EVEN AND ODD GRAPH HOMOLOGY

(The commutative case)

by

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Abstract

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Adviser: Professor Dennis Sullivan

In this paper we prove that odd commutative graph complex is quasi isomorphic to its subcomplex spanned by loop-less graphs (which is much smaller in size), provided the differential contracts only non-loop edges. In the case that the differential is allowed to contract loops as well as non-loop edges, we show that odd commutative graph homology vanishes in all dimensions. We also define the notion of an apple tree complex, and by analyzing the spectral sequence or its geometric realization we show that every apple tree complex is acyclic. We also observe that this spectral sequence contains rooted Lie trees as either the bottom or the top row of its E_1-term. From this new approach we relate the geometric realization of the Lie operad to the geometric realization of certain poset which is known to have a homotopy type of a wedge of spheres of appropriate dimension. Out of this we deduce a new proof that the homology of Lie operad is concentrated in the top degree. From the acyclicity of apple tree complexes, as a corollary, we deduce that even and odd commutative graph complexes are quasi-isomorphic with their quotients modulo the sub-complex of exactly 1-connected graphs. Even and odd graph homologies are calculated in low dimensions; various notions of orientations and their relations are discussed.

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To my family and Heather.

Contents

Chapter 1

Graph Complexes

1.1 Introduction

Graph homology was introduced by M. Kontsevich in [1] and in [2]. He ingeniously reduced the problem of the computing homology of three infinite dimensional Lie algebras into the homology of some smaller "invariant" subcomplexes. Then using classical invariant theory, he interpreted the generators of these complexes as graphs with certain orientations (and decorations). From this the term graph complex and graph homology/cohomology was coined. These three infinite dimensional Lie algebras are actually certain Lie subalgebras of the Lie algebra of derivations of three types of free algebras which are: "Lie", "Associate", and "Commutative". Using work of Culler-Vogtmann [7] he showed that the "Lie" graph complex, corresponding to the "Lie" case, calculates the homology of outer automorphisms group of free groups. Using the work of R. Penner [28], Strebel [34], Harer [29], Mumford, and Thurston on combinatorial models for the moduli space of Riemann surfaces he showed that the "associative" graph complex , which is also called ribbon or fat graph complex, calculates the rational homology of moduli space of Riemann surfaces. The "commutative" graph complex gives invariants of odd dimensional manifolds.

Kontsevich's graph complexex have been generalized to the contexts of cyclic operads by Ginzburg-Kapranov [16] and by M. Markl [14], and to the context of modular operads by Getzler-Kapranov [17]. Getzler-Kapranov have a functor which they called Feynman transform from any differential graded algebra into graph complexes

Graph complexes (if one allows the graph to be disconnected) have a Hopf algebra structure. The product is the disjoint union of graphs, and the coproduct is defined in the standard way so that the connected graphs are exactly the primitive elements. There is an implicit Lie algebra and Lie coalgebra structure on graph complexes, as they are isomorphic to the invariant part of the Chevalley-Eilenberg complex of the aforementioned Lie algebras, under the action of symplectic Lie algebra, $sp(2n)$, which posses those structures. Conant and Vogtmann defined these structures on graph complexes explicitly and further showed they are not compatible, in the sense that they do not define a Lie bialgebra structure on these graph complexes. However they showed (as it can be easily seen) they are compatible on the smaller subcomplex spanned by 1-particle graphs (graphs without an edge whose removal causes the graph to become disconnected). They further showed that in the Lie and associative case the inclusion of this subcomplex into the big complex induces an isomorphism in homology. They showed that this however is not true in the commutative case. We show here that in the commutative case the graph homology can be calculated modulo the subcomplex spanned by graphs with a separating vertex (a vertex that its removal disconnects the graph). That is the map from commutative graph complex to commutative graph complex mod this subcomplex, induces an isomorphism in homology. Conant, Gerlits and Vogtmann have also independently proved this [9].

In this thesis we present a few other facts about the commutative graph complex in both the even and odd case. The list of the new results of this thesis is contained in the abstract and therefore we do not repeat them here.

1.2 Graphs

Definition 1.2.1. By a graph we mean a finite (connected) 1-dimensional CW complex Γ. A 0-cell is called a vertex, and a 1-cell is called an edge. An edge that is attached only to one vertex is called loop, otherwise it is called a non-loop edge. The set of vertices (0-cells), edges (1-cells), and loop-edges are respectively denoted by $V(\Gamma)$, $E(\Gamma)$, and $L(\Gamma)$. Note that $E(\Gamma) \supseteq L(\Gamma)$. We use $|V(\Gamma)|$, $|E(\Gamma)|$, and $|L(\Gamma)|$ for the cardinality of these sets. The valence of a vertex v is the number of half-edges incident to v. We call $|E(\Gamma)| - |V(\Gamma)| + 1$, the **rank** of Γ, which is the same thing as the number of circuits (or first Betti number) in the graph or equivalently

$$rank(\Gamma) := n = \dim H_1(\Gamma, \mathbb{Q}).$$

Alternatively, one can use (the so-called) half-edge language. A graph Γ is a set $F(\Gamma)$ of half-edges, consisting of $2n$ elements, along with two partitions of this set into disjoint unions of subsets. The first partition, $E(\Gamma)$, is called the set of edges and consists of n two-elements subsets of $F(\Gamma)$. The second partition, $V(\Gamma)$, is called the set of vertices and consists of arbitrary non-empty sub-sets; cardinality of each subset is called the valence of that vertex. An edge is called loop if both of its half-edges belong to one vertex.

An automorphism of the graph Γ is an arbitrary permutation on the set $F(\Gamma)$, that respects the structures of both $V(\Gamma)$ and $E(\Gamma)$. We use $Aut(\Gamma)$ to denote the group of automorphisms of Γ.

Convention: in this note by a graph we mean an isomorphism class of graphs, i.e., we identify two graphs that are isomorphic. So for example by the phrase " vector space generated by graphs with such and such property" we mean "vector space generated by isomorphism classes of graphs with such and such property". We still sometime redundantly continue to use the term "isomorphism class".

1.3 Orientations

Given a set of n elements $X = \{x_1, ..., x_n\}$, by a labelling of it we mean any linear ordering of its elements. There are $n!$ such labellings. The symmetric group on n elements, Σ_n, acts on this set of labellings by reordering the linear order. Let us put an equivalence relation on the set of labellings, L, by calling two labellings equivalent if they differ by an even permutation.

We define an orientation for X, with $|X| \geq 2$, to be one of the two orbits or cosets in L/A_n, where A_n is the group of even permutations on n elements. For a set with one element, there is only one labelling; therefore we define the orientation to be either 1 or -1. Equivalently an orientation for the set X can be defined to be a choice of an orientation for the vector space $\mathbb{Q}\{X\}$, generated by elements of X over the field of rational numbers. An orientation for the latter vector space is usually expressed as the choosing of one of the two canonical basis for the one dimensional vector space $\bigwedge^n \mathbb{Q}\{X\}$ (i.e. $1 = x_1 \wedge x_2 ... \wedge x_n$ or $-1 = -x_1 \wedge x_2 ... \wedge x_n$). Note that any 1-1 map $f : X \to X$ induces a map on $\bigwedge^n \mathbb{Q}\{X\}$ which is either identity or minus identity, and in the later case we call f orientation reversing.

Also note that an *or* for X induces an *or* on its 1-element smaller subsets as follows: if $x_1 \wedge x_2 ... \wedge x_n$ is the *or* for the set X, the induced *or* on X/x_i would be $(-1)^{n-i} x_1 \wedge x_2 ... \widehat{x_i} ... \wedge x_n$ when $\widehat{x_i}$ means x_i is deleted. Similarly it induces an *or* on its 1-element supersets. The induced *or* on $X \cup \{x_{n+1}\}$ would be $x_1 \wedge x_2 ... \wedge x_n \wedge x_{n+1}$

Given a graph Γ we can have the following notions of orientation:

*V: an *or* for the set of its vertices (or C_0 for the vector space of 0-chains).
*V_o: an *or* for the set of its odd valence vertices.
* V_e: an *or* for the set of its even valence vertices.
*E : an *or* for the set of its edges.
*C: choosing an *or* for set of edges coming to each vertex up to negating any two.
* H_1: choosing an *or* for the vector space $H_1(\Gamma, \mathbb{Q})$.
* A: choosing an arrow (a direction) on each edge up to reversing any two.
*F: an *or* for the set of half edges or flags.
*C_1: an *or* for the vector space of 1-chains.

Proposition 1.3.1. *With the tensor product of orientation the above set of orientations form an abelian group with exponent 2, (i.e., each element has order 2) satisfying the following relations:*

$V_e \otimes V_o = V;\qquad A = F;\qquad C = V_o \otimes F$

$C_1 = E \otimes A;\qquad H_1 = V \otimes C_1 = (C_0 \otimes C_1)$

Proof. • $V_e \otimes V_o = V$ is trivial.

• $C_1 = E \otimes A$: C_1 is simply choosing a labelling for all oriented edges such that if one switches the labels of two oriented edges and keep everything else fixed or if one changes the direction of an arrow on one edge and keep all other arrows and labels fixed, the orientation changes. $E \otimes A$ has exactly the same meaning.

• $A = F$: any labelling of flags induces a linear order on the two flags of each edge and determines an arrow on the edge by assuming the arrow is pointed toward the bigger label; if we switch the labels of the two flags of an edge that inflicts a minus sign on both F and A. Vice versa with any choosing of arrows on the edges, and any labelling of edges $e_1, e_2, ..., e_n$ we will get a labelling for the flags as follows: to the tail flag of the edge that has label 1 assign 1 and to its head flag assign $n+1$; to the edge labeled by 2 assign 2 to the tail and $n+2$ to the head, and so forth. This labelling of flags up to even permutation is independent of the labelling of edges. For example if we change labels 1 and 2 of e_1 and e_2, it causes the changing of ordered pairs $(1, n)$ and $(2, n+1)$, because this is an even permutation it does not change F. Now if one changes an arrow on an edge e_k this causes the linear order in $(k, k+n)$ to reverse ,i.e., to $(k+n, k)$, which inflicts a minus sign in F. Thus the two notions are equivalent.

• $C = V_o \otimes F$: let us rewrite C in $\bigwedge$ language. Label the vertices of Γ by $1, 2, ..., k$ and let F_i be the vector space generated by the set of flags (half edges) incident to vertex i. Set $dim(F_i) = n_i$ = valence of i-th vertex. Then an *or* for this set would be an *or* for 1-dimensional space $\bigwedge^{n_i} F_i$, and a C orientation would be an *or* for $\bigwedge^{n_1} F_1 \otimes \bigwedge^{n_2} F_2 \otimes ... \otimes \bigwedge^{n_k} F_k$. We can see that if n_i is even then $\bigwedge^{n_i} F_i$ is an even tensor which commutes with other tensors without inflicting a sign. Thus this orientation does not depend on labellings of even vertices, proving our assertion.

- $H_1 = C_0 \otimes C_1$: this states that an orientation on first homology of the graph is equivalent to the product of the orientations for 0-chains and 1-chains. We will give a slightly intuitive arguments here. Any graph is homotopy equivalent to a bouquet of circles and this statement is clear for a bouquet of circles, as all it says is that orientation of first homology is the same thing as putting an arrow on each circle and labelling them, up to total even number of changes of labels and arrows. Any graph can be obtained from a bouquet of circles by a series of non-loop edge insertion. Equivalently (in a dual picture) any graph can be mapped to bouquet of circles by a series of non-loop edge contraction as shown below.

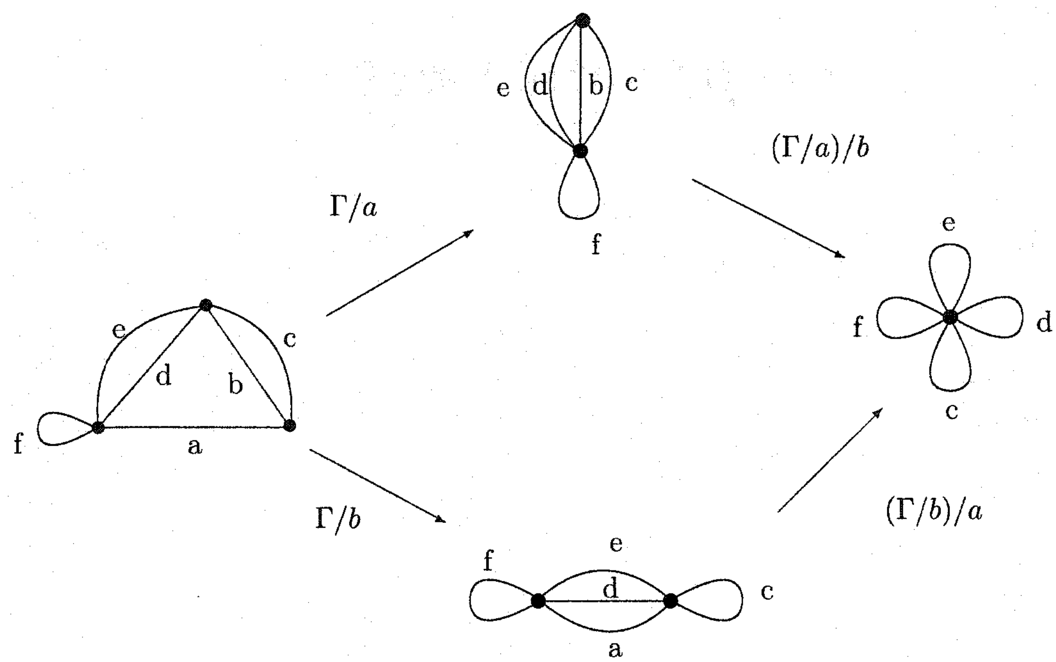


Figure 1. Every graph can be mapped to a bouquet of circles with the same homotopy type by a series of non-loop edge contraction.

Now observe that:

$$C_0/v \otimes C_1/e = H_1/e \;\Leftrightarrow\; C_0 \otimes C_1 = H_1$$

when C_0 is the canonical orientation obtained from C_0/v, i.e., if $v_1 \wedge v_2 ... \wedge v_k$ is the *or* for C_0/v then $v_1 \wedge v_2 ... \wedge v_k \wedge v$ is the induced *or* on C_0 and likewise C_1/e induces an *or* on C_1; $H_1/e = H_1$. In other words suppose vertices are labeled by $1, .., k$ and edges by $1, ..., n$. We blow up a vertex, say v_m (with label m), create two new vertices out of it and insert a new edge, label one of the new vertices by m and the other by $k+1$, and the new edge by $n+1$ and orient the new edge towards the vertex with the higher labelling. It is clear that it does not matter which one of the two is given the new labelling, the $C_0 \otimes C_1$ is the same in either case and is independent of the choice. Thus H_1 orientation induces an *or* on C_0 and C_1 such that $C_0 \otimes C_1$ is well defined. So let us recapitulate the proof:

1-$H_1(\Gamma, \mathbb{Q}) \longmapsto H_1(\Gamma/e_1, \mathbb{Q}) \longmapsto H_1(\Gamma/(e_1, e_2), \mathbb{Q}) \longmapsto ...$ when $\longmapsto$ means that there is a canonical isomorphism of the two vector spaces that of course induces a canonical isomorphism on orientations.

2-$C_0 \otimes C_1 \longmapsto C_0/e_1 \otimes C_1/e_1 \longmapsto C_0/(e_1, e_2) \otimes C_1/(e_1, e_2) \longmapsto ...$

Here by C_0/e_1 means the orientation induced on the set of vertices out of collapsing the edge e_1, and so on.

3- It is clear that $H_1 = C_0 \otimes C_1$ for a bouquet of circles

□

Proposition 1.3.2. $E \otimes H_1 = V \otimes A$

Proof. By looking at the relation above we see $H_1 = C_1 \otimes V$ and $C_1 = E \otimes A \Rightarrow H_1 = (E \otimes A) \otimes V$. Now let us multiply both side by E, we get $E \otimes H_1 = E \otimes E \otimes A \otimes V$, but $E^2 = 1$ and this proves the assertion.

□

It is often easier to work with the latter when doing actual calculations (when one needs to know if a given graph has an orientation reversing automorphism), for the obvious reason that it is easier to work with arrows than labelling and orienting loops and keeping track of their changes under an automorphism.

Proposition 1.3.3. *For a graph which is trivalent at each vertex, C is the same as choosing a cyclic ordering at each vertex up to reversing any two. That is for such a graph $C = V \otimes A = E \otimes H_1$.*

Proof. In general for a vertex of valence n there are $(n-1)!$ cyclic orderings for the set of its incident edges, while there are only two *or* for this set, so they are not the same thing. For $n = 3$ however $(3-1)! = 2$, and the two cyclic ordering if we label the three incoming edges by 1,2,3 will be $(123) = (231) = (312)$ and $(132) = (321) = (213)$; linear ordering (231) is obtained from linear ordering (123) by an even permutation so they are in the same *or* class, while linear ordering(132) is obtained by an odd permutation from linear ordering (123) and so they are in different *or* classes. Thus any cyclic ordering is the same thing as an *or* class. This explains the first part of the statement. For the second part from above we know $C = V_o \otimes F$ and $F = A$, for a pure trivalent graph $V_o = V$ thus proving $C = V \otimes A = E \otimes H_1$. □

We will use this fact in 7.0.56, when constructing non-trivial cycles in the top degree of the even graph complex with rank n.

Definition 1.3.1. Even and odd orientations. Orientation of a graph Γ for us in the rest of this manuscript will almost always be either E or $E \bigotimes H_1$. We call $E \bigotimes H_1$ and E respectively **"even"** and **"odd"** orientation following M. Kontsevich. Also as shown above in 1.3.2 the $E \bigotimes H_1$ is the same as $V \bigotimes A$.

