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OBSTRUCTIONS TO COHERENCE

Natural Noncoherent Associativity and Tensor Functors

by

NOSON S. YANOFSKY

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

OBSTRUCTIONS TO COHERENCE:

Natural Noncoherent Associativity and Tensor Functors

by

Noson S. Yanofsky

Advisor: Professor Alex Heller.

In the past few years, categorical coherence questions have arisen in many diverse branches of mathematics like quantum groups, knot theory and proof theory. In this paper, we study what happens when coherence fails. We consider categories with a tensor product and a natural associativity isomorphism that does not necessarily satisfy the pentagon coherence requirements (called Associative Categories). Categorical versions of associahedra, $\mathbf{A}_n$, are constructed (called Catalan groupoids). The objects correspond to associations of letters, and morphisms correspond to reassociations. These groupoids are used in the constructions of the free associative category. They are also used in the construction of the theory of associative categories (given as a 2-sketch). Generators and relations are given for the fundamental group, $\pi(\mathbf{A}_n)$, of the of the Catalan groupoids – thought of as a simplicial complex. These groups are shown to be more then just free groups on the number of pentagons. Each associative category, $\mathbf{B}$, has related fundamental groups $\pi(\mathbf{B}_n)$ and homomorphisms $\pi(P_n) : \pi(\mathbf{A}_n) \longrightarrow \pi(\mathbf{B}_n)$. If the images of the

$\pi(P_n)$ are trivial, i.e. there is only one associativity path between any two objects, then the category is coherent. Otherwise the images of $\pi(P_n)$ are obstructions to coherence. Some progress is made to classifying noncoherence of associative categories.

Functors between associative categories that do not necessarily satisfy the hexagon coherence requirement are dealt with. Much of the same constructions are also done for this coherence problem. The fundamental groups of the associative categories are shown to be related to the fundamental groups of the functors between them.

Acknowledgments

This thesis would not have been possible without the help of a few special people. My dissertation advisor, Professor Alex Heller, was always there for me. His teaching and enthusiasm excited my interest from the beginning. He is a unique model of vision and clarity. His help with this thesis was immense. I thank him for the warm encouragement and patience. I also thank him for the abundance of entertaining and enlightening "schmoozing" over the years. I am forever indebted.

I am also deeply indebted to the entire Mathematics Department of the Graduate Center for creating an environment for which such work is possible. In particular, I would like to thank my colleague and friend Mirco Mannucci. His unbridled fervor is an inspiration to all who know him.

Other debts accrued along the way. In my last year as a Brooklyn College undergraduate, I had the privilege to work on a math project under Professor Chaya Gurewitz. This experience was invaluable to me and kindled my interest in the magical world of mathematics. She has always been there with unfailing support and advice.

To all my friends and family who felt ignored or abandoned while I was wrestling with these "undisciplined diagrams", I apologize. I thank them for standing by my side.

My parents have been a never-ending source of confidence and sustenance. They gave me the intellectual, economic and emotional support as well as the necessary independence needed to make this thesis come into being.

Finally, I would like to thank my dear wife, Malky, for her patience, unfaltering kindness and especially her wonderous smile!

Contents

0.1 Introduction

And new Philosophy calls all in doubt,
The element of fire is quite put out;
The sun is lost, and the earth, and no man's wit
Can well direct him where to look for it.
And freely men confess that this world's spent
When in the planets, and the firmament
They seek so many new; then see that this
Is crumbled out again to his atomies.
'Tis all in pieces, all coherence gone ...

An Anatomie of the World [1611]
John Donne

The history of coherence theory has its beginnings in homotopy theory. In 1963, J. Stasheff [22] investigated the conditions in which an H-space has a homotopy associative multiplication. At around the same time, D.B.A. Epstein [3] came across some associativity questions while dealing with Steenrod operations. With these papers in mind, S. Mac Lane then wrote his classic paper on coherence [16]. He abstracted the problem to the following categorical question: Given a category **B** and a tensor product on it $\otimes : \mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{B}$ that is associative up to a natural isomorphism $\beta_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$ when is there a unique canonical map between two specified formal combinations of objects? In other words, what conditions on $\mathbf{B}, \otimes$ and β insure that all combinations of identities and β between any two objects are the same. Mac Lane answered the question by giving the following condition:

$$\begin{tikzcd}
 & A((BC)D) \arrow[rr, "\beta_{A,B\otimes C,D}"] & & (A(BC))D \arrow[dr, "\beta_{A,B,C}\otimes Id_D"] & \\
A(B(CD)) \arrow[ur, "Id_A\otimes\beta_{B,C,D}"] \arrow[drr, "\beta_{A,B,C\otimes D}"'] & & & & ((AB)C)D \\
 & & (AB)(CD). \arrow[urr, "\beta_{A\otimes B,C,D}"'] & &
\end{tikzcd}$$

If this diagram commutes for every four objects A, B, C and D, then there is only one canonical map between any two objects. This was proved by considering complexes whose vertices were formal combinations of n objects and whose morphisms were formal combinations of β. Mac Lane showed that these complexes were made of squares that commute by naturality and pentagons that commute by hypotheses. Hence, between any two objects there is only one such morphism. An associativity isomorphism that satisfies the above condition is called coherent and categories with a coherent associativity isomorphism (and a unitary requirement) were later called monoidal or tensor categories. Mac Lane finished his paper by asking and answering the same question when the tensor product is also assumed to be commutative up to isomorphism or "symmetric" i.e. there exists a natural isomorphism $\gamma_{A,B} : A \otimes B \longrightarrow B \otimes A$.

D.B.A. Epstein [4] solved the next major coherence question. Given two monoidal categories $(\mathbf{B}, \otimes, \beta)$ and $(\mathbf{B}', \otimes', \beta')$ (he assumed symmetric) and a functor $F : \mathbf{B} \longrightarrow \mathbf{B}'$ with a natural isomorphism $\sigma : F(A) \otimes F(B) \longrightarrow F(A \otimes B)$, when is there a unique canonical morphism between two specified formal combinations of objects in the image of F. In other words, given two morphisms in $\mathbf{B}'$ made of composing and tensoring morphisms of the form $F(\beta)$, β' and σ, when are they the same morphisms? Epstein answered that if

$$\begin{tikzcd}
FA \otimes' (FB \otimes' FC) \arrow[r, "\beta'_{FA,FB,FC}"] \arrow[d, "Id_{FA} \otimes' \sigma_{A,B}"'] & (FA \otimes' FB) \otimes' FC \arrow[d, "\sigma_{A,B} \otimes Id_{FC}"] \\
FA \otimes' F(B \otimes C) \arrow[d, "\sigma_{A,B \otimes C}"'] & F(A \otimes B) \otimes' FC \arrow[d, "\sigma_{A \otimes B,C}"] \\
F(A \otimes (B \otimes C)) \arrow[r, "F(\beta_{A,B,C})"] & F((A \otimes B) \otimes C).
\end{tikzcd}$$

commutes then any two such morphisms are the same. He showed that any such complex was made of pentagons (that commute because $\mathbf{B}$ and $\mathbf{B}'$ are tensor categories), squares (that commute by naturality) and hexagons (that commute by hypothesis.)

People went on to study other coherence questions. There were cartesian closed categories, monoidal closed categories, distributive categories etc. In recent times, coherence problems have arisen in many new and exciting areas of mathematics and mathematical physics. Quantum groups, quantum field theories, linear logic, knot theory etc. are just a few of the diverse areas that now deal with coherence problems. (See Chapter 5 for more examples and some references to current coherence problems.) Basically, categories with structure and coherence conditions are seen as a type of higher dimensional algebras and such algebras are ubiquitous in the mathematics done today.

What seems to have been left out is what happens if a coherence condition fails. Is there any structure that can still be recovered? Is there a hierarchy of coherence conditions? The prototypical example of a situation in which coherence fails is when the category is $\mathbf{R}-\mathbf{Mod}$, the category of left R modules for an arbitrary ring R, the $\otimes$ is the usual tensor product and the β is the *un*usual

$$\beta(a \otimes (b \otimes c)) = -1((a \otimes b) \otimes c).$$

Given this β the above pentagon does not commute. Since β is an isomorphism, we can express this noncommutativity by saying that starting from $A \otimes (B \otimes (C \otimes D))$ and going clockwise around the pentagon, we do not get the identity map. However there is some structure left, namely, going around the pentagon twice does give the identity. The higher complexes do not commute. However, we may be able to say something about the structure of the higher complexes. The goal of this thesis is to explore this structure.

We turn back to homotopy theory to study this higher structure. For every natural number n, we construct a category $\mathbf{A_n}$ whose objects correspond to associations of n letters and whose morphisms correspond to legal instances of reassociations. Since the reassociations are isomorphisms, the $\mathbf{A_n}$'s are in fact groupoids. We call these groupoids the Catalan groupoids.

As with all groupoids, they can be thought of as simplicial complexes: the objects are 0-cells, the morphisms are 1-cells and commuting part will be a 2-cell. One of the main parts of this thesis is to calculate the fundamental groups of the $\mathbf{A_n}$'s and their quotients. If a quotient of the $\mathbf{A_n}$'s is indiscrete (i.e. one morphism between any two objects), then it is called coherent. If there is more then one morphism, then we have an obstruction to coherence.

The second part of this thesis is an exploration of the tensor functor coherence problem. We construct $\mathbf{M_n}$ as the groupoid analogous to $\mathbf{A_n}$. Similar constructions are done to these groupoids. Here is a detailed plan of this thesis.

Chapter 1 begins by defining an associative category. An associative category is a category with a bifunctor that is associative up to a isomorphism that does not necessarily satisfy the above pentagon condition. Examples of such categories are given. The Catalan groupoids, (the $\mathbf{A_n}$s) are constructed. The $\mathbf{A_n}$s are used to construct the free associative category on one generator, $\bar{\mathbf{A}}$. The universal properties of $\bar{\mathbf{A}}$ are proved. We then go on to show that all the $\mathbf{A_n}$ together have the structure of an operad. This operad is used in the construction of $\mathbf{A}$, the 2-sketch of the theory of associative categories. The chapter ends with a short discussion of associative categories that have units.

In Chapter 2, the fundamental groups of each of the $\mathbf{A_n}$ is calculated. To each $\mathbf{A_n}$ we assign a maximal indiscrete subgroupoid $\mathbf{T_n}$, the categorical analogue of assigning a maximal tree to a simplicial complex. The fundamental groups are then obtained by "moding out" the $\mathbf{T_n}$. A way of describing the morphisms of $\mathbf{A_n}$ which are the generators of the group is introduced. The general scheme for the generators and relations are provided and the first seven groups are presented. The seventh group is shown not to be a free group. All higher groups are non-free groups. The chapter ends with a discussion of quotients of the 2-sketch of associative categories. An attempt is made to classify the failure of coherence for both unital and nonunital associative categories.

Chapter 3 looks at the tensor functor coherence problem. Multiplicative functors are tensor functors that do not necessarily satisfy the above hexagon condition. To every multiplicative functor (F, σ) we associate a groupoid $\bar{\mathbf{F}}\sigma$ called the mapping funnel of F. The $\mathbf{M_n}$ groupoids are recursively constructed. The coproduct of all the $\mathbf{M_n}$, called $\bar{\mathbf{M}}$, is shown to be the free mapping funnel. Universal properties of $\bar{\mathbf{M}}$ are proven. The $\mathbf{M_n}$ are combined to create M, a type of 2-sketch of the theory of mapping funnels.

Chapter 4 gives the general scheme for the generators and relations of the fundamental groups of the $\mathbf{M_n}$. Some preliminaries are dealt with and then the scheme is given.

The thesis ends with a chapter that lists some of the possible applications of this work and ways we can go further in the study of the failure of coherence. That Chapter can be read as a continuation of this introduction.

Chapter 1

Associative Categories

1.1 Definitions and examples

Definition 1.1 (Associative Category) *An Associative category is a category* $\mathbf{B}$*, a bifunctor* $\otimes_{\mathbf{B}} : \mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{B}$ *called "tensor", and a natural isomorphism*

$$\beta_{\mathbf{B},\otimes} : \otimes_{\mathbf{B}} \circ (Id_{\mathbf{B}} \times \otimes_{\mathbf{B}}) \longrightarrow \otimes_{\mathbf{B}} \circ (\otimes_{\mathbf{B}} \times Id_{\mathbf{B}})$$

i.e. for every A, B, C *in* $\mathbf{B}$ *an isomorphism*

$$\beta_{\mathbf{B},\otimes,\mathbf{A},\mathbf{B},\mathbf{C}} : A \otimes_{\mathbf{B}} (B \otimes_{\mathbf{B}} C) \longrightarrow (A \otimes_{\mathbf{B}} B) \otimes_{\mathbf{B}} C$$

called the "reassociation".

We reserve the right to abandon the subscripts when there is no concern for ambiguity. Discussions of a unit of the tensor will be left for the end of the chapter. The important point is that we do not make any coherence requirements.

In general, all categories have a composition of morphisms that is associative, however, the name "associative category" is used here because the most interesting feature about our categories, is that their tensors are associative up to isomorphism.

Examples of associative categories abound. Any monoidal category of [17] (also called a tensor category in the literature e.g. [8]) is automatically an associative category. A noncoherent example of an associative category is **R – Mod**, the category of R modules for a commutative ring R. The tensor product is the usual tensor product of modules and the reassociation

$$\beta_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$$

is defined as

$$a \otimes (b \otimes c) \longmapsto \zeta(a \otimes b) \otimes c$$

where $\zeta = -1$ (see [16]) or ζ is, for example, the fifth root of unity.

The above example can be abstracted and put into the language of quantum groups. Let $A = (A, \Delta, \varepsilon)$ be an algebra with a comultiplication and a counit. Let Φ be an invertible element in $A \otimes A \otimes A$ such that

$$(Id \otimes \Delta)(\Delta(a)) = \Phi((\Delta \otimes Id)(\Delta(a)))\Phi^{-1},$$

for all $a \in A$. Then the category, **A – Mod**, of A modules, has the structure of an associative category. The tensor product of two modules is constructed using the comultiplication and the associativity isomorphism is given as

$$\beta_{A,B,C}(a \otimes (b \otimes c)) = \Phi((a \otimes b) \otimes c).$$

If Φ further satisfies

$$[(Id \otimes Id \otimes \Delta)(\Phi)][(\Delta \otimes Id \otimes Id)(\Phi)] = [\Phi_{234}][(Id \otimes \Delta \otimes Id)(\Phi)][\Phi_{123}]$$

where $\Phi_{123} = \Phi \otimes 1$ and $\Phi_{234} = 1 \otimes \Phi$, then $\mathbf{A} - \mathbf{Mod}$ is, in fact, a coherent monoidal category (with proper concern given to units). Such an algebra is called a Drinfeld algebra [21] or a quasi-bialgebra [10] (again, care must be given to units).

We will construct $\bar{\mathbf{A}}$, the free associative category on one generator. Roughly speaking, the objects in the category will be associations of n letters for any positive integer n (the elements of the free "anomic" algebra with one binary operation – sometimes called "magma" – on one generator). Morphisms are called reassociations. The tensor product, $\otimes_{\bar{\mathbf{A}}}$, will concatenate two associations (ie. multiplication in the anomic algebra). The tensor product of reassociations will be defined similarly.

1.2 The A_n

For each positive integer n, we will construct the category – actually a groupoid – $\mathbf{A_n}$ which has as objects associations of n letters and as morphisms reassociations. The free associative category will then be the disjoint union of all the $\mathbf{A_n}$ i.e.

$$\bar{\mathbf{A}} = \coprod_{n \in N+} \mathbf{A_n}.$$

These groupoids will be called the Catalan groupoids. We remind the reader that the Catalan numbers

$$c_n = \frac{\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}}{n}$$

are the number of associations of n letters with no ambiguity in the multiplication (see e.g. [7]). These groupoids are the categorical version of what people who study finite complexes call associahedra (e.g. [26]).

The categories $\mathbf{A_n}$ are "built up" inductively in a manner not unrelated to the way Stasheff's complexes K_n are "built up" ([22]). We let $\mathbf{A_1} = \mathbf{1}$, the trivial category with one object and one identity morphism. Now assume that each of the $\mathbf{A_1}$,$\mathbf{A_2}, \ldots, \mathbf{A_{n-1}}$ is defined. We define $\mathbf{A_n}$ with the following pushout:

$$\begin{tikzcd}
\coprod\limits_{i+j+k=n} \mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \dot{\mathbf{I}} \arrow[r, "W_{i,j,k}"] \arrow[d, hook] & \coprod\limits_{a+b=n} \mathbf{A_a} \times \mathbf{A_b} \arrow[d, "U_{a,b}"] \\
\coprod\limits_{i+j+k=n} \mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \mathbf{I} \arrow[r, "V_{i,j,k}"] & \mathbf{A_n}
\end{tikzcd} \tag{1.1}$$

where i, j, k, a and b range over all positive integers and $\mathbf{I}$ (respectively $\dot{\mathbf{I}}$) is the indiscrete (resp. discrete) category with two objects, 0 and 1. The left hand vertical map is the obvious inclusion. The map $W_{i,j,k}$ is defined for each i, j, and k as follows: let f, g, h be objects in $\mathbf{A_i}, \mathbf{A_j}, \mathbf{A_k}$ respectively then

$$W_{i,j,k}(f,g,h,t) = \begin{cases} f, U_{j,k}(g,h) & : \quad t = 0 \\ U_{i,j}(f,g), h & : \quad t = 1 \end{cases}$$

where the U's are defined from the previous pushouts. $W_{i,j,k}$ is defined for morphisms similarly.

A discussion of what this pushout does is in order. What are the possible ways of associating an n letter word? Each such word must have an "outermost" multiplication i.e. the last two segments of the word to be associated. There are a letters to the left of this multiplication and b letters to the right of this multiplication. Each of these smaller words are also associated. Hence the category $\mathbf{A_a} \times \mathbf{A_b}$. This "outermost" multiplication can occur anywhere within the word, hence the coproduct. All these smaller categories also have reassociations and they are carried over to the to the new $\mathbf{A_n}$. There are, however, new reassociations that are handled by the left-hand side of the pushout. Reassociations are concerned with three smaller words, hence the $\mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k}$. There are two ways of associating these three words into one whole word. $W_{i,j,k}$ maps these three words into the two ways of associating them. The pushout connects these two ways with isomorphisms.

Let's look at the first few $\mathbf{A_n}$. The objects of $\mathbf{A_n}$ are to be thought of as associations of n letters. We write the letters of the associations with non-boldface letters $A, B, C \ldots$ etc. The letters should not be thought of as objects in a category. Rather they are variables or "place holders". We write

$A \otimes (B \otimes C)$ as a shorthand for the functor $(-) \otimes ((-) \otimes (-)) = \otimes \circ (Id \times \otimes)$ from a category cubed to itself. Reassociations will be written as α. We also show how each $\mathbf{A_n}$ is "built up" from the lower groupoids. This is done by writing $\mathbf{A_a} \times \mathbf{A_b}$ for its image under $U_{a,b}$. Similarly for the image of $\mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \mathbf{I}$ under $V_{i,j,k}$.

- $\mathbf{A_1}$ was defined to be $\mathbf{1}$. The reader can think of the category as looking like

$$\mathbf{A_1} = A$$

  where A is just a variable that corresponds to the single identity functor.

- $\mathbf{A_2}$ is defined from the following pushout:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & \mathbf{1} \times \mathbf{1} = \mathbf{1} \arrow[d] \\
\emptyset \arrow[r] & \mathbf{A_2}
\end{tikzcd}$$

  The left side of the pushout is a vacuous coproduct. So

$$\mathbf{A_2} = AB$$

- $\mathbf{A_3}$

$$\begin{tikzcd}
\begin{matrix} \mathbf{A_1} \times \mathbf{A_2} \\ A(BC) \end{matrix} \arrow[r, "\mathbf{A_1}\times\mathbf{A_1}\times\mathbf{A_1}\times\mathbf{I}", "\alpha"'] & \begin{matrix} \mathbf{A_2} \times \mathbf{A_1} \\ (AB)C \end{matrix}
\end{tikzcd}$$

- $\mathbf{A_4}$

$$\begin{tikzcd}
& A((BC)D) \arrow[rr, "\mathbf{A_1\times A_2\times A_1\times I}"] & & (A(BC))D \arrow[dr, "\mathbf{A_3\times A_1}"] & \\
A(B(CD)) \arrow[ur, "\mathbf{A_1\times A_3}"] \arrow[drr, "\mathbf{A_1\times A_1\times A_2\times I}"'] & & & & ((AB)C)D \\
& & \begin{matrix}\mathbf{A_2}\times\mathbf{A_2}\\ (AB)(CD)\end{matrix} \arrow[urr, "\mathbf{A_2\times A_1\times A_1\times I}"'] & &
\end{tikzcd}$$

Note: This diagram does *not* commute.

