Affine automorphism groups of surfaces of infinite type

by

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Abstract

AFFINE AUTOMORPHISM GROUPS OF SURFACES OF INFINITE TYPE

by

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In this dissertation, examples of flat surfaces with infinite genus and finite area are constructed. It is shown that the automorphism groups of these surfaces are non-elementary Fuchsian groups of the second kind. In certain cases the automorphism group is infinitely generated. This gives an explicit construction of Teichmüller disks in the Teichmüller space of a surface of infinite genus X, stabilized by large sub-groups of the mapping class group of X.

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To my mother, my father, Sepideh and Elham

CONTENTS

INTRODUCTION

It is well-known that there is a two-to-one homomorphism from the mapping class group of the torus onto the Fuchsian group $PSL(2,\mathbb{Z})$. This is because any mapping class can be represented as an affine automorphism of a flat torus with finite area. In general one can start with a flat metric of finite area on an arbitrary surface and consider the group of all automorphisms of this flat surface which are locally affine with respect to the charts of the flat structure. Since different choices of charts may change the affine representatives up to a translation or a half rotation, there is an element of $PSL(2,\mathbb{R})$ assigned to any such automorphism. An alternative way to describe this subgroup is to start with a a pair (X,φ) of a Riemann surface X and an integrable holomorphic quadratic differential φ on X. Any such pair defines a one complex dimensional slice D_φ of the Teichmüller space of X. D_φ equipped with the restriction of the Teichmüller metric is totally geodesic and isometric to the hyperbolic plane, it is thus called a Teichmüller disk. Therefore the stabilizer of the disk D_φ in the mapping class group of X can be regarded as a subgroup of $PSL(2,\mathbb{R})$. This group is denoted by $Aut(X,\varphi)$.

Generically $Aut(X,\varphi)$ is trivial. On the other hand Thurston constructed non-elementary examples. His construction starts with a pair of multi-twists on a pair of intersecting multi-curves on a surface of finite type. Then he finds a flat structure that realizes these two multi-twists as affine automorphisms. The composition of these two *parabolic* automorphisms produces a *hyperbolic* automorphism which represents a pseudo-Anosov homeomorphism. The automorphism group for this flat surface is a Fuchsian group of the first kind commensurable with $SL(2,\mathbb{Z})$ (see [11]). In addition, Veech found out that for any $g > 0$, there are Riemann surfaces of genus g, equipped with flat structures, where automorphism groups are lattices not commensurable with $SL(2,\mathbb{Z})$ (see [4] and [12]).

Here we investigate the existence of Riemann surfaces of infinite topological type with non-elementary automorphism groups. Although the mapping class group of such a surface is rich, it is very hard to realize an element of the mapping class group as an affine automorphism of a flat structure with finite area. We introduce a one parameter family of flat surfaces $(X_\alpha, \varphi_\alpha)$ $0 < \alpha < 1$, such that $\Gamma_\alpha = Aut(X_\alpha, \varphi_\alpha)$ is non-elementary for all rational values of α. The surface X_α has infinite genus and one end. Using an extremal length argument we show that the Fuchsian group G_α obtained from the uniformization of X_α is of the first kind.

For surfaces of finite type, the mapping class group always acts discontinuously on the Teichmüller space. Therefore the automorphism group is a Fuchsian group. It is not obvious that the automorphism groups of surfaces of infinite type should also be Fuchsian (discrete) groups.

Let us denote the Teichmüller disk generated by the flat surface X_α by D_α. It turns out that for $X_\alpha, \alpha = \frac{1}{n}$, Γ_α is a Fuchsian group of the second kind. In fact Γ_α contains a pseudo-Anosov (actually Anosov) automorphism Φ_n and a pair of parabolic automorphisms Δ_0 and Δ_1 such that $\Delta_\alpha^1 \circ \Delta_\alpha^0 = \Phi_n$. The Riemann surface Σ_α obtained from the action of Γ_α on D_α is a surface of genus zero with two cusps and one hole. The length of the geodesic representative of the homotopy class of curves separating the hole from the rest of the surface, which is the same as the translation length of Φ_n, is $\ln n$. $X_{\frac{1}{2}}$ doubly covers a surface of genus zero with infinitely many cusps and with the same automorphism group. The map induced by Φ_2 on this surface, is a generalized pseudo-Anosov map known as the *full horseshoe map* (see [3]). When α is rational but is not equal to $\frac{1}{n}$, Σ_α is conformally equivalent to a disk with infinitely many points removed. The removed points accumulate to a Cantor set on the boundary of the disk.

In section 1 we introduce the definitions and concepts involved in the construction and study of the automorphism groups. These concepts are introduced in terms of flat surfaces and their affine automorphisms as well as in language of the Teichmüller theory. In section 2 we study parabolic automorphisms of flat surfaces. Section 3 introduces the family X_α and its topological and analytic properties. Section 4 involves the study of the parabolic automorphisms of X_α. Finally in Section 5 we find Γ_α and Σ_α explicitly. We use *sliding* which is analogue of earthquaking in the context of flat surfaces to give a description of Σ_α when $\alpha \neq \frac{1}{n}$. Section 6 is devoted to more general constructions and some questions that arise from this work.

1. TEICHMÜLLER DISKS AND FLAT STRUCTURES

Teichmüller space: Let X be any Riemann surface whose universal covering surface is the unit disk Δ and let ∂X be its ideal boundary. Define $\bar{X}$ to be $X \cup \partial X$. Let $QC(X)$ be the group of homeomorphisms of X onto itself. We define $QC_0(X)$ to be the normal subgroup of $QC(X)$ consisting those elements of $QC(X)$ whose extensions to $\bar{X}$

are homotopic to identity by a homotopy fixing ∂X. Quasiconformal homeomorphisms $f_1 : X \longrightarrow Y_1$ and $f_2 : X \longrightarrow Y_2$ are equivalent if $f_2 \circ f \circ f_1^{-1} : Y_1 \longrightarrow Y_2$ is a conformal map for some f in $QC_0(X)$. The Teichmüller space of X , denoted by $Teich(X)$, is the space of equivalence classes of all quasiconformal homeomorphisms with domain X. The class of $id : X \longrightarrow X$ is called the basepoint of $Teich(X)$.

The Teichmüller distance d_T between two classes $[f]$ and $[g]$ in $Teich(X)$ is defined to be

$$d_T([f],[g]) = \frac{1}{2} \inf \log K(f_1 \circ (g_1)^{-1}),$$

where the infimum is taken over all quasiconformal maps f_1 and g_1 in the equivalence classes $[f]$ and $[g]$ and $K(h)$ is the global quasiconformal dilatation of h.

Beltrami coefficients : A tensor μ that in any local coordinate z has the form $f(z)\frac{d\bar{z}}{dz}$ for a bounded measurable function f is called a Beltrami form. As usual, we identify two Beltrami forms if they are equal almost everywhere. Note that for different expressions f_1 and f_2 of μ in neighborhoods U_1 and U_2 with local coordinates z_1 and z_2, value of $|f_1(z)|$ coincides almost everywhere in $U_1 \cap U_2$ with the value of $|f_2(z)|$. Therefore it makes sense to define

$$||\mu||_\infty = \sup\{|f(z)|\};$$

where the supremum is taken over all local coordinates z. We denote by $L_\infty(X)$ the Banach space of the Beltrami forms μ for which $||\mu||_\infty$ is bounded, and we let $M(X)$ be the unit ball in this Banach space.

Any quasiconfomal homeomorphism $f : X \longmapsto f(X)$ defines a Beltrami coefficient μ_f in $\mathbf{M}(X)$ which is equal to $\frac{\bar{\partial}(w \circ f)}{\partial(w \circ f)}$ on $f^{-1}(v)$ for the local coordinate w defined on $V \subset Y$. Conversely, because of the *measurable Riemann mapping theorem* [1], any element μ of $M(X)$ is the Beltrami coefficient of some quasiconformal map f from X to some other Riemann surface $f(X)$. Therefore the quotient map $\Phi : \mathbf{M}(X) \rightarrow \mathit{Teich}(X)$ induces an equivalence relation on $\mathbf{M}(X)$ that gives an alternative way to describe the Teichmüller space of X. We denote the equivalence class of μ by $[\mu]$ and recognize it as an element in $\mathit{Teich}(X)$. The complex manifold structure on $\mathbf{M}(X)$ induces the complex manifold structure on $\mathit{Teich}(X)$ through the quotient map Φ.

Mapping class group : $\mathbf{MCG}(X)$ is defined to be the quotient $QC(X)/QC_0(X)$. Any quasiconformal homeomorphism f in $QC(X)$ induces maps $\rho_f : \mathit{Teich}(X) \rightarrow \mathit{Teich}(X)$ and $\sigma_f : \mathbf{M}(X) \rightarrow \mathbf{M}(X)$ by

$$\rho_f([g]) = [g \circ f^{-1}] \qquad \mathit{and} \qquad \sigma_f(\mu_g) = \mu_{g \circ f^{-1}}.$$

Both maps respect the complex and metric structure of the corresponding spaces and they satisfy $\Phi \circ \sigma_f = \rho_f \circ \Phi$. The map $[f] \longmapsto \rho_f$ factors through $\mathbf{MCG}(X)$, producing a homomorphism:

$$\Theta : \mathbf{MCG}(X) \rightarrow \mathit{Aut}(\mathit{Teich}(X)).$$

Quadratic differentials: A tensor ϕ that in any local coordinate z has the form $f(z)dz^2$, where f is a measurable function, is called a quadratic differential. For any such differential, $|\phi|$ is the two-form on X whose restriction to the domain corresponding to the coordinate z is of the form $|f|dxdy$. The norm of ϕ is defined by:

$$||\phi|| = \int\int_X |\varphi|$$

A quadratic differential ϕ is called holomorphic if all the functions f are holomorphic and it is called integrable if $||\phi||$ is finite. We denote the space of all integrable holomorphic quadratic differentials on X

equipped with the above norm by $\mathbf{Q}(X)$. $\mathbf{Q}(X)$ is a complex Banach space.

Teichmüller disks and their automorphism groups: For any φ in $\mathbf{Q}(X)$ and any t in the unit disk Δ , $t\frac{|\varphi|}{\varphi}$ is an element of $\mathbf{M}(X)$. The Teichmüller disk defined by φ is the locus of points in $Teich(X)$ defined by:

$$D_{\varphi} = \{\Phi(t\frac{|\varphi|}{\varphi}) : t \in \Delta\}$$

By Teichmüller's theorem, the Teichmüller metric restricted to D_{φ} is isometric to the Poincaré Metric on Δ.

We denote the subgroup of all elements F in $Aut(Teich(X))$ such that $F = \Theta(h)$ for some h in $\mathbf{MCG}(X)$ and $F(D_{\varphi}) = D_{\varphi}$ by $Aut(D_{\varphi})$. Any element of $Aut(D_{\varphi})$ acts on D_{φ} as an isomorphism. Therefore by fixing an identification of D_{φ} with the upper half plane one can identify $Aut(D_{\varphi})$ with a subgroup of $\mathbf{PSL}(2, \mathbb{R})$. Note that any two choices of this identification lead to conjugate subgroups. We regard $Aut(D_{\varphi})$ as a conjugacy class of a subgroup of $\mathbf{PSL}(2, \mathbb{R})$.

Now we discuss a more geometric description of $Aut(D_{\varphi})$:

Flat structures on surfaces Let X be a surface and E a discrete subset of points on it. A *flat structure* $\mathfrak{F}$ on X is an atlas of charts on $X \setminus E$ compatible with the orientation of X whose transition functions on overlaps of domains of two charts z and w are of one of the following types:

$$w = z + c \qquad or \qquad w = -z + c \qquad c \in \mathbb{C}.$$

The pair $(X, \mathcal{F})$ is called a flat surface. Since all the transition functions are Euclidean isometries, any flat surface inherits a metric of constant zero curvature on $X \setminus E$. We denote this metric by $g_{\mathcal{F}}$ and the corresponding distance function by $d_{\mathcal{F}}$. The area 2-form on the plane induces an area 2-form on X. We assume that the total area of X is finite. The transition functions are also holomorphic. Hence a flat structure determines a complex structure.

Moreover any transition function sends a line to a parallel line. Therefore for any direction θ, Euclidean lines in that direction foliate $X \setminus E$. Moreover this foliation, which we will denote by $F_{(\theta, \mathcal{F})}$, is equipped with a transverse measure locally defined by the Lebesgue measure in the perpendicular direction to the direction of the foliation.

In general the metric $g_{\mathcal{F}}$ is not complete. To make it complete, it must be extended to points of E and possibly also to another set E' of exceptional points.

We call points in $E \bigcup E'$ singularities of the flat surface. We will use the following terms to describe certain types of singularities:

(1) *Removable singularities*: These are singularities with a neighborhood isometric to a Euclidean disk. The flat structure can be extended to these singularities. We always assume that the flat structure is extended to its removable singularities and therefore there are no such singularities.

(2) *Isolated cone singularities* with cone angle $k\pi$ for a positive integer k: this means that in a neighborhood of z, the metric can be written in polar coordinates (r, θ) as follows:

$$g_{\mathcal{T}}^2 = (dr)^2 + (\frac{k}{2} r d\theta)^2$$

When $k = 1$ a neighborhood of the singularity z looks like a neighborhood of the origin in the cone obtained from the upper half-plane H by gluing the positive real axis to the negative real axis using the map $z \mapsto -z$. In this case the point is called a *pole*.

When $k > 1$ a neighborhood of the singularity z can be described by gluing k copies of upper half-plane H, ordered cyclically so that the positive real axis on one is glued to the negative real axis on the next. In this case the singularity is called a *zero of order* $k - 2$.

(3) *essential singularities*: Neighborhoods of this type of singularity contain *double spirals*. Double spirals can be described as follows. Let R_i be the rectangle

$$R_i = \{z \in \mathbb{C} : -\epsilon_i \leq \Re e(z) \leq \epsilon_i', \ 0 \leq \Im m(z) \leq a_i\} \ , i \in \mathbb{Z}$$

for positive real numbers a_i, ϵ_i and ϵ_i' such that $\epsilon_i' = \epsilon_{i+1}$. Gluing $\{0 \leq z \leq \epsilon_i'\}$ on H_i to $\{-\epsilon_{i+1} \leq z \leq 0\}$ on H_{i+1} for all i we obtain a double spiral. All of the 0-points of each rectangle R_i are identified to one point, which we call the *origin* of the double spiral. A neighborhood of an essential singularity contains a union of at most countably many double spirals, and the origins of these double spirals are identified to a single point. We use the term *index* for the total number of copies of double spirals in a neighborhood of an essential singularity . In all the examples discussed here, the essential singularities have finite index. Since the area of the flat surface is assumed to be finite, $\sum_{i=-\infty}^{\infty} \epsilon_i \alpha_i < \infty$ for any double spiral.

It can be shown that a flat structure with finite area and a fixed direction θ is the same object as an integrable holomorphic quadratic differential. In the next step we describe how Teichmüller disks and their automorphism groups can be defined in the setting of flat structures.

Action of SL(2, ℝ) : For any flat surface $(X, \mathfrak{F})$ and any affine map F of the plane, one can define a new flat surface $(X, F(\mathfrak{F}))$. The affine chart maps are maps of the form $F \circ \varphi$ where φ is a chart map of $\mathfrak{F}$. Since affine maps preserve parallel lines, the new chart system satisfies the condition on transition functions so this new chart system yields an affine structure.

Two flat surfaces $(X, \mathfrak{F}_1)$ and $(X, \mathfrak{F}_2)$ are called equivalent if for some orientation preserving Euclidean isometry A, $(X, \mathfrak{F}_1) = (X, A(\mathfrak{F}_2))$. From now on by $(X, \mathfrak{F})$ we mean the equivalence class of flat structures. Also for the affine surfaces of finite area we normalize the area to be equal to one: therefore one can restrict F to be an area preserving linear map. Now we consider

$$\mathfrak{D}_{(X,\mathfrak{F})} = \{(X, F(\mathfrak{F})) : F \in \mathit{Aff}(2, \mathbb{R})\} = \{(X, A(\mathfrak{F})) : A \in SL(2, \mathbb{R})\}$$

In other words $\mathfrak{D}_{(X,\mathfrak{F})}$ consists of all non-trivial affine deformations of $(X, \mathfrak{F})$. One can show that in the correspondence between integrable holomorphic quadratic differentials and flat surfaces, the Teichmüller disk D_ϕ corresponds to $\mathfrak{D}_{(X,\mathfrak{F})}$ where $(X, \mathfrak{F})$ is the flat surface obtained from the quadratic differential ϕ.

Any orientation preserving $\mathbb{R}$-linear map L acting on $\mathbb{C}$ can be written as $L(z) = w_1 z + w_2 \bar{z}$ where w_1 and w_2 are complex. Since L is assumed to be orientation preserving, $|w_1|^2 - |w_2|^2 > 0$ and therefore $\mu_L = \frac{w_2}{w_1}$ is in the unit disk.

The Teichmüller metric on D_o corresponds to the metric $d_{\mathfrak{F}}(\mathfrak{F}_1, \mathfrak{F}_2) = \ln K(L) = \ln \frac{1+|\mu_L|}{1-|\mu_L|}$ on $\mathfrak{D}_{(X,\mathfrak{F})}$, where L is any linear map such that $L(\mathfrak{F}_1) = \mathfrak{F}_2$. Again $\mathfrak{D}_{(X,\mathfrak{F})}$ together with this metric is isometric to the hyperbolic plane.

Affine automorphism groups of flat structures: We denote the group of orientation preserving affine homeomorphisms of a flat surface $(X, \mathfrak{F})$ by $Aff(X, \mathfrak{F})$. This is the group of all homeomorphisms h of X such that at any point $x \in X \setminus E$ and any local charts φ_1 defined on the neighborhood U_1 of x and φ_2 defined on the neighborhood $U_2 = h(U_1)$ of $h(x)$, the composition $\varphi_2 \circ h \circ \varphi_1^{-1}$ is restriction of an affine map A to $\varphi_1(U_1)$. For an affine map A, $Der(A)$ is defined to be its derivative or linear part. For an $h \in Aff(X, \mathfrak{F})$, $Der(h)$ is defined to be the derivative of any of the affine maps representing h around any point of X. The definition of the flat surfaces implies that $Der(h)$ is well defined and independent of the choice of the representative affine map or the chosen point. On the other hand, two equivalent representatives of the flat surface X, give rise to two conjugate linear maps for $Der(h)$. $Aff(X, \mathfrak{F})$ acts on $\mathfrak{D}_{(X,\mathfrak{F})}$ by $h(\mathcal{E}) = Der(A)(\mathcal{E})$ where A is any affine map representing h in a chart system and $\mathcal{E}$ is a flat structure in $\mathfrak{D}_{(X,\mathfrak{F})}$.

We will be interested in the image of $Aff(X, \mathfrak{F})$ under the map $\Pi = \pi \circ Der$ in $PSL(2, \mathbb{R})$ where $\pi : SL(2, \mathbb{R}) \to PSL(2, \mathbb{R})$ is the natural projection with kernel $\{\pm Id\}$. We denote this subgroup of $PSL(2, \mathbb{R})$ by $Aut(X, \mathfrak{F})$. It acts on $\mathfrak{D}_{(X,\mathfrak{F})}$ as a group of isometries of the hyperbolic plane. Different choices of flat structures in $\mathfrak{D}_{(X,\mathfrak{F})}$ or choosing different representative of one class of flat structures in $\mathfrak{D}_{(X,\mathfrak{F})}$ produce conjugate affine automorphism groups. Therefore when we speak of $Aut(X,\mathfrak{F})$ it is a conjugacy class of groups in $PSL(2,\mathbb{R})$.

Realization of the boundary of Teichmüller disks as space of directions in the plane: For a surface of finite topologic type with genus g and n punctures it is known that there is a natural compactification of *Teich*(X) using the space of projective classes of measured foliations on X. The compactified space is homeomorphic to the unit ball of dimension $6g-6+2n$. The closure of a Teichmüller disk D_ϕ in this compactified space consists of a one parameter family of projective measured foliations which may be described precisely as projective classes of those measured foliations equal to $F_{(\theta,\mathfrak{F})}$ for a flat surface $(X,\mathfrak{F})$ defined by an element in D_ϕ and a direction θ. Hence one can identify ∂D_ϕ with space of planar directions $\mathbb{R}P(2)$. This identification depends on the choice of the flat structure and the representative for that flat structure. More precisely $F_{(\theta,\mathfrak{F})} = F_{(A(\theta),A(\mathfrak{F}))}$ for an affine map A. Even though the theory of compactification of the Teichmüller space is not developed for surfaces of infinite type, the boundary of a Teichmüller disk D_ϕ, still can be realized as a one parameter family of projective measured foliations $F_{(\theta,\mathfrak{F})}$. Moreover by fixing a representative of a flat surface defined by an element in D_ϕ one can identify this boundary with the space of planar directions $\mathbb{R}P(2)$.