Let us also mention that $or(\Gamma)$ can be thought of as a $\mathbb{Z}_2$ module. $Aut(\Gamma)$ acts on this module, and if the action is non-trivial, we call the graph, Γ, non-orientable.

1.4 Complexes of Graphs

Suppose we have a set of isomorphism classes of oriented graphs, such that if class (Γ, or) is in the set, then so is $(\Gamma, -or)$ and $(\Gamma/e, or/e)$ for any edge (and/or loop) e in Γ, where by Γ/e we mean graph obtained from Γ by collapsing edge e and by or/e we mean the orientation that is induced from Γ on Γ/e. Furthermore assume the orientation for graphs in this set is defined in such a way that the two different ways of inducing an orientation from Γ on $\Gamma/\{e_i, e_j\}$ leads to opposite orientations, i.e.,

$$(or/e_i)/e_j = -(or/e_j)/e_i. \tag{1.4.1}$$

Then by graph complex, GC, generated on this set we mean the vector space generated on this set over $\mathbb{Q}$ ($\mathbb{Q}$ could be replaced with $\mathbb{R}$,$\mathbb{C}$, $\mathbb{Z}_2$,... depending on the need) and with the imposed relation of

$$(\Gamma, -or) = -(\Gamma, or) \tag{1.4.2}$$

and with the differential d defined as the following:

$$d(\Gamma, or) := \sum_{e_i \in \Gamma} (\Gamma/e_i, or/e_i).$$

Proposition 1.4.1. $d^2 = 0$.

Proof. Note that $d^2(\Gamma, or) = \sum_{(e_i,e_j)\in\Gamma}((\Gamma/e_i)/e_j, (or/e_i)/e_j)$, where the sum is over all ordered pair of edges of Γ, and $(\Gamma/e_i)/e_j$ means first collapsing e_i and then e_j. We have $(\Gamma/e_i)/e_j = (\Gamma/e_j)/e_i$ as a combinatorial graph, however by choosing of the orientation we have $(or/e_i)/e_j = -(or/e_j)/e_i$; therefore the sum on the right cancels out in pairs. □

Remark 1.4.2. *Note that the relation (1.2) kills all non-orientable classes, for if* (Γ, or) *is representative of one such class that has an isomorphism that reverses the orientation then by definition (of isomorphism class) we must have* $(\Gamma, or) = (\Gamma, -or)$ *and by the (1.2) this is equal to* $-(\Gamma, or)$*, thus yielding* $2(\Gamma, or) = 0$*; consequently over a field with characteristic not equal to 2 we have* $(\Gamma, or) = 0$.
For example with the even orientation, $A \otimes V$*, any graph that has a loop is zero, because such a graph has an automorphism that flips the loop (which is considered non-trivial), and fixes the rest of the graph, thus changing the arrow on the loop which changes the orientation. Similarly with the odd orientation any graph that has multiple edges (between two vertices), or multiple loops at a vertex, is zero. For the automorphism that switches two such multiple edges, or multiple loops, changes the orientation. For this reason the differential never creates new loops, because creating new loop is possible only out of collapsing a non-loop edge that belongs to a group of multiple edges; but such a graph is already zero.*

Over $\mathbb{Z}_2$ the orientation does not play a role and $d^2 = 0$ is satisfied automatically.

Proposition 1.4.3. *Orientations* E *and* $A \otimes V$ *defined above satisfy the 1.4.1 property.*

Proof. E: Suppose the edges of (Γ, or) are labelled by $1, 2, ..., n$ and edge a to be contracted to get $(\Gamma/a, or/a)$, which is a term in $d(\Gamma, or)$. By permuting an even permutation on the labels (if we have more than 3 edges, otherwise using a transition and a sign) we make n the label of edge a. Now if we want to contract two edges in two steps (as in d^2), first edge a then edge b to get $(\Gamma/a)/b$, we make their labelling n and $n-1$ respectively. On the other hand to get $(\Gamma/b)/a$ by contracting b first and then a, we have to make their labelling n and $n-1$ respectively, which can be obtained from the same configuration of labels by the transition $(n-1, n)$ (to switch the labels of a and b). The transition $(n-1, n)$ is an odd permutation, thus proving (*).

$A \otimes V = E \otimes H_1$: H_1 orientation carries along when one contracts non-loop edges. This means for H_1 orientation we have $(or/e_i)/e_j = (or/e_j)/e_i$. The E orientation however satisfies 1.4.1. Consequently $E \otimes H_1$ in whole satisfies 1.4.1. □

Definition 1.4.1. Graph homology by definition is the homology of this complex, i.e.:

$$H_*(GC) := \frac{\ker d}{\text{im } d}$$

Remark 1.4.4. *If we let our differential collapse only non-loop edges, i.e., $d = d_E$, which is the case unless otherwise stated, then one can grade GC by the number of vertices $|V|$, or by the number of edges $|E|$, or both $(|V|, |E|)$ to get a bigrading. However if we also allow collapsing loops, then $|V|$ is no longer a proper grading, but the number of loops $|L|$, or total number of non-loop edges and loops, or both $(|E|, |L|)$ would be an appropriate grading/bigrading. Any grading of GC would also induce a grading on $H_*(GC)$. The grading for the rest of this paper is going to be by the number of the vertices, unless otherwise stated.*

Definition 1.4.2. Even (resp. odd) $GC_{\succeq 1}$ is the complex spanned by isomorphism classes of all connected oriented graphs with even (resp. odd) orientation.

Definition 1.4.3. $GC_{\succeq 2}$ is complex of isomorphism classes of all connected oriented graphs that have valence at least 2 at each vertex.

Remark 1.4.5. *Let the orientation of graphs in each complex discussed below be either the odd or the even. If we allow only collapsing non-loop edges, then $d = d_E$*

preserves the rank of the graphs. This means that we have different graph complexes corresponding to different ranks. Now note that for a graph Γ of rank n we have:

$$|V(\Gamma)| - |E(\Gamma)| = \chi(\Gamma) = 1 - dimH_1(\Gamma) = 1 - \beta_1(\Gamma) = 1 - n. \quad (1.4.3)$$

If we assume our graphs are at least trivalent then we will have $3|V(\Gamma)| \leq 2|E(\Gamma)|$. We will get:

$$|V(\Gamma)| \leq -3\chi(\Gamma) = 2n - 2 \quad (1.4.4)$$

$$|E(\Gamma)| \leq -2\chi(\Gamma) = 3n - 3 \quad (1.4.5)$$

These two inequalities show that our graph complexes are finitely generated and at the same time also provide us with the highest grading(s).

Definition 1.4.4. GC^1 is the complex generated by oriented graphs of rank one (or equivalently $|V(\Gamma)| - |E(\Gamma)| = 0$), and with no univalent vertex. The only such graphs are n-gons. They have valence 2 at each vertex and for this reason we also sometimes denote this complex by GC_2.

Definition 1.4.5. • $GC^n_{\geq 1}$: is the complex spanned by isomorphism classes of oriented connected graphs of rank n.

• $GC^n_{\geq 2}$: is the complex spanned by isomorphism classes of oriented connected graphs of rank n, with no univalent vertices

Definition 1.4.6. GC^n for $n \geq 2$ is the complex spanned by isomorphism classes of connected oriented graphs of rank n, which have valence three or more at each vertex.

This means that GC^n will be of the following form:

$$0 \longrightarrow GC^{2n-2,3n-3} \xrightarrow{d} GC^{2n-3,3n-4} \xrightarrow{d} \dots \longrightarrow GC^{1,n} \longrightarrow 0$$

where $GC^{i,j}$ is the subspace of GC^n generated by graphs that have i vertices and j edges. However in the rest of this writing we prefer to use only one grading (by the number of the vertices).

Remark 1.4.6. *In [3] Kontsevich claims and gives evidence that in the above complexes with the odd orientation, showing the triviality of the homology in spot $(n+2, 2n+1)$ for all n, would imply* ***Formality Conjecture.***

In 3.0.27 we show that in "odd" case $H_k(GC^n) = 0$, for $n \gg k$.

Definition 1.4.7. $GC_{\geq 3} := \bigoplus_{n=2} GC^n$,i.e., it is the complex spanned by isomorphism classes of connected oriented graphs that have valence three or higher at each vertex.

Remark 1.4.7. *$Aut(\Gamma)$ acts on $E(\Gamma) = \{e_1, ..., e_n\}$, and thus induces an equivalence relation (partition) on this set; namely e_i and e_j are in the same equivalence class if there exists an element σ in $Aut(\Gamma)$ such that $\sigma(e_i) = e_j$. Call these equivalence classes by X_1,..., X_k and their cardinality by m_1,...,m_k respectively, and let e_{i_1},..., e_{i_k} be an arbitrary representative of each class respectively. Then $m_j = \frac{|Aut(\Gamma)|}{|Aut_j(\Gamma)|}$, where $Aut_j(\Gamma)$ is the subgroup of $Aut(\Gamma)$ that stabilizes e_{i_j}. Cardinality of $Aut_j(\Gamma)$ does not depend on the choice of the representative for X_j, as the stabilizers of any two such representatives are two conjugates subgroups of $Aut(\Gamma)$. If (Γ, or) is a non-zero or equivalently orientable class, one can then write:*

$$d(\Gamma, or) = m_1.(\Gamma/e_{i_1}, or/e_{i_1}) + \dots + m_k.(\Gamma/e_{i_k}, or/e_{i_k})$$

This observation facilitates our computation in subsequent chapters, for instead of being required to examine the act of differential on each edge individually, we only need to examine it on each orbit class of edges (under the action of group of automorphisms of the graph).

Chapter 2

Homologies of $GC_{\succeq 2}$ and $GC_{\succeq 1}$

The following fact is mentioned in [1], with only a (vague) outline of a proof. It holds in both the even and odd case.

Proposition 2.0.8.

$$H_*(GC_{\succeq 2}) = H_*(GC_{\succeq 3}) \oplus H_*(GC_2) = \bigoplus_{n=1}^{\infty} H_*(GC^n)$$

Proof. For any graph Γ in $GC_{\succeq 2}$ one can write: $d(\Gamma) = d_2(\Gamma) + d_3(\Gamma)$, where d_3 collapses only those edges, which both of their incident vertices have valence three or more; while d_2 collapses those edges which one or both of their incident vertices have valence two. This decomposition of d leads to the spectral sequence in figure [2] for $(GC_{\succeq 2}, d)$, where v is the number of vertices with valence three or higher and b is the number of bivalent vertices.

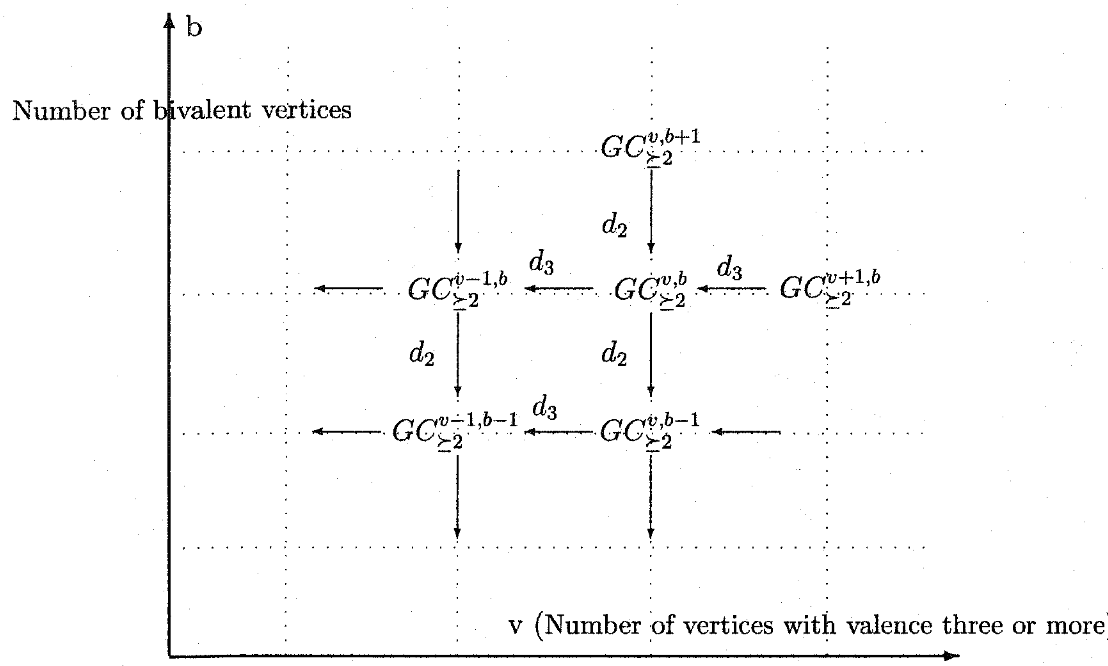


Figure 2.

In the above spectral sequence $GC_{\succeq 2}^{v,b}$ is the vector space generated by oriented graphs with v vertices of valence three or more and b bivalent vertices. Note that the row zero is nothing but $GC_{\succeq_3}$, which is a subcomplex of $GC_{\succeq 2}$. Similarly the zero column (at nodes $(0,b)$) is generated by pure bivalent iso. classes of graphs and there is only one such class, namely the class represented by an n-gon, at each node. We will show that except the zero column all other columns are acyclic in positive b gradings (i.e., the only possible non-zero homology of each column lives in the node $(v,0)$ of the column).

Consider column v: any graph in node (v,b) is obtained by planting b bivalent vertices on the edges of a graph living in node $(v,0)$. This means that column v can be decomposed into the direct sum of finitely many subcomplexes over the finite set of at least trivalent graphs with v vertices. Let C_Γ denote the subcomplex obtained by planting bivalent vertices on the edges of one such graph, Γ.

Lemma 2.0.9. *$H_k(C_\Gamma, d_2) = 0$ if $k > 0$.*

Proof. First consider two vertices A and B on the graph Γ and start adding vertices on the edge connecting A and B as depicted below.

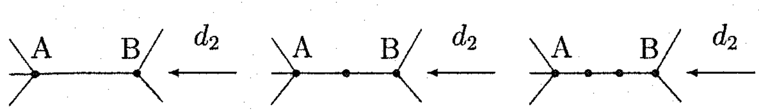


Figure 3.

Let the grading be the number of bivalent vertices on AB. As H_1 part in the "even" ($E \otimes H_1$) orientation, plays no role here (or carries along); in both the "even" and "odd" case E (labelling of the edges up to even permutation) is the orientation. This way we get a complex, (C, d_2), which is 1-dimensional at each grading $i \geq 0$, generated by G_i, where G_i is the graph obtained from AB by planting i bivalent vertices. (C, d_2) is acyclic in all positive gradings as the following computation shows:

$$
\begin{aligned}
&d(G_0) = 0 && \\
&d(G_2) = G_1 && d(G_1) = 0 \\
&d(G_4) = G_3 && d(G_3) = 0 \\
&\dots &&
\end{aligned}
$$

Now $(C_\Gamma, d_2) \cong (C \otimes ... \otimes C)/Aut(\Gamma)$, where the tensor product is over the set of the edges of Γ. By Kunneth formula and the fact that with coefficients in $\mathbb{Q}$, finite groups have no homology, we have :

$$
\begin{aligned}
H_*((C_\Gamma, d_2), \mathbb{Q}) &= H_*((C \otimes ... \otimes C)/Aut(\Gamma), \mathbb{Q}) \\
&= [H_*(C \otimes ... \otimes C, \mathbb{Q})]^{Aut(\Gamma)} \\
&= [H_*(C, \mathbb{Q}) \otimes ... H_*(C, \mathbb{Q})]^{Aut(\Gamma)}
\end{aligned}
$$

thus proving $H_n((C_\Gamma, d_2), \mathbb{Q}) = 0$ for $n > 0$, and $H_0((C_\Gamma, d_2), \mathbb{Q}) = \mathbb{Q} < \Gamma >$.

□

Therefore except the zero column all others are acyclic. The E_1 terms of the big complex is row zero plus homology of column zero. Hence the E_2 terms of the complex is homology of row zero and homology of column zero. The E_∞ terms also will be exactly the same as E_2 terms, as higher differentials of the spectral sequence are all zero. This completes the proof. □

Remark 2.0.10. *This proof not only holds for both the even and odd (commutative) graph complexes, but also holds in the case of ribbon graph homology. A ribbon or a fat graph is a graph with an additional fixed cyclic ordering at each vertex. For a* *bivalent vertex there is only one such cyclic ordering, so it does not impose any new constraint, and the same proof holds. This fact, that graphs with bivalent vertices do not contribute to the homology, also holds for the case of complexes of rooted leaf-labeled trees, ordinary non-decorated and non-rooted trees, and apple trees which will be introduced later in this manuscript.*

Now let us examine $GC_{\succeq 1}$, the connected graph complex. One can write $GC_{\succeq 1} = \bigoplus_{n=0}^{\infty} GC_{\succeq 1}^{n}$ where $GC_{\succeq 1}^{i}$ is the subcomplex of $GC_{\succeq 1}$ generated by oriented graphs of rank i.

Proposition 2.0.11. *$GC_{\succeq 1}$ is acyclic, i.e., $H_i(GC_{\succeq 1}) = 0$ for $i \geq 0$.*

Proof. First note that $GC_{\succeq 1}^{0}$ is nothing but tree complex. This sub-complex is acyclic for the following reason: with an argument like the one used in the previous theorem we may assume that there are no bivalent vertices. The only difference in such an argument would be that one or both of the two end vertices, A and B, might be univalent here; we do not let any of the two to be bivalent however. Therefore we may assume all vertices are either univalent or at least trivalent. Now we notice that in any tree that has at least two edges and no bivalent vertex, there are always two exterior edges with a common vertex. There is an automorphism of the tree that interchanges these two exterior edges and fixes the rest of the tree. This automorphism reverses the orientation, for it changes the labelling of the edges (or vertices) by an odd permutation. Thus any such tree is zero in the complex. If the tree has less than than two edges then it is either a tree with a single vertex and no edge, or it is a tree with two vertices and one edge connecting the two. The differential sends the latter tree to the former by collapsing the edge, therefore there is no non-trivial homology here either. This proves the assertion for $GC_{\succeq 1}^{0}$.

