= e^{-rt}(PFG_y v + (1-G)(rPF-(1-F)C)) + pu + qv \\ H_2 &= e^{-rt}(QGF_x u + (1-F)(rQG-(1-G)D)) + \sigma u + sv \end{aligned}$$

We assume that the controls $u(t)$ and $v(t)$ are not bounded. They maximize H1 and H2 respectively. We have then

$$(10.5) \qquad \begin{aligned} p &= e^{-rt} (1-F)(1-G)\ C_u \\ s &= e^{-rt} (1-F)(1-G)\ D_v \end{aligned}$$

The costate variables verifies

$$(10.6) \qquad \begin{aligned} \dot{p} &= -e^{-rt} (PF_xG_yv + (1-G)(rP+C)\ F_x) \\ \dot{s} &= -e^{-rt} (QF_xG_yu + (1-F)(rQ+D)\ G_y) \end{aligned}$$

(the equations for q and σ are not needed here). Combining the time derivatives of (10.5) with (10.4) and (10.6) we obtain the following final result

$$(10.7)\qquad \begin{aligned} C_{uu}\,\dot{u} &= h(uC_u - C) + (r + kv)(C_u - hP) \\ D_{vv}\,\dot{v} &= k(vD_v - D) + (r + hu)(D_u - kQ) \end{aligned}$$

The transversality conditions are

$$\begin{aligned} p(T) &= e^{-rT}\,PF_x\,(1-G)\,\Big|_{t=T} \\ s(T) &= e^{-rT}\,QG_y\,(1-F)\,\Big|_{t=T} \end{aligned}$$

Combining the above with (10.5) we obtain the final transversality conditions:

$$(10.8)\qquad \begin{aligned} C_u - hP &= 0 \\ D_v - kQ &= 0 \end{aligned} \qquad \text{for} \quad t = T$$

The above results are summarized by

Proposition 4. The Nash equilibrium strategy (u,v) of the patent race game verifies the system of differential equations

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ C_{uu}\,\dot{u} &= h(uC_u - C) + (r + kv)(C_u - hP) \\ D_{vv}\,\dot{v} &= k(vD_v - D) + (r + hu)(D_v - kQ)\ , \end{aligned}$$

the initial and transversality conditions being

$$\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0 \\ C_u - h(x)\,P &= 0 \\ D_v - k(y)\,Q &= 0\,. \end{aligned} \qquad \text{for} \quad t = T$$

First, let us compare this result with the one obtained in the previous section (proposition 1, equation (9.4)). We see that the effect of the competition on the optimal R & D investment strategy of a firm is equivalent to an increase of the discount rate by a term equal to the product of its competitor's hazard rate and its own rate of investment in R & D - we mean instantaneous rates. This observation, however, is not valid when the reward has a constant net present value, as we shall see later.

The effect of competition is best expressed by the term $kv(C_u - hP)$. In the sequel, subscripts c and nc are used to indicate the R & D competitive environment and non-competitive (monopolistic) environment. The above remark leads to

Proposition 5. Let u_c and u_{nc} be the optimal growth rate of a firm's rate of acquisition of additional relevant knowledge in a competitive and non-competitive environment respectively. We have at equilibrium

$$\overset{\circ}{u}_c = \overset{\circ}{u}_{nc} + \overset{\circ}{w} \tag{10.9}$$

where

$$\overset{\circ}{w} = kv(C_u - hP) / C_{uu} \tag{10.10}$$

$\overset{\circ}{w}$ represents the competition effect. In the expression of $\overset{\circ}{w}$, k and v are the rival's instantaneous hazard rate and rate of acquisition of additional knowledge.

It is not possible to analyze rigorously the competition effect since the systems (9.4) and (10.7) with their two-boundary conditions can only be solved numerically. However, we provide below

A Heuristic Analysis of the Competition Effect.

Let $C_u - hP = z(t)$.

Since Cuu is positive, we derive from the transversality condition (a) $\dot{w}(t=T) = z(t=T) = 0$

(b) $\dot{u}(t=T) > 0$; ($\dot{u}$ can be $\dot{u}_c$ or $\dot{u}_{nc}$)

Examine the sign of z through its time derivative

$$\dot{z} = C_{uu}\dot{u} - Ph'u$$

If $x(T)$ is large so that $h'(x(T))$ is small then $\dot{z}(T)$ is positive. Since $z(T) = 0$, $z(t)$ is negative in the neighborhood of T, t being less than T. As t decreases from T, if $\dot{z}(t)$ increases then $z(t)$ remains negative. In the other case, if $\dot{z}(t)$ decreases and eventually becomes negative, $z(t)$ may become positive but only after a certain period of time (see fig. 10.1). The first conclusion of our analysis is: if $h'(x(T))$ is small, there exists a continuous time domain with upper bound T in which the competition effect $\dot{w}$ is negative.

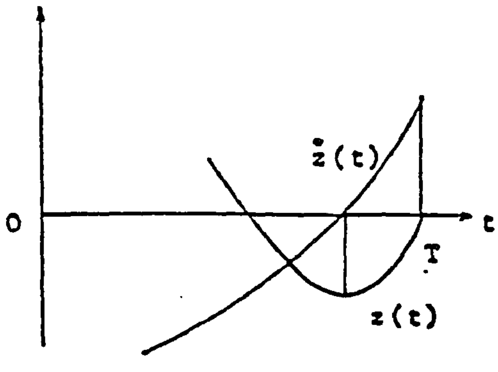


fig. 10.1

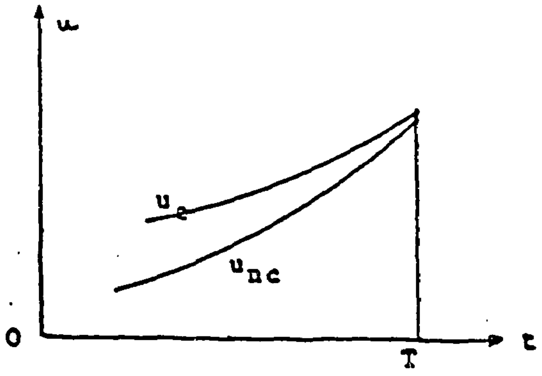


fig. 10.2

How does this conclusion help to compare u_c and unc?

If T is large so that $x_c(T)$ and $x_{nc}(T)$ are large enough then

$h_c(x(T))$ and $h_{nc}(x(T))$ are close and consequently $u_c(T)$ and $u_{nc}(T)$ are close. From the first conclusion above, we derive that the curve $u_c(t)$ is flatter than the curve $u_{nc}(t)$ and the $u_c(t)$ is above $u_{nc}(t)$ since $u_c(T)$ and $u_{nc}(T)$ are close (see fig. 10.2). Our second conclusion is: there exists a continuous time domain with upper bound T in which a firm engaging in a (long) R & D project increases its stock of knowledge at a faster rate when there is competition than in absence of competition.

When this domain covers the interval [0,T], the rate of acquisition of additional knowledge is continuously larger in case of competition than in absence of competition. The gap between u_c and u_{nc} decreases in time to become small at T.

<u>Case of Constant Present Value of Reward</u>. The pay-off becomes

$$U = \int_0^T (1-F)(1-G)(Phu-e^{-rt}C)\,dt$$

Using the same methodology as in the previous section, we arrive to the following result

$$(10.11) \quad C_{uu}\,\mathring{u} = h(uC_u - C) + rC_u + kv(C_u - e^{-rt}hP).$$

The transversality condition is

$$C_u(u(T)) - hPe^{-rT} = 0\ .$$

As mentioned earlier, the competition effect in this case of constant present value of reward is no longer equivalent to an increase of the discount rate. However it can still be expressed by

$$(10.12) \qquad \overset{\circ}{w} = kv(C_u - e^{-rt}hP) \,/\, C_{uu}$$

and we still have (10.9)

$$\overset{\circ}{u}_c = \overset{\circ}{u}_{nc} + \overset{\circ}{w}$$

We notice the similarity of the expressions (10.12) and (10.10) of the competition effect in the cases of constant present value of reward and constant current value of reward.