Trichotomy of affine automorphisms: Thurston classified homeomorphisms of surfaces of finite type (see [5] and [11]) . Although this theory doesn't extend directly to surfaces of infinite type, there is an analogue for affine automorphisms of flat surfaces.

According to the number of linearly independent real eigenvectors of its projection by Π to $Aut(X, \mathfrak{F})$, an element in $Aff(X, \mathfrak{F})$ must be exactly one of the following:

(1) *Elliptic automorphisms* are those elements in $Aff(X, \mathfrak{F})$ such that their Π-image has no real eigenvalue. It turns out that in this case they must be conformal automorphisms of finite order on a Riemann surface in D_Φ. The Π-image of these automorphisms acts on D_Φ as an elliptic isometry with fixed point inside D_Φ.

(2) *Parabolic automorphisms* are those elements in $Aff(X, \mathfrak{F})$ whose Π projection have only one direction of eigenvectors. For surfaces of finite type it can be shown that for any such automorphism δ there is a positive integer N and a finite collection of disjoint closed curves γ_i and positive integers n_i such that f^N is homotopic to composition of n_i Dehn twists along γ_i. Moreover all the twists are either left twists or right twists. In particular parabolic automorphisms are always reducible. For surfaces of infinite type one can not necessarily find an iterate of a parabolic automorphism which is a composition of Dehn twists on disjoint curves (see section 3). However in the case of infinite type surfaces, still there is a maximal collection of possibly infinite, non-homotopic, disjoint closed curves invariant under the action of δ. δ induces a parabolic isometry of D_Φ (regarded as a copy of the hyperbolic plane). The direction fixed by $\Pi(\delta)$

corresponds to the only fixed point of the corresponding isometry on ∂D_ϕ. We postpone more detailed study of this case to section 3.

(3) *Hyperbolic automorphisms* are those elements in $Aff(X, \mathfrak{F})$ whose Π-images in $Aut(X, \mathfrak{F})$ have two distinct real eigenvalues. For any such automorphism Ψ the directions of these two linearly independent eigenvectors generate two foliations F^s_Ψ and F^u_Ψ which are invariant under the automorphism. There will be a real number $\lambda_\Psi > 1$ such that locally the action of Ψ on $(X, \mathfrak{F})$ can be described by the action of the linear map $(v_s, v_u) \longmapsto (\lambda_\Psi v_s, \lambda_\Psi^{-1} v_u)$, where v_s and v_u are any two vectors respectively in the directions of F^s_Ψ and F^u_Ψ. $\Pi(\Psi)$ induces a hyperbolic isometry of D_ϕ with two fixed points on ∂D_ϕ corresponding to F^s_Ψ and F^u_Ψ. The axis of $\Pi(\Psi)$ is the Teichmüller geodesic in D_ϕ joining these two fixed points on the boundary; it consists of all the flat surfaces $(X, \mathfrak{F})$ for which the two foliations $F_{(\theta_s, \mathfrak{F})}$ and $F_{(\theta_u, \mathfrak{F})}$ representing F^s_Ψ and F^u_Ψ are perpendicular. On this axis $d_T([X], [\Psi(X)])$ is minimized among all $X \in D_\phi$. For surfaces of finite type the hyperbolic automorphisms are pseudo-Anosov homeomorphisms and it is known that on their axis $d_T([X], [\Psi(X)])$ is minimized among all $X \in Teich(X)$. This fact is conjectured to be true for surfaces of infinite type too.

2. CYLINDERS AND MULTI-TWISTS

In this section we study the parabolic affine automorphisms. Throughout this section we assume $(X, \mathfrak{F})$ is a fixed flat surface corresponding to a pair (X, ϕ) of a Riemann surface X and an integrable holomorphic q.d. ϕ on X and $f \in Aff(X, \mathfrak{F})$ is an affine automorphism of X.

Definition 2.1. *A* ***ϕ-cylinder*** *(or $\mathfrak{F}$-cylinder) on X is a connected component of the union of all regular closed ϕ-geodesics parallel to a given one. Any such closed ϕ-geodesic is called a* ***generator*** *of the ϕ-cylinder. A ϕ-cylinder is said to be in the direction λ if any of its generators are in that direction.*

It is easy to show that any two generators for a ϕ-cylinder have the same length. Since by definition ϕ-cylinders are maximal. both of their boundary components contain at least one singularity.

Definition 2.2. *The* ***modulus*** *of a ϕ-cylinder $\mathcal{C}$ is the ratio of the Euclidean length of the shortest curve joining its two boundary components to the Euclidean length of any of its generators. We denote it by $Mod(\mathcal{C})$; it is equal to the reciprocal to the extremal length of the family of all curves lying in $\mathcal{C}$ and homotopic to the generator of $\mathcal{C}$.*

Definition 2.3. *We call a direction λ a* ***Jenkins-Strebel*** *direction for ϕ if almost every point of X lies on a closed ϕ-geodesic in direction λ. Any such direction defines a splitting of X into ϕ-cylinders in direction λ which intersect only on their boundaries. We call a Jenkins-Strebel direction finite if there are only finitely many such cylinders. Otherwise, it is called infinite.*

Definition 2.4. *A* ***saddle connection*** *in direction λ is a ϕ-geodesic segment that joins a singularity to a singularity (possibly the same one) and has no interior singularities.*

Proposition 2.5. *(a) Any ϕ-geodesic infinite in at least one direction, is dense in an open subset of X. (b) Suppose there is no saddle connection in direction λ. Then any ϕ-geodesic in direction λ is dense in X.*

Proof. See Lemma 1.7 and the proof of Theorem 1.8 in [9]. In [9] the proposition is proved only for surfaces of finite type but the proof only uses the finiteness of the φ-area of X. ☐

Proposition 2.6. *The invariant direction of a parabolic affine automorphism f of X is a Jenkins-Strebel direction.*

Proof. Let λ be the only invariant direction of $Der(f)$. If there is at least one ϕ-geodesic l in direction λ which is infinite in at least one direction, the restriction of $Der(f)$ on l is the identity. But by Proposition 2.5, l should be dense in an open subset of X. Hence $Der(f)$ is equal to the identity on an open subset of X. This contradicts the assumption that f is parabolic. Therefore any ϕ-geodesic in direction λ is either a saddle connection or a closed ϕ-geodesic. This implies that λ is a Jenkins-Strebel direction. ☐

Definition 2.7. *We call a homeomorphism f of X, a left (right)* ***multi-twist*** *if there is a collection of non-homotopic, disjoint closed curves γ_i and positive integers n_i such that f is homotopic to the composition of n_i left (right) Dehn twists around the γ_i. A homeomorphism is called a* ***virtual multi-twist*** *if for a positive integer N, f^N is a multi-twist. A parabolic automorphism f with invariant direction λ is a virtual multi-twist if there is a positive integer N such that for all the cylinders $\mathcal{C}$ in the direction λ, $f^N(\mathcal{C}) = \mathcal{C}$ and the restriction of f^N*

to the boundary of $\mathcal{C}$ *is the identity . The order of* f *is the smallest positive integer* N *satisfying the condition of the definition.*

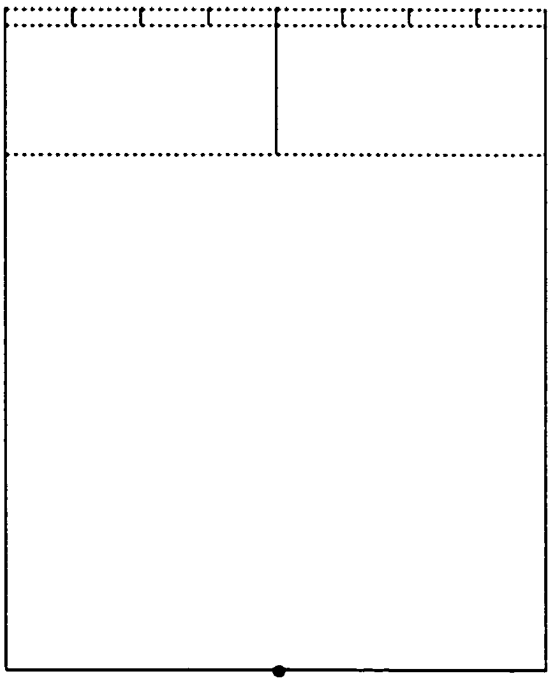


FIGURE 1. The first three rows of the flat surface X in the Example 3.8.

EXAMPLE 2.8. *(See figure 1) Here we describe an example of a parabolic automorphism which is not a virtual multi-twist. It is defined on a flat surface* X, *conformally equivalent to the complement of a Cantor set of bounded geometric type in the sphere (see* [8] *for the definition).* X *is constructed as follows. We start with the unit square and identify its vertical sides via a horizontal translation to get a cylinder* $\mathcal{C}_1$.

We also identify the right half of the lower side with its left half using the 180° *rotation through its midpoint. Then we add two rectangles of height* $\frac{1}{4}$ *and width* $\frac{1}{2}$ *on the top of the* $\mathcal{C}_1$ *and identify the two vertical sides of each of them using horizontal translations to get two cylinders* $\mathcal{C}^i_2, i = 1, 2$. *In the* n*th step we add* 2^n *rectangles of width* $l_n = 2^{-\frac{n(n+1)}{2}}$ *and height* $h_n = 2^{-n} l_n$ *to the top of each of the* 2^{n-1} *cylinders from the* $(n-1)$*st step. Then we identify their two vertical sides using horizontal translations to get the cylinders* $\mathcal{C}^i_n, 1 \leq i \leq 2^n$. *This process gives us a flat surface conformally equivalent to the complement of a Cantor set of bounded geometric type in the sphere. The linear map* $(x, y) \mapsto (x + \frac{1}{2}y, y)$ *induces a* $2^{-(n+1)}$ *turn twist in the cylinders of the* n*th row, which permutes the cylinders of the next row. It is easy to see that this map defines a parabolic automorphism of* X *which is not a virtual multi-twist.*

Proposition 2.9. *Let* $f \in \mathrm{Aff}(X, \mathcal{F})$ *be parabolic and let* λ *be its unique Jenkins-Strebel invariant direction. Then:*

(a) For any cylinder $\mathcal{C}$ *in direction* λ, *there is a positive integer* N *such that* $f^N(\mathcal{C}) = \mathcal{C}$ *and the restriction of* f^N *to the boundary of* $\mathcal{C}$ *is the identity.*

(b) If there is a positive integer M *such that there are at most* M *cylinders in direction* λ *with the same area, then* f *is virtual multi-twist of order at most equal to* M'*; where* M' *is the least common multiple of all the numbers not greater than* M.

Proof. (a) Given a cylinder $\mathcal{C}$ in direction λ, $f(\mathcal{C})$ is also a cylinder in the same direction and with the same area. Any two such cylinders are either the same or they have no intersection on their interiors. Since the total area of X is finite, there is an integer n_1 such that f^{n_1} maps $\mathcal{C}$ onto itself. Now on the boundary of $\mathcal{C}$ any saddle connection is mapped by f^{n_1} to another saddle connection with the same length. Since the total length of the two boundary components of $\mathcal{C}$ is finite, there is a positive integer n_2 which is a multiple of n_1, such that f^{n_2} fixes that given saddle connection and therefore the boundary component it belongs to. By taking another saddle connection and using the same argument one can show that for a multiple N of n_2, $f^N(\mathcal{C}) = \mathcal{C}$ and f^N fixes the boundary of $\mathcal{C}$ point-wise. This proves *(a)*.

(b) Let M and M' be as in the part *(b)*. Then for an arbitrary cylinder $\mathcal{C}$, since there are at most M cylinders with the same area, there must be an integer $n < M$ such that $f^n(\mathcal{C}) = \mathcal{C}$. Therefore $f^{M'}(\mathcal{C}) = \mathcal{C}$ for any arbitrary cylinder $\mathcal{C}$ in direction λ. Having stabilized all the cylinders, $f^{M'}$ must fix all the points on their common boundaries point-wise and therefore f is a virtual multi-twist of order at most M'. □

DEFINITION 2.10. *We call a parallelogram $\mathcal{P}$ a* ***sub-parallelogram of a cylinder*** $\mathcal{C}$ ***in direction*** λ *if it is a subset of* $\mathcal{C}$ *and two of its opposite sides are respectively subsets of two saddle connections on the two boundary components of* $\mathcal{C}$. *We call these two sides* ***bases***. *The modulus of such a* $\mathcal{P}$ *is defined to be the ratio of the Euclidean lengths of its height and its base and is denoted by* $Mod(\mathcal{P})$. *A sub-parallelogram is said to be* ***maximal*** *if at least one of its bases is a saddle connection.*

DEFINITION 2.11. *For a parabolic affine map* f *with invariant direction* λ, *we define* $\tau(f)$ *to be the cotangent of the angle between the vectors*

$f(v_1)$ and $f(v_2)$, for any counterclockwise perpendicular pair of vectors (v_1, v_2) with v_1 in direction λ and v_2 of length one.

In other words if l_1 and l_2 are two lines and if (v_1, v_2) are as in the definition 3.10 such that translation by vector v_2 of line l_1 is the line l_2, then $\tau(f)$ is the subtraction of the translation induced by f on line l_1 from the translation induced by f on line l_2 both measured in direction of vector v_1.

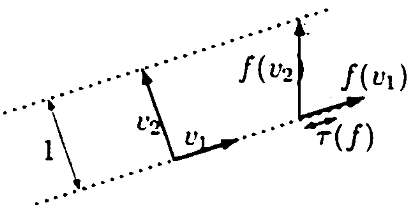


FIGURE 2. For a parabolic affine map f with invariant direction λ, $\tau(f)$ measures the difference between the displacements by f along two lines in direction λ which are distance 1 apart. The two dotted lines are in direction λ.

PROPOSITION 2.12. *For any parabolic f and arbitrary g in $PSL(2, \mathbb{R})$:*

(a) $\tau(f^n) = n\tau(f)$ for any $n \in \mathbb{Z}$.

(b) If f is a left (right) twist, $\tau(f)$ is negative (positive). Moreover $\tau(g \circ f \circ g^{-1})$ and $\tau(f)$ have the same signs.

Proof. part *(a)* is trivial. For part *(b)* one should notice that for two parabolic linear maps f_1 and f_2 conjugated by an orientation preserving linear map L and counter-clockwise pairs of vectors (v_1, v_2) for f_1 and $(L(v_1), L(v_2))$ for f_2 such that v_1 is an eigenvector of f_1 (and therefore $L(v_1)$ is an eigenvector of f_2) either they both increase the angle between the corresponding pair of vectors or they both decrease that angle. □

Proposition 2.13. *If p is a parabolic automorphism of X with invariant direction λ and if at least one cylinder $\mathcal{C}$ in direction λ contains a maximal sub-parallelogram $\mathcal{P}$ with modulus M and Euclidean area A, then $\tau(Der(p)) > \frac{A}{(A+1)M}$.*

Proof. Let N be the smallest positive integer for which $f^N(\mathcal{P})$ intersects itself. Since $\Pi(f)$ is not equal to the identity, the difference between the displacements of the two bases of $\mathcal{P}$ in direction λ measured in the universal covering flat surface corresponding to X is more than the width of $\mathcal{P}$. This implies $N\tau(f) = \tau(f^N) > \frac{1}{M}$. On the other hand, since the total area of the surface is equal to one and all the first $N-1$ f-images of $\mathcal{P}$ are disjoint, we have $(N-1)A < 1$. This implies that $\frac{1}{N} > \frac{A}{A+1}$ and therefore $\tau(\Pi(p)) > \frac{A}{(A+1)M}$. □

Corollary 2.14. *The subgroup of $Aut(X, \mathcal{F})$ consisting of all the parabolic elements with invariant direction λ, is either trivial or an infinite cyclic group.*

Proof. Otherwise there must be a sequence p_n of parabolic automorphisms of X, all fixing direction λ that converges to the identity. This would imply $\tau(Der(p_n)) \to 0$ as $n \to \infty$. Applying p_n to any maximal sub-parallelogram of a cylinder in direction λ with modulus M and area A would require $\frac{A}{(A+1)M}$ to be less than $\tau(Der(p_n))$ for any n. Therefore AM can not be a non zero finite number which is a contradiction. □

In corollary 2.19 we will give a necessary and sufficient condition for a Jenkins-Strebel direction to be an invariant direction of a virtual multi twist.

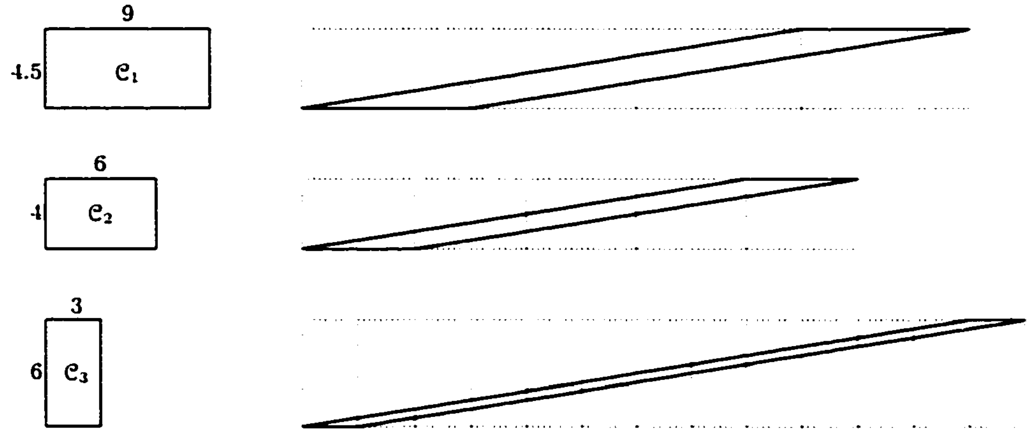


FIGURE 3. The rectangles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ on the left respectively have moduli $\frac{1}{2}$, $\frac{2}{3}$ and 2. The numbers written by each side show the lengths of those sides and the opposite vertical sides of each rectangle are assumed to be identified by horizontal translations. There is an affine map that does three twists on $\mathcal{C}_1$, four twists on $\mathcal{C}_2$ and 12 twists on $\mathcal{C}_3$ leaving their boundaries point-wise fixed. The parallelograms on the right show the action of this affine map on the corresponding cylinders.

DEFINITION 2.15. *We say a Jenkins-Strebel direction λ is* ***of bounded type*** *if fixing any λ-cylinder $\mathcal{C}$, there is a positive integer N such that for any λ-cylinder $\mathcal{C}'$, the ratio $\frac{Mod(\mathcal{C})'}{Mod(\mathcal{C})}$ is rational number $\frac{p}{q}$ with $q < N$.*

It is easy to see that being of bounded type does not depend on the choice of the cylinder $\mathcal{C}$.