For $GC^n_{\geq 1}$, $n \geq 1$ we get a spectral sequence in the first quadrant. In the row zero at the spot $(i, 0)$ the vector space is generated by graphs with no univalent vertices and with i vertices of valence two or more. The column L_i above this spot is generated by graphs, which are formed by planting trees at the vertices of graphs lying in this spot. Any vertex of the tree and not just the exterior or univalent ones, could be fused or glued with the vertices of the graph. The vertical grading on the column L_i is by the sum of the numbers of edges of all these planted trees. To rephrase this, over any graph Γ in spot $(i, 0)$ there lies a subcomplex which is isomorphic to $(GC^0_{\geq 1} \bigotimes ... \bigotimes GC^0_{\geq 1})/Aut(\Gamma)$. Now by the fact that $GC^0_{\geq 1}$ is acyclic and the same argument as the one in the previous proposition, the assertion follows. □

Chapter 3

Complexes of apple trees

Definition and Construction 3.0.12. *Let X be a category. Let $A = \{a_1, ..., a_n\}$ be a set of n, $n \geq 2$, objects in this category. We call A a set of apples. Let (T_1, or) be a connected graph of rank 0 (i.e., a connected tree) equipped with a labelling of its edges by numbers 1,2,..., $|E(T_1)|$ up to even permutations. If there exists a map, $f : A \to V(T_1)$, with the property that every vertex in T_1 with valence less than three is in the image of f (other vertices may or may not be in the image), then we call (A, f, T_1, or) an ordered oriented apple tree. There is an action of $Aut(T_1)$ on the tree, T_1, and its set of vertices, $V(T_1)$. Therefore there is an action of $Aut(T_1)$ on the set of (A, f, T_1, or)'s (by changing f). Denote the set of the equivalence classes of this action by $A(T_1, or)$. Define $A(T_1, or)$ to be the empty set if there is no quadruple with the above property (if the tree T_1 has more than n vertices with valence less than three then there is no such quadruple). For a fixed A, define AT' to be the vector space spanned by the union of all these sets (over the set of isomorphism classes of connected oriented trees). Let AT be the quotient of AT' by the subspace spanned by the relations, $(A, f, T_1, -or) = -(A, f, T_1, or)$. Let the grading on AT be by the* *number of the edges of the underlying tree. The differential d is defined by*

$$d(A, f, T_1, or) = \sum_{e_i} (A, f/e_i, T_1/e_i, or/e_i)$$

where the sum is over the set of the edges of T_1, *and* f/e_i *is the induced map from* A *to* $V(T_1/e_i)$ *obtained from* f *in the following natural way.* f/e_i *maps the union of the preimages of the two vertices that were fused together (as the result of the contraction of* e_i*), to the newly formed vertex. The preimage of any other vertex under* f/e_i *is the same as under* f*. Thus we get a complex of the following form:*

$$0 \longrightarrow AT_{2n-3} \xrightarrow{d} AT_{2n-4} \xrightarrow{d} \dots \longrightarrow AT_0 \longrightarrow 0$$

The highest grading is $2n-3$, *as there is no map with the above property from* A *to any tree* T_1, *with* $|E(T_1)| > 2n-3$.

Remark 3.0.13. *There is an action of symmetric group on* n *elements,* Σ_n*,on* A, *by permuting the apples. Therefore it induces an action on* AT, *the apple tree complex on* A. *Suppose some of the apples are isomorphic, then one can define an equivalence relation,* $\sim$, *on the set of apples,* A, *by setting* $a_i \sim a_j$ *if* a_i *is isomorphic to* a_j. *This gives a partition of* A *into the set of the equivalence classes,* $A_1, A_2, ..., A_k$, *with cardinalities respectively,* $n_1, n_2, ..., n_k$. *One can define the group of symmetry of* A *(up to canonical isomorphism) to be* $\Sigma_{n_1} \times \Sigma_{n_2} ... \times \Sigma_{n_k}$, *where* $\times$ *is the direct product of groups. This is a subgroup of* Σ_n *which acts on apple tree complex. The action of this subgroup induces an equivalence relation on the set of generators of the apple tree complex, namely two apple trees are equivalent if and only if they differ by an element of this subgroup. The* ***symmetrized*** *apple tree complex (if there is some nontrivial symmetry in the set of apples) by definition is the quotient of the apple tree complex by this induced equivalence relation.*

Remark 3.0.14. *In the rest of this section in order to make the discussion easier to understand and to make the analogy with apple trees more real, we make the following changes. Instead of following the above formal discussion, for each apple assigned to a vertex we draw a loop at the vertex with the name of the apple on the loop to make them look like actual apples.*

Example 3.0.15. *The apple tree complex on the set of two apples $\{a, b\}$, shown by its generators in each degree, is:*

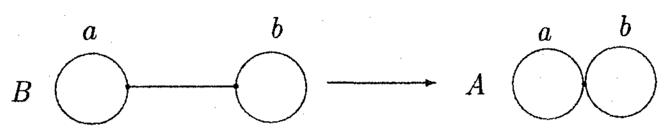


Figure 4.

we have $d(B) = A$ which shows T^2 is acyclic.

Example 3.0.16. *For the set of three apples $\{a, b, c\}$ we get the following complex (shown on its non-zero generators in each grading):*

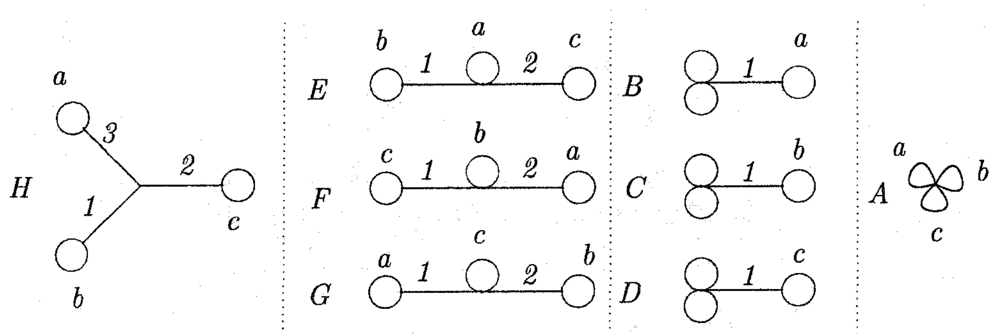


Figure 5. Generators (and a choice of an orientation for each) of AT^3 in each degree.

We have: $d(H) = E + F + G$, $d(E) = C - D$, $d(F) = B - C$, $d(G) = B - C$,

$d(B) = d(C) = d(D) = A$. This again shows that this complex is acyclic.

Example 3.0.17. *In the previous example if we have $a \sim b \sim c$, then the symmetrized apple tree complex, shown on its non-zero generators in each degree, will be as follows:*

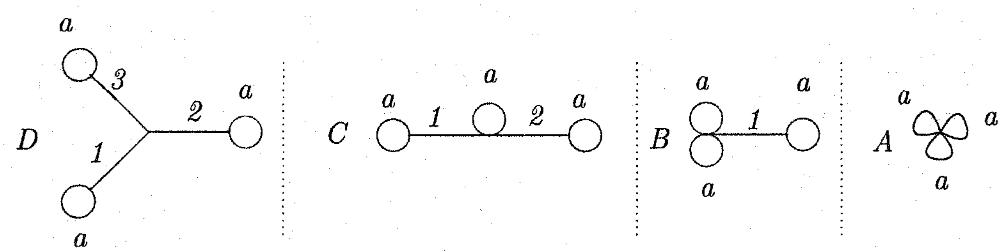


Figure 6.

It is an easy check to see that this is acyclic.

Example 3.0.18. *For the set $A = \{a, b, c, d\}$ of four apples, the generators of the complex will be as in figure [7]. Here for simplicity of drawing in Latex (and also to give a hint about some kind of weight decomposition) we drop the loop symbol and instead write a number next to each vertex indicating the number of apples at that vertex. The second number is the number of different apple trees which have the same underlying tree and the same "distribution" (of apple) type (the "distribution" type of an apple tree remains invariant under the action of Σ_n and the combinatorial type under isomorphism of trees).*

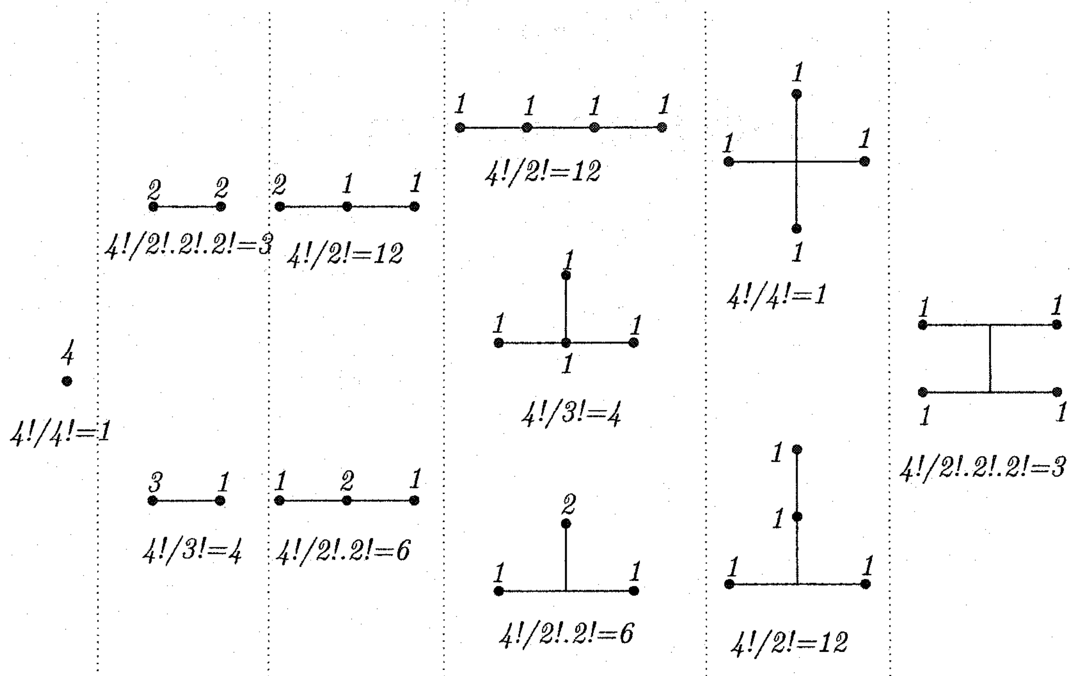


Figure 7. Generators of the complex of AT^4 in each degree.

The Euler characteristic of the complex is $3-(1+12)+(12+4+6)-(12+6)+(3+4)-1=0$, *which suggests this complex might be acyclic as well.*

This pattern is not accidental, as the following theorem shows:

Theorem 3.0.19. *Every apple tree complex is acyclic, i.e.,* $H_i(AT^n, \mathbb{Q}) = 0$ *for* $i \geq 0$.

Proof. Assume $n = |A| \geq 3$; the case $n = 2$ was considered above. We show the compl

$$0 \longrightarrow AT_{2n-3} \xrightarrow{d} AT_{2n-4} \xrightarrow{d} \dots \longrightarrow AT_0 \longrightarrow 0$$

is acyclic by decomposing d into two parts, namely $d = d_1 + d_2$, where d_1 collapses external edges and d_2 collapses internal edges, both to be defined below. An external edge is an edge that on one side is attached to only one apple and to no other edge, i.e, its removal divides the apple tree into two pieces, one piece only a single apple (not attached to any edge) and the second piece an apple tree with $n-1$ apples. All non-external edges are defined to be internal. It is clear that $d_1^2 = 0$ and $d_2^2 = 0$. This decomposition of the differential gives the following spectral sequence:

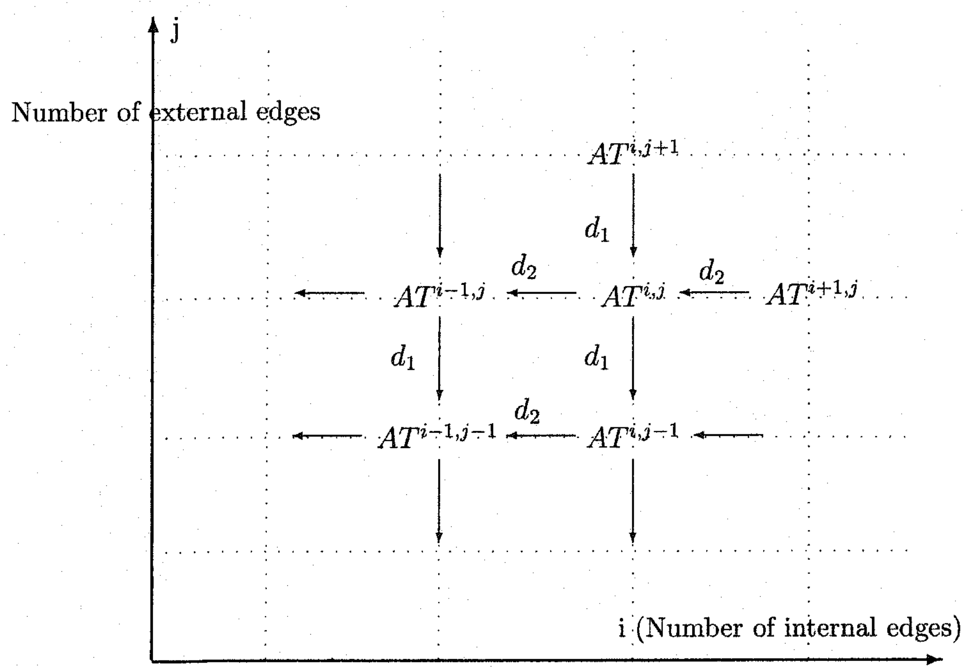


Figure 8.

We will prove this spectral sequence collapses at E_1 terms by showing that each column is acyclic. This in turn will prove the claim that the total complex is acyclic. Each column, say the i-th column, with the differential d_1 is a complex on its own.

We show this complex (column) is the direct sum of one or more subomplexes over the set of different (isomorphism classes of) apple trees sitting on nod $AT^{i,0}$ (i.e. apple trees with no external edge and i internal edges). The generators of one such subcomplex is obtained by all possible ways of putting or inserting one or more external edges on one such apple tree. Thus any such subcomplex is isomorphic to the product of complexes 'A_k', $1 \leq k \leq n$, over the set of apples of that apple tree, where the complex A_k, by its non-zero generators (in degree 0 and 1), is:

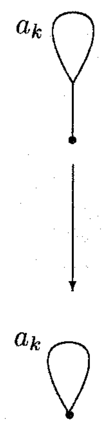


Figure 9.

Clearly $H_*(A_k) = 0$. By Kunneth formula we have $H_*(subcomplex, d_1) = H_*(A_1) \otimes \dots \otimes H_*(A_k) = 0$. □

Remark 3.0.20. *In the case that some of the apples are isomorphic, the symmetrized apple complex also will be acyclic. For it will be the quotient of its non-symmetrized apple tree complex, AT^n, with the finite symmetry group of the set of apples. Let S denote this symmetry group. We consider the homology groups with coefficient in $\mathbb{Q}$, and since any finite group such as S, has no homology with coefficient in $\mathbb{Q}$, we have:*

$$H_*(AT^n/S, \mathbb{Q}) = [H_*(AT^n, \mathbb{Q})]^S = [0]^S = 0$$

To understand this proof better let us go back to the case $n = 4$ in the example 3.0.18, and write out all the generators in each non-zero spot of the spectral sequence. It will look like the following:

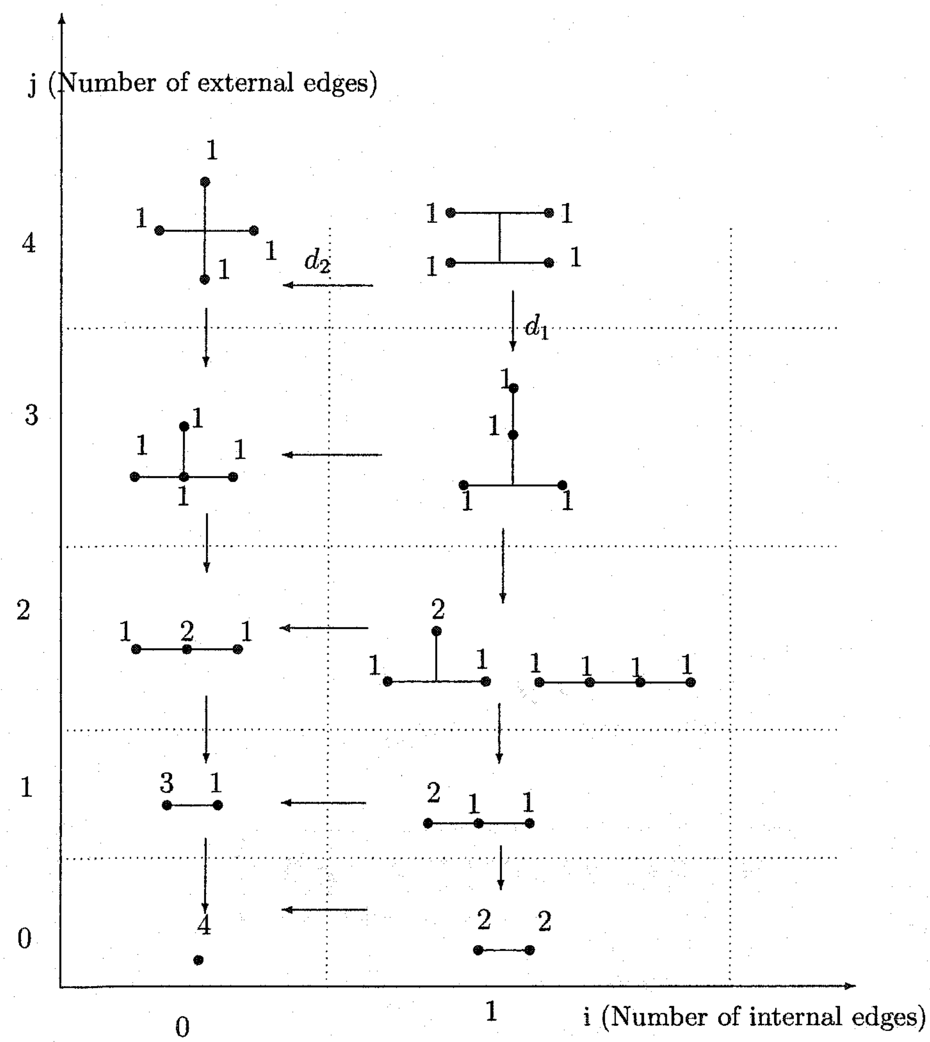


Figure 10. Spectral sequence for AT^4 at E_1-term, by the generators.