- $\mathbf{A_5}$ is complicated, and we will print it twice: first with just the vertices and edges. Some of the names of the edges are left out in order to make it more readable.

$$\begin{tikzcd}
 & & & (A(BC))(DE) \arrow[rrr, "\alpha_{A,B,C}\otimes Id_{DE}"] \arrow[dr, "\alpha_{A(BC),D,E}"] & & & ((AB)C)(DE) \arrow[ddd, "\alpha_{(AB)C,D,E}"] \\
 & A((BC)(DE)) \arrow[urr, "\alpha_{A,BC,DE}"] \arrow[dr, "Id_A\otimes\alpha_{BC,D,E}"] & & & & ((A(BC))D)E \arrow[ddr, "(\alpha_{A,B,C}\otimes Id_D)\otimes Id_E"] & \\
 & & A(((BC)D)E) \arrow[rr, "\alpha_{A,(BC)D,E}"] & & (A((BC)D))E \arrow[ur, "\alpha_{A,BC,D}\otimes Id_E"] & & \\
A(B(C(DE))) \arrow[uur, "Id_A\otimes\alpha_{B,C,DE}"] \arrow[ddr] \arrow[ddd, "\alpha_{A,B,C(DE)}"] & & & & & & (((AB)C)D)E \\
 & & A((B(CD))E) \arrow[uu, "Id_A\otimes(\alpha_{B,C,D}\otimes Id_E)"] \arrow[rr, "\alpha_{A,B(CD),E}"'] & & (A(B(CD)))E \arrow[uu, "(Id_A\otimes\alpha_{B,C,D})\otimes Id_E"'] \arrow[dr] & & \\
 & A(B((CD)E)) \arrow[ur, "Id_A\otimes\alpha_{B,CD,E}"] \arrow[drr, "\alpha_{A,B,(CD)E}"] & & & & ((AB)(CD))E \arrow[uur, "\alpha_{AB,C,D}\otimes Id_E"] & \\
(AB)(C(DE)) \arrow[rrr, "Id_{AB}\otimes\alpha_{C,D,E}"'] \arrow[uuuuuurrrrrr, bend left=60, "\alpha_{AB,C,DE}"] \arrow[uuuuuurrrrrr, bend right=60, "\alpha_{AB,C,DE}"'] & & & (AB)((CD)E) \arrow[urr, "\alpha_{AB,CD,E}"] & & &
\end{tikzcd}$$

Notice the map $\alpha_{AB,C,DE}$ is shown twice: once going around the top and once going around the bottom. There is only one such map and if the reader "bends" the page, he will see that $\mathbf{A}_5$ is really a sphere.

We also print it with the subgroupoids marked off.

$$\begin{tikzcd}
 & & & (A(BC))(DE) \arrow[rrr, "\mathbf{A}_3\times\mathbf{A}_2"] \arrow[rr, "\mathbf{A}_3\times\mathbf{A}_1\times\mathbf{A}_1\times\mathbf{I}"] \arrow[rrd, "\mathbf{A}_3\times\mathbf{A}_1\times\mathbf{A}_1\times\mathbf{I}"'] & & & ((AB)C)(DE) \arrow[ddd, "\alpha_{(AB)C,D,E}"] \\
 & A((BC)(DE)) \arrow[rru, "\mathbf{A}_1\times\mathbf{A}_2\times\mathbf{A}_2\times\mathbf{I}"'] \arrow[rd, "Id_A\otimes\alpha_{BC,D,E}"] & & & & ((A(BC))D)E \arrow[rdd, "(\alpha_{A,B,C}\otimes Id_D)\otimes Id_E"] & \\
 & & A(((BC)D)E) \arrow[rr, "\alpha_{A,(BC)D,E}"] & & (A((BC)D))E \arrow[ru, "\alpha_{A,BC,D}\otimes Id_E"] & & \\
A(B(C(DE))) \arrow[ruu, "Id_A\otimes\alpha_{B,C,DE}"] \arrow[ddd, "\alpha_{A,B,C(DE)}"'] \arrow[rdd] & \mathbf{A}_1\times\mathbf{A}_4 & & \mathbf{A}_1\times\mathbf{A}_3\times\mathbf{A}_1\times\mathbf{I} & & \mathbf{A}_4\times\mathbf{A}_1 & (((AB)C)D)E \\
 & & A((B(CD))E) \arrow[uu] \arrow[rr, "\alpha_{A,B(CD),E}"'] & & (A(B(CD)))E \arrow[uu] \arrow[rd] & & \\
 & A(B((CD)E)) \arrow[ru, "Id_A\otimes\alpha_{B,CD,E}"'] \arrow[rrd, "\mathbf{A}_1\times\mathbf{A}_1\times\mathbf{A}_3\times\mathbf{I}"'] & & & & ((AB)(CD))E \arrow[ruu, "\alpha_{AB,C,D}\otimes Id_E"'] & \\
(AB)(C(DE)) \arrow[rrr, "\mathbf{A}_2\times\mathbf{A}_3"'] \arrow[uuuuuurrrrrr, bend left=60, "\mathbf{A}_2\times\mathbf{A}_1\times\mathbf{A}_2\times\mathbf{I}"] \arrow[uuuuuurrrrrr, bend right=60, "\mathbf{A}_2\times\mathbf{A}_1\times\mathbf{A}_2\times\mathbf{I}"'] & & & (AB)((CD)E) \arrow[rru, "\mathbf{A}_2\times\mathbf{A}_2\times\mathbf{A}_1\times\mathbf{I}"] & & &
\end{tikzcd}$$

$\mathbf{A_1} \times \mathbf{A_1} \times \mathbf{A_3} \times \mathbf{I}$ in the lower left corner describes the lower left quadrilateral. Similarly for $\mathbf{A_3} \times \mathbf{A_1} \times \mathbf{A_1} \times \mathbf{I}$.

1.3 The free associative category

We are now ready to define the free associative category on one generator.

$$(\bar{\mathbf{A}} = \coprod_{n \in N^+} \mathbf{A_n}, \otimes_{\bar{\mathbf{A}}}, \bar{\alpha}).$$

The tensor product, $\otimes_{\bar{\mathbf{A}}}$, is defined as follows: given $\phi : f \longrightarrow f'$ in $\mathbf{A_q}$, $\gamma : g \longrightarrow g'$ in $\mathbf{A_r}$ and $\eta : h \longrightarrow h'$ in $\mathbf{A_s}$ then $f \otimes_{\bar{\mathbf{A}}} g = U_{q,r}(f, g)$ in $\mathbf{A_{q+r}}$. The tensor of morphisms is defined as follows: $\phi \otimes_{\bar{\mathbf{A}}} \gamma = U_{q,r}(\phi, \gamma)$ in $\mathbf{A_{q+r}}$. Since $U_{q,r}$ is a functor, $\otimes_{\bar{\mathbf{A}}}$ is defined. The reassociation is given by

$$\bar{\alpha}_{f,g,h} = V_{q,r,s}(Id_f, Id_g, Id_h, \iota) : f \otimes_{\bar{\mathbf{A}}} (g \otimes_{\bar{\mathbf{A}}} h) \longrightarrow (f \otimes_{\bar{\mathbf{A}}} g) \otimes_{\bar{\mathbf{A}}} h$$

where ι is the unique nontrivial isomorphism in $\mathbf{I}$. Naturality means that the following diagram commutes

$$\begin{tikzcd}
f \otimes_{\bar{\mathbf{A}}} (g \otimes_{\bar{\mathbf{A}}} h) \arrow[r, "\bar{\alpha}_{f,g,h}"] \arrow[d, "\phi \otimes_{\bar{\mathbf{A}}} (\gamma \otimes_{\bar{\mathbf{A}}} \eta)"'] & (f \otimes_{\bar{\mathbf{A}}} g) \otimes_{\bar{\mathbf{A}}} h \arrow[d, "(\phi \otimes_{\bar{\mathbf{A}}} \gamma) \otimes_{\bar{\mathbf{A}}} \eta"] \\
f' \otimes_{\bar{\mathbf{A}}} (g' \otimes_{\bar{\mathbf{A}}} h') \arrow[r, "\bar{\alpha}_{f',g',h'}"'] & (f' \otimes_{\bar{\mathbf{A}}} g') \otimes_{\bar{\mathbf{A}}} h'.
\end{tikzcd}$$

The simple observation that $\phi \otimes_{\bar{\mathbf{A}}} (\gamma \otimes_{\bar{\mathbf{A}}} \eta) = V_{q,r,s}(\phi, \gamma, \eta, Id_0)$ with a similar identity for the right vertical map and the fact that $V_{q,r,s}$ is a functor shows that the square indeed commutes.

We emphasize that the reassociation $\bar{\alpha}$ is not coherent.

The following construction will be important for both this chapter and Chapter 3.

There is a well-known categorical principle that in any category $\mathbf{C}$, $Hom_{\mathbf{C}}(X, G)$ inherits the structure of the codomain object. If a category $\mathbf{B}$ has some higher-order structure (we remain suitably ambiguous) then, loosely speaking,

$$\coprod_{n\in \mathbf{N}^+} \mathbf{Hom}_{\mathbf{Cat}}(\mathbf{X}^{\mathbf{n}}, \mathbf{B})$$

inherits this structure. In our case we set $\mathbf{X} = \mathbf{B}$. We claim that if $\mathbf{B}$ has the structure of an associative category then so does

$$\bar{\mathbf{B}} = \coprod_{n\in \mathbf{N}^+} \mathbf{Hom}_{\mathbf{Cat}}(\mathbf{B}^{\mathbf{n}}, \mathbf{B}).$$

Since this construction will be used often, we feel obliged to go through the gory details at least once. The tensor and reassociation are defined as follows. Let $f : \mathbf{B}^{\mathbf{q}} \longrightarrow \mathbf{B}$, $g : \mathbf{B}^{\mathbf{r}} \longrightarrow \mathbf{B}$ and $h : \mathbf{B}^{\mathbf{s}} \longrightarrow \mathbf{B}$. Then

$$f \otimes_{\bar{\mathbf{B}}} g = \otimes_{\mathbf{B}} \circ (f \times g) : \mathbf{B}^{\mathbf{q}} \times \mathbf{B}^{\mathbf{r}} \xrightarrow{f\times g} \mathbf{B} \times \mathbf{B} \xrightarrow{\otimes_{\mathbf{B}}} \mathbf{B}.$$

The reassociation

$$\begin{tikzcd}
\mathbf{B}^{\mathbf{q+r+s}} \arrow[rr, bend left, "f\otimes_{\bar{\mathbf{B}}}(g\otimes_{\bar{\mathbf{B}}}h)"] \arrow[rr, bend right, "(f\otimes_{\bar{\mathbf{B}}}g)\otimes_{\bar{\mathbf{B}}}h"'] \arrow[rr, phantom, "\Downarrow \bar{\beta}_{f,g,h}"] & & \mathbf{B}
\end{tikzcd}$$

is set to

$$\bar{\beta}_{f,g,h}(b_1, b_2, \ldots, b_{q+r+s}) = \beta_{f(b_1,\ldots,b_q),g(b_{q+1},\ldots,b_{q+r}),h(b_{q+r+1},\ldots,b_{q+r+s})}.$$

We have proven the following.

Lemma 1.1 *Given an associative category* $(\mathbf{B}, \otimes_{\mathbf{B}}, \beta_{\mathbf{B}})$*, then* $(\bar{\mathbf{B}}, \otimes_{\bar{\mathbf{B}}}, \bar{\beta})$ *also has the an associative category structure.*

Definition 1.2 *Given an associative category* $(\mathbf{B}, \otimes_{\mathbf{B}}, \beta_{\mathbf{B}})$*, the category of iterates of* $\otimes_{\mathbf{B}}$*, denoted* $\overline{\mathbf{It}(\mathbf{B}, \otimes_{\mathbf{B}}, \beta_{\mathbf{B}})}$ *or* $\mathbf{It}(\bar{\mathbf{B}})$*, is the associative subcategory of* $(\bar{\mathbf{B}}, \otimes_{\bar{\mathbf{B}}}, \bar{\beta})$ *generated by* $Id_B \in \mathbf{Hom}(\mathbf{B}, \mathbf{B})$.

$\mathbf{It}(\bar{\mathbf{B}})$ can be looked at as a disjoint union $\coprod_{n \in \mathbf{N}^+} \mathbf{It}(\mathbf{B})_n$ where $\mathbf{It}(\mathbf{B})_n$ is constructed recursively. $\mathbf{It}(\mathbf{B})_1 = Id_{\mathbf{B}}$. $\mathbf{It}(\mathbf{B})_n$ has as objects $\otimes_{\mathbf{B}} \circ (f \times g)$ where f is an object in $\mathbf{It}(\mathbf{B})_a$, g is an object in $\mathbf{It}(\mathbf{B})_b$ and $a + b = n$. Morphisms of $\mathbf{It}(\mathbf{B})_n$ are generated like the objects, however, there are also morphisms inherited from the associative category structure of $\bar{\mathbf{B}}$.

Definition 1.3 (Strict Tensor Functor) *Given two associative categories* $(\mathbf{B}, \otimes, \beta)$ *and* $(\mathbf{B}', \otimes', \beta')$, *a strict tensor functor between them is a functor* $F : \mathbf{B} \longrightarrow \mathbf{B}'$ *satisfying the following two requirements:*

i) $F(A \otimes B) = F(A) \otimes' F(B)$ *and*

ii)$F(\alpha_{A,B,C}) = \alpha'_{FA,FB,FC}$

Proposition 1.1 (Universality of $\bar{\mathbf{A}}$) *For every associative category* $(\mathbf{B}, \otimes, \beta)$, *there is a unique strict tensor functor*

$$P^{\mathbf{B}} : (\bar{\mathbf{A}} = \coprod \mathbf{A}_n) \longrightarrow (\mathbf{It}(\bar{\mathbf{B}}) = \coprod \mathbf{It}(\mathbf{B})_n)$$

such that

i) $* \in \mathbf{A}_1 \longmapsto Id_{\mathbf{B}} \in \mathbf{It}(\mathbf{B})_1$

ii) $* \in \mathbf{A}_2 \longmapsto \otimes \in \mathbf{It}(\mathbf{B})_2$

iii) ι *(the isomorphism in* $\mathbf{A}_3$*)* $\longmapsto$ β *(the isomorphism in* $\mathbf{It}(\mathbf{B})_3$*)*

Proof. Any functor $S : \bar{\mathbf{A}} \longrightarrow \mathbf{It}(\bar{\mathbf{B}})$ has as its source a coproduct and hence can be described by its components $S_n : \mathbf{A}_n \longrightarrow \mathbf{It}(\bar{\mathbf{B}})$. Since we require the first summand $\mathbf{A}_1$ to go into the first summand, $\mathbf{It}(\mathbf{B})_1$, and S is a strict tensor functor, a quick induction shows that each S_n actually lands in the

n-th summand of $\mathbf{It}(\bar{\mathbf{B}})$ i.e. $\mathbf{It}(\mathbf{B})_\mathbf{n}$. By the requirements of the theorem P_1, P_2 and P_3 are forced. We construct and show uniqueness of P_n with the following argument. Assume $P_1, P_2, \ldots, P_{n-1}$ are defined. Then P_n is the unique functor making the following pushout:

$$\begin{tikzcd}
\coprod_{i+j+k=n} \mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \dot{\mathbf{I}} \arrow[r, "W_{i,j,k}"] \arrow[d, hook] & \coprod_{a+b=n} \mathbf{A_a} \times \mathbf{A_b} \arrow[d, "U_{a,b}"] \arrow[ddr, "\varphi_{a,b}"] & \\
\coprod_{i+j+k=n} \mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \mathbf{I} \arrow[r, "V_{i,j,k}"] \arrow[drr, "\psi_{i,j,k}"] & \mathbf{A_n} \arrow[dr, dashed, "P_n"] & \\
 & & \mathbf{It}(\mathbf{B})_\mathbf{n}
\end{tikzcd}$$

commutative where

$$\varphi_{a,b}(f,g) = P_a(f) \otimes_{\bar{\mathbf{B}}} P_b(g) = P_2(*)(P_a(f) \times P_b(g)$$

$$\psi_{i,j,k}(f,g,h,0) = P_i(f) \otimes_{\bar{\mathbf{B}}} (P_j(g) \otimes_{\bar{\mathbf{B}}} P_k(h)) = P_3(0)(P_i(f) \times P_j(g) \times P_k(h))$$

$$\psi_{i,j,k}(f,g,h,1) = (P_i(f) \otimes_{\bar{\mathbf{B}}} P_j(g)) \otimes_{\bar{\mathbf{B}}} P_k(h) = P_3(1)(P_i(f) \times P_j(g) \times P_k(h))$$

$$\psi_{i,j,k}(\phi,\gamma,\eta,\iota) = \bar{\beta}_{P_i(\phi),P_j(\gamma),P_k(\eta)} = P_3(\iota)(P_i(\phi) \times P_j(\gamma) \times P_k(\eta)).$$

A simple diagram trace shows that the outer square commutes. The pushout insures that there is a unique map $P_n : \mathbf{A_n} \longrightarrow \mathbf{It}(\mathbf{B})_\mathbf{n}$.

In order to show that P is a strict tensor, let f and g be objects in $\mathbf{A_a}$

and $\mathbf{A_b}$ respectively. The following square commutes.

$$\begin{tikzcd}
f,g \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & U_{a,b}(f,g) \arrow[ddd, mapsto] \\
& \bar{\mathbf{A}} \times \bar{\mathbf{A}} \arrow[r, "\otimes_{\bar{\mathbf{A}}}"] \arrow[d, "P\times P"'] & \bar{\mathbf{A}} \arrow[d, "P"] & \\
& \bar{\mathbf{B}} \times \bar{\mathbf{B}} \arrow[r, "\otimes_{\bar{\mathbf{B}}}"'] & \mathbf{It}(\bar{\mathbf{B}}) & \\
P_a(f), P_b(g) \arrow[rrr, mapsto] & & & \begin{array}{c} P_n(U_{a,b}(f,g)) = \\ P_a(f) \otimes_{\bar{\mathbf{B}}} P_b(g) \end{array}.
\end{tikzcd}$$

The lower right-hand equality holds because that is just the upper right-hand triangle in the previous pushout diagram. Similar arguments are needed for morphisms. P's uniqueness follows from the fact that φ and ψ are used in the proof of P being strict.□

1.4 The 2-sketch of associative categories

In our discussion, when we refer to partitions of a set X, we mean an *ordered* set of disjoint nonempty subsets of X. The following fact will be helpful for notation: Every (ordered) partition of t objects into $n \leq t$ disjoint subsets can be written as an order-preserving surjection $\pi : t \longrightarrow n$.

In order to define the 2-sketch, we need to give the $\mathbf{A_n}$ an operad structure. Basically an operad is a way of combining associations. Given an association of n letters and n associations of $m_1, m_2, \ldots, m_n$ letters, an operad makes a new association of $t = \sum m_i$ letters by considering each of the m_i letters to be one unit of the original association. We basically follow May's [19] definition of a topological A_∞ operad. We leave out his requirements for the unit and the symmetric groups operation on the operad.

Proposition 1.2 $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}, \ldots$ *has the structure of an operad.*

Proof. We need to define a functor

$$Q_{n,t,\pi} : \mathbf{A_n} \times \mathbf{A_{m_1}} \times \mathbf{A_{m_2}} \times \cdots \times \mathbf{A_{m_n}} \longrightarrow \mathbf{A_t}$$

for all n, $t \geq n$ and partitions $\pi : t \longrightarrow n$ where $m_i = |\pi^{-1}(i)|$. We define the $Q_{n,t,\pi}$'s by induction on n. For $n = 1$, and for any $t \geq 1$ there is a unique $\pi : t \longrightarrow 1$. We set

$$Q_{1,t,\pi} = Proj : \mathbf{A_1} \times \mathbf{A_t} \longrightarrow \mathbf{A_t}.$$

Now assume every $Q_1, Q_2, \ldots, Q_{n-1}$ is defined for each t and partition π. We define $Q_{n,t,\pi}$ for a given t and π on 0)objects and 1)morphisms.

0) Let f be an object in $\mathbf{A_n}$ and $g_1, g_2, \ldots, g_n$ be objects in $\mathbf{A_{m_1}}, \ldots, \mathbf{A_{m_n}}$ respectively. Since each object f in $\mathbf{A_n}$ has the property that $f = U_{a,b}(f_1, f_2)$ $(a + b = n)$ for unique a, b, f_1 and f_2. We let

$$Q_{n,t,\pi}(f, g_1, g_2, \ldots, g_n) = \tag{1.2}$$

$$U_{a,b}(Q_{a,t_1,\pi_1}(f_1, g_1, g_2, \ldots, g_a), Q_{b,t_2,\pi_2}(f_2, g_{a+1}, g_{a+2}, \ldots, g_{a+b})).$$

where

$$t_1 = \sum_{i=1}^{a} m_i = \sum_{i=1}^{a} |\pi^{-1}(i)|, \qquad t_2 = \sum_{i=a+1}^{a+b} m_i = \sum_{i=a+1}^{a+b} |\pi^{-1}(i)|$$

and π_1 amd π_2 are the restrictions of π to a and b.

Since $Q_{n,t,\pi}$ is a functor, it suffices to define it on the generating morphisms of $\mathbf{A_n}$. 1)morphisms in $\mathbf{A_n}$ of the form $\phi = U_{a,b}(\phi_1, \phi_2)$ are done similarly.