Numerical Examples. A FORTRAN program in interactive mode was prepared to provide numerical solutions to the system (10.7). (see appendix E.) In our examples, the cost functions are c_1u^2 and d_1v^2, and the hazard rates $c_2^2x/(1+c_2x)$ and $d_2^2y/(1+d_2y)$.

The input parameters of the program are:

- discount rate r
- project life T
- current values of reward P, Q
- cost function parameters c1, d1
- hazard rate parameters c2, d2
- initial stocks of knowledge x0, y0.

Figure (10.3) represents the time-varying rate of investment of a firm competing with an identical firm (uc) and in absence of competition (u_{nc}). Fc and Fnc are the cumulative distribution function of the firm and FI is the

CDF of the industry in case of competition:

$F_I = 1 - (1 - F_c)(1 - G_c)$.

Figures (10.4) and (10.5) show the growth paths (x(t),y(t)) the rates of investment u, v and the CDF F, G of two identical firms with different initial stocks (x_0, y_0). Notice that the first diagonal is the growth path for the case $x_0 = y_0$.

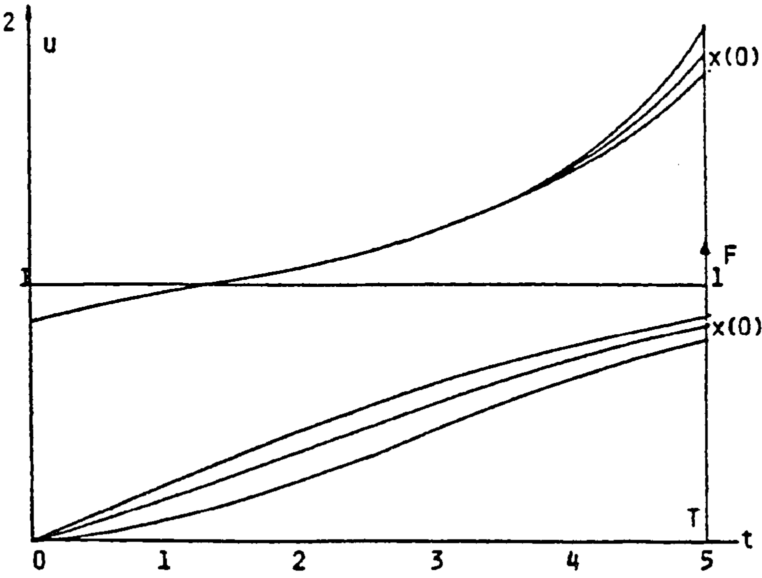
Figures (10.6) and (10.7) show the growth paths (x,y), the investment rates u,v and the CDF F, G for the two cases of same and different prizes, the firms' initial stocks being different.

These results show that:

1. The growth rates of u and v are generally small at the beginning of the project and become larger toward the end of the project life.
2. Other characteristics being equal, the leader (having an edge on initial stock over his rival) invests slightly more than its rival and continuously has a better chance of making the invention than his rival.
3. The prize is a great incentive for the firm to spend on R & D. A firm being behind its rival in R & D would invest more to catch up with and even to overtake its rival if the reward that it may obtain is larger than the one that its competitor expects.

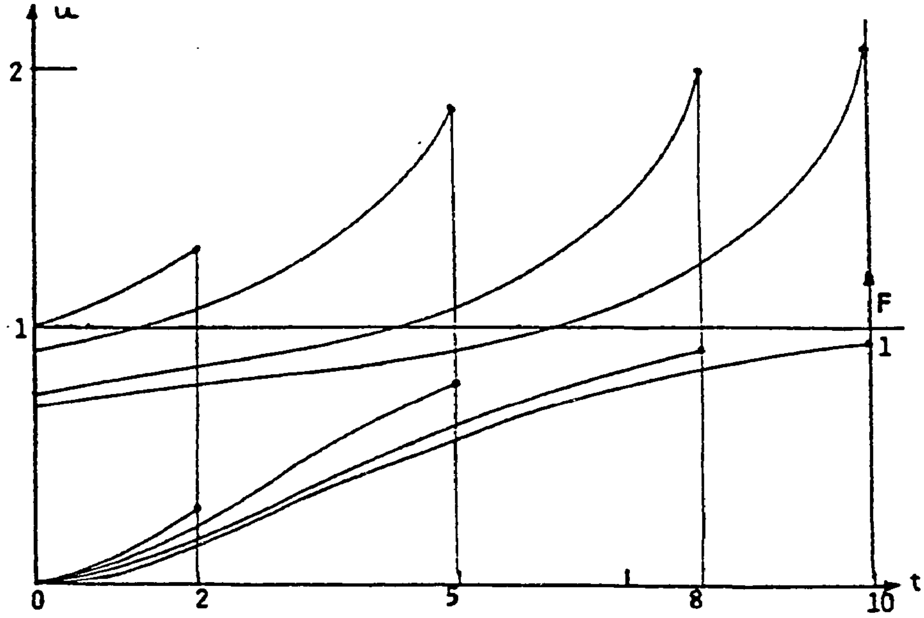
11. CONCLUSION.

In this part, we modified Reinganum's two-firm dynamic model of R & D expenditures under uncertainty to incorporate the firms' economic characteristics. To the model, they are exogeneous variables. The modified model provides possibilities to the challenger to catch up with the current leader and also to have a better chance of being the winner. The equations of motion in the model are rather simple, although they can only be solved numerically in the general case. We could isolate the competition effect by comparing the equations of motion in the cases of competition and absence of competition. Finally, by a heuristic analysis and by numerical examples, we came to the conclusion obtained with simple models that competition provides incentives to increase R & D expenditures under uncertainty.



SINGLE FIRM: EFFECT OF INITIAL VALUE

Fig. 9.1



SINGLE FIRM: EFFECT OF PLANNED PROJECT LIFE

Fig. 9.2

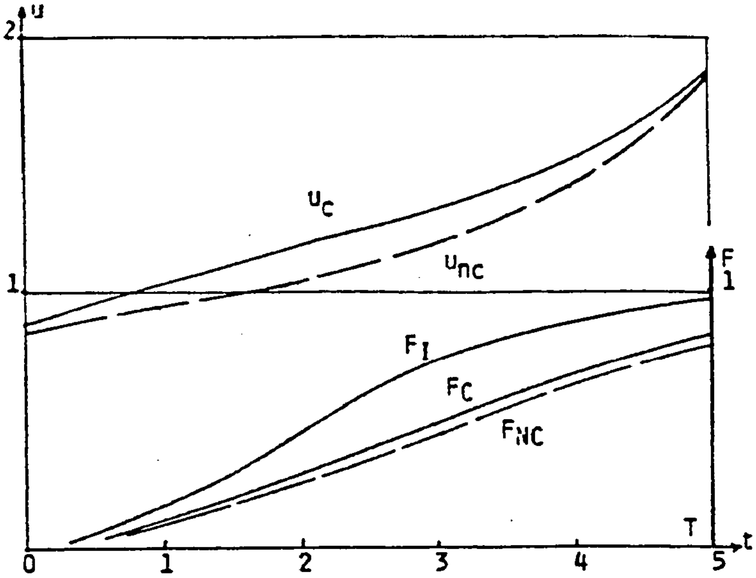
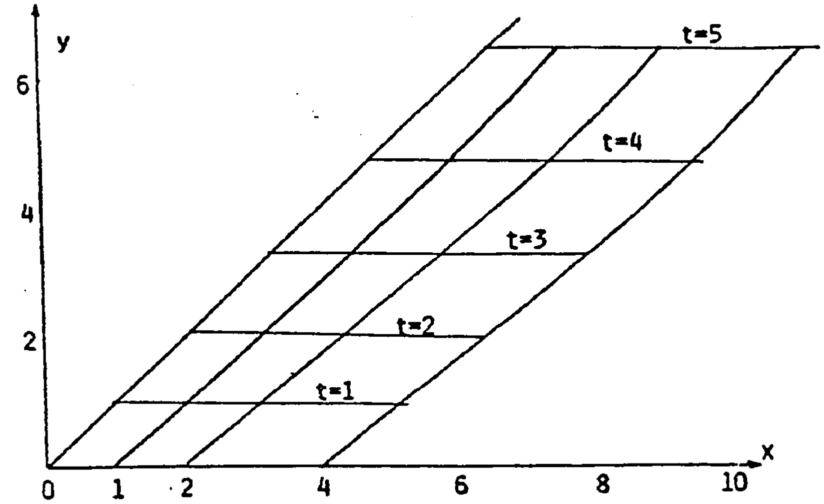


