On Embedding Models of Arithmetic into Reduced Powers

by

Juliette Kennedy

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1996

UMI Number: 9630472

This manuscript has been read and accepted by the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

5/2/1996
Date

Chair of Examining Committee

5/2/96
Date

Isaac Chavel
Executive Officer

Attila Máté

Melvin Fitting

R. Parikh
Rohit Parikh

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Acknowledgements

For very stimulating discussions about mathematics I thank Roman Kossak, Mirco Mannucci, Kenneth McAloon, my advisor Attila Máté, Rohit Parikh, and Burton Randol, also for his unstinting support during this project. I thank Sidney Raffer; his substantial contribution to this work is explicitly acknowledged in these pages. Finally, I especially acknowledge the generous help of Stanley Tennenbaum, who told me of his original construction and suggested the investigation of componentwise behavior. I am very grateful for his constant support and encouragement.

This thesis is dedicated to my husband, Vladimír Březina, and to my mother, Poppy Kennedy.

1 Introduction

In 1934, Thoralf Skolem[1] constructed a family of nonstandard models of true arithmetic without making use of the completeness or compactness theorems for first order logic. Skolem's models came to be known as definable ultrapowers. This terminology is suggestive, but it can be misleading. The ultrapower construction was not known, or at least it was not published in 1934, and Skolem did not employ it.[2] Moreover, the current widespread use of ultrapowers has obscured the original meaning of Skolem's proof. Skolem's models are not ultrapowers. They are substructures of a certain structure $\mathcal{N}$ that will be the central topic of this dissertation. Skolem's discovery, from our point of view, was that $\mathcal{N}$ contains nonstandard models of true arithmetic. This is interesting because $\mathcal{N}$ has a much simpler definition than any nonstandard model of true arithmetic constructed by means of the completeness theorem. The domain of $\mathcal{N}$ and its operations occur naturally in mathematics.

We now define $\mathcal{N}$ and for historical reasons describe Skolem's construction. Let $\mathbb{N}^{\mathbb{N}}$ be all functions from the natural numbers to the natural numbers. We define an equivalence relation on $\mathbb{N}^{\mathbb{N}}$ as follows: Two functions f and g are equivalent if the equation $f(x) = g(x)$ holds for all but finitely many natural numbers x. Let $[f]$ be the equivalence class of the function f.

[1]see [20]

[2]The fundamental theorem on ultraproducts due to Łoś was published in 1955. See [14].

We add and multiply equivalence classes by the rules

$$[f] + [g] = [f + g] \text{ and } [f] \cdot [g] = [f \cdot g].$$

Equivalence classes are partially ordered by the relation

$$[f] \leq [g] \text{ iff for all but finitely many } x, \ f(x) \leq g(x).$$

The structure $\mathcal{N}$ has domain all equivalence classes $[f]$, and is equipped with the addition, multiplication and order relation we have just defined.

Skolem found nonstandard models of true arithmetic inside $\mathcal{N}$ by constructing a function g from the natural numbers to the natural numbers with the following property: If S is any set of natural numbers defined by an arithmetic formula, then either $g(x)$ is an element of S for all but finitely many x, or $g(x)$ fails to be an element of S for all but finitely many x. Assume that we have such a function g. Let $f_1, f_2, \ldots$ be all arithmetic functions from the natural numbers to the natural numbers. Then Skolem showed, by an induction on the complexity of formulas, that the set of all equivalence classes of the form $[f_i \circ g]$ gives a substructure M of $\mathcal{N}$ satisfying true arithmetic. For a very similar argument, see the statement and proof of Theorem 7.6.

Skolem's models do not exhaust the isomorphism types of countable models of true arithmetic. Thus it is reasonable to ask if there are other countable models of true arithmetic inside $\mathcal{N}$, not isomorphic to any of those given by Skolem's construction. Skolem did not pursue this problem, perhaps because he did not have the structure $\mathcal{N}$ specifically in view. In the early 1970's, Stanley Tennenbaum proved that every countable model of true arithmetic is present, up to isomorphism, in $\mathcal{N}$. He also proved a second, analogous theorem for nonnegative parts of discretely ordered rings. (See Theorem 3.2.) Thus he created an entirely new approach to the study of Models of Arithmetic.

A model of arithmetic contained in $\mathcal{N}$ is a more concrete object than one given by the completeness theorem. The elements of such a model are, on the one hand, integer-valued functions, and on the other hand, objects that belong to a certain element type in a model of arithmetic. What is the connection between the two? For example, if a function f takes on only prime values, and if $[f]$ belongs to a model of true arithmetic contained in $\mathcal{N}$, must $[f]$ be prime in that model? The study of the connection between satisfaction and *componentwise satisfaction* is one of the two main topics of this thesis. The other is this: What numerical properties must a function have in order to belong to a model of a given subtheory of true arithmetic? For example, can the identity function belong to a model of true arithmetic? As we shall see, the answer is no. We will see that the functions that belong to particular subtheories of true arithmetic are of a kind very similar to the function g of Skolem's proof.[3]

[3] See the definition of cohesiveness in section 7.

2 Preliminaries

1. Semantics, Syntax

We assume the reader is familiar with the notion of a first order language. We say that M is an *L-structure* if M consists of a non-empty set, called the *domain* of M, together with the following:

(i) For each n-ary function symbol f, a function $f^M : M^n \longrightarrow M$, where M^n denotes the n-times repeated Cartesian power of M with itself. We will use M interchangeably for the structure and the domain when no confusion is possible.

(ii) For each n-ary relation symbol R a subset $R^n \subseteq M^n$.

Let A and B be two L-structures. A *homomorphism* $f : A \longrightarrow B$ is a map from the domain of A to the domain of B such that:

(i) For every n-ary function symbol g and every n-tuple $\vec{a}$ belonging to A, $f(g^A(a_1,\ldots,a_n)) = g^B(f(a_1),\ldots,f(a_n))$.

(ii) For every n-ary relation symbol R and for every n-tuple $\vec{a}$ belonging to A, $< a_1,\ldots,a_n > \in R^A \Rightarrow < f(a_1),\ldots,f(a_n) > \in R^B$.

An *embedding* is a homomorphism with the additional property that

(ii)' For every n-ary relation symbol R and for every n-tuple $\vec{a}$ belonging to A, $< a_1, \ldots, a_n >$ belongs to $R^A \Leftrightarrow < f(a_1), \ldots, f(a_n) >$ belongs to R^B.

We say of two L-structures A and B that B is an *extension* of A, or A is a *substructure* of B, denoted $A \subseteq B$, if:

(i) the domain of A is a non-empty subset of the domain of B,

(ii) the operations of A are the operations of B restricted to the domain of A,

(iii) A is closed under the functions of B.

We will require some syntactic notions before proceeding further:

We fix a language L. Assuming the definition of an L-term, we say that a formula of L is an *atomic formula* if it is an equation of the form $s = t$, where s and t are terms, or a formula of the form $R(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms. An *open* formula is a formula with no quantifiers occurring in it. A variable occurring in a formula is *bound* if it falls under the scope of some quantifier which occurs in the formula. A *sentence* is a formula all of whose variables are bound. A *theory* is a set of sentences.

For a given L-structure M, the theory of M, denoted $Th(M)$, is the set of all sentences in the language of arithmetic true in the structure M. Two L-structures A and B are said to be *elementarily equivalent*, denoted $A \equiv B$, if $Th(A) = Th(B)$. For a theory T, we say that M is a *model* of T, denoted $M \models T$, if every sentence ϕ of T is true in M. We say that T *proves* ϕ, denoted $T \vdash \phi$, if ϕ holds in every model of T. We will use the phrase "two formulas are equivalent Mod T" to mean that the theory T proves their equivalence.

We now define the notion of elementary embedding. Let A and B be L-structures and let $f : A \longrightarrow B$ be an embedding with the property that $A \models \phi(\vec{a}) \Rightarrow B \models \phi(f(\vec{a}))$ for all L-formulas ϕ and all tuples $\vec{a}$ belonging to A. Then f is said to be an *elementary embedding* and A is said to be an *elementary substructure* of B.

The last of the model-theoretic notions we will require are the notions of filter and ultrafilter, and the notion of a reduced product:

Let I be a non-empty set and let $\mathcal{B}$ be a boolean algebra of subsets of I, i.e., $\mathcal{B}$ contains I and $\emptyset$ and is closed under intersection and complement. A *filter* in $\mathcal{B}$ is a non-empty subset $\mathcal{F} \subseteq \mathcal{B}$ which is closed under intersection and superset. A filter is said to be *proper* if it is not $\mathcal{B}$. A filter is called *principal* if it is the set of all sets that contain an element of I. Otherwise the filter is called *non-principal.* A proper filter $\mathcal{F}$ in $\mathcal{B}$ is called *maximal* if there is no proper filter in $\mathcal{B}$ containing $\mathcal{F}$. An *ultrafilter* $\mathcal{U}$ over I is a maximal proper filter.

We can now define the notion of reduced product:

We fix a language L. Let I be a non-empty set and let $\mathcal{F}$ be a proper filter over I. Let M_i be a family of L-structures indexed by I. Let $M = \prod_{i \in I} M_i$ be the Cartesian product of the sets M_i. We will define an equivalence relation on the elements of M as follows: Let f, g be two functions belonging to M. Then $f \sim_{\mathcal{F}} g$ iff $\{i \in I : f(i) = g(i)\} \in \mathcal{F}$. It is easily verified that this is an equivalence relation. Let $\prod_{\mathcal{F}} M_i$ be the following structure:

(i) the domain of $\prod_{\mathcal{F}} M_i$ is the set of equivalence classes $[f]$ of M induced by $\mathcal{F}$.

(ii) Let R be an n-ary relation symbol of L. The interpretation of R in $\prod_{\mathcal{F}} M_i$ is the relation $R^{\mathcal{F}}$ such that

$$< [f_1], \ldots, [f_n] > \in R^{\mathcal{F}} \text{ if and only if } \{i \in I : R^{M_i}(f_1(i), \ldots, f_n(i))\} \in \mathcal{F}.$$

(iii) Let f be an n-ary function symbol of L. The interpretation of f in $\prod_{\mathcal{F}} M_i$ is the function $f^{\mathcal{F}}$, where $f^{\mathcal{F}}(i) = f^{M_i}(f_1(i), \ldots, f_n(i))$, for all $i \in I$.

That these definitions are well-defined is left to the reader. When all the M_i's are equal to a set M', this is called the *reduced power of M' modulo $\mathcal{F}$*. In case $\mathcal{F}$ is an ultrafilter, we say that the associated reduced product (reduced power) is an ultraproduct (resp. ultrapower).

We state without proof a corollary of the fundamental theorem on ultrapowers, due to Łoś:

THEOREM 1. *Let $\mathcal{U}$ be an ultrapower of an L-structure M. Then $\mathcal{U}$ is elementarily equivalent to M.*

2. Theories

Let LA, the language of arithmetic, be the first order language with non-logical symbols $+, \cdot, 0, 1, \leq$. $\mathbb{N}$ denotes the standard LA structure.

An existential quantifier which occurs in a given LA-formula is said to be a *bounded existential quantifier* if it is of the form $\exists x \leq t$, where this is an abbreviation for $\exists x(x \leq t \wedge \cdots)$. A universal quantifier is said to be a *bounded universal quantifier* if it is of the form $\forall x \leq t$, where t is any LA-term not containing x and where this is abbreviates $\forall x(x \leq t \rightarrow \cdots)$. An LA-formula is *bounded* if all of its quantifiers are bounded. It is customary to let Δ_0 denote the class of all bounded formulas. We define the classes of Σ_n and Π_n formulas inductively as follows: $\Sigma_0 = \Pi_0 = \Delta_0$. If ϕ is a Σ_n formula, then $\forall\vec{x}\phi$ is Π_{n+1}; if ϕ is a Π_n formula, then $\exists\vec{x}\phi$ is Σ_{n+1}.

