# $W$-INFINITY AND $w$-INFINITY GAUGE THEORIES,

# AND

# UNIVERSALITY

# IN RANDOM MATRIX MODELS

by

ANDREW R. KAVALOV

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UMI
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# Contents

# Chapter 1

# INTRODUCTION

• $W_\infty$ algebra and its so-called classically contracted $w_\infty$ algebra appeared recently in various problems in physics, in particular in the study of $c = 1$ string theory [1] and quantum Hall system [2], [3]. The gauge theory based on these algebras also appeared in these studies [4], [5]. In fact the same algebras and the gauge theories based on them had been proposed previously as the theories [6], [7], [8] relevant to the large $N$ limit of $SU(N)$ gauge theories [9]. But it seems to us that not enough studies have been done to differentiate the formal and dynamic aspect of $W_\infty$ and $w_\infty$ theories. In view of the recent developments we study in this chapter this subject as systematically as possible by using the technique developed in the study of quantum Hall system [2], [5].

The $W_\infty$ algebra is a commutator algebra of Hermitian operators of one harmonic oscillator [10]. It is an infinite-dimensional Lie algebra. If we choose a set of linearly independent real function of $z$ and $\bar{z}$ as the parameters of $W_\infty$ group, the structure constants of the algebra are expressed in terms of Moyal bracket [11]. Replacing the Moyal bracket by a Poisson bracket we define the $w_\infty$ algebra. It is an algebra of area-preserving diffeomorphisms. As an introduction we discuss this issue in section 2.1 together with the so-called classical contraction procedure by which the $W_\infty$ algebra is transformed to the $w_\infty$ algebra.

$W_\infty$ gauge theory is a gauge field theory of $W_\infty$ as an internal symmetry algebra. The $W_\infty$ gauge potential is a space-time dependent Hermitian operator of harmonic oscillator. In the coherent state representation it is a function of $z$, $\bar{z}$, which we call the color space coordinates, and $x^\mu$ $(\mu = 1, 2, \ldots d)$, the space time coordinates. Thus, we can express the $W_\infty$ gauge theories in terms of $d+2$ dimensional local fields. The interactions of the fields are necessarily non-local in the color space in $W_\infty$ theories, but they are local in $w_\infty$ theories. In section 2.2 we define $W_\infty$ theories as $d+2$ dimensional field theories with non-local interactions and $w_\infty$ gauge theories as $d+2$ dimensional local field theories. Since the $W_\infty$ algebra is closely related to the $w_\infty$ algebra, the gauge theories based on these algebras may also be closely related. In order to see the relationship at classical Lagrangian level, we introduce the $l \to 0$ contraction procedure by which we derive the $w_\infty$ gauge theories from the corresponding $W_\infty$ gauge theories. The procedure consists of the introduction of a length scale $l$ in the color space, an appropriate scale transformation of the fields, and the $l \to 0$ limit. In this section we also introduce matter fields analogous to the quark fields and the Higgs fields.

The $W_\infty$ algebra can be considered as an $N \to \infty$ limit of the $SU(N)$ algebra [10]. Therefore, the $W_\infty$ gauge theory or its variation $w_\infty$ gauge theory [7], [12] might be used for the large $N$ gauge theory. Since the $w_\infty$ gauge theory is a local theory and much easier to be handled, it is important to determine whether this theory can serve for the large $N$ gauge theory or not. For this purpose we solve $d = 1$ gauge theory of scalar field exactly in section 2.3, which reveals also the quantum mechanical relationship between the $W_\infty$ theory and the $w_\infty$ theory. In $d = 1$ exists only the time component of the gauge potential and the pure gauge theory is trivial but it constrains the states of scalar field to be gauge invariant. The $W_\infty$ theory becomes essentially $N = \infty$ limit of $d = 1$ gauged matrix model. On the other hand the $w_\infty$ model becomes an infinite number of non-interacting quantum mechanical systems. We use the collective field method [13] to solve these theories. The spectrum of these two theories are in general different and coincides with the $N = \infty$ limit of $SU(N)$ theory only for $W_\infty$ model. Based on this result we conclude that $W_\infty$ gauge theory can but $w_\infty$ gauge theory cannot serve for the purpose of large $N$ gauge theory.

• In chapter 2 we develop the general formalism for constructing $W_\infty$ gauge theories. In chapter 3 we propose gauge invariant observables for these theories - $W_\infty$ Wilson loops. We solve $W_\infty$ two dimensional Yang-Mills on the cylinder exactly. After appropriate coupling constant renormalization ($g^2N \equiv g_c^2$ - fixed, $N \to \infty$, where $N$ is the volume of the color space) solution agree with $N \to \infty$ limit of $SU(N)$ Yang-Mills [20]. This theory is equivalent to a $W_\infty$ one-dimensional unitary matrix model.

• Random matrix theory was originally introduced by Wigner and studied in detail by Dyson and Metha to investigate statistical properties of energy levels of heavy nuclei [25]. It has been applied to various fields recently such as quantum chaos [26], quantum dots or 2d discretized gravity [27]. It has also been applied to quantum transport problem of mesoscopic wires reviewed in [28].

In the above mentioned contexts universal behavior of correlation functions of eigenvalues has been discussed. There are two types of universalities, one for a short distance behavior (where correlation functions oscillate rapidly) [30] and the other for a smoothed correlator of distance scale larger than the rapid oscillation [31]. In chapter 4 we study the latter universal behavior of correlation functions.

The universal large scale behavior of density correlation functions was pointed out by Brézin and Zee who stressed its importance in disordered systems though it was already derived by Ambjorn, Jurkiewicz and Makeenko [32] in the context of 2d discretized gravity. They considered a random matrix theory for large matrices $M$, whose statistical weight is given by

$$\frac{1}{Z}e^{-N\mathrm{tr}V(M)} \tag{1.1}$$

and showed that, in the large matrix limit, the connected two point correlation function of density of eigenvalues $\langle\rho(x)\rho(y)\rangle_c$ has a universal form which has no explicit dependence on the potential $V$. For a symmetric potential it is

$$-\frac{1}{2\pi^2\lambda(x-y)^2}\frac{a^2-xy}{\sqrt{(a^2-x^2)(a^2-y^2)}} \tag{1.2}$$

where $a$ and $-a$ are the end points of the density distribution and $\lambda = 1/2, 1, 2$ corresponding to orthogonal, unitary or symplectic ensembles (see Appendix C, page 62). This universal form has been calculated in various methods [32], [31], [33], [34], [37]. and its $1/N$ corrections were also studied for general ensembles [32], [35], [36].

Physical implication of this universality to the universal conductance fluctuation of mesoscopic wire is discussed in a recent review by Beenaker [28]. Universality here means that fluctuation of conductance $\delta G$ in a metallic regime is of order $e^2/h$, independent of sample size or disorder strength. This strong suppression of conductance fluctuation was first pointed out by Altshuler [38] and Lee and Stone [39]. Also, when magnetic field is applied, variance $(\delta G)^2$ of the conductance decreases exactly by a factor of two compared to the fluctuation without magnetic field. The system can be modeled by random matrix theory for the transmission matrix [28]. Since the variance of conductance is proportional to the connected density-density correlation function, the above universality of conductance fluctuation is mapped to the universality of correlation functions in random matrix theory. Looking back to the equation (1.2), first it is $O(N^0)$ and independent of the system size ($N$) or details of disorder ($V$). Next it decreases exactly by a factor of two when magnetic field is applied ($\lambda$ is changed from $1/2$ to 1 by applying magnetic field.) It is quite nice that such a simple model as the random matrix theory can explain some of important features of complicated systems.

Since the main guiding principle of random matrix theory is randomness and symmetry, it is natural to ask whether this universality still holds for a more general ensemble which is invariant under symmetry rotation. Simple generalization of the potential (1.1) is to add products of traces to the statistical weight of matrices;

$$\frac{1}{Z} e^{-N\mathrm{tr}V(M) - \mathrm{tr}W_1(M)\cdot \mathrm{tr}W_2(M)}. \tag{1.3}$$

This ensemble is invariant under $M \to UMU^{-1}$ where $U$ is an orthogonal, unitary or symplectic matrix correspondingly. The ensemble with this generalized potential was studied in [40] in the context of 2d gravity. Universality in this ensemble was discussed by [41], [42]. Brézin and Zee [41] argued that this model is equivalently described by an ensemble with an effective single trace potential $V_{eff}$ and concluded that the universality still holds.

In chapter 4 we study this generalized ensemble and obtain density of eigenvalues and its correlation functions explicitly. In section 4.1, we review a collective field theory approach to an ensemble with a single trace potential and show how the universal behavior of correlation functions emerges. In section 4.2, we generalize it to an ensemble with a multi trace potential (containing products of two traces). We show that the universality is broken and the correlation function is no longer universal in the strong sense. Finally in discussion, we analize why the argument by Brézin and Zee [41] does not hold for correlation functions.

In Appendices A to C we prove useful formulas.

# Chapter 2

# $W_\infty$ AND $w_\infty$ GAUGE THEORIES AND CONTRACTION

We present a general method of constructing $W_\infty$ and $w_\infty$ gauge theories in terms of $d+2$ dimensional local fields. In this formulation the $W_\infty$ gauge theory Lagrangians involve non-local interactions, but the $w_\infty$ theories are entirely local. We discuss the so-called classical contraction procedure by which we derive the Lagrangian of $w_\infty$ gauge theory from that of the corresponding $W_\infty$ gauge theory. In order to discuss the relationship between quantum $W_\infty$ and quantum $w_\infty$ gauge theory we solve $d=1$ gauge theory models of a scalar field exactly by using the collective field method. Based on this we conclude that the $W_\infty$ gauge theory can be regarded as the large $N$ limit of the corresponding $SU(N)$ gauge theory once an appropriate coupling constant renormalization is made, while the $w_\infty$ gauge theory cannot be. [1]

[1] These results have been published in [18]

## 2.1 $W_\infty$ and $w_\infty$ Algebra.

We define the $W_\infty$ algebra as a commutator algebra of all Hermitian operators $\xi(\hat{a},\hat{a}^\dagger)$ in the Hilbert space of a harmonic oscillator [2]. A convenient parametrization for these operators is achieved by using a real function $\xi(z,\bar{z})$ as

$$\xi(\hat{a},\hat{a}^\dagger) = \ddagger\xi(z,\bar{z})|_{\substack{z=\hat{a}\\ \bar{z}=\hat{a}^\dagger}}\ddagger = \int d^2z e^{-|z|^2}|z\rangle\xi(z,\bar{z})\langle z|, \tag{2.1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are standard creation and annihilation operators [17] and $\ddagger\ \ddagger$ stands for the anti-normal-order symbol, i.e., all the creation operators stand to the right of the annihilation operators. The last expression (2.1) is in the coherent state representation (see Appendix A, page 58). Obviously the product of two $\xi$'s is not anti-normally ordered and we bring it to the anti-normal-order form by using the commutation relation $[\hat{a},\hat{a}^\dagger]=1$. We obtain (see A.5)

$$\xi_1(\hat{a},\hat{a}^\dagger)\xi_2(\hat{a},\hat{a}^\dagger) = \ddagger\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\partial_{\bar{z}}^n\xi_1(z,\bar{z})\partial_z^n\xi_2(z,\bar{z})|_{\substack{z=\hat{a}\\ \bar{z}=\hat{a}^\dagger}}\ddagger, \tag{2.2}$$

from which the following commutation relation follows

$$[\xi_1(\hat{a},\hat{a}^\dagger),\xi_2(\hat{a},\hat{a}^\dagger)] = i\{\!\{\xi_1,\xi_2\}\!\}(\hat{a},\hat{a}^\dagger), \tag{2.3}$$

where $\{\!\{\xi_1,\xi_2\}\!\}$ is a Moyal bracket [11] defined by

$$\{\!\{\xi_1,\xi_2\}\!\}(z,\bar{z}) \equiv i\sum_{n=1}^{\infty}\frac{(-)^n}{n!}\left(\partial_z^n\xi_1(z,\bar{z})\partial_{\bar{z}}^n\xi_2(z,\bar{z})-\partial_{\bar{z}}^n\xi_1(z,\bar{z})\partial_z^n\xi_2(z,\bar{z})\right). \tag{2.4}$$

The commutation relation (2.3) is that of the $W_\infty$ Lie algebra in the fundamental representation. The $W_\infty$ is an infinite-dimensional Lie group with parameters being a set of linearly independent real functions $\xi(z,\bar{z})$. The generators of $W_\infty$ are the linear functionals of $\xi(z,\bar{z})$. Thus we write for arbitrary representation:

$$[\rho[\xi_1],\rho[\xi_2]] = i\rho[\{\!\{\xi_1,\xi_2\}\!\}], \tag{2.5}$$

where $\rho$ is the generator of $W_\infty$ group.

The Lie algebra of $w_\infty$ , the area-preserving diffeomorphisms, is defined by the commutation relation

$$[\rho[\xi_1],\rho[\xi_2]] = i\rho[\{\xi_1,\xi_2\}], \tag{2.6}$$

where $\{\xi_1,\xi_2\}(z,\bar{z}) \equiv -i\Big(\partial_z\xi_1(z,\bar{z})\partial_{\bar{z}}\xi_2(z,\bar{z}) - \partial_{\bar{z}}\xi_1(z,\bar{z})\partial_z\xi_2(z,\bar{z})\Big)$ is the Poisson bracket.

It is well known [2] that one can obtain the $w_\infty$ algebra from the $W_\infty$ by a contraction. To explain it let us introduce a length scale $l$ in the $z,\bar{z}$ space, which we call the color space, and set

$$z = \frac{1}{\sqrt{2}l}(\sigma_x + i\sigma_y), \qquad \bar{z} = \frac{1}{\sqrt{2}l}(\sigma_x - i\sigma_y). \tag{2.7}$$

The Poisson bracket is the leading surviving term of the Moyal bracket in the $l \to 0$ limit. To be more specific we set $\xi(z,\bar{z}) = l^{-2}\xi(\vec{\sigma})$ and obtain

$$\begin{aligned}\lim_{l\to 0} l^2\{\!\{\xi_1,\xi_2\}\!\}(z,\bar{z}) &= \partial_{\sigma_x}\xi_1(\vec{\sigma})\partial_{\sigma_y}\xi_2(\vec{\sigma}) - \partial_{\sigma_y}\xi_1(\vec{\sigma})\partial_{\sigma_x}\xi_2(\vec{\sigma}) \\ &\equiv \epsilon^{ij}\partial_i\xi_1(\vec{\sigma})\partial_j\xi_2(\vec{\sigma}) \\ &\equiv \{\xi_1,\xi_2\}(\vec{\sigma}) \end{aligned} \tag{2.8}$$

In this chapter we call this procedure as the $l \to 0$ contraction.

## 2.2 $W_\infty$ and $w_\infty$ Gauge Invariant Lagrangians

The $W_\infty$ gauge theory is a gauge field theory of $W_\infty$ as an internal symmetry algebra.

Let us discuss pure Yang-Mills theory first. We introduce a gauge potential $\hat{\mathcal{A}}_\mu$ which is a Hermitian operator in the harmonic oscillator Hilbert space as well as a function of space time:

$$\hat{\mathcal{A}}_\mu(x) \equiv \mathcal{A}_\mu(x, \hat{a}, \hat{a}^\dagger) = \int d^2z e^{-|z|^2} |z\rangle \mathcal{A}_\mu(x, z, \bar{z}) \langle z|. \tag{2.9}$$

The action is given by

$$S_{YM} = -\frac{1}{4g^2} \int d^dx \ \mathrm{tr}(\hat{\mathcal{F}}_{\mu\nu} \hat{\mathcal{F}}^{\mu\nu}), \tag{2.10}$$

where $\hat{\mathcal{F}}_{\mu\nu}$ is the field strength defined by

$$\hat{\mathcal{F}}_{\mu\nu} = \partial_\mu \hat{\mathcal{A}}_\nu - \partial_\nu \hat{\mathcal{A}}_\mu - i[\hat{\mathcal{A}}_\mu, \hat{\mathcal{A}}_\nu]. \tag{2.11}$$

and $tr$, for any operator $A(\hat{a}, \hat{a}^\dagger)$ defined as follows:

$$tr A(\hat{a}, \hat{a}^\dagger) = \int d^2z e^{-|z|^2} \langle z| A(\hat{a}, \hat{a}^\dagger) |z\rangle \tag{2.12}$$

(see Appendix A, page 59).