Fig. 10.3

Fig. 10.4



FIRMS' OPTIMAL GROWTH PATH.
EFFECT OF INITIAL STOCK

Fig. 10.5

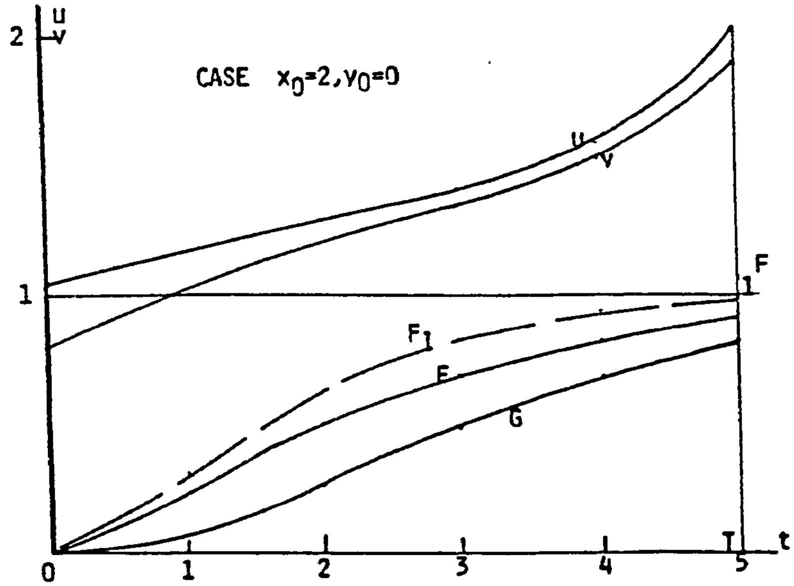
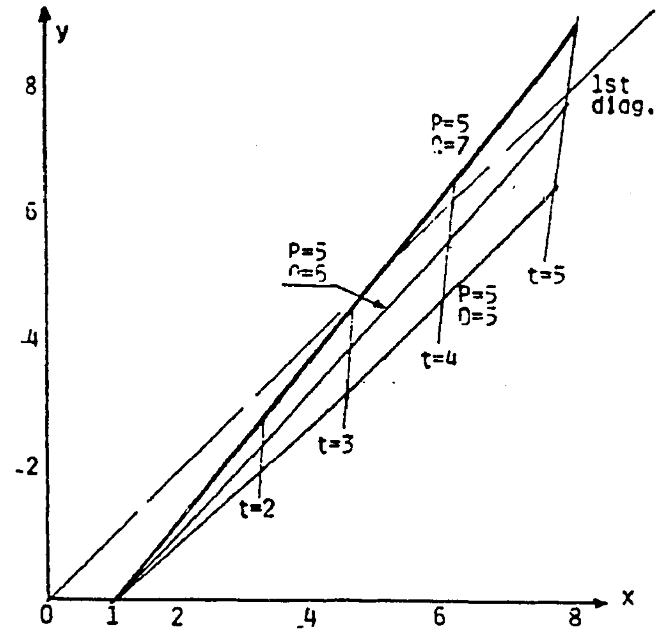
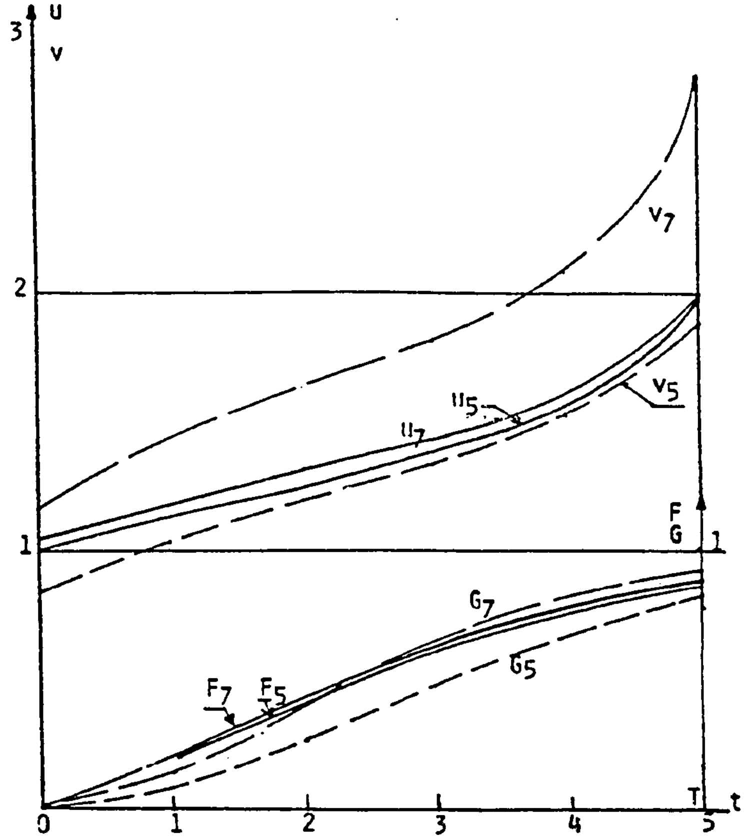


Fig. 10.6



FIRMS' OPTIMAL GROWTH PATH.
REWARD VALUE EFFECT.

Fig. 10.7



APPENDIX

APPENDIX A

Proof of Proposition 1.

The objective is to maximize $U = e^{-rt}(\pi(x(t)) - u(t))dt$

The control u maximizes the Hamiltonian

$H = e^{-rt}\pi(x) + (p-e^{-rt})u$ where $\mathring{p} = -\frac{\partial H}{\partial x} = -e^{-rt}\ \pi_x(t)$.

A necessary condition is $\mathring{x} = u = \lambda$ if $p - e^{-rt} > 0$

$\mathring{x} = u = 0$ if $p - e^{-rt} < 0$.

x is a piecewise linear, non decreasing function of t.

Let $\delta(t) = p(t) - e^{-rt}$. Its derivative $\mathring{\delta}(t) = e^{-rt}(r-\pi_x)$.

The transversality condition is (see Arrow [1]):

$\lim_{t\to\infty} p(t) = 0$ or $\lim_{t\to\infty} \delta(t) = 0$.

$\pi_x(x(t))$ is a decreasing function of t since $\pi_{xx}(x) < 0$.

If $\pi_x(x(t=0)) \leq r$, then $\mathring{\delta}(t) \leq 0$ and $\delta(t)$ non decreasing in t. Since $\delta(\infty) = 0$, we must have $\delta(t) < 0$ which means that the firm does not invest at all.On the other hand, if $\pi_x(x(t=0)) > r$, then $\delta(t)$ is decreasing at $t=0$ and as long as $\pi_x(x) > r$. Let T be the time such that $\pi_x(x(T)) = r$. For $t \geq T$, $\mathring{\delta}(t \geq T) \geq 0$ or $\delta(t \geq T)$ is non decreasing. But since $\delta(\infty)=0$, we must have $\delta(t \geq T) \leq 0$ which means that (a) for $t \geq T$, $u=0$, $\delta(t)=0$ and x unchanged and (b) for $t<T$, $\delta(t)>0$ and $u=\lambda$.