The *arithmetic hierarchy* is the class of Σ_n and Π_n formulas defined in the language of arithmetic.

We define an $\exists_n$ formula in the language of arithmetic to be a Σ_n formula whose unbounded quantifiers prefix an open formula. An $\forall_n$ formula is defined similarly from Π_n formulas.

We shall be concerned with the following theories: The $\forall_1$-$Th(\mathbb{N}$, Π_1-$Th(\mathbb{N})$, and the Π_2-$Th(\mathbb{N})$, defined respectively as the set of all $\forall_1$, Π_1 and Π_2 formulas true in the standard LA structure $\mathbb{N}$. $Th(\mathbb{N})$ is the theory "true arithmetic," i.e., the set of all the first order sentences in the language of arithmetic true in the standard structure $\mathbb{N}$.

We shall also be concerned with the theory PA^-, which consists of the following axioms ($x < y$ abbreviates $x \leq y \wedge x \neq y$):

1.) $\forall x, y, z((x + y) + z = x + (y + z))$

2.) $\forall x, y(x + y = y + x)$

3.) $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$

4.) $\forall x, y(x \cdot y = y \cdot x)$

5.) $\forall x, y, z(x \cdot (y + z) = x \cdot y + x \cdot z)$

6.) $\forall x((x + 0 = x) \wedge (x \cdot 0 = 0))$

7.) $\forall x(x \cdot 1 = x)$

8.) $\forall x, y, z((x < y \wedge y < z) \rightarrow x < z)$

9.) $\forall x\, x \leq x$

10.) $\forall x, y(x < y \vee x = y \vee y < x)$

11.) $\forall x, y, z(x < y \rightarrow x + z < y + z)$

12.) $\forall x, y, z(0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z)$

13.) $\forall x, y(x < y \rightarrow \exists z(x + z = y)$

14.) $0 < 1 \wedge \forall x(x > 0 \rightarrow x \geq 1)$

15.) $\forall x(x \geq 0)$

Thus the theory PA^- is the theory of nonnegative parts of discretely ordered rings. For interesting examples of these models see [8].

We will need the notion of a function given in lambda notation: Let $[-x-]$ be an expression which defines, for any integer value of x, at most one corresponding value. Then $\lambda x.[-x-]$ denotes the partial function:

$$\{\langle u, v\rangle : [-x-] \text{ defines the value } v \text{ when } u \text{ is substituted for } x\}.$$

Finally, we mean by the MRDP Theorem the result of Matiyasevich, Robinson, Davis and Putnam, which states that every recursively enumerable subset of the natural numbers has a Diophantine definition.

The convention governing our reference to Theorems, Corollaries, etc. within the thesis is this: In section 3, we will refer to Theorem 4 of section 3 as Theorem 4. In section 3 we will refer to Theorem 9 of section 5 as Theorem 5.9.

3 The Basic Construction

Let LA, the language of arithmetic, be the first order language of signature $\{+,\cdot,0,1,\leq\}$. If $\mathbb{N}$ is the standard LA structure, let $\mathcal{N}$ be the reduced power (of LA structures) $\mathbb{N}^\omega/\mathcal{F}$, where $\mathcal{F}$ is the cofinite filter in the boolean algebra of subsets of $\mathbb{N}$. Let **A** be the standard LA structure with domain all nonnegative real algebraic numbers, and let $\mathcal{A}$ be the reduced power $\mathbf{A}^\omega/\mathcal{F}$. We shall see that every countable model of PA^- is contained, up to isomorphism, in $\mathcal{A}$. Moreover, a countable model of PA^- appears, up to isomorphism, as a substructure of $\mathcal{N}$ if and only if it is Diophantine correct, i.e., a model of the $\forall_1$-$Th(\mathbb{N})$.

If f is a function from $\mathbb{N}$ to $\mathbb{N}$, let $[f]$ denote the equivalence class of f in $\mathcal{N}$. We use a similar notation for $\mathcal{A}$. When no confusion is possible, we will use f and $[f]$ interchangeably. Central to this thesis is the following

DEFINITION. If $\phi(x_1,\ldots,x_n)$ is an LA formula, and $f_1,\ldots,f_n$ are in $\mathcal{N}$, then we say $\phi(f_1,\ldots f_n)$ *holds componentwise* or is *true componentwise* (in $\mathcal{N}$) if, for all sufficiently large i, $\mathbb{N}\models\phi(f_1(i),\ldots,f_n(i))$.

As we will see, there are substructures of $\mathcal{N}$ for which componentwise truth and the usual satisfaction relation are identical. Let M be such a substructure. Then in M, the LA formulas satisfied by an element $[f]$ are completely determined by the LA formulas satisfied by the numbers $f(1),f(2),\ldots$. For example, $[f]$ is prime in M iff for large enough n, the numbers $f(n)$ are standard primes. Note that M must satisfy $Th(\mathbb{N})$, since all sentences holding in $\mathbb{N}$ hold componentwise. Now in any model M of $Th(\mathbb{N})$, a total recursive function g from $\mathbb{N}$ to $\mathbb{N}$ has a unique continuation g^M to M, via any Σ_1 formula defining the graph of g over $\mathbb{N}$. Because satisfaction and componentwise truth coincide in M, we get a very concrete picture of the behavior of g^M: If $[f]$ is in M, then $g^M([f]) = [g \circ f]$.

Such a strong relation between truth and componentwise truth is far from typical. We shall be concerned with the properties of substructures of $\mathcal{N}$ which imply and are implied by such relations. We shall characterize the LA theories that prove the MRDP Theorem in terms of the componentwise behavior of Δ_0 formulas in models of those theories. We shall establish a connection between the componentwise behavior of LA formulas, and the preservation of those formulas in extensions. And we shall also give characterizations of models of PA^-, Π_1-$Th(\mathbb{N})$, and Π_2-$Th(\mathbb{N})$ in terms of componentwise behavior.

We begin by presenting the two embedding theorems of Stanley Tennenbaum, which represent countable models of PA^- by means of sequences of real numbers. The proofs are slightly modified versions of the proofs communicated to us by him.

THEOREM 1 (TENNENBAUM). Let M be a countable Diophantine correct model of PA^-. Then M can be embedded in $\mathcal{N}$.

PROOF. Let $m_1, m_2, \ldots$ be the distinct elements of M. Let $P_1, P_2, \ldots$ be all polynomial equations over $\mathbb{N}$ in the variables $x_1, x_2, \ldots$ such that $M \models P_i(x_1/m_1, x_2/m_2, \ldots)$. Each system of equations $P_1 \wedge \cdots \wedge P_n$ has a solution in M. Thus, by Diophantine correctness, there is a sequence of natural numbers $v_1(n), v_2(n), \ldots$ for which

$$\mathbb{N} \models P_1 \bigwedge \cdots \bigwedge P_n(x_1/v_1(n), x_2/v_2(n), \ldots).$$

Note that if the variable x_i doesn't appear in $P_1 \wedge \cdots \wedge P_n$, then the choice of $v_i(n)$ is completely arbitrary.

Our embedding $h : M \longrightarrow \mathcal{N}$ is given by:

$$m_i \longmapsto [\lambda n. v_i(n)].$$

In the figure below, the i-th row is the solution in integers to $P_1 \wedge \cdots \wedge P_n$, and the i-th column "is" $h(m_i)$.

	m_1	m_2	$\ldots$	m_n	$\ldots$
P_1	$v_1(1)$	$v_2(1)$	$\ldots$	$v_n(1)$	$\ldots$
P_2	$v_1(2)$	$v_2(2)$	$\ldots$	$v_n(2)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	
P_n	$v_1(n)$	$v_2(n)$	$\ldots$	$v_n(n)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Note that if m_e is the element 0^M of M, then the polynomial equation $x_e = 0$ appears as one of the P's. It follows that, for n sufficiently large, $v_e(n) = 0$. Thus $h(0^M)$ is the equivalence class of the zero function. Similarly, h maps every standard integer of M to the class of the corresponding constant function.

We show that h is a homomorphism. Suppose $M \models m_i + m_j = m_k$. Then the polynomial $x_i + x_j = x_k$ must be one of the P's, say P_r. If $n \geq r$, then by construction $v_i(n) + v_j(n) = v_k(n)$. Hence $h(m_i) + h(m_j) = h(m_k)$, as required. A similar argument works for multiplication. Suppose $M \models m_i \leq m_j$. By an axiom of PA^-, for some k, $M \models m_i + m_k = m_j$. Thus, as we've shown, $h(m_i) + h(m_k) = h(m_j)$. It follows from the definition of the relation $\leq$ in $\mathcal{N}$ that $h(m_i) \leq h(m_j)$.

To see that h is one to one, suppose that $m_i \neq m_j$. Since in models of PA^- the order relation is total, we may assume that $m_i < m_j$. Again by the axioms of PA^-, we can choose m_k such that $m_i + m_k + 1 = m_j$. As we have shown, $h(m_i) + h(m_k) + h(1) = h(m_j)$. Since h(1) is the class of the constant function 1, it follows that $h(m_i) \neq h(m_j)$. □

COROLLARY. Let M be a countable model of the $\forall_1$-$Th(N)$. Then M can be embedded in $\mathcal{N}$.

PROOF. The models of the $\forall_1$-$Th(N)$ are precisely the substructures of models of $Th(N)$. (See the remark following Lemma 6.1.) Thus, M extends to a model of PA^-, which can be embedded in $\mathcal{N}$ as in Theorem 1. □

Before turning to the theorem for the non-Diophantine correct case, we observe first that the given embedding depends upon a particular choice of enumeration $m_1, m_2, \ldots$ of M, since different enumerations will in general produce different polynomials. We also note that different choices of solution yield different embeddings. Also, as we shall see below, we need not restrict ourselves to Diophantine formulas: we can carry out the construction for LA formulas of any complexity which hold in M.

We state next the non-Diophantine correct case of the theorem:

THEOREM 2 (TENNENBAUM). Let M be a countable model of PA^-. Then M can be embedded in $\mathcal{A}$.

PROOF. Given an enumeration $m_1, m_2, \ldots$ of M, we form conjunctions of polynomial equations P_n exactly as before. We wish to produce solutions of $P_1 \wedge \cdots \wedge P_n$ in the nonnegative algebraic reals for each n. We proceed as follows: The model M can be embedded in a real closed field F by a standard construction. (Embed M in an ordered integral domain, then form the (ordered) quotient field, and then the real closure. See [9].) Choose k so large that $x_1, \ldots, x_k$ are all the variables that occur in the conjunction $P_1 \wedge \cdots \wedge P_n$. The sentence $\exists x_1 \ldots x_k(P_1 \wedge \cdots \wedge P_n \wedge x_1 \geq 0 \wedge x_2 \geq 0 \cdots \wedge x_k \geq 0)$ is true in M, hence in F. It is a theorem of Tarski that the theory of real closed fields is complete. (See [5]). Thus, this same sentence must be true in the field of real algebraic numbers. This means we can choose nonnegative algebraic real numbers $v_1(n), v_2(n) \ldots$ satisfying the conjunction $P_1 \wedge \cdots \wedge P_n$. Let $h : M \longrightarrow \mathcal{A}$ be given by

$$m_i \longmapsto [\lambda n.v_i(n)].$$

The proof that h is a homomorphism, and furthermore an embedding, proceeds exactly as before, once we note that the equivalence classes all consist of nonnegative sequences of real algebraic numbers. □

REMARK. Under any of the embeddings given above, if $M \models PA^-$ then nonstandard elements of M are mapped to equivalence classes of functions tending to infinity. Why? If f is a function in the image of M, and f does not tend to infinity, then choose an integer k such that f is less than k infinitely often. Since $M \models PA^-$, either $[f] \leq [k]$ or $[k] \leq [f]$. The second alternative contradicts the definition of $\leq$ in $\mathcal{N}$. Hence $[f] \leq [k]$, i.e., $[f]$ is standard.