Therefore rewriting (2.10) in the coherent state representation we obtain:

$$S_{YM} = -\frac{1}{4g^2} \int\int d^dx d^2z \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \partial_z^n \mathcal{F}_{\mu\nu}(x, z, \bar{z}) \partial_{\bar{z}}^n \mathcal{F}^{\mu\nu}(x, z, \bar{z}), \tag{2.13}$$

where

$$\mathcal{F}_{\mu\nu}(x, z, \bar{z}) = \partial_\mu \mathcal{A}_\nu(x, z, \bar{z}) - \partial_\nu \mathcal{A}_\mu(x, z, \bar{z}) + \{\!\{\mathcal{A}_\mu, \mathcal{A}_\nu\}\!\}(x, z, \bar{z}). \tag{2.14}$$

This action is invariant under the $W_\infty$ gauge transformation:

$$\delta\hat{A}_\mu(x) = \partial_\mu\hat{\xi}(x)+i[\hat{\xi}(x),\hat{A}_\mu(x)], \quad \delta\mathcal{A}_\mu(x,z,\bar{z}) = \partial_\mu\xi(x,z,\bar{z})-\{\!\{\xi,\mathcal{A}_\mu\}\!\}(x,z,\bar{z}). \tag{2.15}$$

In the coherent sate representation the gauge fields are formally $d+2$ dimensional local fields, $d$ for space time and 2 for color space. However, the interactions are non-local since the action involves derivatives of infinite order.

In the action (2.13) we no longer have a damping factor $e^{-|z|^2}$ because it is a trace expression. Therefore we have to restrict the field configurations so that we can integrate by parts in the color space. We define our $W_\infty$ gauge theories as $d+2$ dimensional field theories such that all the fields and their derivatives vanish at $z=\infty$.

Next, let us introduce a fermion field, which is a fundamental representation of $W_\infty$ , namely a field which transforms as a bra or ket vector in the Hilbert space of harmonic oscillator:

$$|\psi(x)\rangle = \int |z\rangle d^2 z e^{-|z|^2}\langle z|\psi(x)\rangle \equiv \int |z\rangle\; d^2 z e^{-|z|^2}\psi(x,\bar{z}). \tag{2.16}$$

We write the action as

$$\begin{aligned} S_F &= \int d^n x \langle\psi(x)|\; \gamma^\mu(i\partial_\mu - \hat{A}_\mu(x))|\psi(x)\rangle \\ &= \int\!\!\int d^n x d^2 z e^{-|z|^2}\bar{\psi}(x,z)\gamma^\mu(i\partial_\mu - \mathcal{A}_\mu(x,z,\bar{z}))\psi(x,\bar{z}), \end{aligned} \tag{2.17}$$

which is invariant under the $W_\infty$ gauge transformation (2.15) and

$$\delta|\psi(x)\rangle = -i\hat{\xi}(x)|\psi(x)\rangle, \qquad \delta\psi(x,\bar{z}) = -i\ddagger\xi(\partial_{\bar{z}},\bar{z})\ddagger\psi(x,\bar{z}), \tag{2.18}$$

where ‡ ‡ indicates that the derivatives are placed on the left of $\bar{z}$.

As a last example of $W_\infty$ gauge theory let us consider next a scalar field which is in an adjoint representation.

$$\hat{M}(x) \equiv M(x, \hat{a}, \hat{a}^\dagger) = \int d^2 z e^{-|z|^2} |z\rangle M(x, z, \bar{z}) \langle z|. \qquad (2.19)$$

The action is given by:

$$\begin{aligned} S_H &= \int d^d x \mathrm{tr} \left[ \frac{1}{2} \left( \partial_\mu \hat{M}(x) - [\hat{A}_\mu, \hat{M}](x) \right) \left( \partial^\mu \hat{M}(x) - [\hat{A}^\mu, \hat{M}](x) \right) - v(\hat{M}) \right] \\ &= \int d^d x \left[ \int d^2 z \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-)^m}{m!} \partial_z^m \left( \partial_\mu M(x, z, \bar{z}) - \{\!\{ A_\mu, M \}\!\}(x, z, \bar{z}) \right) \times \right. \\ & \qquad \left. \times \partial_{\bar{z}}^m \left( \partial^\mu M(x, z, \bar{z}) - \{\!\{ A^\mu, M \}\!\}(x, z, \bar{z}) \right) \right] - \mathrm{tr} v(\hat{M}), \\ v(\hat{M}) &= \sum_n g_n \hat{M}^n, \qquad dim(g_n) = d \left( \frac{n}{2} - 1 \right) - n. \end{aligned} \qquad (2.20)$$

Here again we require that the fields and their $z, \bar{z}$ derivatives should fall off to zero at $z = \infty$. We can check that this action is invariant under the $W_\infty$ gauge transformation (2.15) and

$$\delta M(x, z, \bar{z}) = \{\!\{ \xi, M \}\!\}(x, z, \bar{z}), \qquad \delta \hat{M}(x) = -i[\hat{\xi}(x), \hat{M}(x)]. \qquad (2.21)$$

Notice again, that the interactions are non-local in the color space.

The quantization of the theory is done by the standard canonical quantization. Although the interactions are non-local in the color space, they are local in the ordinary space. Accordingly there is no problem for the quantization. We shall see it explicitly in an example in the next section.

Let us discuss next the contraction procedure which will allow us to obtain the $w_\infty$ gauge invariant actions from $W_\infty$ ones. For the fields in the adjoint representation such as $A_\mu$ and $M$ this procedure is straightforward. As we have mentioned earlier, by introducing two real coordinates $\sigma_x$ and $\sigma_y$ as in (2.7) and by taking the $l \to 0$ limit we reduce Moyal bracket to Poisson bracket. If we simultaneously rescale the fields and the coupling constants as

$$\begin{aligned}
&\mathcal{A}_\mu(x, z, \bar{z}) = l^{-2} A_\mu(x, \vec{\sigma}) \\
&M(x, z, \bar{z}) = (2\pi)^{\frac{1}{2n}} l M(x, \vec{\sigma}) \\
&g^2 = \bar{g}^2 l^{-6}, \quad g_n = \bar{g}_n (2\pi)^{\frac{1}{2}} l^{2-n},
\end{aligned} \tag{2.22}$$

we obtain that the dominant term in a power series expansion in $l$, in $YM$ action (2.13) and in the action for the scalar field (2.20) is *one*. Indeed, field strength $\mathcal{F}_{\mu\nu}(x, z, \bar{z})$ and integration measure $d^2 z$ is of the order $l^{-2}$, infinite sum in these actions can be dropped out, therefore integrands are of order $l^{-6}$ in $YM$ case and *one* in scalar field case. This $l^{-6}$ power cancels out with $l^6$ power from coupling constant $g^2$. Therefore in $l \to 0$ limit we obtain the following $w_\infty$ gauge invariant $d+2$ dimensional local field theory:

$$\begin{aligned}
S_{YM} = -\frac{1}{4\bar{g}^2} \int d^d x d^2 \vec{\sigma} F_{\mu\nu}(x, \vec{\sigma}) F^{\mu\nu}(x, \vec{\sigma}), \\
F_{\mu\nu}(x, \vec{\sigma}) = \partial_\mu A_\nu(x, \vec{\sigma}) - \partial_\nu A_\mu(x, \vec{\sigma}) + \epsilon^{ij} \partial_i A_\mu(x, \vec{\sigma}) \partial_j A_\nu(x, \vec{\sigma});
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
S_H = \int d^d x d^2 \vec{\sigma} \Big[ \frac{1}{2} (\partial_\mu M(x, \vec{\sigma}) - \epsilon^{ij} \partial_i A_\mu(x, \vec{\sigma}) \partial_j M(x, \vec{\sigma})) \times \\
\times (\partial^\mu M(x, \vec{\sigma}) - \epsilon^{ij} \partial_i A^\mu(x, \vec{\sigma}) \partial_j M(x, \vec{\sigma})) - \bar{v}(M) \Big], \\
\bar{v}(M) = \sum_n \bar{g}_n M^n(x, \vec{\sigma}).
\end{aligned} \tag{2.24}$$

Here again we require that the fields vanish at $\vec{\sigma} = \infty$.

Also, setting $\xi(x, z, \bar{z}) = l^{-2} \xi(x, \vec{\sigma})$ in the equations for the gauge transformation (2.15) for $YM$ and (2.21) for scalar field, and extracting the dominant terms in the power series expansion in $l$ we obtain the following $w_\infty$ gauge transformation:

$$\delta \mathcal{A}^\mu(x,\vec{\sigma}) = \partial^\mu \xi(x,\vec{\sigma}) - \epsilon^{ij}\partial_i \xi(x,\vec{\sigma})\partial_j \mathcal{A}^\mu(x,\vec{\sigma})$$
$$\delta M(x,\vec{\sigma}) = \epsilon^{ij}\partial_i \xi(x,\vec{\sigma})\partial_j M(x,\vec{\sigma}) \qquad (2.25)$$

Even though the $w_\infty$ transformations (2.25) are obtained from $W_\infty$ transformations (2.15) and (2.21) by the $l \to 0$ limit, we can independently check that the actions (2.24) are really invariant by the $w_\infty$ gauge transformation (2.25). We remark that the second equation of (2.25) can be written as

$$\delta M(x,\vec{\sigma}) = M(x,\vec{\sigma} + \delta\vec{\sigma}(x,\vec{\sigma})) - M(x,\vec{\sigma}), \qquad \delta\sigma^i(x,\vec{\sigma}) = -\epsilon^{ij}\partial_j \xi(x,\vec{\sigma}), \qquad (2.26)$$

which is a local area-preserving coordinate transformation.

As we mentioned earlier the damping factor $e^{-|z|^2}$ cancels out in Lagrangians for the fields in the adjoint representation such as $\mathcal{A}_\mu$ and $M$, due to the property of the trace in coherent state representation. But it remains there for the fields in fundamental representation, such as Fermi field (see (2.17)). Plugging in $z$ and $\bar{z}$ from equation (2.7) to the action (2.17) we obtain that $e^{-|z|^2} = e^{-\frac{\sigma^2}{2l^2}}$ factor is dominant in the $l \to 0$ limit. Therefore, a naive limit leads to the trivial result ($S_F \equiv 0$). Fermi field $\psi(x,z)$ which is the fundamental representation of $W_\infty$ , goes to Fermi field $\psi(x,\bullet)$ which is the fundamental representation of $w_\infty$ algebra in the $l \to 0$ limit. It is unknown to us how to construct such a representation. This, we believe, is the source of above mentioned difficulty.

We should mention that the YM lagrangian (2.23) had already been written down in the literature [7], [8].

## 2.3 One Dimensional Scalar Field Theory: One Dimensional $W_\infty$ and $w_\infty$ Matrix Model

In the previous section we presented a general method for constructing $W_\infty$ gauge theory and then we obtained the $w_\infty$ gauge theories by using the $l \to 0$ contraction procedure from $W_\infty$ theories. On the algebra level (see page 9) the contraction procedure ($W_\infty \to w_\infty$ ) requires throwing away an infinite set of generators of $W_\infty$ algebra shrinking it to much "smaller" (but still infinite) set of $w_\infty$ generators and therefore drastically reduces a symmetry of the gauge theory based on this algebras.

Several questions arise. Since $W_\infty$ group can be considered as an $N = \infty$ limit of $SU(N)$ group, can one use the $W_\infty$ gauge theory for the large $N$ limit of $SU(N)$ gauge theory, especially for the large $N$ QCD [9]? In the large $N$ QCD one takes the $N \to \infty$ limit keeping $g^2N$ finite. Since in $W_\infty$ theories $N$ is already at infinity, how can one implement the large $N$ QCD condition ? Treating color coordinates $\vec{\sigma}$ in $w_\infty$ theory (2.24) as additional space coordinates we have derived for $d = 1 + 2$ dimensional theory propagator and vertexes and using the standard perturbation diagrammatic calculation showed that the coupling constants are multiplicatively renormalized to absorb the infinite volume of color space. Analogous calculations in $W_\infty$ theory are complicated by highly non-local structure of the vertexes. Therefore in $W_\infty$ gauge theory the question arises whether or not one can implement the

QCD condition as a multiplicative renormalization of coupling constants ? Also, as shown in the previous section it is possible to obtain $w_\infty$ theory from $W_\infty$ theory by contraction. This shows the relationship between these theories as classical theories. On the classical level that procedure allows one to go smoothly from the solutions of the classical equations of motion for $W_\infty$ theory to those for $w_\infty$ theory. How about quantum theory ? Is there any physical region where one can use the $w_\infty$ gauge theory for large $N$ QCD ? In this section we address these questions by solving the simplest one-dimensional case exactly.

Since in $d = 1$ only time component of gauge field exists and the pure gauge field model becomes trivial, we consider the gauge invariant scalar field model. This theory may be thought of as a gauged one dimensional $W_\infty$ (or $w_\infty$ ) matrix model [14]. We solve it by using the collective field method [13]. Since we can carry out the discussions entirely in parallel for both $W_\infty$ and $w_\infty$ models, we present the corresponding expressions simultaneously and put label $a$ for $W_\infty$ model and label $b$ for $w_\infty$ model.

One dimensional $W_\infty$ and $w_\infty$ matrix model Lagrangians are given by (compare with (2.20) and (2.24) respectively):

$$\begin{aligned} L_a &= \mathrm{tr}\left[\frac{1}{2}(\partial_t \hat{M} - [\hat{A}_0, \hat{M}])^2 - v(\hat{M})\right] \\ &= \frac{1}{2}\int d^2z \Big(\partial_t M(t,z,\bar{z}) - \{\!\{A_0, M\}\!\}(t,z,\bar{z})\Big) e^{\partial_z \partial_{\bar{z}}} \Big(\partial_t M(t,z,\bar{z}) - \{\!\{A_0, M\}\!\}(t,z,\bar{z})\Big) \\ &- \mathrm{tr} v(\hat{M}), \\ L_b &= \int d\vec{\sigma} \left[\frac{1}{2}\Big(\partial_t M(t,\vec{\sigma}) - \epsilon^{ij}\partial_i A_0(t,\vec{\sigma})\partial_j M(t,\vec{\sigma})\Big)^2 - v(M)\right]. \end{aligned} \tag{2.27}$$

Using $W_\infty$ (and $w_\infty$ ) gauge invariance of the actions (2.27) let us fix the gauge

$\mathcal{A}_0 = 0$. The canonical quantization leads to the following Hamiltonians:

$$H_a = \int P(z,\bar{z})\partial_t M(z,\bar{z};t)d^2z - L = \int d^2z \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\partial_z^n P(z,\bar{z})\partial_{\bar{z}}^n P(z,\bar{z}) + \mathrm{tr}v(\hat{M}),$$
$$H_b = \int P(\vec{x})\partial_t M(\vec{x};t)d^2x - L = \int d\vec{\sigma}\left(\frac{1}{2}P(\vec{\sigma})^2 + v(M(\vec{\sigma}))\right), \quad (2.28)$$

where $P$'s are the canonical momentum operators conjugated to $M$'s, with the following commutation relations:

$$[\hat{M}(z,\bar{z}),\hat{P}(z',\bar{z}')] = i\delta^{(2)}(z-z'),$$
$$[\hat{M}(\vec{\sigma}),\hat{P}(\vec{\sigma}')] = i\delta(\vec{\sigma}-\vec{\sigma}'). \quad (2.29)$$

In addition there is an equation of motion of the field $\mathcal{A}_0$ which we impose as an operator constraints on the state vector $|\Psi\rangle$:

$$\hat{\Pi}_\xi|\Psi\rangle \equiv \int d^2z \{\!\{\xi,\hat{M}\}\!\}\hat{P}(z,\bar{z})|\Psi\rangle = 0,$$
$$\hat{\Pi}_\xi|\Psi\rangle \equiv \int d\vec{\sigma}\epsilon^{ij}\partial_i\xi(\vec{\sigma})\partial_j\hat{M}(\vec{\sigma})\hat{P}(\vec{\sigma})|\Psi\rangle = 0, \quad (2.30)$$

which simply state that the wave function (or the state vector) is gauge invariant and depends only on gauge invariant singlet variables. Therefore in order to solve the problem we choose the following $W_\infty$ (and $w_\infty$ ) invariant collective field as dynamical variables:

$$\phi(x) = \mathrm{tr}\delta(x - M(\hat{a},\hat{a}^\dagger)) \quad (\text{for } W_\infty),$$
$$\phi(x) = \int d\vec{\sigma}\delta(x - M(\vec{\sigma})) \quad (\text{for } w_\infty) \quad (2.31)$$