Proposition 1 can be shown differently. U is a function of of the stopping time T: $U = \int^T e^{-rt}(\pi(\xi+\lambda t)-\lambda)dt + \int_T^\infty e^{-rt}\pi(\xi+\lambda T)dt$

$\frac{\partial U}{\partial T} = 0$ when $\pi_x(\xi+\lambda T) = r$.

If T is not finite then the firm invests forever. Both x and $\pi(x)$ are unbounded. We assume that this is not the case.

APPENDIX B

Transversality Conditions in Optimal Control

(see Kirk [5])

The performance to be maximized being

$$U = h(x(t_f),t_f) + \int_0^{t_f} g(x(t),u(t),t)\,dt$$

where x and u are vectors, the boundary conditions are

$$(3.1)\quad \begin{aligned} 0 = {} & \{\frac{\partial h}{\partial x}(x^*(t_f),t_f) - p^*(t_f)\}.\ \delta x_f \\ & + \{H(x^*(t_f),u^*(t_f),p^*(t_f),t_f) + \frac{\partial h}{\partial t}(x^*(t_f),t_f)\}.\delta t_f \end{aligned}$$

For free terminal time, the coefficient of δt_f in the above expression is zero. For free terminal state, the vector defined by the first bracket in the above expression is zero. For a given terminal surface, the boundary condition is that the first inner product is zero. In our case,

$$(3.2)\quad \begin{aligned} U &= \frac{e^{-rt}}{r}\pi(x_f,y_f) + \int_0^{t_f} e^{-rt}\{\pi(x,y) - u\}\,dt \\ V &= \frac{e^{-rt}}{r}\phi(x_f,y_f) + \int_0^{t_f} e^{-rt}\{\phi(x,y) - v\}\,dt \end{aligned}$$

The free terminal time conditions are

$$(B.3)\quad \begin{aligned} (p - e^{-r\vartheta} u + qv &= 0 \\ (\sigma + (s - e^{-r\vartheta} &= 0 \end{aligned}$$

The free terminal state conditions are

$$(B.4)\quad \begin{aligned} p - e^{-rt} &= \frac{e^{-rt}}{r}\left(\pi_x(x_f,y_f) - r\right) \\ s - e^{-rt} &= \frac{e^{-rt}}{r}\left(\phi_y(x_f,y_f) - r\right) \end{aligned}$$

If the given terminal surface is defined by $\phi_y(x,y) - r = 0$ then $\delta x_f = (1, -\frac{\phi_{xy}}{\phi_{yy}})$ and the transversality conditions become

$$(3.5) \qquad \begin{aligned} (\frac{e^{-rt}}{r}\pi_x - p) + (\frac{e^{-rt}}{r}\pi_y - q)(-\frac{\phi_{xy}}{\phi_{yy}}) &= 0 \\ (\frac{e^{-rt}}{r}\phi_x - \sigma) + (\frac{e^{-rt}}{r}\phi_y - s)(-\frac{\phi_{xy}}{\phi_{yy}}) &= 0 \end{aligned}$$

Combined with the free terminal time conditions (B.3), they become

$$(3.6) \qquad \begin{aligned} (p-e^{-rt})(u\phi_{xy}+v\phi_{yy}) &= e^{-rt}v((\pi_x - \ )\phi_{yy}-\pi_y\phi_{xy}) \\ (s-e^{-rt})(u\phi_{xy}+v\phi_{yy}) &= -e^{-rt}\,u\phi_x\phi_{yy} \end{aligned}$$

Similarly, if the terminal surface is defined by

$$\lambda\phi_x + \mu(\phi_y - r) = 0$$

(the early stop line), the transversality conditions will be

$$(3.7) \qquad \begin{aligned} & r(p-e^{-rt})[(\lambda\phi_{xx}+\mu\phi_{xy})\frac{u}{v} + (\lambda\phi_{xy}+\mu\phi_{yy})]\, e^{rt} \\ & \quad = (\pi_x - r)(\lambda\phi_{xy}+\mu\phi_{yy}) - \pi_y(\lambda\phi_{xx}+\mu\phi_{xy}) \\ & r(s-e^{-rt})[(\lambda\phi_{xx}+\mu\phi_{xy}) + (\lambda\phi_{xy}+\mu\phi_{yy})\frac{v}{u}]\, e^{rt} \\ & \quad = (\phi_y - r)(\lambda\phi_{xx}+\mu\phi_{xy}) - \phi_x(\lambda\phi_{xy}+\mu\phi_{yy}) \ . \end{aligned}$$

APPENDIX C

An Example of Optimal Capital Investment Strategy in Production

1. An instantaneous profit function.

Assuming a linear demand function when the price P and quantity Q belong to a certain domain : $P = A - BQ$.

Also assuming that both firms' production functions are identical $Q^x = Cx^\alpha$, $Q^y = Cy^\alpha$, where C is positive and independent of the capital stocks x,y.

The instantaneous profit function of firm X is

$$\pi(x,y) = PQ^x - Rx - E$$

where R is the price of capital for the industry and E some fixed cost of firm X. $\pi(x,y)$ has a simple expression in the cases $\alpha = 1$ and $\alpha = 0.5$.

For $\alpha = 1$, $\pi(x,y) = x(b - cx - cy) - E$

For $\alpha = .5$, $\pi(x,y) = x^{.5}(b - cx^{.5} - dy^{.5}) - E$,

where in both cases, b, c, and d are positive constants independent of (x,y).

2.The switch lines.

In the simple case of $\alpha = 1$, the switch line SF is defined by the following equations

$$t = (x_G - x_H) / \lambda = (y_G - y_H) / \mu$$

$$\dot{\delta}(t) = \left\{r - b + c\left(2x_H + y_H + \frac{2\lambda+\mu}{r}\right)\right\}\frac{1-e^{-rt}}{r} - \frac{2\lambda+\mu}{r}bte^{-rt}$$

$$\delta(t) = (b - r - 2cx_G)\frac{\mu}{\lambda+2\mu}\;\frac{e^{-rt}}{r}$$

The switch lines FT and TT_1 can be defined by similar relations. In the case of $\alpha = 0.5$ only numerical solution can be provided.

3.Non discounting case.

Let Σ be the intersection of the IGP_S and R_1. It can be found from the above equations (case $\alpha = 1$) that

$$\lim_{r \to 0} \delta(t_\Sigma) = -\frac{2\lambda + \mu}{288} b^2 c < 0$$

which means that the optimal path through Σ is vertical and firm X is not investing at Σ. In this special non discounting case with $\alpha = 1$ the optimal path can have no, one, or two switch points depending on the initial state just as in the case of discounting. Numerical solution (obtained through a FORTRAN program) for the case $\alpha = 0.5$ also shows that the optimal path of the non discounting limiting game has up to two switch points.

The numerical solution to the discounting case with either α equal 1 or .5 also shows that there is only one optimal path arriving on S, the Stackelberg point on R_2.