REMARK. Let F be a countable ordered field. Then F is embedded in $\mathbb{R}^\omega/\mathcal{F}$, where $\mathbb{R}$ is the field of real algebraic numbers. The proof is mutatis mutandis the same as in Theorem 2, except that due to the presence of negative elements we must demonstrate differently that the mapping obtained is one to one. But this must be the case, since every homomorphism of fields has this property. (See [9].)

REMARK. For any pair of LA structures A and B satisfying PA^-, if A is countable and if A satisfies the $\forall_1$-$Th(B)$ then there is an embedding of A into $B^\omega/\mathcal{F}$. In particular, if M is a model of PA^-, then every countable extension of M satisfying the $\forall_1$-$Th(M)$ can be embedded in $M^\omega/\mathcal{F}$.

4 Componentwise Behavior and Open Formulas

Let M be a substructure of $\mathcal{N}$, and let ϕ be a formula with parameters from M. If ϕ holds in M we cannot conclude that ϕ holds componentwise. For example, let $f : N \longrightarrow N$ be the function taking even numbers to 0 and odd numbers to 1. Let M be the substructure of $\mathcal{N}$ generated by $[f]$. If ϕ is the formula $[f] \neq [0]$ then ϕ holds in M, but does not hold componentwise. It is just as easy to find examples where the converse implication fails. What, then, is the relation between componentwise truth and truth in a substructure of $\mathcal{N}$? We shall study classes of formulas ϕ and classes of substructures M of $\mathcal{N}$ for which componentwise truth and satisfaction are connected in interesting ways.

Let M be a substructure of $\mathcal{N}$. An atomic formula with parameters from M holds in M iff it holds in $\mathcal{N}$ iff it holds componentwise. From this, it follows that an atomic formula prefaced by a string of existential quantifiers that holds in M will hold componentwise.

As the example above shows, it is not in general true that open formulas that are true in M hold componentwise. However, we have

PROPOSITION 1. Suppose M is a substructure of $\mathcal{N}$ that satisfies PA^-. Then an open formula with parameters from M is true in M iff it holds componentwise.

PROOF. Suppose ϕ is an open formula with parameters from M, and suppose ϕ holds in M. The formula ϕ is equivalent, in any model of PA^-, to an atomic formula prefaced by a string of existential quantifiers. (Using the axioms for a total order, and the axiom $\forall y \forall x \leq y \exists z x + z = y$, one converts inequalities into equations prefaced by strings of existential quantifiers. One then combines conjunctions and disjunctions of equations into a single equation.) Thus, by a previous remark, because ϕ holds in M it must hold componentwise. Conversely, if ϕ holds componentwise, then the open formula $\sim\phi$ cannot hold in M, else, as we have already established, it would hold componentwise. Hence ϕ must hold in M. □

From this, we have

PROPOSITION 2. Suppose M is a substructure of $\mathcal{N}$ satisfying PA^-, and ϕ is a formula with parameters from M. If ϕ is $\exists_1$ and $M \models \phi$ then ϕ holds componentwise. If ϕ is $\forall_1$ and ϕ holds componentwise then $M \models \phi$.

PROOF. The first assertion follows from Proposition 1. As for the second, suppose ϕ is $\forall_1$ and ϕ fails to hold in M. Then the $\exists_1$ formula $\sim\phi$ holds in M, hence holds componentwise. Hence ϕ cannot hold componentwise. □

We note that not every substructure of $\mathcal{N}$ satisfies PA^-, or the $\forall_1$-$Th(\mathbb{N})$ for that matter. For example, $\mathcal{N}$ itself fails to satisfy the $\forall_1$ sentence asserting that the order relation is total. Moreover, there are substructures of $\mathcal{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$ but not satisfying PA^-. An example is the substructure M of $\mathcal{N}$ generated by the identity function f on $\mathbb{N}$. To see that $M \models \forall_1$-$Th(\mathbb{N})$, we observe that M is isomorphic to $\mathbb{N}[x]$, the semiring of polynomials over $\mathbb{N}$ in the variable x. But this structure $\mathbb{N}[x]$ appears up to an isomorphism in any nonstandard model of $Th(\mathbb{N})$, as the substructure generated by a nonstandard element. This means M extends to a model of $Th(\mathbb{N})$. Since universal formulas are downward preserved, we have shown that $M \models \forall_1\text{-}Th(\mathbb{N})$. On the other hand, M fails to satisfy PA^- since, in M, f has no predecessor.

However, we have

PROPOSITION 3. If M is a substructure of $\mathcal{N}$ and $M \models PA^-$, then $M \models \forall_1\text{-}Th(\mathbb{N})$.

PROOF. Suppose $\phi(\vec{x})$ is open, and $N \models \forall\vec{x}\phi(\vec{x})$. Let $\vec{f}$ be a tuple from M. Then $\phi(\vec{f})$ holds componentwise. Thus, by Proposition 1, $\phi(\vec{f})$ holds in M. Since $\vec{f}$ was chosen arbitrarily from M, it follows that $M \models \forall\vec{x}\phi(\vec{x})$, as required. □

For the next result, we need the following

DEFINITION. Let M be a substructure of $\mathcal{N}$. Let Φ be a set of LA formulas. We say that Φ *behaves componentwise* in M if for all tuples $\vec{f}$ from M, and for all formulas ϕ from Φ, $M \models \phi(\vec{f})$ iff $\phi(\vec{f})$ holds componentwise.

For which substructures of $\mathcal{N}$ do the open formulas behave componentwise?

THEOREM 4 (KENNEDY-RAFFER). Let M be a substructure of $\mathcal{N}$. Then the following are equivalent: (i) The open formulas behave componentwise in M. (ii) $M \models \forall_1\text{-}Th(\mathbb{N})$. (iii) M extends to a model of PA^- included in

$\mathcal{N}$.

PROOF.

(i) implies (ii): Assume (i), and let ϕ be an open formula such that $\mathbb{N} \models \forall \vec{x}\phi$. Suppose it was the case that $M \models \exists \vec{x} \sim \phi$. Choose a tuple $\vec{f}$ from M such that $M \models \sim \phi(\vec{x}/\vec{f})$. Then the latter holds componentwise, so there is a tuple of integers $\vec{n}$ such that $\mathbb{N} \models \sim \phi(\vec{x}/\vec{n})$. But this contradicts $\mathbb{N} \models \forall \vec{x}\phi$.

(ii) implies (iii): Assume (ii). Let M' be

$$\{a \in \mathcal{N} : \exists b, c \in M \ s.t. \ a + b = c\}.$$

Then M' is a substructure of $\mathcal{N}$ and M' contains M. We will show that $M' \models PA^-$. As a substructure of $\mathcal{N}$, M' is automatically a partially ordered semiring[4]. So we have to check that the order on M' is total and discrete (meaning that two consecutive elements cannot bound a third), and that M' is closed under nonnegative differences. Note that $\forall_1$ sentences express that the order relation is total and discrete in M. First we will show that the ordering is total in M'. If a and b are in M', choose elements r, s, r' and s' in M such that $a + r = s$ and $b + r' = s'$. Then $a + r + s' = b + r' + s$. Now $r + s'$ and $r' + s$ are in M, so they are comparable: We will suppose that $r' + s \leq r + s'$. It follows that the formula $a \leq b$ holds componentwise, hence holds in M'. This proves that M' is totally ordered. To prove discreteness, suppose (keeping the same notation) that $a \leq b \leq a + 1$. From the relations

[4]A semiring is a structure of type $(+, \cdot, 0, 1)$ obeying all the axioms for a commutative ring except the one requiring the existence of additive inverses.

$a + r = s$ and $b + r' = s'$ we conclude that $s + r' \leq s' + r \leq s + r' + 1$. Since M is discretely ordered, it follows that either $s' + r = s + r'$, in which case $a = b$, or $s' + r = s' + r + 1$, in which case $a + 1 = b$. This proves that the order on M' is discrete. As for nonnegative differences, suppose, (with the same notation) that $a \leq b$. Choose c in $\mathcal{N}$ so that $a + c = b$. We have to show that c is in M': But the equation $a + r + s' = b + r' + s$ implies that that $r + s' = c + r' + s$. Thus c is indeed in M'.

(iii) implies (i): Suppose M extends to a model M' of PA^- included in $\mathcal{N}$. Let ϕ be an open formula with parameters from M, and suppose ϕ holds in M. We have to check that ϕ holds componentwise. But this follows from Proposition 1, and the fact that ϕ holds in M'. Conversely, suppose ϕ holds componentwise. Then $\sim \phi$ cannot hold in M, else it would hold in M' and therefore would hold componentwise. Hence ϕ holds in M. □

5 Componentwise Behavior and Σ_1 Formulas

The requirement that a substructure of $\mathcal{N}$ satisfy PA^- is not strong enough to insure that the Δ_0 formulas behave componentwise. For example, let $\mathbb{Z}[t]^+$ be the semiring of all polynomials in t over $\mathbb{Z}$ with positive leading coefficients. We order this semiring by the rule $p(t) \leq q(t)$ if the leading coefficient of $q(t) - p(t)$ is nonnegative. We then obtain an LA structure satisfying PA^-. If f is any function from $\mathbb{N}$ to $\mathbb{N}$, and if f tends to infinity, then $\mathbb{Z}[t]^+$ can be embedded in $\mathcal{N}$ via the map

$$p(t) \mapsto [|p(f)|],$$

where '$| \, |$' denotes absolute value. (The function $p(f)$ is eventually nonnegative. The point of the absolute value is to make sure it is always nonnegative.) For example, take f to be the identity function. Let $\phi(x)$ be the Δ_0 formula "x is not even and x is not odd." Then $\phi(f)$ holds in $\mathbb{Z}[t]^+$, but does not hold componentwise. On the other hand, we have:

PROPOSITION 1. If M is a substructure of $\mathcal{N}$, and $M \models MRDP + PA^-$, then the Δ_0 formulas behave componentwise in M.

Before proving the proposition, we need the following

DEFINITION. We say that an LA structure M satisfies the MRDP Theorem if every Δ_0 formula is equivalent, over M, to an $\exists_1$ formula.