Then we change variables from $P(z,\bar{z})$, $M(z,\bar{z})$ to $\pi(x)$, $\phi(x)$, where $\pi(x)$ is, *by definition*, a canonical momentum conjugate to $\phi(x)$:

$$[\pi(x),\phi(x)] = -i\delta(x-x') \quad (2.32)$$

The wave function is a functional of the gauge invariant collective coordinates $\Psi = \Psi[\phi(x)]$. Acting on the wave function and using a chain rule:

$$\hat{P}\ \Psi[\phi(x)] = i\int dy\ [\hat{P},\phi(y)]\ \hat{\pi}(y)\ \Psi[\phi] \tag{2.33}$$

we can change all $\hat{P}$'s in Hamiltonians (2.28) to $\hat{\pi}$'s. Therefore we need to compute the following $\Omega(x,x')$ and $\omega(x)$ (see [13] or Appendix B, page 60 for details):

$$\begin{aligned} \Omega_a(x,x') &\equiv -\int d^2z \sum_{n=0}^{\infty}\frac{1}{n!}[\partial_{\bar{z}}^n P(z,\bar{z}),\phi(x)][\partial_z^n P(z,\bar{z}),\phi(x')] \\ \Omega_b(x,x') &\equiv -\int d\vec{\sigma}[P(\vec{\sigma}),\phi(x)][P(\vec{\sigma}),\phi(x')] \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} \omega_a(x) &\equiv \int d^2z \sum_{n=0}^{\infty}\frac{1}{n!}[\partial_{\bar{z}}^n P(z,\bar{z}),[\partial_z^n P(z,\bar{z}),\phi(x)]] \\ \omega_b(x) &\equiv \int d\vec{\sigma}[P(\vec{\sigma}),[P(\vec{\sigma}),\phi(x)]] \end{aligned} \tag{2.35}$$

These calculations can be easely done in momentum space. First, straightforward calculations lead to:

$$\begin{aligned} &[P(z,\bar{z}),e^{-ikM(\hat{a},\hat{a}^\dagger)}] = -e^{-|z|^2}k\int_0^1 d\tau e^{-ik\tau M(\hat{a},\hat{a}^\dagger)}|z\rangle\langle z|e^{-ik(1-\tau)M(\hat{a},\hat{a}^\dagger)}, \\ &[P(z,\bar{z}),\phi(k)] = -ke^{-|z|^2}\langle z|e^{-ikM(\hat{a},\hat{a}^\dagger)}|z\rangle, \\ &[P(z,\bar{z}),[P(w,\bar{w}),\phi(k)]] = k^2e^{-|z|^2-|w|^2}\int_0^1 d\tau\langle w|e^{-ik\tau M(\hat{a},\hat{a}^\dagger)}|z\rangle\langle z|e^{-ik(1-\tau)M(\hat{a},\hat{a}^\dagger)}|w\rangle. \end{aligned} \tag{2.36}$$

where $\phi(k) \equiv e^{-ikM(\hat{a},\hat{a}^\dagger)}$ is the Fourier transform of $\phi(x)$. Then using the following identity:

$$e^{-|z|^2}\sum_n \frac{1}{n!}((\partial_z-\bar{z})^n|z\rangle)((\partial_{\bar{z}}-z)^n\langle z|) = 1. \tag{2.37}$$

which expresses the completeness property [2] of the generators of $W_\infty$ fundamental representation (see A.3 for proof) we finally obtain:

$$\Omega_a(x,x') = \Omega_b(x,x') = \partial_x\partial_{x'}[\delta(x-x')\phi(x)], \tag{2.38}$$

and

$$\begin{aligned} \omega_a(x) &= 2\partial_x[\phi(x)G(x;\phi)], \\ \omega_b(x) &= -\kappa^2\partial_x^2\phi(x), \end{aligned} \tag{2.39}$$

where $\kappa^2 = \delta^2(0)$, and $G(x;\phi) = P\int \mathrm{dx}'\frac{\phi(x')}{x-x'}$. Notice that we obtained the same expression for $\Omega$ for both theories but quite different expressions for $\omega$.

In the collective field theory the hermiticity requirement of the Hamiltonian leads to the following equation for the Jacobian $J$ of change of variables:

$$\omega(x) + 2\int dx'\Omega(x,x')C(x') = 0, \quad C(x) = \frac{1}{2}\frac{\delta}{\delta\phi(x)}J, \tag{2.40}$$

Using (2.38) and (2.39) and assuming $\partial_x\phi(-\infty) = \phi(-\infty)\partial_x C(-\infty) = 0$ we obtain

$$\begin{aligned} \partial_x C_a(x) &= G(x;\phi), \\ \partial_x C_b(x) &= -\frac{1}{2}\kappa^2\frac{\partial_x\phi(x)}{\phi(x)} = -\frac{1}{2}\kappa^2\partial_x \ln\phi(x). \end{aligned} \tag{2.41}$$

Also in the collective field theory kinetic energy part of the hermitian Hamiltonian is given by:

$$K = \frac{1}{2}\int\int dx dx'[\pi(x)\Omega(x,x')\pi(x') + C(x)\Omega(x,x')C(x')], \tag{2.42}$$

[2] It can be considered as a generalization of the completeness property of $SU(N)$ generators to the case of $N=\infty$.

Since $\Omega_a = \Omega_b$ the first term in (2.42) is the same for both theories:

$$\frac{1}{2}\int\int dx dx' \pi(x)\Omega(x,x')\pi(x') = \frac{1}{2}\int\int dx dx' \pi(x)\partial_x\partial_{x'}[\delta(x-x')\phi(x)]\pi(x') = \int dx \frac{1}{2}\left(\partial_x\pi(x)\right)^2 \phi(x) \quad (2.43)$$

The second term, however, is different and equal to

$$\frac{1}{2}\int\int dx dx'\, C(x)\,\Omega(x,x')\,C(x') = \frac{1}{2}\int dx\, \left(\partial_x C_a(x)\right)^2 \phi(x) = \frac{1}{2}\int dx\, G(x;\phi)^2\phi(x) = \frac{\pi^2}{6}\int dx\, \phi(x)^3 \quad (2.44)$$

for $W_\infty$ theory and

$$\frac{1}{2}\int\int dx dx'\, C(x)\,\Omega(x,x')\,C(x') = \frac{1}{2}\int dx\, \left(\partial_x C_b(x)\right)^2 \phi(x) = \frac{1}{8}\kappa^4\frac{(\partial_x\phi(x))^2}{\phi(x)} \quad (2.45)$$

for $w_\infty$ . Collecting all results together we obtain the following Hamiltonians:

$$H_a = \int dx \left(\frac{1}{2}(\partial_x\pi(x))^2\phi(x) + \frac{\pi^2}{6}\phi(x)^3 + v(x)\phi(x)\right) - e\left(\int dx\phi(x) - N\right) \quad (2.46)$$

$$H_b = \int dx \left(\frac{1}{2}(\partial_x\pi(x))^2\phi(x) + \frac{1}{8}\kappa^4\frac{(\partial_x\phi(x))^2}{\phi(x)} + \tilde{v}(x)\phi(x)\right) - e\left(\int dx\phi(x) - L^2\right) \quad (2.47)$$

where $e$ is a Lagrange multiplier to insure

$$\int dx\,\phi(x) = \mathrm{tr}1 \equiv N \qquad (\text{for } W_\infty), \quad (2.48)$$

$$\int dx\,\phi(x) = \int d\vec{\sigma} \equiv L^2 \qquad (\text{for } \omega_\infty) \quad (2.49)$$

which follows from the definition of collective field (2.31). Notice that eventually we have to take $N \to \infty,\ \kappa \to \infty,\ L \to \infty$. For this purpose we first make the following scale transformations:

$$x \to N^{1/2}x, \quad \phi(x) \to N^{1/2}\phi(x), \quad \pi(x) \to N^{-1}\pi(x), \quad e \to Ne \quad (2.50)$$

(for $W_\infty$ ) and

$$x \to \kappa x, \quad \phi(x) \to L^2\kappa^{-1}\phi(x), \quad \pi(x) \to L^{-2}\pi(x), \quad e \to \kappa^2 e \qquad (2.51)$$

(for $w_\infty$ ), which preserve the canonical commutation relation (2.32). We obtain

$$H_a = \int dx \left[\frac{1}{2N^2}(\partial_x\pi(x))^2\phi(x) + N^2\left(\frac{\pi^2}{6}\phi(x)^3 + u(x)\phi(x) - e\phi(x)\right)\right] + N^2 e \qquad (2.52)$$

$$H_b = \int dx \left[\frac{1}{2N^2}(\partial_x\pi(x))^2\phi(x) + N^2\left(\frac{1}{8}\frac{(\partial_x\phi(x))^2}{\phi(x)} + \tilde{u}(x)\phi(x) - e\phi(x)\right)\right] + N^2 e \qquad (2.53)$$

where for $w_\infty$ we set $N \equiv L\kappa$ and

$$u(x) = \sum_n N^{(\frac{n}{2}-1)} g_n x^n \equiv \sum_n \alpha_n x^n, \qquad (2.54)$$

$$\tilde{u}(x) = \sum_n \kappa^{n-2}\tilde{g}_n x^n = \sum_n (\kappa l)^{n-2} g_n x^n \equiv \sum_n \tilde{\alpha}_n x^n. \qquad (2.55)$$

We know from the previous study [13],[15] that the $1/N$ expansion of Hamiltonians (2.52) and (2.53) is a standard semi-classical expansion. We prove this statement in the next two sections. Here, let us simply quote the results. In the $N \to \infty$ limit the excitation spectrum is finite provided that $u(x)$ is finite and given by

$$H = \frac{1}{2}\sum_{n=0}^{\infty}(p_n^2 + \omega_n^2 q_n^2), \qquad [q_n, p_m] = i\delta_{nm}, \qquad (2.56)$$

where for $W_\infty$ :

$$\omega_n = n\pi/T, \quad T = \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{dx'}{\sqrt{2(\tilde{e}_0 - u(x'))}}, \quad \frac{1}{\pi}\int_{\tilde{x}_1}^{\tilde{x}_2} dx'\sqrt{2(\tilde{e}_0 - u(x'))} = 1, \qquad (2.57)$$

and for $w_\infty$ :

$$\omega_n = E_n - E_0, \qquad \left(-\frac{1}{2}\partial_x^2 + \tilde{u}(x)\right)\chi_n(x) = E_n\chi_n(x) \tag{2.58}$$

### 2.3.1 Solution of $W_\infty$ Matrix Model

As we promised above, let us do a semi-classical expansion of the Hamiltonian (2.52) [13] [15]. In large $N$ limit last two terms are dominant, therefore the classical field configuration can be found by solving classical equations of motion. It is particularly simple for this case since variation with respect to $\phi_0(x)$ of the second term in (2.52) leads to the simple algebraic equation:

$$\frac{\pi^2}{2}\phi_0(x)^2 + u(x) - e = 0 \tag{2.59}$$

And variation with respect to lagrange multiplyer $e_0$ leads to

$$\int dx\ \phi_0(x) = 1. \tag{2.60}$$

Let us restrict ourselves to *one cut* solution of the system (2.59) and (2.60).

$$\phi_0(x) = \frac{1}{\pi}\sqrt{2(e_0 - u(x))}, \qquad x_1 \leq x \leq x_2 \tag{2.61}$$

and $\phi_0(x) = 0$ otherwise. $x_1$ and $x_2$ are turning points defined from the following conditions: $u(x_1) = u(x_2) = e_0$ and lagrange multiplyer $e_0$ can be found from (2.60):

$$\frac{1}{\pi}\int_{x_1}^{x_2} dx\sqrt{2(e_0 - u(x))} = 1. \tag{2.62}$$

$1/N$ expansion is now straightforward. First, let us set:

$$\phi(x) = \phi_0(x) + \frac{1}{N}\eta(x), \qquad \pi(x) = N\tilde{\pi}(x), \qquad e = e_0 + \frac{1}{N^2}\lambda \quad (2.63)$$

Then plugging back into the equation (2.52) and expanding the Hamiltonian up to the order $1/N$ we obtain

$$H_a = N^2 E_0 + H_{coll} + o(\frac{1}{N}), \quad (2.64)$$

where $E_0 = E_0(\phi_0, e_0)$ is a *constant* ground state energy, and $H_{coll}$ is a Hamiltonian of the collective excitations.

$$H_{coll} = \frac{1}{2}\int_{x_1}^{x_2} dx\ \phi_0(x)\left((\partial_x\tilde{\pi}(x))^2 + \pi^2\ \eta(x)^2\right) - \lambda\int_{x_1}^{x_2} dx\ \eta(x) \quad (2.65)$$

Let us introduce the new variable [15] [13]

$$\xi(x) \equiv \int_{x_1}^{x} \frac{dy}{\phi_0(y)}, \qquad 0 \le \xi \le T \equiv \int_{x_1}^{x_2} \frac{dy}{\phi_0(y)} \quad (2.66)$$

and set

$$\begin{aligned} \eta(x) &\equiv \frac{1}{\phi_0(x)}\sum_{n=1}^{\infty}(\frac{2}{T})\ cos(\frac{n\pi\xi}{T})\ (\frac{n\pi}{T})\ q_n, \\ -\partial_x\tilde{\pi}(x) &\equiv \frac{1}{\phi_0(x)}\sum_{n=1}^{\infty}(\frac{2}{T})\ sin(\frac{n\pi\xi}{T})\ p_n \end{aligned} \quad (2.67)$$

where $[q_n, p_m] = i\delta_{nm}$ are canonical field variables. Plugging back equations (2.67) into the collective Hamiltonian (2.65) and integrating over $x$ we obtain (2.56) and (2.57). (QED).

### 2.3.2 Solution of $W_\infty$ Matrix Model, II

Although we solved this model in the straightforward fashion (used in [15]) starting from (2.52), we solve it below in a slightly different way which illuminates the dynamical group structure of the theory.

We start with the Hamiltonian (2.46) with the constraint (2.48):

$$H = \int dx \left(\frac{1}{2}(\partial_x \pi(x))^2 \phi(x) + \frac{\pi^2}{6}\phi(x)^3 + v(x)\phi(x)\right), \quad \int dx \phi(x) = \mathrm{tr}1 = N. \tag{2.68}$$

Using $y_\pm(x) = \pm\pi\phi(x) - \partial_x\pi(x)$ one [16] writes the Hamiltonian and the constraint as

$$H = \frac{1}{2\pi}\int dx \int_{y_-(x)}^{y_+(x)} dy \left(\frac{1}{2}y^2 + v(x)\right), \qquad \frac{1}{2\pi}\int dx \int_{y_-(x)}^{y_+(x)} dy = N, \tag{2.69}$$

where $y_\pm(x)$ 's satisfy the commutation relation

$$[y_\pm(x), y_\pm(x')] = \mp 2\pi i \delta'(x - x'). \tag{2.70}$$

We diagonalize this Hamiltonian by a canonical transformation. In the integral of (2.69) we change variables from $y, x$ to $e, \xi$ such that $e = \frac{1}{2}y^2 + v(x)$. The action integral is the generator of transformation

$$S(x, e) = \int^x y dy = \pm \int^x \sqrt{2(e - v(x'))}dx', \tag{2.71}$$

accordingly $\xi = \frac{\partial S}{\partial e} = \pm \int^x \frac{dx'}{\sqrt{2(e - v(x'))}}$, where $\pm$ is for positive and negative $\xi$ respectively. The boundary of the phase space is transformed from $y_\pm(x)$ to $e(\xi)$

$$e(\xi) = \frac{1}{2}y_\pm^2(x) + v(x) \tag{2.72}$$

and the Hamiltonian and the constraint are given by

$$H = \frac{1}{2\pi}\oint d\xi \int^{e(\xi)} e de, \qquad \frac{1}{2\pi}\oint d\xi \int^{e(\xi)} de = N. \tag{2.73}$$

We expand $e(\xi)$ around the minimum configuration $e_0$ of $H$: $e(\xi) = e_0 + \delta e(\xi)$, where $e_0$ is given by a solution of

$$\frac{1}{\pi}\int_{x_1}^{x_2} \sqrt{2(e_0 - v(x))}dx = N, \tag{2.74}$$

where $x_1$ and $x_2$ are the turning points.

Since we see from (2.74) that $e_0$ is of order $N$, we may assume $e_0 \gg \delta e(\xi)$ and we may approximate $\xi$ on the boundary by $\xi(x) = \pm\int_{x_1}^{x} \frac{dx'}{\sqrt{2(e_0 - v(x'))}}$ and the half period by

$$T = \int_{x_1}^{x_2} \frac{dx'}{\sqrt{2(e_0 - v(x'))}} = \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{dx'}{\sqrt{2(\tilde{e}_0 - u(x'))}}, \tag{2.75}$$

where $\tilde{e}_0 = N^{-1}e_0$, $\tilde{x}_1 = N^{-\frac{1}{2}}x_1$ and $u(x) = N^{-1}v(N^{\frac{1}{2}}x)$ (see (2.54)).