APPENDIX D

R & D EXPENDITURES OF A SINGLE FIRM :

A FORTRAN PROGRAM

```
ASU004   INPUT VSPC LIBRARY OPENED.

ASU001   NOLIST PARAMETER IN EFFECT FOR THIS STEP.

 AUTH 6050102

ASU091   RETURN CODE = 00.

 EXPORT  DIF4 TO (SYSPRINT)
     10 50 CONTINUE
     20 PRINT 900
     30 900 FORMAT(/,' ENTER A, T, P, C, Q, XO - FREE FORMAT')
     40 READ(6,*) A,T,P,C,Q,XO
     50 PRINT 910,A,T,P,C,Q,XO
     60 910 FORMAT(/,' A = ',F4.2,'  T = ',F4.1,'  P = ',F4.1,'  C = ',F4.2,    -
     70 '  Q =',F4.1,'  XO = ',F4.1,/)
     80 ****** INITIALIZATION
     90 BETAXO=C**2*XO/(C*XO+1.)
    100 UO=BETAXO*P/Q
    110 PRINT 911,XO,UO,BETAXO
    120 911 FORMAT (' XO, UO, BETAXO : ',3F9.4,/)
    130 PRINT 912
    140 912 FORMAT (' ENTER INITIAL UO')
    150 READ(6,*) UO
    160 DT=.1
    170 N=(T+.1)/DT
    180 DUIO=0.1
    190 NF=0
    200 TO=0.
    210 U=UO
    220 ***END INITIALIZATION. BEGIN ADJUST UO.
    230 110 CONTINUE
    240 X=XO
    250 X3=XO
    260 U2=U
    270 U3=U2+.02
    280 T1=-DT*.5
    290 COST=0.
    300 *****COMPUTE X,U,TCU,PAYOFF & DELTA U FROM T=0 TO TC BASED ON UO
    310 DO 100 M=1,N
    320 BETAX=C**2*X/(C*X+1.)
    330 BETAX3=C**2*X3/(C*X3+1.)
    340 DU=DT*(A*U+.5*BETAX*U**2-A*P*BETAX/Q)
    350 DU3=DT*(A*U3+.5*BETAX3*U3**2-A*P*BETAX3/Q)
    360 U=U+DU
    370 U3=U3+DU3
    380 X=X+U*DT
    390 X3=X3+U3*DT
    400 U1=U-DU/2.
```

```
410 X1=X-(U*DT)/2.
420 T1=T1+DT
430 F=1.-(C*X1+1.)*EXP(C*(X0-X1))/(C*X0+1.)
440 COST=COST+EXP(-A*T1)*(A*P*F-(1.-F)*.5*Q*U1**2)*DT
450 100 CONTINUE
460 TCU=P/Q*BETAX
470 TCU1=TCU-.001
480 TCU2=TCU+.001
490 PAYOFF=P*F*EXP(-A*T)+COST
500 DUI1=.02*(TCU-U)/(U3-U)*1.5
510 IF(U.GT.TCU1.AND.U.LT.TCU2) GOTO 130
520 U=U2+DUI1
530 GOTO 110
540 130 CONTINUE
550 U=U2
560 PAYOFF=0.
570 PRINT 880
580 880 FORMAT('   T         X         U      DIST.F    DENS.F    BETA',  -
590 '    TCU  PAYOFF,/)
600 T2=0.
610 T1=-.05
620 X=X0
630 DO 140 M=1,N,10
640 DO 141 MM=1,10
650 BETAX=C**2*X/(C*X+1.)
660 DU=DT*(A*U+.5*BETAX*U**2-A*P/Q*BETAX)
670 U=U+DU
680 X=X+U*DT
690 T2=T2+DT
700 TCU=P/Q*BETAX
710 U1=U-DU/2.
720 X1=X-(U*DT)/2.
730 T1=T1+DT
740 F=1.-(C*X1+1.)*EXP(C*(X0-X1))/(C*X0+1.)
750 BETAX1=(C**2*X1)/(C*X1+1.)
760 DENS=BETAX1*(1-F)
770 PAYOFF=PAYOFF+EXP(-A*T1)*(P*DENS*U1-(1.-F)*.5*Q*U1**2)*DT
780 IF(M.EQ.1 .AND. MM.EQ.1) PRINT 830,T2,X,U,F,DENS,BETAX1,TCU,PAYOFF
790 141 CONTINUE
800 PRINT 830,T2,X,U,F,DENS,BETAX1,TCU,PAYOFF
810 830 FORMAT(F6.2,7F9.4)
820 140 CONTINUE
830 PRINT 850
840 850 FORMAT(/,' CONTINUE ? ENTER 1')
850 READ(6,*) NCONT
860 IF(NCONT.EQ.1) GOTO 50
870 END

ASU091   RETURN CODE = 00.


ASU095   END OF SERVICE PROGRAM JOB STEP.
```

```
15:47:52 05/05/87 6050102
DIF4 05/05/87 15:48:01

ENTER A, T, P, C, Q, X0 - FREE FORMAT
?

A = 0.10  T =  5.0  P =  5.0  C = 0.50  Q = 1.0  X0 =  0.

X0, U0, BETAX0 :     0.        0.        0.

ENTER INITIAL U0
?
  T         X         U       DIST.F     DENS.F     BETA      TCU       PAYOFF

 0.10    0.0873    0.8731    0.0002    0.0107    0.0107    0.        -0.0329
 1.00    0.9107    0.9484    0.0702    0.1402    0.1507    0.7243    -0.0204
 2.00    1.9106    1.0462    0.2382    0.1835    0.2408    1.1863     0.3515
 3.00    3.0295    1.1876    0.4372    0.1682    0.2988    1.4818     0.8099
 4.00    4.3345    1.4204    0.6285    0.1264    0.3403    1.6926     1.2090
 5.00    5.9703    1.8581    0.7915    0.0778    0.3731    1.8577     1.4901

CONTINUE ? ENTER 1
?

ENTER A, T, P, C, Q, X0 - FREE FORMAT
?

A = 0.10  T =  5.0  P =  5.0  C = 0.50  Q = 1.0  X0 =  2.0

X0, U0, BETAX0 :    2.0000    1.2500    0.2500

ENTER INITIAL U0
?
  T         X         U       DIST.F     DENS.F     BETA      TCU       PAYOFF

 0.10    2.0890    0.8904    0.0111    0.2499    0.2528    1.2500    0.0716
 1.00    2.9207    0.9535    0.2127    0.2321    0.2948    1.4638    0.7056
 2.00    3.9249    1.0515    0.4243    0.1898    0.3297    1.6409    1.3218
 3.00    5.0547    1.2053    0.6088    0.1397    0.3570    1.7789    1.8004
 4.00    6.3931    1.4710    0.7601    0.0911    0.3798    1.8936    2.1370
 5.00    8.1200    1.9957    0.8765    0.0494    0.4002    1.9960    2.3426

CONTINUE ? ENTER 1
?

ENTER A, T, P, C, Q, X0 - FREE FORMAT
?

A = 0.10  T =  5.0  P =  5.0  C = 0.50  Q = 1.0  X0 =  3.0

X0, U0, BETAX0 :    3.0000    1.5000    0.3000

ENTER INITIAL U0
?
  T         X         U       DIST.F     DENS.F     BETA      TCU       PAYOFF

 0.10    3.0894    0.8937    0.0134    0.2977    0.3018    1.5000    0.0930
 1.00    3.9227    0.9545    0.2414    0.2502    0.3298    1.6420    0.8522
 2.00    4.9278    1.0527    0.4616    0.1909    0.3545    1.7671    1.5166
 3.00    6.0605    1.2103    0.6430    0.1339    0.3750    1.8702    1.9990
 4.00    7.4091    1.4871    0.7863    0.0840    0.3929    1.9601    2.3230
```

```
 5.00   9.1666   2.0436   0.8933   0.0437   0.4096   2.0439   2.5134

CONTINUE ? ENTER 1
?

ENTER A, T, P, C, Q, X0 - FREE FORMAT
?

A = 0.10  T =  2.0  P =  5.0  C = 0.50  Q = 1.0  X0 =  0.

X0, U0, BETAX0 :     0.        0.        0.

ENTER INITIAL U0
?
  T        X         U       DIST.F    DENS.F     BETA      TCU      PAYOFF

 0.10   0.1025   1.0246   0.0003   0.0125   0.0125   0.       -0.0454
 1.00   1.0743   1.1283   0.0929   0.1530   0.1686   0.8117   -0.0482
 2.00   2.2892   1.2978   0.3054   0.1829   0.2633   1.2979    0.3756

CONTINUE ? ENTER 1
?

ENTER A, T, P, C, Q, X0 - FREE FORMAT
?

A = 0.10  T =  8.0  P =  5.0  C = 0.50  Q = 1.0  X0 =  0.

X0, U0, BETAX0 :     0.        0.        0.

ENTER INITIAL U0
?
  T        X         U       DIST.F    DENS.F     BETA      TCU      PAYOFF

 0.10   0.0730   0.7297   0.0002   0.0090   0.0090   0.       -0.0230
 1.00   0.7577   0.7833   0.0510   0.1254   0.1322   0.6339   -0.0039
 2.00   1.5686   0.8331   0.1781   0.1779   0.2165   1.0654    0.3025
 3.00   2.4301   0.8861   0.3348   0.1809   0.2720   1.3483    0.7014
 4.00   3.3517   0.9540   0.4917   0.1583   0.3115   1.5488    1.0795
 5.00   4.3563   1.0521   0.6335   0.1251   0.3414   1.7001    1.3910
 6.00   5.4876   1.2077   0.7538   0.0900   0.3654   1.8213    1.6249
 7.00   6.8307   1.4784   0.8507   0.0576   0.3858   1.9242    1.7858
 8.00   8.5716   2.0175   0.9242   0.0307   0.4045   2.0178    1.8823

CONTINUE ? ENTER 1
?

ENTER A, T, P, C, Q, X0 - FREE FORMAT
?

A = 0.10  T = 10.0  P =  5.0  C = 0.50  Q = 1.0  X0 =  0.

X0, U0, BETAX0 :     0.        0.        0.

ENTER INITIAL U0
?
  T        X         U       DIST.F    DENS.F     BETA      TCU      PAYOFF

 0.10   0.0687   0.6866   0.0001   0.0084   0.0084   0.       -0.0203
 1.00   0.7121   0.7346   0.0457   0.1205   0.1262   0.6051   -0.0006
 2.00   1.4686   0.7732   0.1610   0.1749   0.2085   1.0256    0.2839

 3.00   2.2606   0.8073   0.3047   0.1829   0.2630   1.3038    0.6592
 4.00   3.0877   0.8441   0.4498   0.1661   0.3018   1.5007    1.0208
 5.00   3.9561   0.8904   0.5819   0.1383   0.3308   1.6478    1.3250
 6.00   4.8802   0.9552   0.6951   0.1078   0.3536   1.7630    1.5615
 7.00   5.8858   1.0536   0.7880   0.0789   0.3723   1.8574    1.7353
 8.00   7.0207   1.2139   0.8620   0.0536   0.3884   1.9382    1.8564
 9.00   8.3769   1.4993   0.9189   0.0327   0.4029   2.0111    1.9353
10.00  10.1584   2.0817   0.9605   0.0165   0.4170   2.0816    1.9805

CONTINUE ? ENTER 1
?
STOP
TIME 0.1 SECS
```