PROOF. Suppose, then, that $M \models MRDP$. Let $\phi(\vec{x})$ be a Δ_0 formula and let $\vec{f}$ be a tuple from M. Suppose $M \models \phi(\vec{f})$. Let $\psi(\vec{x})$ be the $\exists_1$

equivalent of $\phi(\vec{x})$ over M. Then $M \models \psi(\vec{f})$. By Proposition 6.1, $\psi(\vec{f})$ holds componentwise. Now the sentence $\sim\forall\vec{x}(\psi(\vec{x}) \rightarrow \phi(\vec{x}))$ is logically equivalent to a Σ_1 formula. If it held in $\mathbb{N}$ then, by the well known "Σ_1-completeness" of PA^- it would hold in M, which it does not. (Every Σ_1 sentence true in $\mathbb{N}$ is provable in PA^-. See [8], chapter 3) Thus $\mathbb{N} \models \forall\vec{x}(\psi(\vec{x}) \rightarrow \phi(\vec{x}))$. It follows that $\phi(\vec{f})$ holds componentwise. In order to complete the proof, we now have to assume that $\phi(\vec{f})$ holds componentwise, and show that it holds in M. But, if it did not, then the above argument carried out with $\sim\phi(\vec{f})$ in place of $\phi(\vec{f})$ would give a contradiction. □

If the Δ_0 formulas behave componentwise in $M \subseteq \mathcal{N}$, what can we say about $Th(M)$?

PROPOSITION 2. Suppose M is a substructure of $\mathcal{N}$ in which the Δ_0 formulas behave componentwise. Then $M \models \Pi_1\text{-}Th(\mathbb{N})$.

PROOF. Suppose $\phi(\vec{x})$ is a Δ_0 formula, and $\mathbb{N} \models \forall\vec{x}\phi(\vec{x})$. Choose $\vec{f}$ from M. Then $\phi(\vec{f})$ holds componentwise, hence it holds in M. But $\vec{f}$ in M was arbitrary. So $M \models \forall\vec{x}\phi(\vec{x})$ □

Next, we consider the componentwise behavior of $\exists_1$ formulas. At this point, it is not clear that there are any substructures of $\mathcal{N}$ in which this class of formulas behaves componentwise. In fact we will see from the following generalization of Theorem 3.1 that there substructures of $\mathcal{N}$ in which truth and componentwise truth coincide for formulas of any prescribed complexity. We will then be able to construct models of the $Th(\mathbb{N})$ in which all first order formulas behave componentwise.

THEOREM 3. Suppose M is a countable LA structure satisfying the Π_{n+1}-$Th(\mathbb{N})$ $(n \geq 0)$. Then there is an embedding h of M into $\mathcal{N}$ such that in the structure $h(M)$ both the Σ_n and the Π_n formulas behave componentwise.

PROOF. The proof proceeds as in Theorem 3.1, except that instead of enumerating just Diophantine formulas, we expand our enumeration to include Σ_n and Π_n formulas:

Let $m_1, m_2, \ldots$ be the distinct elements of M. Let $\phi_1, \phi_2, \ldots$ be all Σ_n and Π_n formulas in the variables $x_1, x_2, \ldots$ such that $M \models \phi_i(x_1/m_1, x_2/m_2, \ldots)$. The Σ_{n+1} sentence

$$\exists \vec{x}(\phi_1 \bigwedge \cdots \bigwedge \phi_k)$$

holds in M. Since $M \models \Pi_{n+1}$-$Th(\mathbb{N})$ this sentence must also hold in $\mathbb{N}$. We choose natural numbers $v_1(k), v_2(k), \ldots$ for which

$$\mathbb{N} \models \phi_1 \bigwedge \cdots \bigwedge \phi_k(x_1/v_1(k), x_2/v_2(k), \ldots).$$

We then define the map $h : M \longrightarrow \mathcal{N}$ by:

$$m_i \longmapsto [\lambda k.v_i(k)].$$

That h is an embedding follows exactly as in Theorem 3.1. To see that the right class of formulas behaves componentwise, let ϕ be a Σ_n formula with parameters from M, and let ϕ^h be the same formula with its parameters mapped to $h(M)$. Suppose that $h(M) \models \phi^h$. We will show that ϕ^h holds componentwise. Choose k so that ϕ has the form $\phi_k(x_1/m_1, x_2/m_2, \ldots)$.

Then ϕ^h has the form $\phi_k(x_1/f_1, x_2/f_2, \ldots)$, where f_i is the function $\lambda k.v_i(k)$. Writing this last formula as $\phi_k(f_1, f_2, \ldots)$, we observe that, for $i \geq k$, $\mathbb{N} \models \phi_k(f_1(i), f_2(i), \ldots)$, which means that ϕ^h holds componentwise. A similar argument shows that Π_n formulas holding in $h(M)$ must hold componentwise. Conversely, if a Π_n formula defined in $h(M)$ holds componentwise, then its negation cannot hold in $h(M)$, or else by the above argument that negation would hold componentwise. □

THEOREM 4. There are substructures of $\mathcal{N}$ satisfying $Th(\mathbb{N})$ for which truth and componentwise truth coincide.

PROOF. Let M be a countable model of $Th(\mathbb{N})$. Embed M in $\mathcal{N}$ as in Theorem 3, letting the ϕ_k's run through all first order formulas holding at $m_1, m_2, \ldots$. The argument of Theorem 3 shows that $h(M)$ is the required substructure of $\mathcal{N}$. □

It follows that there are substructures of $\mathcal{N}$ in which the class of $\exists_1$ formulas behaves componentwise: Pick an arbitrary countable model M of the Π_2-$Th(\mathbb{N})$, and use Theorem 3 to construct an isomorphic copy in $\mathcal{N}$. In order to give some properties of these substructures, we need

DEFINITION. Let C be a class of L-structures, where L is a first order language. A structure M in C is said to be existentially closed (with respect to C) if any existential formula with parameters in M that holds in some extension of M belonging to C already holds in M. We shall be concerned only with the case that L is LA, and C is all models of $\forall_1$-$Th(\mathbb{N})$. We will then refer to M as simply e.c.

The following proposition appears implicitly in [6]:

PROPOSITION 5. Every e.c. structure M satisfies Π_2-$Th(\mathbb{N})$.

PROOF. One first shows that every e.c. structure M satisfies the $\forall_2$-$Th(\mathbb{N})$. To this end, let $\Psi = \forall x \exists \vec{y} \phi(\vec{x}, \vec{y})$ be an $\forall_2$ sentence which holds in $\mathbb{N}$. Since the model M is e.c. it satisfies the $\forall_1$-$Th(\mathbb{N})$. Hence by the remark following Lemma 6.1, M extends to a model M' of $Th(\mathbb{N})$. The model M' satisfies Ψ, since $\mathbb{N}$ does. Now let $\vec{a}$ be a tuple from M. It suffices to show that $M \models \exists \vec{y} \phi(\vec{a}, \vec{y})$. But the structure M' satisfies $\exists \vec{y} \phi(\vec{a}, \vec{y})$. Therefore, since M is e.c., M must also satisfy $\exists \vec{y} \phi(\vec{a}, \vec{y})$. But $\vec{a}$ was arbitrary. Therefore $M \models \forall \vec{x} \exists \vec{y} \phi(\vec{x}, \vec{y})$, as required. Now the Π_2-$Th(\mathbb{N})$ is an inductive theory (see [3]). The Chang-Loś-Suszko theorem states that inductive theories are $\forall_2$ axiomatizable. (See [1].) But the $\forall_2$ sentences belonging to the Π_2-$Th(\mathbb{N})$ are precisely the $\forall_2$-$Th(\mathbb{N})$. It follows that the $\forall_2$-$Th(\mathbb{N})$ proves the Π_2-$Th(\mathbb{N})$. □

PROPOSITION 6. Suppose M is an e.c. substructure of $\mathcal{N}$. Then the class of $\exists_1$ formulas behaves componentwise in M.

PROOF. Let ϕ be an $\exists_1$ formula defined in M, and suppose $M \models \phi$. By Proposition 5, $M \models PA^-$. Thus, by Proposition 4.2, ϕ holds componentwise. Conversely, suppose ϕ holds componentwise. Let $\mathcal{G}$ be a non-principal ultrafilter in the boolean algebra of subsets of $\mathbb{N}$, and let $\mathbb{N}^\omega/\mathcal{G}$ be the ultrapower determined by $\mathcal{G}$. The LA structure $\mathbb{N}^\omega/\mathcal{G}$ is a model of $Th(\mathbb{N})$.

(This follows from a theorem of Loś. See Section 2.) Now there is a natural homomorphism $h : \mathcal{N} \longrightarrow \mathbb{N}^\omega/\mathcal{G}$ given by

$$[f] \mapsto \langle f \rangle,$$

where $\langle f \rangle$ is the class of the function f in $\mathbb{N}^\omega/\mathcal{G}$. The mapping h is injective on M: This is a consequence of the non-principality of $\mathcal{G}$, and the fact that if the formula $[f] \neq [g]$ holds in M, then it must hold componentwise. Let ϕ have the form $\exists x \psi(x, g)$, where g is in $\mathcal{N}$. (For simplicity, we assume ϕ has one quantifier and one parameter, but our argument is perfectly general.) Since ϕ holds componentwise, there is a function k from $\mathbb{N}$ to $\mathbb{N}$ such that $\psi(k, g)$ holds componentwise. Again because $\mathcal{G}$ is non-principal, it follows that $\psi(\langle k \rangle, \langle g \rangle)$, and hence $\exists x \psi(x, \langle g \rangle)$ hold in $\mathbb{N}^\omega/\mathcal{G}$. But $h(M)$ is e.c. (since h is injective), hence $\exists x \psi(x, \langle g \rangle)$ holds in $h(M)$. Thus ϕ holds in M, as required. □

REMARK. The converse of Proposition 6 is false: There are substructures of $\mathcal{N}$ which are not existentially closed, but in which the class of $\exists_1$ formulas behaves componentwise. In fact, let M be any countable nonstandard model of $Th(\mathbb{N})$. Using the proof of Corollary 4, we can find an isomorphic copy M' of M in $\mathcal{N}$ in which all formulas behave componentwise. But by a theorem of Rabin (Theorem 9 of this section), no nonstandard model of $Th(\mathbb{N})$ is e.c.

REMARK. The embedding used in Proposition 6 is injective on every substructure of $\mathcal{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$. Since the ultrafilter $\mathcal{G}$ used there was arbitrary, it follows that every countable model of the $\forall_1$-$Th(\mathbb{N})$ can be embedded in every ultrapower $\mathbb{N}^\omega/\mathcal{G}$.

Our next result gives models of $Th(\mathbb{N})$ in which the $\exists_1$ formulas fail to behave componentwise.

PROPOSITION 8. Every countable nonstandard model M of $Th(\mathbb{N})$ has an isomorphic copy in $\mathcal{N}$ under which some $\exists_1$ formula fails to behave componentwise.

PROOF. Let M be a countable nonstandard model of $Th(\mathbb{N})$. Let S be a simple subset of $\mathbb{N}$. This means S is an recursively enumerable set whose complement is infinite and contains no infinite recursively enumerable set. By the MRDP Theorem, there is an $\exists_1$ formula $\phi(x)$ defining S over $\mathbb{N}$. The sentence $\forall x \exists y(y > x \wedge \sim\phi(x))$ holds in $\mathbb{N}$, hence in M. Therefore there is a nonstandard element m of M such that $M \models \sim\phi(m)$. Let $m = m_1, m_2, \ldots$ be the distinct elements of M. We embed M in $\mathcal{N}$ as in Theorem 5.1. In the notation of that theorem, for all n, it is possible to choose $v_1(n)$ in S. Why? For each n, the formula in the single variable x_1, given by

$$\exists x_2, x_3, \ldots (P_1 \bigwedge \cdots \bigwedge P_n)$$

defines over $\mathbb{N}$ a recursively enumerable set S_n. Moreover, S_n is infinite: Since m_1 is nonstandard, there are P_k's of the form $x_1 = x_i + r$ for arbitrarily large integers r. It follows from the fact that S is simple that each S_n meets S, so we can choose $v_1(n)$ in S for all n, as asserted. But then we obtain an embedding h such that the $\exists_1$ formula $\phi(h(m_1))$ holds componentwise. Since

$\phi(m)$ fails to hold in M, and since h is an embedding, the formula $\phi(h(m_1))$ does not hold in $h(M)$. So $\phi(h(m_1))$ does not behave componentwise in $h(M)$. □

Propositions 6 and 8 imply:

THEOREM 9 (RABIN, 1962)[5].