PROPOSITION 2.16. *If a Jenkins-Strebel direction λ is of bounded type, there is an infinite cyclic subgroup of $Aut(X, \mathcal{F})$ consisting of all parabolic elements with invariant direction λ. Moreover any automorphism in that subgroup is a virtual multi-twist.*

Proof. First we label all of the countably many cylinders in direction λ. We denote the modulus of $\mathcal{C}_1$ by M. Then $Mod(\mathcal{C}_i) = \frac{p_i}{q_i}M$ where all q_is are less than a positive integer N. Now letting Q be the least common multiple of all q_i's, the affine map that induces Q twists on $\mathcal{C}_1$ and is equal to the identity on its boundary components will have the same linear part (derivative) as the one that induces $Q\frac{p_i}{q_i}$ twists on $\mathcal{C}_i$ and is equal to the identity on the boundary of $\mathcal{C}_i$. Since all these maps coincide on the common boundaries and they have the same derivative, they give rise to a globally defined affine autmorphism. Evidently λ is the only invariant direction of its linear part. This shows that the subgroup of all the parabolic automorphisms of X with invariant direction λ is not trivial and therefore by corollary 2.14 is an infinite cyclic group. Since at least one element in this group is a virtual multi-twist all the elements are virtual multi-twists. ☐

Proposition 2.17. *If there is a sequence of cylinders $\mathcal{C}_i$ in direction λ containing sub-parallelograms $\mathcal{P}_i$ such that $Mod(\mathcal{P}_i) \to 0$ as $i \to \infty$, then there is no virtual multi-twist parabolic automorphism with invariant direction λ.*

Proof. Suppose f is a parabolic automorphism with invariant direction λ which is a virtual multi-twist of order N. Then f^N stabilizes all the cylinders and $N\tau(Der(f)) = \tau(Der(f^N)) \geq \frac{1}{Mod(\mathcal{P}_i)}$ for any i. This implies that $\tau(Der(f))$ is larger than any positive number which is a contradiction. ☐

Proposition 2.18. *If there are at least two cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ in direction λ such that the ratio of their moduli is irrational, then there is no parabolic automorphism with invariant direction λ.*

Proof. By part *(a)*, of Proposition 2.9 if there is such a parabolic automorphism, there are positive integers N_1 and N_2 such that $f^{N_i}(\mathcal{C}_i) = \mathcal{C}_i$ and f^{N_i} fixes the boundary of $\mathcal{C}_i$ point-wise for $i = 1, 2$. Taking N to be their least common multiple, f^N stabilizes both cylinders and fixes their boundaries point-wise. This implies that there are integers P_1 and P_2 such that $N\tau(f^N) = \tau(f^N) = \frac{P_i}{Mod(\mathcal{C}_i)}$ for $i = 1, 2$. Therefore $\frac{P_1}{Mod(\mathcal{C}_1)} = \frac{P_2}{Mod(\mathcal{C}_2)}$ which is a contradiction. Hence there is no such automorphism. ☐

COROLLARY 2.19. *λ is the invariant direction of a virtual multi-twist if and only if λ is a Jenkins-Strebel direction of bounded type.*

Proof. By the Proposition 2.16, if λ is a Jenkins-Strebel direction of bounded type, it is the invariant direction of a virtual multi-twist. Conversely assume f is a virtual multi-twist with the invariant direction λ. There is a positive integer N such that f^N stabilizes all the cylinders in the direction λ and therefore fixes their boundaries point-wise. Fixing a λ-cylinder $\mathcal{C}$, for any arbitrary λ-cylinder $\mathcal{C}'$ by the Proposition 2.18, $\frac{Mod(\mathcal{C}')}{Mod(\mathcal{C})}$ is rational. f^N has to induce m full twists on $\mathcal{C}$ and m' full twists on $\mathcal{C}'$. Therefore $N\tau(f) = \tau(f^N) = \frac{m}{Mod(\mathcal{C})} = \frac{m'}{Mod(\mathcal{C}')}$. Hence $\frac{Mod(\mathcal{C}')}{Mod(\mathcal{C})} = \frac{m'}{m}$. This shows that λ is a Jenkins-Strebel direction of bounded type. ☐

PROPOSITION 2.20. *Suppose λ is a Jenkins-Strebel direction of bounded type for a flat surface X and a geodesic segment l of finite length in direction κ intersects infinitely many cylinders in direction λ without intersecting any singularities. Then κ is not a Jenkins-Strebel direction of bounded type for X. In particular, it is not the invariant direction of any virtual multi-twist of X.*

Proof. Suppose l is inside the closure of a κ-cylinder $\mathcal{C}$. There are infinitely many λ-cylinders intersecting l and therefore intersecting $\mathcal{C}$. Labeling them, we denote these cylinders by $\mathcal{C}_i$ and the Euclidean distance between their two boundary components by d_i. Since the length of the generator of $\mathcal{C}$ is finite, we have $\sum_{i=1}^{\infty} d_i < \infty$ and hence $d_i \to 0$ as $i \to \infty$. The portions of $\mathcal{C}_i$ that pass through $\mathcal{C}$, are sub-parallelograms all with lengths more than the height of $\mathcal{C}$ and height d_i which goes to zero. Therefore their moduli converge to zero. Thus λ cannot be a Jenkins-Strebel direction of bounded type. □

3. CONSTRUCTION OF X_α AND D_α

We shall construct a one parameter family of Riemann surfaces X_α equipped with a holomorphic quadratic differential Φ_α for any α in $(0,1)$. We start with a square $\mathcal{S} = ABA'C$, centered at the origin in the complex plane such that its sides have length one and the diagonal BC is on the real line. Set $B_0 = B$ and $C_0 = C$. For $i \geq 1$ define B_i (respectively B_{-i}, C_i and C_{-i}) to be the point on the interval BA (respectively BA', CA and CA') such that the length of AB_i (respectively $A'B_{-i}$, AC_i and $A'C_{-i}$) is α^i. We identify B_iB_{i+1} with $C_{-(i+1)}C_{-i}$ for any $i \in \mathbb{Z}$ using the translation $z \mapsto z + (C_{-(i+1)} - B_i)$. Through these identifications all points B_{2k+1} and C_{2k} for $k \in \mathbb{Z}$ are identified; their equivalence class is denoted by P. Similarly all points B_{2k} and C_{2k+1} for $k \in \mathbb{Z}$ are identified ; their equivalence class is denoted by Q. Excluding these points together with A and A' from $\mathcal{S}$ we denote the resulting subset of $\mathcal{S}$ by $\mathcal{P}_\alpha$, the space obtained after applying the identifications by X_α and the projection map from $\mathcal{P}_\alpha$ to X_α by p. For any point x inside $\mathcal{P}_\alpha$, any disk inside $\mathcal{S}$ and containing

x projects to a disk neighborhood of x in X_α. The flat structure of the plane induces a flat structure on that neighborhood in X_α. For any point x on the boundary of $\mathcal{S}$, which is not one of the removed points, there is a neighborhood U of $\hat{x} = p(x)$ such that $p^{-1}(U)$ is the union of two semi-disks in $\mathcal{P}_\alpha$. Identification along the corresponding boundary intervals makes U isometric to a disk and again the flat structure of the plane on the two semi-disks induces a flat structure on U. Hence X_α is equipped with a flat structure $\mathcal{F}_\alpha$. This flat structure does not have any singularities and its completion contains only one essential singularity of index two; its origin corresponds to all the points in classes P and Q and also to the two points A and A'.

Since all the identifications are in the form $z \longmapsto z + b$ for some $b \in \mathbb{C}$, dz^2 induces a quadratic differential Φ_α on X_α which is holomorphic with respect to the conformal structure on X_α induced by the flat structure $\mathcal{F}_\alpha$. Φ_α is also integrable with $\|\Phi_\alpha\| = 1$ (area of the square $\mathcal{S}$). We define $\mathcal{D}_\alpha$ to be the Teichmüller disk determined by Φ_α. The relation between holomorphic integrable quadratic differentials and flat structures with finite area gives an explicit description of $\mathcal{D}_\alpha$. Starting with any parallelogram $\Pi = ABA'C$ centered at the origin with area one, such that BC is on the real line and following the construction of X_α with a given α, one obtains a Riemann surface $X_{\Pi,\alpha}$ and a holomorphic quadratic differential $\phi_{\Pi,\alpha}$ with its induced flat structure. Since for any three points E,F and G on a line, an affine map preserves the ratio $\frac{EF}{EG}$, the element in $SL(2, \mathbb{R})$ that maps the square $\mathcal{S}$ in the construction of X_α onto Π, respects the identifications. It therefore gives rise to an affine homeomorphism from X_α onto $X_{\Pi,\alpha}$. On the other hand for any element f in $SL(2,\mathbb{R})$, $f(\mathcal{S}) = \Pi$ generates a surface $X_{\Pi,\alpha}$. Since rotating Π does not change the flat structure of $X_{\Pi,\alpha}$ composing f with a rotation one can always assume that the diagonal BC of Π is on the real axis. This shows that $\mathcal{D}_\alpha = \{X_{\Pi,\alpha}\}$ where Π is a parallelogram centered at origin with a diagonal BC on the real axis and area one.

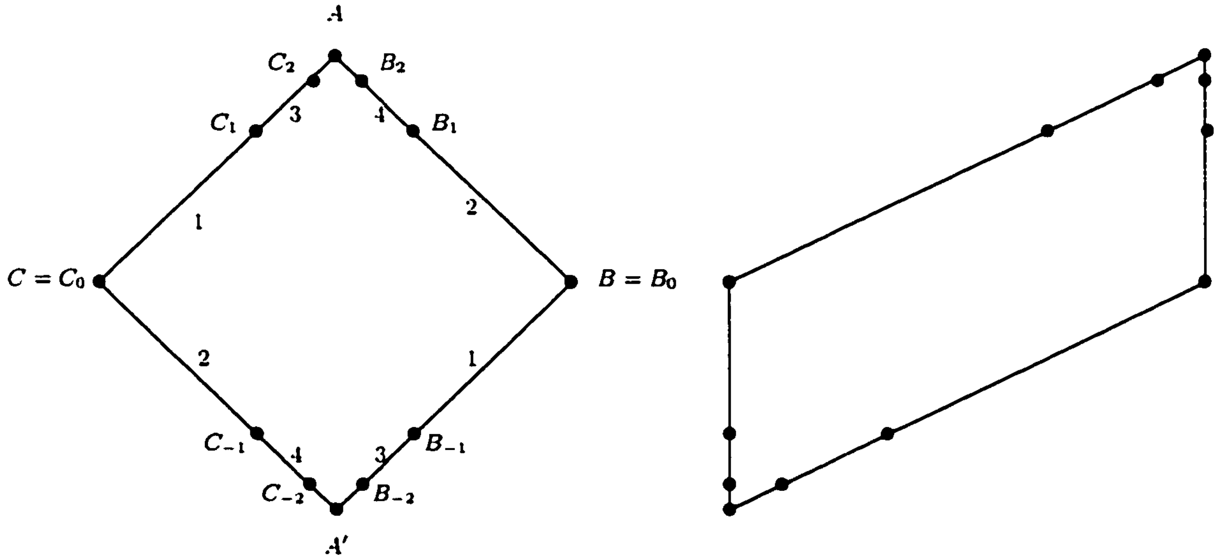


FIGURE 4. Left: $\mathcal{S}$ and some of the identifications that lead to the construction of X_α for $\alpha = \frac{1}{3}$. The segments with the same lables are identified. Right: the parallelogram Π that leads to the construction of $X_{\Pi,\alpha} \in \mathcal{D}_\alpha$ for $\alpha = \frac{1}{3}$.

X_α has one orientation preserving affine automorphism of order two, induced by $z \longmapsto -z$. We denote this special automorphism by ι. There are also two orientation reversing affine automorphisms ι_1 and ι_2 of order two, induced respectively by reflection through the line joining A to A' (the imaginary axis) and the line joining C to B (the real axis). Clearly $\iota_1 \circ \iota_2 = \iota_2 \circ \iota_1 = \iota$.

We can show the following topological properties of the underlying surface of X_α:

Proposition 3.1. *The underlying surface of X_α is a surface of infinite genus with one end.*

Proof. We denote the sub-surface of X_α obtained by cutting along C_nB_n and $C_{-n}B_{-n}$ and discarding the parts above C_nB_n and below $C_{-n}B_{-n}$ by $\hat{X}^n_\alpha$. Since C_nB_n and $C_{-n}B_{-n}$ are parallel we can identify them by a vertical translation and since the new identification is a translation there will be a flat structure $\mathcal{F}^n_\alpha$ on the quotient space X^n_α. Also dz^2 induces a non-zero quadratic differential Φ^n_α, holomorphic with respect to the corresponding conformal structure. The conformal structure on X^n_α and Φ^n_α can be extended to the two points on the boundary of X^n_α corresponding to the two equivalence classes of points B_i and C_i for $-n \leq i \leq n$. These two points are zeros of order $n+1$ on the extended quadratic differential. We denote the extended surface and extended quadratic differential respectively by X^n_α and Φ^n_α as well. Therefore X^n_α is a compact Riemann surface of genus n and $\hat{X}^n_\alpha$ which is obtained from that surface by cutting along one closed interval, is a sub-surface of X_α. Hence X_α has sub-surfaces with arbitrarily large genus. This shows that X_α is a surface of infinite genus.

Now we introduce the simple closed curves γ^n_α and $\bar{\gamma}^n_\alpha$ for any $n \geq 0$. We shall use the careful geometric construction of these curves in the next proposition. Since they have the topological properties required to resume the proof of Proposition 3.1, we introduce them at this point.(see figure 5)

We denote the point Z on the line segment $[X, Y]$ such that $\frac{XZ}{XY} = (\frac{1}{2})^a$, by $\mathcal{C}_a([X, Y])$. First we define the following arcs: ζ^n_α for $n > 0$ is the horizontal line segment that joins the midpoint of $[B_{n-1}, B_n]$ to the midpoint of $[C_{n-1}, C_n]$ and $\bar{\zeta}^n_\alpha$ for $n > 0$ is the horizontal line segment joining $\mathcal{C}_2([B_{n-1}, B_n])$ to $\mathcal{C}_2([C_{n-1}, C_n])$. For $n < 0$ we define:

$\zeta^n_\alpha = \iota_2(\zeta^{-n}_\alpha)$

$\bar{\zeta}^n_\alpha = \iota_2(\bar{\zeta}^{-n}_\alpha)$

ξ^n_α is the vertical line segment joining $\mathcal{C}_n([B, B_1])$ to $\mathcal{C}_n([B, B_{-1}])$ and we denote the length of ξ^1_α by δ. For $0 < k < n$, $\xi^n_{k,\alpha}$ is defined to be the arc obtained by the horizontal line segment of length $\alpha^n\delta$ starting from $\mathcal{C}_{n-k}([B_{k-1}, B_k])$ going to the left followed by the vertical line segment of the same length going upwards and ending at $\mathcal{C}_{n-k}([B_k, B_{k+1}])$.We define:

$\eta^n_\alpha = \iota_1(\xi^n_\alpha)$

$\xi^n_{k,\alpha} = \iota_2(\xi^n_{-k,\alpha})$ for $-n < k < 0$

$\eta^n_{k,\alpha} = \iota_1(\xi^n_{k,\alpha})$ for $0 < k < n$

$\eta^n_{k,\alpha} = \iota_2(\eta^n_{-k,\alpha})$ for $-n < k < 0$

Now for $n > 0$ we define:

$\gamma^n_\alpha = p\,[\zeta^n_\alpha \cup \zeta^{-n}_\alpha \cup \xi^n_\alpha \cup \eta^n_\alpha \cup \bigcup_{k=1}^{n-1}(\xi^n_{k,\alpha} \cup \xi^n_{-k,\alpha} \cup \eta^n_{k,\alpha} \cup \eta^n_{-k,\alpha})]$

and

$\bar{\gamma}^n_\alpha = p\,[\bar{\zeta}^n_\alpha \cup \bar{\zeta}^{-n}_\alpha \cup \xi^{n+1}_\alpha \cup \eta^{n+1}_\alpha \cup \bigcup_{k=1}^{n-1}(\xi^{n+1}_{k,\alpha} \cup \xi^{n+1}_{-k,\alpha} \cup \eta^{n+1}_{k,\alpha} \cup \eta^{n+1}_{-k,\alpha})]$

These are closed curves on X_α. The order of arcs on the curves, however is different from the one in the above unions.

Now we finish the proof of Proposition 3.1. γ^n_α divides X_α into two connected components, a compact component K^n_α and a non-compact component U^n_α. Any compact subset of X_α avoids a neighborhood of all the omitted points B_i, C_i, A and A'. Thus it must be inside one K^n_α for sufficiently large n. This shows that the complement of any compact subset of X_α has only one non compact component and therefore X_α has only one end. ☐

Now we use the curves γ_α^n and $\bar{\gamma}_\alpha^n$ defined in the proof of Proposition 3.1 to study the analytic properties of X_α:

PROPOSITION 3.2. *X_α is a Riemann surface of the first kind.*

REMARK: The conformal structure on X_α determines a unique hyperbolic structure. The corresponding holonomy group can be realized as a Fuchsian group acting discontinuously on the hyperbolic disk. The meaning of proposition 3.2 is that the limit set of that group is the whole unit circle.

Proof. We start by constructing a sequence of annuli A_α^n, $n > 0$ exiting any arbitrary compact subset of X_α, such that the sum of the reciprocals of their moduli is divergent. Let A_α^n be the annulus between γ_α^n and $\bar{\gamma}_\alpha^n$. It is easy to see that the two curves are homotopic and therefore the region between them forms an annulus inside X_α. Now we use the Euclidean metric defined by the $\mathcal{F}$-structure on X_α to find a lower bound for the extremal length of the family Λ_n of arcs joining the two boundary components of A_α^n. A_α^n can be regarded as a union of the following pieces:

- $2(n-1)$ V-shaped regions, surrounded by $\xi_{\pm k,\alpha}^n$ and $\xi_{\pm(k+1),\alpha}^n$ and sides AB or $A'B$. We denote the Euclidean area of each of these by a_n.

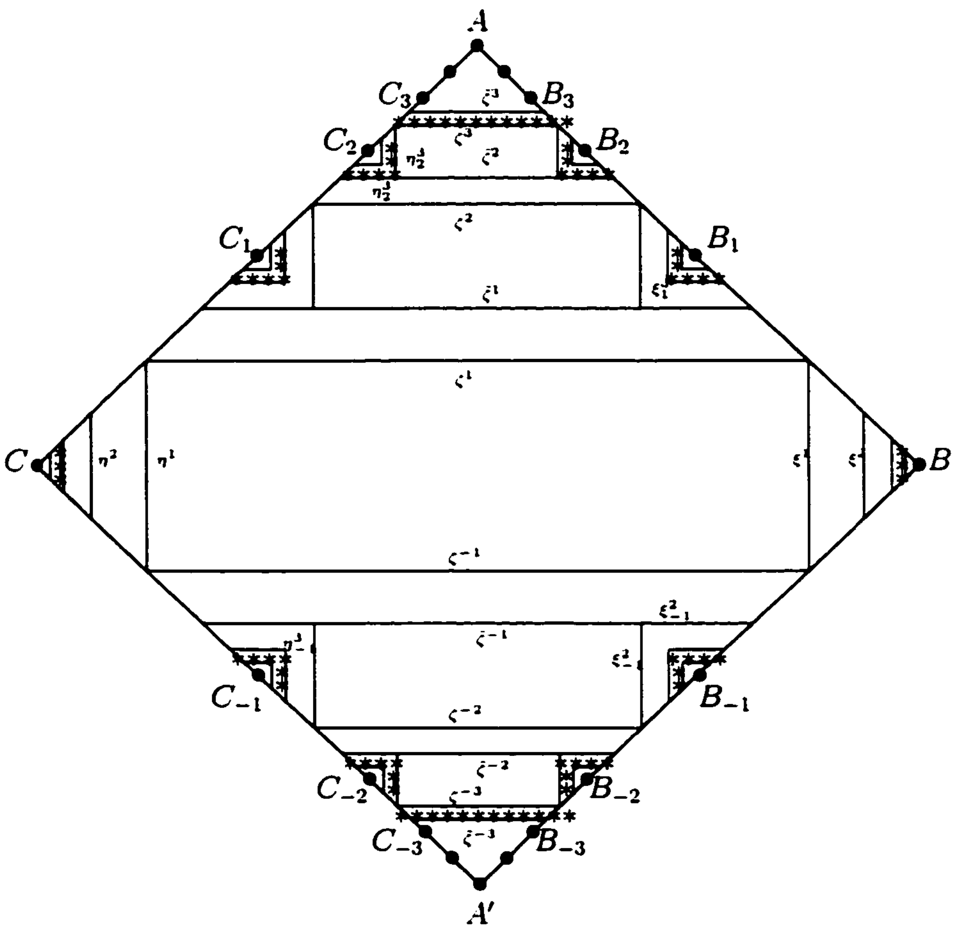


FIGURE 5. Some of the curves and annuli introduced in the proofs of Propositions 4.1 and 4.2 for $\alpha = \frac{1}{2}$. The shaded region shows areas contained in A_2. In all the labels the subscripts α have been omitted.

- $2(n-1)$ V-shaped regions surrounded by $\eta^n_{\pm k,\alpha}$ and $\eta^n_{\pm(k+1),\alpha}$ and sides AC or $A'C$. Each of these also has Euclidean area equal to a_n

- Two trapezoidal regions surrounded by $\zeta_{\pm n}$ and $\bar{\zeta}_{\pm n}$ and sides of $\mathcal{S}$. We denote the Euclidean area of each of them by b_n.
- Two trapezoidal regions respectively surrounded by ξ^n_α and ξ^{n+1}_α and sides AB and $A'B$ and by η^n_α and η^{n+1}_α and sides AC and $A'C$. We denote the Euclidean area of each of them by c_n.

Since these regions cover A^n_α and they intersect only on their boundaries we have: $\mathcal{A}_n = 4(n-1)a_n + 2b_n + 2c_n$ where $\mathcal{A}_n$ is the Euclidean area of A^n_α.