We will give another proof for this which will be based on the geometric realization of these complexes.

Definition 3.0.8. Poset and geometric realization. A poset $(S, \preceq)$ is a finite partially ordered set, i.e., a finite set S equipped with a relation "$\preceq$" which is reflexive, transitive, and antisymmetric. A chain in S is a totally ordered subset of S; its length is the number of elements minus 1.

Given a partially ordered set $(S, \preceq)$ one can form a simplicial complex called the geometric realization of S, denoted $|S|$, as follows: The elements $s \in S$ are the 0-simplices of $|S|$, and in general to any n-tuples $(s_0, ..., s_n)$ of elements of S, forming a chain, $s_0 \preceq s_1 \preceq ... \preceq s_n$, it is assigned an n-simplex. There is an obvious natural identification which makes this into a simplicial complex. A poset is called n-spherical if its realization is n-dimensional and $(n-1)$-connected.

Fact 3.0.21. *The geometric realization of a poset that has a unique minimal (resp. maximal) element is a contractible simplicial complex, for the whole complex is a cone on the vertex corresponding to this minimal (resp. maximal) element.*

Remark 3.0.22. *The geometric realization of the posets we study in this manuscript are actually the first barycentric subdivisions of what we will show in our pictures. This does not of course make any difference in the homotopy type and the (simplicial) homology of the geometric realization. It is often easier to see the relation of the differential in the graph complexes we consider to the boundary map in the simplicial complexex of what we have presented here.*

For apple trees on three apples we get the following picture which is clearly contractible.

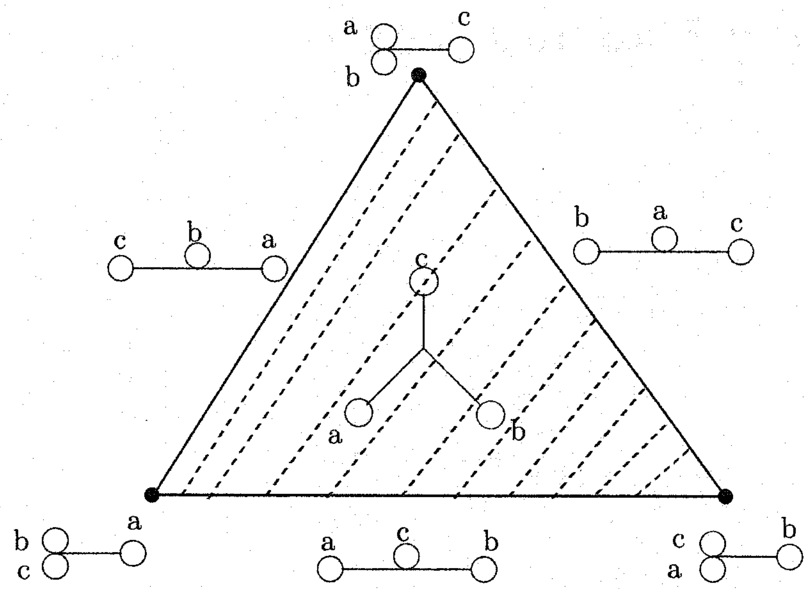


Figure 11. Geometric realization of AT^3.

For trees with four apples we get the following geometric realization:

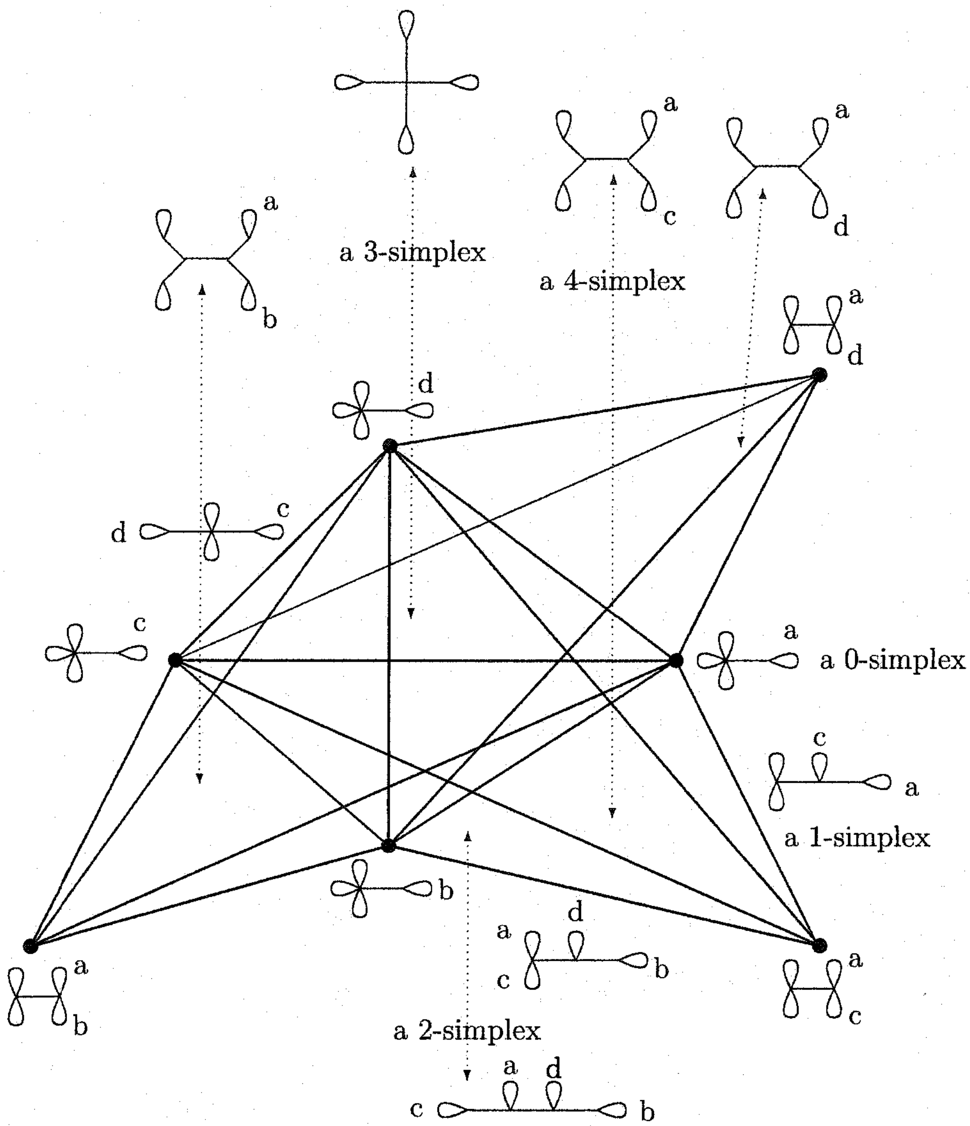
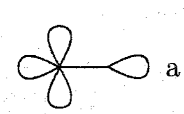
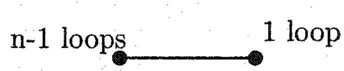


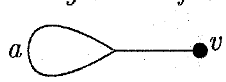
Figure 12. Geometric realization of apple tree complex on 4 apples a, b, c, and d.

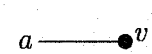
In figure [12] above we have three 4-simplices sharing a 3-simplex (solid tetrahedron) as a codimension one face. Note that the vertex  is attached to all other vertices in this simplicial complex, thus the whole space is a cone on this vertex and hence contractible. This situation repeats in higher dimensions as well. The geometric realization of AT^n will be a simplicial complex, GAT^n, of dimension $2n-4$ (with an action of the symmetric group on n generators on it). All the $2n-4$ dimensional simplices share a face of dimension $n-1$, corresponding to the apple tree with n external edges and no internal edge. GAT^n is contractible, for one can see that

a vertex of type 

can be connected through an edge to all other vertices. Thus everything can be retracted along these edges to this single vertex, that is, the whole complex is a cone on this vertex.

Remark 3.0.23. *Note that the top and the bottom row in the above spectral sequence are exactly the complex of non-rooted labeled trees with n leaves. To see this one has to forget about the loop or apple picture and recall that the apples are objects in any category, in the case of the top row one thinks of an apple as simply an input or a label, or pictorially think of the*

external edge 

as 

In the case of the bottom row, let an apple be a labelled edge, attached on one side to the underlying (combinatorial) tree.

If we fix one of the leaves as the root, the top or the bottom row then will be isomorphic to the complex of rooted non-planar (Lie) trees with $n-1$ leaves. In the next chapter we show the homology of this complex, i.e., the row zero (R_0, d_2), or the n-th row (R_n, d_2), is concentrated in the top degree and is $(n-2)!$ dimensional.

Definition 3.0.9. Let $GC^m_{\succeq 1}$ be as in 1.4.5, the connected rank m even (resp. odd) graph complex. Let A as in 3.0.12 be a set of n apples, $n > 1$ if $m = 0$, and $n \geq 1$ if $m > 0$. One can repeat the construction in 3.0.12 with obvious modifications to get, $AGC^m_{\succeq 1}$, the rank m even (resp. odd) apple graph complex. It is easy to see that $AGC^m_{\succeq 1}$ is finite dimensional with the highest grading of $3m - 3 + 2n$. The apple tree complex with this new notation is then just $AGC^0_{\succeq 1}$.

In the proof of theorem 3.0.19, to show that the columns were acyclic, we did not use any particular property of the trees. The same proof can be exactly repeated to get:

Theorem 3.0.24. *$AGC^m_{\succeq 1}$ is acyclic.*

As a corollary we can prove the following fact about "odd" graph complex, when collapsing loops is not allowed:

Definition 3.0.10. (GC^n_L, d_E) is the subcomplex of odd (GC^n, d_E) spanned by (those) oriented graphs that have loops. Similarly $(GC^L_{\succeq 3}, d_E)$ is the subcomplex of $(GC_{\succeq 3}, d_E)$ spanned by oriented graphs that have loops.

Proposition 3.0.25. $H_*(GC^n, d_E) = H_*(GC^n/GC^n_L, d_E)$.

Proof. Recall that the differential, d_E, neither collapses nor creates loops. Thus d_E preserves the number of loops. This means that under d_E we have the following direct sum decomposition: $GC^n = \bigoplus_{l=0}^{l=n} GC^n_l$, where GC^n_l is the subcomplex spanned by graphs with l loops. For $l \geq 1$, GC^n_l is isomorphic to a quotient of apple tree complex, $AGC^{n-l}_{\succeq 1}$, (A is set of l loops) by a finite symmetry group. Dividing by such a finite symmetry group is necessary, as in GC^n_l the loops are also labelled (up to an even permutation) and their labelling counts in the orientation. For example a graph with two loops at a vertex is zero, while in the $AGC^{n-l}_{\succeq 1}$ that is not the case, unless one imposes some new relations arising from this. Consider the sign representation of Σ_l on the set A, $\{a_1, ..., a_l\}$, of l loops. GC^n_l is the quotient of $AGC^{n-l}_{\succeq 1}$ by this action. Therefore once again by the same standard argument, that the quotient of an acyclic differential graded vector space over $\mathbb{Q}$ by a finite group is acyclic, we have that GC^n_l is acyclic. This finishes the proof. □

Theorem 3.0.26. *$(GC_{\succeq 3}, d_E) \cong (GC_{\succeq 3}/GC^L_{\succeq 3}, d_E)$, i.e, the quotient map, $GC_{\succeq 3} \to GC_{\succeq 3}/GC^L_{\succeq 3}$ induces an isomorphism in the homology.*

Proof. Recall $(GC_{\succeq 3}, d_E) = \bigoplus_{n=2}(GC^n, d_E)$ as direct sum of (sub)complexes. Therefore by the previous proposition applied to each summand on the right, the claim follows. □

Proposition 3.0.27. *In odd GC^n, $H_k(GC^n, d_E) = 0$ for $k < (3 + \sqrt{8n+1})/2$.*

Proof. By the previous proposition it is safe to restrict our attention to the subcomplex spanned by the loop-less graphs. In the odd graph complex any graph with multiple edges is zero. Thus the (non-zero) graphs that generate this complex are simple. A simple graph with k vertices can have at most $\binom{k}{2}$ edges. On the other hand for any connected graph, Γ, with k vertices and with rank n we have $|E(\Gamma)| - |V(\Gamma)| = |E(\Gamma)| - k = n - 1$. Therefore the following holds:

$$|E(\Gamma)| = k + n - 1$$

$$|E(\Gamma)| \leq k(k-1)/2$$

From which we obtain $k^2 - 3k - 2n + 2 \geq 0$. Using the quadratic formula we get $k \geq (3 + \sqrt{8n+1})/2$, which proves the claim. Note that this in fact shows that the loop-less odd graph complex is zero in the gradings lower than this number (not just the homology). □

This observation (with a slight modification) actually shows that all the odd graph complexes defined in chapter one namely $GC_{\succeq 1}$, $GC_{\succeq 2}$, $GC_{\succeq 3}$ are finite dimensional in each degree or grading.

In [1] Kontsevich writes:

" We conjecture for all 3 cases (or 6, if one takes into account also odd versions) stable homology of poisson algebras are finite dimensional"

He then adds: " this conjecture has a non-trivial consequence that the difference between the virtual and actual rational Euler characteristic for moduli spaces of open curves tends to $+\infty$ when the genus tends to infinity"

The proposition 3.0.27 above proves the case of "odd commutative Poisson algebras", i.e., it proves that the stable Lie algebra homology $H_k(C_\infty) = \lim_n H_k(C_n)$ is finite dimensional for the infinite dimensional super Lie algebra of symplectic vector fields that vanish at the origin on infinite dimensional super Euclidean space $\mathbb{R}^{\infty|\infty}$. See [1] and [3].

Proving this conjecture for "even commutative" case seems to be much harder. In 7.0.54 we show that this is indeed true for $k \leq 3$ ($k = 1, 2$ are easy to check).

Chapter 4

Tree Homology

Definition 4.0.11. Let T^n be the vector space generated by isomorphism classes of "oriented" (non-planar) trees that have n labeled leaves and one unlabeled leaf called root (with no bivalent vertices). Find the quotient of this vector space by the usual relation, $(T, or) = -(T, -or)$. Grade this complex by the number of interior edges. Let the orientation be the labelling of interior edges up to even permutation. Let the differential be summation over all trees (with the induced orientation) which are obtained out of collapsing an inner edge. We get a complex of the following form,

$$0 \longrightarrow T^n_{n-2} \xrightarrow{d} T^n_{n-3} \xrightarrow{d} \dots \longrightarrow T^n_0 \longrightarrow 0$$

Which is called Lie (or sometimes non-planar) tree complex. We are interested in the homology of this complex. We show that the homology of this complex is located in the top degree and is $(n-1)!$ dimensional.

Remark 4.0.28. *The fact that the homology of this complex is in its top degree, is proved using Hodge theory (over the field $\mathbb{C}$) in [15]. V. Ginzburg and M. Kapranov* *have given a purely algebraic proof for this using the theory of operad (and Koszul dulaity for operad) in [16].*

Definition 4.0.12. Let $L(x_1, ..., x_n)$ be the free Lie algebra over $\mathbb{Q}$ generated by $x_1, ..., x_n$. Let $Lie(n) \subseteq L(x_1, ..., x_n)$ be the linear subspace spanned by all bracket monomials containing each x_i exactly once. These monomials are not linearly independent, because of the Jacobi identity. $Lie(n)$ is an invariant subspace under the action of the symmetric group Σ_n on $L(x_1, ..., x_n)$, which is by permutation of x_i's. It is known that as a vector space (but not necessarily a representation of Σ_n):

$$Lie(n) \cong \bigoplus_{(n-1)!} \mathbb{Q}$$

Example 4.0.29. T^3.

T^3 has the following 4 generators:

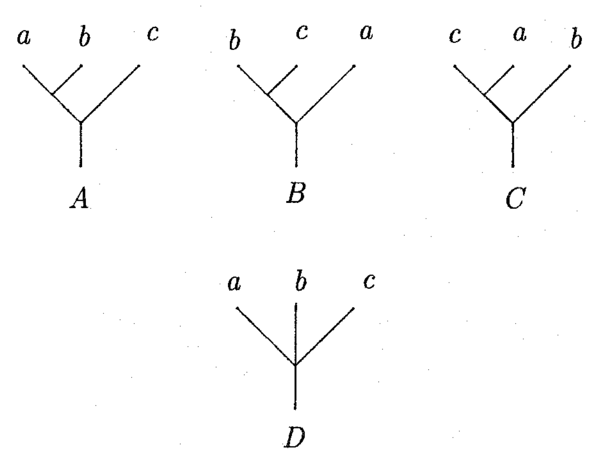


Figure 13.