1')morphisms in $\mathbf{A_n}$ of the form $\phi = V_{i,j,k}(\phi_1, \phi_2, \phi_3, \iota)$ are handled in the following manner: Let

$$t_i = \sum_{x=1}^{i} m_x, \qquad t_j = \sum_{x=i+1}^{i+j} m_x, \qquad t_k = \sum_{x=i+j+1}^{i+j+k} m_x.$$

Then

$$Q_{n,t,\pi}(\phi, \gamma_1, \gamma_2, \ldots, \gamma_n) = \tag{1.3}$$

$$V_{t_i,t_j,t_k}(Q_{i,t_i,\pi_i}(\phi_1, \gamma_1, \ldots, \gamma_i), Q_{j,t_j,\pi_j}(\phi_2, \gamma_{i+1}, \ldots, \gamma_{i+j}), Q_{k,t_k,\pi_k}(\phi_3, \gamma_{i+j+1}, \ldots, \gamma_n), \iota)$$

An example of the way the operad works with associations is in order. Let $n = 4$ and $t = 13$. Let $f \in \mathbf{A_4}$ correspond to the following association: $A[[BC]D]$. (We use square brackets because the contrast is easier to see. There is no distinction made.) Let g_1, g_2, g_3 and g_4 correspond to the following associations:

$$A(BC), \quad D, \quad (EF)((GH)I), \quad ((JK)L)M.$$

Then $Q(f, g_1, g_2, g_3, g_4)$ corresponds to

$$A(BC)[[D((EF)((GH)I))][((JK)L)M].$$

For an example of the way reassociations are handled, let ϕ correspond to the reassociation $A[[BC]D] \longrightarrow [AB][CD]$ and

$$\begin{tikzcd}
A(BC) \arrow[d, "\gamma_1"] & D \arrow[d, "\gamma_2"] & (EF)((GH)I) \arrow[d, "\gamma_3"] & (JK)L)M \arrow[d, "\gamma_4"] \\
(AB)C & D & ((EF)(GH))I & J(K(LM)).
\end{tikzcd}$$

Then the operad will produce:

$$\begin{tikzcd}
A(BC)[[D((EF)((GH)I))](JK)L)M] \arrow[d, "Q(\phi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)"] \\
{[((AB)C)D][(((EF)(GH))I)(J(K(LM)))]}.
\end{tikzcd}$$

In order for the $Q_{n,t,\pi}$'s to have an operad structure, the following "associativity" lemma must be true. We abandon all unnecessary subscripts for the benefit of the reader. $\vec{f}$ is used to mean a sequence of f's of arbitrary length. The lemma for morphisms is left to the reader.

Lemma 1.2 (Associativity of Q) *Let h be an object in $\mathbf{A_n}$. Let g_i be objects $\mathbf{A_{m_i}}$. We have the following equality:*

$$Q(h, Q(g_1, \vec{f}), \ldots, Q(g_n, \vec{f})) = Q(Q(h, g_1, \ldots, g_n), \vec{f})$$

Proof. By induction on n. If $n = 1$ then $h = * \in \mathbf{A_1}$ and

$$LHS = Q(*, Q(g_1, f, \ldots, f)) = Q(g_1, f, \ldots, f) = Q(Q(*, g_1), f, \ldots, f) = RHS.$$

Assume the lemma is true for all $a < n$. Then $h \in \mathbf{A_n}$ and $h = U_{a,b}(h_a, h_b)$ for unique a, b, h_a and h_b. And

$$\begin{aligned}
LHS &=_1 Q(h, Q(g_1, \vec{f}), \ldots, Q(g_n, \vec{f})) \\
&=_2 Q(U_{a,b}(h_a, h_b), Q(g_1, \vec{f}), \ldots, Q(g_n, \vec{f})) \\
&=_3 U_{a,b}(Q_a(h_a, Q(g_1, \vec{f}), \ldots, Q(g_a, \vec{f})), Q_b(h_b, Q(g_{a+1}, \vec{f}), \ldots, Q(g_{a+b}, \vec{f}))) \\
&=_4 U_{a,b}(Q(Q_a(h_a, g_1, \ldots, g_a), \vec{f}), Q(Q_b(h_b, g_{a+1}, \ldots, g_{a+b}), \vec{f})) \\
&=_5 Q(U(Q_a(h_a, g_1, \ldots, g_a), Q_b(h_b, g_{a+1}, \ldots, g_{a+b})), \vec{f}) \\
&=_6 Q(Q(U_{a,b}(h_a, h_b), g_1, \ldots, g_{a+b}), \vec{f}) \\
&=_7 Q(Q(h, g_1, \ldots, g_n), \vec{f}) \\
&= RHS.
\end{aligned}$$

$=_2$ and $=_7$ are from the definition of h. $=_3$, $=_5$ and $=_6$ are from definition of Q. $=_4$ is from the induction hypothesis.□

A 2-sketch (called an algebraic 2-sketch in [6]) is a strict tensor 2-category whose underlying category (0-cells and 1-cells) is a sketch i.e. a sketch that is enriched over **Cat**. An algebra F for a 2-sketch $\mathbf{G}$ is a strict tensor 2-functor $F : \mathbf{G} \longrightarrow \mathbf{Cat}$.

At last, we are ready to define the 2-sketch, $\mathbf{A}$, of the theory of associative categories. $\mathbf{A}$ is a strict tensor 2-category. The objects are the positive natural numbers. In order for $\mathbf{A}$ to be a 2-category, it must be enriched over Cat i.e. every hom set must be a category and composition must be a functor. Given any two positive integers n and k, we define the category

$$\mathbf{Hom_A(n,k)} = \coprod_{n_1+n_2+\cdots+n_k=n} \mathbf{A_{n_1}} \times \mathbf{A_{n_2}} \times \cdots \times \mathbf{A_{n_k}} = \coprod_{\pi:n\to k} \prod_{i=1}^{k} \mathbf{A}_{|\pi^{-1}(\mathbf{i})|}$$

where π ranges over all partitions of n into k parts. Notice that

$$\mathbf{Hom_A(n,k)} = \begin{cases} \mathbf{A_n} & : \ k=1 \\ (\mathbf{A_1})^{\mathbf{k}} = \mathbf{A_1} & : \ k=n \\ \emptyset & : \ k>n \end{cases} .$$

Each object of $\mathbf{Hom_A(n,k)}$ corresponds to a partial association of n letters into k parts. Each of the k parts is totally associated. For example, a typical object in $\mathbf{Hom(10,4)}$ looks like this

$$(AB), \quad C(DE))F, \quad G, \quad H(IJ).$$

This object corresponds to the partition $10 = 2+4+1+3$. We write objects of $\mathbf{Hom_A(n,k)}$ as

$$f = (f_1, f_2, \ldots, f_k)$$

where each f_i is an object of $\mathbf{A}_{|\pi^{-1}(\mathbf{i})|}$. Morphisms of $\mathbf{Hom_A(n,k)}$ correspond to reassociations of partial associations. Since $\mathbf{Hom_A(n,k)}$ is made up of a disjoint union, there are only reassociations of partial associations *of the same partition.* Morphisms are written as

$$\phi = (\phi_1, \phi_2, \ldots, \phi_k).$$

Composition: Consider the following situation:

$$\begin{tikzcd}
n \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "f'"{name=M}] \arrow[r, bend right=60, "f''"'{name=D}] & k \arrow[r, bend left=60, "g"{name=U2}] \arrow[r, "g'"{name=M2}] \arrow[r, bend right=60, "g''"'{name=D2}] & l.
\arrow[Rightarrow, from=U, to=M, "\phi"]
\arrow[Rightarrow, from=M, to=D, "\phi'"]
\arrow[Rightarrow, from=U2, to=M2, "\gamma"]
\arrow[Rightarrow, from=M2, to=D2, "\gamma'"]
\end{tikzcd}$$

Let $f = (f_1, f_2, \ldots, f_k)$ with its corresponding partition $\pi_f : n \longrightarrow k$. Let $g = (g_1, g_2, \ldots, g_l)$ with its partition $\pi_g : k \longrightarrow l$. Then horizontal composition, $\circ_H$, is defined as

$$g \circ_H f = h = (h_1, h_2, \ldots, h_l)$$

with corresponding partition $\pi_h = \pi_g \circ \pi_f : n \longrightarrow l$, where

$$h_i = Q_{|\pi_g^{-1}(i)|,|\pi_h^{-1}(i)|,\pi_f|}(g_i, f_{i_1}, f_{i_2}, \ldots, f_{i_s}).$$

The i_j range over $\pi_h^{-1}(i)$ and $\pi_f|$ is a restriction of π_f to this subset. An example is called for. Let $n = 10$, $k = 4$ and $l = 2$. Let f correspond to

$$(A(BC)), \quad D, \quad (E(F(GH))), \quad (IJ).$$

Let g correspond to

$$[AB], \quad [CD].$$

Then $g \circ_H f$ corresponds to

$$[(A(BC)) \quad D], \qquad [(E(F(GH))) \quad (IJ)].$$

Horizontal composition of 2-cells are also done with the operad Q and the details are left to the reader. Associativity of horizontal composition follows from lemma 1.2. We leave the details to the reader, however, it is obvious once you look at what the Q's are constructed to do.

Vertical composition of 2-cells, $\circ_V$, is much simpler. Let $\phi = (\phi_1, \phi_2, \ldots, \phi_k)$ be of a reassociation of a particular partition and let $\phi' = (\phi'_1, \phi'_2, \ldots, \phi'_k)$ be of the *same* partition, then

$$\phi' \circ_V \phi = (\phi'_1 \circ \phi_1, \phi'_2 \circ \phi_2, \ldots, \phi'_k \circ \phi_k).$$

Associativity of vertical composition is obvious.

In order for **A** to be an honest 2-category, horizontal and vertical composition must "commute" i.e.

Lemma 1.3

$$(\gamma' \circ_H \phi') \circ_V (\gamma \circ_H \phi) = (\gamma' \circ_V \gamma) \circ_H (\phi' \circ_V \phi) : g \circ_H f \longrightarrow g' \circ_H f'.$$

Proof. LHS=$(\lambda_1, \lambda_2, \ldots, \lambda_l)$ where

$$\lambda_i = Q_{|\pi_{\gamma'}^{-1}(i)|,m,\pi_{\phi'}|}(\gamma'_i, \phi'_{i_1}, \phi'_{i_2}, \ldots, \phi'_{i_m}) \circ Q_{|\pi_{\gamma}^{-1}(i)|,m,\pi_{\phi}|}(\gamma_i, \phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_m}).$$

RHS=$(\rho_1, \rho_2, \ldots, \rho_l)$ where

$$\rho_i = Q_{|\pi_{\gamma}^{-1}(i)|,m,\pi_{\phi}|}(\gamma'_i \circ \gamma_i, \phi'_{i_1} \circ \phi_{i_1}, \phi'_{i_2} \circ \phi_{i_2}, \ldots, \phi'_{i_m} \circ \phi_{i_m}).$$

Since $\pi_\gamma = \pi_{\gamma'}$ and $\pi_\phi = \pi_{\phi'}$, these two Q's are actually the same functor and by the functoriality of Q, $\lambda_i = \rho_i$ and hence LHS = RHS. □

Proposition 1.3 **A** *has a strict 2-tensor structure,* $\otimes_{\mathbf{A}}$.

Proof.

0) 0-cells: $n \otimes_{\mathbf{A}} m = n + m$.

1) 1-cells: Let $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_m)$ be two 1-cells, then $f \otimes_{\mathbf{A}} g = (f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m)$.

2) 2-cells: This is done the same way as 1-cells. □

In order for the following definition to make sense, we must remind ourselves that **Cat**, the 2-category of categories, functors and natural transformations, has a strict 2-tensor structure: product. (We parenthetically note that product is only coherently associative but we think of it - perhaps in error - as strict because of the usual coherence theories.)

Definition 1.4 *Let* $\mathbf{Hom}_{\mathbf{st}\otimes}(\mathbf{A}, \mathbf{Cat})$ *be the category (we forget, for now, its higher-order structure) whose objects are the strict tensor 2-functors*

$$R : \mathbf{A} \longrightarrow \mathbf{Cat}$$

i.e. the 2-functors that satisfy

0) $R(n \otimes_{\mathbf{A}} m) = R(n) \times R(m)$,

1) $R(f \otimes_{\mathbf{A}} g) = R(f) \times R(g)$ *and*

2) $R(\phi \otimes_{\mathbf{A}} \gamma) = R(\phi) \times R(\gamma)$.

Morphisms are strict 2-natural transformations

$$F : R \Longrightarrow S$$

i.e. 0) For every 0-cell n *in* $\mathbf{A}$ *there is a functor* $F(n) : R(n) \longrightarrow S(n)$ *such that* $F(n \otimes_{\mathbf{A}} m) = F(n) \times F(m)$.

1) For every 1-cell $f : n \longrightarrow m$ in $\mathbf{A}$ the following square commutes "on the nose"

$$\begin{tikzcd}
R(n) \arrow[r, "F(n)"] \arrow[d, "R(f)"'] & S(n) \arrow[d, "S(f)"] \\
R(m) \arrow[r, "F(m)"'] & S(m).
\end{tikzcd}$$

2) For every 2-cell $\phi : f \Longrightarrow f'$ in $\mathbf{A}$, the following square commutes "on the nose"

$$\begin{tikzcd}
R(n) \arrow[r, "F(n)"] \arrow[d, bend right=60, "R(f)"'] \arrow[d, bend left=60, "R(f')"] \arrow[d, phantom, "\overset{\Rightarrow}{R(\phi)}"] & S(n) \arrow[d, bend right=60, "S(f)"'] \arrow[d, bend left=60, "S(f')"] \arrow[d, phantom, "\overset{\Rightarrow}{S(\phi)}"] \\
R(m) \arrow[r, "F(m)"'] & S(m)
\end{tikzcd}$$

Definition 1.5 *The category* $\mathbf{Assoc}-\mathbf{Cat}_{\mathbf{st}}$ *has as objects associative categories and as morphisms, strict tensor functors.*

Proposition 1.4 (A as 2-sketch of associative categories) *The category* $\mathbf{Hom}_{\mathbf{st}\otimes}(\mathbf{A},\mathbf{Cat})$ *is equivalent to* $\mathbf{Assoc}-\mathbf{Cat}_{\mathbf{st}}$.

(This is actually a weak form of a theorem that will be proven at the end of Chapter 3. Strict tensor functors do not show up in the "natural world" and we have to weaken our notion of morphism in both categories in order to make the equivalence more meaningful.)

Proof. The proof calls for only a few minutes of staring at the definitions. We will not go through all the hideous details; but shall point the way. Given a strict tensor 2-functor $R : \mathbf{A} \longrightarrow \mathbf{Cat}$ we set the underlying category, $\mathbf{B}$, to be $R(1)$. Set $\otimes_{\mathbf{B}} = R(f)$ where f is the unique morphism from 2 to 1 in $\mathbf{A}$. $\beta_{\mathbf{B}} = R(\iota)$ where ι is the unique nontrivial isomorphism in $\mathbf{A}_3$.

To a strict 2-natural transformation $F : R \Longrightarrow S$ we assign a strict tensor functor (also called F) $F : R(1) \longrightarrow S(1)$. To every positive natural number $n = 1 + 1 + \cdots + 1$, there is a functor

$$[F(n) = F(1)^n] : [R(n) = R(1)\times\cdots\times R(1) = R(1)^n] \longrightarrow [S(1)\times\cdots\times S(1) = S(1)^n].$$

If we let $R(1) = \mathbf{B}$ and $S(1) = \mathbf{B}'$, then the above line looks like the more familiar

$$F^n : \mathbf{B}^n \longrightarrow \mathbf{B}'^n.$$

The two commuting diagrams in the definition of strict 2-natural transformations correspond to the two requirements for a functor to be a strict tensor functor. □

1.5 Associative categories with units

Definition 1.6 *An associative category with a unit is an associative category* $(\mathbf{B}, \otimes_{\mathbf{B}}, \beta)$ *with a distinguished object* $I \in \mathbf{B}$ *and the following two natural isomorphisms:*

$$L_A : I \otimes A \longrightarrow A \qquad R_A : A \otimes I \longrightarrow A.$$

There are times when the following coherence condition will be important. An associative category with a unit that in which the following diagram commutes is said to have *unital coherence*:

$$\begin{tikzcd}
(A\otimes I)\otimes B \arrow[rr, "\beta_{A,I,B}"] \arrow[dr, "R_A\otimes Id_B"'] & & A\otimes(I\otimes B) \arrow[dl, "Id_A\otimes L_B"] \\
& A\otimes B &
\end{tikzcd}$$

If the L_A and R_A are identity natural isomorphisms then we say the associative category is *unital strict.*

The L_A's and R_A's are morphisms connecting associations of $n+1$ letters to associations of n letters. In the free associative category, they are isomorphisms from the objects of $\mathbf{A_{n+1}}$ to the objects of $\mathbf{A_n}$. These connections can be formalized with a new inductive scheme of pushouts, however we will not do this here. In this new formalism, we generate a new sequence of groupoids $\mathbf{A}'_0, \mathbf{A}'_1, \ldots, \mathbf{A}'_n, \ldots$. Notice that this time we have a $\mathbf{A}'_0$ whereas there is no $\mathbf{A}_0$. Each $\mathbf{A}'_n$ is actually a subgroupoid of $\mathbf{A}'_{n+1}$. The free associative category with unit will not be the coproduct of the $\mathbf{A}'_n$ rather it will be the colimit i.e.

$$\bar{\mathbf{A}}' = Colim_{n\geq 0}\mathbf{A}'_n.$$

The details of the structure of $\bar{\mathbf{A}}'$ and its universal properties are straightforward.

The $\mathbf{A}'_n$ also have the structure of an operad and this operad is used to construct $\mathbf{A}'$, the 2-sketch of associative categories with units. There is one interesting difference that is worth pointing out. For **A** we had

$$\mathbf{Hom_A(n,k)} = \coprod_{n_1+n_2+\cdots+n_k=n} \mathbf{A_{n_1}} \times \mathbf{A_{n_2}} \times \cdots \times \mathbf{A_{n_k}} = \coprod_{\pi:n\to k} \prod_{i=1}^{k} \mathbf{A}_{|\pi^{-1}(i)|}.$$

where the n_i's are positive numbers and, equivalently, the partitions π's are surjective. Here — when talking about units — we allow all nonnegative n_i's and nonsurjective π's. $|\pi^{-1}(i)|$ can equal 0 and we would get $\mathbf{A}'_0$ which corresponds to the unit.

Chapter 2

The Fundamental Groups of the $\mathbf{A_n}$

The goal of this chapter is to calculate the fundamental group of each of the $\mathbf{A_n}$ and its quotients. If the fundamental group of a quotient of $\mathbf{A_n}$ is trivial, then there is only one path between any two vertices (objects) in the complex (groupoid) and we call it "coherent". If it is not coherent, what are the possible structures?

2.1 Maximal indiscrete subgroupoids

In order to determine the fundamental group of a simplicial complex, one can use the method of maximal trees. We employ a method analogous to maximal trees. For a general introduction to maximal tree, we refer the reader to Rotman [20] (he makes a distinction between a simplicial complex and a topological space and calls the fundamental group of a complex an "edgepath group.") For a more advanced treatment of the subject see [15].

Given any complex K, to find the fundamental group of K, denoted $\pi(K)$, one associates a maximal tree, T_K, to K. A tree in K is a connected subcomplex of K which has no circuits. A maximal tree in K is a tree in K contained in no larger tree or, equivalently, a tree that contains all vertices of K. $\pi(K)$ is then given by the following presentation

- Generators
    1. All edges (u, v) in K.
- Relations
    1. $(u, v) = e$ if (u, v) is in T_K.
    2. $(u, v)(v, w) = (u, w)$ if u, v, w lie in a simplex of K.

There is a standard theorem that $\pi(K)$ is (up to conjugation) invariant under a change of the maximal tree.

We are dealing with the Catalan groupoids and shall assign to each groupoid - the categorical analog of a maximal tree - a maximal indiscrete subgroupoid (henceforth MIS). A MIS is a subgroupoid with the same objects and exactly one isomorphism between each ordered pair of objects. So, for each $\mathbf{A_n}$ we shall construct a MIS $\mathbf{T_{A_n}}$ (or $\mathbf{T_n}$). It must be stressed that $\mathbf{T_n}$ is an MIS and not a tree. There may, in fact, be circuits in our MIS but they correspond to a commuting part of the groupoid. (Both the language of trees and the language of categories will be employed. "vertex" and "object" will be interchanged, as will "edge" and "morphism".)

The image of $V_{i,j,k}$ in the pushout 1.1 on page 10 has objects that are in the image of $U_{i,j+k}$ and $U_{i+j,k}$. So the image of $\mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \mathbf{I}$ under $V_{i,j,k}$, loosely speaking, only contributes new morphisms. Each of these morphisms are natural to one another i.e. they are two parallel sides of a square that commutes under naturality. This can be seen by considering the following situation. Let $\phi : f \longrightarrow f'$ in $\mathbf{A_i}$, $\gamma : g \longrightarrow g'$ in $\mathbf{A_j}$ and $\eta : h \longrightarrow h'$ in $\mathbf{A_k}$. Then the following square commutes out of the functoriality of $V_{i,j,k}$

$$\begin{tikzcd}
U(\mathbf{A_i} \times \mathbf{A_{j+k}}) & U(\mathbf{A_{i+j}} \times \mathbf{A_k}) \\
f \otimes (g \otimes h) \arrow[r, "V_{i,j,k}(Id_f, Id_g, Id_h, \iota)"] \arrow[d, "V_{i,j,k}(\phi,\gamma,\eta,Id_0)"'] & (f \otimes g) \otimes h \arrow[d, "V_{i,j,k}(\phi,\gamma,\eta,Id_1)"] \\
f' \otimes (g' \otimes h') \arrow[r, "V_{i,j,k}(Id_{f'}, Id_{g'}, Id_{h'}, \iota)"'] & (f' \otimes g') \otimes h'.
\end{tikzcd}$$

The left (respectively right) side of the diagram is in the image of $\mathbf{A_i} \times \mathbf{A_{j+k}}$ under $U_{i,j+k}$ (resp. $\mathbf{A_{i+j}} \times \mathbf{A_k}$ under $U_{i+j,k}$.) The entire diagram is contained in $\mathbf{A_n}$ where $n = i + j + k$. The point is that the entire image of $V_{i,j,k}$ is really a set of edges of the $\mathbf{A_n}$ and of its $\mathbf{T_n}$. We shall denote this set of edges as $< i, j, k >$. So $< i, j, k >$ can be thought of as a set of morphisms

$$f \otimes (g \otimes h) \xrightarrow{<i,j,k>} (f \otimes g) \otimes h.$$

one for each f, g and h. We may denote the above element of $< i, j, k >$ as $< i, j, k >_{f,g,h}$

The MIS's are defined inductively. $\mathbf{T_1} = \mathbf{A_1} = \mathbf{1}$. Assume $\mathbf{T_1}, \mathbf{T_2}, \ldots, \mathbf{T_{n-1}}$ are defined, then $\mathbf{T_n}$ is constructed from the following now-familiar pushout.

$$\begin{tikzcd}
\coprod\limits_{i+1+k=n} \mathbf{T_i} \times \mathbf{T_1} \times \mathbf{T_k} \times \dot{\mathbf{I}} \arrow[r, "W_{i,1,k}|"] \arrow[d, hook] & \coprod\limits_{a+b=n} \mathbf{T_a} \times \mathbf{T_b} \arrow[d, "U_{a,b}|"] \\
\coprod\limits_{i+1+k=n} \mathbf{T_i} \times \mathbf{T_1} \times \mathbf{T_k} \times \mathbf{I} \arrow[r, "V_{i,1,k}|"] & \mathbf{T_n}
\end{tikzcd}$$

where $W_{i,1,k}|$ is the restriction of $W_{i,j,k}$ of our original pushout. (This is intuitive but actually too swift because we have not yet shown that $\mathbf{T_n}$ is a subcategory of $\mathbf{A_n}$. So define $W|$ in a similar manner to the way W was on page 11). $\mathbf{T_n}$ contains only one class of morphisms between each component. This corresponds to moving the outermost parentheses one place to the left.