Since the commutation rules of $e(\xi)$'s are given by

$$\begin{aligned} [e(\xi), e(\xi')] &= \left[\frac{1}{2}y_\pm^2(x) + v(x), \frac{1}{2}y_\pm^2(x') + v(x')\right] \\ &= \mp 2\pi i y_\pm(x) y_\pm(x')\delta'(x - x') \\ &\approx -2\pi i\delta'(\xi - \xi'), \end{aligned} \tag{2.76}$$

if we define $\theta$ and $r(\theta)$ by $\theta = \omega\xi$, $\quad r(\theta) = \omega^{-1}\delta e(\xi)$, $\quad \omega = \frac{\pi}{T}$, we obtain

$$[r(\theta), r(\theta')] = -2\pi i\delta'(\theta - \theta') \tag{2.77}$$

The normal mode expansion of $r(\theta)$ is given by $r(\theta) = \sqrt{\frac{2}{\omega}}\sum_{n>0}(\sin(n\theta)p_n + n\omega\cos(n\theta)q_n)$, and leads to the following Hamiltonian:

$$H = E_0 + H_{\text{coll}},$$

$$H_{\text{coll}} = \oint \frac{d\xi}{2\pi} \int_0^{\delta e(\xi)} ede = \omega \oint \frac{d\theta}{2\pi} \frac{1}{2} (r(\theta))^2 = \frac{1}{2} \sum_n (p_n^2 + \omega_n^2 q_n^2)),$$
$$\omega_n = n\omega = n\frac{\pi}{T}. \tag{2.78}$$

It is obvious from (2.74) and (2.75) that in the large $N$ limit $T^{-1}$ is finite provided that $u(x)$ is finite. In the double scaling limit one of the turning points goes to infinity so that $T \to \infty$ and we obtain the continuous spectrum (chiral Boson).

The result for $W_\infty$ model exactly coincides with the $N \to \infty$ limit of the matrix model discussed in [13],[15]. Of course it is already expected from the equations (2.38) and (2.39) since these equations coincide with the ones obtained in [13]. The infinite volume of color space (i.e. $N$) is absorbed by the multiplicative renormalization into the coupling constants (see (2.55)).

The reason why we could solve the $W_\infty$ model by a canonical transformation is that a dynamical $w_\infty$ algebra exists in the physical Hilbert space and the Hamiltonian is a generator of the algebra. More general: $\rho[\xi]$'s defined by

$$\rho[\xi] = \oint \frac{d\theta}{2\pi} \int^{\hat{r}(\theta)} dr \xi(r, \theta), \tag{2.79}$$

satisfy the $w_\infty$ commutation relation (2.6) [2].

$$[\rho[\xi_1], \rho[\xi_2]] = i\rho[\{\xi_1, \xi_2\}], \tag{2.80}$$

Let us prove (2.80). Choosing basis $\xi_1(r, \theta) = r^{k_1 - 1} e^{in\theta}$ and $\xi_2(r, \theta) = r^{k_2 - 1} e^{im\theta}$ we get:

$$[\hat{\rho}(\xi_1), \hat{\rho}(\xi_2)] \;=\; \oint \frac{d\theta_1 d\theta_2}{4\pi^2} e^{in\theta_1} e^{im\theta_2} \frac{1}{k_1 k_2} [\hat{r}^{k_1}(\theta_1), \hat{r}^{k_2}(\theta_2)] =$$

$$
\begin{aligned}
&= \oint \frac{d\theta_1 d\theta_2}{4\pi^2 k_1} e^{in\theta_1} e^{im\theta_2} \sum_{s=0}^{k_1-1} \hat{r}^s(\theta_1)[\hat{r}(\theta_1), \hat{r}(\theta_2)]\hat{r}^{k_2-1}(\theta_2)\hat{r}^{k_1-1-s}(\theta_1) = \\
&= i\oint \frac{d\theta}{2\pi k_1} \frac{\partial}{\partial\theta}\Big(e^{in\theta} e^{im\theta_2} \sum_{s=0}^{k_1-1} \hat{r}^s(\theta)\hat{r}^{k_2-1}(\theta_2)\hat{r}^{k_1-1-s}(\theta)\Big)_{\theta_2=\theta} \qquad (2.81)
\end{aligned}
$$

Using the following equation

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\hat{r}^{\alpha}(\theta) &= \alpha\hat{r}'(\theta)\hat{r}^{\alpha-1}(\theta) - \frac{\lambda}{2}\alpha(\alpha-1)\hat{r}^{\alpha-2}(\theta) \\
\lambda &\equiv [\hat{r}'(\theta), \hat{r}(\theta_1)] = -2\pi i\delta''(\theta-\theta_1)|_{\theta_1=\theta} \qquad (2.82)
\end{aligned}
$$

for the right hand side of (2.80) we obtain

$$
\begin{aligned}
[\hat{\rho}(\xi_1), \hat{\rho}(\xi_2)] &= A + B + C; \\
A &\equiv -n\oint \frac{d\theta}{2\pi} e^{i(n+m)\theta}\hat{r}^{k_1+k_2-2}(\theta), \\
B &\equiv i(k_1-1)\oint \frac{d\theta}{2\pi} e^{i(n+m)\theta}\hat{r}'(\theta)\hat{r}^{k_1+k_2-3}(\theta), \\
C &\equiv -i\frac{\lambda}{2}(k_1-1)(k_1+k_2-3)\oint \frac{d\theta}{2\pi} e^{i(n+m)\theta}\hat{r}^{k_1+k_2-4}(\theta).
\end{aligned}
\qquad (2.83)
$$

And for the left hand side of (2.80) we obtain:

$$
\begin{aligned}
i\rho\Big(\{\xi_1, \xi_2\}\Big) &= i\oint \frac{d\theta}{2\pi}\int^{\hat{r}(\theta)} dr \frac{\partial}{\partial r}\Big[\frac{\partial}{\partial\theta}(e^{in\theta}r^{k_1-1})(e^{im\theta}r^{k_2-1})\Big] \\
&\qquad -\frac{\partial}{\partial\theta}\Big[\frac{\partial}{\partial r}(e^{in\theta}r^{k_1-1})(e^{im\theta}r^{k_2-1})\Big] = \\
&= A + \frac{i(k_1-1)}{(k_1+k_2-2)}\oint \frac{d\theta}{2\pi} e^{i(n+m)\theta}\frac{\partial}{\partial\theta}\hat{r}^{k_1+k_2-2} = \\
&= A + B + C. \qquad (2.84)
\end{aligned}
$$

(QED)

As we have shown the above $W_\infty$ effective Hamiltonian (2.52) is actually an element of the $w_\infty$ spectrum generating algebra [1]. However this

$w_\infty$ symmetry is not directly related to the original $W_\infty$ gauge symmetry of the Lagrangian. Indeed, the original $W_\infty$ gauge symmetry acts trivially in the physical Hilbert space, since all wave functions are functionals of gauge invariant collective coordinate $\phi(x)$. From the other side the dynamical $w_\infty$ transformations are realized non-trivially in the physical Hilbert space.

### 2.3.3 Solution of $w_\infty$ Matrix Model

We first obtain the field configuration at which the potential energy of (2.53) is minimum. The equations are $-\frac{1}{4}\partial\left(\frac{\partial\phi}{\phi}\right)-\frac{1}{8}\left(\frac{\partial\phi}{\phi}\right)^2+\tilde{u}(x)=e$ and $\int dx\phi(x)=1$. For the variable $\varphi(x)=\sqrt{\phi(x)}$ the first equation is the Scrödinger equation $\left(-\frac{1}{2}\partial^2+\tilde{u}(x)\right)\varphi(x)=e\varphi(x)$ and the second equation is the normalization of the wave function. Therefore we consider an orthonormal set of eigenfunction $\chi_n$ with eigenvalue $E_n$. We set $\phi(x)=\chi_0^2(x)+\frac{1}{\sqrt{N}}\eta(x)$, $\quad \pi(x)=\sqrt{N}\zeta(x)$ and expand the Hamiltonian. We obtain

$$H=\frac{1}{2}\int dx\left[\chi_0^2(x)(\partial\zeta(x))^2+\frac{1}{4}\left(\frac{(\partial\eta(x))^2}{\chi_0^2(x)}+(2\partial^2\ln\chi_0)\frac{\eta^2(x)}{\chi_0^2(x)}\right)\right] \tag{2.85}$$

$\eta(x)$ and $\zeta(x)$ satisfy the canonical commutation relation:

$$[\partial\zeta(x),\eta(x')]=-i\delta'(x-x') \tag{2.86}$$

and the constraint $\int dx\eta(x)=0$. The normal mode expansion is now straightforward:

$$\eta(x)=\chi_0(x)\sum_{n=1}^{\infty}\sqrt{2\omega_n}\chi_n(x)\,q_n,\qquad \zeta(x)=\chi_0^{-1}(x)\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\omega_n}}\chi_n(x)\,p_n. \tag{2.87}$$

and

$$H = \frac{1}{2}\sum_{n=0}^{\infty}(p_n^2 + \omega_n^2 q_n^2), \qquad [q_n, p_m] = i\delta_{n,m}\ , \qquad \omega_n = E_n - E_0. \qquad (2.88)$$

As we see in (2.57) the energy spectrum of $W_\infty$ model is that of bosons with equally spaced frequencies irrespective of their interactions. For the $w_\infty$ model the equally spaced spectrum occurs only in the free theory, when $\tilde{\alpha}_n = 0$ for $n \neq 2$ (which corresponds to quadratic potential $u(x) = x^2/2$). It appears that in quantum theory the $l \to 0$ contraction of $W_\infty$ theory is a free $w_\infty$ theory. In general, for arbitrary potential $u(x)$, the spectrum of $w_\infty$ theory is essentially different from spectrum of $W_\infty$ theory. One should not be surprised, since we have mentioned before the contraction procedure drastically changes the underlying symmetry of the theory, and therefore changes also the physical picture of the theory. Another words, taking $l \to 0$ limit first and then quantizing the theory is not the same as quantizing the $W_\infty$ theory first and then taking the $l \to 0$ limit. Therefore, it is not possible to learn $W_\infty$ theory by studying $w_\infty$ theory.

# Chapter 3

# GAUGE INVARIANTS FOR $W_\infty$ YANG-MILLS

In the previous chapter we developed the general formalism for constructing $W_\infty$ gauge theories. In the present chapter we propose gauge invariant observables for these theories - $W_\infty$ Wilson loops. We solve $W_\infty$ two dimensional Yang-Mills theory on the cylinder exactly. After appropriate coupling constant renormalization ($g^2 N \equiv g_c^2$ - fixed, $N \to \infty$, where $N$ is the volume of the color space) solution agree with $N \to \infty$ limit of $SU(N)$ Yang-Mills [20]. This theory is equivalent to a $W_\infty$ one-dimensional unitary matrix model.

## 3.1 Introduction and Summary

In chapter 2 (see also [18]) we started systematic approach to gauge theories with gauge symmetry based on the $W_\infty$ algebra. $W_\infty$ is a commutator algebra of all Hermitian operators in the Hilbert space of a Harmonic oscillator. The gauge theories which possess this symmetry as an internal symmetry we call

$W_\infty$ gauge theories. In this chapter we construct gauge invariant observables (Wilson loops) for these theories, and solve $1+1$ dimensional Yang-Mills theory exactly.

Let us recall that for the pure $W_\infty$ Yang-Mills theory we have (2.10), (2.11) [18]:

$$\begin{aligned} S_{YM} &= -\frac{1}{4g^2}\int d^dx\ \mathrm{tr}(\hat{\mathcal{F}}_{\mu\nu}\hat{\mathcal{F}}^{\mu\nu}), \\ \hat{\mathcal{F}}_{\mu\nu} &= \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i[\hat{A}_\mu,\hat{A}_\nu] \end{aligned} \tag{3.1}$$

where a gauge potential $\hat{A}_\mu(x,\hat{a},\hat{a}^\dagger)$ is a Hermitian operator in the harmonic oscillator Hilbert space as well as a function of space-time. In the coherent state representation (see Appendix A, page 58) these equations can be rewritten as follows (compare with (2.13) and (2.14)):

$$\begin{aligned} S_{YM} &= -\frac{1}{4g^2}\int\int d^dx d^2z \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\partial_z^n\mathcal{F}_{\mu\nu}(x,z,\bar{z})\partial_{\bar{z}}^n\mathcal{F}^{\mu\nu}(x,z,\bar{z}), \\ \mathcal{F}_{\mu\nu}(x,z,\bar{z}) &= \partial_\mu\mathcal{A}_\nu(x,z,\bar{z}) - \partial_\nu\mathcal{A}_\mu(x,z,\bar{z}) + \{\!\{\mathcal{A}_\mu,\mathcal{A}_\nu\}\!\}(x,z,\bar{z}). \end{aligned} \tag{3.2}$$

To integrate by parts in the color space we have to restrict the field configurations to those fields which vanish together with their derivatives at $z=\infty$. This action is invariant under the $W_\infty$ gauge transformation (2.15):

$$\delta\hat{A}_\mu(x) = \partial_\mu\hat{\xi}(x) + i[\hat{\xi}(x),\hat{A}_\mu(x)], \quad \delta\mathcal{A}_\mu(x,z,\bar{z}) = \partial_\mu\xi(x,z,\bar{z}) - \{\!\{\xi,\mathcal{A}_\mu\}\!\}(x,z,\bar{z}). \tag{3.3}$$

The interaction in the coherent state representation is non-local in the color space, since the action involves derivatives of infinite order. However it is local in the ordinary space-time. Therefore the quantization can be consistently done.

In section 3.2 we construct $W_\infty$ and $w_\infty$ gauge invariant Wilson lines in arbitrary dimensions. In sections 3.3 we quantize $1+1$ $W_\infty$ Yang-Mills theory to test our results. We prove that it reduces to the one-dimensional $W_\infty$ unitary matrix model [20].

## 3.2 $W_\infty$ Wilson lines

Let us consider a space-time curve $C \equiv \{x^\mu(\tau) | \tau \in [0,1]\}$. We will define a $W_\infty$ Wilson line as a path ordered exponent:

$$\hat{W}(C) \equiv Pe^{-i\int_0^1 d\tau \hat{A}_\mu(x(\tau))\dot{x}^\mu(\tau)} = Pe^{-i\int_0^1 d\tau \int d^2z e^{-|z|^2} |z\rangle \mathcal{A}_\mu(x(\tau),z,\bar{z})\dot{x}^\mu(\tau)\langle z|} \quad (3.4)$$

where we use the coherent state representation for operator $\hat{A}_\mu$ (2.9). This is a formal definition. It defines the $W_\infty$ group element which is naively divergent [22], and therefore requires further specification of the regularization procedure, which manifestly guarantees that (3.4) actually exists. One way to regularize (3.4) is to consider the following discretized version:

$$\hat{W}(C) = \prod_{k=0}^{K-1}\Big(1 - i\hat{A}_\mu(x(\tau_k))\dot{x}^\mu(\tau_k)\Delta\tau_k\Big), \qquad \Delta\tau_k \equiv \frac{1}{K} \quad (3.5)$$

and take (eventually) $K \to \infty$. Or instead, (rewriting (3.4) in coherent state representation) we can show that this definition is actually finite or superrenormalizable. For this purpose let us divide the $[0,1]$ interval into $n$ equal parts: $0 = \tau_1, \tau_2, ..., \tau_n, \tau_{n+1} = 1$. Then the $W(C)$ can be written as:

$$\hat{W}(C) = \lim_{n\to\infty}\int ... \int d^2z_1 ... d^2z_n e^{-|z_1|^2 - ... - |z_n|^2} |z_1\rangle(1 - i\mathcal{A}_\mu(x(\tau_1), z_1, \bar{z}_1)\dot{x}^\mu(\tau_1)d\tau_1) \times$$

$$
\begin{aligned}
&\times \; e^{\bar{z}_1 z_2} \ldots\ldots e^{\bar{z}_{n-1} z_n}(1 - i\mathcal{A}_\mu(x(\tau_n), z_n, \bar{z}_n)\dot{x}^\mu(\tau_n) d\tau_n)\langle z_n| = \\
&= \; \lim_{n\to\infty} \int \ldots \int d^2 z_1 \ldots d^2 z_n e^{-|z_1|^2 - \ldots - |z_n|^2} e^{\bar{z}_1 z_2 + \ldots + \bar{z}_{n-1} z_n} \times \\
&\times \; |z_1\rangle e^{-i\mathcal{A}_\mu(x(\tau_1), z_1, \bar{z}_1)\dot{x}^\mu(\tau_1)d\tau_1 - \ldots - i\mathcal{A}_\mu(x(\tau_n), z_n, \bar{z}_n)\dot{x}^\mu(\tau_n)d\tau_n}\langle z_n| = \\
&= \; \lim_{n\to\infty} \int \ldots \int d^2 z_1 \ldots d^2 z_{n+1} e^{\bar{z}_1 \frac{(z_2 - z_1)}{d\tau_1} d\tau_1 + \ldots + \bar{z}_n \frac{(z_{n+1} - z_n)}{d\tau_n} d\tau_n} \times \\
&\times \; |z_1\rangle e^{-i\mathcal{A}_\mu(x(\tau_1), z_1, \bar{z}_1)\dot{x}^\mu(\tau_1)d\tau_1 - \ldots - i\mathcal{A}_\mu(x(\tau_n), z_n, \bar{z}_n)\dot{x}^\mu(\tau_n)d\tau_n} e^{-|z_{n+1}|^2}\langle z_{n+1}| \qquad (3.6)
\end{aligned}
$$

where on the last line $1 = \int d^2 z_{n+1} e^{-|z_{n+1}|^2}|z_{n+1}\rangle\langle z_{n+1}|$ is inserted. Since $z_1 = z(\tau_1) = z(0)$ and $z_{n+1} = z(\tau_{n+1}) = z(1)$, one can easily take the limit (keeping the endpoints fixed) to obtain:

$$
\begin{aligned}
\hat{W}(C) &= \int D^2 z(\tau)|z(0)\rangle e^{i\int_0^1 \mathcal{L}(\tau)d\tau} e^{-|z(1)|^2}\langle z(1)|, \\
\mathcal{L}(\tau) &\equiv \bar{z}(\tau)\frac{1}{i}\frac{d}{d\tau}z(\tau) - \mathcal{A}_\mu(x(\tau), z(\tau), \bar{z}(\tau))\dot{x}^\mu(\tau). \qquad (3.7)
\end{aligned}
$$

(3.7) means that we can express Wilson lines in terms of the one-dimensional quantum mechanical system with the Hamiltonian $\mathcal{A}_\mu(x(\tau), z(\tau), \bar{z}(\tau))\dot{x}^\mu(\tau)$. This is nothing but the Bargman-Fock representation for one-dimensional path integral [23] and in fact it is superrenormalizable. (3.5) or (3.7) will be our working definition of $W_\infty$ Wilson lines. In a different context for a finite $N$ similar expression has been derived in [24].