Let us calculate the homology of this complex:

We have $d(A) = d(B) = d(C) = D$*. This means that* $A - B, B - C$*, and* $C - A$ *are all cycles; however there is one relation between them which is* $(A - B) + (B - C) + (C - A) = 0$*. This shows that* $H_1(T^3, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ *and* $H_0(T^3, \mathbb{Q}) = 0$*.*

Example 4.0.30. T^4.

There are 26 different rooted labeled trees with 4 inputs, which are the generators of the tree complex T^4*. For brevity in the following picture we will only write the 5 different combinatorial types that they divide into. We write a number next to each of these 5 combinatorial type trees indicating the number of labeled trees of that type :*

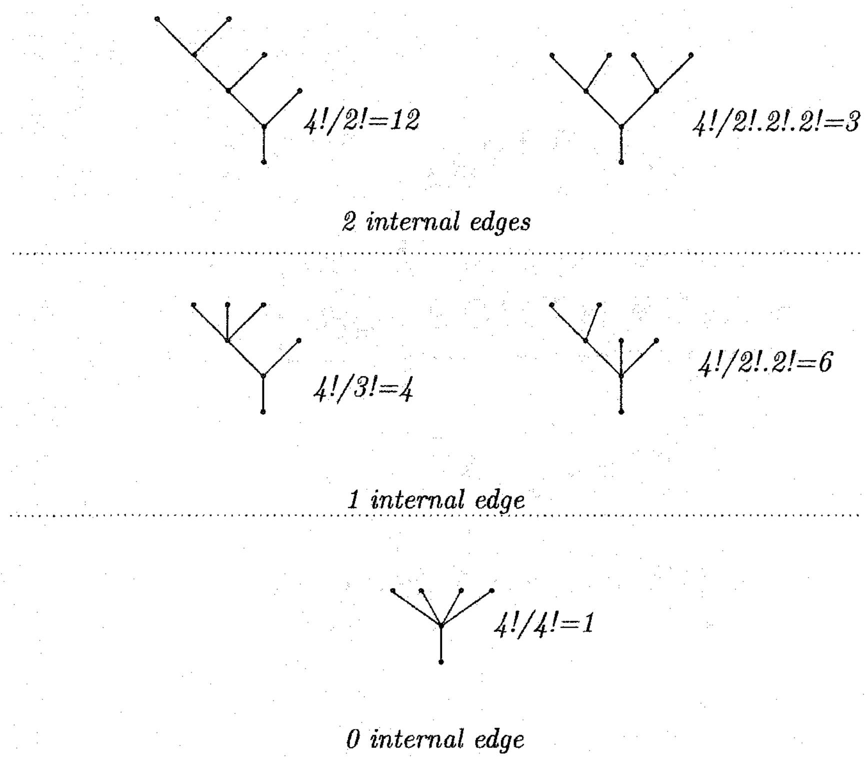


Figure 14. Rooted leaf-labeled (non-planar) trees with 4 inputs.

The Euler characteristic of this complex is $(12+3)-(4+6)+1=6=(4-1)!$.

We prove that the tree homology of this complex is indeed located in the top degree and is 6 dimensional, but instead of giving an algebraic proof we will give a geometric proof based on the geometric realization of this complex which can be more illuminating for higher dimensions. We think of a tree with one internal edge as a vertex; a tree with two internal edges as a 1-face or simply an edge; a tree with three internal edges as a two simplex or 2-face; etc. The partial relation is that the simplex associated to tree T_1 is a codimension k face of the simplex associated to tree T_2 if T_1 is obtained from T_2 by collapsing k internal edges. Then our differential in above becomes exactly the standard boundary operator in simplicial chain complex setting. The tree homology of our T^n becomes the reduced homology (because we throw away the tree with no internal edge!) of the simplicial complex obtained this way (with a shift of grading by one!). We call the geometric realization of tree complex T^n by GT^n. There is an obvious action of symmetric group on T^n and GT^n. This action commutes with the differential, hence it descends to the homology.

Example 4.0.31. GT^3. *It is the following simplicial complex which simply consists of three disjoint points:*

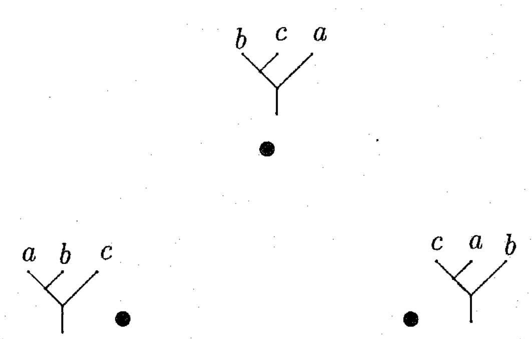


Figure 15. Geometric realization of rooted leaf-labelled trees with 3 inputs
It is a wedge of two zero-spheres

We have $H_1(T^3, \mathbb{Q}) = \widetilde{H}_0(GT^3, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$.

Example 4.0.32. GT^4; *it is a 1-dimensional simplicial complex (i.e. nothing but a graph) with 10 vertices and 15 edges. We have* $V - E = dim(H_0(GT^4, \mathbb{Q})) -$

$dim(H_1(GT^4, \mathbb{Q}))$; since $H_0(GT^4, \mathbb{Q}) = \mathbb{Q}$ it follows that $H_1(GT^4, \mathbb{Q}) = \mathbb{Q}^6$. This implies that $H_2(T^4, \mathbb{Q}) = \mathbb{Q}^6$ and $H_1(T^4, \mathbb{Q}) = H_0(T^4, \mathbb{Q}) = 0$

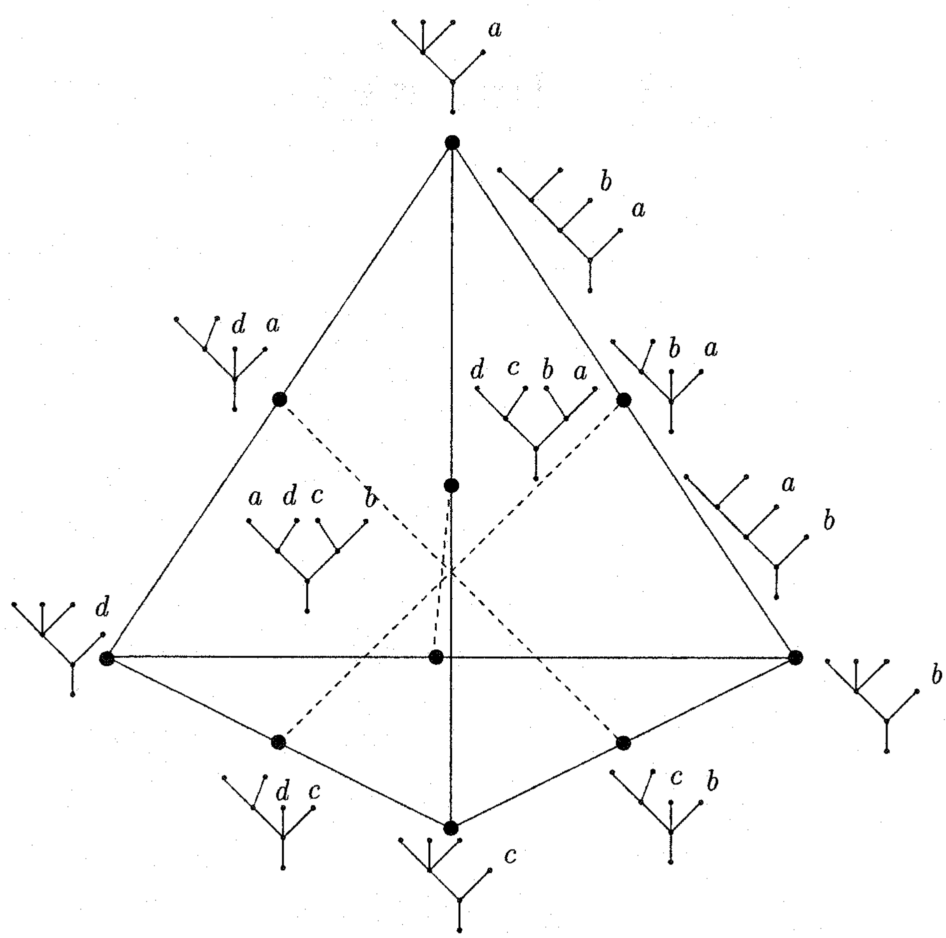


Figure 16. Geometric realization of rooted leaf-labelled trees with 4 inputs. It is homotopy equivalent to a wedge of six 1-spheres

In the following we will show by induction that the geometric realization has the homotopy type of wedges (joint at a point) of some number of spheres, to be more exact: $GT^n \cong \vee_{(n-1)!} S^{n-3}$; and this therefore provides a new proof that the homology of Lie operad is concentrated in the top degree.

Remark 4.0.33. *The idea of the proof arose from reading a paper by M. Bridson K. Vogtmann [8] and observing that the simplicial realization they have for the partial ordered set of thick partitions is precisely geometric realization of the bottom row of our spectral sequence in the previous section which in turn by the remark in 3.0.23 is the same thing as our rooted tree complex.*

Definition 4.0.13. Suppose (B_1, C_1) is an un-ordered partition of our apple complex $A = (a_1, ..., a_n)$, into two parts, each part containing at least two elements. Two such partitions (B_1, C_1) and (B_2, C_2) are called compatible if one part in one partition is a subset of a part in the second partition (for example if $B_1 \subset B_2$, which in turn also implies that $C_2 \subset C_1$).

We associate to any partition (B_1, C_1) a vertex, and a chain of pairwise compatible partitions, $(B_1, C_1), (B_2, C_2), ..., (B_m, C_m)$, an $m-1$ simplex. Note that these partition can be thought of as those apple trees with one edge, living in the $(2,0)$ spot of our spectral sequence in the previous chapter. Also note that if two partitions are compatible then there exists an apple tree with two edges in our sense (or a 1-simplex) which connects the two. And if there are three partitions (or vertices) pairwise compatible, then there is an apple tree with three edges or a two simplex having these three partitions as its vertices. All we want to say here is that a k-simplex in this new sense is the same thing as a k-simplex in the sense that appeared before. This new approach however is nicer in the sense that we only deal with the vertices.

Theorem 4.0.34. $GT^{n-1} \cong \vee_{(n-2)!} S^{n-4}$.

Proof. Induction on number of apples n. For $n = 3$ as we saw geometric realization for trees with three inputs was three points or wedges of the two 0-spheres (equivalently there are only three partitions on a set of four elements, each two non-compatible). For $n = 4$ we saw geometric realization in previous figure, a graph with 6 circuits which is a wedge of 6 circles or 1-spheres. Assume $n \geq 5$. Let size of a partition (A_1, B_1) be the cardinality of the smaller part. For simplicity let us call our apples just by $1, 2, ..., n$. Consider the partition $w = (12, 34...n)$. Let G_1 be simplicial subcomplex generated by all those partitions compatible with w. This subcomplex has dimension $n - 4$ and is a cone on the minimal element w, and hence contractible. Next define G_i, $i \geq 2$ to be simplicial span of G_1 and all those partitions, (A_1, B_1), non-compatible with w for which $1 \in A_1$ and $2 \in B_1$, and $|A_1| \geq n - i$. This gives a filtration of the GT^{n-1} by decreasing size of the partitions non-compatible with w, i.e., $G_1 \subset G_2 \subset ... \subset G_{n-3} \subset G_{n-2} = GT^{n-1}$. Note that all G_i's, have the same dimension, $n - 4$ (because G_1 and G_{n-2} have this dimension). For $k \leq n - 3$, G_{k-1} is deformation retract of G_k. Let us assume this for the moment. Since G_1 was contractible, this means the G_{n-3} is contractible. What are the partition not in G_{n-3} ? Those are precisely partitions of the form $(1p, 2...m)$ where $3 \leq p \leq n$ (if $p = n$ then $m = n - 1$ otherwise $m = n$). There are $n - 2$ such partitions. Let us consider one such partition, say $u = (1n, 24...n)$ for simplicity. What is the $Link(u) \cap G_{n-3}$? It is the simplicial span of those partitions (X, Y) compatible with partitions of the form $(1nj, 2...m)$ and the size of X at least three and containing both 1 and n. In all of these later partitions and those compatible with them 1 and n are on the same side, so if identify $\overline{1n}$ with a new element say a_1, these gives us an identification of $Link(u) \cap G_{n-3}$ with the simpicial realization for the set of thick partition on the set $a_1, 2, 3, .., n-1$ which is isomorphic to GT^{n-2}. But GT^{n-2} by induction hypothesis is homotopy equivalent to $\vee_{(n-3)!}S^{n-5}$. This shows that $G_{n-3} \cup Star(u) = Susp(G_{n-3} \cap Linku) = Susp(\vee_{(n-3)!}S^{n-5}) \cong \vee_{(n-3)!}S^{n-4}$ (note that both $Star(u)$ and G_{n-3} are contractible and $n-4$ dimensional and disjoint). We had $n-3$ partitions which were like u (not in G_{n-3}), including u itself. Any two such partitions are incompatible and hence theirs stars are disjoint, each of this u-type partition contributes $(n-4)!$ spheres, all of these spheres share a contractible space, G_{n-3}, which finishes the proof. One can give a more formal proof using Mayer-Vietoris and Van Kampen for this last part.

It is then left to prove the claim that G_i deformation retracts onto G_{i-1} for $i \leq n-3$:

Assume $x = (E, F)$ is one thick partition which is not in G_1 and $1 \in E$. Suppose $x = (E, F)$ is in $G_i - G_{i-1}$, this means $|E| = n - i \geq 3$ by the definition of G_i's. Then $Link(x) \cap G_{i-1}$ is a cone on vertex (partition) $\overline{x}$ where, $\overline{x} := (E-1, F \cup 1)$ ($\overline{x}$ is still a thick partition and it is in fact in G_1 which is a subcomplex of G_{i-1}). Thus $Link(x) \cap G_{i-1}$ is contractible. No two vertices of $G_i - G_{i-1}$ are compatible, thus G_i collapses into G_{i-1}. □

The simplicial complexes obtained here are similar to the Stasheff's associahedron A_n which are geometric realization or polytope of poset of (isomorphism classes of) planar rooted trees with n fixed leaves. A_n is homoemorphic to an $n-2$ dimensional disk. They are called an associahedron, for they are an operad that continuously parameterizes the possible ways to multiply n loops in any loop space, or in general in any space that has a morally associative structure (or strongly homotopy associative space), see [33].

It is easy to see geometric realization of A_n, $|A_n|$, is an $n-2$ disk. The partial order is the same as in the non-planar or Lie case, i.e., $T_1 \preceq T_2$ if and only if T_1 could be obtained from T_2 by collapsing some internal edges. The tree with no internal edge is the minimal element, hence $|A_n|$ is contractible. The longest chain has length $n-2$, since this is the maximum number of internal edges a rooted tree with n leaves could have, hence $|A_n|$ is $n-2$ dimensional. In the figure [17] we see the geometric realization of A_n for $n = 2, 3, 4$, and 5.

Definition 4.0.14. Let PT^n, for $n > 1$, denote the complex generated over $\mathbb{Q}$ by isomorphism classes of oriented planar rooted trees with n labeled leaves, with the same orientation and differential as the non-planar case, and divide by the same relation, i.e., $(T, or) = -(T, -or)$.

Proposition 4.0.35.

$$H_i(PT^n) = \begin{cases} n!\mathbb{Q}, & \text{if } \ i = n-2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For a labeled non-planar rooted tree with n leaves (labeled by different inputs) and no internal edge, there are obviously $n!$ different ways (up to isotopy) to be imbedded into the plane. Thus PT^n is direct sum of $n!$ different subcomplexes over such different imbeddings. Each of these subcomplexes as explained above has a geometric realization that is homeomorphic to an $n-2$ disk. The differential in PT^n corresponds to the differential in the simplicial complex, with the difference that grading i in the former goes to $n-2-i$ in the latter. Therefore we have:

$$H_i(PT^n) = \bigoplus_{n!} H_{n-2-i}(|A_n|, \mathbb{Q}) = \bigoplus_{n!} H_{n-2-i}(point, \mathbb{Q})$$

which proves the claim.