APPENDIX E

R & D EXPENDITURES IN A DUOPOLY :

A FORTRAN PROGRAM

```
ASU004   INPUT VSPC LIBRARY OPENED.

ASU001   NOLIST PARAMETER IN EFFECT FOR THIS STEP.

 AUTH 6050102

ASU091   RETURN CODE = 00.

 EXPORT  DIF03 TO (SYSPRINT)
     10 COMMON A,T,DT,P,PB,CX,QX,CY,QY
     20 700 CONTINUE
     30 PRINT 900
     40 900 FORMAT(' ENTER A,T,PX,CX,QX,X0,PY,CY,QY,Y0 - FREE FORMAT')
     50 READ(6,*) A,T,P,CX,QX,X0,PB,CY,QY,Y0
     60 PRINT 901,A,T,P,PB
     70 UMAX=CX*P/QX
     80 VMAX=CY*PB/QY
     90 PRINT 902,CX,QX,X0,BETAX0,UMAX
    100 PRINT 903,CY,QY,Y0,BETAY0,VMAX
    110 901 FORMAT(' A, T, PX, PY                :',4F9.2)
    120 902 FORMAT(' CX, QX, X0, BETAX0, UMAX :',3F9.2,2F9.4)
    130 903 FORMAT(' CY, QY, Y0, BETAY0, VMAX :',3F9.2,2F9.4)
    140 PRINT 904
    150 904 FORMAT(' ENTER U0, V0')
    160 READ(6,*) U0,V0
    170 PRINT 891,U0,V0
    180 891 FORMAT(' INPUT U0/V0 :',2F9.4)
    190 ******* INITIALIZATION*******
    200 DT=.1
    210 N=(T+.04)/DT
    220 T0=0.
    230 U=U0
    240 V=V0
    250 110 CONTINUE
    260 X=X0
    270 Y=Y0
    280 PAYX=0.
    290 PAYY=0.
    300 T1=-DT*.5
    310 XU=X
    320 XV=X
    330 YU=Y
    340 YV=Y
    350 U2=U
    360 V2=V
    370 *****VARIATION OF U0/V0 - IF CHANGE DU2/V2, MUST CHANGE LINES 910/920 ***
    380 UU=U2+.00001
    390 VU=V2
    400 UV=U2
```

```
VV=V2+.00001
******* FIND U0, V0 SO THAT U=TCU, V=TCV ******
DO 100 M=1,N
CALL DELTA(X,U,BETAX,DELTU,Y,V,BETAY,DELTV)
U=U+DELTU
V=V+DELTV
X=X+U*DT
Y=Y+V*DT
T1=T1+DT
X1=X-U*DT*.5
U1=U-DELTU*.5
Y1=Y-V*DT*.5
V1=V-DELTV*.5
EPX=EXP(CX*(X0-X1))
EPY=EXP(CY*(Y0-Y1))
FX=1.-(CX*X1+1.)/(CX*X0+1.)*EPX
FY=1.-(CY*Y1+1.)/(CY*Y0+1.)*EPY
***** VARIATION OF INITIAL VALUE OF U0 & V0 ************
CALL DELTA(XU,UU,BTUU,DELTUU,YU,VU,BTVU,DELTVU)
CALL DELTA(XV,UV,BTUV,DELTUV,YV,VV,BTVV,DELTVV)
UU=UU+DELTUU
UV=UV+DELTUV
VU=VU+DELTVU
VV=VV+DELTVV
XU=XU+UU*DT
XV=XV+UV*DT
YU=YU+VU*DT
YV=YV+VV*DT
****** COMPUTATION OF EXPECTED PAY-OFF BASED ON INITIAL U0,V0 *****
DENSX=BETAX*(1.-FX)
DENSY=BETAY*(1.-FY)
PINTX=(P*FX*DENSY*V1+(1.-FY)*(A*P*FX-(1.-FX)*.5*QX*U1**2))*DT
PINTY=(PB*FY*DENSX*U1+(1.-FX)*(A*PB*FY-(1.-FY)*.5*QY*V1**2))*DT
PAYX=PAYX+PINTX*EXP(-A*T1)
PAYY=PAYY+PINTY*EXP(-A*T1)
960 FORMAT(/,' FX,DFX,PAYOFF-X/Y ',I4,3F9.4,3X,3F9.4)
100 CONTINUE
TCU=P/QX*BETAX
TCV=PB/QY*BETAY
PAYX=PAYX+P*(1.-FY)*FX*EXP(-A*T)
PAYY=PAYY+PB*(1.-FX)*FY*EXP(-A*T)
DIFFER=ABS(TCU-U)+ABS(TCV-V)
871 FORMAT(/,' ABS(TCU-U)+ABS(TCV-V) =',F10.5)
910 FORMAT(' INIT.U, U, TCU, X, PAYOFF X :',5F10.4)
911 FORMAT('    SAME FOR Y / V             :',5F10.4)
IF(DIFFER.LT..01) GOTO 120
DENO=(UU-U)*(VV-V)-(UV-U)*(VU-V)
AU=((TCU-U)*(VV-V)-(TCV-V)*(VU-V))/DENO
AV=((TCV-V)*(UU-U)-(TCU-U)*(UV-U))/DENO
U=U2+AU*.00001*0.7
V=V2+AV*.00001*0.7
GOTO 110
120 CONTINUE
```

```
 940 PRINT 871,DIFFER
 950 PRINT 910,U2,U,TCU,X,PAYX
 960 PRINT 911,V2,V,TCV,Y,PAYY
 970 ***** U0/V0 FOUND - SOLVE SYSTEM OF DIF.EQUATIONS *******
 980 U=U2
 990 V=V2
1000 PAYX=0.
1010 PAYY=0.
1020 PRINT 913
1030 913 FORMAT(//,'     T       X|Y       U|V DIST F|G DENS F|G BETA X|Y',    -
1031 '     TCU|V  P-O X|Y   UDOT|VDOT X-Y|U-V',/)
1040 T2=0
1050 PRINT 892,T2,X0,U
1060 PRINT 893,Y0,V
1070 892 FORMAT(F5.1,2F9.4)
1080 893 FORMAT(5X,2F9.4,/)
1090 T2=0.
1100 T1=-DT*.5
1110 X=X0
1120 Y=Y0
1130 DO 140 M=1,N,10
1140 DO 141 MM=1,10
1150 CALL DELTA(X,U,BETAX,DELTU,Y,V,BETAY,DELTV)
1160 UDOT=DELTU/DT
1170 VDOT=DELTV/DT
1180 U=U+DELTU
1190 V=V+DELTV
1200 X=X+U*DT
1210 Y=Y+V*DT
1220 T1=T1+DT
1230 X1=X-U*DT*.5
1240 U1=U-DELTU*.5
1250 Y1=Y-V*DT*.5
1260 V1=V-DELTV*.5
1270 EPX=EXP(CX*(X0-X1))
1280 EPY=EXP(CY*(Y0-Y1))
1290 FX=1.-(CX*X1+1.)/(CX*X0+1.)*EPX
1300 FY=1.-(CY*Y1+1.)/(CY*Y0+1.)*EPY
1310 FS=1.-(1.-FX)*(1.-FY)
1320 T2=T2+DT
1330 TCU=P/QX*BETAX
1340 TCV=PB/QY*BETAY
1350 CALL BETA(X1,CX,BETAX1)
1360 CALL BETA(Y1,CY,BETAY1)
1370 DENSX=BETAX1*(1.-FX)
1380 DENSY=BETAY1*(1.-FY)
1390 PINTX=P*DENSX*U1-(1.-FX)*.5*QX*U1**2
1400 PINTY=PB*DENSY*V1-(1.-FY)*.5*QY*V1**2
1410 PAYX=PAYX+PINTX*(1.-FY)*EXP(-A*T1)*DT
1420 PAYY=PAYY+PINTY*(1.-FX)*EXP(-A*T1)*DT
1430 DIFXY=X-Y
1440 DIFUV=U-V
1450 141 CONTINUE
```

```
1460 PRINT 861,T2,X,U,FX,DENSX,BETAX1,TCU,PAYX,UDOT,DIFXY
1470 PRINT 862,Y,V,FY,DENSY,BETAY1,TCV,PAYY,VDOT,DIFUV
1480 PRINT 863,FS
1490 861 FORMAT(F5.1,9F9.4)
1500 862 FORMAT(5X,9F9.4)
1510 863 FORMAT(23X,F9.4)
1520 140 CONTINUE
1530 PRINT 710
1540 710 FORMAT(//,' CONTINUE? ENTER 1')
1550 READ(6,*) NRE
1560 IF(NRE.EQ.1) GOTO 700
1570 END
1580 ***
1590 ***
1600 SUBROUTINE BETA(X,C,BET)
1610 BET=C**2*X/(1.+C*X)
1620 RETURN
1630 END
1640 ***
1650 ***
1660 SUBROUTINE DELTA(X,U,BEX,DU,Y,V,BEY,DV)
1670 COMMON A,T,DT,P,PB,CX,QX,CY,QY
1680 CALL BETA(X,CX,BEX)
1690 CALL BETA(Y,CY,BEY)
1700 DU=DT/QX*((A+BEY*V)*(QX*U-BEX*P)+BEX*QX*.5*U**2)
1710 DV=DT/QY*((A+BEX*U)*(QY*V-BEY*PB)+BEY*QY*.5*V**2)
1720 RETURN