We also have: $a_n = \alpha^{n-1}a_1$, $b_n = \alpha^{n-1}b_1$ and $c_n = \alpha^{n-1}c_1$. Hence

$$\mathcal{A}_n = 4(n-1)\alpha^{n-1}a_1 + 2\alpha^{n-1}b_1 + 2\alpha^{n-1}c_1$$

On the other hand if l_n is the minimum Euclidean length of Λ_n . we have $l_n = \alpha^{n-1}l_1$. Therefore

$$\text{Extremal length of } \Lambda_n \geq$$

$$\geq \frac{\alpha^{n-1}l_1}{4(n-1)\alpha^{n-1}a_1 + 2\alpha^{n-1}b_1 + 2\alpha^{n-1}c_1} =$$

$$= \frac{l_1}{4(n-1)a_1 + 2(b_1 + c_1)}$$

This shows that the sum of the reciprocals of moduli of the A^n_α's is divergent. Now we consider the family Λ_K of all curves that start from a compact subset K of X_α and leave any other compact subset. Any such curve goes through A^n_α for all large n. Taking the test metric which is equal to zero outside the union of all the A^n_α's and is equal to d_n on A^n_α, where d_n is Euclidean metric dz^2 re-scaled so that the area of A^n_α with respect to d_n is equal to the d_n-length of the shortest curve joining the two boundary components of A^n_α, one can see that the extremal length of Λ_K is infinite for any K.

Now consider the hyperbolic uniformization of X_α. If the corresponding covering group is not of the first kind there is an interval I of discontinuity on the unit circle. We let the closed interval J be a proper sub-interval of I such that $g(J) \cap J = \emptyset$ for any g not equal to the identity in the covering group . Now if U is the half space bounded by J and the hyperbolic geodesic L joining the two endpoints of J, then $g(U) \cap U = \emptyset$ for any g not equal to the identity in the covering group. Let K' be any hyperbolic geodesic segment on L which is small enough so that no two different points on it are equivalent by the action of the covering group. Also let Γ' be the family of arcs joining a point on K' to a point on J. The projection map from the hyperbolic disk to X_α maps K' homeomorphically onto a compact subset K of X_α and any curve in Γ' onto a curve in Γ_K. Therefore the extremal length of the family Γ' is larger than or equal to the extremal length of the family Γ_K which is infinite. This shows the extremal length of Γ' is infinite which is not possible because Γ' is conformally equivalent to the family of curves joining two opposite sides of a rectangle. This shows that the existence of an interval of discontinuity leads to a contradiction. Hence X_α is a surface of the first kind. ☐

We denote the affine automorphism group of X_α by Γ_α. In the next section we will take the first step in understanding this group by introducing a countable set of parabolic elements.

4. JENKINS-STREBEL DIRECTIONS ON X_α AND CORRESPONDING MULTI-TWISTS

In this section we introduce a countable family of Jenkins-Strebel directions. Each of them generates infinitely many cylinders whose stabilizers form a cyclic parabolic subgroup of Γ_α.

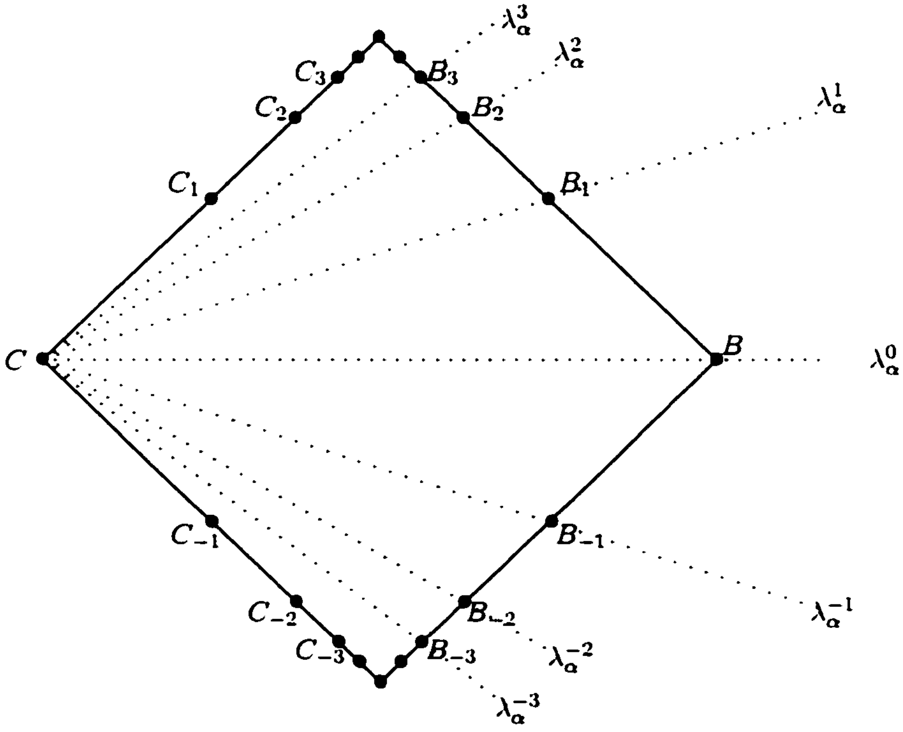


FIGURE 6. Directions λ_α^i.

For an integer n, let λ_α^n be the direction of the line joining C to B_n on X_α. Let λ_α^∞ and $\lambda_\alpha^{-\infty}$ be the directions of the lines joining C respectively to A and A' on X_α. We denote the foliation in direction λ_α^n for $n \in \mathbb{Z} \cup \{\pm\infty\}$ by F_α^n.

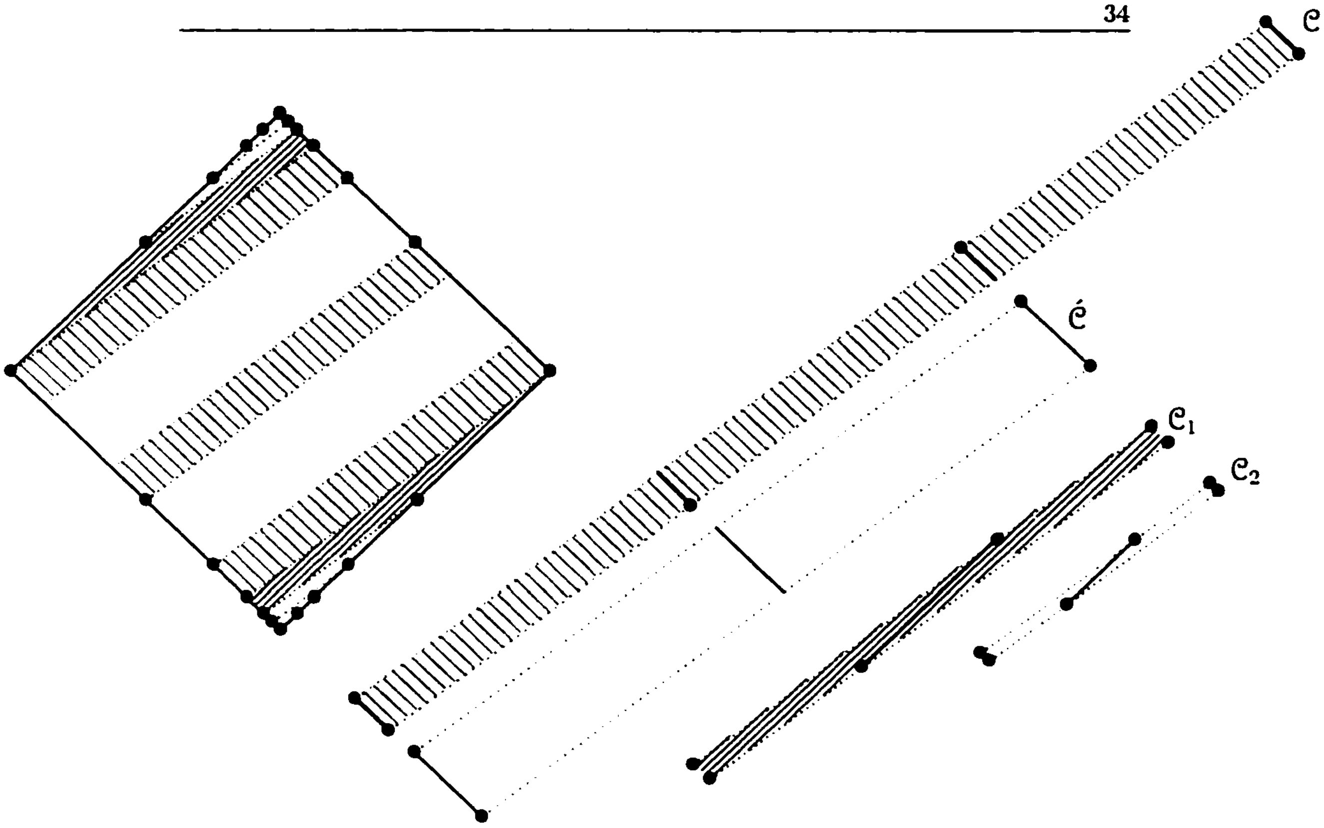


FIGURE 7. Cylinders for $\alpha = \frac{1}{2}$ and direction λ_α^3. There are two cylinders $\mathcal{C}$ and $\mathcal{C}'$ in the middle and infinitely many cylinders $\mathcal{C}_i$ with modulus equal to M. The modulus of $\mathcal{C}$ is M and the modulus of $\mathcal{C}'$ is $3M$. $\mathbf{D}_\alpha^3$ acts by one twist around the generators of $\mathcal{C}_i$ and $\mathcal{C}$ and three twists around the generator of $\mathcal{C}'$.

PROPOSITION 4.1. *For a rational number* $0 < \alpha = \frac{p}{q} < 1$ *and any* $n \in \mathbb{Z}$ *there is a cyclic subgroup of* $Aff(X_\alpha, \mathcal{F}_\alpha)$, *consisting of all the parabolic automorphisms in* $Aff(X_\alpha, \mathcal{F}_\alpha)$ *with invariant direction* λ_α^n.

Proof. The two segments CB_n and $C_{-n}B$ (see figure 7) divide $\mathcal{S}$ into three pieces: the middle parallelogram, and two other pieces. The latter ones are formed of infinitely many cylinders $\mathcal{C}_i$, each made up of two trapezoids $C_{i-1}B_{n+i-1}B_{n+i}C_i$ and $B_{1-i}C_{1-n-i}C_{-n-i}B_{-i}$ when $n \geq 0$ and $B_{i-1}C_{n+i-1}C_{n+i}B_i$ and $C_{1-i}B_{1-n-i}B_{-n-i}C_{-i}$ otherwise. Since all the trapezoids are similar, the $\mathcal{C}_i$s are also similar and all have modulus equal to the same number M. We denote the height of $\mathcal{C}_1$ by H and the length of its generator by L. The middle parallelogram piece can be decomposed into finitely many cylinders with height equal to an integral multiple of $\frac{H}{q}$ and the length of their generator equal to an integral multiple of $\frac{L}{1+\alpha}$. This shows that there are finitely many more cylinders whose moduli are rational multiples of M. Apparently λ_α^n is of bounded type and therefore by Proposition 2.16 there is an infinite cyclic subgroup of $Aut(X, \mathfrak{F})$ consisting of parabolic elements with invariant direction λ_α^n. □

Let $\mathbf{D}_\alpha^n$ be the one of the two generators of the cyclic subgroup of all the parabolic automorphisms in $Aff(X_\alpha, \mathfrak{F}_\alpha)$ with invariant direction λ_α^n that satisfies $\tau(\mathbf{D}_\alpha^n) > 0$. We denote $\Pi(\mathbf{D}_\alpha^n)$ in $Aut(X_\alpha, \mathfrak{F}_\alpha)$ by f_α^n. For future reference we will give a precise description of the automorphisms $\mathbf{D}_\alpha^0$ and $\mathbf{D}_\alpha^1$.

Proposition 4.2. *(a) For any $\alpha \in (0, 1)$, $\mathbf{D}_\alpha^0$ acts by one right twist along all the generators of the cylinders in the direction λ_α^0.*

(b) For $\alpha = \frac{p}{q}$, with p and q positive integers with no common factor and $p < q$, $\mathbf{D}_\alpha^1$ acts by p right twists along all but one of the generators of the cylinders in the direction λ_α^1 and $p+q$ right twists along that one cylinder.

Proof. (See figure 8) *(a)* In the direction λ_α^0, the saddle connections C_iB_i, $i \in \mathbb{Z}$ divide $\mathcal{S}$ into pairs of isometric trapezoids, each pair forming a cylinder such that all the cylinders are similar to each other and therefore have the same modulus. This shows that the affine maps that

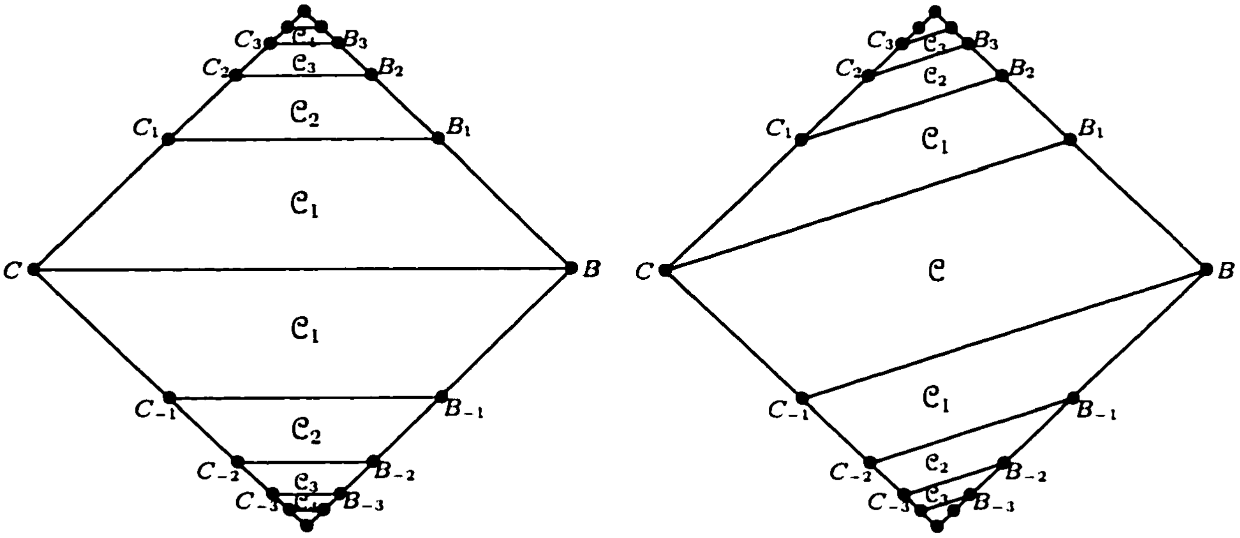


FIGURE 8. For $\alpha = \frac{1}{2}$ Left: the cylinders $\mathcal{C}_i$ in the direction λ_α^0 formed by pairs of trapezoidal regions. all have the same modulus. Right: In the direction λ_α^1, the cylinders $\mathcal{C}_i$ are formed by pairs of trapezoidal regions, all with the same modulus and a single cylinder $\mathcal{C}$ formed by the middle parallelogram with modulus three times the modulus of any $\mathcal{C}_i$.

induce one twist along each of these cylinders fixing their boundaries point-wise have the same derivative and they can be pieced together to form a global affine automorphism. Since any two of these cylinders have different areas, by part *(b)* of Proposition 2.9, this automorphism must be the generator $\mathbf{D}_\alpha^0$ of the maximal subgroup of all parabolic automorphisms in $Aff(X_\alpha, \mathcal{F}_\alpha)$ fixing the direction λ_α^0.

*(b)*As in the proof of Proposition 4.1, there is a sequence of cylinders $\mathcal{C}_i$, $i \geq 1$ in the direction λ_α^1. All of them have modulus equal to a number M. In the middle parallelogram $CC_{-1}BB_1$ there is only one

cylinder, $\mathcal{C}$ with height equal to $\frac{1}{\alpha} = \frac{q}{p}$ times the height of $\mathcal{C}_1$ and the length of the generator equal to $\frac{1}{1+\alpha} = \frac{q}{p+q}$ times the length of the generator of $\mathcal{C}_1$. Therefore $Mod(\mathcal{C}) = \frac{(p+q)}{p}M$. This shows that there is a parabolic automorphism acting by p twists on the generators of all the $\mathcal{C}_i$s followed by $p+q$ twists on the generator of $\mathcal{C}$. Again since any two of these cylinders have different areas, by part *(b)* of Proposition 2.9 , this automorphism must be the generator $\mathbf{D}_\alpha^1$ of the maximal subgroup of all parabolic automorphisms in $Aff(X_\alpha, \mathcal{F}_\alpha)$ fixing the direction λ_α^1. □

Definition 4.3. *A subgroup of $PSL(2,\mathbb{R})$ is called* ***elementary*** *if its limit set contains at most two points on the unit circle. Elementary subgroups of $PSL(2,\mathbb{R})$ are precisely those containing an Abelian normal subgroup of finite index.*

Corollary 4.4. *For any rational number $\alpha \in (0,1)$, $Aut(X_\alpha, \mathcal{F}_\alpha)$ is not an elementary subgroup of $PSL(2,\mathbb{R})$.*

Proof. For any given rational number $\alpha \in (0,1)$, the Π images of $\mathbf{D}_\alpha^0$ and $\mathbf{D}_\alpha^1$ in $Aut(X_\alpha, \mathcal{F}_\alpha)$ are two parabolic elements f_α^0 and f_α^1 in $PSL(2,\mathbb{R})$ with different fixed points z_α^0 and z_α^1 on the boundary of the hyperbolic plane. Then $f_\alpha^1 \circ f_\alpha^0 \circ f_\alpha^{1^{-1}}$ has $f_\alpha^1(z_0)$ as its fixed point which is different from the other two fixed points. This means that $Aut(X_\alpha, \mathcal{F}_\alpha)$ has at least three distinct limit points and therefore can not be an elementary subgroup of $PSL(2,\mathbb{R})$. □

Proposition 4.5. *For any direction λ that makes an angle more than $\frac{\pi}{4}$ with the horizontal direction, there is no virtual multi-twist parabolic automorphism of X_α with invariant direction λ.*

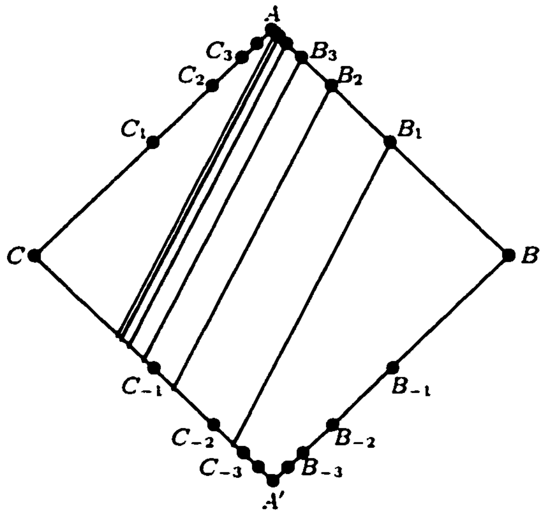


FIGURE 9. For directions that make an angle more than $\frac{\pi}{4}$ with the horizontal direction, even if they are Jenkins-Strebel directions, the existence of sub-parallelograms whose moduli approach zero does not let those directions be the invariant directions of any parabolic automorphism of X_α. Here α is taken to be $\frac{1}{2}$.

Proof. For any such direction, there is a geodesic segment in that direction that intersects infinitely many horizontal cylinders. Therefore by Proposition 2.20 no such direction can be the invariant direction of a virtual multi-twist of X_α. (see figure 9) ☐

In all the cases where the Teichmüller space of X is finite dimensional, the mapping class group acts discontinuously. The restriction of the stabilizer of a Teichmüller disk to that disk will also act discontinuously and therefore $Aut(X, \mathfrak{F})$ will be a Fuchsian group (See [12]). Here we use Proposition 4.5 to prove that $Aut(X_\alpha, \mathfrak{F}_\alpha)$ is a Fuchsian group and also to show that its limit set is not the whole circle at infinity.