□

Remark 4.0.36. *A better account of the material of this chapter based on the theory of operads will appear later (this chapter has not been carefully edited).*

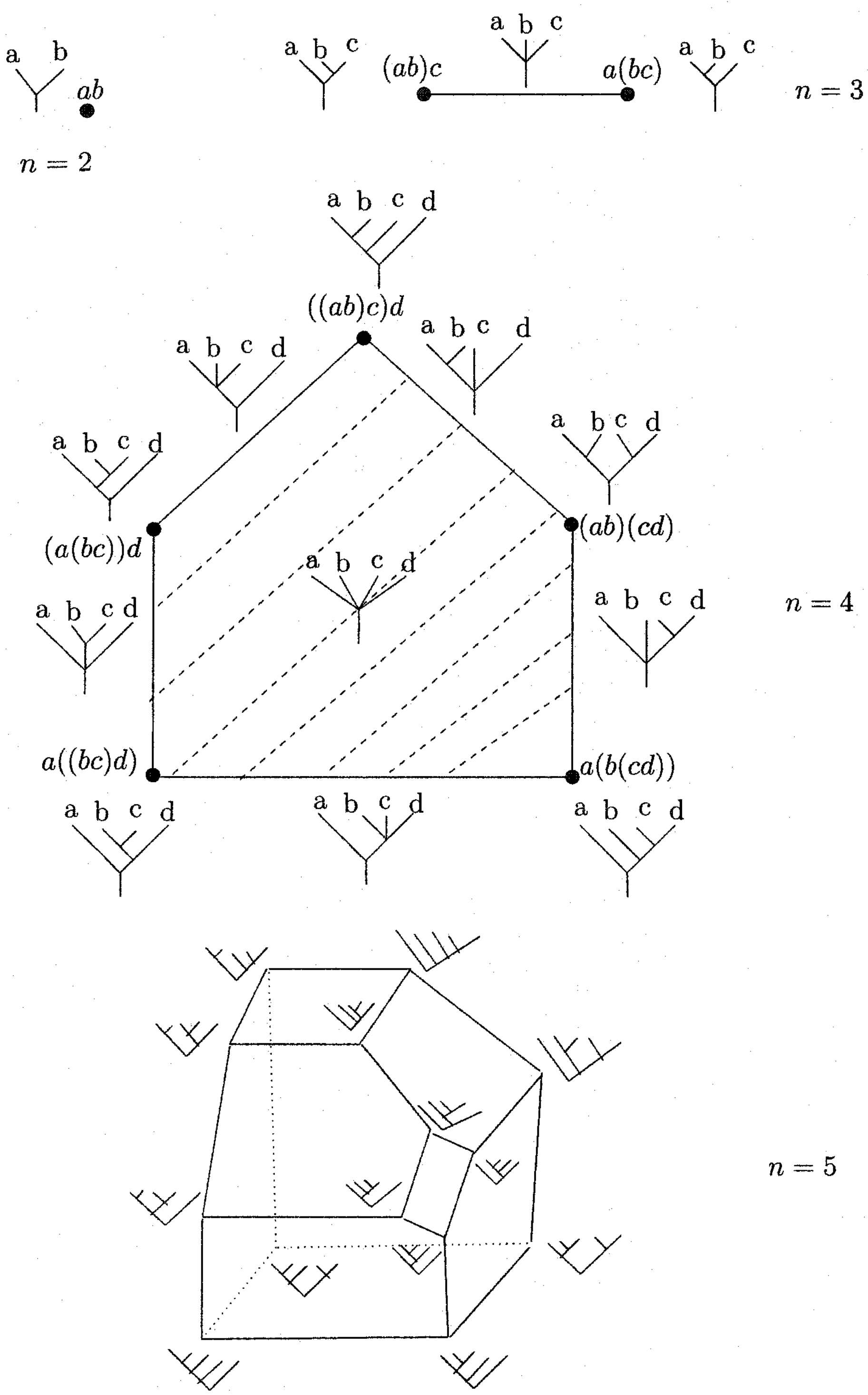


Figure 17. Geomteric realization of planar rooted trees (Stasheff associahedron's) for $n = 2, 3, 4$ and 5.

Chapter 5

Odd Graph Complex with the $d_E + d_L$ differential

In this part we consider a subcomplex of the odd graph complex with the full differential, i.e., $(GC_{\succeq 1}; d_E + d_L)$, and show that its graph homology vanish. Here d_E is the usual differential that collapses only non-loop edges, while d_L is a new differential that collapses only loops.

Fact 5.0.37. *Note that by 2.0.11, i.e., that $H_*(GC_{\succeq 1}; d_E) = 0$ and by the standard knowledge of spectral sequences we have that $H_*(GC_{\succeq 1}; d_E + d_L) = 0$.*

Now consider the following subcomplex of $(GC_{\succeq 1}; d_E + d_L)$:

Definition 5.0.15. $(\widehat{GC}_{\succeq 3}; d_E + d_L)$ is spanned by those oriented graphs that have "valence" at least three at each vertex, where loops are not counted towards the "valence" (i.e, if we remove all the loops from the graph, no univalent or bivalent vertex is created)

Theorem 5.0.38. *(with D. Sullivan) Odd graph complex is acyclic, that is* $H_*(\widehat{GC}_{\succeq 3}; d_E + d_L) = 0$.

Proof. Given a graph G without any loop in $\widehat{GC}_{\succeq 3}$, we can start adding loops to each vertex of G and get new graphs. We can generate a vector space by these graphs. We can then grade this vector space by number of the loops. The orientation as in the odd case is the labelling of all non-loop edges and loops up to even permutations. With the choosing of this orientation any graph that has two or more loops at a vertex is zero. Therefore there is a highest grading which is number of the vertices of the graph, $|V(G)|$. Now consider the operator d_L which is defined $d_L(\Gamma, or) = \sum_{l_i \in \Gamma}(\Gamma/l_i, or/l_i)$ where the sum is over all loops in Γ. It is clear that $d_L^2 = 0$. Hence we have a finite dimensional graded complex which we call (g, d_L). We have :

Lemma 5.0.39. $H_*(g, d_L) = 0$

Proof. First consider the following complex C:

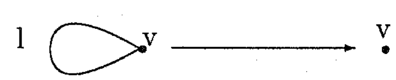


Figure 18.

Clearly we have $H_*(C, d_L) = 0$.

Now, as illustrated in figure [19] below:

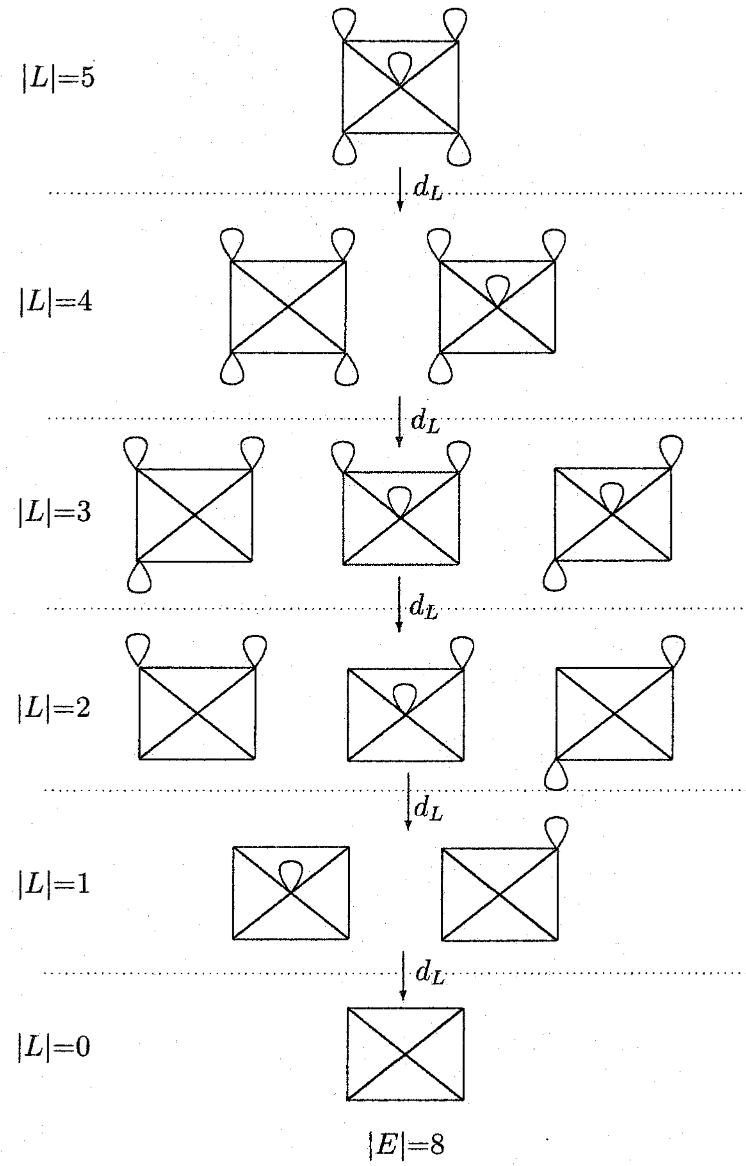


Figure 19. Generators of the complex g, formed out of adding loops to the loopless graph G.

we have $(g, d_L) \cong (C \otimes ... \otimes C)/Aut(G)$, where the product is over the set of vertices of the graph G. Once again by the Kunneth formula we have:

$$H_*(g, d_l) = H_*((C \otimes ... \otimes C)/Aut(G), \mathbb{Q})$$
$$= [H_*(C) \otimes ... \otimes H_*(C)]^{Aut(G)} = [0]^{Aut(G)} = 0$$

□

Now we are ready to prove the theorem:

Let us recapitulate. The number of non-loop edges and number of loops in a graph give us a bi-grading, and $d = d_E + d_L$ by its way of definition decomposes into two parts d_E plus d_L, where d_L collapses loops and d_E collapses non-loop edges. We have $d_L^2 = 0$ and $d_E^2 = 0$, and $d_L d_E + d_E d_L = 0$. These leads to the following spectral sequence:

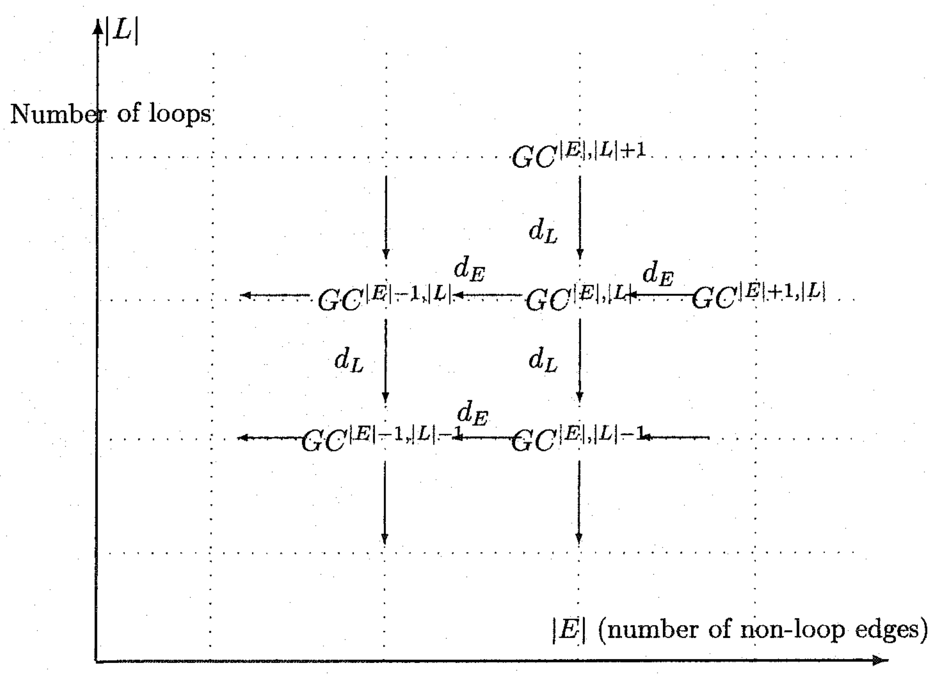


Figure 20.

where $GC^{|E|,|L|}$ is the vector space generated by oriented graphs with $|E|$ edges and $|L|$ loops. To get the E_2 terms of this complex we first take the homology of each column, i.e., with respect to the d_L, and take the homology of these new formed rows with respect to the d_E. But each column can be decomposed into the direct sum of finitely many subcomplexes over all loop-less graphs (orientable and non-orientable, both!) sitting in node $(|E|, 0)$, the homology of each of those sub-columns vanishes by lemma, so the homology of each column is zero and hence the E_2 term is zero and this implies that the total homology of the complex is zero. The reason we have to include the non-orientable (i.e., zero) graphs in the first row is that when we add loops to them, they may become orientable and hence non-zero. □

Chapter 6

Acyclicity of the exactly 1-connected graph subcomplex

In this part we show that graph homology in both the even and odd case can be calculated modulo the subcomplex spanned by graphs which have a separating vertex, explained below. Let us first make the following definition.

The following is the usual **combinatorial** definition of n-connectedness for a graph (see [22] or [23]).

Definition 6.0.16. A graph with k vertices is connected if any two of its vertices are joined by a chain of its edges. A graph is n-connected if it is connected, and removing from it any $n-1$ vertices together with all incident edges, we obtain again a connected graph (with $k-n+1$ vertices).

The following is the/our **topological** definition of n-connectedness of a graph (viewed as a 1-dimensional cell complex).

Definition 6.0.17. A graph is n-connected ($n \geq 1$), if it is connected, has at least

$n-1$ vertices, and after removing any $n-1$ vertices from it still remains connected (as a topological space).

The above two definitions are similar and almost but not exactly equivalent. For example with the first definition the following two graphs have the same degree of connectivity (one or two, depending on whether the empty set is considered connected or not). Graph A consists of a single vertex and a loop attached to it, and graph B consists of a single vertex and two or more loops attached to it. With the topological definition of connectivity however the graph A is 2-connected, while the graph B is only 1-connected. **We stick to the second definition.**

Definition 6.0.18. We define C_1 to be the sub-complex of $GC_{\succeq 3}$ spanned by those graphs that are exactly 1-connected, and C_2 the linear subspace of $GC_{\succeq 3}$ spanned by two or higher connected graphs.

We obviously have $GC = C_1 + C_2$, as the sum of vector spaces. Note that C_1 is a subcomplex but C_2 is not, as illustrated in the figure [21] below.

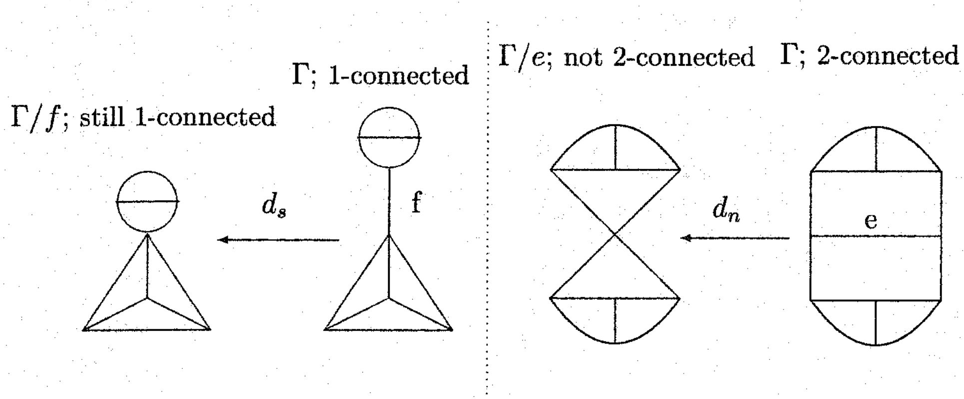


Figure 21.

Proposition 6.0.40. *$H_*(C_1, d) = 0$, i.e., in both the even and odd case, the subcomplex spanned by exactly 1-connected graphs does not contribute to the homology of the graph complex.*

Proof. We show this by decomposing $d = d_s + d_n$ where d_s collapses separating edges and d_n collapses edges that do not separate. We will obtain the following double complex:

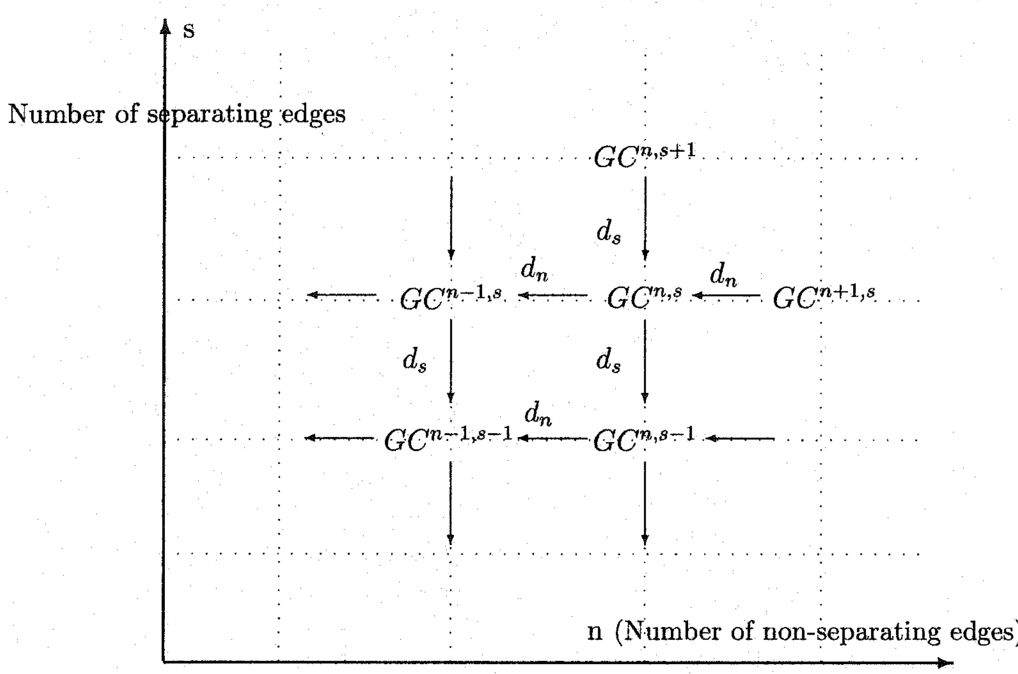


Figure 22.

We show that the homology of each column is zero by decomposing it into the finite direct sum of subcomplexes over the set of graphs (the non-orientable ones included as well) sitting in $GC^{n,0}$ and showing that the homology of each of these subcomplexes is zero. It will then follow that $E_1(GC^{n,*}, d_s) = 0$. Consequently all the E_2 terms will be zero, and therefore the total homology of the double complex will vanish as well.

Now it remains to prove the claim that the homology of each subcomplex of a column is zero. Every 1-connected graph is a tree of the 2-connected (sub)graphs; the 2C (2-connected) subgraphs will be those connected components that one gets if all the separating edges are removed.