1730 END

ASU091   RETURN CODE = 00.

ASU095   END OF SERVICE PROGRAM JOB STEP.

ASU099   HIGHEST RETURN CODE IN JOB STEP = 00.
```

```
15:29:18 05/05/87 6050102
DIFD3 05/05/87 15:29:23
ENTER A,T,PX,CX,QX,X0,PY,CY,QY,Y0 - FREE FORMAT
?
A, T, PX, PY                 :      0.10      5.00      5.00      5.00
CX, QX, X0, BETAX0, UMAX :      0.50      1.00      0.        0.        2.5000
CY, QY, Y0, BETAY0, VMAX :      0.50      1.00      0.        0.        2.5000
ENTER U0, V0
?
INPUT U0/V0 :   0.9000   0.9000

ABS(TCU-U)+ABS(TCV-V) =    0.00803
INIT.U, U, TCU, X, PAYOFF X :    0.9098     1.8924     1.8964     6.4733     0.7991
  SAME FOR Y / V             :    0.9098     1.8924     1.8964     6.4733     0.7991


   T       X|Y       U|V DIST F|G DENS F|G BETA X|Y    TCU|V  P-O X|Y   UDOT|VDOT X-Y|U-V

 0.      0.        0.9098
         0.        0.9098

 1.0     0.9763    1.0395    0.0789    0.1456    0.1580    0.7592  -0.0334    0.1476    0.
         0.9763    1.0395    0.0789    0.1456    0.1580    0.7592  -0.0334    0.1476    0.
                             0.1517
 2.0     2.0964    1.1851    0.2711    0.1839    0.2523    1.2431   0.2895    0.1422    0.
         2.0964    1.1851    0.2711    0.1839    0.2523    1.2431   0.2895    0.1422    0.
                             0.4687
 3.0     3.3597    1.3287    0.4900    0.1587    0.3111    1.5434   0.5788    0.1503    0.
         3.3597    1.3287    0.4900    0.1587    0.3111    1.5434   0.5788    0.1503    0.
                             0.7399
 4.0     4.7826    1.5122    0.6813    0.1118    0.3509    1.7460   0.7345    0.2281    0.
         4.7826    1.5122    0.6813    0.1118    0.3509    1.7460   0.7345    0.2281    0.
                             0.8984
 5.0     6.4733    1.8924    0.8274    0.0657    0.3806    1.8964   0.7946    0.5877    0.
         6.4733    1.8924    0.8274    0.0657    0.3806    1.8964   0.7946    0.5877    0.
                             0.9702

CONTINUE? ENTER 1
?
ENTER A,T,PX,CX,QX,X0,PY,CY,QY,Y0 - FREE FORMAT
?
A, T, PX, PY                 :      0.10      5.00      5.00      5.00
CX, QX, X0, BETAX0, UMAX :      0.50      1.00      2.00      0.        2.5000
CY, QY, Y0, BETAY0, VMAX :      0.50      1.00      0.        0.        2.5000
ENTER U0, V0
?
INPUT U0/V0 :   1.0500   0.8000

ABS(TCU-U)+ABS(TCV-V) =    0.00767
INIT.U, U, TCU, X, PAYOFF X :    1.0543     2.0366     2.0373     9.0100     1.7094
  SAME FOR Y / V             :    0.7919     1.8896     1.8965     6.4746     0.5157


   T       X|Y       U|V DIST F|G DENS F|G BETA X|Y    TCU|V  P-O X|Y   UDOT|VDOT X-Y|U-V

 0.      2.0000    1.0543
         0.        0.7919

 1.0     3.1208    1.1751    0.2559    0.2251    0.3025    1.5007   0.7433    0.1192    2.1881
         0.9327    1.0334    0.0727    0.1418    0.1529    0.7328  -0.0302    0.2012    0.1417
```

```
                              0.3100
  2.0    4.3596    1.2901     0.5005     0.1704     0.3411     1.6975     1.2801     0.1132     2.2963
         2.0633    1.2030     0.2648     0.1839     0.2502     1.2319     0.1997     0.1504     0.0871
                              0.6328
  3.0    5.7144    1.4108     0.6910     0.1141     0.3692     1.8398     1.5615     0.1336     2.3690
         3.3454    1.3462     0.4876     0.1591     0.3105     1.5404     0.3858     0.1450     0.0646
                              0.8416
  4.0    7.2152    1.5913     0.8248     0.0684     0.3905     1.9479     1.6771     0.2410     2.4337
         4.7815    1.5212     0.6811     0.1119     0.3509     1.7458     0.4760     0.2174     0.0701
                              0.9441
  5.0    9.0100    2.0366     0.9138     0.0352     0.4083     2.0373     1.7146     0.7274     2.5354
         6.4746    1.8896     0.8275     0.0657     0.3807     1.8965     0.5077     0.5781     0.1470
                              0.9851

CONTINUE? ENTER 1
?
ENTER A,T,PX,CX,QX,X0,PY,CY,QY,Y0 - FREE FORMAT
?
A, T, PX, PY                 :       0.10      5.00      5.00      5.00
CX, QX, X0, BETAX0, UMAX :       0.50      1.00      4.00   0.        2.5000
CY, QY, Y0, BETAY0, VMAX :       0.50      1.00      0.     0.        2.5000
ENTER U0, V0
?
INPUT U0/V0 :   1.0900    0.7500

ABS(TCU-U)+ABS(TCV-V) =    0.00765
INIT.U, U, TCU, X, PAYOFF X :     1.0919     2.1104     2.1161    11.2340     2.0216
  SAME FOR Y / V              :     0.7470     1.8991     1.8972     6.4839     0.4254


   T       X|Y       U|V DIST F|G DENS F|G BETA X|Y    TCU|V  P-O X|Y   UDOT|VDOT X-Y|U-V

  0.     4.0000    1.0919
         0.        0.7470

  1.0    5.1662    1.2235     0.3185     0.2448     0.3593     1.7902     1.0282     0.1209     4.2532
         0.9130    1.0285     0.0700     0.1400     0.1505     0.7207    -0.0287     0.2252     0.1950