If M is a nonstandard model of $Th(\mathbb{N})$ then M is not e.c.

PROOF. Assume, on the contrary, that M is e.c. Then any countable elementary substructure of M is also e.c. (This is a consequence of the joint embedding property for models of the $\forall_1$-$Th(\mathbb{N})$. (See [6].) Thus we can assume that M is countable. By Proposition 8, M has an isomorphic copy in $\mathcal{N}$ in which some $\exists_1$ formula fails to behave componentwise. But by Proposition 6, the $\exists_1$ formulas behave componentwise in every countable e.c. substructure of $\mathcal{N}$, a contradiction. □

If M is a substructure of $\mathcal{N}$, and if the $\exists_1$ formulas behave componentwise in M, what theory must M satisfy?

PROPOSITION 10. If the $\exists_1$ formulas behave componentwise in the substructure M of $\mathcal{N}$, then $M \models \Pi_2$-$Th(\mathbb{N})$. Conversely, if $M \models \Pi_2$-$Th(\mathbb{N})$, then the $\exists_1$ formulas behave componentwise in some isomorphic copy of M in $\mathcal{N}$.

PROOF. Assume the $\exists_1$ formulas behave componentwise in M. First, we shall show that $M \models \forall_2$-$Th(\mathbb{N})$. Let $\phi(\vec{x}, \vec{y})$ be an open formula, and suppose

[5] Rabin's 1962 proof did not make use of the MRDP Theorem. See [15]

$\mathbb{N} \models \forall\vec{x}\exists\vec{y}\phi(\vec{x},\vec{y})$. Let $\vec{f}$ be a tuple from M. Then the formula $\exists\vec{y}\phi(\vec{f},\vec{y})$ holds componentwise, hence it holds in M. Hence $M \models \forall\vec{x}\exists\vec{y}\phi(\vec{x},\vec{y})$, as required. But it is known that $\forall_2$-$Th(\mathbb{N})$ proves the Π_2-$Th(\mathbb{N})$.[6] The second assertion follows from Theorem 3. □

Finally, we note that if M is a substructure of $\mathcal{N}$, and if, in M, the $\exists_1$ formulas behave componentwise, then so must the Σ_1 formulas: For by the previous proposition, $M \models \Pi_2$-$Th(\mathbb{N})$. Each Σ_1 formula is equivalent to an $\exists_1$ formula over $\mathbb{N}$, and this equivalence persists in M, since it is itself Π_2. Thus these two classes of formulas behave componentwise in exactly the same substructures of $\mathcal{N}$.

[6]Note that this is not a simple consequence of the MRDP Theorem, since it is not obvious that $\forall_1$-$Th(\mathbb{N})$ proves the MRDP Theorem.

6 Componentwise Behavior and Absoluteness

We shall describe a connection between the componentwise truth of formulas in a substructure M of $\mathcal{N}$, and the satisfaction of that formula in extensions of M. This will lead to a characterization of the LA theories T which prove the MRDP Theorem in terms of the componentwise behavior of Δ_0 formulas in models of those theories.

We begin with some model theory. We fix a language L. Let $M, M' \ldots$ be L structures, and let $\phi(\vec{x}), \psi(\vec{x}), \ldots$ be L formulas.

DEFINITION. We use $Res^T_\phi(\vec{x})$ to denote the set of all $\forall_1$ formulas $\psi(\vec{x})$ such that

$$T \vdash \forall\vec{x}(\phi(\vec{x}) \rightarrow \psi(\vec{x}))$$

LEMMA 1. Let $\vec{m}$ be a tuple from M. M has an extension M' such that $M' \models T + \phi(\vec{m})$ iff $M \models Res^T_\phi(\vec{m})$.

PROOF. See [6], Theorem 5.1.1. In Hodges's statement of the theorem, ϕ is assumed to be an $\exists_1$ formula. However, his proof does not require or use this assumption.

The following special case of Lemma 1 will be useful: If L is LA, and T is $Th(\mathbb{N})$, and ϕ is the formula $0 = 0$, then the Lemma asserts that M has an extension to a model of $Th(\mathbb{N})$ iff $M \models \forall_1\text{-}Th(\mathbb{N})$.

We will also need the following

DEFINITION. We say a formula ϕ is absolute for models of T if for all models $M \subseteq M'$ of T, and for all tuples $\vec{m}$ from M, we have $M \models \phi(\vec{m})$ iff $M' \models \phi(\vec{m})$.

Lemma 2 and Proposition 3 are implicitly stated in [8]:

LEMMA 2. The formula $\phi = \phi(\vec{x})$ is absolute for models of T iff $\phi(\vec{x})$ is equivalent to both an $\exists_1$ and to an $\forall_1$ formula Mod T.

PROOF. Suppose ϕ is absolute for models of T and suppose the conclusion is false. Replacing ϕ by $\sim \phi$ if necessary, we may assume that ϕ is not equivalent to any $\forall_1$ formula Mod T. Let $\vec{a}$ be a tuple of constant symbols not in L. Let Γ be the set of formulas

$$T + Res_{\phi}^{T}(\vec{a}) + \sim\phi(\vec{a})$$

Γ is finitely satisfiable, since otherwise, Mod T, some finite conjunction from $Res_{\phi}^{T}(\vec{a})$ would imply $\phi(\vec{a})$. But then the converse implication would hold by definition of $Res_{\phi}^{T}(\vec{a})$. So this would make ϕ equivalent, Mod T, to an $\forall_1$ formula, a contradiction. Let M be a model of Γ. Then $M \models T + \sim \phi(\vec{a})$. But also $M \models Res_{\phi}^{T}(\vec{a})$, hence, by Lemma 1, M has an extension to $M' \models T + \phi(\vec{a})$. This contradicts the absoluteness of ϕ. $\square$

This lemma suggests the general problem of characterizing, for a theory T, the class of formulas absolute for models of T. For example, If T is $Th(\mathbb{N})$, then an LA formula is absolute for models of T iff it defines a recursive set over $\mathbb{N}$. (This follows from Lemma 2 and the MRDP Theorem.) Another example: if T is the Π_1-$Th(\mathbb{N})$, then every formula absolute for T must be equivalent, Mod T, to a Δ_0 formula. (See [7], page 223.) Whether or not all Δ_0 formulas are absolute for T is not known.

We say that an LA theory T proves the MRDP Theorem if for every Σ_1 formula σ there is an $\exists_1$ formula ϕ such that $T \vdash \phi \leftrightarrow \sigma$. Lemma 2 gives us a model-theoretic description of the LA theories proving the MRDP Theorem:

PROPOSITION 3. An LA theory T proves the MRDP Theorem iff Δ_0 formulas are absolute for models of T.

PROOF. Use Lemma 2. □

DEFINITION. Let T be an LA theory containing the $\forall_1$-$Th(\mathbb{N})$. We say that T is Δ_0-coherent if in every substructure of $\mathcal{N}$ satisfying T the Δ_0 formulas behave componentwise.

We can now prove:

THEOREM 4 (KENNEDY-RAFFER). Suppose T is an LA theory containing $\forall_1 - Th(\mathbb{N})$. T proves the MRDP Theorem iff T is Δ_0-coherent.

PROOF. Suppose T proves the MRDP Theorem. Let $M \subseteq \mathcal{N}$ be a model of T. We must show that Δ_0 formulas behave componentwise in M. Let $\delta(\vec{x})$ be a Δ_0 formula, and let $\sigma(\vec{x})$ be its $\exists_1$ equivalent Mod T. The Π_2 sentence $\Phi \overset{def}{=} \forall\vec{x}(\sigma(\vec{x}) \leftrightarrow \delta(\vec{x}))$ holds in M. We claim that it also holds in $\mathbb{N}$: If not, then, instantiating the $\exists_2$ sentence $\sim\Phi$, we obtain a Π_1 sentence holding in $\mathbb{N}$ but not in M. But, since M proves both the MRDP Theorem and also the $\forall_1$-$Th(\mathbb{N})$, it follows that M proves $\Pi_1 - Th(\mathbb{N})$. This contradicts the fact that the sentence Φ holds in $\mathbb{N}$ but not in M. We have now shown that

Φ holds in $\mathbb{N}$.

Now suppose $\vec{f}$ is a tuple from M, and $M \models \delta(\vec{f})$ We wish to show that the formula $\delta(\vec{f})$ holds componentwise. But then M satisfies $\sigma(\vec{f})$: Hence by Proposition 4.2, the formula $\sigma(\vec{f})$ holds componentwise. It follows from the fact that Φ holds in $\mathbb{N}$ that $\delta(\vec{f})$ holds componentwise, as required.

Conversely, assume T is Δ_0-coherent. By Proposition 3, it suffices to show that Δ_0 formulas are absolute for models of T. Suppose that A and B are models of T, and $A \subseteq B$. Let $\phi(\vec{x})$ be a Δ_0 formula. Choose a tuple $\vec{a}$ from A, and suppose that $A \models \phi(\vec{a})$. We have to show that $B \models \phi(\vec{a})$. Let A' be a countable elementary substructure of A containing $\vec{a}$, and extend A' to a countable elementary substructure B' of B. It is enough to show that $B' \models \phi(\vec{a})$. Since B' is a countable model of the $\forall_1$-$Th(\mathbb{N})$, it is isomorphic to a substructure of $\mathcal{N}$. Replacing B' with an isomorphic copy does not affect the absoluteness of ϕ: Henceforth we shall regard B' as included in $\mathcal{N}$. Since A' is an elementary substructure of A, the formula $\phi(\vec{a})$ holds in A', and A' satisfies T. Since T is Δ_0-coherent, $\phi(\vec{a})$ must hold componentwise in A'. Therefore, since $A' \subseteq B'$, $\phi(\vec{a})$ holds componentwise in B'. But B' is also a model of T. Thus by Δ_0 coherence $\phi(\vec{a})$ holds in B'. $\Box$

The following theorem relates componentwise behavior to truth in an extension. Let T be any consistent LA theory including the Π_1-$Th(\mathbb{N})$:

THEOREM 5 (KENNEDY-RAFFER). Suppose A is a countable model of T, and $\phi(\vec{x})$ is a Δ_0 formula. Let $\vec{a}$ be a tuple from A. The formula $\phi(\vec{a})$ holds in some extension of A to a model of T iff there is an embedding $h : A \longrightarrow \mathcal{N}$

such that $\phi(h\vec{a})$ holds componentwise.

PROOF. Suppose $A \subseteq B$, and $B \models T + \phi(\vec{a})$ We can assume B is countable. By Theorem 5.3, there is an embedding $h : B \longrightarrow \mathcal{N}$ such that Δ_0 formulas with parameters in $h(B)$ behave componentwise. The restriction of h to A is the required embedding.