Now inductively define

$$L_n : \mathbf{T_n} \longrightarrow \mathbf{A_n}.$$

$L_1 = Id_1 : \mathbf{T_1} \longrightarrow \mathbf{A_1}$. Assuming $L_1, L_2, \ldots, L_{n-1}$ are defined, L_n is then constructed from the following diagram:

$$\begin{tikzcd}
\coprod_{i+j+k=n} \mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \dot{\mathbf{I}} \arrow[rrr, "W_{i,j,k}"] \arrow[ddd, hook] & & & \coprod_{a+b=n} \mathbf{A_a} \times \mathbf{A_b} \arrow[ddd, "U_{a,b}"] \\
& \coprod_{i+1+k=n} \mathbf{T_i} \times \mathbf{T_1} \times \mathbf{T_k} \times \dot{\mathbf{I}} \arrow[ul] \arrow[r] \arrow[d, hook] & \coprod_{a+b=n} \mathbf{T_a} \times \mathbf{T_b} \arrow[ur] \arrow[d, "U_{a,b}|"] & \\
& \coprod_{i+1+k=n} \mathbf{T_i} \times \mathbf{T_1} \times \mathbf{T_k} \times \mathbf{I} \arrow[dl] \arrow[r] & \mathbf{T_n} \arrow[dr, dashed] & \\
\coprod_{i+j+k=n} \mathbf{A_i} \times \mathbf{A_j} \times \mathbf{A_k} \times \mathbf{I} \arrow[rrr, "V_{i,j,k}"] & & & \mathbf{A_n}
\end{tikzcd}$$

The miters of the diagram are made of (co)products of L_a. The upper and left-hand trapezoid commute because the miters are basically inclusions and the parallel morphisms are defined the same way. A diagram chase shows that $\mathbf{A_n}$ satisfies the inner pushout condition and hence there is a unique map $L_n : \mathbf{T_n} \longrightarrow \mathbf{A_n}$.

In order for $\mathbf{T_n}$ to be a MIS of $\mathbf{A_n}$, L_n must be bijective on objects (maximal) and for every t, t' in $\mathbf{T_n}$, $Hom_{\mathbf{T_n}}(t, t') = *$, the one object set(indiscrete).

L_n can be shown to be bijective on objects with a short inductive proof. The base case is true by definition. The inductive step follows from the fact that the right hand trapezoid commutes, $U_{a,b}$ is bijective on objects and the product of bijective-on-objects functors (the miters) is bijective-on-objects. So going around the right hand trapezoid are only functors that are bijective-on-objects.

In order to show that $Hom(t, t') = *$, we must be a bit more subtle. This is again a proof by induction on n. The base case is trivial. Assume $\mathbf{T_a}$ and

$\mathbf{T_b}$ are indiscrete subgroupoids. Then the product of indiscrete subgroupoids are indiscrete subgroupoids and hence $\mathbf{T_a} \times \mathbf{T_b}$ is an indiscrete subgroupoid. Since there is only one class of morphisms ($< i, 1, k >$) connecting these indiscrete subgroupoids, the entire $\mathbf{T_n}$ is indiscrete.

A few diagrams of the MIS are called for. We shall display the generators of the groupoid of the first few $\mathbf{T_n}$ and the way they sit in $\mathbf{A_n}$. For each $\mathbf{A_n}$ there are three types of generating morphisms:

1. those not in $\mathbf{T_n}$ - denoted

$$\bullet \longrightarrow \bullet\,.$$

2. those in $\mathbf{T_n}$ within $\mathbf{T_a} \times \mathbf{T_b}$ for some a and b - denoted

$$\bullet \Longrightarrow \bullet\,.$$

3. those in $\mathbf{T_n}$ of the form $< i, 1, k >$ - denoted

$$\bullet \Rrightarrow \bullet\,.$$

The first two maximal trees are simple $\mathbf{T_1} = \mathbf{A_1} = \mathbf{1}$, $\mathbf{T_2} = \mathbf{A_2} = \mathbf{1}$.

- $\mathbf{T_3}$

$$\begin{tikzcd}
\begin{array}{c}\mathbf{A_1} \times \mathbf{A_2} \\ A(BC)\end{array} \arrow[r, Rrightarrow, "\mathbf{A_1 \times A_1 \times A_1 \times I}", "\alpha"'] & \begin{array}{c}\mathbf{A_2} \times \mathbf{A_1} \\ (AB)C\end{array}
\end{tikzcd}$$

- $\mathbf{T_4}$

$$\begin{tikzcd}
& A((BC)D) \arrow[rr, "\mathbf{A_1\times A_2\times A_1\times I}"] & & (A(BC))D \arrow[dr, Rightarrow, "\mathbf{A_3\times A_1}"] & \\
A(B(CD)) \arrow[ur, Rightarrow, "\mathbf{A_1\times A_3}"] \arrow[drr, Rightarrow, "\mathbf{A_1\times A_1\times A_2\times I}"'] & & & & ((AB)C)D \\
& & \substack{\mathbf{A_2\times A_2} \\ (AB)(CD)} \arrow[urr, Rightarrow, "\mathbf{A_3\times A_1\times A_1\times I}"'] & &
\end{tikzcd}$$

- $\mathbf{T_5}$

$$\begin{tikzcd}
& & & (A(BC))(DE) \arrow[rrr, Rightarrow, "\alpha_{A,B,C}\otimes Id_{DE}"] \arrow[drr, Rightarrow, "\alpha_{A(BC),D,E}"] & & & ((AB)C)(DE) \arrow[ddd, Rightarrow, "\alpha_{(AB)C,D,E}"] \\
& A((BC)(DE)) \arrow[urr, "\alpha_{A,BC,DE}"] \arrow[dr, Rightarrow, "Id_A\otimes\alpha_{BC,D,E}"] & & & & ((A(BC))D)E \arrow[ddr, Rightarrow, "(\alpha_{A,B,C}\otimes Id_D)\otimes Id_E"'] & \\
& & A(((BC)D)E) \arrow[rr, "\alpha_{A,(BC)D,E}"] & & (A((BC)D))E \arrow[ur, "\alpha_{A,BC,D}\otimes Id_E"] & & \\
A(B(C(DE))) \arrow[uur, Rightarrow, "Id_A\otimes\alpha_{B,C,DE}"'] \arrow[ddr, Rightarrow] \arrow[ddd, Rightarrow, "\alpha_{A,B,C(DE)}"'] & & & & & & (((AB)C)D)E \\
& & A((B(CD))E) \arrow[uu, Rightarrow, "Id_A\otimes(\alpha_{B,C,D}\otimes Id_E)"] \arrow[rr, "\alpha_{A,B(CD),E}"'] & & (A(B(CD)))E \arrow[uu, Rightarrow, "(Id_A\otimes\alpha_{B,C,D})\otimes Id_E"] \arrow[dr, Rightarrow] & & \\
& A(B((CD)E)) \arrow[ur, "Id_A\otimes\alpha_{B,CD,E}"] \arrow[drr, Rightarrow, "\alpha_{A,B,(CD)E}"'] & & & & ((AB)(CD))E \arrow[uur, Rightarrow, "\alpha_{AB,C,D}\otimes Id_E"'] & \\
(AB)(C(DE)) \arrow[rrr, Rightarrow, "Id_{AB}\otimes\alpha_{C,D,E}"'] \arrow[uuuuuurrrrrr, Rightarrow, bend left=60, "\alpha_{AB,C,DE}"] \arrow[uuuuuurrrrrr, Rightarrow, bend right=60, "\alpha_{AB,C,DE}"'] & & & (AB)((CD)E) \arrow[urr, "\alpha_{AB,CD,E}"] & & &
\end{tikzcd}$$

Each vertex is reached by $\mathbf{T_n}$. The only circuits are naturality squares. The long top map is the same as the long bottom map. The single-line arrows have more then one letter in the center. The double-line arrows have only one letter in the center, but the map is tensored i.e. old.

The triple-line arrows have only one letter in the center and are new i.e. not tensored. Bask in the glorious symmetry of this diagram!

Remark. There is nothing canonical about our MIS. We could have chosen as our MIS morphisms those morphisms of the form $< 1, j, k >$, or some other scheme. This would have made a different MIS but we would get –up to conjugation– the same group at the end.

As in maximal trees, we must now "mod-out" all the morphisms in the MIS. This is done with the following pushout in the category of categories or groupoids:

$$\begin{tikzcd}
\mathbf{T_n} \arrow[r, "!"] \arrow[d, "L_n"'] & \mathbf{1} \arrow[d] \\
\mathbf{A_n} \arrow[r] & \pi(\mathbf{A_n}, \mathbf{T_n}).
\end{tikzcd}$$

$\pi(\mathbf{A_n}, \mathbf{T_n})$ can be thought of as the fundamental group of $\mathbf{A_n}$ relative to Tn. Since $\mathbf{T_n}$ will not be changed in this thesis, we shorten $\pi(\mathbf{A_n}, \mathbf{T_n})$ to $\pi(\mathbf{A_n})$. The fact that L_n is surjective on objects, shows that $\pi(\mathbf{A_n})$ is a one object category. Since all the morphisms in $\pi(\mathbf{A_n})$ come from $\mathbf{A_n}$ or $\mathbf{1}$ which only have isomorphisms, $\pi(\mathbf{A_n})$ is a group. Every morphism in the MIS is sent to the identity of $\pi(\mathbf{A_n})$. The second type of relation comes out of the pushout and the way $\mathbf{T_n}$ "sits in" $\mathbf{A_n}$. We feel secure enough to call this group the fundamental group of the $\mathbf{A_n}$.

2.2 Presentation of the groups

The presentation of $\pi(\mathbf{A_n})$ will be given by two sets: A set of generators, G_n, and a set of relations, R_n. We represent the operation of the group as $*$ and the trivial element as e. The generators of the group will be of the form $X < i,j,k >$ where X is an element of the free monoid on two generators λ and ρ (the monoid operation is represented as concatenation). Since the monoid is free, the length of an element is a well defined concept. Intuitively new generators in G_n are of the form $< i,j,k >$ where there is no prefix. Some generators are from old components and the monoid is used to describe which old component the generator is from. If the generator is from the right side of $\mathbf{A_a} \times \mathbf{A_b}$ then we have a blanks on the left (λ). On the other hand, if the generator is on the left, we have b blanks to the right (ρ). In the latter case, for example, the generator might have come even come from an earlier component and more λ's and ρ's will be tagged on to the front of the generator. For each generator $X < i,j,k >$ of G_n, the length of X added to the sum of i,j and k will equal n. In other words, a generator first started in $\mathbf{A_{i+j+k}}$ and we use the free monoid to describe how it sits in $\mathbf{A_n}$. In order to facilitate writing out the generators, we define a function $\#$. Given X an element of the free monoid and G_a, a set of generators, we let

$$X\#G_a = \{Xy : y \in G_a\}.$$

The relations are given as a set of elements of the form $A = B$ for which we mean the set of elements of the form $A * B^{-1}$ is equal to e. Given X an element of the free monoid and R_a, a set of relations,

$$X\#R_a = \{Xy_1 * Xy_2 * \cdots * Xy_m : y_1 * y_2 * \cdots * y_m \in R_a\}.$$

We begin by discussing the generators and relations and then give a formal definition. The next section has a few calculations carried out.

- Generators. What are the generators of the group? They are the morphisms (edges) in the $\mathbf{A_n}$. They form a set G_n. There are two types of such morphisms:
  1. Morphisms from the old components. This corresponds to the upper right-hand corner of the pushout 1.1 of Chapter 1.
  2. New morphisms between the old components. This corresponds to the lower left-hand corner of the pushout 1.1 of Chapter 1.
- Relations. There are several types of relations:
  1. Those from commuting squares.
  2. Set new morphisms that are in the MIS to the identity of the group. i.e. $< i, j, k >= e$ if $j = 1$.
  3. Old relations from the old components.
  4. Relations from the fact that we use a product of groupoids $\mathbf{A_a} \times \mathbf{A_b}$. As in topological spaces, generators of product groupoids commute.

Some more words on the first type of relation are needed. The only commuting parts of $\mathbf{A_n}$ arise in the following situation. Let $\phi : f \longrightarrow f'$ in $\mathbf{A_i}$, $\gamma : g \longrightarrow g'$ in $\mathbf{A_j}$ $\eta : h \longrightarrow h'$ in $\mathbf{A_k}$ and

$$\begin{tikzcd}
f \otimes (g \otimes h) \arrow[r, "<i,j,k>"] \arrow[d, "\phi\otimes(\gamma\otimes\eta)"'] & (f \otimes g) \otimes h \arrow[d, "(\phi\otimes\gamma)\otimes\eta"] \\
f' \otimes (g' \otimes h') \arrow[r, "<i,j,k>"'] & (f' \otimes g') \otimes h'.
\end{tikzcd}$$

This square commutes if the following diagram commutes:

$$\begin{tikzcd}
f \otimes (g \otimes h) \arrow[rr, "<i,j,k>"] \arrow[dd, "\phi\otimes(Id_g\otimes Id_h)"'] & & (f \otimes g) \otimes h \arrow[dd, "(\phi\otimes Id_g)\otimes Id_h"] \\
& (a) & \\
f' \otimes (g \otimes h) \arrow[rr, "<i,j,k>"] \arrow[dd, "Id_{f'}\otimes(\gamma\otimes Id_h)"'] & & (f' \otimes g) \otimes h \arrow[dd, "(Id_{f'}\otimes\gamma)\otimes Id_h"] \\
& (b) & \\
f' \otimes (g' \otimes h) \arrow[rr, "<i,j,k>"] \arrow[dd, "Id_{f'}\otimes(Id_{g'}\otimes\eta)"'] & & (f' \otimes g') \otimes h \arrow[dd, "(Id_{f'}\otimes Id_{g'})\otimes\eta"] \\
& (c) & \\
f' \otimes (g' \otimes h') \arrow[rr, "<i,j,k>"] & & (f' \otimes g') \otimes h'.
\end{tikzcd}$$

Remember that $< i, j, k >$ represents a set of morphisms and each of the squares commute.

Squares of these type generate all commuting squares. Lets look at square (b) in depth. The left vertical map, $Id_{f'} \otimes (\gamma \otimes Id_h)$, is denoted by $\lambda^i \rho^k y$ for some $y \in G_j$. Similarly, the right vertical map is denoted $\rho^k \lambda^i y$. Since the square commutes – i.e. all four edges are "in the same simplex"– there is the relation:

$$< i, j, k > * \rho^k \lambda^i y = \lambda^i \rho^k y * < i, j, k > .$$

The multiplication is written as regular multiplication rather then morphism composition. Since $< i, j, k >$ denotes an *iso*morphism, in fact, a whole set of isomorphisms, we can take inverses. Thus the above equation looks like

$$< i, j, k > * \rho^k \lambda^i y * < i, j, k >^{-1} = \lambda^i \rho^k y.$$

This looks like the formula for an HNN-extensions. (For an exposition of HNN-extensions see [20] and/or [15].) This is made more apparent by looking at the categorical constructions of HNN-extensions. Given a group, G, and two subgroups, A, B with an isomorphism $f : A \longrightarrow B$ between them, the HNN-extension, H, is given as the following pushout in the category of groupoids:

$$\begin{tikzcd}
A \times \dot{I} \arrow[r, "W"] \arrow[d, hook] & G \arrow[d] \\
A \times I \arrow[r] & H
\end{tikzcd}$$

where $W(a, o) = a$ and $W(a, 1) = f(a)$. Our pushout is then just a much more complicated version of this. The $< i, j, k >$'s are to be thought of as the new generators that extend all the old groups.

There is a special case of the above situation. When $j = 1$ the $< i, j, k >$ generator is set to e and the above equation looks like

$$\rho^k \lambda^i y = \lambda^i \rho^k y.$$

This is the relation for a free product with amalgamation of groups. This can be thought of as changing the I in the lower left-hand corner of the above pushout into **1**, the trivial one object groupoid (with the left vertical map as the projection on A.)

We have only looked at the square (b). There are, however, similar equations for the other two types of boxes and they are given in the scheme.

We give the inductive scheme for the generators and relations. Note that although all the morphisms from the old components are generators, most of them get set to e by the old relations. Throughout the scheme, i, j and k are positive integers. $G_1 = R_1 = \emptyset$. Assume we have all the G_k and R_k for $k \leq n-1$ then G_n and R_n is given by:

- Generators. $G_n =$

  1.
  $$\{[(\lambda^i \# G_{n-i}) \cup (\rho^{n-i} \# G_i) : i = 1, 2, \ldots, n-1\} \cup$$

  2.
  $$\{< i, j, k >: i + j + k = n\}.$$

- Relations. $R_n =$

1. For each $< i, j, k >$ such that $i + j + k = n$ we have the unions of the following relations.

   (a)

   $$\{< i, j, k > *\rho^k \rho^j x* < i, j, k >^{-1} = \rho^{j+k} x : x \in G_i\}$$

   (b)

   $$\{< i, j, k > *\rho^k \lambda^i y* < i, j, k >^{-1} = \lambda^i \rho^k y : y \in G_j\}$$

   (c)

   $$\{< i, j, k > *\lambda^{i+j} z* < i, j, k >^{-1} = \lambda^i \lambda^j z : z \in G_k\}$$

2.

   $$\{< i, 1, k >= e : i + 1 + k = n\} \cup$$

3.

   $$\{[(\lambda^i \# R_{n-i}) \cup (\rho^{n-i} \# R_i) : i = 1, 2, \ldots, n-1\} \cup$$

4.

   $$\{\rho^{n-i} x * \lambda^i y = \lambda^i y * \rho^{n-i} x : x \in G_i, y \in G_{n-i}\}$$

2.3 The first few groups

In order to give the presentations in an orderly fashion, we conform to the following conventions:

- The old edges are listed in columns below the names of the components that contributed them. The new edges are listed at the bottom.

- If an old edge was set to e, we do not list it in further groups. For example $< 1,1,1 >$ will be listed in G_3. Since it is set to e, we do not list it or any of its "progeny" (e.g. $\lambda < 1,1,1 >$ or $\rho\lambda^3\rho < 1,1,1 >$.)
- We do not list relations about edges that were set to e.
- Since the first nontrivial relation is in R_6 we do not list a relation of type 3 until R_7.
- Since the first nontrivial generator is in G_4, the first time we have a relation of type 4 is in R_8. Due to the fact that G_8 and R_8 will not be listed in this paper, we feel obliged to give this relation of type 4:

$$\rho^4 < 1,2,1 > *\lambda^4 < 1,2,1 >= \lambda^4 < 1,2,1 > *\rho^4 < 1,2,1 > .$$

- If two generators are set to be equal, then, in the future, we only list one of them. We usually choose the shorter name e.g. $\lambda^2\rho < 1,2,1 >$ rather then $\lambda\lambda\rho < 1,2,1 >$.

Here are the groups:

G_3

$\mathbf{A_1 \times A_2}$	$\mathbf{A_2 \times A_1}$
$< 1,1,1 >$	

R_3

$< 1,1,1 >= e$

So $\pi(\mathbf{A_3})$ is the trivial group.

G_4

$\mathbf{A_1} \times \mathbf{A_3}$	$\mathbf{A_2} \times \mathbf{A_2}$	$\mathbf{A_3} \times \mathbf{A_1}$
$\lambda < 1,1,1 >$		$\rho < 1,1,1 >$
$< 1,2,1 >$		
$< 1,1,2 >, < 2,1,1 >$		

R_4

$< 1,1,2 >= e$
$< 2,1,1 >= e$

$\pi(\mathbf{A_4})$ is the free group generated by the generator $< 1,2,1 >$.

G_5

$\mathbf{A_1} \times \mathbf{A_4}$	$\mathbf{A_2} \times \mathbf{A_3}$	$\mathbf{A_3} \times \mathbf{A_2}$	$\mathbf{A_4} \times \mathbf{A_1}$
$\lambda < 1,2,1 >$			$\rho < 1,2,1 >$
$< 1,3,1 >$			
$< 1,2,2 >, < 2,2,1 >$			
$< 1,1,3 >, < 2,1,2 >, < 3,1,1 >$			

R_5

$< 1,1,3 >= e$
$< 2,1,2 >= e$
$< 3,1,1 >= e$

$\pi(\mathbf{A}_5)$ is the free group generated by five generators. There are six pentagons in $\mathbf{A}_5$ but they are linked up in such a way that there are only five generators. This is similar to the fact that although the cube has six faces (squares) there are only five generators i.e. only five faces must commute in order for the entire cube to commute.