COROLLARY 4.6. *$Aut(X_\alpha, \mathfrak{F}_\alpha)$ is a Fuchsian group of the second kind.*

Proof. Suppose F_θ is the measured foliation obtained from parallel Euclidean lines in the direction θ on X_α and I is the interval on the boundary of $\mathfrak{D}_\alpha$ corresponding to those measured foliations F_θ with direction θ making an angle more than $\frac{\pi}{4}$ with the horizontal direction. Then Proposition 4.5 states that the image under Π of no virtual multi-twist parabolic element in $Aff(X_\alpha, \mathfrak{F}_\alpha)$ has a fixed point in I. If the intersection of I with the limit set of $Aut(X_\alpha, \mathfrak{F}_\alpha)$ is non-empty, there must be an element $f \in Aut(X_\alpha, \mathfrak{F}_\alpha)$ with an attracting fixed point in I. Now let f_α^0 be the Π image of D_α^0 and z_α^0 be its only fixed point. Then for a large integer N, $f^N(z_\alpha^0)$ is in I and therefore the parabolic element $f^N \circ f_\alpha^0 \circ f^{-N}$ which is the image under Π of the virtual multi-twist parabolic automorphism has a fixed point in I which is a contradiction. Now since the limit set of $Aut(X_\alpha, \mathfrak{F}_\alpha)$ does not intersect I, $Aut(X_\alpha, \mathfrak{F}_\alpha)$ must act discontinuously on a neighborhood of I in $\mathfrak{D}_\alpha$. Invoking the fact that a subgroup of $PSL(2, \mathbb{R})$ acts discontinuously on the hyperbolic plane if and only if it acts discontinuously on an open subset of the hyperbolic plane, this completes the proof of the assertion that $Aut(X_\alpha, \mathfrak{F}_\alpha)$ is a Fuchsian group of the second kind. ☐

5. DESCRIPTION OF Γ_α AND Σ_α

Now that we have shown $\Gamma_\alpha = Aut(X_\alpha, \mathfrak{F}_\alpha)$ is a Fuchsian group of the second kind, we plan to describe its structure and also to give a geometric description of the surface Σ_α, obtained by forming the quotient of the hyperbolic plane $\mathfrak{D}_\alpha$ by the action of Γ_α. Σ_α is naturally equipped with a hyperbolic metric and since the group Γ_α is of the second kind its hyperbolic area is infinite.

The geometry of Σ_α varies significantly as α varies. One can observe that there are three different cases:

Case I: $\alpha = \frac{1}{n}$ for a positive integer n. The main characteristic of this case is the existence of a pseudo-Anosov automorphism $\mathbf{P}_n$ with fixed points corresponding to the foliations F_α^∞ and $F_\alpha^{-\infty}$ that are obtained from Euclidean lines respectively parallel to the sides AC and AB of $\mathcal{S}$ in X_α. If we stretch $\mathcal{S}$ in the direction λ_α^∞ by a factor of n and squeeze it in the direction $\lambda_\alpha^{-\infty}$ by a factor of $\frac{1}{n}$ and then cut and paste the pieces of the resulting rectangle we obtain another square similar to $\mathcal{S}$ (see figure 10). $h = \Pi(\mathbf{P}_n)$ is a hyperbolic element in $PSL(2,\mathbb{R})$ with two fixed points, Z_a the attracting fixed point corresponding to the measured foliation F_α^∞ and Z_r the repelling fixed point corresponding to the measured foliation $F_\alpha^{-\infty}$. Z_a and Z_r divide the boundary of $\mathcal{D}_\alpha$ into two open intervals. One, the interval I, is the interval in the proof of Corollary 4.6 which does not intersect the limit set of Γ_α. The axis of h in $\mathcal{D}_\alpha$ consists of all the surfaces X in $\mathcal{D}_\alpha$ where the two foliations F_α^∞ and $F_\alpha^{-\infty}$ are perpendicular to each other. Evidently one can obtain a representative of each of them, using an arbitrary rectangle of area one (instead of $\mathcal{S}$), finding the points and making identifications in a procedure similar to the one in the construction of X_α.

One interesting feature of $\mathbf{P}_n$ is the way it acts on the directions λ_α^n. This feature actually leads to one important relation between $\mathbf{P}_n$ and the $\mathbf{D}_\alpha^i$s.

Proposition 5.1. *For* $\alpha = \frac{1}{n}$, $\mathbf{P}_n \circ \mathbf{D}_\alpha^i \circ \mathbf{P}_n^{-1} = \mathbf{D}_\alpha^{i+2}$.

Proof. (See figure 10) One can easily observe that $\mathbf{P}_n$ shifts down the singular points on the two left boundary sides of $\mathcal{S}$ by one and also shifts

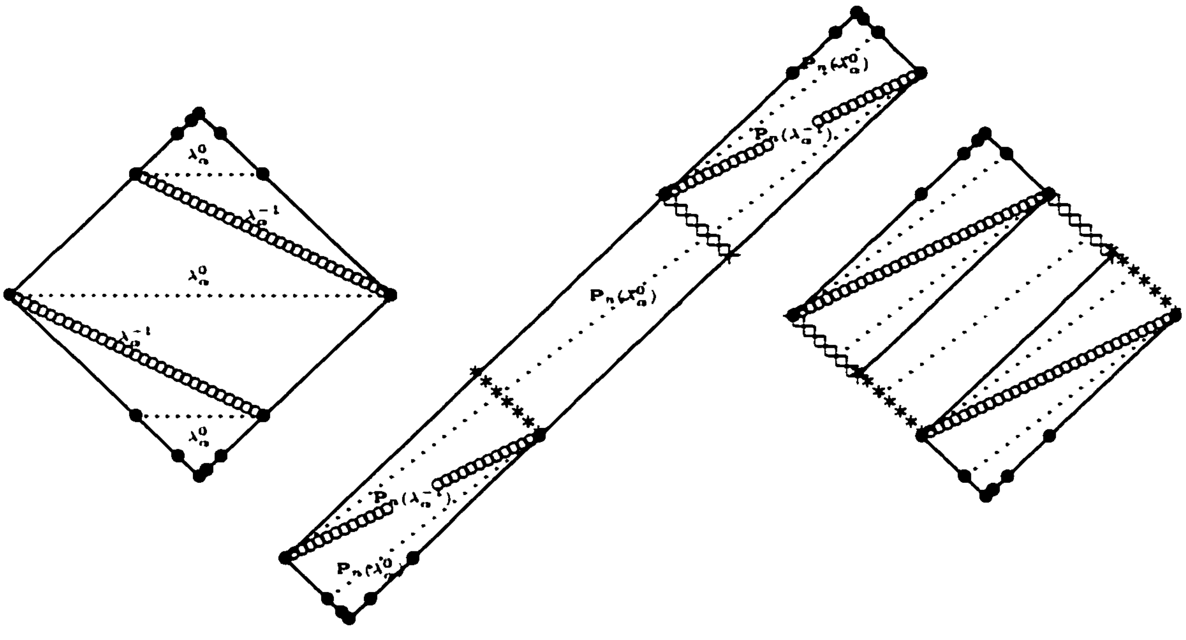


FIGURE 10. *The pseudo-Anosov automorphism* $\mathbf{P}_\alpha$ *on* X_α *for* $\alpha = \frac{1}{3}$. Stretching X_α in the direction λ_α^∞ by a factor of 3 and squeezing in the direction $\lambda_\alpha^{-\infty}$ by a factor of $\frac{1}{3}$, we obtain an affine automorphism of X_α. In order to see that there is such a self map, one can cut along the segments denoted by stars and crosses and paste along the portions of sides that get identified on X_α. $\mathbf{P}_\alpha$ sends the direction λ_α^{-1} (shown by small circles) to the direction λ_α^1 and the direction λ_α^0 to the direction λ_α^2 on its image.

up the singular points on the two right boundary sides also by one. In other words, the point C_i goes to C_{i-1} and B_i goes to B_{i+1}. This implies that $\mathbf{P}_n(\lambda_\alpha^i) = \lambda_\alpha^{i+2}$. This means that $\mathbf{P}_n$ induces a correspondence between the cylinders in the direction λ_α^i and those in the direction λ_α^{i+2}. Hence there is an isomorphism between the group of all parabolic automorphisms with invariant direction λ_α^i and the group of parabolic automorphisms with invariant direction λ_α^{i+2} via conjugation by $\mathbf{P}_n$. Noting that $\mathbf{D}_\alpha^i$ and $\mathbf{D}_\alpha^{i+2}$ both can be described as the one of the two generators for the corresponding cyclic subgroups, satisfying $\tau(\mathbf{D}^i_\alpha) > 0$ and $\tau(\mathbf{D}^{i+2}_\alpha) > 0$, and using part *(b)* of Proposition 2.12, we get $\mathbf{P}_n \circ \mathbf{D}^i_\alpha \circ \mathbf{P}^{-1}_n = \mathbf{D}^{i+2}_\alpha$. ☐

Proposition 5.2. *For* $\alpha = \frac{1}{n}$, $\mathbf{D}^1_\alpha \circ \mathbf{D}^0_\alpha = \mathbf{P}_n$.

Proof. (See figure 11) $\mathbf{D}^0_\alpha$ maps the direction $\lambda^{\infty}_{\frac{1}{n}}$ to the direction of the line joining C to the midpoint of BB_1. This line intersects the vertex F of the parallelogram BB_1EF which is congruent to the parallelogram BB_1CC_{-1}. On the cylinder $\mathcal{C}$ defined by the parallelogram BB_1CC_{-1}, $\mathbf{D}^1_\alpha$ acts by $(n+1)$ twists. Suppose $C_{-1}CHG$ is the parallelogram with $C_{-1}G$ $(n-1)$ times longer than BC_{-1} but with HG the same length as CC_{-1}. The triangles $\triangle CHG$ and $\triangle CB_1A$ are similar because their angles at vertices H and B_1 are equal. Moreover the relations $(n-1)B_1C = CH$ and $AB_1 = \frac{1}{n}AB$ imply that $(n-1)AB_1 = GH$. Since CH and B_1C are co-linear, AC and CG must also be co-linear. Therefore $\mathbf{D}^1_\alpha$ maps the segment $CF = \mathbf{D}^0_\alpha(CC_1)$ to the segment CG which is in the same direction as CC_1 but n times longer. Therefore $\mathbf{D}^1_\alpha \circ \mathbf{D}^0_\alpha$ fixes the direction $\lambda^{\infty}_{\frac{1}{n}}$, stretching any vector in that direction by a factor of n. One can similarly show that the inverse $\mathbf{D}^{0^{-1}}_\alpha \circ \mathbf{D}^{1^{-1}}_\alpha$ also fixes the direction $\lambda^{-\infty}_{\frac{1}{n}}$, stretching any vector in that direction by a factor of n. This shows that $\mathbf{D}^1_\alpha \circ \mathbf{D}^0_\alpha = \mathbf{P}_n$. ☐

This proposition implies there are at most two conjugacy classes of parabolic elements f^n_α. In fact these two conjugacy classes are different because the cylinders in the direction λ^0_α all have two saddle connections on each of their boundary components but the middle cylinder in direction λ^1_α has only one saddle connection on each of its boundary components. Any automorphism conjugating $\mathbf{D}^0_\alpha$ and $\mathbf{D}^1_\alpha$ has to map

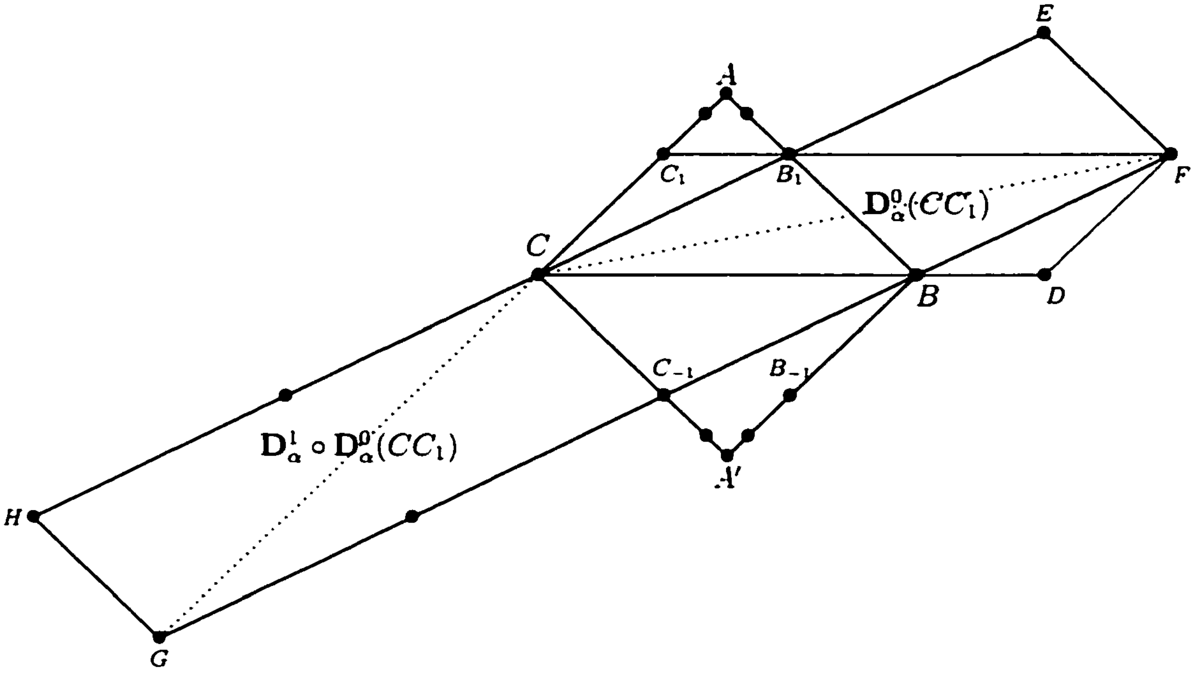


FIGURE 11. $\mathbf{D}^1_\alpha \circ \mathbf{D}^0_\alpha = \mathbf{P}_n$

the middle cylinder in direction λ^1_α to a cylinder in direction λ^0_α, and that is impossible. This shows that the surface Σ_α has at least two cusps. In order to complete the understanding of Σ_α and Γ_α we need the following proposition.

Here we describe X_α in yet a different way which will be useful in what follows. By applying a linear map one can make the two directions λ^0_α and λ^1_α perpendicular to each other. Then following the steps of Figure 10, one obtains a region composed of squares with sides of lengths α^i attached to rectangles with sides of lengths α^i and α^{i+1} and then followed by a square with sides of lengths α^{i+1} and so on. Any two opposite sides or portions of sides with equal lengths are identified by vertical or horizontal translations. We call this description of X_α, its *rectangular model.*(see figure 12)

For $\alpha = \frac{1}{n}$, in the standard basis $\{(1,0),(0,1)\}$ we have the following linear representations for $\Pi(\mathbf{D}^0_\alpha)$ and $\Pi(\mathbf{D}^1_\alpha)$:

$$\Pi(\mathbf{D}^0_\alpha) = f^0_\alpha = \begin{pmatrix} 1 & 1+\alpha \\ 0 & 1 \end{pmatrix} \qquad \Pi(\mathbf{D}^1_\alpha) = f^1_\alpha = \begin{pmatrix} 1 & 0 \\ \frac{\alpha+1}{\alpha} & 1 \end{pmatrix}$$

Therefore the group generated by $\Pi(\mathbf{D}^0_\alpha)$ and $\Pi(\mathbf{D}^1_\alpha)$ regarded as a Fuchsian group is conjugate to the group generated by $z \longmapsto z + \alpha + 1$ and $z \longmapsto \frac{\alpha z}{(\alpha+1)z+\alpha}$. This in turns is conjugate to:

$$G_\alpha = \langle f : z \longmapsto z + 1, g_\alpha : z \longmapsto \frac{\alpha z}{(\alpha+1)^2 z + \alpha} \rangle.$$

It is easy to see that the hyperbolic surface obtained from the quotient of the hyperbolic plane by the action of G_α, for $0 < \alpha < 1$, is a surface S_α of genus zero with two cusps and one hole. The simple closed curve that represents the homotopy class of the hole is generated by the hyperbolic element P_α defined by $z \longmapsto \frac{\alpha(z+1)}{(\alpha+1)^2(z+1)+\alpha}$ and its length is equal to $-\ln(\alpha)$.

Corollary 5.3. *(a) The group $\Gamma_{\frac{1}{n}}$ is the free group generated by the two parabolic elements $f^0_\alpha = \Pi(\mathbf{D}^0_\alpha)$ and $f^1_\alpha = \Pi(\mathbf{D}^1_\alpha)$. (b) The surface $\Sigma_{\frac{1}{n}}$ is a surface of genus zero with two cusps and one hole, such that the length of the closed geodesic representing the homotopy class of the hole is $\ln n$.(see figure 13)*

Proof. The group generated by $\Pi(\mathbf{D}^0_\alpha)$ and $\Pi(\mathbf{D}^1_\alpha)$ is a Fuchsian group conjugate to $G_{\frac{1}{n}}$. Since Γ_α contains $\langle \Pi(\mathbf{D}^0_\alpha), \Pi(\mathbf{D}^1_\alpha) \rangle$, $\Sigma_{\frac{1}{n}}$ must be covered by $S_{\frac{1}{n}}$. If a covering transformation $f : S_{\frac{1}{n}} \longrightarrow S_{\frac{1}{n}}$ has a

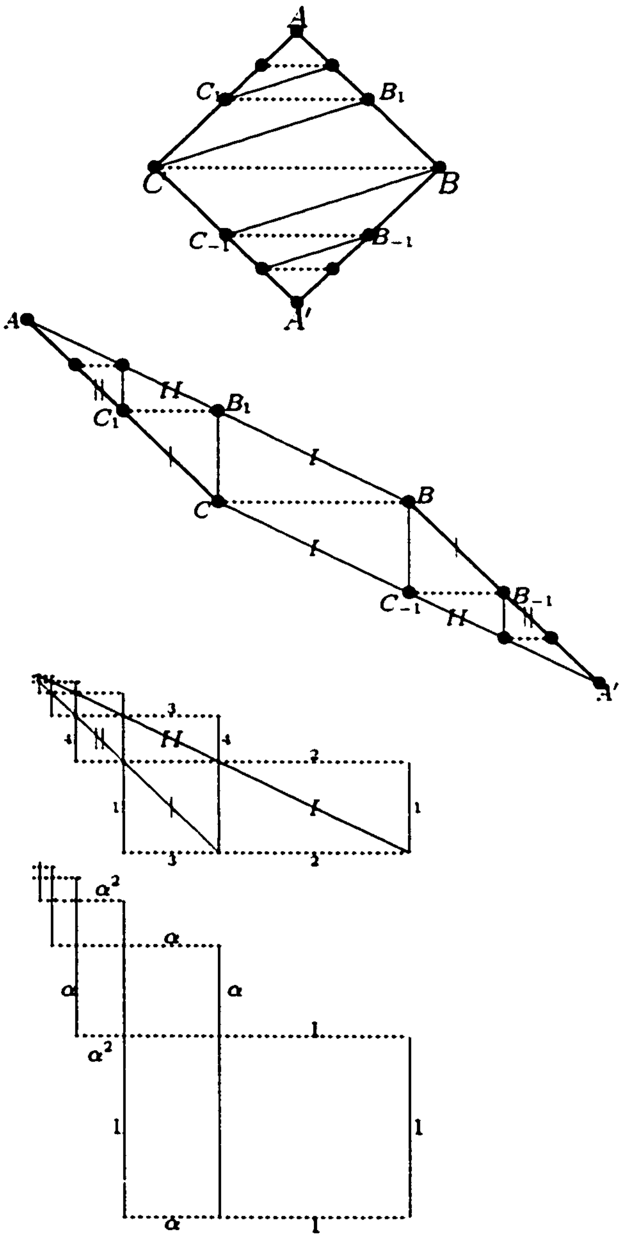


FIGURE 12. The rectangular model of $X_{\frac{1}{2}}$.

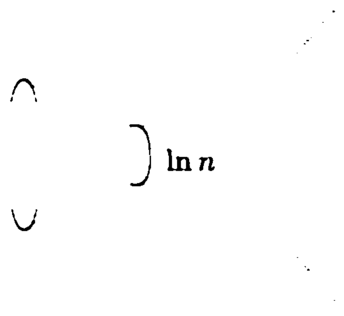


FIGURE 13. When $\alpha = \frac{1}{n}$. the surface$\Sigma_{\frac{1}{n}}$ is the surface of genus zero with two cusps and one hole. such that the length of the closed geodesic representing the homotopy class of the hole is $\ln n$.

hyperbolic or parabolic lift to the hyperbolic plane. it must have a fixed point on the conformal boundary of $S_{\frac{1}{n}}$. This fixed point is either a parabolic fixed point at one of the two cusps or it is a point on the conformal boundary of $S_{\frac{1}{n}}$ corresponding to the hole. The first case is ruled out because the generators of the two cusps are primitive in the automorphism group. If f has a fixed point z on portion of the conformal boundary corresponding to the hole. one of the lifts of z must be on the interval I in the proof of the Corollary 4.6. which is impossible. The only possibility for an elliptic covering transformation $f : S_{\frac{1}{n}} \rightarrow S_{\frac{1}{n}}$ is the homeomorphism of $S_{\frac{1}{n}}$ that switches the two cusps. This homeomorphism does not induce an automorphism in $\Gamma_{\frac{1}{n}}$. because otherwise. it would have conjugated $\mathbf{D}_{\alpha}^{0}$ and $\mathbf{D}_{\alpha}^{1}$ and one of its conjugates would have mapped the cylinders in the direction λ_{α}^{0} to the ones in the direction λ_{α}^{1} which is impossible. Therefore $\Gamma_{\frac{1}{n}} = G_{\frac{1}{n}}$ and the corollary is true. □

Case II: α is a rational number not equal to $\frac{1}{n}$: The main difference between this case and the first case is that for $\alpha \neq \frac{1}{n}$ the hyperbolic automorphism with fixed directions λ_α^∞ and $\lambda_\alpha^{-\infty}$, which correspond to the end-points of the maximal interval I of discontinuity of the action of Γ_α, does not exist. It turns out that in the absence of this automorphism, whose iterates conjugate $\mathbf{D}_\alpha^i$s into two conjugacy classes for $\alpha = \frac{1}{n}$, no two of these parabolic automorphisms are conjugate.