Conversely, assume we have an embedding $h : A \longrightarrow \mathcal{N}$ such that $\phi(h\vec{a})$ holds componentwise, and suppose, in order to obtain a contradiction, that there is no extension of A to a model of $T + \phi(\vec{a})$. By Lemma 1, we must have $A \not\models Res_\phi^T(\vec{a})$. Choose a formula $\psi \in Res_\phi^T(\vec{a})$ such that $A \models \sim\psi(\vec{a})$. Then $h(A) \models \sim\psi(h\vec{a})$. But $\sim\psi(h\vec{a})$ is logically equivalent to an $\exists_1$ formula. Thus, by Proposition 4.2, the formula $\sim\psi(h\vec{a})$ holds componentwise. But the definition of Res_ϕ^T implies that T proves the sentence

$$\forall\vec{x}(\phi(\vec{x}) \rightarrow \psi(\vec{x})).$$

Since A is a model of T, A must satisfy this sentence. But, T contains the Π_1-$Th(\mathbb{N})$, so this sentence holds in $\mathbb{N}$. It follows that $\sim\phi(h\vec{a})$ holds componentwise, a contradiction. □

REMARK. We can prove a similar version of Theorem 5, in which T is a theory containing the Π_{n+1}-$Th(\mathbb{N})$, and ϕ is a Σ_n or Π_n formula. We can also take T to be $Th(\mathbb{N})$, in which case the theorem holds for all first order formulas ϕ.

REMARK. Theorem 5 provides yet another route to Rabin's theorem (Theorem 5.9), which states that a nonstandard model M of $Th(N)$ cannot be existentially closed. As in Theorem 9 we can assume M is countable. By Proposition 3.8, there is an embedding $h : M \longrightarrow \mathcal{N}$ such that, in $h(M)$, some $\exists_1$ formula ϕ holds componentwise but does not hold. By Theorem 5, ϕ must hold in some extension of M satisfying $Th(\mathcal{N})$, so M cannot be existentially closed.

The right to left direction of Theorem 5 has a stronger statement and a more constructive proof, which is of independent interest:

THEOREM 7. Suppose A is a substructure of $\mathbb{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$, and ϕ is a formula with parameters from A. If ϕ holds componentwise, then A extends to a model of $Th(\mathbb{N}) + \phi$.

PROOF. Let $\mathcal{G}$ be a non-principal ultrafilter in the boolean algebra of subsets of $\mathbb{N}$, and let $\mathbb{N}^\omega/\mathcal{G}$ be the ultrapower determined by $\mathcal{G}$. The LA structure $\mathbb{N}^\omega/\mathcal{G}$ is a model of $Th(\mathbb{N})$. If g is a function on $\mathbb{N}$, let $\langle g \rangle$ be the equivalence class of g in $\mathbb{N}^\omega/\mathcal{G}$. The function $h : \mathcal{N} \longrightarrow \mathbb{N}^\omega/\mathcal{G}$ given by:

$$[f] \mapsto \langle f \rangle$$

is an embedding of A in $\mathbb{N}^\omega/\mathcal{G}$. (See the proof of Proposition 5.6.) Let ϕ^h be the formula ϕ with h applied to its parameters. Since ϕ holds componentwise, and $\mathcal{G}$ contains the cofinite filter, ϕ^h holds in $\mathbb{N}^\omega/\mathcal{G}$. $\square$

The Π_2-$Th(\mathbb{N})$ proves the MRDP Theorem, since equivalences between Δ_0 formulas and Diophantine formulas are Π_2. It follows from Theorem 4 that the Π_2-$Th(\mathbb{N})$ is Δ_0-coherent. This means in any substructure of $\mathcal{N}$ satisfying the Π_2-$Th(\mathbb{N})$, Δ_0 formulas behave componentwise. Now this fact does not have any immediate connection with Diophantine formulas. This raises the question whether there are properties of $\mathcal{N}$ which enforce componentwise behavior of Δ_0 formulas in these models. We observe that if we were to establish, without appealing to the MRDP Theorem, the existence of such properties, then this would yield a new proof of that theorem.

In fact, the MRDP Theorem is equivalent to an (apparently) weaker fact about componentwise behavior, which we now describe. We first give a definition of what it means for a model to be Δ_0-coherent.

DEFINITION. Let M be a countable model of the $\forall_1$-$Th(\mathbb{N})$. We say that M is Δ_0 coherent if given any substructure M' of $\mathcal{N}$ isomorphic to M, the Δ_0 formulas behave componentwise in M'.

We shall establish a connection between Δ_0-coherence in a model of $Th(\mathbb{N})$ and sets of exponential growth: A set D of pairs of natural numbers is said to be of exponential growth if it has the following two properties:

(i) $\langle x, y\rangle \in D$ implies $y \leq x^x$.

(ii) For each n there exists $\langle x, y\rangle \in D$ such that $y > x^n$.

In 1952 Julia Robinson proved that if there is a Diophantine set of exponential growth then every recursively enumerable set is Diophantine. (See [16].) The following theorem gives a construction of a set of exponential growth based on the assumption of the existence of a countable, Δ_0-coherent model of $Th(\mathbb{N})$. It does not presuppose the 1970 result of Matiyasevich, Robinson, Davis and Putnam that every recursively enumerable set of natural numbers is Diophantine. We note then, that if we were to establish the existence of such a model without making use of the MRDP Theorem, we would obtain a new proof of that theorem.

THEOREM 9. There exists a nonstandard model of $Th(\mathbb{N})$ which is Δ_0-coherent iff every recursively enumerable subset of $\mathbb{N}$ is Diophantine.

PROOF. Suppose every recursively enumerable subset of $\mathbb{N}$ is Diophantine. It suffices to show that if M is a countable model of $Th(\mathbb{N})$ contained in $\mathcal{N}$, then the Δ_0 formulas behave componentwise in M. Let $\phi(\vec{x})$ be a Δ_0 formula, and let $\vec{a}$ be a tuple from M. Suppose that M satisfies $\phi(\vec{a})$. We have to show that $\phi(\vec{a})$ holds componentwise. The set $\phi^{\mathbb{N}}$ is recursively enumerable, hence diophantine. Let $\psi(\vec{x})$ be a Diophantine formula such that $\psi^{\mathbb{N}} = \phi^{\mathbb{N}}$. The sentence

$$\forall \vec{x}(\phi(\vec{x}) \leftrightarrow \psi(\vec{x})) \tag{1}$$

holds in $\mathbb{N}$, hence also in M. Thus $M \models \psi(\vec{a})$. By Proposition 4.2, the formula $\psi(\vec{a})$ holds componentwise. Since the sentence (1) holds in $\mathbb{N}$, it follows that $\phi(\vec{a})$ holds componentwise, as required. Conversely, if $\phi(\vec{a})$ holds componentwise in M, then it must hold in M, else repeating the above argument with $\sim\phi$ in place of ϕ we would obtain a contradiction. We have shown that every countable model of $Th(\mathbb{N})$ is Δ_0-coherent.

For the converse, let M be a Δ_0-coherent nonstandard model of $Th(\mathbb{N})$. We will show that the exponential function is Diophantine. The theorem will then follow by a result of Julia Robinson. By way of contradiction, suppose the exponential function has no Diophantine equivalent. Let $\phi(x, y)$ be a Δ_0

formula defining the exponential function in $\mathbb{N}$, i.e., $\mathbb{N} \models \phi(x,y) \leftrightarrow 2^x = y$. Because the exponential function is total in M, there must be nonstandard elements a, $b \in M$ for which $M \models \phi(a,b)$. The model M is embeddable into $\mathcal{N}$. Using the notation of Theorem 3.1, fix one such embedding h. Without loss of generality, we can assume that $a = m_1$ and $b = m_2$. We shall show that, if n is sufficiently large, then the Diophantine formula

$$D_n(x_1, x_2) = \exists x_3, x_4, \ldots (P_1 \bigwedge \ldots \bigwedge P_n)$$

defines, over $\mathbb{N}$, a subset of the graph of the exponential function $x_2 = 2^{x_1}$. Suppose not. Then for infinitely many i there would be integers r_i and s_i such that $N \models D_i(r_i, s_i)$, and $2^{r_i} \neq s_i$. But then we could use these counterexamples r_i and s_i to embed M in $\mathcal{N}$ in such a way that the exponential function fails to hold componentwise, contrary to our assumption that M is Δ_0-coherent. Thus, we have shown that, for n large enough, $D_n(x_1, x_2)$ defines over $\mathbb{N}$ a subset of the graph of the exponential function. This subset must be infinite, since a and b are nonstandard. We have shown the existence of a Diophantine set of exponential growth. The fact that every recursively enumerable set is Diophantine follows from this, by Julia Robinson's theorem. See [16]. □

REMARK. The assumptions made in Theorem 9 can be significantly weakened. In order to establish that every recursively enumerable subset of natural numbers is Diophantine we need only exhibit a countable, Δ_0-coherent model of $\forall_1$-$Th(\mathbb{N})$ containing a nonstandard pair of exponentially related elements, rather than a model of $Th(\mathbb{N})$ with those properties. The only property of M required to establish the theorem in the other direction is that M satisfy the Π_2-$Th(\mathbb{N})$.

REMARK. In the discussion following Proposition 4.2 we saw that the substructure M of $\mathcal{N}$ generated by the identity function is a model of $\forall_1$-$Th(\mathbb{N})$ (but not a model of PA^-). Therefore, since the models of the $\forall_1$-$Th(\mathbb{N})$ are precisely the substructures of $Th(\mathbb{N})$[7], M must extend to a model M' of $Th(\mathbb{N})$. We claim that M' cannot be included in $\mathcal{N}$. Why? In Theorem 4 of this section we proved that substructures of $\mathcal{N}$ which satisfy MRDP $+\forall_1$-$Th(\mathbb{N})$ are Δ_0-coherent. Therefore M' is Δ_0-coherent. Let f be the identity function and let $\phi(x)$ be the Δ_0 formula "x is not even and x is not odd." The formula $\phi(f)$ holds in M', but clearly cannot hold componentwise. This contradicts the Δ_0-coherence of M', proving the claim. This observation raises the following question: What is a necessary and sufficient condition for a substructure of $\mathcal{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$ to extend to a model of $Th(\mathbb{N})$ inside $\mathcal{N}$? We do not know the answer to this question.

[7]see Lemma 1

7 Cohesiveness

In this section we answer the following question: Let T be one of the theories $\forall_1$-$Th(\mathbb{N})$, Π_2-$Th(\mathbb{N})$, $Th(\mathbb{N})$. For which functions f from $\mathbb{N}$ to $\mathbb{N}$ is there a model of T inside $\mathcal{N}$ which contains f? We will not answer this question for general LA theories T, but we will indicate the form that such an answer might take. We need a

DEFINITION. Let $\mathcal{L}$ be a subset of $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. Let f be a function from $\mathbb{N}$ to $\mathbb{N}$. We say that f is $\mathcal{L}$-cohesive if for all s in $\mathcal{L}$, the function f either eventually assumes values in s or eventually assumes values outside of s.[8]

For example, if $\mathcal{L}$ is the set of all finite subsets of $\mathbb{N}$, then a function f is $\mathcal{L}$-cohesive iff it is eventually constant or tends to infinity. If $\mathcal{L}$ is the set of all recursive sets of integers, then an $\mathcal{L}$-cohesive function is eventually prime or eventually composite, eventually even or eventually odd, and so on. In this case we will refer to f as r-cohesive. In the the case that $\mathcal{L}$ is the set of recursively enumerable sets of integers, we will refer to f as simply cohesive.[9]

As a first application of this definition, we establish

PROPOSITION 1. Let $\mathcal{L}$ be the set of finite subsets of $\mathbb{N}$. A function f is contained in a substructure of $\mathcal{N}$ satisfying $\forall_1$-$Th(\mathbb{N})$ iff f is $\mathcal{L}$-cohesive.