There is one unwritten trivial relation

$$< 1,3,1 > *\rho\lambda < 1,1,1 > * < 1,3,1 >^{-1} = \lambda\rho < 1,1,1 > .$$

However since $X < 1,1,1 >$ is set to e, the relation is superfluous. This relation corresponds to the center square in $\mathbf{A}_5$.

G_6

<table>
<tr><th>$\mathbf{A_1 \times A_5}$</th><th>$\mathbf{A_2 \times A_4}$</th><th>$\mathbf{A_3 \times A_3}$</th><th>$\mathbf{A_4 \times A_2}$</th><th>$\mathbf{A_5 \times A_1}$</th></tr>
<tr><td>$\lambda\lambda < 1,2,1 >$</td><td>$\lambda^2 < 1,2,1 >$</td><td></td><td>$\rho^2 < 1,2,1 >$</td><td>$\rho\lambda < 1,2,1 >$</td></tr>
<tr><td>$\lambda\rho < 1,2,1 >$</td><td></td><td></td><td></td><td>$\rho\rho < 1,2,1 >$</td></tr>
<tr><td>$\lambda < 1,3,1 >$</td><td></td><td></td><td></td><td>$\rho < 1,3,1 >$</td></tr>
<tr><td>$\lambda < 1,2,2 >$</td><td></td><td></td><td></td><td>$\rho < 1,2,2 >$</td></tr>
<tr><td>$\lambda < 2,2,1 >$</td><td></td><td></td><td></td><td>$\rho < 2,2,1 >$</td></tr>
<tr><td colspan="5">$< 1,4,1 >$
$< 1,3,2 >, < 2,3,1 >$
$< 1,2,3 >, < 2,2,2 >, < 3,2,1 >$
$< 1,1,4 >, < 2,1,3 >, < 3,1,2 >, < 4,1,1 >$</td></tr>
</table>

R_6

$< 1,4,1 > *\rho\lambda < 1,2,1 > * < 1,4,1 >^{-1} = \lambda\rho < 1,2,1 >$
$< 1,1,4 > *\lambda^2 < 1,2,1 > * < 1,1,4 >^{-1} = \lambda\lambda < 1,2,1 >$
$< 4,1,1 > *\rho\rho < 1,2,1 > * < 4,1,1 >^{-1} = \rho^2 < 1,2,1 >$
$< 1,1,4 >= e$ $< 2,1,3 >= e$ $< 3,1,2 >= e$ $< 4,1,1 >= e$

The first HNN-extension is the first non-trivial relation that we have. The second relation is an amalgamation because $< 1,1,4 >= e$ and so the relation reduces to

$$\lambda^2 < 1,2,1 >= \lambda\lambda < 1,2,1 > .$$

This essentially shows the associativity of the free monoid on two generators. Similarly for the third relation. So $\pi(\mathbf{A}_6)$ is a group with 22 generators (12 old + 10 new.) Two pairs of old generators are set equal to each other and four new generators are set to e. We are left with 16 generators (10 old + 6 new.) There is one nontrivial relation on these generators. However the group is isomorphic to the free group on 15 generators.

G_7

<table>
<tr><th>$\mathbf{A}_1 \times \mathbf{A}_6$</th><th>$\mathbf{A}_2 \times \mathbf{A}_5$</th><th>$\mathbf{A}_3 \times \mathbf{A}_4$</th><th>$\mathbf{A}_4 \times \mathbf{A}_3$</th><th>$\mathbf{A}_5 \times \mathbf{A}_2$</th><th>$\mathbf{A}_6 \times \mathbf{A}_1$</th></tr>
<tr><td>$\lambda\lambda^2 < 1,2,1 >$</td><td>$\lambda^2\lambda < 1,2,1 >$</td><td>$\lambda^3 < 1,2,1 >$</td><td>$\rho^3 < 1,2,1 >$</td><td>$\rho^2\lambda < 1,2,1 >$</td><td>$\rho\lambda^2 < 1,2,1 >$</td></tr>
<tr><td>$\lambda\lambda\rho < 1,2,1 >$</td><td>$\lambda^2\rho < 1,2,1 >$</td><td></td><td></td><td>$\rho^2\rho < 1,2,1 >$</td><td>$\rho\lambda\rho < 1,2,1 >$</td></tr>
<tr><td>$\lambda\lambda < 1,3,1 >$</td><td>$\lambda^2 < 1,3,1 >$</td><td></td><td></td><td>$\rho^2 < 1,3,1 >$</td><td>$\rho\lambda < 1,3,1 >$</td></tr>
<tr><td>$\lambda\lambda < 1,2,2 >$</td><td>$\lambda^2 < 1,2,2 >$</td><td></td><td></td><td>$\rho^2 < 1,2,2 >$</td><td>$\rho\lambda < 1,2,2 >$</td></tr>
<tr><td>$\lambda\lambda < 2,2,1 >$</td><td>$\lambda^2 < 2,2,1 >$</td><td></td><td></td><td>$\rho^2 < 2,2,1 >$</td><td>$\rho\lambda < 2,2,1 >$</td></tr>
<tr><td>$\lambda\rho^2 < 1,2,1 >$</td><td></td><td></td><td></td><td></td><td>$\rho\rho^2 < 1,2,1 >$</td></tr>
<tr><td>$\lambda\rho\lambda < 1,2,1 >$</td><td></td><td></td><td></td><td></td><td>$\rho\rho\lambda < 1,2,1 >$</td></tr>
<tr><td>$\lambda\rho < 1,3,1 >$</td><td></td><td></td><td></td><td></td><td>$\rho\rho < 1,3,1 >$</td></tr>
<tr><td>$\lambda\rho < 1,2,2 >$</td><td></td><td></td><td></td><td></td><td>$\rho\rho < 1,2,2 >$</td></tr>
<tr><td>$\lambda\rho < 2,2,1 >$</td><td></td><td></td><td></td><td></td><td>$\rho\rho < 2,2,1 >$</td></tr>
<tr><td>$\lambda < 1,4,1 >$</td><td></td><td></td><td></td><td></td><td>$\rho < 1,4,1 >$</td></tr>
<tr><td>$\lambda < 1,3,2 >$</td><td></td><td></td><td></td><td></td><td>$\rho < 1,3,2 >$</td></tr>
<tr><td>$\lambda < 2,3,1 >$</td><td></td><td></td><td></td><td></td><td>$\rho < 2,3,1 >$</td></tr>
<tr><td>$\lambda < 1,2,3 >$</td><td></td><td></td><td></td><td></td><td>$\rho < 1,2,3 >$</td></tr>
<tr><td>$\lambda < 2,2,2 >$</td><td></td><td></td><td></td><td></td><td>$\rho < 2,2,2 >$</td></tr>
<tr><td>$\lambda < 3,2,1 >$</td><td></td><td></td><td></td><td></td><td>$\rho < 3,2,1 >$</td></tr>
<tr><td colspan="6">$< 1,5,1 >$
$< 1,4,2 >, < 2,4,1 >$
$< 1,3,3 >, < 2,3,2 >, < 3,3,1 >$
$< 1,2,4 >, < 2,2,3 >, < 3,2,2 >, < 4,2,1 >$
$< 1,1,5 >, < 2,1,4 >, < 3,1,3 >, < 4,1,2 >, < 5,1,1 >$</td></tr>
</table>

R_7

$$< 1,5,1 > *\rho\lambda\lambda < 1,2,1 > * < 1,5,1 >^{-1} = \lambda\rho\lambda < 1,2,1 >$$

$$< 1,5,1 > *\rho\lambda\rho < 1,2,1 > * < 1,5,1 >^{-1} = \lambda\rho\rho < 1,2,1 >$$

$$< 1,5,1 > *\rho\lambda < 1,3,1 > * < 1,5,1 >^{-1} = \lambda\rho < 1,3,1 >$$

$$< 1,5,1 > *\rho\lambda < 1,2,2 > * < 1,5,1 >^{-1} = \lambda\rho < 1,2,2 >$$

$$< 1,5,1 > *\rho\lambda < 2,2,1 > * < 1,5,1 >^{-1} = \lambda\rho < 2,2,1 >$$

$$< 1,4,2 > *\rho^2\lambda < 1,2,1 > * < 1,4,2 >^{-1} = \lambda\rho^2 < 1,2,1 >$$

$$< 2,4,1 > *\rho\lambda^2 < 1,2,1 > * < 2,4,1 >^{-1} = \lambda^2\rho < 1,2,1 >$$

$$< 1,2,4 > *\lambda^3 < 1,2,1 > * < 1,2,4 >^{-1} = \lambda\lambda^2 < 1,2,1 >$$

$$< 4,2,1 > *\rho\rho^2 < 1,2,1 > * < 4,2,1 >^{-1} = \rho^3 < 1,2,1 >$$

$< 1,1,5 >$:

$$\lambda^2\lambda < 1,2,1 > = \lambda\lambda^2 < 1,2,1 >$$

$$\lambda^2\rho < 1,2,1 > = \lambda\lambda\rho < 1,2,1 >$$

$$\lambda^2 < 1,3,1 > = \lambda\lambda < 1,3,1 >$$

$$\lambda^2 < 1,2,2 > = \lambda\lambda < 1,2,2 >$$

$$\lambda^2 < 2,2,1 > = \lambda\lambda < 2,2,1 >$$

$< 5,1,1 >$:

$$\rho\rho\lambda < 1,2,1 > = \rho^2\lambda < 1,2,1 >$$

$$\rho\rho^2 < 1,2,1 > = \rho^2\rho < 1,2,1 >$$

$$\rho\rho < 1,3,1 > = \rho^2 < 1,3,1 >$$

$$\rho\rho < 1,2,2 > = \rho^2 < 1,2,2 >$$

$$\rho\rho < 2,2,1 > = \rho^2 < 2,2,1 >$$

$< 2,1,4 >$:

$$\lambda^3 < 1,2,1 > = \lambda^2\lambda < 1,2,1 >$$

$< 4,1,2 >$:

$$\rho^2\rho < 1,2,1 > = \rho^3 < 1,2,1 >$$

$$< 1,1,5 > = \rho$$

There are 57 (= 42 old + 15 new) generators. After amalgamations and setting some of the new generators to e we have 42 = 32 old + 10 new generators. There are 11 relations on these 42 generators.

If you combine all the relations you get

$$\rho^3 < 1,2,1 >= \rho\rho^2 < 1,2,1 >= \rho^2\rho < 1,2,1 >$$

i.e. ρ is associative. Similarly for λ. The relation

$$< 4,2,1 > *\rho\rho^2 < 1,2,1 > * < 4,2,1 >^{-1} = \rho^3 < 1,2,1 >$$

then reduces to

$$< 4,2,1 > *\rho^3 < 1,2,1 > * < 4,2,1 >^{-1} = \rho^3 < 1,2,1 >$$

or

$$< 4,2,1 > *\rho^3 < 1,2,1 >= \rho^3 < 1,2,1 > * < 4,2,1 > .$$

This relation can actually be seen. Consider the following diagram of associations and reassociations.

$$\begin{tikzcd}
{(A(B(CD)))[E[FG]]} \arrow[r] \arrow[d] & {(A((BC)D))[E[FG]]} \arrow[r, "\rho^3<121>"] \arrow[d] & {((A(BC))D)[E[FG]]} \arrow[r] \arrow[d] & {(((AB)C)D)[E[FG]]} \arrow[d] & {((AB)(CD))[E[FG]]} \arrow[l] \arrow[d] & {} \arrow[l] \\
{(A(B(CD)))[[EF]G]} \arrow[r] \arrow[d, "<421>"'] & {(A((BC)D))[[EF]G]} \arrow[r, "\rho^3<121>"] \arrow[d, "<421>"'] & {((A(BC))D)[[EF]G]} \arrow[r] \arrow[d, "<421>"'] & {(((AB)C)D)[[EF]G]} \arrow[d, "<421>"'] & {((AB)(CD))[[EF]G]} \arrow[l] \arrow[d, "<421>"'] & {} \arrow[l] \\
{[(A(B(CD)))[EF]]G} \arrow[r] \arrow[d] & {[(A((BC)D))[EF]]G} \arrow[r, "\rho^3<121>"] \arrow[d] & {[((A(BC))D)[EF]]G} \arrow[r] \arrow[d] & {[(((AB)C)D)[EF]]G} \arrow[d] & {[((AB)(CD))[EF]]G} \arrow[l] \arrow[d] & {} \arrow[l] \\
{[[(A(B(CD)))E]F]G} \arrow[r] & {[[(A((BC)D))E]F]G} \arrow[r, "\rho^3<121>"] & {[[((A(BC))D)E]F]G} \arrow[r] & {[[(((AB)C)D)E]F]G} & {[[((AB)(CD))E]F]G} \arrow[l] & {} \arrow[l] \\
{[(A(B(CD)))E][FG]} \arrow[r] \arrow[u] & {[(A((BC)D))E][FG]} \arrow[r, "\rho^3<121>"] \arrow[u] & {[((A(BC))D)E][FG]} \arrow[r] \arrow[u] & {[(((AB)C)D)E][FG]} \arrow[u] & {[((AB)(CD))E][FG]} \arrow[l] \arrow[u] & {} \arrow[l] \\
{(A(B(CD)))[E[FG]]} \arrow[r] \arrow[u] & {(A((BC)D))[E[FG]]} \arrow[r, "\rho^3<121>"] \arrow[u] & {((A(BC))D)[E[FG]]} \arrow[r] \arrow[u] & {(((AB)C)D)[E[FG]]} \arrow[u] & {((AB)(CD))[E[FG]]} \arrow[l] \arrow[u] & {} \arrow[l]
\end{tikzcd}$$

The top row and the bottom row are the same. As are the left side and the right side. Each square commutes out of naturality. The whole thing is a torus. As with every torus, the generators commute. The two generators have been marked off. Going around $\rho^3 < 1,2,1 >$ and then going around $< 4,2,1 >$ is the same as doing it vice versa.

The important point is that the two stared relations show commutativity of generators. These generators do not show up in any other relations and hence the group is not free! The higher groups contain this group and hence they are also not free.

2.4 Fundamental groups of quotients

In order to look at quotients of associative categories, we must define congruences that respect the associative category structure.

Definition 2.1 *A congruence on an associative category* $(\mathbf{B}, \otimes, \beta)$ *is an equivalence relation* $\sim$ *for each Hom set of morphisms of* $\mathbf{B}$ *such that*

1. *composition is respected;*
2. *the* $\otimes$ *is respected, i.e. if* $f \sim f'$ *and* $g \sim g'$*, then* $f \otimes g \sim f' \otimes g'$*;*
3. *naturality is respected, i.e. for any* $f : A \longrightarrow A'$*,* $g : B \longrightarrow B'$ *and* $h : C \longrightarrow C'$*, we have*

$$((f \otimes g) \otimes h) \circ \beta_{A,B,C} \quad \sim \quad \beta_{A',B',C'} \circ (f \otimes (g \otimes h)).$$

By the customary universal algebra tricks, to every congruence on $(\mathbf{B}, \otimes, \beta)$ there is a unique associative category $(\tilde{\mathbf{B}}, \tilde{\otimes}, \tilde{\beta})$ and a unique strict tensor functor $\Pi : \mathbf{B} \longrightarrow \tilde{\mathbf{B}}$ that satisfies universal properties. $\tilde{\mathbf{B}}$ is isomorphic on objects. Morphisms are equivalence classes of morphisms of $\mathbf{B}$, denoted $[f]$. The tensor product is defined as usual: $[f]\tilde{\otimes}[g] = [f \otimes g]$. The associativity is $\tilde{\beta}_{A,B,C} = [\beta_{A,B,C}]$. Condition 3 insures naturality of the associativity.

The next step is to look at quotients of the 2-sketch $\mathbf{A}$ of the theory of associative categories. Since $\mathbf{A}$ is a strict tensor 2-category, we must define what congruences of such categories are like. Our only interest is when 2-cells of $\mathbf{A}$ are set equal to each other, not 0-cells or 1-cells. So we have the following definition.

Definition 2.2 *A 2-congruence on a strict tensor 2-category* $(\mathbf{C}, \otimes)$ *is an equivalence relation* $\sim$ *on each set of 2-cells between any two 1-cells of* $\mathbf{C}$ *such that*

1. *the* $\otimes$ *is respected, i.e. if* $\phi \sim \phi'$ *and* $\gamma \sim \gamma'$*, then* $(\phi \otimes \gamma) \sim (\phi' \otimes \gamma')$*;*
2. *vertical composition is respected, i.e. if* $\phi \sim \phi'$ *and* $\gamma \sim \gamma'$*, then*

$$(\phi \circ_V \gamma) \sim (\phi' \circ_V \gamma');$$

3. *horizontal composition is respected (similar to 2).*

By generalized universal algebra, to every 2-congruence on a strict tensor 2-category $(\mathbf{C}, \otimes)$ there is a unique strict tensor 2-category $(\tilde{\mathbf{C}}, \tilde{\otimes})$ and a unique strict tensor 2-functor $\Pi : \mathbf{C} \longrightarrow \tilde{\mathbf{C}}$ that satisfies the usual universal properties. $\tilde{\mathbf{C}}$ is isomorphic on 0-cells and 1-cells. 2-cells are equivalence classes of 2-cells of $\mathbf{C}$, denoted $[\phi]$. The tensor product, vertical composition and horizontal composition are defined as usual. The "commutativity" of vertical composition and horizontal composition in $\tilde{\mathbf{C}}$, falls out of the fact that $\mathbf{C}$ has this property. Given such a strict tensor 2-category and strict tensor 2-functor (that is isomorphic on 0-cells and 1-cells) we get such a 2-congruence.

We shall now examine what a 2-congruence is like in our 2-sketch $\mathbf{A}$. The fact that the 2-congruence preserves tensor product means that if $\phi = (\phi_1, \ldots, \phi_k) \sim \phi' = (\phi'_1, \ldots, \phi'_{k'})$ and $\gamma = (\gamma_1, \ldots, \gamma_l) \sim \gamma' = (\gamma'_1, \ldots, \gamma'_{l'})$ then

$$\phi \otimes \gamma = (\phi_1, \ldots, \phi_k, \gamma_1, \ldots, \gamma_l) \sim (\phi'_1, \ldots, \phi'_{k'}, \gamma'_1, \ldots, \gamma'_{l'}) = \phi' \otimes \gamma'.$$

There is however one interesting fact here. Since we are interested in associative categories in $\mathbf{Cat}$ i.e. strict tensor 2-functors F from $(\mathbf{A}, \otimes)$ to $(\mathbf{Cat}, \times)$, we have the following

$$F(\phi) \times F(\gamma) = F(\phi \otimes \gamma) = F(\phi' \otimes \gamma') = F(\phi') \times F(\gamma'),$$

and hence using usual properties of the product in $\mathbf{Cat}$ $F(\phi) = F(\phi')$ and $F(\gamma) = F(\gamma')$. Such functors determine quotients and hence if $(\phi \otimes \gamma) \sim (\phi' \otimes \gamma')$ then $\phi \sim \phi'$ and $\gamma \sim \gamma'$. By a small inductive argument it suffices to discuss only individual components of the 2-cells. Hence we shall talk about $\phi_i \sim \phi'_i$ where they are both elements of $\mathbf{A}_{\mathbf{n}_i}$ for some n_i. Hence we have, that every 2-congruence on $\mathbf{A}$ induces a congruence on each of the $\mathbf{A}_\mathbf{n}$.

Vertical composition of 2-cells in **A** is simply composition in each $\mathbf{A}_{\mathbf{n}_i}$.

Finally we are left with horizontal composition which is the heart of the matter. We remind the reader that horizontal composition is defined with the use of the operad Q. A review of how the operad is defined and employed in Section 1.4 would be beneficial at this time. If $\phi \sim \phi'$ and $\gamma \sim \gamma'$ then

$$\Lambda = (\phi \circ_H \gamma) \sim (\phi' \circ_H \gamma') = \Delta.$$

By the argument above, this means that

$$\Lambda_i = Q(\phi_i, \gamma_{n_1}, \ldots, \gamma_{n_s}) \sim Q(\phi'_i, \gamma'_{n_1}, \ldots, \gamma'_{n_s}) = \Delta_i$$

where the subscripts of Q are abandoned since they are the same Q.

Between every two 1-cells of **A** there is a distinct 2-cell, denoted $\check{\gamma}$, that has all its components $\check{\gamma}_i$ in the MIS $\mathbf{T}_{\mathbf{n}_i}$. We call such a 2-cells a"branch" (branches make up trees.) Given two branches $\check{\gamma}$ and $\check{\gamma}'$, their tensor product $\check{\gamma} \otimes \check{\gamma}'$ is also a branch. Similarly for vertical composition. "Branchness" is also preserved by horizontal composition because it is preserved by the operad Q i.e.

$$Q(\check{\phi}, \check{\gamma}_{n_1}, \ldots, \check{\gamma}_{n_s}) = \check{\phi}'.$$

This is shown by an inductive proof on the construction of the Q's and from the fact that the $V_{i,j,k}$ are built up from earlier $V_{i,j,k}$.

For every congruence, there is a unique equivalence class between every two 1-cells, namely, those 2-cells congruent to the unique branch 2-cell. We denote this equivalence class by $[\check{\phi}]$.