                              0.3662
  2.0    6.4521    1.3349     0.5760     0.1614     0.3807     1.8989     1.6266     0.1071     4.4049
         2.0471    1.2107     0.2618     0.1839     0.2492     1.2265     0.1700     0.1563     0.1242
                              0.6870
  3.0    7.8481    1.4496     0.7533     0.0981     0.3977     1.9847     1.8986     0.1287     4.5098
         3.3384    1.3557     0.4864     0.1594     0.3103     1.5390     0.3230     0.1443     0.0939
                              0.8733
  4.0    9.3868    1.6303     0.8672     0.0546     0.4115     2.0545     1.9994     0.2483     4.6037
         4.7831    1.5288     0.6813     0.1118     0.3509     1.7459     0.3935     0.2154     0.1015
                              0.9577
  5.0   11.2340    2.1104     0.9380     0.0263     0.4238     2.1161     2.0296     0.8033     4.7501
         6.4839    1.8991     0.8281     0.0655     0.3808     1.8972     0.4169     0.5880     0.2113
                              0.9893

CONTINUE? ENTER 1
?
ENTER A,T,PX,CX,QX,X0,PY,CY,QY,Y0 - FREE FORMAT
?
A, T, PX, PY                 :       0.10      5.00      5.00      5.00
CX, QX, X0, BETAX0, UMAX :       0.50      1.00      1.00   0.        2.5000
CY, QY, Y0, BETAY0, VMAX :       0.50      1.00      0.     0.        2.5000
ENTER U0, V0
?
```

```
INPUT U0/V0 :    1.0000    0.8300

ABS(TCU-U)+ABS(TCV-V) =    0.00785
INIT.U. U, TCU, X, PAYOFF X :     1.0120      1.9857      1.9805      7.8229      1.4014
  SAME FOR Y / V              :     0.8341      1.8942      1.8968      6.4787      0.6076

   T       X|Y       U|V DIST F|G DENS F|G BETA X|Y    TCU|V  P-O X|Y   UDOT|VDOT X-Y|U-V

 0.     1.0000    1.0120
        0.        0.8341

 1.0    2.0747    1.1294    0.1950    0.2022    0.2511    1.2379   0.4706    0.1240    1.1251
        0.9496    1.0369    0.0751    0.1433    0.1549    0.7431  -0.0316    0.1809    0.0925
                            0.2555
 2.0    3.2719    1.2522    0.4247    0.1772    0.3080    1.5285   0.9415    0.1218    1.1946
        2.0772    1.1968    0.2674    0.1839    0.2511    1.2366   0.2296    0.1468    0.0553
                            0.5785
 3.0    4.5930    1.3804    0.6266    0.1295    0.3467    1.7254   1.2291    0.1396    1.2403
        3.3526    1.3400    0.4888    0.1589    0.3108    1.5419   0.4492    0.1470    0.0404
                            0.8091
 4.0    6.0651    1.5625    0.7800    0.0824    0.3748    1.8678   1.3586    0.2374    1.2807
        4.7844    1.5187    0.6815    0.1118    0.3509    1.7461   0.5602    0.2225    0.0437
                            0.9299
 5.0    7.8229    1.9857    0.8876    0.0446    0.3972    1.9805   1.4035    0.6786    1.3442
        6.4787    1.8942    0.8278    0.0656    0.3807    1.8968   0.6006    0.5865    0.0915
                            0.9806

CONTINUE? ENTER 1
?
ENTER A,T,PX,CX,QX,X0,PY,CY,QY,Y0 - FREE FORMAT
?
A, T, PX, PY               :      0.10      5.00      5.00      7.00
CX, QX, X0, BETAX0, UMAX :      0.50      1.00      1.00   0.        2.5000
CY, QY, Y0, BETAY0, VMAX :      0.50      1.00      0.     0.        3.5000
ENTER U0, V0
?
INPUT U0/V0 :    1.0500    1.1700

ABS(TCU-U)+ABS(TCV-V) =    0.00517
INIT.U. U, TCU, X, PAYOFF X :     1.0490      1.9884      1.9933      8.0661      1.1706
  SAME FOR Y / V              :     1.1660      2.8450      2.8452      8.9755      1.2931

   T       X|Y       U|V DIST F|G DENS F|G BETA X|Y    TCU|V  P-O X|Y   UDOT|VDOT X-Y|U-V

 0.     1.0000    1.0490
        0.        1.1660

 1.0    2.1209    1.1841    0.2038    0.2021    0.2538    1.2508   0.4621    0.1418    0.7962
        1.3247    1.4398    0.1307    0.1674    0.1926    1.2992   0.1146    0.2323   -0.2557
                            0.3078
 2.0    3.3796    1.3174    0.4430    0.1737    0.3118    1.5472   0.8785    0.1260    0.5014
        2.8782    1.6390    0.4075    0.1727    0.2915    2.0152   0.7303    0.1797   -0.3217
                            0.6700
 3.0    4.7651    1.4419    0.6480    0.1234    0.3506    1.7448   1.0885    0.1290    0.1508
        4.6143    1.8175    0.6602    0.1178    0.3467    2.4118   1.1040    0.1901   -0.3757
                            0.8804
 4.0    6.2889    1.6030    0.7977    0.0765    0.3782    1.8849   1.1629    0.2066   -0.2719
        6.5608    2.0788    0.8325    0.0640    0.3818    2.6620   1.2449    0.3620   -0.4759

                            0.9661
 5.0    8.0661    1.9884    0.8980    0.0408    0.3997    1.9933   1.1821    0.6529   -0.9094
        8.9755    2.8450    0.9346    0.0267    0.4077    2.8452   1.2831    1.3711   -0.8566
                            0.9933

CONTINUE? ENTER 1
?
STOP
TIME 0.3 SECS
```

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