PROOF. Suppose f is $\mathcal{L}$-cohesive. Either f is eventually constant or f

[8]There is an analogous definition of cohesive sets of integers in [17], page 231. A function f tending to infinity is cohesive in our sense iff it has a cohesive range in Rogers' sense.

[9]This conforms to Rogers' definitions for sets.

tends to infinity. In the first case, the substructure of $\mathcal{N}$ generated by f is isomorphic to the standard model, which satisfies the required theory. Assume f tends to infinity. Then the substructure generated by f is isomorphic to the polynomial semiring $\mathbb{N}[x]$. The proof of Theorem 4.4 shows that this is a model of $\forall_1$-$Th(\mathbb{N})$.

Conversely, assume f is contained in a model of $\forall_1$-$Th(\mathbb{N})$. In the discussion following Theorem 3.1, we show that f is either eventually constant or tends to infinity. □

The next sequence of lemmas is devoted to proving an analog of Proposition 1 for Π_2-$Th(\mathbb{N})$.

LEMMA 2. Let $M \subseteq \mathcal{N}$ be a model of the Π_2-$Th(\mathbb{N})$. If $[f] \in M$, then the function f is r-cohesive.

PROOF. Let R be a recursive set of natural numbers, and let S be the complement of R in $\mathbb{N}$. Making use of the MRDP Theorem, let $\rho(x)$ and $\sigma(x)$ be $\exists_1$ formulas defining R and S in the structure $\mathbb{N}$. We note that M satisfies the sentence $\forall x(\rho(x) \vee \sigma(x))$, since this sentence holds in $\mathbb{N}$, and is logically equivalent to a Π_2 sentence. Thus M satisfies one of the formulas $\rho(f)$, $\sigma(f)$. In the first case, by Proposition 4.2, $\rho(f)$ holds componentwise. In the second case $\sigma(f)$ holds componentwise. But this means either f is eventually in R or f is eventually in S. Thus f is r-cohesive. □

LEMMA 3. Let M be a substructure of $\mathcal{N}$ which satisfies the $\forall_1$-$Th(\mathbb{N})$. Suppose further that M is closed under the componentwise application of recursive functions, i.e., given any recursive function $g : \mathbb{N} \longrightarrow \mathbb{N}$, if $[f] \in M$ then $[g \circ f] \in M$. If $[f]$ is an element of M, then f is r-cohesive.

PROOF. Let R be a recursive set of natural numbers with characteristic function χ_R. We observe that $[\chi_R \circ f] \in M$, since M is closed under the componentwise application of recursive functions. Since M satisfies $\forall_1$-$Th(\mathbb{N})$, it follows from Proposition 1 that the function $\chi_R \circ f$ is $\mathcal{L}$-cohesive, where $\mathcal{L}$ is the set of finite subsets of $\mathbb{N}$. By the discussion preceding Proposition 1, $\chi \circ f$ is eventually constant. Hence f is eventually in R or eventually in the complement of R, as required. □

Cohesiveness is related in a natural way to componentwise behavior:

LEMMA 4. Let M be a substructure of $\mathcal{N}$ which satisfies the $\forall_1$-$Th(\mathbb{N})$. Suppose that M is closed under the componentwise application of recursive functions from $\mathbb{N}$ to $\mathbb{N}$. Then the Δ_0 formulas behave componentwise in M.

PROOF. We note first that if $g : \mathbb{N}^m \rightarrow \mathbb{N}$ is recursive (for any $m \geq 0$) and $f_1, \ldots, f_m$ are in M, then so is $g(f_1, \ldots, f_m)$. To see this, let $\sigma_n(x_1, \ldots, x_n)$ be the iterated Cantor pairing function, defined inductively as follows.

$$\sigma_1(x) = x$$

$$\sigma_2(x, y) = \tfrac{(x+y)(x+y+1)}{2} + y$$

$$\sigma_n(x_1, \ldots, x_n) = \sigma_2(x_1, \sigma_{n-1}(x_2, \ldots x_n)) \quad \text{for } n \geq 2$$

The function $2^m \cdot \sigma_m$ is a polynomial σ'_m over $\mathbb{N}$. It follows that the function $\sigma'_m(f_1, \ldots, f_m)$ is in M. Let $g' : \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$g'(x) = \begin{cases} g(x_1, \ldots, x_m) & \text{if } x = \sigma'_m(x_1, \ldots, x_m) \\ 0 & \text{if } x \text{ is not in the range of } \sigma'_m \end{cases}$$

Then the function $g' \circ \sigma'_m(f_1, \ldots, f_m)$ is in M, i.e., $g(f_1, \ldots, f_m)$ is in M, as required.

We now turn to the proof that Δ_0 formulas behave componentwise in M. We proceed by induction on the complexity of the Δ_0 formula ϕ.

If ϕ is atomic then it is immediate that ϕ behaves componentwise. If ϕ is a conjunction then the componentwise behavior of ϕ follows immediately by induction.

Assume now that ϕ has the form $\sim \psi(\vec{x})$. For this case we will require the Cantor projection functions $\pi_i(x)$, which we define as follows:

$$y = \pi_i(x) \quad \text{iff} \quad \exists w_1, \ldots, w_{i+1} \leq x(x = \sigma_{i+1}(w_1, \ldots, w_{i+1}) \wedge y = w_i).$$

The π_i are total recursive functions from $\mathbb{N}$ to $\mathbb{N}$ satisfying

$$\pi_i(\sigma_{i+1}(x_1, \ldots, x_{i+1})) = x_i.$$

Thus, by the closure property of M, if $[f]$ is in M then so is $[\pi_i \circ f]$.

Now suppose $M \models \sim \psi(f_1, \ldots, f_k)$. Then it is not the case that $M \models \psi(f_1, \ldots, f_k)$. Therefore by the induction hypothesis it is not the case that $\psi(f_1, \ldots, f_k)$ holds componentwise. This means, for infinitely many n,

$$\mathbb{N} \models \sim \psi(f_1(n), \ldots, f_k(n)).$$

By the closure property of M, the function $f(x)$ defined by $\sigma_n(f_1(x), \ldots, f_n(x))$ is an element of M. Hence f is r-cohesive, and $\pi_i(f(x)) = f_i(x)$ for all $i \leq n$. But the formula $\sim \psi(\pi_1(x), \ldots, \pi_k(x))$ defines a recursive subset of $\mathbb{N}$ containing $f(n)$ for infinitely many n. It follows that f is eventually in this set. Hence, the formula $\sim \psi(f_1, \ldots, f_k)$ holds componentwise, as required.

Conversely, suppose that $\sim \psi(f_1, \ldots, f_k)$ holds componentwise. Then it is not the case that the formula $\psi(f_1, \ldots, f_k)$ holds componentwise. Thus, by induction, the formula $\sim \psi(f_1, \ldots, f_k)$ must hold in M.

Finally, suppose ϕ has the form $\exists x \leq y \ \psi(x, y, z_1, \ldots, z_m)$, and suppose that

$$M \models \exists x \leq f \ \psi(x, f, g_1, \ldots, g_m).$$

Then

$$M \models \psi(h, f, g_1, \ldots, g_m) \wedge h \leq f$$

for some $h \in M$. By the induction hypothesis and the componentwise behavior of atomic formulas, the formula

$$\psi(h, f, g_1, \ldots, g_m) \wedge h \leq f$$

holds componentwise. But this means the formula

$$\exists x \leq f \ \psi(x, f, g_1, \ldots, g_m)$$

holds componentwise.

Conversely suppose the formula $\exists x \leq f \ \psi(x, f, g_1, \ldots, g_m)$ holds componentwise. We wish to show that

$$M \models \exists x \leq f\ \psi(x, f, g_1, \ldots, g_m).$$

Define $r : \mathbb{N}^{m+1} \to \mathbb{N}$ as follows: $r(u, v_1, \ldots, v_m)$ is the least x such that $\mathbb{N} \models x \leq u \wedge \psi(x, u, v_1, \ldots, v_m)$, if there is such an x, and 0 otherwise. The function r is total recursive. Let s be defined by

$$s(n) = r(f(n), g_1(n) \ldots, g_m(n)).$$

Then $[s]$ is in M, by the closure property of M and the note at the beginning of this lemma. But we have defined s so that the formula $\psi(s, f, g_1, \ldots, g_m) \wedge s \leq f$ holds componentwise. Thus, by induction,

$$M \models \psi(s, f, g_1, \ldots, g_m) \wedge s \leq f.$$

Therefore

$$M \models \exists x \leq f \psi(x, f, g_1, \ldots, g_m),$$

as required. □

We mention the following characterization of models of $\Pi_2 - Th(\mathbb{N})$:

THEOREM 5 (KENNEDY-RAFFER). Let M be a countable substructure of $\mathcal{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$. Then $M \models \Pi_2$-$Th(\mathbb{N})$ iff M is closed under the componentwise application of total recursive functions.

PROOF. Suppose $M \models \Pi_2$-$Th(\mathbb{N})$ and suppose $g : \mathbb{N} \to \mathbb{N}$ is total recursive. Let f be an element of M. We wish to show that $g \circ f$ is in

M. Using the MRDP Theorem, we choose an $\exists_1$ formula $\delta(x,z)$ defining the graph of g in $\mathbb{N}$. That is, for all x and y in $\mathbb{N}$,

$$\mathbb{N} \models \delta(x,y) \text{ iff } y = g(x).$$

The Π_2 formula $\forall x \exists! y \delta(x,y)$ holds in $\mathbb{N}$, hence it also holds in M. Choose a function h in M such that $M \models \delta(f,h)$. Since M satisfies $\forall_1\text{-}Th(\mathbb{N})$, by Proposition 4.2, the formula $\delta(f,h)$ holds componentwise. Also the formula $\delta(f, g \circ f)$ holds componentwise. Since δ defines in $\mathbb{N}$ the graph of a function, it follows that $[h] = [g \circ f]$. Thus $[g \circ f]$ is in M, as required.

Conversely, suppose M has the stated closure property. Let $\phi(\vec{x},\vec{y})$ be a Δ_0 formula and let Φ be the sentence $\forall \vec{x} \exists \vec{y} \phi(\vec{x},\vec{y})$. We assume that Φ holds in $\mathbb{N}$, and we wish to show that it holds in M. Choose $f_1,\ldots,f_m$ arbitrarily from M. It suffices to show that $M \models \exists \vec{y} \phi(f_1,\ldots,f_m,\vec{y})$. To this end, we define the function h on $\mathbb{N}^m$ by:

$$h(\vec{x}) = \text{the least } y \text{ such that } \mathbb{N} \models \phi(\vec{x}, \pi_1(y), \ldots, \pi_n(y)),$$

where the functions π_i are defined in Lemma 4. Because $\mathbb{N}$ satisfies Φ, h is a total function. Moreover h is recursive. Now let f be the function defined on $\mathbb{N}$ by

$$f(x) = h(f_1(x).\ldots,f_m(x)).$$

By the remarks at the beginning of the proof of Lemma 4, the function f must be in M. Thus, by the assumed closure property of M, the functions $\pi_1 \circ f, \ldots, \pi_n \circ f$ are all in M. But, since Φ holds in $\mathbb{N}$, the formula

$$\phi(f_1,\ldots,f_m,\pi_1 \circ f,\ldots,\pi_n \circ f)$$

holds componentwise. By Lemma 4, this formula holds in M. Therefore $M \models \exists \vec{y} \phi(f_1, \ldots, f_m, \vec{y})$, and the proof is complete. □

We can now answer the question, concerning Π_2-$Th(\mathbb{N})$, stated at the beginning of this section:

THEOREM 6. Let f be a function from $\mathbb{N}$ to $\mathbb{N}$. Then f is contained in some substructure of $\mathcal{N}$ satisfying Π_2-$Th(\mathbb{N})$ iff f is r-cohesive.