Note that the endpoints of the curve $C$ enter (3.7) asymmetrically. It will be very important for us later that $\mathcal{L}(\tau)$ (as follows from (3.6)) is an analytic function of $z_{n+1} = z(1)$. Because of that equation (3.7) is very convenient for the coupling with other operators from the right hand side since it allows to use property (A.8) of the coherent state representation. However, this asymmetry is artificial and can be easily removed by noting that in the same way as we work out expression (3.7) we can work out the following expression:

$$\hat{W}(C) = \int D^2 z(\tau) e^{-|z(0)|^2} |z(0)\rangle e^{i\int_0^1 \mathcal{L}'(\tau)d\tau} \langle z(1)|,$$
$$\mathcal{L}'(\tau) \equiv -\frac{1}{i}\frac{d\bar{z}(\tau)}{d\tau} z(\tau) - \mathcal{A}_\mu(x(\tau), z(\tau), \bar{z}(\tau))\dot{x}^\mu(\tau). \qquad (3.8)$$

Where now in opposition to $\mathcal{L}(\tau)$, $\mathcal{L}'(\tau)$ is an anti-analytic function of $\bar{z}_1 = \bar{z}(0)$, so that this expression is perfectly adopted for the couplings with other operators from the left hand side (using (A.8)). We will refer to these expressions interchangeably. Note that: $\mathcal{L}(\tau) = \mathcal{L}'(\tau) - i\frac{d}{d\tau}|z(\tau)|^2$.

Several properties should be checked for further reference:

### 3.2.1 Group Property

Suppose the curve $C$ consists of two parts: $C_1 \equiv \{x^\mu(\tau)|\tau \in [0,t]\}$ and $C_2 \equiv \{x^\mu(\tau)|\tau \in [t,1]\}$. Formally: $C = C_1 \circ C_2$. Then for the product of two Wilson lines we get:

$$\hat{W}(C_1)\hat{W}(C_2) = \int D^2 z(\tau)|z(0)\rangle e^{i\int_0^t \mathcal{L}(\tau)d\tau} e^{-|z(t)|^2}$$
$$\times \int D^2 w(\tau) e^{\bar{z}(t)w(t)} e^{i\int_t^1 \mathcal{L}(\tau)d\tau} e^{-|w(1)|^2} \langle w(1)| =$$
$$= \int D^2 z(\tau)|z(0)\rangle e^{i\int_0^1 \mathcal{L}(\tau)d\tau} e^{-|z(1)|^2} \langle z(1)| = \hat{W}(C_1 \circ C_2) \qquad (3.9)$$

where integral $\int d^2 z(t)$ ... is manifestly taken using representation (3.6) and the property (A.8).

### 3.2.2 Endpoint Derivatives

Let us define the following endpoint derivative:

$$\frac{\delta}{\delta x^\mu(t)}\hat{W}(C_1) \equiv \lim_{\delta C \to 0} \frac{\hat{W}(C_1 + \delta C) - \hat{W}(C_1)}{\delta C} \qquad (3.10)$$

where $\delta C \equiv \{\dot{x}^\mu(t)dt\}$ is the infinitesimal line added to the end of the line $C_1$ (defined above in (3.9)). Using (3.2) one can easely see that:

$$\frac{\delta}{\delta x^\mu(t)}\hat{W}(C_1) = -i\hat{W}(C_1)\hat{A}_\mu(x(t)) \tag{3.11}$$

Or instead, one can directly differentiate $\hat{W}(C_1)$ using representation (3.7) and property (A.8) to show that (3.11) is valid. The same way the derivative of the beginning of the line $C_2$ leads to:

$$\frac{\delta}{\delta x^\mu(t)}\hat{W}(C_2) \equiv \lim_{\delta C\to 0}\frac{\hat{W}(C_2) - \hat{W}(\delta C + C_2)}{\delta C} = i\hat{A}_\mu(x(t))\hat{W}(C_2), \tag{3.12}$$

where now $\delta C$ is the line element added to the begining of $C_2$.

### 3.2.3 Gauge Invariance

Let us prove first the gauge covariance of the Wilson line for the infinitesimal line element $c$ connecting points $x^\mu$ and $x^\mu + dx^\mu$:

$$\begin{aligned}
&\hat{W}(c)\hat{\xi}(x+dx) - \hat{\xi}(x)\hat{W}(c) = \\
&= (1 - i\hat{A}_\mu(x)dx^\mu)(\hat{\xi}(x) + \partial_\rho\hat{\xi}(x)dx^\rho) - \hat{\xi}(x)(1 - i\hat{A}_\mu(x)dx^\mu) = \\
&= (\partial_\mu\hat{\xi}(x) + i[\hat{\xi}(x), \hat{A}_\mu(x)])dx^\mu = \delta\hat{A}_\mu(x)dx^\mu = i\delta\hat{W}(c)
\end{aligned} \tag{3.13}$$

Using group property (3.9) one can now generalize this result to arbitrary curve C, dividing it into the set of infinitesimal curves and aplying (3.13):

$$\delta\hat{W}(C) = -i(\hat{W}(C)\hat{\xi}(x(1)) - \hat{\xi}(x(0))\hat{W}(C)) \tag{3.14}$$

However, one can prove the same result directly from (3.6) (or (3.7) and (3.8)) noting that for arbitrary variation $\delta\hat{A}_\mu(x)$:

$$\delta\hat{W}(C) = -i\int_C dx^\mu \hat{W}(C_1)\delta\hat{A}_\mu(x)\hat{W}(C_2) \tag{3.15}$$

where $C = C_1 \circ C_2$ as in (3.9). For gauge transformation using (3.11) and (3.12)) we get

$$\begin{aligned} \delta\hat{W}(C) &= -i\int_C dx^\mu \hat{W}(C_1)(\partial_\mu\hat{\xi} + i[\hat{\xi}, \hat{A}_\mu])\hat{W}(C_2) = \\ &= -i\int_C dx^\mu \frac{\delta}{\delta x^\mu}(\hat{W}(C_1)\hat{\xi}\hat{W}(C_2)) = \\ &= -i\Big(\hat{W}(C)\hat{\xi}(x(1)) - \hat{\xi}(x(0))\hat{W}(C)\Big) \end{aligned} \tag{3.16}$$

Closing line ($x^\mu(1) = x^\mu(0)$) and taking trace over coherent state representation leads to the $W_\infty$ gauge invariant Wilson loop:

$$W(C) = \int D^2 z(\tau) e^{i\oint \mathcal{L}(\tau)d\tau} \tag{3.17}$$

Anti-normal ordering of the $\hat{W}(C)$ (see (3.7) or (3.8)) leads to the following coherent state representation for the $W_\infty$ Wilson lines.

$$\begin{aligned} &\hat{W}(C) = W(C, \hat{a}, \hat{a}^\dagger) = \int d^2 z e^{-|z|^2} |z\rangle W(C, z, \bar{z})\langle z|, \\ &W(C, z, \bar{z}) = e^{-\partial^2} e^{-|z|^2} \langle z|\hat{W}(C)|z\rangle \end{aligned} \tag{3.18}$$

(see (A.6) and (A.7)). Then we can rewrite (3.9) and (3.16) in the coherent state representation as follows:

$$W(C_1, z, \bar{z}) * W(C_2, z, \bar{z}) = W(C_1 \circ C_2, z, \bar{z}), \tag{3.19}$$

$$\delta W(C, z, \bar{z}) = -i\Big(W(C, z, \bar{z}) * \xi(x(1), z, \bar{z}) - \xi(x(0), z, \bar{z}) * W(C, z, \bar{z})\Big) \tag{3.20}$$

## 3.3 $W_\infty$ Two Dimensional Yang-Mills

Going back to (3.2) let us restrict ourselves to the $d = 1+1$ dimensional Yang-Mills theory with the spatial dimensions compactified to the circle of length $L$. In this section we solve this theory exactly and show that it is equivalent to the one dimensional unitary matrix model [20]. We use the same collective field approach as in previous chapter. Canonically quantizing this theory in the $\hat{A}_0(x) = 0$ gauge we get the following Hamiltonian:

$$H = \frac{g^2}{2}\int dx d^2 z \sum_{n=0}^{\infty} \frac{1}{n!}\partial_z^n \hat{P}(x,z,\bar{z})\partial_{\bar{z}}^n \hat{P}(x,z,\bar{z}), \tag{3.21}$$

where $\hat{P}(x,z,\bar{z})$ is the canonical momentum operator conjugated to $\mathcal{A}_1(x,z,\bar{z})$:

$$[\mathcal{A}_1(x,z,\bar{z}), \hat{P}(y,z',\bar{z}')] = i\delta^{(2)}(z-z')\delta(x-y) \tag{3.22}$$

The Gauss Law constraint effectively resolved on functions which are functionals of gauge invariant collective coordinate:

$$\phi_k = \mathrm{tr}(\hat{W}^k) \tag{3.23}$$

with one dimensional Wilson line $\hat{W} \equiv \hat{W}_0^L$ given by (see (3.7)):

$$\hat{W}_a^b = \int_{z(a)}^{z(b)} D^2 z(x) |z(a)\rangle e^{i\int_a^b \mathcal{L}(x,z(x),\bar{z}(x))dx} e^{-|z(b)|^2} \langle z(b)|,$$
$$\mathcal{L}(x,z(x),\bar{z}(x)) \equiv \bar{z}(x)\frac{1}{i}\frac{d}{dx}z(x) - \mathcal{A}_1(x,z(x),\bar{z}(x)). \tag{3.24}$$

Note that since W is the unitary matrix (rather then Hermitian) $k$ in (3.23) is an integer and can not be promoted to all real values.

In this case straightforward calculations (see Appendix B, page 60) [13] lead to the following results for $\Omega$ and $\omega$:

$$\begin{aligned}\Omega(k,k') &= -\int_0^L dx \int d^2z \sum_{n=0}^{\infty} \frac{1}{n!}[\partial_{\bar{z}}^n \hat{P}(x,z,\bar{z}), \phi_k][\partial_z^n \hat{P}(x,z,\bar{z}), \phi_{k'}] = \\ &= -Lkk'\phi_{k+k'}. \qquad (3.25)\\ \omega(k) &= \int_0^L dx \int d^2z \sum_{n=0}^{\infty} \frac{1}{n!}[\partial_{\bar{z}}^n \hat{P}(x,z,\bar{z}), [\partial_z^n \hat{P}(x,z,\bar{z}), \phi_k]] = \\ &= Lk\sum_{s=0}^{k-1} \phi_{s+1}\phi_{k-s-1} \qquad (3.26)\end{aligned}$$

And after Fourier transform:

$$\begin{aligned}\Omega(x,x') &\equiv \frac{1}{4\pi^2}\sum_k\sum_s e^{ikx}e^{iks}\Omega(k,k') = L\partial_x\partial_{x'}\Big(\delta_p(x-x')\phi(x')\Big) \qquad (3.27)\\ \omega(x) &\equiv \frac{1}{2\pi}\sum_k e^{ikx}\omega(k) = L\partial_x\Big(\phi(x)G_p(x;\phi)\Big) \qquad (3.28)\end{aligned}$$

where

$$\begin{aligned}\phi(x) &\equiv \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}\phi_k, \qquad \delta_p(x) \equiv \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx} \qquad (3.29)\\ G_p(x;\phi) &\equiv P\int_0^{2\pi} dy\ \phi(y)\ cot\Big(\frac{x-y}{2}\Big) \qquad (3.30)\end{aligned}$$

All dependence upon space dimension is factorized here into a prefactor $L$ in front of $\Omega$ and $\omega$. This is a crucial observation. It follows therefore, that up to multiplicative prefactor $L$ this theory is equivalent to the $W_\infty$ unitary matrix model.

Hermiticity requirement of the Hamiltonian leads to the following equation for the Jacobian $J$ of change of variables (compare to (2.40)):

$$\omega(x) + 2\int_0^{2\pi} dx'\Omega(x,x')C(x') = 0, \quad C(x) = \frac{1}{2}\frac{\delta}{\delta\phi(x)}J, \qquad (3.31)$$

Using (3.27) and (3.28) we obtain:

$$\partial_x C(x) = \frac{1}{2} G_p(x;\phi), \tag{3.32}$$

Since in the collective field theory the kinetic energy part of the hermitian Hamiltonian is given by (2.42) we obtain the following answer for $H$:

$$H = g^2 L \int_0^{2\pi} dx \left( \frac{1}{2} (\partial_x \pi(x))^2 \phi(x) + \frac{\pi^2}{6} \phi(x)^3 \right) - e \left( \int_0^{2\pi} dx \phi(x) - N \right) \tag{3.33}$$

where $e$ is a Lagrange multiplier to insure

$$\int_0^{2\pi} dx \phi(x) = \mathrm{tr} 1 \equiv N \tag{3.34}$$

This is a collective Hamiltonian of one dimensional $W_\infty$ unitary matrix model. The same result can be obtained using a slightly different approach of [20] presented in next section.

Eventually we have to take $N \to \infty$. Therefore let us renormalize the coupling constant $g^2 N \equiv g_c^2$ and to keep it finite as $N \to \infty$. Then $H = N^2 E_0 + H_{coll} + o(1/N)$, where $E_0$ is the energy of the ground state field configuration given by $\phi_0 = \frac{1}{2\pi}$ and $e_0 = \frac{1}{4} g_c^2 L \pi$. To prove this expression let us consider small (of the order 1) fluctuationes around ground state fields:

$$\phi(x) = N\phi_0 + \epsilon(x), \qquad e = N e_0 + \lambda, \tag{3.35}$$

Plugging back into (3.33) we obtain the following Hamiltonian:

$$H_{coll} = \frac{g_c^2 L}{2\pi} \int_0^{2\pi} dx \left[ \frac{1}{2} (\partial_x \pi(x))^2 + \frac{\pi^2}{2} \epsilon(x)^2 \right] - \lambda \int_0^{2\pi} dx \epsilon(x). \tag{3.36}$$

The normal mode expansion is now straightforward:

$$\pi(x) = \sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx) p_n, \qquad \epsilon(x) = -\sqrt{2} \sum_{n=1}^{\infty} n \sin(nx) q_n \tag{3.37}$$

$$H_{coll} = \frac{g_c^2 L}{2}\sum_{n=1}^{\infty}(p_n^2 + \omega_n^2 q_n^2), \qquad [q_n, p_m] = i\delta_{nm}, \qquad \omega_n = n\pi. \quad (3.38)$$

The spectrum of the theory is that of $SU(N)$, $N \to \infty$ unitary matrix model. This is in complete agreement with our statement from the last chapter, that large $N$ limit of gauge theories corresponds to the $W_\infty$ theory.