Proposition 5.4. *For $\alpha = \frac{p}{q}, 1 < p < q$, p and q positive integers, no two parabolic elements $\Pi(\mathbf{D}_\alpha^i)$ and $\Pi(\mathbf{D}_\alpha^j)$ for $i \neq j, i, j \geq 0$ are conjugate in Γ_α. Therefore Σ_α has infinitely many cusps.*

Proof. (See figure 14) Suppose the automorphism $f \in \Gamma_\alpha$ conjugates $\mathbf{D}_\alpha^i$ and $\mathbf{D}_\alpha^j$. Then it should map the direction λ_α^i to λ_α^j. Since affine maps preserve the ratios of the lengths of two segments in one direction, the lists of the lengths of saddle connections in the two directions λ_α^i and λ_α^j have to be the same up to multiplication by a positive factor.

For any $k \in \mathbb{Z}$, X_α is cut into two pieces by the two saddle connections CB_k and $C_{-k}B$: the finite type piece which is the rectangular region $CB_kC_{-k}B$ with the side identifications and the rest of the surface which is of infinite type. Inside the interior of the finite type piece there are k saddle connections joining a C_l to a $B_{l'}$. The rest of the saddle connections in the direction λ_α^k, are in the infinite type piece or are one of the two saddle connections CB_k and $C_{-k}B$ on the common boundary of the two pieces. Listing the Euclidean lengths of the saddle connections in the direction λ_α^k, we get k lengths corresponding to the saddle connections in the interior of the finite type piece, all of them

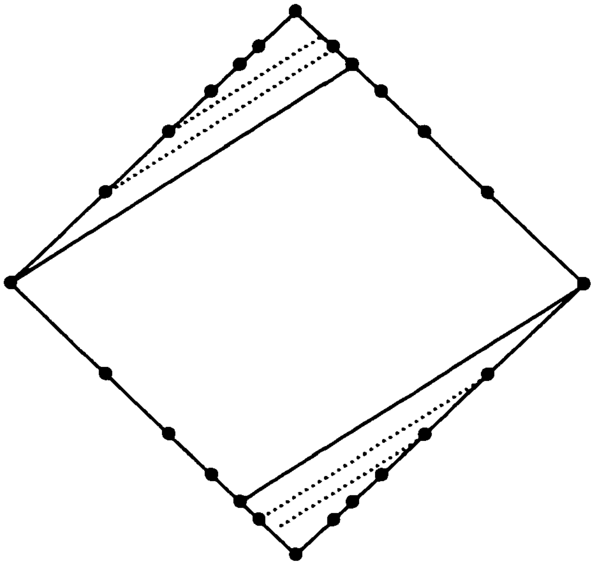


FIGURE 14. The saddle connections CB_k and $C_{-k}B$ in the direction λ_α^k divide X_α into two pieces; one of finite type which contains k saddle connections in direction λ_α^j of lengths an integer multiple of the length of CB_k and an infinite type piece which contains saddle connections in direction λ_α^j of lengths α^n times the length of CB_k. When $\alpha \neq \frac{1}{n}$ this implies that $\mathbf{D}_\alpha^i$ and $\mathbf{D}_\alpha^j$ are not conjugate for $i \neq j$, $i, j > 0$.

an integer multiple of $|CB_k|$, and the rest of the lengths which are the numbers $\alpha^m|CB_k|$, each listed twice. Up to a factor the above list is equivalent to a list of k integers and the numbers α^m, $m = 0, 1, \ldots$ each listed twice. Evidently for two different values of k, the list of the lengths of the saddle connections in the direction λ_α^k are not equivalent up to a positive factor. Hence for $i \neq j$, $\mathbf{D}_\alpha^i$ and $\mathbf{D}_\alpha^j$ are not conjugate in $Aff(X_\alpha, \mathcal{F}_\alpha)$.

□

In order to describe Γ_α explicitly when $\alpha \neq \frac{1}{n}$, we need a slightly different surface from X_α, which although doesn't have finite area, supports a large automorphism group. We denote it by $\hat{X}_\alpha$. It is constructed in a similar way as the rectangular model of X_α is constructed only it does not stop from below. It turns out that any automorphism of X_α, induces an automorphism of $\hat{X}_\alpha$. Later we will see how to detect which automorphisms of $\hat{X}_\alpha$ can be induced from an automorphism of X_α.

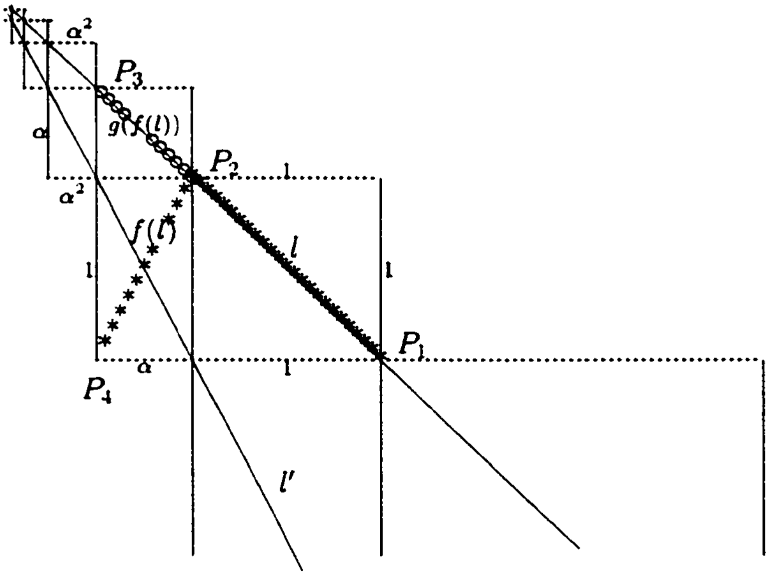


FIGURE 15. The flat surface $\hat{X}_{\frac{1}{2}}$. The pattern of rectangles and squares continues in the same way bellow the picture. The total area is infinite. The hyperbolic automorphism $g \circ f$ fixes the two directions l and l'. One can see that by keeping track of a segment P_1P_2 under this map. $g \circ f(P_1P_2) = g(P_2P_4) = P_2P_3$.

Let $f_\alpha, g_\alpha \in PSL(2, \mathbb{R})$ be the following:

$$f_\alpha = \begin{pmatrix} 1 & 1+\alpha \\ 0 & 1 \end{pmatrix} \qquad g_\alpha = \begin{pmatrix} 1 & 0 \\ \frac{\alpha+1}{\alpha} & 1 \end{pmatrix}$$

Now on $\hat{X}_\alpha$ all the horizontal cylinders as well as all the vertical cylinders have equal moduli. Therefore f and g induce parabolic automorphisms which respectively fix the horizontal and the vertical directions and do one twist on every horizontal and vertical cylinder. $g \circ f$ induces a hyperbolic automorphism with fixed directions l and l' (see figure 15). Let J be the interval of directions between the directions of l and l'. Any such direction intersects infinitely many horizontal cylinders. Therefore by Proposition 2.20. the interior of J corresponds to a maximal interval of discontinuity for the action of $Aut(\hat{X}_\alpha)$: the "area preserving" affine automorphism group of $\hat{X}_\alpha$, on the boundary of the Teichmüller disk $\hat{\mathfrak{D}}_\alpha$ defined by the flat structure of $\hat{X}_\alpha$. Now, following the proof of the Corollary 5.3, it can be shown that $Aut(\hat{X}_\alpha)$ is $\langle f, g\rangle$: the free group generated by f and g.

Remark: $\hat{X}_\alpha$ has many non-area preserving affine automorphisms. One that we'll denote by s, shifts horizontal and vertical cylinders by one and in local coordinates it can be represented by $s(z) = \alpha(z - O)$ where O is the accumulation point of the singularities at the intersection of l and l' in the plane. There are others that conjugate f and g and switch vertical and horizontal cylinders. Any one of them can be obtained from a fixed one followed by s^n for some integer n. In fact, the existence of such non-area preserving automorphisms implies that there is no area preserving automorphism conjugating f and g. In other words there is no horizontal cylinder whose area is the same as a vertical cylinder. This is a part of the proof of Corollary 5.3 that can not be done in a similar way and needs this extra area argument. For future use we denote the automorphism that sends any horizontal cylinder $\mathcal{C}$ to the vertical cylinder with less area and intersecting $\mathcal{C}$, by

r. Note that there is also an area preserving automorphism of order two which is in the kernel of Π and is defined in a similar way to the way ι was defined for X_α in section 4. It is much harder to see it in the rectangular model. One has to note that a half-turn rotation on all of the individual squares and rectangles forming $\hat{X}_\alpha$, induces a self map. We denote this map by $\hat{\iota}$ and we have:

$$\hat{\iota} \circ s = r^2 \qquad \textit{and} \qquad s^2 = r^4 \tag{5.5}$$

The key point in finding Γ_α for $\alpha = \frac{1}{n}$ was that we found a group of automorphisms that can not be enlarged. In order to find Γ_α for $\alpha = \frac{p}{q}$, $p \neq 1$, we need to extend it to a larger group. Instead of looking at the stabilizer of one Teichmüller disk in the mapping class group of X_α, we will consider the stabilizer of a family of Teichmüller disks.

Definition 5.6. *For a flat surface $(X, \mathfrak{F})$, and a closed Euclidean geodesic γ with Euclidean length L and a real number t, we define $\mathcal{E}_{t\gamma}(X)$ to be the flat surface obtained by cutting $(X, \mathfrak{F})$ along γ and gluing it back after sliding one side tL units along γ to the left when viewed from the other side.*

Note that in the definition, sliding one side to the left when viewed from the other side, does not depend on the choice of one of the two sides. When t is an integer, $\mathcal{E}_{t\gamma}(X)$ is isometric to X.

Lemma 5.7 (Sliding Lemma). *Let X be a flat surface and λ a Jenkins-Strebel direction. Let γ be a closed geodesic not in the direction λ. Then:*

(1) *λ is a Jenkins-Strebel direction for $\mathcal{E}_{t\gamma}(X)$ iff t is a rational number.*

(2) *If λ is a Jenkins-Strebel direction of bounded type for X and t is rational and if the ratio of the heights of any two λ cylinders is rational, then λ is also a Jenkins-Strebel direction of bounded type for $\mathcal{E}_{t\gamma}(X)$. Moreover the ratios of the heights of any two λ cylinders in $\mathcal{E}_{t\gamma}(X)$ are also rational.*

Proof. (1) Let ζ be any arbitrary geodesic of X in the direction λ which intersects γ at point x. Fix an orientation on ζ. ζ is either a closed geodesic or part of a saddle connection. Let $t = \frac{p}{q}$ be a rational number and let x_k, for $0 \leq k \leq q$, be the point obtained from moving x a distance of ktL on γ to the left (with respect to the orientation of ζ) where L is the length of γ. In particular $x_0 = x_q = x$. Let ζ_k be the λ-geodesic that intersects γ at x_k. After sliding along γ, any ζ_k is cut at x_k and it is glued to ζ_{k+1}. In other words, inside $\mathcal{E}_{t\gamma}(X)$ when travelling along λ-geodesics, exiting ζ_k at x_k one enters ζ_{k+1}. If all the ζ_ks are are closed geodesics in X, then together they form one closed geodesic of $\mathcal{E}_{t\gamma}(X)$. Otherwise, there will be a sequence of successive ζ_ks. such that the first and last of them are saddle connections in X, the rest of them are closed geodesics of X and they together form a saddle connection in $\mathcal{E}_{t\gamma}(X)$. As a result any λ-geodesic in X that intersects γ turns into either a saddle connection or a closed geodesic of $\mathcal{E}_{t\gamma}(X)$. The λ-geodesics in X that don't intersect γ, remain unchanged. Therefore λ is a Jenkins-Strebel direction for $\mathcal{E}_{t\gamma}(X)$.

Now assume t is irrational and ζ is an oriented λ-geodesic which intersects γ at x. Define x_k to be the point that is a distance of ktL on γ to the left of x. Let ζ_k be the λ-geodesic that intersects γ at x_k. Since there are only finitely many saddle connections inside X in the direction λ that intersect γ, we can choose x so that none of the ζ_k's are saddle connections. Inside $\mathcal{E}_{t_\gamma}(X)$, when travelling along λ-geodesics, exiting ζ_k at x_k one enters ζ_{k+1} and therefore the ζ_k's together form one λ-geodesic of infinite length. Therefore λ is not a Jenkins-Strebel direction for $\mathcal{E}_{t_\gamma}(X)$.

(2) Choose an orientation on γ and suppose x^i, $0 \leq i \leq N$, $x^0 = x^N$, are the intersection points of the λ saddle-connections with γ ordered as their cyclic order on γ. Since the ratios of the heights of the λ-cylinders intersecting γ in X are rational, the ratios $\frac{|x^i x^{i+1}|}{|x^j x^{j+1}|}$ are also rational. For $t = \frac{p}{q}$, let x^i_k be the point That is a distance of ktL on γ to the left of x^i. Let ζ^i_k, $1 \leq k \leq q$ be the λ-geodesic that intersects γ at x^i_k. Then ratio of the distances of successive pairs $(x^i_k, x^{i'}_{k'})$ is rational. Therefore ratios of the heights of the pairs of λ-cylinders in $\mathcal{E}_{t_\gamma}(X)$, are rational.

On the other hand since the ratio of the moduli of any pair of λ-cylinders in X is rational, the ratio of the lengths of the generators of any pair of λ-cylinders of X should also be rational. The generators of the λ-cylinders in $\mathcal{E}_{t_\gamma}(X)$, intersecting γ are obtained from gluing some of ζ^i_k to each other. Hence the length of each of them is a rational multiple of the length of a λ-cylinder in X. Therefore the ratio of the lengths of the generators of any pair of λ-cylinders in $\mathcal{E}_{t_\gamma}(X)$is also rational. Since except for finitely many of them, the λ-cylinders of $\mathcal{E}_{t_\gamma}(X)$ and X are the same, and the ratio of the moduli of any pair of them is rational, λ is a Jenkins-Strebel direction of bounded type on $\mathcal{E}_{t_\gamma}(X)$. □

On the rectangular model of X_α, $\Pi(\mathbf{D}_\alpha^0) = f_\alpha$ and $\Pi(\mathbf{D}_\alpha^1) = g_\alpha^p$. We will fix α and avoid using the index α in what follows. The linear map g induces one twist on every vertical cylinder except for one. We denote the curve representing the homotopy class of a generator of this cylinder by γ. On that exceptional vertical cylinder, g induces $\frac{p+q}{p}$ of a full twist. More precisely there is an affine automorphism from X_α to $\mathcal{E}_{(\frac{p+q}{p})\gamma}(X)$ whose image under Π is g. Keeping the difference between this automorphism and its linear representative g implicit, we denote the automorphism by g as well. This map is continuous everywhere except for the points on γ. We denote $\mathcal{E}_{i(\frac{p+q}{p})\gamma}(X)$ by $X_\alpha^{g^i}$. g acts on the set $\{X_\alpha^{g^i}, 0 \leq i \leq p-1\}$ as a cyclic permutation.

Although $\mathbf{D}_\alpha^0$ acts on X as a self-map, there is no self-map of $X_\alpha^{g^i}$, $1 \leq i \leq p-1$, whose linear representative is f. The sliding lemma, however, implies that the horizontal direction is a Jenkins-Strebel direction of bounded type for each $X_\alpha^{g,i}$. In fact f defines an automorphism on all but finitely many cylinders in the horizontal direction. On each of the finitely many exceptional cylinders f^j defines a non-integral rational sliding. Let $X_\alpha^{g^i f^j}$ be the flat surface obtained from $X_\alpha^{g^i}$ by the rational sliding defined be f^j. For each i, there is a smallest positive integer N^{g^i}, such that $X_\alpha^{g^i f^{N^{g^i}}} = X_\alpha^{g^i}$.

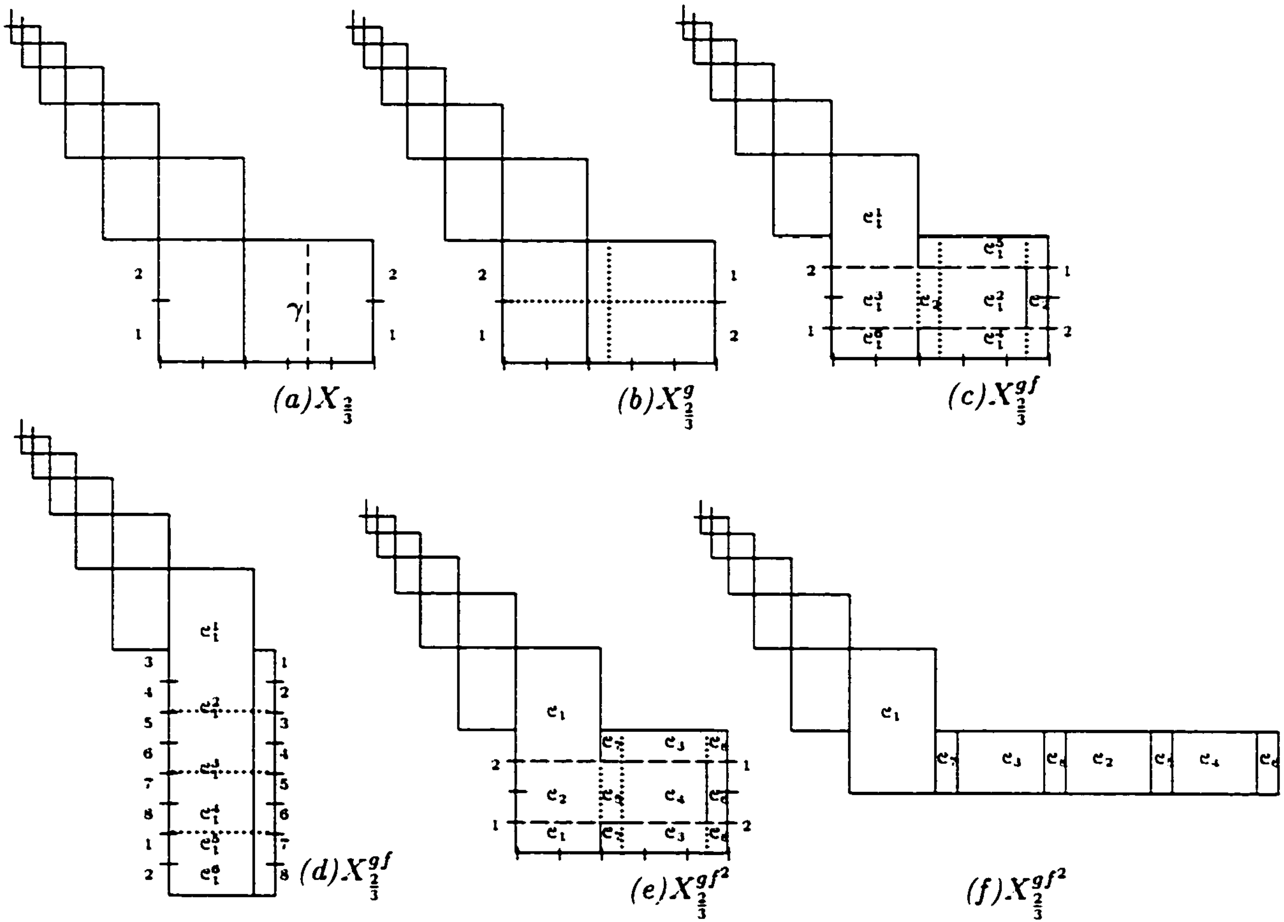
For a word w in the free group generated by f and g, X_α^w can be defined as follows. Suppose w' is a word ending with f and $w = w'g$. When w' ends with g and $w = w'f$, X_α^w can be defined by a similar construction obtained by switching f with g and vertical with horizontal. Also assume $X_\alpha^{w'}$ is already defined and satisfies the following:

- The horizontal and vertical directions both are Jenkins-Strebel directions of bounded type on $X_\alpha^{w'}$ and the ratio of the heights of any pair of horizontal or vertical cylinders is rational.
- $X_\alpha^{w'}$ as a flat surface differs from X_α only on finitely many vertical and horizontal cylinders.

g defines an affine automorphism on all but finitely many vertical cylinders of $X_\alpha^{w'}$. On any of those finitely many vertical cylinders g induces a rational sliding. Let X_α^w be the flat surface obtained by composition of these finitely many slidings. One should notice that since the generators of these cylinders are disjoint the order of the composition does not matter. By the sliding lemma the horizontal direction is again a Jenkins-Strebel direction of finite type for X_α^w. The moduli of the vertical cylinders remain unchanged, hence the horizontal direction is also a Jenkins-Strebel direction of finite type for X_α^w. Clearly X_α^w as a flat surface differs from $X_\alpha^{w'}$ and therefore from X_α on only finitely many vertical and horizontal cylinders. If w ends with g, we denote the smallest positive integer N for which $X_\alpha^{wf^N} = X_\alpha^w$ by N^w. When w ends with f, N^w is the smallest positive integer N for which $X_\alpha^{wg^N} = X_\alpha^w$.