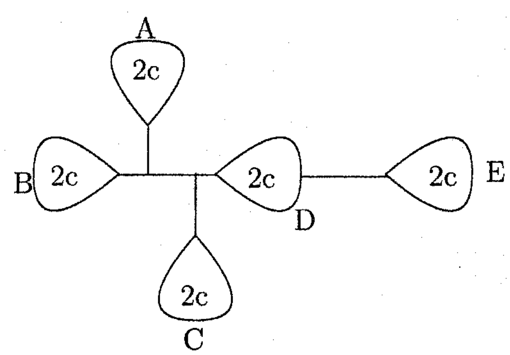


Figure 23. A 1-connected graph with 6 separating edges.

If we collapse all the separating edges of these exactly 1-connected graphs, we get graphs with no separating edge but with one or more separating vertices.

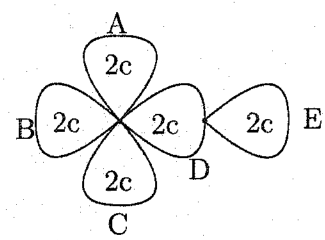


Figure 24. A 1-connected graph with no separating edge.

The graphs obtained this way each reside in $GC^{n,0}$ for some n. Consider one such graph. If we have only one separating vertex in this graph and blow it up under the dual of d_s (which is the coboundary operator in the dual complex), we get graphs which have one or more separating edges. The span of these graphs (with the differential d_s) is an apple tree complex (with apples being 2-connected components). Therefore it is an acyclic subcomplex (and direct summand) of the column. If we have more than one separating vertex and blow up two such vertices we get graphs like the one in figure [23], which does not exactly look like an apple tree. However as d_s is a derivation, and as the subcomplex obtained by the span of graphs created by the blow up of only one vertex is acyclic, and by the general fact that follows, the span of such 1-connected graphs also is an acyclic subcomplex of the column.

Suppose (A, d_a) and (B, d_b) are two differential graded algebras, which are acyclic in all dimensions ($H_i(A, d_a) = 0$ and $H_i(B, d_b) = 0$ for all i's); then $(A \otimes B, d_a \otimes 1 + 1 \otimes d_b)$ is again a differential graded algebra, acyclic in all dimensions.

□

Theorem 6.0.41. *Both the even and odd graph complexex are quasi isomorphic with their quotients mod exactly 1-connected graphs. i.e.* $GC_{\geq 3} \cong (GC_{\geq 3})/C_1$.

This theorem, for the even case only, was also independently obtained by J. Conant, F. Gerlits and K. Vogtmann [9].

The previous proof can be applied almost word by word to the odd $GC_{\succeq 3}/GC^L$, defined in 3.0.10 to get the following, where C_1' here is the subcomplex of $GC_{\succeq 3}/GC^L$ spanned by 1-connected graphs.

Corollary 6.0.42. *The odd graph complex is quasi isomorphic to its subcomplex spanned by the loop-less graphs and the latter is quasi isomorphic with its quotient mod exactly 1-connected graphs, i.e.,* $(GC_{\succeq 3}, d_E) \cong (GC_{\succeq 3}/GC^L, d) \cong ((GC_{\succeq 3}/GC^L)/C_1', d)$.

Remark 6.0.43. *Note that we have* $GC^L \subset C_1 \subset GC_{\succeq 3}$*, as the inclusion of subcomplexes, therefore* $(GC_{\succeq 3}/C_1) \subset (GC_{\succeq 3}/GC^L) \subset GC_{\succeq 3}$*. By 6.0.41 and functorial properties of the homology this gives a new proof for 3.0.26, i.e.,* $GC_{\succeq 3} \cong GC_{\succeq 3}/GC^L$.

Chapter 7

Calculations

In this part we compute even and odd graph homologies of GC^n, defined in the first chapter, up to $n \leq 4$. The explicit generators and non-trivial cycles (if any) in each case is given. We additionally show that in the odd case, $H_3(GC_{\succeq 3}) = H_3(GC_{\succeq 2}) = 0$ and $H_4(GC_{\succeq 3}) = H_4(GC_{\succeq 2}) = \mathbb{Q}$; and in the even case, $H_3(GC_{\succeq 3}) = 0$ and $H_3(GC_{\succeq 2}) = \mathbb{Q}$. This last case affirmatively verifies a conjecture by Kontsevich in [1] for $k = 3$ that $H_k(GC_{\succeq 3})$ is finite dimensional for all k's ($k = 2$ is an easy check). We proved this in odd case for all k's in 3.0.27. This also proves $H_3(C_\infty, \mathbb{Q})$ is finite dimensional where C_∞ is the infinite dimensional Lie algebra defined in [1]

Example 7.0.44. *"Even" GC^1:*

Recall from 1.4.4 that the generators of this complex are n-gon's. If $n \equiv 1 \mod 4$, then a symmetry like the one shown in the figure [25] for $n = 1$ (symmetry with respect to any line that passes through one vertex and bisects the n-gon), keeps the V orientation fixed (because it swipes $\frac{n-1}{2}$, an even integer, pair of vertices) and changes A. For an even number n, a $360/n$ rotation like the one shown in figure [21] keeps A *part in $A \otimes V$ orientation fixed, and it changes the V part by the sign of the rotation which is $(-1)^{(n-1)}$, thus changing (V and) $A \otimes V$ orientation. For $n \equiv 3 \mod 4$ there is no isomorphism or symmetry that changes the orientation.*

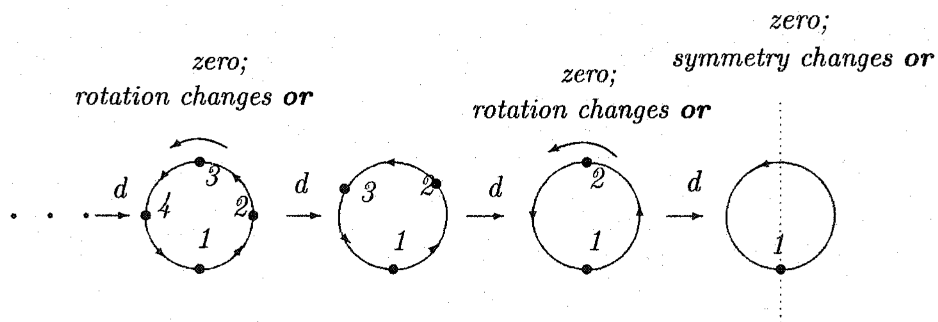


Figure 25. Even GC^1 / Even GC_2.

Therefore by mere orientation consideration (not the boundary operator) we get:

$$H_n(GC^1) = \begin{cases} \mathbb{Q}, & \text{if } n \equiv 3 \mod 4 \\ 0, & \text{otherwise.} \end{cases}$$

Remark 7.0.45. *The primitive homology of $sp(\infty)$ is known to be the same, i.e. $PrimH_n(sp(\infty), \mathbb{Q}) = H_n(GC^1)$*

Example 7.0.46. *Odd GC^1 :*

This time the only n-gon's that do not vanish (by orientation consideration) as depicted below in figure [26], are those for which $n \equiv 1 \mod 4$.

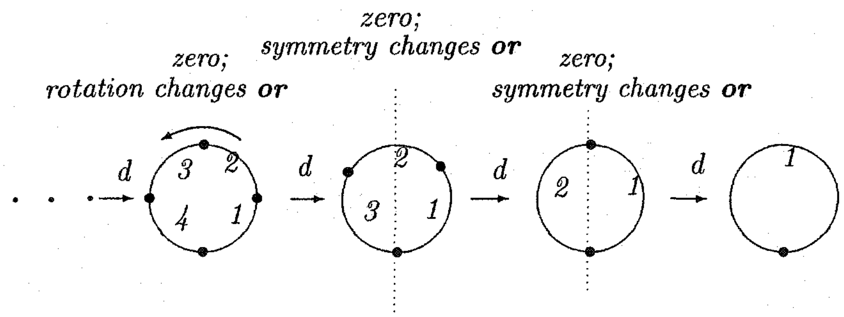


Figure 26. Odd GC^1 / *Odd* GC_2.

Thus we have:

$$H_n(GC^1) = \begin{cases} \mathbb{Q}, & \textit{if} \quad n \equiv 1 \mod 4 \\ 0, & \textit{otherwise.} \end{cases}$$

Isomorphism classes of connected graphs of rank two, with valence three or more at each vertex, are as follows:

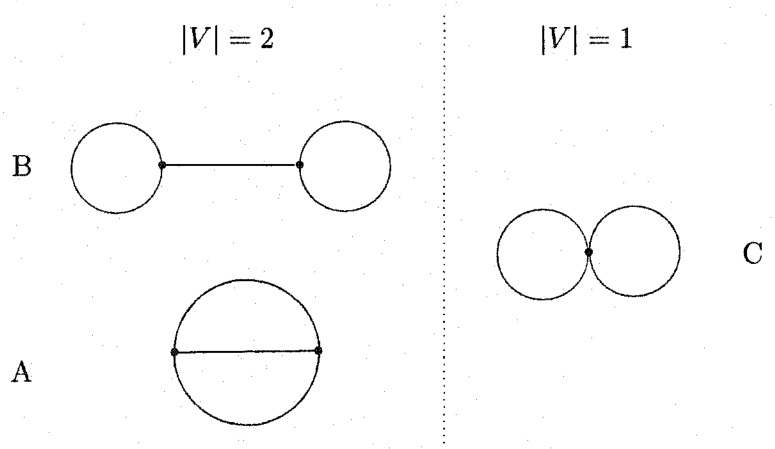


Figure 27. List of all connected graphs, up to isomorphism, of rank two with valence three or more at each vertex.

Example 7.0.47. *"Even" GC^2:*

The three possible candidates for the generators of this complex are shown in the above figure. However graphs B and C are non-orientable because they have loops, and hence zero. Graph A does not have an orientation reversing isomorphism and therefore it is the only generator of GC^2. Consequently it is a cycle. This shows that $H_2(GC^2) = \mathbb{Q}$ and $H_i(GC^2) = 0$ for $i \neq 2$.

Example 7.0.48. *"Odd" GC^2:*

All the three graphs in above figure which are the only possible candidates for the generators of this complex are zero by orientation consideration. Graph A is zero because it has multiple edges. Graph C is zero because it has multiple loops, and graph B is zero because the automorphism that switches its two vertices reverses its orientation. Thus the odd GC^2 is zero, as is its homology.

Isomorphism classes of connected graphs of rank three, and valence at least three at each vertex, are shown in the following picture:

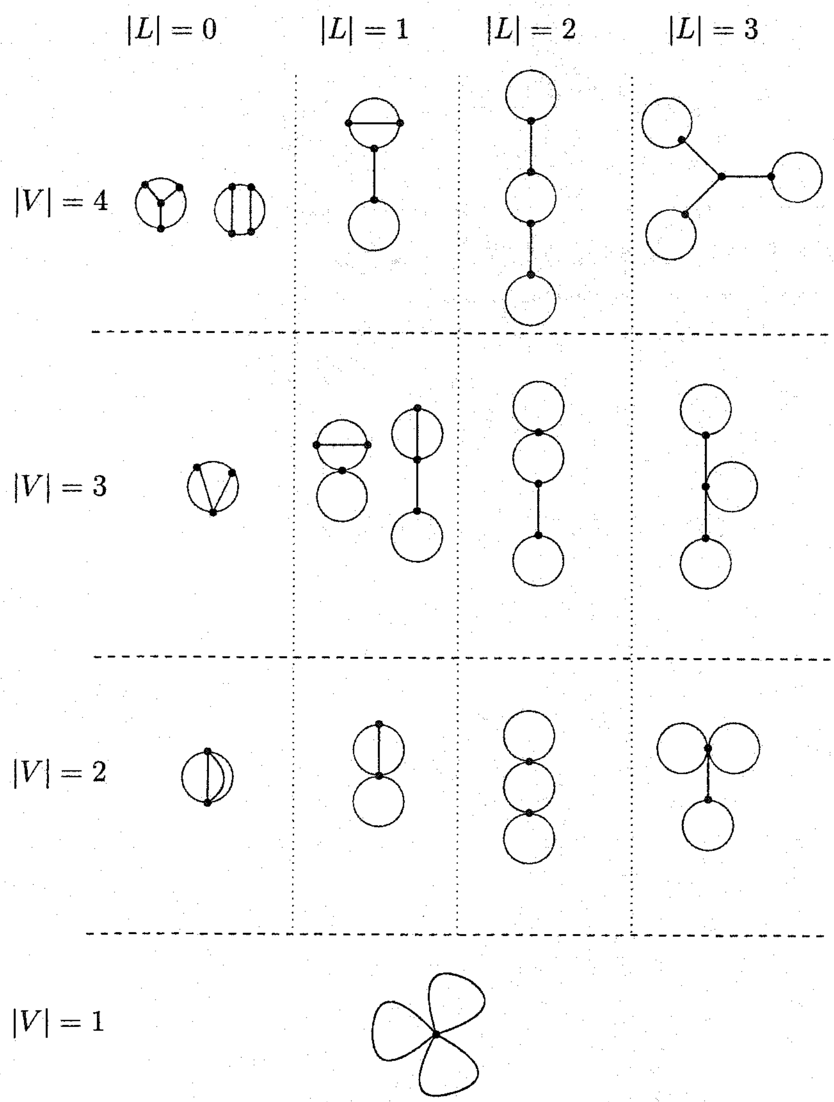


Figure 28. List of all connected graphs (up to isomorphism) of rank 3 with valence three or more at each vertex

Example 7.0.49. *"Even" GC^3 :*

All possible different combinatorial candidates for the set of generators of this graph *complex are in the above table. Since with even orientation graphs with loops are zero, we only need to look at the first column (i.e., column $|L| = 0$). In that column the only graph with two vertices has an orientation reversing isomorphism, namely the isomorphism that exchanges the two vertices and keeps all the edges fixed (it changes four arrows on the edges which has even parity, and changes the two vertices which has odd parity, thus changing $A \otimes V$ altogether); and hence it is zero. This means that our complex is as follows:*

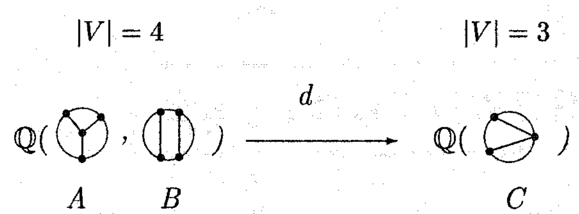


Figure 29. Even GC^3, by its non-zero generators

Graphs A, B and C are non-zero. One can check this by looking at the generators of automorphism group of each graph. For example the automorphism group of graph A is Σ_4, the symmetric group on four generators. It is generated by transitions, i.e., the automorphisms that switch two vertices and keep the other two fixed. Any such automorphism changes both A and V part of the orientation by a sign, and hence preserves the product. It is easy to see $d(A) = \pm 6C$ and $d(B) = \pm 2C$ ($\pm$ depending on the choice of orientation of each graph), so that means $A \pm 3B$ is a cycle. This means that $H_4(GC^3) = \mathbb{Q}$. Homologies in all other dimensions for this complex are zero.

Example 7.0.50. *"Odd" GC^3:*

By looking at the list of all the possible candidates in figure [28], and disregarding all those with multiple edges or multiple loops at a vertex, we will be left with only three candidates for the set of generators of this complex. By looking at the generators of the automorphism group of each graph we will see that none of them change the orientation. Thus these three graphs are actual generators of this complex:

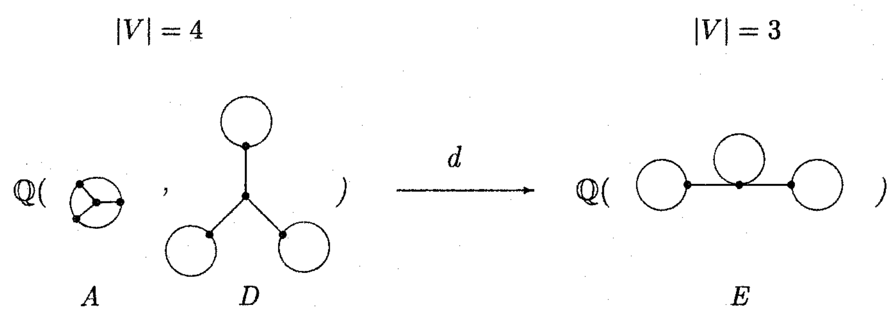


Figure 30. The odd GC^3, by its non-zero generators

To calculate the homology of this complex we first see that $d(A) = 0$, for by collapsing any edge in A we will get a graph with multiple edges. It is also easy to see that $d(D) = \pm 3E$. This shows that the complete graph on four vertices is the only non-trivial cycle; $H_4(GC^3) = \mathbb{Q}$ and $H_i(GC^3) = 0$ for $i \neq 4$.