Given an associative category $(\mathbf{B}, \otimes, \beta)$, there are unique strict tensor functors $P_n : \mathbf{A}_\mathbf{n} \longrightarrow \mathbf{It}(\mathbf{B})_\mathbf{n}$ as was shown in Section 3 of Chapter 1. We use P_n in the following diagram.

$$\begin{tikzcd}
\mathbf{T}_\mathbf{n} \arrow[r, "!"] \arrow[d, "L_n"'] & \mathbf{1} \arrow[r, equal] \arrow[d] & \mathbf{1} \arrow[dd] \\
\mathbf{A}_\mathbf{n} \arrow[r] \arrow[d, "P_n"'] & \pi(\mathbf{A}_\mathbf{n}) \arrow[dr, dashed, "\pi(P_n)"] & \\
\mathbf{It}(\mathbf{B})_\mathbf{n} \arrow[rr] & & \pi(\mathbf{It}(\mathbf{B})_\mathbf{n}).
\end{tikzcd}$$

Both squares are pushouts. The inner square is familiar from Section 1. The result of the outer pushout is $\pi(\mathbf{It}(\mathbf{B})_\mathbf{n})$ which is shortened to $\pi(\mathbf{B}_\mathbf{n})$ (this should not seem so strange since the tensor product and the reassociation of $\mathbf{It}(\mathbf{B})_\mathbf{n}$ is basically the same as the tensor product and the reassociation of $\mathbf{B}$.) The induced arrow is denoted by $\pi(P_n)$. The functorial properties of $\pi(-)$ will not be discussed here. Since the left hand vertical map is surjective on objects, $\pi(\mathbf{B}_\mathbf{n})$ is a group and we call it the fundamental group of $\mathbf{B}_\mathbf{n}$. We must stress that P_n is not necessarily full (surjective on morphisms). A typical example is when $\mathbf{B}$ is a *strict* tensor category with β being something other then the identity. Then $\mathbf{A}_3$ is the indiscrete category with two objects whereas the $\mathbf{It}(\mathbf{B})_3$ has one object and - by composition - infinite morphisms. Our interest lies not in $\pi(\mathbf{B}_\mathbf{n})$ but in the image of $\pi(P_n)$ since there are to be found the morphisms that are of concern to coherence.

Now for a short discussion of the way Q works. $Q(\phi, \gamma, \ldots, \gamma)$ takes ϕ and for each letter that ϕ reassociates, puts in other letters. How many other letters? One or more, depending on what partition we have. Now if ϕ is a generating morphism of the form $\phi = V_{i,j,k}(\phi_i, \phi_2, \phi_3)$ then

$$Q(\phi, \gamma, \ldots, \gamma) = V_{i',j',k'}(\phi_i', \phi_2', \phi_3')$$

by the definition of Q given in Section 1.4. This can be denoted in our group theoretic notation as

$$Q(< i, j, k >, \gamma, \ldots, \gamma) =< i', j', k' >$$

where $i' \geq i, j' \geq j$ and $k' \geq k$. Similarly if $\phi = \phi_a \otimes \phi_b$ then

$$Q(\phi, \gamma, \ldots, \gamma) = Q(\phi_a, \gamma, \ldots \gamma) \otimes Q(\phi_b, \gamma, \ldots \gamma).$$

In group theoretic notation we have

$$Q(X < i, j, k >, \gamma, \ldots, \gamma) = X' < i', j', k' >$$

where $i' \geq i, j' \geq j, k' \geq k$ and $length(X') \geq length(X)$.

The important point is that every 2-cell (of $\mathbf{A}$ or 1-cell of $\mathbf{A_n}$) that is a branch, goes to e, the identity of $\pi(\mathbf{B_n})$.

Putting this all together we have the following.

Theorem 2.1 *If $j \geq 2$, $i+j+k = n$ and $\pi(P_n)(< i, j, k >) = e$ (the identity of the* $\mathrm{It}(\mathbf{B})_\mathbf{n}$ *), then for all $i' \geq i$, $j' \geq j$ and $k' \geq k$ with $i' + j' + k' = n'$ we have $\pi(P_{n'})(< i', j', k' >) = e$.*

This theorem is a proper generalization of the Mac Lane's coherence theorem and we have:

Corollary 2.1 (Mac Lane's Coherence Theorem) *If* $\pi(P_4)(<1,2,1>) = e$, *then for all* i', j' *and* k' *with* $i'+j'+k' = n'$ *we have* $\pi(P_{n'})(<i',j',k'>) = e$ *and hence* $\pi(P_{n'})$ *is the trivial map for all* n'.

Proof. Set $i = 1, j = 2$ and $k = 1$ in the above theorem. The fact that $\pi(P_{n'})$ is trivial for all n' comes from the fact that all the generators of $\pi(\mathbf{A_n})$ are of the form $X < i,j,k >$ where X is an element of the free monoid on λ, ρ. If $< i,j,k > = e$, then $X < i,j,k > = Xe = e$□

The conclusion is that there are many forms of noncoherence but we can classify them as a type of a tree. If going around a certain loop in $\mathbf{A_n}$ is set to equal the identity, then there are implications for the higher $\mathbf{A_n}$. However, there is none for the lower (till next section).

2.5 Quotients of unital associative categories

Since we have not given a formal construction of $\bar{\mathbf{A}}'$, the free associative category with a unit or of $\mathbf{A}'$ the theory of associative categories with units, we shall not give a formal discussion of the fundamental groups. However we shall give an intuitive discussion of it. All congruences must also take into account the morphisms $L, R : \mathbf{A_n} \longrightarrow \mathbf{A_{n-1}}$. Hence there are, in fact, fewer congruences if we insist upon units. Here is another way of looking at it.

Assume there is a loop in $\mathbf{It(B)_n}$ we can write it as

$$f_1 \xrightarrow{\phi_1} f_2 \xrightarrow{\phi_2} \cdots \longrightarrow f_n \xrightarrow{\phi_n} f_1.$$

Where the f_i's are associations of any n objects in $\mathbf{B}$. In particular, one of the objects may be the unit object I. Let f_i^* be the association f_i without that unit object. f_i^* is then an association of $n-1$ objects. Hence we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{It(B)_n} & f_1 \arrow[r, "\phi_1"] \arrow[d] & f_2 \arrow[r, "\phi_2"] \arrow[d] & \cdots \arrow[r] \arrow[d] & f_n \arrow[r, "\phi_n"] \arrow[d] & f_1 \arrow[d] \\
\mathbf{It(B)_{n-1}} & f_1^* \arrow[r, "\phi_1^*"] & f_2^* \arrow[r, "\phi_2^*"] & \cdots \arrow[r] & f_n^* \arrow[r, "\phi_n^*"] & f_1^*.
\end{tikzcd}$$

where the vertical maps are identities tensored with L or R. The squares commute only if we assume the identity triangle coherence requirement discussed in Section 1.5. If the top loop is the identity, then the bottom loop must also be the identity. In other words, when there are no units a relation in $\mathbf{A_n}$ causes a relation in $\mathbf{A_{n+t}}$. In contrast, when there are units, relations in $\mathbf{A_{n+t}}$ *go down* to the lower $\mathbf{A_n}$. Hence there are fewer congruences possible.

Chapter 3

Multiplicative Functors

3.1 Definitions and examples

Definition 3.1 (Multiplicative Functor) *A multiplicative functor, (F, σ), between two associative categories $(\mathbf{B}, \otimes, \beta)$ and $(\mathbf{B}', \otimes', \beta')$ is a functor $F : \mathbf{B} \longrightarrow \mathbf{B}'$ and a natural isomorphism*

$$\sigma_F : \otimes' \circ (F \times F) \longrightarrow F \circ \otimes$$

i.e. for every A, B in $\mathbf{B}$, an isomorphism

$$\sigma_{F,A,B} : FA \otimes' FB \longrightarrow F(A \otimes B).$$

We do not make the usual coherence requirement that the hexagon (see Introduction) commutes. Examples of such functors are readily available. If $\mathbf{B}$ and $\mathbf{B}'$ are monoidal and the hexagon diagram commutes, the functor is called a tensor functor (there are also requirements for units, however, units are not discussed at all in this chapter.) The term "multiplicative functor" is used in [21] (Pg.171). They use such functors - with units - in their discussion of Drinfeld algebras (also called quasi-algebras in [10]) and Majid algebras.

Given two multiplicative functors $(F,\sigma) : (\mathbf{B},\otimes,\beta) \longrightarrow (\mathbf{B}',\otimes',\beta')$ and $(G,\tau) : (\mathbf{B}',\otimes',\beta') \longrightarrow (\mathbf{B}'',\otimes'',\beta'')$, their composition is given as $(G \circ F, \tau @ \sigma)$ where

$$(\tau @ \sigma)_{A,B} = G(\sigma_{A,B}) \circ \tau_{FA,FB}$$

i.e.

$$(\tau @ \sigma)_{A,B} : GF(A) \otimes'' GF(B) \xrightarrow{\tau_{FA,FB}} G(FA \otimes' FB) \xrightarrow{G(\sigma_{A,B})} GF(A \otimes B).$$

Associativity is tedious but obvious. The identity property of multiplicative functors is only obvious.

Definition 3.2 *The category* **Assoc - Cat** *has as objects associative categories and as morphisms, multiplicative functors. (This paper will not deal with the higher order structure of this category i.e. natural transformations).*

Given two associative categories, $(\mathbf{B},\otimes,\beta)$ and $(\mathbf{B}',\otimes',\beta')$, we will work with the following associative categories:

$$\bar{\mathbf{B}} = \coprod_{n\in\mathbf{N}} \mathbf{Hom}(\mathbf{B}^n,\mathbf{B}), \qquad \bar{\mathbf{B}'} = \coprod_{n\in\mathbf{N}} \mathbf{Hom}(\mathbf{B}'^n,\mathbf{B}'), \qquad \mathbf{B}\bar{\mathbf{B}}' = \coprod_{n\in\mathbf{N}} \mathbf{Hom}(\mathbf{B}^n,\mathbf{B}')$$

and the respective associative subcategories $\mathbf{It}(\bar{\mathbf{B}})$ and $\mathbf{It}(\bar{\mathbf{B}}')$ of the first two. We will create an analogous associative subcategory of the third. The reader is reminded that if $(\mathbf{B}, \otimes, \beta)$ is an associative category, then there is also a unique strict tensor functor $P^{\mathbf{B}} : \bar{\mathbf{A}} \longrightarrow \mathbf{It}(\bar{\mathbf{B}})$.

Let (F, σ) be a multiplicative functor, then we have the following not necessarily commutative diagram:

$$\begin{tikzcd}
 & \bar{\mathbf{A}} \arrow[dl, "P^{\mathbf{B}}"'] \arrow[dr, "P^{\mathbf{B}'}"] & \\
\mathbf{It}(\bar{\mathbf{B}}) \arrow[dr, "F\circ -"'] & & \mathbf{It}(\bar{\mathbf{B}}') \arrow[dl, "\{-\circ F^n\}"] \\
 & \mathbf{B}\bar{\mathbf{B}}'. &
\end{tikzcd}$$

The lower right hand map is the map induced by the coproduct of all $-\circ F^n$ that takes a functor $f : \mathbf{B}'^{\mathbf{n}} \longrightarrow \mathbf{B}$ to $f \circ F^n : \mathbf{B}^{\mathbf{n}} \longrightarrow \mathbf{B}'^{\mathbf{n}} \longrightarrow \mathbf{B}'$. The lower right functor is a strict tensor functor, whereas the lower left hand functor need not be. This can be shown with a diagram trace, however, one can think of it as a higher order version of something much more familiar: let G and G' be group objects in **Set** and X and X' be sets. If $f : X \longrightarrow X'$ is a set map, then $Hom(f, G) : Hom(X', G) \longrightarrow Hom(X, G)$ is a group homomorphism. However, if $g : G \longrightarrow G'$ is a map of sets, then $Hom(X, g) : Hom(X, G) \longrightarrow Hom(X, G')$ need not be a group homomorphism.

A typical element of the image of the right hand is $F(A \otimes (B \otimes C))$. An element of the image of the left hand map is $FA \otimes' (FB \otimes' FC)$.

To every multiplicative functor, (F, σ), we associate a groupoid $\bar{\mathbf{F}}\sigma$ called the *mapping funnel* of (F, σ). $\bar{\mathbf{F}}\sigma$ is the associative subcategory of $\mathbf{B}\bar{\mathbf{B}}'$ generated by

1. $F \circ P_n^{\mathbf{B}}(\mathbf{A_n})$ for all n and
2. if $f = U_{a,b}(f_a, f_b)$, the isomorphism

$$FP_a^{\mathbf{B}}(f_a) \otimes' FP_b^{\mathbf{B}}(f_b) \xrightarrow{\sigma_{P(f_a),P(f_b)}} FP_n^{\mathbf{B}}(f) \cdot$$

Lets look more carefully at the mapping funnel. It is made in the following manner:

$$\bar{\mathbf{F}}\sigma = \coprod_{n \in \mathbf{N}} (\mathbf{F}\sigma)_{\mathbf{n}}$$

where each $(\mathbf{F}\sigma)_{\mathbf{n}}$ is a subgroupoid of $\mathbf{Hom}(\mathbf{B^n}, \mathbf{B}')$ "built up" with the following inductive scheme. $(\mathbf{F}\sigma)_{\mathbf{1}} = F : \mathbf{B} \longrightarrow \mathbf{B}'$. Assume $(\mathbf{F}\sigma)_{\mathbf{1}}, (\mathbf{F}\sigma)_{\mathbf{2}}, \ldots, (\mathbf{F}\sigma)_{\mathbf{n-1}}$ is defined, then $(\mathbf{F}\sigma)_{\mathbf{n}}$ has as objects

1. $FP_n^{\mathbf{B}}(f)$ where $f \in \mathbf{A_n}$
2. If $f_a \in (\mathbf{F}\sigma)_{\mathbf{a}}$ and $f_b \in (\mathbf{F}\sigma)_{\mathbf{b}}$ where $a + b = n$, then $\otimes' \circ (f_a \times f_b)$ denoted $f_a \otimes' f_b$.

Morphisms of $(\mathbf{F}\sigma)_{\mathbf{n}}$ are

1. $FP_n^{\mathbf{B}}(\alpha)$ where $\alpha \in \mathbf{A_n}$
2. If $\phi_a \in (F\sigma)_a$ and $\phi_b \in (F\sigma)_b$ where $a + b = n$, then $\otimes' \circ (\phi_a \times \phi_b)$ denoted $\phi_a \otimes' \phi_b$

3. Associativity isomorphisms inherited from the $\mathbf{B}\bar{\mathbf{B}}'$ structure

4. If $f = U_{a,b}(f_a, f_b)$, the isomorphism

$$FP_a^{\mathbf{B}}(f_a) \otimes' FP_b^{\mathbf{B}}(f_b) \xrightarrow{\sigma_{P(f_a),P(f_b)}} FP_n^{\mathbf{B}}(f) \cdot$$

3.2 The $\mathbf{M_n}$

We shall construct $\bar{\mathbf{M}}$, the free mapping funnel on one functor. $\bar{\mathbf{M}}$ will be an associative category that will be a groupoid. Objects will correspond to "associations of images". A typical element may correspond to

$$F((A \otimes B) \otimes C) \otimes' (F(D) \otimes' F(E \otimes G)).$$

The morphisms will correspond to reassociations of the source tensor, $\otimes$, reassociations of the target tensor, $\otimes'$, and "funneling into F" i.e. $FA \otimes' FB \longrightarrow F(A \otimes B)$. Formally, the objects will be associations of associated letters. That is, objects will be elements of the free anomic algebra with one "outer" binary operation, $\otimes'$. This algebra will not be over a set, rather, it will be over the elements of a free anomic algebra with one "inner" binary operation, $\otimes$. An example of such an object is

$$((A \otimes B) \otimes C) \otimes' (D \otimes' (E \otimes G))$$

which will correspond to the above "association of images". Warning: This is not the same as elements of the free "anomic" algebra with *two* binary operations on one generator. $A \otimes' (B \otimes C)$ is an association of associated letters and it corresponds to $FA \otimes' F(B \otimes C)$. In contrast, $A \otimes (B \otimes' C)$, an element of the free "anomic" algebra with two binary operations does not correspond to any "association of images".

Morphisms correspond to reassociations of both binary multiplications and "funneling" an outer multiplication into an inner multiplication. All the morphisms are isomorphisms so we are dealing with a groupoid. The tensor product is just taking two associations of associated letters and multiplying them with an outer multiplication.

For each positive integer n, we will construct a groupoid $\mathbf{M_n}$ which has as objects associations of associated letters of length n. Morphisms are as explained above. The free mapping funnel, $\bar{\mathbf{M}}$, will then be the disjoint union of all the $\mathbf{M_n}$ i.e.

$$\bar{\mathbf{M}} = \coprod_{n \in N^+} \mathbf{M_n}.$$

The groupoids $\mathbf{M_n}$ are created inductively. Let $\mathbf{M_1} = \mathbf{1}$. Assuming each of the $\mathbf{M_1}, \mathbf{M_2}, \ldots, \mathbf{M_{n-1}}$ has been constructed, then $\mathbf{M_n}$ is created in the following manner. Construct the following pushout:

$$\begin{tikzcd}
\coprod_{i+j+k=n} \mathbf{M_i} \times \mathbf{M_j} \times \mathbf{M_k} \times \dot{\mathbf{I}} \arrow[r, "W'_{i,j,k}"] \arrow[d, hook] & \coprod_{a+b=n} \mathbf{M_a} \times \mathbf{M_b} \arrow[d, "U'_{a,b}"] \\
\coprod_{i+j+k=n} \mathbf{M_i} \times \mathbf{M_j} \times \mathbf{M_k} \times \mathbf{I} \arrow[r, "V'_{i,j,k}"] & \mathbf{H_n}
\end{tikzcd} \tag{3.1}$$

where i, j, k, a and b range over all positive integers. The left hand vertical map is the obvious inclusion. The map $W'_{i,j,k}$ is defined for each $i, j,$ and k

as follows: let f, g, h be objects in $\mathbf{M_i}, \mathbf{M_j}, \mathbf{M_k}$ respectively then

$$W'_{i,j,k}(f,g,h,t) = \begin{cases} f, U'_{j,k}(g,h) & : \quad t = 0 \\ U'_{i,j}(f,g), h & : \quad t = 1 \end{cases}$$

where the U's are defined from the previous pushouts. $W'_{i,j,k}$ is defined for morphisms similarly. Once this pushout is constructed, we use the result, $\mathbf{H_n}$, to get the following pushout:

$$\begin{tikzcd}
\coprod_{a+b=n} \mathbf{A_a} \times \mathbf{A_b} \times \dot{\mathbf{I}} \arrow[r, "Z_{a,b}"] \arrow[d, hook] & \mathbf{H_n} \coprod \mathbf{A_n} \arrow[d, "X_n"] \\
\coprod_{a+b=n} \mathbf{A_a} \times \mathbf{A_b} \times \mathbf{I} \arrow[r, "Y_{a,b}"] & \mathbf{M_n}
\end{tikzcd} \tag{3.2}$$

where $Z_{a,b}$ is defined for each a and b as follows: let f, g be in $\mathbf{A_a}, \mathbf{A_b}$ respectively then

$$Z_{a,b}(f,g,t) = \begin{cases} U'_{a,b}(X_a(f), X_b(g)) \in \mathbf{H_n} & : \quad t = 0 \\ U_{a,b}(f,g) \in \mathbf{A_n} & : \quad t = 1 \end{cases}$$

$U_{a,b}$ is defined in Chapter 1, Section 2.

An explanation is needed. We want to funnel n distinct images into one image. We already know how to do it for $1, 2, \ldots, n-1$ letters. How do we do it for n letters? We split the n letters into two smaller parts and funnel all the images into the two images. This is the upper right hand corner of the top pushout. There are, however, reassociations between these different ways. This is taken care of by the left side of pushout 3.1. Once we have the two images, we still have to make it into one image. This is done by pushout 3.2. The $\mathbf{H_n}$ contains the double images. The $\mathbf{A_n}$ corresponds to the single images. The left side of pushout 3.2 connects them. $X_a(f)$ is the bottom - i.e. the single image - of $\mathbf{M_a}$. $U'_{a,b}(X_a(f), X_b(g))$ is the bottom of $\mathbf{H_n}$.

Following the conventions of Chapter 1, let's list the first few $\mathbf{M_n}$. In order to make it more readable, we use F as a way of grouping the images. The inner multiplication is shown as concatenation and the outer one is shown as $\otimes$.