### 3.3.1 $W_\infty$ Two Dimensional Yang-Mills, II

In this section we derive equation (3.33) using different approach of [20]. In $\hat{A}_0(x) = 0$ gauge we have the following Hamiltonian and the classical constraint:

$$H = \frac{1}{2g^2}\int dx tr(\dot{\hat{A}}_1(x))^2, \qquad \partial_x \dot{\hat{A}}_1(x) - i[\hat{A}_1, \dot{\hat{A}}_1](x) = 0. \qquad (3.39)$$

Let us consider

$$\hat{V}(x) \equiv \hat{W}_0^x \dot{\hat{A}}_1(x) \hat{W}_x^L \qquad (3.40)$$

where $\hat{W}_a^b$ as in (3.24). Using endpoint derivatives (3.11) and (3.12) one can easily see that on the constraint (3.39) the following property holds:

$$\frac{d}{dx}\hat{V}(x) = \hat{W}_0^x(\partial_x \dot{\hat{A}}_1(x) - i[\hat{A}_1, \dot{\hat{A}}_1](x))\hat{W}_x^L = 0, \qquad (3.41)$$

Therefore $\hat{V}(x)$ is independent of $x$: $\hat{V}(x) = \hat{V}(0) = \hat{V}(L)$, $\dot{\hat{A}}_1(x) = \hat{W}_x^0 \dot{\hat{A}}_1(0) \hat{W}_0^x$ and using (A.10) we obtain that:

$$H = \frac{1}{2g^2}\int_0^L dx tr(\dot{\hat{A}}_1(x))^2 = \frac{1}{2g^2}\int_0^L dx tr(\dot{\hat{A}}_1(0))^2 = -\frac{1}{2Lg^2} tr(\hat{W}^{-1}\dot{\hat{W}})^2, \qquad (3.42)$$

where $\hat{W} \equiv \hat{W}_0^L$, constrained to the equation: $[\hat{W}, \dot{\hat{W}}] = 0$ [21] [20]. This constraint reduces the whole configuration space to the eigenvalue subspace of $W$

[21]. Thus the whole system is equivalent to the one-dimensional $W_\infty$ gauged matrix model. Introducing Hermitian operator $\hat{M}$ as follows $\hat{W} = e^{-iM(\hat{a},\hat{a}^\dagger)}$ and quantizing the theory we finally get:

$$H = \frac{g^2 L}{2}\int d^2 z P(z,\bar{z}) e^{-\partial_z \partial_{\bar{z}}} P(z,\bar{z}), \tag{3.43}$$

$$[M(z',\bar{z}'), P(z,\bar{z})] = i\delta^{(2)}(z-z'). \tag{3.44}$$

The last equation defined only locally since it does not respect periodicity ($\theta_i \to \theta_i + 2\pi n_i$, where $\theta_i$ is any of eigenvalues) of $W$'s Lie algebra. To resolve this problem let us consider following collective coordinates:

$$\phi(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx}\mathrm{tr}(e^{-iM(\hat{a},\hat{a}^\dagger)})^n. \tag{3.45}$$

and assume that the wave function depends upon the eigenvalues of $W$ through $\phi(x)$ only. In contrast with hermitian matrix model case the sum here is over the integers only and can not be promoted to all real values because of the above mentioned periodicity. Note that $\phi(x)$ is gauge invariant on a circle since according to (3.14): $\delta\hat{W} = i[\hat{\xi}, \hat{W}]$. Then we change variables from $P(z,\bar{z})$, $M(z,\bar{z})$ to $\pi(x)$, $\phi(x)$, where $\pi(x)$ is a canonical momentum conjugate to $\phi(x)$:

$$[\pi(x), \phi(x')] = -i\delta_p(x-x') \tag{3.46}$$

The standard procedure to do this is to compute $\Omega(x,x')$ and $\omega(x)$. The definitions and the results are

$$\Omega(x,x') \equiv -\int d^2 z \sum_{n=0}^{\infty}\frac{1}{n!}[\partial_{\bar{z}}^n P(z,\bar{z}), \phi(x)][\partial_z^n P(z,\bar{z}), \phi(x')] = \partial_x \partial_{x'}[\delta_p(x-x')\phi(x)], \tag{3.47}$$

$$\omega(x) \equiv \int d^2z \sum_{n=0}^{\infty} \frac{1}{n!}[\partial_{\bar{z}}^n P(z,\bar{z}), [\partial_z^n P(z,\bar{z}), \phi(x)]] = 2\partial_x[\phi(x) G_p(x;\phi)], \tag{3.48}$$

and therefore

$$H = g^2 L \int_0^{2\pi} dx \left( \frac{1}{2} (\partial_x \pi(x))^2 \phi(x) + \frac{\pi^2}{6} \phi(x)^3 \right) - e \left( \int_0^{2\pi} dx \phi(x) - N \right) \tag{3.49}$$

# Chapter 4

# BREAKDOWN OF UNIVERSALITY IN RANDOM MATRIX MODELS

In this chapter we discuss universal behavior of two and higher order density-density correlation functions of matrix models.

We calculate the smoothed correlators for a large random matrix model with a potential containing products of two traces $\mathrm{tr}W_1(M) \cdot \mathrm{tr}W_2(M)$ in addition to a single trace $\mathrm{tr}V(M)$. The connected correlation function of density eigenvalues receives corrections besides the universal part derived by Brézin and Zee and it is no longer universal in a strong sense. [1]

[1]These results appear in [48].

## 4.1 Single Trace Matrix Model

In this section we review how to calculate the density of eigenvalues and its correlations in random matrix theory in the large $N$ limit($N$ is the size of the matrices) and show how the universal form of a two-point correlation function emerges. We consider a matrix model with an ordinary single trace potential in the Collective Field approach [43]. This approach is easily generalized to a multi trace potential, discussed in the next section.

The free energy $F[V]$ is defined as follows:

$$e^{-N^2F[V]} \equiv \int \frac{dM}{Vol}\, e^{-N\mathrm{tr}V(M)} = \int dx_1...dx_N\, \Delta^{2\lambda}\, e^{-N\sum_{i=1}^{N} V(x_i)} \tag{4.1}$$

where $Vol$ is the volume of gauge symmetry group, $\Delta = \prod_{1\leq i<j\leq N}(x_i - x_j)$ is the Van der Monde determinant, and $\lambda = 1/2, 1$ or $2$ for orthogonal, unitary or symplectic ensembles correspondingly. The partition function is invariant under orthogonal, unitary or symplectic rotations and the matrix integral can be reduced to integrals over its eigenvalues $x_i$ ($i = 1 \sim N$). Normalized density of eigenvalues is defined by

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i). \tag{4.2}$$

Connected density-density correlation functions can be obtained from the free energy by taking variational derivatives with respect to $V(x)$:

$$\langle \rho(x_1)\rho(x_2)...\rho(x_m)\rangle_c = \frac{(-)^{m-1}}{N^{2m-2}} \frac{\delta}{\delta V(x_1)} \frac{\delta}{\delta V(x_2)} \cdots \frac{\delta}{\delta V(x_m)}\, F[V]. \tag{4.3}$$

It is obvious now that the leading non-vanishing term for an $m$-point (unnormalized) density correlation function is $O(N^{2-m})$. The standard procedure of collective field theory is to rewrite integrals over eigenvalues in terms of a functional integral over density $\rho(x)$. Inserting *one* to the equation (4.1)

$$1 = \int D\rho(x) \prod_x \delta\Big(\rho(x) - \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i)\Big), \tag{4.4}$$

we obtain up to an overall constant (see Appendix C, page 62)

$$e^{-N^2 F[V]} = \int D\rho(x) d\sigma \; e^{N^2\lambda \int dx dy \rho(x) ln|x-y| \rho(y) - N^2 \int dx \rho(x) V(x) + i\sigma\left(\int dx \rho(x) - 1\right)} \; J[\rho]. \tag{4.5}$$

The Lagrange multiplier $\sigma$ is introduced to impose the constraint that an integral of $\rho(x)$ is normalized to one. We are interested in the large $N$ behavior and the Jacobian $J[\rho]$ can be neglected in this limit (see [44] for details). The resulting integral over $\rho$ can be evaluated by the steepest descent method which requires solving the following equations of motions [43]:

$$0 = 2N^2\lambda \int d\xi \; ln|x - \xi| \rho_0(\xi) - N^2 V(x) + i\sigma_0, \tag{4.6}$$

$$0 = \int dx \; \rho_0(x) - 1. \tag{4.7}$$

By differentiating the first equation we have the equation of BIPZ [46]

$$P \int d\xi \; \frac{\rho_0(\xi)}{x - \xi} = \frac{V'(x)}{2\lambda} \tag{4.8}$$

and it determines the stationary value $\rho_0(x)$. We assume here that $\rho_0(x)$ is equal to *zero* for $x < a$ or $x > b$ (*one-cut* from $a$ to $b$). Then the solution of this Cauchy integral equation is given by [45]:

$$\rho_0(x) = \frac{1}{\pi} \frac{1}{\sqrt{(x-a)(b-x)}} \left\{ \frac{1}{2\pi\lambda} P \int_a^b d\xi \; \frac{\sqrt{(\xi - a)(b - \xi)}}{\xi - x} V'(\xi) \; + \; 1 \right\} \tag{4.9}$$

The second term in the curly bracket is a solution of a homogeneous integral equation (i.e set the r.h.s. of eq. (4.8) *zero*) and its coefficient is determined such that $\rho_0(x)$ satisfies equation (4.7), while the first term does not contribute to the integral of eq.(4.7) (see Appendix C, page 64).

Equation (4.6) defines $\sigma_0[V]$ as a functional of $V$ and, of course is independent of $x$ even though it enters to the solution manifestly. Unknowns left are the positions of the end points of the cut $[a, b]$. By choosing boundary conditions $\rho_0(a) = 0$ and $\rho_0(b) = 0$ we get:

$$\rho_0(a) = 0 \quad \Longleftrightarrow \quad \frac{1}{2\pi\lambda}\int_a^b d\xi \sqrt{\frac{b-\xi}{\xi-a}}\, V'(\xi) + 1 = 0, \tag{4.10}$$

$$\rho_0(b) = 0 \quad \Longleftrightarrow \quad -\frac{1}{2\pi\lambda}\int_a^b d\xi \sqrt{\frac{\xi-a}{b-\xi}}\, V'(\xi) + 1 = 0. \tag{4.11}$$

Equations (4.10) and (4.11) determine $a = a[V]$ and $b = b[V]$ as functionals of $V(x)$, and should be solved first. This completes the solution $\rho_0(x)$ through the equation (4.9). For a simple case $V(x) = x^2/2$ this procedure reproduces the famous Wigner semi-circle solution.

The dependence of $\rho_0(x)$ on $V(x)$ comes both explicitly and implicitly through the end points and in order to find connected correlation functions (4.3) we have to know $V$-dependence of the end points. However we have an important relation that, given the boundary conditions (4.10) and (4.11), the solution (4.9) satisfies

$$\frac{\partial}{\partial a}\rho_0(x) = \frac{\partial}{\partial b}\rho_0(x) = 0. \tag{4.12}$$

This can be checked by straightforward calculations. Therefore upon variation of $\rho_0(x)$ with respect to $V(x)$ only the manifest dependence on $V(x)$

in (4.9) is relevant, while $V$-dependences of the boundaries $a$ and $b$ are cancelled out. Since the two-point correlation function is obtained from $\rho_0(x)$ by taking a variational derivative, this is essentially the statement of the universality of a two-point correlation function:

$$N^2\langle\rho_0(x)\rho_0(y)\rangle_c = -\frac{\delta}{\delta V(y)}\rho_0(x) = \frac{1}{2\pi^2\lambda}\frac{\partial}{\partial y}\left\{\sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}}\frac{P}{y-x}\right\} \tag{4.13}$$

All dependence on the potential $V(x)$ is implicit only through the end-points $a$ and $b$. Here $P$ stands for the principal value: $P\frac{1}{x} = \lim_{\epsilon\to 0} x/(x^2+\epsilon^2)$. This result has been obtained in various papers by various methods [31], [32], [33], [34], [36], [37].

Collecting all the results together we obtain for the free energy $F[V]$ the following expression (in the $N\to\infty$ limit):

$$F[V] = \int_{a[V]}^{b[V]} dx\ \rho_0(x)V(x) - \lambda\int_{a[V]}^{b[V]} dx\ dy\rho_0(x)ln|x-y|\ \rho_0(y). \tag{4.14}$$

As expected, density of eigenvalues is given by the saddle point solution;

$$\begin{aligned}\langle\rho(x)\rangle_c &= \frac{\delta F[V]}{\delta V(x)}\\ &= \rho_0(x) + \int_{a[V]}^{b[V]} d\xi\ \frac{\delta\rho_0(\xi)}{\delta V(x)}(V(\xi) - 2\lambda\int dy\ ln|x-y|\ \rho_0(y))\\ &= \rho_0(x) - \frac{i\sigma_0}{N^2}\int_{a[V]}^{b[V]} d\xi\ \langle\rho_0(\xi)\rho_0(x)\rangle_c = \rho_0(x).\end{aligned} \tag{4.15}$$

In principle $1/N$ corrections can be calculated by evaluating the Jacobian $J[\rho]$ [44] and fluctuationes around the saddle point.

## 4.2 Multi Trace Matrix Model

Let us now generalize our approach to a multi trace case. We consider an ensemble with the following statistical weight

$$\frac{1}{Z}\exp\Big(-N\mathrm{tr}V(M)-\frac{1}{2}\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}\mathrm{tr}W_\alpha(M)\mathrm{tr}W_\beta(M)\Big). \tag{4.16}$$

In this case, after integrating over the angular variables the free energy is given by:

$$e^{-N^2F[V]} \equiv \int dx_1...dx_N\ \Delta^{2\lambda}\ e^{-N\sum_{i=1}^{N}V(x_i)-\frac{1}{2}\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}\sum_{i,j=1}^{N}W_\alpha(x_i)W_\beta(x_j)} \tag{4.17}$$

and as before connected $m$-point correlation functions can be obtained by taking functional derivatives with respect to $V(x)$ (see (4.3)). Expressing $F[V]$ in terms of the density $\rho(x)$ we obtain:

$$\begin{aligned} e^{-N^2F[V]} = \int D\rho(x)d\sigma\ e^{N^2\lambda\int dxdy\rho(x)ln|x-y|\rho(y)-N^2\int dx\rho(x)V(x)}\times \\ \times e^{-N^2\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}\int dx\ \rho(x)W_\alpha(x)\int dy\ \rho(y)W_\beta(y)+i\sigma\left(\int dx\rho(x)-1\right)}J[\rho] \end{aligned} \tag{4.18}$$

with the same Jacobian $J[\rho]$ as in the previous section. Again in the leading order in $N$ we can set $J[\rho]=1$. The steepest descent equations are:

$$0 = 2N^2\lambda\int d\xi ln|x-\xi|\rho_0(\xi)-N^2V(x)-N^2\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}W_\alpha(x)c_\beta+i\sigma_0, \tag{4.19}$$

$$0 = \int dx\ \rho_0(x)-1 \tag{4.20}$$

where $c_\beta$'s are constants

$$c_\beta \equiv \int dx\ \rho_0(x)W_\beta(x) \tag{4.21}$$

which are determined self-consistently later. Differentiating the equation (4.19) we have a generalized equation of BIPZ

$$P\int d\xi\, \frac{\rho_0(\xi)}{x-\xi} = \frac{V'(x)}{2\lambda} + \frac{1}{2\lambda}\sum_{\alpha,\beta=1}^{K} \omega_{\alpha\beta} W'_\alpha(x) c_\beta \tag{4.22}$$

and it determines the stationary value $\rho_0(x)$. Let us consider again a *one-cut solution,* i.e $\rho_0(x)$ is equal to *zero* if $x < a$ or $x > b$. Then the general solution is given by [45]:

$$\rho_0(x) = \int_a^b d\xi\; \hat{G}(x,\xi)\Big\{V(\xi) \;+ \sum_{\alpha,\beta=1}^{K} \omega_{\alpha\beta} W_\alpha(\xi) c_\beta\Big\} + \;\frac{1}{\pi}\frac{1}{\sqrt{(x-a)(b-x)}} \tag{4.23}$$

where $\hat{G}(x,y)$ is a differential operator defined by

$$\hat{G}(x,y) \equiv \frac{1}{2\pi^2\lambda}\left(\sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}}\frac{P}{y-x}\right)\frac{\partial}{\partial y}. \tag{4.24}$$

The coefficient of the homogeneous part (the second term) of $\rho_0(x)$ is determined so as to satisfy the constraint (4.20).