Lemma 5.8. *For any* $0 < \alpha < 1$,

1. *Any fixed point of a virtual multi-twist in* Γ_α *is also a parabolic fixed point of* $\langle f, g\rangle$.
2. *The limit set of* Γ_α *is a subset of the limit set of* $\langle f, g\rangle$.



(a) $X_{\frac{2}{3}}$ (b) $X_{\frac{2}{3}}^{g}$ (c) $X_{\frac{2}{3}}^{gf}$

(d) $X_{\frac{2}{3}}^{gf}$ (e) $X_{\frac{2}{3}}^{gf^2}$ (f) $X_{\frac{2}{3}}^{gf^2}$

FIGURE 16. For $\alpha = \frac{2}{3}$, *(a)* shows X_α. g_α induces a $\frac{5}{2}$ twist on the curve γ therefore it induces a map from X_α to X_α^g. X_α^g is shown in *(b)*. The labels show the identifications on the parallel sides. The horizontal cylinder with maximum area in X_α now has turned into a horizontal cylinder twice longer and half as high as before. Therefore f induces a $\frac{1}{4}$ twist on this cylinder which can be represented by a sliding along the curve shown with two broken segments in *(c)*. Following the track of cylinders and with some cutting and pasting, we get the representation shown in *(d)* for X_α^{gf}. $\mathcal{C}_1^i$s show different pieces of one vertical cylinder with their order on that cylinder. Similar analysis of cylinders for $X_\alpha^{gf^2}$ shown in *(e)* leads to the representation shown in *(f)* for this surface. Note that many vertical cylinders labelled by $\mathcal{C}_i$, together form one tall vertical cylinder.

Proof. (1) By Proposition 2.20, only finitely many cylinders $\mathcal{C}_i$, $1 < i < N$ in a direction κ fixed by a virtual multi-twist in Γ_α, can intersect the horizontal cylinder $\mathcal{C}$ with the largest area. Again by Proposition 2.20, any one of the $\mathcal{C}_i$'s intersects at most finitely many horizontal cylinders. Let $\mathcal{C}'$ be the horizontal cylinder with the largest area that doesn't intersect any of $\mathcal{C}_i$'s and suppose $\mathcal{C}'_i$, $1 < i < N'$ are all the κ-cylinders that intersect $\mathcal{C}'$. We denote the horizontal cylinder in $\hat{X}_\alpha$, with the same area as $\mathcal{C}'$, by $\mathcal{C}'$ as well. Since no $\mathcal{C}'_i$ intersects $\mathcal{C}$, the pattern of the κ-cylinders in $\hat{X}_\alpha$ intersecting $\mathcal{C}'$ is exactly the same as the pattern of $\mathcal{C}'_i$'s. We therefore denote the κ-cylinders in $\hat{X}_\alpha$ intersecting $\mathcal{C}'$ again by $\mathcal{C}'_i$, $1 < i < N'$. The image of any horizontal or κ-cylinder under the shift automorphism s is another horizontal or κ-cylinder with the same modulus. Moreover $\bigcup_{i \in \mathbb{Z}} s^i(\mathcal{C}') = \hat{X}_\alpha$. Therefore $\{s^i(\mathcal{C}'_j) | i \in \mathbb{Z}$, $1 \leq j \leq N\}$ lists, with possible repetitions, all the κ-cylinders in $\hat{X}_\alpha$ and $\{M_j = Mod(\mathcal{C}'_j), 1 < j < N'\}$ lists the moduli of all the κ-cylinder in $\hat{X}_\alpha$. Since κ is the invariant direction of a virtual multi-twist in Γ_α, the moduli of $\mathcal{C}'_i$, $1 < i < N'$ are rational multiples of each other. Therefore κ is a Jenkins-Strebel direction of bounded type for $\hat{X}_\alpha$ and by Proposition 2.16, κ corresponds to the fixed point of a virtual multi-twist in $Aut(\hat{X}_\alpha) = \langle f, g \rangle$.

(2) This is an immediate consequence of part (1); but we prove it without using part (1). The limit set of $\langle f, g \rangle$ is the complement of its domain of discontinuity on $\partial\hat{\mathfrak{D}}_\alpha$. The domain of discontinuity itself is the union of all the images of the interior of the interval J under the elements of $\langle f, g \rangle$. Typical point in the domain of discontinuity of $\langle f, g \rangle$ is in the form $\kappa = w(\lambda)$ for a direction λ in J and w an element of $\langle f, g \rangle$. There is a line l in the direction λ that leaves any sub-surface of finite

type in X_α. $w(l)$ is a line in X^w_α and it leaves any sub-surface of finite type. Since X^w_α as a flat surface differs from X_α only on finitely many vertical and horizontal cylinders, there is a line l' in X_α in the direction of κ which leaves infinitely many horizontal cylinders. Now it follows from Proposition 2.20 that κ is not the fixed direction of any parabolic element in Γ_α. Therefore the domain of discontinuity of $\langle f, g\rangle$ does not include any parabolic fixed points of Γ_α. Since the parabolic fixed points of Γ_α are dense in its limit set and the domain of discontinuity of $\langle f, g\rangle$ is open, it follows that the domain of discontinuity of $\langle f, g\rangle$ does not contain any point in the limit set of Γ_α. Therefore the limit set of Γ_α is contained in the limit set of $\langle f, g\rangle$. ☐

We define the subgroup $\mathcal{G}_\alpha$ of $\langle f, g\rangle$, the free group generated by f and g to be

$$\mathcal{G}_\alpha = \{w \in \langle f, g\rangle : X^w_\alpha = X_\alpha\}.$$

Proposition 5.9. *For α rational and not equal to $\frac{1}{n}$, $\Gamma_\alpha = \mathcal{G}_\alpha$.*

Proof. It is clear from the construction that $\mathcal{G}_\alpha \subseteq \Gamma_\alpha$. Also the construction of $\mathcal{G}_\alpha$ shows that it consists of all the elements in $\langle f, g\rangle$ which are in Γ_α. Therefore in order to prove the proposition we will have to show that any element of Γ_α is automatically in $\langle f, g\rangle$. Suppose h is an element of Γ_α. We use $\acute{\lambda}^n_\alpha$ for the directions corresponding to λ^n_α in the rectangular model. In particular $\acute{\lambda}^0_\alpha$ and $\acute{\lambda}^1_\alpha$ are respectively the horizontal and vertical directions. Since f is also in Γ_α, the parabolic element $f' = h^{-1} \circ f \circ h$ with invariant direction $h(\acute{\lambda}^0_\alpha)$ is also in Γ_α. By Lemma 5.8, $h(\acute{\lambda}^0_\alpha)$ has to be an invariant direction of a primitive parabolic element p in $\langle f, g\rangle$. There are two possibilities: either p is conjugate to g or conjugate to f inside $\langle f, g\rangle$. The first possibility is ruled out because all but finitely many of the $h(\hat{\lambda}_\alpha^0)$-cylinders in X_α are isometric to the $h(\hat{\lambda}_\alpha^0)$-cylinders in $\hat{X}_\alpha$ and those have the same areas as the corresponding $(\hat{\lambda}_\alpha^1)$-cylinders in $\hat{X}_\alpha$. Since the vertical cylinders in $\hat{X}_\alpha$ are mapped to the horizontal ones via the non-area-preserving map r, no vertical cylinder has the same area as a horizontal cylinder. Therefore no automorphism of X_α can map the horizontal cylinders to cylinders in the invariant direction of a parabolic automorphism conjugate to g in $\langle f, g\rangle$.

The second possibility means that there is an element w in $\langle f, g\rangle$, such that $w^{-1} \circ f \circ w = p$. This together with $f' = h^{-1} \circ f \circ h$ implies that $p' = h \circ w^{-1}$ is parabolic and fixes the horizontal direction. In other words $p' \circ w$ is an automorphism of X_α. Equivalently, p' induces an affine map from X_α^w to X_α. From the construction of X_α^w it is clear that no horizontal sliding can map it to X_α unless if $w = f^n$ and $p' = f^m$ for some integers n and m. Hence $h = f^{m+n}$ is in $\langle f, g\rangle$. This completes the proof of the Proposition. □

Proposition 5.10. *For* $0 < \alpha < 1$ *a rational number in the reduced form* $\frac{p}{q}$ *and* $p \neq 1$, Σ_α *is conformally equivalent to an open disk with countably many isolated points removed. The points accumulate to a Cantor set on the boundary of the disk.*

Proof. Let $\hat{\Sigma}_\alpha$ be the Riemann surface obtained from the action of $\langle f, g\rangle$ on the hyperbolic plane. It is a surface of genus zero with one hole and two cusps. The length of the geodesic representing the homotopy class of the hole is $\ln \frac{1}{\alpha}$. $\Gamma_\alpha \subset \langle f, g\rangle$ induces a covering of the Riemann surface $\hat{\Sigma}_\alpha$ by Σ_α. As is shown in figure 17, this covering can

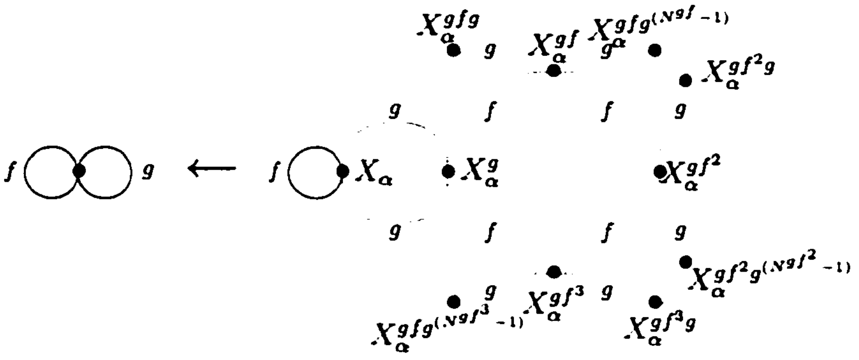


FIGURE 17. $\Gamma_\alpha \subset \langle f, g\rangle$ induces a covering of the Riemann surface $\hat{\Sigma}_\alpha$ by Σ_α. This covering can be described by the above picture. The covering maps all the segments with label f or g respectively to the arc with label f or g in the figure eight that represents the free group generated by f and g.

be described by infinitely many circles each representing one cusp of Σ_α attached to others in the following pattern. Any point of tangency of two circles represents an X_α^w for a $w \in \langle f, g\rangle$. On any circle there are N^w points of tangency with labels of $X_\alpha^{wa^i}$ where $a \in \{f, g\}$ and a is different from the letter to the right of w. Any segment between two consecutive points of tangency on one circle is labelled by the letter a. The two cusps of $\hat{\Sigma}_\alpha$ are represented by two circles labelled by f and g and tangent at one point. The covering maps all the points of tangency to the point of tangency in the figure eight (to the left of the arrow in figure 17) and the arcs with label a to the arc with the same label in figure eight. from this picture it is clear that Σ_α is a surface of genus zero with infinitely many cusps accumulating to a connected conformal boundary. The cusps accumulate to a compact subset of the conformal boundary. Now let I_1 and I_2 be two non intersecting closed intervals on the conformal boundary of $\hat{\Sigma}_\alpha$ and let Λ be the family of arcs inside $\hat{\Sigma}_\alpha$ that join I_1 to I_2 and separate the two cusps. This family has finite extremal length. By choosing a right lift of this family we can separate any two accumulation points on the conformal boundary of Σ_α. Since any such lift has a one-to-one inverse, it defines a family of curves with finite extremal length and therefore two intervals on the boundary of Σ_α separating the given pair of accumulation points. Therefore the set of accumulation points is totally disconnected. There are no isolated points in the set of accumulation points because any such point would be formed by a connected sequence of circles going out to infinity each with only two points of tangency. This is not possible because at any step in the construction of X^w's one new cylinder is changed. New identifications on this cylinder create at least one cylinders with modulus less than half of the modulus of the newly distorted cylinder. Hence the set of accumulation points is a Cantor set. $\square$

Remark: It is known (see [6]) that if a flat surface X of finite type is a flat finite covering of a flat once punctured torus T then its automorphism group is commensurable with $PSL(2, \mathbb{Z})$. Here a flat finite covering is a covering that in local charts of the flat surface is of the form $\pm z + a$. This fact can be proved as follows. When a surface is a finite covering of a flat once punctured torus it can be tiled with several copies of one parallelogram so that all the singularities lie on the boundaries of the parallelograms. Any parallelogram is the image of another one by a translation. Then by an affine change of coordinates one can turn the parallelograms into squares of area one with horizontal and vertical sides. Finally using the sliding lemma it is easy to see that the two elements of $PSL(2,\mathbb{R})$ $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ induce affine maps from X to another surface obtained by composition of certain slidings along certain horizontal and vertical closed curves. The group G generated by A and B is $PSL(2,\mathbb{Z})$. The image X^w of X under the map induced by any word w in terms of A and B can be geometrically described by a surface obtained from X by permuting the square blocks. Therefore there are finitely many possibilities for X^ws and $Aut(X)\bigcap G$ has finite index in G. Hence $Aut(X)$ is commensurable with $PSL(2,\mathbb{Z})$.

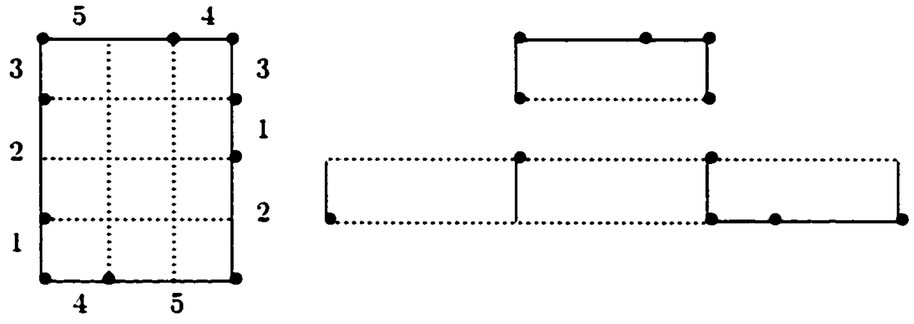


FIGURE 18. A flat surface X of genus 2 with one cylinder in the vertical direction and two cylinders in the horizontal direction. The sides with equal labels are being identified. X is finitely covered by a flat punctured torus with a flat covering map. $Aut(X)$ is commensurable with $PSL(2,\mathbb{Z})$.

Case III: α is irrational: When α is irrational there is no parabolic automorphism in any of the directions λ_α^i for $i \neq 0$. One reasonable question to ask is whether there is any automorphism other than the parabolic automorphisms $(\mathbf{D}_\alpha^0)^n$ for some $n \in \mathbb{Z}$. In order to answer this question we need to study the geometric limits of the surfaces $X^{w}_{\alpha_n}$ as rational parameters α_n approach the given irrational number α.

Proposition 5.11. *When α is irrational, Γ_α is the cyclic group $\langle f^0_\alpha \rangle = \langle f_\alpha \rangle$.*

Proof. Suppose $\Gamma_\alpha \neq \langle f_\alpha \rangle$. Then there must be an automorphism $h \in \Gamma_\alpha$, $h \neq f_\alpha{}^N$. $p_\alpha = h \circ f_\alpha \circ h^{-1}$ is a parabolic automorphism not in $\langle f_\alpha \rangle$. Let λ be the invariant direction of p_α. p_α induces one twist in all the λ-cylinders of X_α and all the λ-cylinders in X_α have the same modulus. By part (2) of Lemma 5.8, λ is the stable direction of a word w representing a parabolic element in $\langle f_\alpha, g_\alpha \rangle$. We may assume w is primitive in $\langle f_\alpha, g_\alpha \rangle$ and therefore it induces one twist on all but finitely many of the λ-cylinders of X_α. Hence $p_\alpha = w \in \langle f_\alpha, g_\alpha \rangle$.

Let $\alpha_n = \frac{p_n}{q_n}$ be a sequence of rational numbers converging to α. Let p_n be the word obtained from p by changing f_α to f_{α_n} and g_α to g_{α_n}. For any n, There is an affine map P_n induced by p_n from X_{α_n} to $X^{p}_{\alpha_n}$. Note that we regard p_n as a word in terms of f_{α_n} and g_α. Now there is a subsequence of α_n, which we also denote by α_n, for which $X^{p}_{\alpha_n}$ converges to a flat surface X and the maps P_n converge to a map $P : X_\alpha \to X$. Here the convergence is the geometric convergence or the convergence of the quadratic differentials. For the definition of geometric convergence and its compactness properties see [10]. All P_n's are affine with linear part equal to p_n. Therefore P is linear and its linear part is equal to p. It maps any saddle connection and cylinder of X_α to a saddle connection and a cylinder of X. Moreover since p induces one full twist on any cylinder of X_α, P is a self map and $X = X_\alpha$. The horizontal direction is the image of the Jenkins-Strebel direction $P^{-1}(\lambda_\alpha^0)$ under the automorphism P. We normalize the rectangular models of the X_{α_n}'s to have area equal to one. Then the intersection U_n of horizontal cylinder with the largest area and the vertical cylinder with the largest area inside X_{α_n} has area α_n. $X_{\alpha_n}^g$ contains horizontal cylinders with height $\frac{1}{p_n}$ inside an open region of area α_n corresponding to U_n. The application of more slidings in the construction of $X_{\alpha_n}^p$ produces horizontal or vertical cylinders inside U_n with height $\frac{K}{p_n}$ where K only depends on the exponents of f_{α_n} and g_α in w. Therefore $X_{\alpha_n}^p$ contains vertical cylinders with height at most equal to $\frac{K}{p_n}$ inside the open region of area α U_n. Since the α_n approach an irrational number, the p_n's tend to infinity. Therefore in the geometric limit $X = X_\alpha$, there is an open region with area $1 - \alpha$, where any horizontal line is dense. This is a contradiction and therefore $\Gamma_\alpha = \langle f_\alpha \rangle$. $\square$

6. AFFINE AUTOMORPHISM GROUPS WITH PARABOLIC ELEMENTS

In this section we generalize the construction of X_α's. In particular we will find a family of examples of flat surfaces where the affine automorphism group is a free group generated by two parabolic elements f and g. We will also be interested in the hyperbolic element gf. In the case of surfaces of finite type, there is no hyperbolic automorphism with a saddle connection in either one of its invariant directions. The automorphism P_n on $X_{\frac{1}{n}}$, however, is an example of a hyperbolic automorphism with dense subset, consisting of the saddle connections in either of the two invariant directions.

The general construction is based on the idea that if the automorphism group of a flat surface is non-elementary and it contains at least one parabolic element, then it must contain infinitely many parabolic elements. By Corollary 1 in [4] the invariant direction of any one of these parabolic automorphisms is an infinite Jenkins-Strebel direction.

We choose an arbitrary pair of parabolic automorphisms. By an affine change of coordinates we may assume that the fixed directions of f and g are respectively the horizontal and vertical directions. Moreover by applying another affine map and using the inverses of f and g instead of f or g if needed we may assume that

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad g = g_a = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$$

for a parameter $a > 0$. It is important to notice that in the rectangular model of X_α, taking f_α and g_α as a pair of parabolic automorphisms, the corresponding value of $a = a(\alpha)$ is $\frac{(\alpha+1)^2}{\alpha}$. When α varies between 0 and 1, the value of $a(\alpha)$ varies between ∞ and 4. On the other hand for any given $a > 0$, $h_a = g_a f$ is a hyperbolic element only if $a > 4$. For $a = 4$, h_a is a parabolic element and for $a < 4$ it is elliptic. When $a > 4$, let $\alpha(a)$ be the unique real number $0 < \alpha < 1$ satisfying $\frac{(\alpha+1)^2}{\alpha} = \alpha + \frac{1}{\alpha} + 2 = a$.

Now we introduce certain building blocks for our construction of flat surfaces. These blocks are similar to the rectangular model of X_αs. Except for one vertical cylinder, f and g will respectively induce one twist on all the horizontal and vertical cylinders.