- $\mathbf{M_1} = \mathbf{1} = F(A)$.
- $\mathbf{M_2}$

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[d, "\mathbf{A_1 \times A_1 \times I}"'] \\
F(AB).
\end{tikzcd}$$

- $\mathbf{M_3}$

$$\begin{tikzcd}
FA \otimes (FB \otimes FC) \arrow[r, "\mathbf{M_1 \times M_1 \times M_1 \times I}"] \arrow[d, "\mathbf{M_1 \times M_2}"'] & (FA \otimes FB) \otimes FC \arrow[d, "\mathbf{M_2 \times M_1}"] \\
FA \otimes F(BC) \arrow[d, "\mathbf{A_1 \times A_2 \times I}"'] & F(AB) \otimes FC \arrow[d, "\mathbf{A_2 \times A_1 \times I}"] \\
F(A(BC)) \arrow[r, "\mathbf{A_3}"] & F((AB)C).
\end{tikzcd}$$

- $\mathbf{M}_4$

$$\begin{tikzcd}
& FA((FBFC)FD) \arrow[rr] \arrow[ddd, dotted] & & (FA(FBFC))FD \arrow[dr] \arrow[ddd, dotted] & \\
FA(FB(FCFD)) \arrow[ur] \arrow[drr] \arrow[ddd] & & & & ((FAFB)FC)FD \arrow[ddd] \\
& & (FAFB)(FCFD) \arrow[urr] \arrow[dddl] \arrow[dddr] & & \\
& FA(F(BC)FD) \arrow[rr, dotted] \arrow[ddd, dotted] & & (FAF(BC))FD \arrow[ddd, dotted] & \\
FA(FBF(CD)) \arrow[dr] \arrow[ddd] & & & & (F(AB)FC)FD \arrow[ddd] \\
& (FAFB)F(CD) \arrow[dddr] & & F(AB)(FCFD) \arrow[ur] \arrow[dddl] & \\
& FAF((BC)D) \arrow[d, dotted] & & F(A(BC))FD \arrow[d, dotted] \arrow[dr, dotted] & \\
FAF(B(CD)) \arrow[ur, dotted] \arrow[d] & F(A((BC)D)) \arrow[rr, dotted] & & F(A(BC))D \arrow[dr, dotted] & F(A(BC))FD \arrow[d] \\
F(A(B(CD))) \arrow[ur, dotted] \arrow[drr] & & F(AB)F(CD) \arrow[d] & & F(((AB)C)D) \\
& & F((AB)(CD)) \arrow[urr] & &
\end{tikzcd}$$

We shall attempt to make this diagram a little more readable. The back left side of the prism is $\mathbf{M}_1 \times \mathbf{M}_3$. The back right side of the prism is $\mathbf{M}_3 \times \mathbf{M}_1$. The center front diamond is $\mathbf{M}_2 \times \mathbf{M}_2$. Both the top and the bottom are $\mathbf{A}_4$. All vertical maps are instances of σ. Horizontal maps are reassociations. The reader is advised not to stare at this Satanic diagram for long periods of time.

We define two functors from $\mathbf{A_n}$ to $\mathbf{M_n}$. One goes to the "top" and one goes to the "bottom" of $\mathbf{M_n}$. $Bot_n : \mathbf{A_n} \longrightarrow \mathbf{M_n}$ is defined as $Bot_n(f) = X_n(f)$. Similarly for morphisms. $Top_n : \mathbf{A_n} \longrightarrow \mathbf{M_n}$ is defined inductively. $Top_1 = Id_1 : \mathbf{1} = \mathbf{A_1} \longrightarrow \mathbf{1} = \mathbf{M_1}$. Assume $Top_1, Top_2, \ldots, Top_{n-1}$ is defined, then Top_n is defined on objects $f = U_{a,b}(f_a, f_b)$ as

$$Top_n(f) = X_n(U'_{a,b}(Top_a(f_a), Top_b(f_b))).$$

Similarly for morphisms of type $\phi = U_{a,b}(\phi_a, \phi_b)$. Morphisms of type $V_{i,j,k}(\phi, \gamma, \eta, \iota)$ are done as follows:

$$Top_n(V_{i,j,k}(\phi, \gamma, \eta, \iota)) = X_n(V'_{i,j,k}(Top_i(\phi), Top_j(\gamma), Top_k(\eta), \iota)).$$

It is not hard to show that Top_n is a well defined functor.

We also define a functor $Out_n : \mathbf{M_n} \longrightarrow \mathbf{A_n}$. This functor takes any association of associated images and sends it to the "bottom" of the funnel i.e. it funnels all its outer multiplications into inner multiplications. The functors are defined inductively. $Out_1 = Id_1 : \mathbf{M_1} = \mathbf{1} \longrightarrow \mathbf{A_1} = \mathbf{1}$. Assume Out_a is defined for $a \leq n-1$ then for objects

$$Out_n(f) = \begin{cases} f' & : \; f = X_n(f'), f' \in \mathbf{A_n} \\ U_{a,b}(Out_a(f_a), Out_b(f_b)) & : \; f = X_n(U'_{a,b}(f_a, f_b)), f_a \in \mathbf{M_a}, f_b \in \mathbf{M_b} \end{cases}.$$

For morphisms, we have

$$Out_n(\phi) = \begin{cases} \phi' & : \; \phi = X_n(\phi'), \phi' \in \mathbf{A_n} \\ U_{a,b}(Out_a(\phi_a), Out_b(\phi_b)) & : \; \phi = X_n(U'_{a,b}(\phi_a, \phi_b)), \phi_a \in \mathbf{M_a}, \phi_b \in \mathbf{M_b} \\ U_{a,b}(Id_f, Id_g) & : \; \phi = Y_{a,b}(Id_f, Id_g, \iota), f \in \mathbf{A_a}, g \in \mathbf{A_b} \end{cases}.$$

Lemma 3.1 *For all n,* $Out_n \circ Top_n = Id_{\mathbf{A_n}}$.

Proof. The proof is inductive. For n=1, it is trivial. For the general case, let $f = U_{a,b}(f_a, f_b)$

$$\begin{aligned} Out_n(Top_n(f)) &= Out_n(X_n(U'_{a,b}(Top_a(f_a), Top_b(f_b)))) \\ &= U_{a,b}(Out_a(Top_a(f_a)), Out_b(Top_b(f_b))) \\ &= U_{a,b}(f_a, f_b) \\ &= f. \quad \square \end{aligned}$$

Lemma 3.2 *For all n,* $Out_n \circ Bot_n = Id_{\mathbf{A_n}}$.

Proof. Clear. □

Lemma 3.3 *For all n,* $Out_n \dashv Bot_n$. *In fact* Out_n *is a reflection.*

Proof. The counit of the adjunction is the identity by the above lemma. The unit for each n, $\eta^n_f : f \longrightarrow Bot_n(Out_n(f))$ is given by the following inductive scheme. The base case is trivial. If $f = X_n(\tilde{f})$ where $\tilde{f} \in \mathbf{A_n}$ then

$$\eta^n_f = Id_f : f \longrightarrow \quad Bot(Out(f)) = Bot(\tilde{f}) = X(\tilde{f}) = f.$$

If $f = X_n(U'_{a,b}(f_a, f_b))$ then by the induction hypothesis, there are maps

$$\eta^a_{f_a} : f_a \longrightarrow Bot_a(Out_a(f_a))$$

and

$$\eta^b_{f_b} : f_b \longrightarrow Bot_b(Out_b(f_b)).$$

Then let

$$\eta_f^n = [X_n(Z_{a,b}(Out(f_a), Out(f_b), \iota)] \circ [X_n(U'_{a,b}(\eta^a_{f_a}, \eta^b_{f_b}))]$$

Where Z is defined after the bottom pushout in the definition of $\mathbf{M_n}$. η_f^n is given by the following composition:

$$f = X_n(U'_{a,b}(f_a, f_b)) \longrightarrow X_n(U'_{a,b}(Bot_a(Out_a(f_a)), Bot_b(Out_b(f_b))))$$

$$= X_n(U'_{a,b}(X_a(Out_a(f_a), X_b(Out_b(f_b)))) \longrightarrow X_n(U_{a,b}(Out(f_a), Out(f_b)))$$

$$= X_n(Out_n(f)) = Bot_n(Out_n(f)).$$

Some explanation: We need a map from any object $f \in \mathbf{M_n}$ to the bottom of the cylinder. There are two possibilities. First, f could already be at the bottom, then let the unit be the identity. In general, f will be somewhere in the middle i.e. in some component $\mathbf{M_a} \times \mathbf{M_b}$. By the induction hypothesis, there are already maps from the middle of these components to the bottom of them. Take the product of those maps to get to the bottom of that $\mathbf{H_n}$. Now all that's left is to get from the bottom of $\mathbf{H_n}$ to the bottom of $\mathbf{M_n}$. That is what the $Z_{a,b}$ does.

The rest of the proof of the adjunction is clear. □

We have the following picture that will stir the hearts of any homotopy theorist who knows about mapping cylinders:

$$\begin{tikzcd}
\mathbf{A_n} \arrow[r, "Top_n"] \arrow[rr, bend left, "Id_{\mathbf{A_n}}"] & \mathbf{M_n} \arrow[r, "Out_n"] & \mathbf{A_n} \arrow[l, bend left, "Bot_n"]
\end{tikzcd} .$$

3.3 The free mapping funnel

We are ready to define the free mapping funnel

$$(\bar{\mathbf{M}} = \coprod \mathbf{M_n}, \otimes_{\bar{\mathbf{M}}}, \alpha_{\bar{\mathbf{M}}}).$$

The tensor product and associativity is defined as follows: Let $\phi : f \longrightarrow f'$ be in $\mathbf{M_i}$, $\gamma : g \longrightarrow g'$ be in $\mathbf{M_j}$ and $\eta : h \longrightarrow h'$ be in $\mathbf{M_k}$. Then

$$f \otimes_{\bar{\mathbf{M}}} g = X_{i+j} U'_{i,j}(f,g).$$

$$\alpha_{\bar{\mathbf{M}},f,g,h} = X_n \circ V'_{i,j,k}(Id_f, Id_g, Id_h, \iota) : f \otimes (g \otimes h) \longrightarrow (f \otimes g) \otimes h.$$

Naturality is from the functoriality of $X_n \circ V'_{i,j,k}$. $\alpha_{\bar{\mathbf{M}}}$ is not coherently associative.

The Top_n (resp. Bot_n): $\mathbf{A_n} \longrightarrow \mathbf{M_n}$ induce functors $\bar{Top}(\bar{Bot})$: $\bar{\mathbf{A}} \longrightarrow \bar{\mathbf{M}}$. $\bar{Top}$ is a strict tensor functor as can be seen from the following commuting square:

$$\begin{tikzcd}
f,g \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & U_{a,b}(f,g) \arrow[ddd, mapsto] \\
& \bar{\mathbf{A}} \times \bar{\mathbf{A}} \arrow[r, "\otimes_{\bar{\mathbf{A}}}"] \arrow[d, "\bar{Top} \times \bar{Top}"'] & \bar{\mathbf{A}} \arrow[d, "\bar{Top}"] & \\
& \bar{\mathbf{M}} \times \bar{\mathbf{M}} \arrow[r, "\otimes_{\bar{\mathbf{M}}}"'] & \bar{\mathbf{M}} & \\
Top_a(f), Top_b(g) \arrow[rrr, mapsto] & & & \begin{array}{c} Top_n(U_{a,b}(f,g)) = \\ X_n(U'_{a,b}(Top_a(f), Top_b(g))) \end{array}
\end{tikzcd}.$$

The lower right-hand equality holds by the definition of Top_n. There is a similar square for morphisms. $\bar{Bot}$ is not a strict tensor functor.

Proposition 3.1 (Universality of $\bar{\mathbf{M}}$) *For any multiplicative functor* (F, σ) : $(\mathbf{B}, \otimes, \beta) \longrightarrow (\mathbf{B}', \otimes', \beta')$ *with associated mapping funnel* $\bar{\mathbf{F}}\sigma$*, there is a unique strict tensor functor*

$$P' : \bar{\mathbf{M}} \longrightarrow \bar{\mathbf{F}}\sigma \hookrightarrow \mathbf{B}\bar{\mathbf{B}}'$$

such that

1. $* \in \mathbf{M}_1 \longmapsto F \in (\mathbf{F}\sigma)_1$
2. f *(the unique nontrivial isomorphism in* $\mathbf{M}_2$*)* $\longmapsto \sigma \in (\mathbf{F}\sigma)_2$
3. $X_3(V'_{1,1,1}(Id_*, Id_*, Id_*, \iota)) \in \mathbf{M}_3 \longmapsto \beta'(F \times F \times F) \in (\mathbf{F}\sigma)_3$
4. *Let* α *be the unique nontrivial isomorphism in* $\mathbf{A}_3$*, then* $X_3(\alpha) \in \mathbf{M}_3 \longmapsto F(\beta) \in (\mathbf{F}\sigma)_3$
5. *The following two triangles commute*

$$\begin{tikzcd}
 & \bar{\mathbf{A}} \arrow[dl, "p^{\mathbf{B}}"'] \arrow[d, shift right, "\bar{Bot}"'] \arrow[d, shift left, "\bar{Top}"] \arrow[dr, "p^{\mathbf{B}'}"] & \\
\mathrm{It}(\bar{\mathbf{B}}) \arrow[dr, "F\circ -"'] & \bar{\mathbf{M}} \arrow[d, "P'"] & \mathrm{It}(\bar{\mathbf{B}}') \arrow[dl, "\{-\circ F^n\}"] \\
 & \bar{\mathbf{F}}\sigma. &
\end{tikzcd}$$

Proof. As in Proposition 1.1 of Section 1.3, P' can be described by its components $P'_n : \mathbf{M_n} \longrightarrow (\mathbf{F}\sigma)_{\mathbf{n}}$,. $P'_1(*) = F$ as needed by requirement 1). P'_2 and P'_3 are forced by requirements 2), 3) and 4). Assume $P'_1, P'_2, \ldots, P'_{n-1}$ are defined, then P'_n is given by the following.

$$\begin{tikzcd}
\coprod_{i+j+k=n} \mathbf{M_i} \times \mathbf{M_j} \times \mathbf{M_k} \times \dot{\mathbf{I}} \arrow[r, "W'_{i,j,k}"] \arrow[d, hook] & \coprod_{a+b=n} \mathbf{M_a} \times \mathbf{M_b} \arrow[d, "U'_{a,b}"] \arrow[rd, "\varphi'_{a,b}"] & \\
\coprod_{i+j+k=n} \mathbf{M_i} \times \mathbf{M_j} \times \mathbf{M_k} \times \mathbf{I} \arrow[r, "V'_{i,j,k}"] \arrow[rrd, "\psi'_{i,j,k}"'] & \mathbf{H_n} \arrow[rd, dashed, "J_n"] & \\
 & & (\mathbf{F}\sigma)_{\mathbf{n}}.
\end{tikzcd}$$

$\varphi'_{a,b}$ is defined on objects as

$$\varphi'_{a,b}(f,g) = P'_a(f) \otimes' P'_b(g).$$

Similarly for morphisms. $\psi'_{i,j,k}$ is defined on objects as

$$\psi'_{i,j,k}(f,g,h,t) = \begin{cases} P'_i(f) \otimes' (P'_j(g) \otimes' P'_k(h)) & : \quad t = 0 \\ (P'_i(f) \otimes' P'_j(g)) \otimes' P'_k(h) & : \quad t = 1 \end{cases}.$$

For morphisms:

$$\psi'_{i,j,k}(\phi,\gamma,\eta,\iota) = \bar{\beta}'_{P'_i(\phi),P'_j(\gamma),P'_k(\eta)}.$$

It is easy to see that the outer square commutes, and this induces a functor $J_n : \mathbf{H_n} \longrightarrow (\mathbf{F}\sigma)_{\mathbf{n}}$. We use this map in the following

$$\begin{tikzcd}
\coprod_{a+b=n} \mathbf{A_a} \times \mathbf{A_b} \times \dot{\mathbf{I}} \arrow[r, "Z_{a,b}"] \arrow[d, hook] & \mathbf{H_n} \coprod \mathbf{A_n} \arrow[d, "X_n"] \arrow[ddr, "\varphi''_n"] & \\
\coprod_{a+b=n} \mathbf{A_a} \times \mathbf{A_b} \times \mathbf{I} \arrow[r, "Y_{a,b}"] \arrow[drr, "\psi''_{a,b}"'] & \mathbf{M_n} \arrow[dr, dashed, "P'_n"] & \\
 & & (\mathbf{F}\sigma)_\mathbf{n}
\end{tikzcd}$$

The maps are defined as follows

$$\varphi''_n = J_n \coprod (F \circ P^{\mathbf{B}}_n),$$

$$\psi''_{i,j,k}(f,g,t) = \begin{cases} F \circ P^{\mathbf{B}}_a(f) \otimes' F \circ P^{\mathbf{B}}_b(g) & : \quad t = 0 \\ F \circ P^{\mathbf{B}}_n(U_{a,b}(f,g)) & : \quad t = 1 \end{cases}$$

And

$$\psi''_{i,j,k}(\phi, \gamma, \iota) = \sigma_{P^{\mathbf{B}}_a(\phi), P^{\mathbf{B}}_b(\gamma)}.$$

The outer square commutes because

$$\begin{aligned}
\varphi''_n(Z_{a,b}(f,g,0)) &= \varphi''_n(U'_{a,b}(X_a(f), X_b(g))) \\
&= J_n(U'_{a,b}(X_a(f), X_b(g))) \\
&= \varphi'_{a,b}(X_a(f), X_b(g)) \\
&= P'_a(X_a(f)) \otimes' P'_b(X_b(g)) \\
&= \varphi''_a(f) \otimes' \varphi''_b(g) \\
&= F \circ P^{\mathbf{B}}_a(f) \otimes' F \circ P^{\mathbf{B}}_b(g) \\
&= \psi''_{a,b}(f,g,0).
\end{aligned}$$

And

$$\begin{aligned}
\varphi''_n(Z_{a,b}(f,g,1)) &= \varphi''_n(U_{a,b}(f,g)) \\
&= F \circ P^{\mathbf{B}}_a(U_{a,b}(f,g)) \\
&= \psi''_{a,b}(f,g,1).
\end{aligned}$$

This induces a functor $P'_n : \mathbf{M_n} \longrightarrow (\mathbf{F}\sigma)_{\mathbf{n}}$. P' is then the functor induced by all the P'_n. P' is a strict tensor functor because the following square commutes:

$$\begin{tikzcd}
f,g \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & X_n(U'_{a,b}(f,g)) \arrow[ddd, mapsto] \\
& \bar{\mathbf{M}} \times \bar{\mathbf{M}} \arrow[r, "\otimes_{\bar{\mathbf{M}}}"] \arrow[d, "P' \times P'"'] & \bar{\mathbf{M}}' \arrow[d, "P"] & \\
& \bar{\mathbf{F}}\sigma \times \bar{\mathbf{F}}\sigma \arrow[r, "\otimes'"'] & \bar{\mathbf{F}}\sigma & \\
P'_a(f), P'_b(g) \arrow[rrr, mapsto] & & & P'_n(X_n(U'_{a,b}(f,g))) = P'_a(f) \otimes' P'_b(g)
\end{tikzcd}.$$

The lower right-hand equality holds because

$$\begin{aligned}
P'_n(X_n(U'_{a,b}(f,g))) &= \varphi''_n(U'_{a,b}(f,g)) \\
&= J_n(U'_{a,b}(f,g)) \\
&= \varphi'_n(f,g) \\
&= P'_a(f) \otimes' P'_b(g).
\end{aligned}$$

The left triangle of part 5) of the proposition commutes because

$$\begin{aligned} P'(\bar{B}ot(f)) &= P'_n(Bot_n(f)) \\ &= P'_n(X_n(f)) \\ &= \varphi''_n(f) \\ &= F \circ P^{\mathbf{B}}_n(f). \end{aligned}$$

The right triangle of part 5) commutes. This can be seen with an inductive proof.

$$P'_1(Top_1(*)) = F \circ Id = P_1 \circ F.$$

Now assume for $a < n$ that $P'_a(Top_a(f)) = P_a(f) \circ F^a$. Then for $f = U_{a,b}(f_a, f_b) \in \mathbf{A_n}$ we have

$$\begin{aligned} P'(T\bar{o}p(f)) &= P'_n(Top_n(f)) \\ &= P'_n(X_n(U'_{a,b}(Top_a(f_a), Top_b(f_b)))) \\ &= \varphi''_n(U'_{a,b}(Top_a(f_a), Top_b(f_b))) \\ &= J_n(U'_{a,b}(Top_a(f_a), Top_b(f_b))) \\ &= \varphi'_{a,b}(Top_a(f_a), Top_b(f_b))) \\ &= P'_a(Top_a(f_a)) \otimes' P'_b(Top_b(f_b)) \\ &= P_a(f_a) \circ F^a \otimes' P_b(f_b) \circ F^b \\ &= \otimes' \circ [(P_a(f_a) \circ F^a) \times (P_b(f_b) \circ F^b)] \\ &= \otimes' \circ [(P_a(f_a) \times P_b(f_b)) \circ (F^a \times F^b)] \\ &= P_n(f) \circ (F^n) \end{aligned}$$

In order to prove the commutativity of the two triangles, we used the functors $\varphi', \psi', \varphi''$ and ψ''. Uniqueness of P' follows from the fact that these and only these maps make the triangles commute. □

3.4 The 2-sketch of mapping funnels

Most of the details of this section will be omitted. They have been written down and the constructions and proofs are similar - but more complicated - to those of Section 1.4. We assume knowledge of Section 1.4 and use the same notation and conventions. Only the major definitions and highlights will be given. The author realizes that he has trespassed far beyond the readers patience.

The objects of the 2-sketch of mapping funnels, $\mathbf{M}$, are the natural numbers. The 1-cells and 2-cells are given as

$$\mathbf{Hom_M(n,k)} = \coprod_{n_1+n_2+\cdots+n_k=n} \mathbf{M_{n_1}} \times \mathbf{M_{n_2}} \times \cdots \times \mathbf{M_{n_k}} = \coprod_{\pi:n\to k} \prod_{i=1}^{k} \mathbf{M}_{|\pi^{-1}(\mathbf{i})|}$$

where π ranges over all partitions of n into k parts. 1-cells are written as $f = (f_1, f_2, \dots, f_k)$. The same for 2-cells. The tensor product of this category is addition of the 0-cells and concatenation of the 1 and 2-cells. Vertical composition is simply composition in each $\mathbf{M_l}$. Horizontal composition is given by the operad Q'. Q' is constructed inductively. First, the base case: We set

$$Q'_{1,t,\pi} : \mathbf{M_1} \times \mathbf{M_t} \longrightarrow \mathbf{M_t}$$

as

$$Q'_{1,t,\pi}(*, f) = X_t(Out_t(f)).$$

Now assume every $Q'_1, Q'_2, \ldots, Q'_{n-1}$ is defined for every t and for every partition π. We define $Q'_{n,t,\pi}$ for a given t and π on objects as

$$Q'_{n,t,\pi}(f, g_1, g_2, \ldots, g_n) =$$

$$\begin{cases} X_t(U'_{t_a,t_b}(Q'_{a,t_a,\pi_a}(f_a, g_1, g_2, \ldots, g_a), Q'_{b,t_b,\pi_b}(f_b, g_{a+1}, \ldots, g_{a+b}))) & : \quad f = X_n(U'_{a,b}(f_a, f_b)) \\ X_t(Q_{n,t,\pi}(\tilde{f}, Out_{m_1}(g_1), \ldots, Out_{m_n}(g_n))) & : \quad f = X_n(\tilde{f}), \tilde{f} \in \mathbf{A_n} \end{cases}$$

There are two possible ways of getting an object f in $\mathbf{M_n}$: from the top or the bottom pushout of the definition of $\mathbf{M_n}$. If f is from the top pushout, then f is made of two parts and we use Q' on each part. If, on the other hand, f is from the bottom pushout and is at the bottom of the funnel, then we "take out" all the g's and use the Q operad.