The constants $c_\beta$'s are still unknown. In order to fix them we plug equation (4.23) into the equation (4.21) and obtain a set of algebraic equations for $c_\beta$'s, which can be written in the following compact form:

$$\sum_{\gamma=1}^{K} \Omega_{\alpha\gamma} c_\gamma = O_\alpha \tag{4.25}$$

where

$$O_\beta \;\equiv\; \int_a^b dxdy W_\beta(x)\hat{G}(x,y)V(y) + \int_a^b dx \frac{W_\beta(x)}{\pi\sqrt{(x-a)(b-x)}} \tag{4.26}$$

$$\Omega_{\beta\gamma} \;\equiv\; \delta_{\beta\gamma} - \sum_{\alpha=1}^{K} \omega_{\alpha\gamma} \int_a^b dxdy\; W_\beta(x)\hat{G}(x,y)W_\alpha(y). \tag{4.27}$$

Assuming further that

$$\det|\Omega| \neq 0 \tag{4.28}$$

equation (4.25) can be inverted

$$c_\alpha = \sum_\beta (\Omega^{-1})_{\alpha\beta} O_\beta \tag{4.29}$$

to give the solution for $\rho_0(x)$:

$$\begin{aligned} \rho_0(x) &= \int_a^b d\xi\ \hat{G}(x,\xi)\Big\{V(\xi) + \sum_{\alpha,\beta,\gamma=1}^{K} W_\alpha(\xi)\omega_{\alpha\beta}(\Omega^{-1})_{\beta\gamma}O_\gamma\Big\} \\ &+ \frac{1}{\pi}\frac{1}{\sqrt{(x-a)(b-x)}} \end{aligned} \tag{4.30}$$

Now the only unknowns left are the end points of the cut $[a,b]$. We can fix them by choosing boundary conditions $\rho_0(a) = 0$ and $\rho_0(b) = 0$. These equations determine $a = a[V,W]$ and $b = b[V,W]$ as functionals of $V(x)$ and $W(x)$'s. Then equation (4.30) provides final expression for $\rho_0(x)$.

As in the previous section one can prove that the solution (4.30) for $\rho_0(x)$ satisfies

$$\frac{\partial}{\partial b}\rho_0(x) = 0. \tag{4.31}$$

In order to prove it, notice that, from equation (4.23) and the boundary conditions $\rho_0(a) = \rho_0(b) = 0$, we get

$$\frac{\partial}{\partial b}\rho_0(x) = \int_a^b d\xi\ \hat{G}(x,\xi) \sum_{\alpha,\beta=1}^{K} \omega_{\alpha\beta}W_\alpha(\xi)\frac{\partial}{\partial b}c_\beta. \tag{4.32}$$

All other contributions cancel out due to the boundary conditions at the end points. Therefore taking derivative of $c_\beta$ in equation (4.21) we get a set of homogeneous algebraic equations, which can be written in the following form

$$\sum_{\gamma=1}^{K} \Omega_{\beta\gamma} \frac{\partial c_\gamma}{\partial b} = 0. \tag{4.33}$$

Since det $|\Omega| \neq 0$, as previously assumed in equation (4.28), we conclude that

$$\frac{\partial c_\gamma}{\partial b} = 0 \tag{4.34}$$

and equation (4.31) is proved. Similarly one can prove that

$$\frac{\partial \rho_0(y)}{\partial a} = \frac{\partial c_\alpha}{\partial a} = 0. \tag{4.35}$$

Therefore upon variation of $\rho_0(x)$ with respect to $V(x)$ only the manifest dependence on $V(x)$ in (4.30) remains. For a connected two point correlation function we obtain:

$$N^2 \langle \rho(x)\rho(y) \rangle_c = -\frac{\delta \rho_0(y)}{\delta V(x)} = \frac{1}{2\pi^2 \lambda} \frac{\partial}{\partial y} \left\{ \sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}} \frac{P}{y-x} \right\}$$
$$- \sum_{\alpha,\beta=1}^{K} \int_a^b d\xi \hat{G}(x,\xi) W_\alpha(\xi)\, \sigma_{\alpha\beta} \int_a^b d\zeta \hat{G}(y,\zeta) W_\beta(\zeta) \tag{4.36}$$

where

$$\sigma_{\alpha\gamma} \equiv \sum_{\beta=1}^{K} \omega_{\alpha\beta} (\Omega^{-1})_{\beta\gamma}. \tag{4.37}$$

Correlation function (4.36) is symmetric in $x$ and $y$ which follows from the symmetry of the matrix $\omega$ and the definition of $\Omega$ (4.27).

The correlation function (4.36) manifestly depends on $W_\alpha$'s and is no longer universal in a strong sense. However it is independent of $V(x)$ and also the short distance behavior is dominated by the first term of the universal form, which is what we call *weak* form of universality.[2] It is $O(N^0)$ and there is still a strong suppression of fluctuation though its amplitude is not universal. And the second term of the correlation function which breaks the universality is not inversly proportional to $\lambda$ generally. But if the second term in the definition of $\Omega$ eq.(4.27) is much larger than *one* the correlation function (4.36) is again inversly proportional to $\lambda$.

The free energy $F[V]$ is written in the following form:

$$F[V] = \int_{a[V,W]}^{b[V,W]} dx\ V(\xi)\rho_0(\xi) + \sum_{\alpha,\beta=1}^{K} \omega_{\alpha\beta} c_\alpha c_\beta - \lambda \int_a^b dx dy \rho_0(x) ln|x-y|\rho_0(y) \tag{4.38}$$

where $c_\alpha$'s are defined in (4.29).

Finally we consider the case that $\omega_{12} = \omega_{21} = 1$ and all other components equal to *zero*. Using equation (4.27) and (4.37) the explicit form of the two-point correlation function is

$$N^2\langle\rho(x)\rho(y)\rangle_c = \frac{1}{2\pi^2\lambda}\frac{\partial}{\partial y}\left\{\sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}}\frac{P}{y-x}\right\}$$
$$-\int_a^b d\xi \hat{G}(x,\xi)W_1(\xi)\ \frac{(W_2\cdot\hat{G}\cdot W_2)}{\det|\Omega|}\int_a^b d\zeta\hat{G}(y,\zeta)W_1(\zeta)$$
$$-\int_a^b d\xi \hat{G}(x,\xi)W_1(\xi)\ \frac{1-(W_1\cdot\hat{G}\cdot W_2)}{\det|\Omega|}\int_a^b d\zeta\hat{G}(y,\zeta)W_2(\zeta)$$
$$-\int_a^b d\xi \hat{G}(x,\xi)W_2(\xi)\ \frac{1-(W_2\cdot\hat{G}\cdot W_1)}{\det|\Omega|}\int_a^b d\zeta\hat{G}(y,\zeta)W_1(\zeta)$$

[2]In the paper by B. Eynard and C. Kristjansen [47] they proved universality of correlation functions for $O(N)$ model on random lattice which can be written as one matrix model with an *infinite* sum of products of two traces. The universality might be recovered for their case since the *infinite* sum may make the determinant eq. (4.28) divergent and $\sigma_{\alpha\gamma}$ in eq.(4.37) *zero*. We would like to thank C. Kristjansen for calling our attention to their papers.

$$-\int_a^b d\xi \hat{G}(x,\xi) W_2(\xi) \frac{(W_1 \cdot \hat{G} \cdot W_1)}{\det|\Omega|} \int_a^b d\zeta \hat{G}(y,\zeta) W_2(\zeta), \qquad (4.39)$$

where

$$\det|\Omega| \equiv (1-(W_2 \cdot \hat{G} \cdot W_1))(1-(W_1 \cdot \hat{G} \cdot W_2)) - (W_1 \cdot \hat{G} \cdot W_1)(W_2 \cdot \hat{G} \cdot W_2).$$

$$(W_\alpha \cdot \hat{G} \cdot W_\beta) \equiv \int_a^b dx dy \; W_\alpha(x) \hat{G}(x,y) W_\beta(y). \qquad (4.40)$$

### 4.2.1 Example: $W_1 = W_2 = g_2 x^2/2 + g_4 x^4/4$

In this section we give an example for an ensemble with a potential $N\mathrm{tr}V(M) + (\mathrm{tr}W(M))^2$ where

$$W(x) = \frac{g_2 x^2}{2} + \frac{g_4 x^4}{4}. \qquad (4.41)$$

We assume that $V(x)$ is a symmetric potential and $a = -b$. From eq. (4.39) we have

$$N^2 \langle \rho(x) \rho(y) \rangle_c = \frac{1}{2\pi^2 \lambda} \frac{\partial}{\partial y} \left\{ \sqrt{\frac{b^2 - y^2}{b^2 - x^2}} \frac{P}{y - x} \right\}$$

$$-\int_{-b}^b d\xi \hat{G}(x,\xi) W(\xi) \frac{2}{1 - 2(W \cdot \hat{G} \cdot W)} \int_{-b}^b d\zeta \hat{G}(y,\zeta) W(\zeta). \qquad (4.42)$$

First it is straightforward to evaluate the integral

$$\int_{-b}^b d\xi \hat{G}(x,\xi) W(\xi) = \frac{1}{2\pi^2 \lambda \sqrt{b^2 - x^2}} \int_{-b}^b d\xi \sqrt{b^2 - \xi^2} \frac{P}{\xi - x} W'(\xi)$$

$$= \frac{1}{2\pi^2 \lambda \sqrt{b^2 - x^2}} \int_{-b}^b d\xi \sqrt{b^2 - \xi^2} \left[ g_2 (1 + x \frac{P}{\xi - x}) + g_4 (\xi^2 + x\xi + x^2 + x^3 \frac{P}{\xi - x}) \right]$$

$$= \frac{1}{2\pi^2 \lambda \sqrt{b^2 - x^2}} \left[ g_2 (\frac{\pi b^2}{2} - \pi x^2) + g_4 (\frac{\pi b^4}{8} + \frac{\pi b^2 x^2}{2} - \pi x^4) \right]. \qquad (4.43)$$

Then by using

$$\int_{-b}^{b} d\zeta \frac{\zeta^{2m}}{\sqrt{b^2-\zeta^2}} = \frac{(2m-1)!!}{2m!!}\pi b^{2m}, \tag{4.44}$$

we can obtain

$$(W\cdot\hat{G}\cdot W) \;=\; -\frac{1}{\lambda}\left[\frac{g_2^2 b^4}{32}+\frac{g_2 g_4 b^6}{32}+\frac{9}{1024}g_4^2 b^8\right]. \tag{4.45}$$

The end points $-b$ and $b$ are determined by the conditions that the density distribution should vanish at the end points. The final answer for the connected density-density correlation is in general not inversely proportional to $\lambda$. This means that, when we apply matrix theory to the problem of conductance fluctuation, the variance $(\delta G)^2$ does not decrease exactly by a factor of two. However, if $|(W\cdot\hat{G}\cdot W)| >> 1$, we can *approximately* write the two point correlation function as

$$\begin{aligned}
N^2\langle\rho(x)\rho(y)\rangle_c &= \frac{1}{2\pi^2\lambda}\frac{\partial}{\partial y}\left\{\sqrt{\frac{b^2-y^2}{b^2-x^2}}\frac{P}{y-x}\right\} \\
&+ \frac{1}{(2\pi^2)^2\lambda\sqrt{(b^2-x^2)(b^2-y^2)}}\left[g_2(\frac{\pi b^2}{2}-\pi x^2)+g_4(\frac{\pi b^4}{8}+\frac{\pi b^2x^2}{2}-\pi x^4)\right]\times \\
&\times\frac{1}{\frac{1}{32}g_2^2b^4+\frac{1}{32}g_2g_4b^6+\frac{9}{1024}g_4^2b^8}\left[g_2(\frac{\pi b^2}{2}-\pi y^2)+g_4(\frac{\pi b^4}{8}+\frac{\pi b^2y^2}{2}-\pi y^4)\right]
\end{aligned} \tag{4.46}$$

and it is again inversely proportional to $\lambda$.

## 4.3 Discussion

In this chapter we have studied density-density correlation functions for random matrix models with a generalized potential containing products of two traces $\mathrm{tr}W_1(M)\cdot\mathrm{tr}W_2(M)$ in addition to a single trace $\mathrm{tr}V(M)$. We have shown that the two point function has no longer the universal form which depends on the potentials only through the end points. This is against the argument by Brézin and Zee [41]. They have considered a special ensemble

$$P(M)=\frac{1}{Z}e^{-N\mathrm{tr}V(M)-[\mathrm{tr}W(M)]^2/2}. \tag{4.47}$$

They argued that the effect of the second term is just to renormalize the potential $V$ to $V+\alpha_0 W$ where $\alpha_0$ is a constant determined by $V$ and $W$ and claimed that the universality for two point correlation function still holds for the above ensemble. Here we briefly review their argument and discuss why it cannot generally hold for higher point correlation functions. By introducing an auxiliary variable $\alpha$ the partition function is written as

$$\begin{aligned} Z(V,W) &= e^{-N^2F[V,W]}=\int dM e^{-(N\mathrm{tr}V(M)+[\mathrm{tr}W(M)]^2/2)} \\ &= \int d\alpha\; e^{N^2\alpha^2/2}e^{-N^2F[V+\alpha W,0]} \end{aligned} \tag{4.48}$$

up to an irrelevant overall factor and the integral over $\alpha$ runs over the imaginary axis. In the large $N$ limit, the integral over $\alpha$ can be evaluated at the saddle point and

$$F[V,W]=F[V+\alpha_0 W,0]-\frac{\alpha_0^2}{2} \tag{4.49}$$

where $\alpha_0$ is determined by

$$\alpha_0=\frac{\partial}{\partial\alpha}F[V+\alpha W,0]|_{\alpha_0}. \tag{4.50}$$

$\alpha_0$ depends on the details of the potentials. We can now obtain its density distribution function:

$$\rho_0(x) \;=\; \frac{\delta F[V,W]}{\delta V(x)}$$

$$
\begin{aligned}
&= \frac{\delta F[V+\alpha_0 W,0]}{\delta V(x)}|_{\alpha_0} + (\frac{\partial F[V+\alpha_0 W,0]}{\partial \alpha_0} - \alpha_0)\frac{\delta \alpha_0}{\delta V(x)} \\
&= \frac{\delta F[V+\alpha_0 W,0]}{\delta V(x)}|_{\alpha_0}. \qquad (4.51)
\end{aligned}
$$

This density distribution is the same as that of an ensemble with an effective potential $\mathrm{tr}V_{eff} = \mathrm{tr}(V + \alpha_0 W)$ as discussed in [41]. In order to obtain the two point correlation function we then take variational derivative of the distribution function with respect to the potential $V(y)$. $\rho_0(x)$ depends on $V(y)$ not only explicitly but implicitly through $\alpha_0$. It becomes

$$
\begin{aligned}
G(x,y) &= -\frac{\delta^2 F[V,W]}{\delta V(x)\delta V(y)} = -\frac{\delta \rho_0(x)}{\delta V(y)} \\
&= -\frac{\delta^2 F[V+\alpha_0 W,0]}{\delta V(x)\delta V(y)}|_{\alpha_0} - \frac{\partial \rho_0(x)}{\partial \alpha_0}\frac{\delta \alpha_0}{\delta V(y)}. \qquad (4.52)
\end{aligned}
$$

The first term is the universal correlation function for an effective potential $V_{eff}$ and actually is independent of the details of the potential. But the second term, which is a product of a function of $x$ and that of $y$, depends on the potential explicitly and the universality is broken down. (It is straightforward to show that the above expression is equal to a special case of that obtained in section 4.2). In this simple case of potential, the extra term is factorized into functions of $x$ and $y$ and we may say that there is still universality in a weak sense. As we discussed in this chapter, if the potential term is more complicated, other products of functions at $x$ and $y$ are added and this weak universality is also gradually broken.

Our result can be also generalized to potentials containing products of more than two traces. The two point correlation function is again given by the universal form plus a sum of products of a function of $x$ and that of $y$.

In this case we have to solve a non-linear equation to obtain these functions.

To conclude we have shown that the two point correlation function is no longer universal in the strong sense and it depends on the details of the potentials. This implies that, when we apply it to the random matrix models for transmission matrix of a quantum wire the conductance fluctuation is not exactly universal. The amplitude of the fluctuation might depend on the system size or disorder strength. Also the variance of the conductance does not decrease exactly by a factor of two when magnetic field is applied.