These building blocks, however, have only finitely many vertical and horizontal cylinders (see figure 20). We start with a square S_0, with sides of length 1, and we identify its two vertical sides to get a cylinder. Any point in S_0 is part of a vertical cylinder C_0. We may as well assume that the vertical side of S_0 on the left is part of the boundary of a rectangle $S_0' \subset S_0$ that forms a vertical cylinder C_0'. Suppose the distance between the two boundary components of C_0' is $0 < t < 1$. If $t = x_0^a = \frac{1}{a}$, then C_0' lies inside C_0 and it covers a strip of width x_0^a. Apparently for no value of t less than x_0^a, the construction can be proceeded so that f and g both define virtual multi-twists. For values of t larger than x_0^a, some parts of C_0' will be left out of C_0. Let y_0^a be the value of t for which $C_0' \setminus C_0$ is a square. One can easily see that in order for that to happen, we must have $ay_0^a - 1 = y_0^a$ or $y_0^a = \frac{1}{a-1}$. When $t = y_0^a$, C_0' is divided into two pieces: one is the intersection with C_0 and the other one is a square S_1 which can form a horizontal cylinder C_1. Moreover f defines one twist on C_1.

In the next step we define the values x_n^a and y_n^a of t inductively. If we cover all of $C_0' \setminus C_0$ with one square S_1, the proportion on the horizontal sides of S_1 that are not covered to the length of the horizontal sides is a number which only depends on t and a. We denote this number by $\Phi_a(t)$. Easy computation shows that:

$$\Phi_a(t) = \frac{(a-1)t - 1}{at - 1}.$$

Let $x_n^a = \Phi_a^{-1}(x_{n-1}^a)$ and $y_n^a = \Phi_a^{-1}(y_{n-1}^a)$. Then starting with $t = x_n^a$ we can get a building block $\mathfrak{X}_n^a$ as follows. $\mathfrak{X}_n^a$ consists of $n+1$ horizontal cylinders C_i, $0 \leq i \leq n$ of modulus 1 and $n+1$ vertical cylinders C_j', $0 \leq j \leq n$ of modulus $\frac{1}{a}$. Any vertical cylinder is covered with exactly

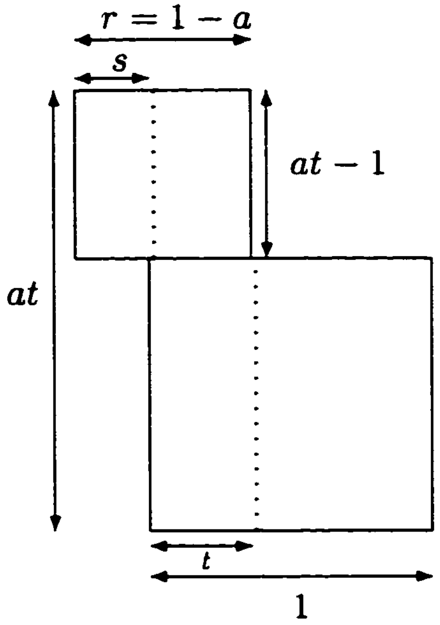


FIGURE 19. $\Phi_a(t) = \frac{s}{r} = \frac{(a-1)t-1}{at-1}$

two horizontal cylinders and any horizontal cylinder except for C_0 is covered with exactly two vertical cylinders. On the boundary of C_0 there are two horizontal segments of length $1 - t$, which can be used to glue $\mathfrak{X}_n^a$ to other pieces. We may also start with $t = y_n^a$ and get the building block $\mathfrak{Y}_n^a$. $\mathfrak{Y}_n^a$ consists of $n + 2$ horizontal cylinders C_i, $0 \leq i \leq n+1$ of modulus 1 and $n+1$ vertical cylinders C'_j, $0 \leq j \leq n$ of modulus $\frac{1}{a}$. Any vertical cylinder is covered with exactly two horizontal cylinders and any horizontal cylinder except for C_0 and C_{n+1} is covered with exactly two vertical cylinders. Again we leave the two horizontal segments of length $1 - t$ on the boundary of C_0 open to glue $\mathfrak{Y}_n^a$ to other pieces. f defines a well-defined automorphism on both $\mathfrak{X}_n^a$ and $\mathfrak{Y}_n^a$ which does one twist on each of the horizontal cylinders C_i. On the other hand g defines an automorphism on all parts of both $\mathfrak{X}_n^a$ and $\mathfrak{Y}_n^a$

except for $C_0'\setminus C_0$. It is not hard to see that $\beta(\alpha) = \frac{\alpha(a)}{\alpha(a)+1}$ is a repelling fixed point of Φ_a and

$$x_0^a < y_0^a < x_1^a < y_1^a < \dots < x_n^a < y_n^a \to \beta(a)$$

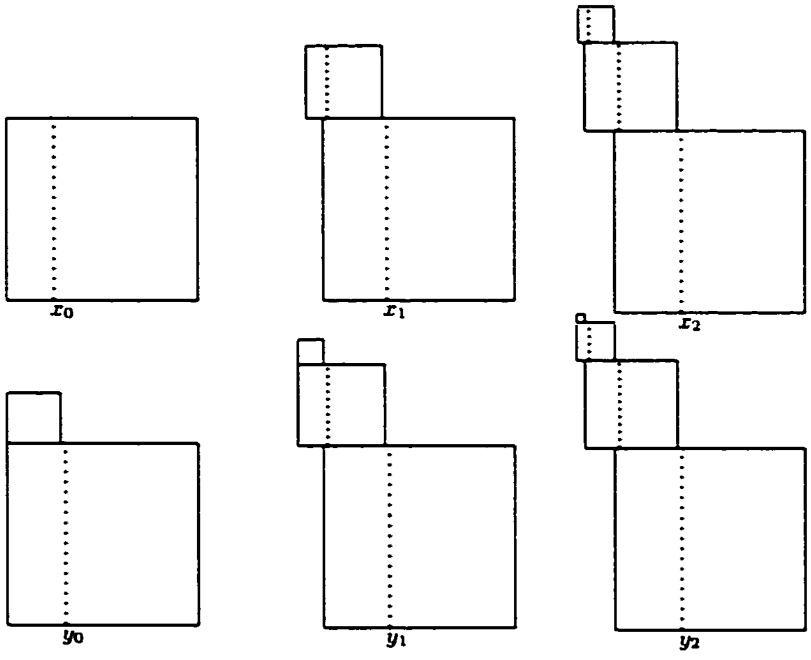


FIGURE 20. The building blocks $\mathfrak{X}_i^a$ and $\mathfrak{Y}_i^a$ for $0 \leq i \leq 2$

Choosing $t = \beta(a)$ one gets a building block $\mathfrak{X}_\infty^a$ with infinitely many vertical and horizontal cylinders. $\mathfrak{X}_\infty^a$ is isometric to an affine distortion of $X_{\alpha(a)}$ except that the two intervals of length $1 - \beta(a)$ on the horizontal boundary components of C_0 are not identified. Again we will use these two openings to glue this block to other pieces.

DEFINITION 6.1. *Provided that*

$$\sum_{i\in I} n_i x_i^a + \sum_{j\in J} m_j y_j^a = 1 \tag{6.2}$$

for finite index subsets I *and* J *of* $\mathbb{Z}\bigcup\{\infty\}$, *and fixing a cyclic order of indices in* $K = I \bigsqcup J$, *we define* X_a^K *to be the flat surface obtained in the following way. Starting with* n_i *copies of* $\mathfrak{X}_i^a$ *for all* $i \in I$ *and* m_i *copies of* $\mathfrak{Y}_j^a$ *for all* $j \in J$, *we identify the cylinder* C_0 *in all of them in such a way that their* C_0' *cylinders go around* C_0 *in the prescribed cyclic order on* K.

Proposition 6.3. *Assuming* $\infty \in K$, $\mathit{Aff}(X_a^K)$ *is the free group generated by* f *and* g. f *induces a multi-twist, twisting once around all the generators of* C_i*s.* g *induces a multi-twist, twisting once around all but finitely many of the generators of* C_i'*s. The element* $h = gf$ *represents a hyperbolic element with translation length* $\ln(\alpha(a))$. h *has a dense subset of saddle connections in both of its invariant directions.*

Proof. The proof is exactly the same as the proof of the Corollary 5.3. ☐

In fact the main objective in the construction of X_a^K is to cut $\hat{X}_\alpha$ to get one piece of infinite topological type with finite area and make changes in a part of finite type so that all the automorphisms of $\hat{X}_\alpha$ remain as automorphisms of the new surface. Not all the values of α, however, satisfy condition (7.2) for some K. In particular (7.2) is an algebraic condition and therefore for a non-algebraic number α, this construction never works. On the other hand for some values of α, there might be more than one pattern K (but always finitely many of them) for which (7.2) is satisfied. In what follows, we study some examples of this construction and we investigate some of the ways one can modify the construction to get other examples. In what follows we represent the pattern K by listing the corresponding $\mathfrak{X}_i^a$s and $\mathfrak{Y}_j^a$s in cyclic order.

Example 6.4. *The pattern* $(\mathfrak{X}_\infty, n\mathfrak{X}_0)$ *for* $n \geq 3$ *leads to the value of* $\frac{1}{n-1}$ *for* α *and the surface* $X_{\frac{1}{n-1}}$ *that we studied in the previous sections.*

Example 6.5. *The pattern* $(3\mathfrak{X}_\infty)$ *leads to the value of* $\frac{1}{2}$ *for* α *and the surface is a three sheeted covering of* $X_{\frac{1}{2}}$.

Example 6.6. *The pattern* $(\mathfrak{X}_\infty, 2\mathfrak{Y}_0)$ *leads to the equation*

$$\frac{\alpha}{\alpha+1} + \frac{2}{\frac{(\alpha+1)^2}{\alpha} - 1} = \beta(a) + \frac{2}{a-1} = 1$$

for the value of α. *The above equation leads to the polynomial equation* $\alpha^2 + \alpha - 1 = 0$ *which has only one root* $0 < \alpha < 1$. *This value of* α *is irrational and therefore this example is different from the examples that were studied in the previous sections.(see figure 21)*

Example 6.7. *The pattern* $(\mathfrak{X}_\infty, \mathfrak{X}_0, \mathfrak{Y}_0, \mathfrak{X}_0)$ *leads to the equation*

$$\frac{\alpha}{\alpha+1} + \frac{2\alpha}{(\alpha+1)^2} + \frac{1}{\frac{(\alpha+1)^2}{\alpha} - 1} = \beta(a) + \frac{2}{a} + \frac{1}{a-1} = 1$$

for the value of α. *The above equation leads to the polynomial equation* $2\alpha^3 + 2\alpha^2 + \alpha - 1 = 0$ *which has only one root* $0 < \alpha < 1$. *(see figure 22)*

Remark[1]: For $a \leq 4$, the value of $\beta(a)$ is larger than $\frac{1}{2}$. One can still define the patterns $\mathfrak{X}_i^a$ and $\mathfrak{Y}_j^a$ but $\mathfrak{X}_\infty^a$ has infinite area (see figure 23).

Remark[2]: In order for the automorphism group to be the free group generated by f and g, it is necessary to have the block $\mathfrak{X}_\infty$. As

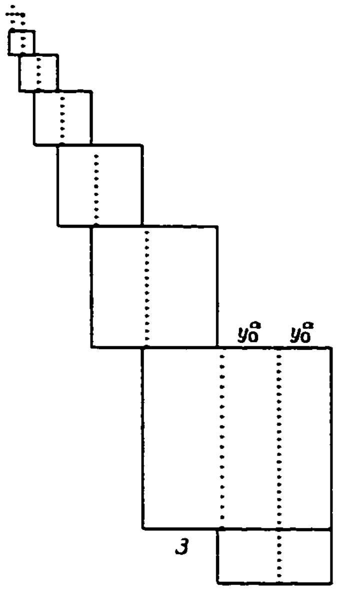


FIGURE 21. The flat surface obtained from the pattern $(\mathcal{X}_\infty, 2\mathcal{Y}_0)$. For this pattern α is the *golden ratio.*

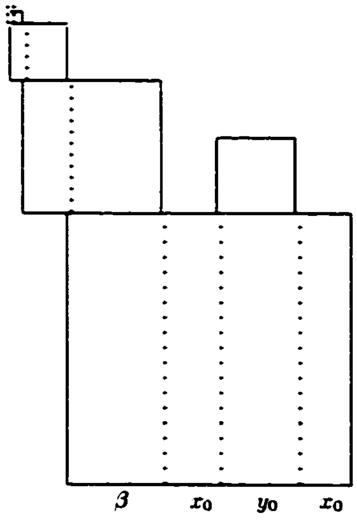


FIGURE 22. The flat surface obtained from the pattern $(\mathcal{X}_\infty, \mathcal{X}_0, \mathcal{Y}_0, \mathcal{X}_0)$. For this pattern α is the only root of the polynomial $2x^3 + 2x^2 + x - 1$.

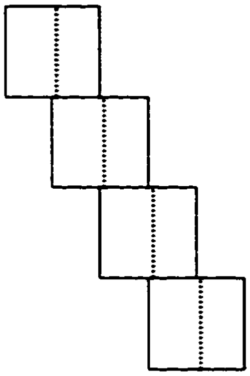


FIGURE 23. For $a = 4$, $\mathcal{X}^{\alpha}_{\infty}$ has infinite area.

an example the pattern $(n\mathcal{X}_0)$ gives rise to a flat torus whose automorphism group is a Fuchsian group of the first kind $PSL(2, \mathbb{Z})$ and therefore contains elements not inside $\langle f, g \rangle$. Using the Remark following Proposition 5.10 one can show that a pattern that does not contain $\mathcal{X}_{\infty}$ will give rise to a flat surface with automorphism group commensurable with $PSL(2, \mathbb{Z})$.

One can use combinations of the patterns discussed above to get more general examples. The following shows how this can be done.

EXAMPLE 6.8. *Figure 24 shows the flat surface representing the pattern* $((\mathcal{X}_{\infty}, \mathcal{X}_0), \mathcal{X}_0, \mathcal{Y}_0, \mathcal{X}_0, (\mathcal{Y}_0, \mathcal{X}_0, \mathcal{Y}_1))$. *The* α *for which the surface is realized is a root of the polynomial equation, obtained from the relation*

$$\Phi_{\alpha}^{-1}(\beta(\alpha) + x_0^{\alpha}) + 2x_0^{\alpha} + y_0^{\alpha} + \Phi_{\alpha}^{-1}(x_0^{\alpha} + y_0^{\alpha} + y_1^{\alpha}) = 1.$$

This polynomial has two roots between zero and one but for one of the two, $\Phi_{\alpha}^{-1}(x_0^{\alpha} + y_0^{\alpha} + y_1^{\alpha})$ *will be a negative number.*

QUESTION 6.9. *For what values of* α, *is it possible to find a flat surface with automorphism group conjugate to* $\langle f, g_{a(\alpha)} \rangle$*?*

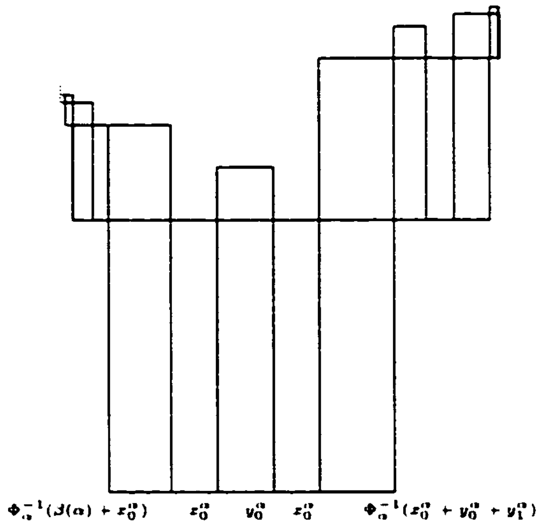


FIGURE 24. The flat surface that realizes the pattern $((\mathfrak{X}_\infty, \mathfrak{X}_0), \mathfrak{X}_0, \mathfrak{Y}_0, \mathfrak{X}_0, (\mathfrak{Y}_0, \mathfrak{X}_0, \mathfrak{Y}_1))$.

QUESTION 6.10. *Is it true that having an automorphism group conjugate to $\langle f, g_{a(\alpha)}\rangle$ for a flat surface is equivalent to the surface containing a copy of $\mathfrak{X}_\infty^a$?*

Hyperbolic automorphisms and thick tree maps It was mentioned in the beginning of this section that the hyperbolic automorphism Φ_2 of $X_{\frac{1}{2}}$, has saddle connections in both of its invariant directions. Φ_2 and its conjugates are the only hyperbolic automorphisms of $X_{\frac{1}{2}}$ with these properties. This fact along with the property that their fixed points are end points of maximal intervals of discontinuity for the action of the automorphism group, causes a distinction between these automorphisms and other hyperbolic automorphisms. X_α for $\alpha \neq \frac{1}{n}$ does not have any automorphism with these properties. Any of the

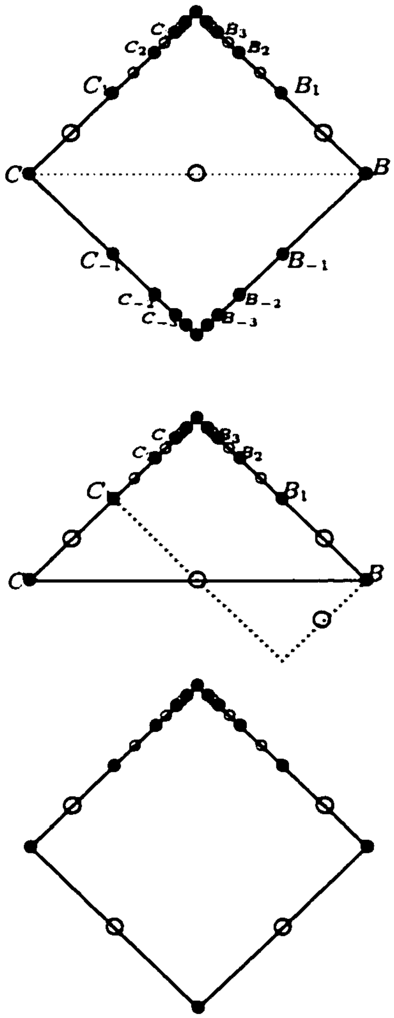


FIGURE 25. $X_{\frac{1}{2}}$ doubly covers a surface of genus zero with a sequence of points and their limit point removed which is demonstrated by the picture at the bottom. The small circle at the middle of the intervals indicate the identification of one half of the interval with the other half by a half-turn rotation. Φ_2 induces the full horse-shoe map.

examples made in this section with the free automorphism group generated by two parabolic automorphisms f and g has the hyperbolic automorphism $h = g \circ f$ with these properties.

X_α has a symmetry ι of order two which is in the kernel of the map Π. $Y_\alpha = X_\alpha/\langle\iota\rangle$ is a sphere with infinitely many punctures accumulating to an essential singularity. This double covering for $\alpha = \frac{1}{2}$ is shown in figure 25. Since ι commutes with all the automorphisms of X_α, Y_α has the same automorphism group Γ_α. In particular when $\alpha = \frac{1}{n}$, Y_α has a hyperbolic automorphism corresponding to Φ_n. The way any automorphism in Γ_α acts as a homeomorphism of Y_α is determined by the *braiding* of its action as a permutation of infinitely many punctures.

When $n = 2$, the action of Φ_2 is exactly the action of the *full horseshoe* map (see [3]). On a symmetric pattern, like the one in example 7.6, also the half-turn rotation induces an automorphism in the kernel of Π that commutes with all the automorphisms. Therefore the surface $(\mathcal{X}_\infty, \mathcal{X}_0, \mathcal{Y}_0, \mathcal{X}_0)$ doubly covers a surface of genus zero with infinitely many punctures. Therefore the map $h = g_\alpha \circ f$ induces a hyperbolic automorphism whose action is determined by the way it acts as an element of the infinite braid group on the sphere with infinitely many points removed. In all the cases, however, the braiding is concentrated in finitely many braids and the rest of the action is only a shift.

QUESTION 6.11. *In what way do the pairs of flat surfaces and their hyperbolic automorphisms constructed above correspond to the Riemann surfaces and generalized pseudo-Anosov maps obtained from thick tree maps (see* [3]*)?*

QUESTION 6.12. *Let $(X, \mathfrak{F})$ be a flat surface and h a hyperbolic automorphism of X. $d_T(Y, h(Y))$ when Y varies over all the surfaces in $\mathfrak{D}_{(X,\mathfrak{F})}$ is minimized for $Y = X$ exactly when X is on the axis of h. The surfaces on the axis of h minimize $d_T(Y, h(Y))$ when Y varies in the Teichmüller space of X. However it is not clear if they are the only surfaces minimizing $d_T(Y, h(Y))$. In other words, one can ask whether the surfaces on the axis of h are absolutely extremal (see* [2]*) for h.*

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