The definition of Q' on morphisms is complicated. There are five possible ways of getting a morphism:

- The upper right-hand of pushout 3.1 i.e. $\phi = X_n(U'_{a,b}(\phi_a, \phi_b)$.
- The lower left-hand of pushout 3.1 i.e. $\phi = X_n(V'_{i,j,k}(\phi_i, \phi_j, \phi_k, \iota))$.
- The $\mathbf{A_n}$ of the upper right hand of pushoutbotpo. There are two ways to be in $\mathbf{A_n}$:
  - i.e. $\phi = X_n(U_{a,b}(\phi_a, \phi_b)$.
  - i.e. $\phi = X_n(V_{i,j,k}(\phi_i, \phi_j, \phi_k, \iota))$.

- The connection from the lower left of pushout 3.2 i.e. $\phi = Y_{a,b}(\phi_a, \phi_b, \iota)$

We define Q' on morphisms in the corresponding 5 ways:

$$Q'_{n,t,\phi}(\phi, \gamma_1, \ldots, \gamma_n) =$$

$$\begin{cases} X_t(U'_{a,b}(Q'_{a,t_a,\pi_a}(\phi_a, \gamma_1, \ldots, \gamma_a), Q'_{b,t_b,\pi_b}(\phi_b, \gamma_{a+1}, \ldots, \gamma_{a+b}))) \\ X_t(V'_{t_i,t_j,t_k}(Q'_{i,t_i,\pi_i}(\phi_i, \gamma_1, \ldots, \gamma_i), Q'_{j,t_j,\pi_j}(\phi_j, \gamma_{i+1}, \ldots, \gamma_{i+j}), Q'_{k,t_k,\pi_k}(\phi_k, \gamma_{i+j+1}, \ldots, \gamma_n), \iota))) \\ X_t(U_{a,b}(Q_{a,t_a,\pi_a}(\phi_a, Out(\gamma_1), \ldots, Out(\gamma_a)), Q_{b,t_b,\pi_b}(\phi_b, Out(\gamma_{a+1}), \ldots, Out(\gamma_{a+b})))) \\ X_t(V_{t_i,t_j,t_k}(Q_{i,t_i,\pi_i}(\phi_i, Out(\gamma), \ldots,), Q_{j,t_j,\pi_j}(\phi_j, \ldots,), Q_{k,t_k,\pi_k}(\phi_k, Out(\gamma), \ldots, Out(\gamma)), \iota))) \\ X_t(Y_{t_a,t_b}(Q_{a,t_a,\pi_a}(\phi_a, Out(\gamma_1), \ldots, Out(\gamma_a)), Q_{b,t_b,\pi_b}(\phi_b, Out(\gamma_{a+1}), \ldots, Out(\gamma_{a+b})), \iota)). \end{cases}$$

Till now, we have basically followed Section 4 of Chapter 1. However mapping funnels are more unruly then associative categories. Whereas there is a lemma concerning the "associativity" of the Q operad, there is no similar lemma for the Q' operad. The two ways of associating Q' are not equal but there is an isomorphism between them. The isomorphism is the counit of the $Bot_n \vdash Out_n$ adjunction. The details are long and tedious.

The lack of associativity of the operad has a major effect on the 2-sketch **M**. Whereas **A** is a strict tensor 2-category, **M**, in contrast, is a strict tensor bicategory i.e. the horizontal composition is associative only up to a 2-cell isomorphism. This can all be said succinctly in the new and powerful language of tricategories [5] (a tricategory with one object is a 2 dimensional category with a tensor product), however, we do not find that this language adds to our understanding of the 2-sketch.

The fact that the 2-sketch is a bicategory has an influence on the category of algebras of the 2-sketch. An algebra for **M**, or equivalently, a mapping funnel, will be a strict tensor 2-functor $F : \mathbf{M} \longrightarrow \mathbf{Cat}$. The problem of associativity means that composition of the mapping funnels will not be associative. This fact should not be surprising to anyone familiar with the properties of a mapping *cylinder* in algebraic topology. Composition of mapping cylinders are associative only up to homotopy. The mapping funnels share many properties of the mapping cylinder.

The maps $Top_n, Bot_n : \mathbf{A_n} \longrightarrow \mathbf{M_n}$ induce the following functors of the 2-sketches

$$TOP, BOT : \mathbf{A} \longrightarrow \mathbf{M}.$$

Given two associative categories, $R, S : \mathbf{A} \longrightarrow \mathbf{Cat}$ a multiplicative functor, F, from R to S is a strict tensor 2-functor (erroneously) called F from $\mathbf{M}$ to $\mathbf{Cat}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{A} \arrow[r, shift left, "TOP"] \arrow[r, shift right, "BOT"'] \arrow[rr, bend left, "R"] \arrow[rr, bend right, "S"'] & \mathbf{M} \arrow[r, "F"] & \mathbf{Cat}.
\end{tikzcd}$$

We note in passing that we could have gone about defining mapping funnels as a type of strong natural 2-isomorphism. We give the definition for the 2-isomorphism needed in this case, rather then an abstract version ([13] has a discussion about similar 2-morphisms, however, we do not abide by their notation or names.) A strong natural 2-isomorphism $F : R \longrightarrow S$ is a pair $F = (F_0, F_1)$ such that for each 0-cell n of $\mathbf{A}$ there is a functor in

Cat $F_0(n) : R(n) \longrightarrow S(n)$ and for each 1-cell $f : n \longrightarrow m$ of **A** there is a natural isomorphism $F_1(f)$ that "completes" the following square

$$\begin{tikzcd}
R(n) \arrow[r, "F_0(n)"] \arrow[d, "R(f)"'] & S(n) \arrow[d, "S(f)"] \arrow[dl, Rightarrow, "F_1(f)"] \\
R(m) \arrow[r, "F_0(m)"'] & S(m).
\end{tikzcd}$$

These functors and transformations must satisfy the following:

1. For any 2-cell $\alpha : f \longrightarrow g$ in **A** we have

$$\begin{tikzcd}
R(n) \arrow[r, "F(n)"] \arrow[d, bend right=60, "R(f)"'] \arrow[d, bend left=60, "R(g)"] \arrow[d, phantom, "\overset{\Rightarrow}{R(\alpha)}"] & S(n) \arrow[d, bend right=60, "S(f)"'] \arrow[d, bend left=60, "S(g)"] \arrow[d, phantom, "\overset{\Rightarrow}{S(\alpha)}"] \arrow[dl, Rightarrow, shift left] \arrow[dl, Rightarrow, shift right] \\
R(m) \arrow[r, "F(m)"'] & S(m)
\end{tikzcd}$$

   We require

$$S(\alpha) \circ F_1(f) = F_1(g) \circ R(\alpha).$$

   (A usual lax 2-natural transformation would demand the existence of a 3-cell between these 2-cells. We demand this 3-cell to be an identity 3-cell.)

2. $F = (F_0, F_1)$ must respect the tensor structure on **A**. i.e.

$$F_0(n + m) = F_0(n) \times F_1(m)$$

and

$$F_1(f \otimes g) = F_1(f) \times F_1(g).$$

3. F must respect horizontal composition. If

$$\begin{tikzcd} n \arrow[r, "f"] \arrow[rr, bend left, "g \circ_H f"] & m \arrow[r, "g"] & l, \end{tikzcd}$$

then we have

$$\begin{tikzcd} R(n) \arrow[r, "F_0(n)"] \arrow[d, "R(f)"'] & S(n) \arrow[d, "S(f)"] \arrow[dl, Rightarrow, "F_1(f)"'] \\ R(m) \arrow[r, "F_0(m)"] \arrow[d, "R(g)"'] & S(m) \arrow[d, "S(g)"] \arrow[dl, Rightarrow, "F_1(g)"'] \\ R(l) \arrow[r, "F_0(l)"'] & S(l). \end{tikzcd}$$

We require that $F_1(g \circ_H f) = F_1(g) \circ_{VB} F_1(f)$ where $\circ_{VB}$ means the usual vertical box composition. Horizontal box composition has to do with composition of such strong natural 2-isomorphisms.

The above definition of a multiplicative functor is equivalent to the one given in section 1 of this chapter. However this definition is not at all helpful in calculating their fundamental groups.

Chapter 4

The Fundamental Groups of the $\mathbf{M_n}$

4.1 Preliminaries

We assume the notations of Chapter 2. The method of Maximal Indiscrete Subgroupoids (MIS) are employed here to calculate the fundamental groups, $\pi(\mathbf{M_n})$. We shall simply give the inductive scheme to get the generators ($\mathcal{G}_n$) and relations ($\mathcal{R}_n$) of these groups. Since the $\mathbf{A_n}$ are used in the construction of the $\mathbf{M_n}$, the groups $\pi(\mathbf{A_n}) =< G_n; R_n >$ will be used in the construction of $\pi(\mathbf{M_n}) =< \mathcal{G}_n; \mathcal{R}_n >$.

The generators are again equivalence classes of morphisms of the groupoid. Reassociation morphisms are either in $\mathbf{A_n}$ and are denoted $< i, j, k >$ as in Chapter 2, or they are in $\mathbf{H_n}$ and they are denoted $[i, j, k]$. The funneling morphisms that connect $\mathbf{H_n}$ to $\mathbf{A_n}$ are denoted $|a,b|$. We employ the same free monoid on two generators (λ and ρ) to show the "history" of the generators.

In order to see the new notation more clearly, we have the following diagrams which mimic the definition of $\mathbf{M_n}$ given in Section 3.2.

$$\begin{tikzcd}
\mathit{Relations} \arrow[r] \arrow[d, hook] & \coprod < (\rho\mathcal{G}) \cup (\lambda\mathcal{G}); (\rho\mathcal{R}) \cup (\lambda\mathcal{R}) > \arrow[d] \\
\coprod_{i+j+k=n} [i,j,k] \arrow[r] & \mathbf{H_n}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathit{Relations} \arrow[r] \arrow[d, hook] & \mathbf{H_n} \coprod < G_n; R_n > \arrow[d] \\
\coprod_{a+b=n} |a,b| \arrow[r] & \pi(\mathbf{M_n}) =< \mathcal{G}_n; \mathcal{R}_n > .
\end{tikzcd}$$

We employ a similar type of MIS for the $\mathbf{M_n}$ as was done for the $\mathbf{A_n}$. For the $\mathbf{A_n}$ our MIS was all generators of the form $< i, 1, k >$. For the MIS of $\mathbf{M_n}$ we use all generators of the form $[i, 1, j]$. That is to say, we "mod out" all generators of the form $[i, 1, k]$. Considering $< i, 1, k >$ and $[i, 1, k]$ as relations is forced on us by the fact that we want Top_n, Bot_n and Out_n to extend to group homomorphisms:

$$\begin{tikzcd}
\pi(\mathbf{A_n}) \arrow[r, shift left, "BOT_n"] \arrow[r, shift right, "TOP_n"'] & \pi(\mathbf{M_n}) \arrow[r, "OUT_n"] & \pi(\mathbf{A_n}).
\end{tikzcd}$$

There are, however, options for the generators that connect $\mathbf{H_n}$ and $\mathbf{A_n}$.

In order to make sure that we have no loops, we must have only one generator that connects these two MIS. We arbitrarily use $|1, n-1|$.

Relations coming from the fact that $[i,j,k]$ is natural are similar to the relations from the fact that $< i,j,k >$ is natural. They are repeated in the scheme. The fact that $|a,b|$ is natural must be handled in a more subtle way. This diagram will help us see what is going on. We draw the $|a,b|$ vertically because that is the way they look in our diagrams of the $\mathbf{M_n}$.

$$\begin{tikzcd}
F(f) \otimes F(g) \arrow[r, "F(\alpha)\otimes F(\beta)"] \arrow[d, "|a{,}b|"'] & F(f') \otimes F(g') \arrow[d, "|a{,}b|"] \\
F(f \otimes g) \arrow[r, "F(\alpha\otimes\beta)"] & F(f' \otimes g')
\end{tikzcd}$$

This commutes iff the following commutes

$$\begin{tikzcd}
F(f) \otimes F(g) \arrow[r, "F(\alpha)\otimes F(Id)"] \arrow[d, "|a{,}b|"'] & F(f') \otimes F(g) \arrow[r, "F(Id)\otimes F(\beta)"] \arrow[d, "|a{,}b|"] & F(f') \otimes F(g') \arrow[d, "|a{,}b|"'] \\
F(f \otimes g) \arrow[r, "F(\alpha\otimes Id)"] & F(f' \otimes g) \arrow[r, "F(Id\otimes\beta)"] & F(f' \otimes g')
\end{tikzcd}$$

Notice the top horizontal map is in $\mathbf{H_n}$ and the bottom one is in $\mathbf{A_n}$. The squares commutes by naturality and on the left side we have

$$|a,b| * \rho^b x * |a,b|^{-1} = \rho^b \tilde{x}$$

for every $x \in G_a$; and $\tilde{x}$ is the its analog in $\mathcal{G}_a$ (convert the pointed brackets to square ones). There is a similar equation for the right-hand side. They are given below.

4.2 Presentation of the groups

The first group, $\pi(\mathbf{M_1})$ is trivial. $\mathcal{G}_1 = \mathcal{R}_1 = \emptyset$. Assume we have all the $\mathcal{G}_k$ and $\mathcal{R}_k$ for $k \leq n-1$, then $\mathcal{G}_n$ and $\mathcal{R}_n$ are given by:

- Generators. $\mathcal{G}_n =$
    1. The old generators from components of $\mathbf{H_n}$:
       $$\{(\lambda^i \# \mathcal{G}_{n-i}) \cup (\rho^{n-i} \# \mathcal{G}_i) : i = 1, 2, \ldots, n-1\}.$$
    2. The new generators of $\mathbf{H_n}$:
       $$\{[i, j, k] : i + j + k = n\}.$$
    3. The generators of $\mathbf{A_n}$:
       $$G_n.$$
    4. The generators that connect $\mathbf{H_n}$ to $\mathbf{A_n}$:
       $$\{|a, b| : a + b = n\}.$$
- Relations. $\mathcal{R}_n =$
    1. The relations from old components of $\mathbf{H_n}$:
       $$\{(\lambda^i \# \mathcal{R}_{n-i}) \cup (\rho^{n-i} \# \mathcal{R}_i) : i = 1, 2, \ldots, n-1\}.$$
    2. The relations for the MIS:
       $$\{[i, 1, k] = e : i + 1 + k = n\}.$$

3. The relations of $\mathbf{A_n}$:

$$R_n.$$

4. The single generator that connects the MIS of $\mathbf{H_n}$ to $\mathbf{A_n}$:

$$|1, n-1| = e.$$

5. Relations from the naturality of $[i, j, k]$: For each $[i, j, k]$ such that $i + j + k = n$ we have the union of the following relations

   (a)

   $$\{[i, j, k] * \rho^k \rho^j x * [i, j, k]^{-1} = \rho^{j+k} x : x \in \mathcal{G}_i\}.$$

   (b)

   $$\{[i, j, k] * \rho^k \lambda^i y * [i, j, k]^{-1} = \lambda^i \rho^k y : y \in \mathcal{G}_j\}.$$

   (c)

   $$\{[i, j, k] * \lambda^{i+j} z * [i, j, k]^{-1} = \lambda^i \lambda^j z : z \in \mathcal{G}_k\}.$$

6. Relations from the naturality of $|a, b|$: For each $|a, b|$ such that $a + b = n$ we have the union of the following relations

   (a)

   $$\{|a, b| * \rho^b x * |a, b|^{-1} = \rho^b \tilde{x} : x \in G_a\}$$

   (b)

   $$\{|a, b| * \lambda^a x * |a, b|^{-1} = \lambda^a \tilde{x} : x \in G_b\}$$

7. Relations from commutativity of generators of $\mathbf{H_n}$.

$$\{\rho^{n-i} x * \lambda^i y = \lambda^i y * \rho^{n-i} x : x \in \mathcal{G}_i, y \in \mathcal{G}_{n-i}\}$$

There is much work left to do in this chapter. We must look at quotients of $\mathbf{M_n}$ and how they interact with quotients of $\mathbf{A_n}$. How they behave in relation to the short (exact?) sequence

$$\pi(\mathbf{A_n}) \xrightarrow{\mathcal{TOP}_n} \pi(\mathbf{M_n}) \xrightarrow{\mathcal{OUT}_n} \pi(\mathbf{A_n})?$$

What information can we recover about an associative category from the multiplicative functors coming in and leaving such a category? There is a standard coherence theorem that any monoidal category is *tensor* equivalent to a strict monoidal category. What happens if we only have an associative category? Can we tell what type of multiplicative functor is needed to "fix up" the category?

Much work remains to be done, but time is short and I must stop here.

Chapter 5

Future Directions

The plan for the future is to continue research in the general area of this thesis. Until now, we have been concerned with associativity and tensor functor coherence conditions. Now, we would like to go on to other coherence problems. However, before going ahead to other areas, we intend to apply some of the results of this paper to the area of quantum groups and Drinfeld algebras.

Most of the second half of [21] is dedicated to calculating the cohomology of Drinfeld algebras (bialgebras that are coassociative only up to a coherent isomorphism.) It would be interesting to study the relationship of these cohomology groups to the cohomology of bialgebras that are coassociative only up to an isomorphism - without a coherence requirement. Such bialgebras arise when representing (in the sense of, say, [25]) an associative category. We feel that the groups presented in Chapter 2 will also be of importance to the study of pentagonal algebras and homotopy associative (Lie) algebras (see [23] .)

We have not closed the door on the study of associativity. Many of the assumptions made in this work can be relaxed for more interesting computations. For example, in this thesis, the reassociations is always considered an *iso*morphism. What happens if we relax this requirement and only ask for a morphism? We would then be working with general categories rather then (Catalan) groupoids. Would we get a fundamental group or a fundamental *monoid*? Similarly for multiplicative functors, we assumed there is an isomorphism $F(A) \otimes' F(B) \longrightarrow F(A \otimes B)$. What happens if we loosen this requirement and ask only for a morphism between them? These questions are not asked merely for the sake of petty generalizations. Rather, these situations are known to arise "in nature". For instance, [25] is concerned with braidings that are not necessarily isomorphisms (termed "pre-braidings".) Also, when using the forgetful functor, $U : \mathbf{R-Mod} \longrightarrow \mathbf{Ab}$ from the module category of an arbitrary commutative ring R to the category of abelian groups, the morphism $U(A) \otimes U(B) \longrightarrow U(A \otimes B)$ is in general not an isomorphism.

The next coherence requirement that we would like to look at is commutativity. This area is of utmost importance to the study of quantum groups, quasi-triangular Hopf algebras, quantum field theory, etc.. [8] has given a three-level hierarchy of coherence requirements for commutativity:

- symmetric

- balanced and
- braided.

Each of these coherence conditions correspond to different algebraic structures. Is this hierarchy complete? Are there intermediate levels? Each one of these coherence rules corresponds to "filling in" part of the permutoassociahedrons ([9]). we would like to look at the fundamental groups of each of the related polytopes and see if there are any other interesting coherence conditions.

Another area of extreme interest is categorical logic and coherence. We would like to look more carefully at the coherence requirements for cartesian closed categories. It has been shown (e.g. [18] for a general survey, [14], [12]) that these coherence requirements are intimately related to the cut-elimination theory which is central to proof theory of (intuitionistic) logic. The goal would be to furnish a classification of categories that fail this coherence condition and hence to see what can be learned about logical systems in which the cut-elimination theorem fails. Coherence requirements have also shown up in the new and exciting area of linear logic. Linear logic deals with – the more general – *monoidal* closed categories (see e.g. [24]).

There are numerous other coherence problems which we can explore using generalizations of methods used in this thesis. For example, we can look at Laplaza's distributivity categories, Shum's tortile tensor categories, Crane and Yetter's Hopf categories etc.. These – and many other areas – are intriguing.

The long term goals are to study categories and n-categories as forms of higher dimensional algebras. Recently, these algebras have become quite popular with algebraic topologists (R. Brown and A. Grothendieck's n-groupoids) and the mathematical world in general. Even mathematical physicists (see e.g. [2] or [1])have started to get interested in such structures. In 1972, Kelley ([11]) formulated the notion of a club in order to present coherence problems. A club is like a presentation of a universal algebraic theory. However, instead of just having operations and identities (or generators and relations), there are *two* levels of operations and identities. Loosely speaking, coherence requirements are second dimensional identities. (From this point of view, the Catalan groupoids constructed in Chapter 1 are the 2-dimensional versions of the Cayley graphs of groups.) In the thesis, we study categories with structures that are "free" or partially "free" of coherence requirements. In regular (1-dimensional) algebras, the notions of a free algebra plays a major role in homological algebra. This thesis may be seen as a slight beginning towards an investigation of what higher dimensional homological algebra might look like.

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