PROOF. Suppose $M \subseteq \mathcal{N}$ is a model of Π_2-$Th(\mathbb{N})$ containing f. By Lemma 2, f is r-cohesive.

Conversely, suppose f is r-cohesive. Let

$$M = \{[g \circ f] : g \text{ is a recursive function from } \mathbb{N} \text{ to } \mathbb{N}\}.$$

We will show that M is a model of Π_2-$Th(\mathbb{N})$. Since the recursive functions are closed under addition and multiplication, M is a substructure of $\mathcal{N}$. Moreover, M is closed under the componentwise application of total recursive functions. We wish to apply Theorem 5, but first we have to prove that M satisfies $\forall_1$-$Th(\mathbb{N})$. By Theorem 4.4, it suffices to show that M is a model of PA^-. Now M inherits from $\mathcal{N}$ the axioms for a partially ordered semiring, since these axioms are $\forall_1$. Thus we need only check that the order relation in M is total and discrete, and that M is closed under nonnegative differences. For totality, suppose g and h are recursive functions, and suppose $M \models \sim [g \circ f] \leq [h \circ f]$. Then f does not eventually assume values in the set $\{n : g(n) \leq h(n)\}$. But f is r-cohesive, and this is a recursive set. Thus f eventually assumes values in $\{n : h(n) \leq g(n)\}$. Thus $M \models [h \circ f] \leq [g \circ f]$.

This proves that $\leq$ is total in M. Discreteness (which asserts that if $x \leq y \leq x+1$ then $x = y$ or $x = y+1$) is proved similarly. As for nonnegative differences, we observe that the function giving the nonnegative difference of two integers is recursive, thus by an argument similar to the one at the beginning of Lemma 4, M is closed under this function. We have shown that M is a model of PA^-. It follows that M satisfies $\forall_1$-$Th(\mathbb{N})$. Thus, by Theorem 5, M satisfies Π_2-$Th(\mathbb{N})$. □

Next, we consider the problem, when is a function f contained in a model of $Th(\mathbb{N})$? Cohesiveness provides a sufficient condition:

THEOREM 7 (KENNEDY-RAFFER). Let f be cohesive. Then f is contained in a substructure $M \subseteq \mathcal{N}$ satisfying $Th(\mathbb{N})$.

PROOF. Let $\phi_1(x), \phi_2(x), \ldots$ be all $\exists_1$ and $\forall_1$ formulas in one free variable such that for each i, f is eventually in $\phi_i^{\mathbb{N}}$. (If $\phi(x)$ is a formula and M is a structure, we use ϕ^M to denote $\{m \in M : M \models \phi(m)\}$.) Let c be a new constant symbol added to LA, the language of arithmetic, and let Γ be the theory

$$Th(\mathbb{N}) \cup \{\phi_1(c), \phi_2(c), \ldots\}.$$

Because f is eventually in each of the sets $\phi_i^{\mathbb{N}}$, we can satisfy any finite subset of Γ in the standard model $\mathbb{N}$, by taking $c^{\mathbb{N}}$ sufficiently large.

Choose, by compactness, a countable model M of Γ. To prove the theorem, we will construct an embedding of M into $\mathcal{N}$ which maps the element c^M to $[f]$.

Let $m_1, m_2, \ldots$ be an enumeration of M for which $c^M = m_1$. Let $P_1, P_2, \ldots$ be a sequence of polynomials which can be used to construct an embedding of M into $\mathcal{N}$, as in Theorem 3.1. That is, for each n,

$$M \models P_n(x_1/m_1, x_2/m_2, \ldots).$$

Let $D_n(x_1)$ be the formula $\exists x_2, x_3, \ldots P_n(x_1, x_2, \ldots, x_k)$. It would be possible to map c^M to $[f]$ using the argument of Theorem 3.1, if we knew that $\mathbb{N} \models D_n(f(n))$ for all sufficiently large n. However, the P_n's may be ordered in such a way that this is impossible. We will show that there is a re-ordering of the P_n's such that an embedding of the required type can be constructed.

We first observe that for each k, f is eventually in $D_k^{\mathbb{N}}$. To see this, suppose otherwise. Then because the function f is cohesive, and because the set $D_k^{\mathbb{N}}$ is recursively enumerable, it must be that f is eventually in $\sim D_k^{\mathbb{N}}$. Thus $\sim D_k$ appears as one of the $\phi's$, say ϕ_i. But for all n, $M \models \phi_n(m_1)$. Thus $M \models \sim D_i(m_1)$, a contradiction.

Now for each integer i, let $P_{i1}, P_{i2}, \ldots$ be an enumeration of the following set of polynomials:

$$\{P_j : j \geq 1 \text{ and for all } n \geq i, f(n) \in D_j^{\mathbb{N}}\}.$$

We have shown that f eventually assumes its values in each of the sets $D_n^{\mathbb{N}}$. It follows that each of the P_n's appears (in fact infinitely often) as one of the P_{ij}'s. Now let

$$Q_n = \bigwedge_{0 \leq i,j, \leq n} P_{ij}(x_1, \ldots, x_n).$$

It follows that for each n,

$$\mathbb{N} \models \exists x_2, x_3, \ldots Q_n(f(n), x_2, x_3, \ldots).$$

Finally, choose, at each stage n, a sequence of natural numbers $v_2(n), v_3(n), \ldots$ such that

$$\mathbb{N} \models Q_1 \bigwedge \cdots \bigwedge Q_n(x_1/f(n), x_2/v_2(n), \ldots).$$

With the Q_n's replacing the P_n's, we then obtain the embedding we set out to construct, i.e., an embedding which sends c^M to $[f]$. □

We do not know a necessary and sufficient condition for a function f to be contained in a substructure of $\mathcal{N}$ satisfying $Th(\mathbb{N})$. We now give an example to show that the condition given in Theorem 7 is not necessary.

PROPOSITION. There is a substructure M of $\mathcal{N}$ satisfying the $Th(\mathbb{N})$, and a function f in M such that f is not cohesive.

PROOF. Let M be a countable nonstandard model of $Th(\mathbb{N})$. We will show that there is an embedding h of M into $\mathcal{N}$ such that under h, an element of M is mapped to a function which is not cohesive. Let the $\exists_1$ formula $\phi(x)$ define a simple set S in $\mathbb{N}$. Arguing as in Proposition 5.8, there is a nonstandard element $m \in M$ such that $M \models \sim\phi(m)$. Using the notation of Theorems 3.1 and 3.2, let $P_1, P_2, \ldots$ be all the polynomial equations over $\mathbb{N}$ such that $M \models P_i(x_1/m_1, x_2/m_2, \ldots)$. We construct our embedding as follows: At stage n, for n odd, choose natural numbers $v_1(n), v_2(n), \ldots$ for which

$$\mathbb{N} \models P_1 \bigwedge \cdots \bigwedge P_n(x_1/v_1(n), x_2/v_2(n), \ldots,) \wedge \phi(v_1(n)).$$

That such a tuple of natural numbers exists follows from the simplicity of $\phi^{\mathbb{N}}$, as in the proof of Proposition 5.8. At stage n, for n even, choose natural numbers $v_1(n), v_2(n), \ldots$ for which

$$\mathbb{N} \models P_1 \bigwedge \cdots \bigwedge P_n(x_1/v_1(n), x_2/v_2(n), \ldots,) \wedge \sim\phi(v_1(n)).$$

That such a tuple of natural numbers exists follows from the fact that we can expand the set of formulas witnessed in the construction of the embedding as in Theorem 5.3. As in Theorems 3.1 and 3.2, we define our embedding $h : M \to \mathcal{N}$ by:

$$m_i \longmapsto [\lambda k.v(i(k)].$$

But then we obtain an embedding h such that $v_1(n) \in S$, if n is even, $v_1(n) \notin S$, if n is odd. But this means the function $v_1(n)$ is not cohesive. □

References

[1] C.C. CHANG & H.J. KEISLER. *Model Theory.* Amsterdam: North Holland, 1973.

[2] M. DAVIS. *Computability and Unsolvability.* New York: Dover Publications, 1973.

[3] J. HIRSCHFELD. Models of arithmetic and recursive functions. *Israel Journal of Mathematics* #20 (1975), 111–126.

[4] J. HIRSCHFELD & W. WHEELER. *Forcing, Arithmetic and Division Rings.* New York: Springer-Verlag, 1975.

[5] W. HODGES. *Model Theory.* Cambridge, U.K.: Cambridge University Press, 1993.

[6] ——. *Building Models by Games.* London Mathematical Society Student Texts:2. Cambridge, U.K.: Cambridge University Press.

[7] R. KAYE. Open induction, Tennenbaum phenomena, and complexity theory. In *Arithmetic, Proof Theory and Computational Complexity,* J. KRAJÍČEK P. CLOTE, ed., 222–237, Oxford, U.K., 1993. Oxford University Press.

[8] RICHARD KAYE. *Models of Peano Arithmetic.* Oxford Logic Guides. Oxford: Oxford University Press, 1991.

[9] S. LANG. *Algebra.* Reading, Massachusetts: Addison Wesley, first edn., 1971.

[10] L. LIPSHITZ. Diophantine correct models of arithmetic. *Proceedings of the American Mathematical Society* #73 (1979), 107–108.

[11] Y. MATIYASEVICH. *Hilbert's Tenth Problem.* Cambridge, Massachusetts: MIT Press, 1993.

[12] K. MCALOON. Models of arithmetic and complexity theory. In *Studies in Complexity Theory*, R. BOOK, ed., Research Notes in Theoretical Computer Science, 119–221. Wiley, 1986.

[13] P. ODIFREDDI. *Classical Recursion Theory.* Amsterdam: North Holland, 1989.

[14] J. ŁOŚ. On the extending of models 1. *Fundamenta Mathematicae* #42 (1955), 38–54.

[15] M. RABIN. Diophantine equations and nonstandard models of arithmetic. In *Logic, Methodology and Philosophy of Science*, A. TARSKI E. NAGEL, P. SUPPES, ed. Stanford, 1962.

[16] J. ROBINSON. Existential definability in arithmetic. *Transactions of the American Mathematical Society* #73 (1952), 437–449.

[17] H. ROGERS. *Theory of Recursive Functions and Effective Computability.* Cambridge, Massachusetts: MIT Press, second edn., 1987.

[18] D. SCOTT. On constructing models for arithmetic. In *Infinitistic Methods*, PERGAMON, ed., New York, 1961.

[19] S. FEFERMAN, D. SCOTT, & S. TENNENBAUM. Models of arithmetic through function rings. *Notices of the American Mathematical Society* (1959), 173–174.

[20] THORALF SKOLEM. Peano's axioms and models of arithmetic. In *Mathematical Interpretations of Formal Systems*, Amsterdam, 1955. North Holland.

[21] C. SMORYŃSKI. Lectures on nonstandard models of arithmetic. In *Logic Colloquium 1982*, A. MARCJA G. LOLLI, G. LONGO, ed., Amsterdam, 1984. North Holland.

[22] R. SOARE. *Recursively Enumerable Sets and Degrees.* New York: Springer-Verlag, 1987.

[23] STANLEY TENNENBAUM. Non-archimedian models of arithmetic. *Notices of the American Mathematical Society* #6 (1959), 270.