# Appendix A. Coherent state representation

We list the definitions and basic properties of the coherent state representation which we use throughout the paper:

$$|z\rangle = e^{\hat{a}^\dagger z}|0\rangle, \quad \langle z| = \langle 0|e^{\hat{a}\bar{z}}, \quad \langle z'|z\rangle = e^{\bar{z}' z}$$
$$\hat{a}|z\rangle = z|z\rangle, \quad \langle z|\hat{a}^\dagger = \langle z|\bar{z}, \quad \int d^2 z e^{-|z|^2}|z\rangle\langle z| = 1, \quad d^2 z \equiv \frac{dRez\ dImz}{\pi} \tag{A.1}$$

There is important identity:

$$e^{-|z|^2}\sum_n \frac{1}{n!}\Big((\partial_z - \bar{z})^n|z\rangle\Big)\Big((\partial_{\bar{z}} - z)^n\langle z|\Big) = 1. \tag{A.2}$$

The (A.2) expresses the completeness property of the generators of $W_\infty$ fundamental representation. It can be considered as a generalization of the completeness property of $SU(N)$ generators to the case of $N = \infty$ . One can prove this identity by multiplying both sides by arbitrary ket vector $|z'\rangle$. Then the right hand side is $|z'\rangle$ and the left hand side is equal to:

$$\begin{aligned} & e^{-|z|^2}\sum_n \frac{1}{n!}(\hat{a}^\dagger - \bar{z})^n|z\rangle(\partial_{\bar{z}} - z)^n e^{\bar{z}z'} \\ = & \sum_n \frac{1}{n!}((\hat{a}^\dagger - \bar{z})(z' - z))^n|z\rangle e^{-|z|^2+\bar{z}z'} \\ = & e^{(\hat{a}^\dagger - \bar{z})(z'-z)}|z\rangle e^{-|z|^2+\bar{z}z'} \\ = & e^{\hat{a}^\dagger z'}e^{-\hat{a}^\dagger z}|z\rangle \\ = & |z'\rangle \qquad \text{(QED)} \end{aligned} \tag{A.3}$$

For a given real function $\xi(z,\bar{z}) = \sum_{m,n}\xi_{mn}z^n\bar{z}^m$ we obtain:

$$\xi(\hat{a},\hat{a}^\dagger) = \sum_{m,n}\xi_{mn}\hat{a}^n(\hat{a}^\dagger)^m = \int d^2 z e^{-|z|^2}\sum_{m,n}\xi_{mn}\hat{a}^n|z\rangle\langle z|(\hat{a}^\dagger)^m = \int d^2 z e^{-|z|^2}|z\rangle\xi(z,\bar{z})\langle z| \tag{A.4}$$

A proof of (2.2) goes as follows.

$$
\begin{aligned}
\xi_1(\hat{a},\hat{a}^\dagger)\xi_2(\hat{a},\hat{a}^\dagger) &= \int d^2 z e^{-|z|^2} \int d^2 z' e^{-|z'|^2} |z\rangle \xi_1(z,\bar{z}) e^{\bar{z}z'} \xi_2(z',\bar{z}')\langle z'| \\
&= \int d^2 z e^{-|z|^2} \int d^2 z' e^{-|z'|^2} |z\rangle \xi_1(z,\bar{z}) \xi_2(-\overleftarrow{\partial}_{\bar{z}},\bar{z}') e^{\bar{z}z'} \langle z'| \\
&= \int d^2 z |z\rangle \xi_1(z,\bar{z}) \xi_2(z-\overleftarrow{\partial}_{\bar{z}},\bar{z}) e^{-|z|^2} \langle z| \\
&= \int d^2 z e^{-|z|^2} |z\rangle \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \partial_{\bar{z}}^n \xi_1(z,\bar{z}) \partial_z^n \xi_2(z,\bar{z}) \langle z| \\
&= \ddagger \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \partial_{\bar{z}}^n \xi_1(z,\bar{z}) \partial_z^n \xi_2(z,\bar{z})|_{\substack{z=\hat{a} \\ \bar{z}=\hat{a}^\dagger}} \ddagger
\end{aligned} \tag{A.5}
$$

For any operator $\rho(\hat{a},\hat{a}^\dagger)$ obtained from $\xi(\hat{a},\hat{a}^\dagger)$ via commutation relation $\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$ equation (A.4) is valid with the function $\xi(z,\bar{z}')$ given by

$$
\xi(z,\bar{z}') = e^{-\partial_z \partial_{\bar{z}'}} e^{-z\bar{z}'} \langle z'|\rho(\hat{a},\hat{a}^\dagger)|z\rangle \tag{A.6}
$$

The product of two anti-normal ordered operators coherent state representation is given by the "star" product [18], [19]:

$$
\begin{aligned}
\xi_1(\hat{a},\hat{a}^\dagger)\xi_2(\hat{a},\hat{a}^\dagger) &= \int d^2 z e^{-|z|^2} |z\rangle \xi_1(z,\bar{z}) * \xi_2(z,\bar{z}) \langle z|, \\
\xi_1(z,\bar{z}) * \xi_2(z,\bar{z}) &= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} (\partial_{\bar{z}}^n \xi_1(z,\bar{z}))(\partial_z^n \xi_2(z,\bar{z}))
\end{aligned} \tag{A.7}
$$

The following useful relations are valid:

$$
\int d^2 z e^{-|z|^2} e^{\bar{z}' z} f(\bar{z}) = f(\bar{z}'), \qquad \int d^2 z e^{-|z|^2} e^{\bar{z} z'} f(z) = f(z') \tag{A.8}
$$

Action (2.11) naturally restricts physical operators to the ones which are square integrable in the coherent state representation. Then for any (not necessary) anti-normal ordered operator $A(\hat{a},\hat{a}^\dagger)$ we define the $tr$ as follows:

$$
tr A(\hat{a},\hat{a}^\dagger) = \int d^2 z e^{-|z|^2} \langle z|A(\hat{a},\hat{a}^\dagger)|z\rangle \tag{A.9}
$$

For the trace of the product of two operators we obtain:

$$tr(A(\hat{a},\hat{a}^\dagger)B(\hat{a},\hat{a}^\dagger)) = \int\int d^2z d^2z' e^{-|z|^2-|z'|^2}\langle z|A(\hat{a},\hat{a}^\dagger)|z'\rangle\langle z'|B(\hat{a},\hat{a}^\dagger)|z\rangle =$$
$$= \int\int d^2z d^2z' e^{-|z|^2-|z'|^2}\langle z'|B(\hat{a},\hat{a}^\dagger)|z\rangle\langle z|A(\hat{a},\hat{a}^\dagger)|z'\rangle = tr(B(\hat{a},\hat{a}^\dagger)A(\hat{a},\hat{a}^\dagger)) \tag{A.10}$$

This proves the cyclicity property of the trace which we have heavily used throghout this work.

# Appendix B. Change of variables in quantum mechanics [3]

We start with a standard form for the Hamiltonian and Schrödinger equation which is given by

$$\hat{H}\psi = \Big(-\frac{1}{2}\sum_{a=1}^{N}\frac{\partial^2}{\partial q^{a\,2}} + V(\mathbf{q})\Big)\psi(q) = E\psi(q) \tag{B.1}$$

We consider a transformation given by

$$q^a \longrightarrow Q^a = f^a(q), \quad q^a = F^a(Q) \tag{B.2}$$

We use the chain rule of differentiation to convert the derivatives with respect to $q$'s into derivatives with respect to $Q$'s.

$$-\frac{1}{2}\sum_a \frac{\partial^2}{\partial q^{a\,2}}\psi(q) = \frac{1}{2}\Big(-i\sum_{ab}\frac{\partial^2 f^b}{\partial q^{a\,2}}\frac{\partial}{i\partial Q^b} + \sum_{abc}\frac{\partial f^b}{\partial q^a}\frac{\partial f^c}{\partial q^a}\frac{\partial}{i\partial Q^b}\frac{\partial}{i\partial Q^c}\Big)\psi(F(Q)) \tag{B.3}$$

[3]For more details see [13].

We define

$$\omega^a(Q) \equiv -\sum_b \frac{\partial^2 Q^a}{\partial q^{b2}} = -\sum_b \frac{\partial^2 f^a}{\partial q^{b2}}, \quad \Omega^{ab}(Q) \equiv \sum_c \frac{\partial Q^a}{\partial q^c}\frac{\partial Q^b}{\partial q^c} = \sum_c \frac{\partial f^a}{\partial q^c}\frac{\partial f^b}{\partial q^c}. \tag{B.4}$$

Then we obtain

$$\hat{H}\psi = \left[\frac{i}{2}(i\sum_a \omega^a(Q)P_a + \sum_{ab}\Omega^{ab}(Q)P_aP_b) + \tilde{V}(Q)\right]\psi(F(Q)), \tag{B.5}$$

where we set $\frac{1}{i}\frac{\partial}{\partial Q^a} = P_a, \quad \tilde{V}(Q) \equiv V(F(Q))$.

The Hamiltonian after the change of variables appears to be non-Hermitian if we take the naive Hermitian conjugate: $P_a{}^\dagger = P_a, Q^{a\dagger} = Q^a$. This is because $H$ is Hermitian in the original $q$-space. But after the change of variables, the $Q$-space should be defined by multiplying the wave function by the square root of the Jacobian in order to satisfy the naive Hermitian conjugation prescription:

$$\int dq\, \psi_1^*(q)\psi_2(q) = \int J(Q)\, dQ\, \psi_1^*(F(Q))\,\psi_2(F(Q)) = \int dQ\, \Psi_1^*(Q)\,\Psi_2(Q), \tag{B.6}$$

where $\Psi(Q) = J^{1/2}(Q)\psi(F(Q))$. Then the Hamiltonian in $Q$-space is obtained by a similarity transformation

$$H_{\text{eff}} = J^{1/2}HJ^{-1/2}, \tag{B.7}$$

which should be Hermitian.

It is difficult to calculate the Jacobian in practice while $\omega$ and $\Omega$ defined by (B.4) are relatively easy to compute. So, it would be nice if $H_{\text{eff}}$ is expressed in terms of $\omega$ and $\Omega$. Notice first $J^\dagger(Q) = J(Q^\dagger) = J(Q)$, accordingly

$J^{1/2}P_aJ^{-1/2} = P_a + iC_a(Q)$, where $C_a(Q) = \frac{1}{2}\frac{\partial}{\partial Q^a}\ln J(Q)$ and $C_a^\dagger = C_a$. We obtain

$$H_{\rm eff} = \frac{1}{2}[\, i\sum_a \omega^a(Q)(P_a + iC_a) + \sum_{ab}\Omega^{ab}(P_a + iC_a)(P_b + iC_b)] + \tilde{V}(Q). \quad \text{(B.8)}$$

Since $H_{\rm eff}$ should be Hermitian, $H_{\rm eff} - H_{\rm eff}^\dagger = i\sum_a\{(\omega^a + 2\sum_b \Omega^{ab}C_b + \sum_b \Omega^{ab}_{,b}), P_a\}_+ = 0$, and by taking a commutator bracket with $Q_a$ we obtain

$$\omega^a \ + \ 2\sum_b \Omega^{ab}C_b \ + \ \sum_b \Omega^{ab}_{,b} \ = \ 0. \quad \text{(B.9)}$$

This is the equation that determines $C_a$. Then $H_{\rm eff}$ is computed as

$$H_{\rm eff} \ = \ \frac{1}{2}\sum_{ab}[P_a\Omega^{ab}P_b \ + \ C_a\Omega^{ab}C_b \ + \ (C_a\Omega^{ab})_{,b}] \ + \ \tilde{V} \quad \text{(B.10)}$$

(B.9) and (B.10) are the main results.

# Appendix C

Let us remined definition and some properties of the ensembles we discuss in this work. For a simplest case it is:

$$P(M) \equiv \frac{1}{Z}e^{-N{\rm tr}V(M)}. \quad \text{(C.1)}$$

(C.1) is the orthogonal ensemble ($\lambda = \frac{1}{2}$, see below (C.3)) when the matrix elements $M_{ij}$ are real numbers. This ensemble is invariant under $M \to UMU^{-1}$, where $U$ is an orthogonal matrix ($U \in O(N)$). If however, matrix elements $M_{ij}$ are a complex numbers, $U$ is an unitary ($U \in U(N)$) than (C.1) is a unitary ensemble ($\lambda = 1$). Finally if $U$ is symplectic and $M_{ij}$ are real quaternion numbers (i.e. $2 \times 2$ matrices of the form $a + ib\sigma_x + ic\sigma_y + id\sigma_z$, where $\sigma_x, \sigma_y$ and $\sigma_z$ are the three Pauli spin matrices and $a, b, c$ and $d$ are real numbers) than ensemble (C.1) is called symplectic ($\lambda = 2$). Physically ($\lambda = 1$) case corresponds to broken time reversal (in underlying theory for which matrix $M$ is random Hamiltonian. This case realize in magnetic field or if there is magnetic impurities). Surely, if spin rotation symmetry is broken one has ($\lambda = 2$) simplectic ensemble.

Let us prove now the equation (4.1) [29]. Consider matrix $V$ which diagonalize the matrix $M$ such that $M = V^{-1}XV$, where $X$ is diagonal matrix with eigenvalues $x_i$. Plugging *one*

$$1 \equiv \int \prod_{i=1}^{N} dx_i' dU \; \delta(UMU^{-1} - X') \; \Delta^{2\lambda}(x') \tag{C.2}$$

into the right hand side of (4.1), and performing integration over $M$, we observe that integration over the angular variables $U$ decouples and cancels out with $Vol$ in (4.1). We did not specify the measure $dU$ in (C.2) to keep it general. However we specify it exactly in (C.3).

Let us now find the $\Delta(x)$. Since only the infinitesimal neighborhood $U = (1 + T)V$ of $V$ contributes to the integral (C.2) we find:

$$1 = \int \prod_{i=1}^{N} dx_i' dU \; \delta(UMU^{-1} - X')\Delta^{2\lambda}(x') = \Delta^{2\lambda}(x) \int \prod_{1 \le i < j \le N} dT_{ij} \, \delta(T_{ij}(x_i - x_j)) \tag{C.3}$$

For orthogonal ensemble integration is over real $T_{ij}$ and therefore $\Delta(x) = \prod_{i<j}(x_i - x_j)$ and $\lambda = 1/2$. For unitary ensemble number of integration doubled since in this case $T_{ij}$'s are complex numbers and therefore $\lambda = 1$.

Finally, for simplectic ensemble integration is over quaternions (four real numbers) and therefore $\lambda = 2$ (QED).

We prove the following identity we have used in equation (4.5):

$$ln(\Delta^2) = \sum_{i\neq j}^{N} ln|x_i - x_j| = N^2 \int_{-\infty}^{+\infty} dxdy\rho(x)ln|x-y|\rho(y) + const. \quad (C.4)$$

Regularizing $\delta$-function in the definition of the collective coordinate $\rho(x)$ (see (4.2)) as follows,

$$\rho(x) = \lim_{\epsilon\to 0} \frac{1}{N} \sum_{i=1}^{N} \frac{(\epsilon/\pi)}{(x-x_i)^2 + \epsilon^2}, \quad (C.5)$$

we obtain

$$N^2 \int_{-\infty}^{+\infty} dxdy\rho(x)ln|x-y|\rho(y) =$$
$$= \lim_{\epsilon\to 0} \sum_{i,j}^{N} \int_{-\infty}^{+\infty} dxdy \frac{\epsilon/\pi}{(x-x_i)^2+\epsilon^2}\, ln|x-y|\, \frac{\epsilon/\pi}{(y-x_j)^2+\epsilon^2}. \quad (C.6)$$

The sum can be separated into two pieces: $i \neq j$ and $i = j$. The $i \neq j$ terms lead to the expression for the Van der Monde determinant. The $i = j$ terms become, by setting $\xi = x - x_i$ and $\zeta = y - x_i$,

$$\lim_{\epsilon\to 0} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} dxdy \frac{\epsilon/\pi}{(x-x_i)^2+\epsilon^2}\, ln|x-y|\, \frac{\epsilon/\pi}{(y-x_i)^2+\epsilon^2} =$$
$$= \lim_{\epsilon\to 0} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} d\xi d\zeta \frac{\epsilon/\pi}{\xi^2+\epsilon^2}\, ln|\xi-\zeta|\, \frac{\epsilon/\pi}{\zeta^2+\epsilon^2} = const \quad (C.7)$$

and equation (C.4) is proved (QED).

Due to the following identity

$$f(y) \equiv \frac{1}{\pi} P \int_a^b dx \frac{1}{\sqrt{(x-a)(b-x)}} \frac{1}{y-x} = 0, \qquad a < y < b, \quad (C.8)$$

the inhomogeneous term in $\rho_0(x)$ (equation (4.9) or equation (4.23) ) does not contribute to an integral (4.7). This can be proved as follows. We first define a function

$$F(z) = \frac{1}{\pi}\int_a^b dx \frac{1}{\sqrt{(x-a)(b-x)}}\frac{1}{z-x}. \tag{C.9}$$

Then it follows

$$f(y) = \frac{1}{2}\Big(F(y+i\epsilon) + F(y-i\epsilon)\Big), \qquad \epsilon \to 0. \tag{C.10}$$

Choosing the square root to be positive on the upper side of the cut (and negative on the lower), the line integral (C.9) can be deformed to a contour integral along a path $C$ circling clockwise the cut between $a$ and $b$. $z$ is outside of this contour. Since the integrand vanishes at infinity the contour can be deformed smoothly to wind counterclockwise around $z$;

$$\begin{aligned} F(z) &= \frac{1}{2\pi}\oint_C dx \frac{1}{\sqrt{(x-a)(b-x)}}\frac{1}{z-x} \\ &= \frac{1}{2\pi}\oint_z dx \frac{1}{\sqrt{(x-a)(b-x)}}\frac{1}{z-x} \\ &= i\frac{1}{\sqrt{(z-a)(b-z)}}. \end{aligned} \tag{C.11}$$

$F(z)$ has different signs on the upper and lower sides of the cut and we get $f(x) = 0$.

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