Isomorphism classes of trivalent connected graphs of rank four are as follows:

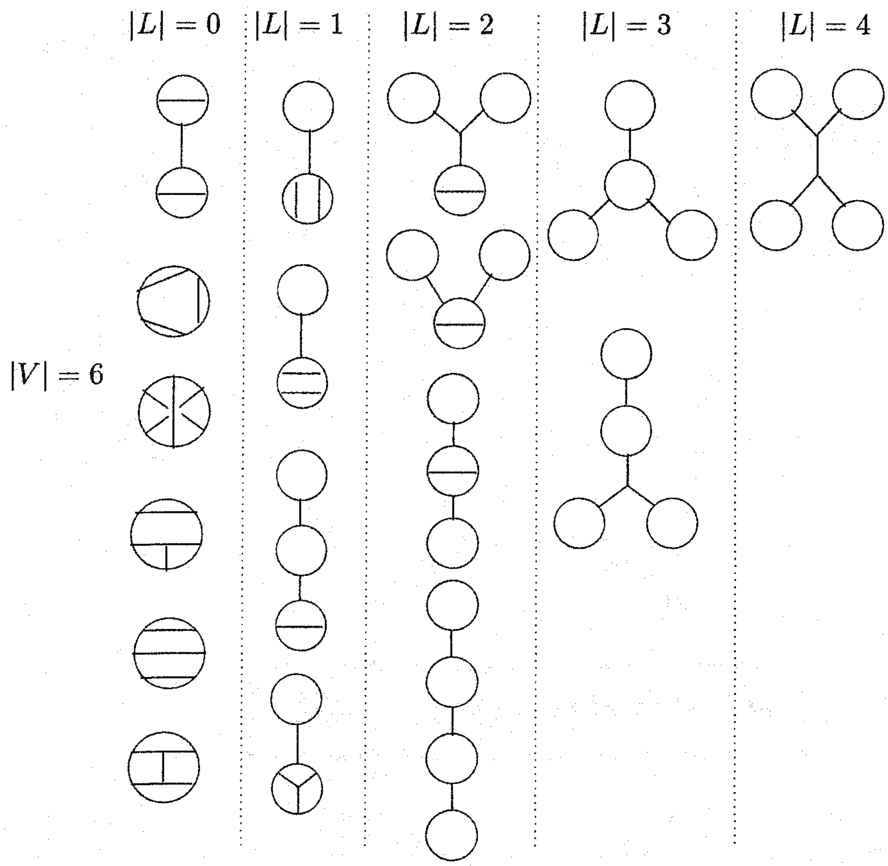


Figure 31. List of all connected trivalent graphs (up to isomorphism) of rank 4.

Example 7.0.51. *"Even" GC^4 :*

By looking at the list of six trivalent graphs without loops (column $|L| = 0$) above, one can see that the first and the third one from the top have orientation reversing automorphisms. The other four however do not have any orientation reversing automorphism, and therefore are nonzero and form a basis for GC^4_6, the subspace of GC^4 in degree six. Similarly there are only three non-zero graphs with 5 vertices, none with

4 vertices, one with 3 vertices and one with 2 vertices. They are shown below:

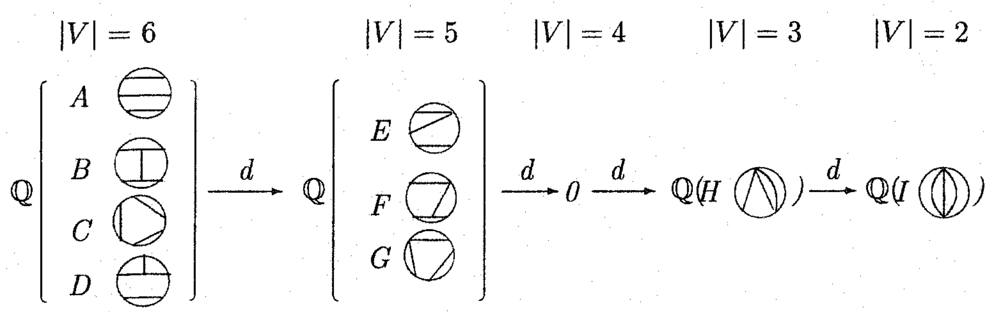


Figure 32. Even GC^4, by its non-zero generators, in each grading.

One can easily see that $d(A) = 4E$, $d(B) = 6F$, $d(C) = 2G$, $d(D) = 4E + 2F + G$ and $d(H) = I$. Choose the orientations of these graphs so that this is true exactly, and not up to a sign. From this it follows that $D - A - 1/3B - 1/3C$ is the only non-zero cycle. This shows that $H_i(GC^4, \mathbb{Q}) = 0$ for $i \neq 6$, and $H_6(GC^4, \mathbb{Q}) = \mathbb{Q}$. The latter group is generated by the following cycle:

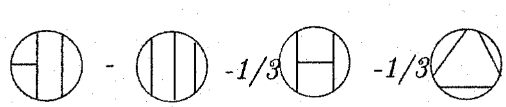


Figure 33. The only non trivial cycle in "even" GC^4.

Example 7.0.52. *"Odd" GC^4 :*

By looking at the all trivalent graphs in figure [31] we first disregard all those that have multiple edges. Then by remembering from theorem 3.0.26 that graphs with loops do not contribute to the homology we will be left with only two graphs shown below. They *are non-orientable because they have orientation reversing automorphisms. Thus all the the trivalent graphs vanish. All the possibly non-zero graphs with fewer vertices will be obtained from these two. The automorphism that kills these two may not kill their offspring a priori. However one easily could check that they also vanish in this case. Those obtained by the others which have multiple edges are automatically zero, since they inherit the multiple edges. Hence the odd* GC^4 *is zero in all dimensions.*

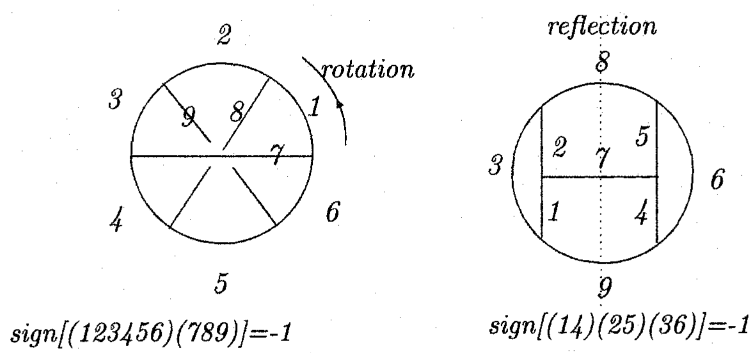


Figure 34.

Example 7.0.53. *In odd graph complex:*

• $H_1(GC_{\succeq 2}) = \mathbb{Q}$, *the only cycle is a loop attached to a vertex; also* $H_1(GC_{\succeq 3}) = 0$.

• $H_2(GC_{\succeq 2}) = H_2(GC_{\succeq 3}) = 0$, *for suppose* Γ *is a graph with two vertices and* m *edges, if* $m \geq 4$ *the graph either has double edges or double loops at a vertex and that makes it non-orientable; if* $m \leq 3$ *then* Γ *either will be a 2-gon or one of the two classes, A or B, in figure [27], and in each case it will be non-orientable.*

•$H_3(GC_{\succeq 2}) = H_3(GC_{\succeq 3}) = 0$; *for by 3.0.27 if* $n \geq 2$ *then* $H_3(GC^n, d_E) = 0$, *and we also previously saw in 7.0.46 that* $H_3(GC^1) = 0$.

• $H_4(GC_{\succeq 2}) = H_3(GC_{\succeq 3}) = \mathbb{Q}$, *the only cycle being the complete graph on 4* *vertices or 1-skeleton of the tetrahedron. Again by 3.0.27 if* $n \geq 4$ *then* $H_4(GC^n, d_E) = 0$. *Thus it is enough to examine only* GC^1, GC^2 *and* GC^3. *By looking at the previous examples, in which these cases were considered, the claim follows.*

Example 7.0.54. *In even graph complex:*

• $H_1(GC_{\succeq 2}) = H_1(GC_{\succeq 3}) = 0$, *because the even graph complex in this degree is zero.*

• $H_2(GC_{\succeq 2}) = H_2(GC_{\succeq 3}) = \mathbb{Q}$, *the only cycle being the class represented by graph A in figure [27], i.e., graph* Θ. *For suppose graph D, shown in the figure [35], is a graph in* GC^n, $n \geq 3$, *with two vertices and* $n+1$ *edges. If* $n+1$ *is even then the automorphism that switches its two vertices reverses its orientation, and hence as an oriented class is zero. If* $n+1$ *is an odd integer, then D is exact. For it is equal to* $d(A_{i,j})$ *where* $A_{i,j}$ *is the graph, shown in figure [35].*

• $H_3(GC_{\succeq 3}) = 0$, *and* $H_3(GC_{\succeq} 2) = \mathbb{Q}$, *the only cycle in the latter being the class represented by a triangle or a 3-gon. To show this, we examine the generators in gradings two, three, and four of all* GC^n*'s. In* GC^2 *and* GC^3 *we previously saw that there is no non-zero cycle in grading three. We can assume* $n \geq 4$. *We notice that any possible non-zero generator in* GC^n, $n \geq 4$, *with 3 vertices has one of the three forms shown in figure [35], where* i, j, m_1, m_2, m_3 *are integers greater than 1 and* n_1 *and* n_2 *are greater than 2. In the last two cases (F and H), it is easily seen that the two classes are exact. The first case* ($A_{i,j}$) *is more subtle however and needs the following analysis. If* $i+j = n+1$ *is an even integer, then the class represented by* $B_{i,j}$ *shown in the figure is non-zero, and* $d(B_{i,j}) = 2A_{i,j}$ *(in this case the graph D represents a zero class). If* $i+j$ *is an odd integer then* $B_{i,j} = 0$, *because it has an orientation reversing automorphism (the one that switches the two simple edges). In* *this case* $d(A_{i,j}) = D \neq 0$. *This shows that* $A_{i,j} - A_{i_1,j_1}$ *is a cycle. All other (possibly non-trivial) cycles in this grading are of this form. We show that this cycle is exact. Assume* $i \geq i_1$, *since* $i + j = i_1 + j_1 = n + 1$ *we have* $d(C_{i_1,i,j}) = A_{i,j} - A_{i_1,j_1}$. *In the case* $i - i_1 = 1$ *the graph* H *also appears in the image of* $C_{i_1,i,j}$ *under the differential* d. *In this case since* $d(G) = H$, *we still have* $d(C_{i_1,i,j} - G) = A_{i,j} - A_{i_1,j_1}$. *This shows there is non-zero cycle in degree three.*

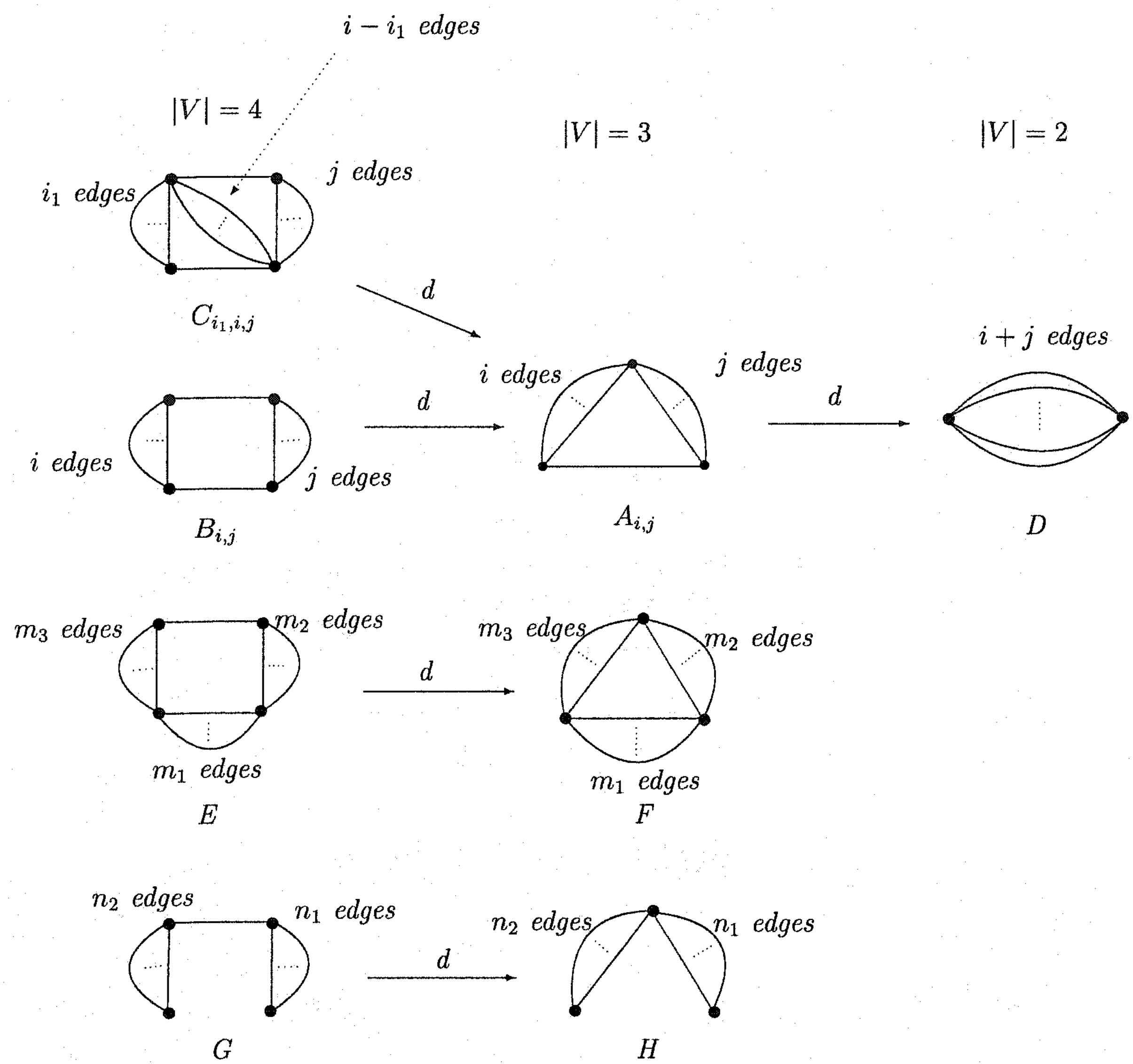


Figure 35. Showing the exactness of the even GC^n for $n \geq 4$ at the degree $|V| = 3$.

- $H_4(GC_{\succeq 3}) = H_4(GC_{\succeq 2}) \supseteq \mathbb{Q}$. *The only cycle seems to be the one obtained in 7.0.49. The calculation is similar to the previous case but naturally involves more scenarios. As it is only partially checked and is rather long we do not include it here.*

The above calculation can be summarized as follows:

Proposition 7.0.55. *The even and odd graph homologies in dimensions 1,2, 3 and 4 are as follows: In the even case:*

$H_i(GC^1) = \mathbb{Q}$ *for* $i \cong 3$ *mod 4 and 0 otherwise.*

$H_i(GC^2) = \mathbb{Q}$ *for* $i = 2$ *and 0 otherwise.*

$H_i(GC^3) = \mathbb{Q}$ *for* $i = 4$ *and 0 otherwise.*

$H_i(GC^4) = \mathbb{Q}$ *for* $i = 6$ *and 0 otherwise.*

$H_1(GC_{\succeq 3}) = H_1(GC_{\succeq 2}) = 0.$

$H_2(GC_{\succeq 3}) = H_2(GC_{\succeq 2}) = \mathbb{Q}.$

$H_3(GC_{\succeq 3}) = 0;\ H_3(GC_{\succeq 2}) = \mathbb{Q}.$

$H_4(GC_{\succeq 3}) = H_4(GC_{\succeq 2}) \supseteq \mathbb{Q}.$

In the odd case:

$H_i(GC^1) = \mathbb{Q}$ *for* $i \cong 1$ *mod 4 and 0 otherwise.*

$H_i(GC^2) = 0$ *in all degrees.*

$H_i(GC^3) = \mathbb{Q}$ *for* $i = 4$ *and 0 otherwise.*

$H_i(GC^4) = 0$ *in all degrees.*

$H_1(GC_{\succeq 3}) = 0; H_1(GC_{\succeq 2}) = \mathbb{Q}.$

$H_2(GC_{\succeq 3}) = H_2(GC_{\succeq 2}) = 0.$

$H_3(GC_{\succeq 3}) = H_3(GC_{\succeq 2}) = 0.$

$H_4(GC_{\succeq 3}) = H_4(GC_{\succeq 2}) = \mathbb{Q}.$

In the proposition above, one may notice that in the even graph complex, the top graph homology is non-zero for n=2,3, and 4. This is actually the case in all higher dimensions. In the following we will see one such construction outlined by M. Kontsevich in [1] for generating such classes.

Proposition 7.0.56. *Any finite dimensional Lie algebra L, equipped with a fixed non-degenerate invariant scalar product $<,>$, defines a sequence of non-zero cycles in the top graph homology of "even" GC^n, i.e., in $H_{2n-2}(GC^n;\mathbb{Q})$.*

Proof. Choose an orthogonal base $\{e_1,...,e_m\}$ in L with respect to $<,>$, i.e., $<e_i,e_j>=0$ if $i\neq j$. The structure constants c_{ij}^k are defined uniquely by the equation $[e_i,e_j]=\sum c_{ij}^k e_k$. From the Jacobi identity, $\sum[[e_i,e_j],e_k]=0$ (when the sum is over cyclic permutation of i,j and k), it follows that $\sum_l(c_{ij}^l c_{lk}^h + c_{jk}^l c_{li}^h + c_{ki}^l c_{lj}^h)=0$. Also from the antisymmetry of the bracket it is clear that $c_{ij}^k=-c_{ji}^k$. From the invariance, i.e., $<[e_i,e_j],e_k>=<e_i,[e_j,e_k]>$, and the orthogonality of the base with respect to $<,>$, it follows that $c_{ij}^k=c_{jk}^i$ and similarly $c_{jk}^i=c_{ki}^j$, thus establishing the cyclic symmetry $c_{ij}^k=c_{jk}^i=c_{ki}^j$

Now, consider any trivalent graph (Γ,or) in the even GC^n. We saw in ?? that for a pure trivalent graph we have $A\otimes V=C$, where C was the choosing of cyclic ordering at each vertex up to reversing any two. Label each edge in Γ with numbers from 1 through m, repetition is allowed. Then to each vertex with adjacent numbers i,j and k assign c_{ij}^k according to the cyclic ordering. Multiply all these numbers (over the set of vertices), and then sum over all the different labelings of Γ. In this way we get a number which we call $\xi(\Gamma,or)$. We have $\xi(\Gamma,-or)=-\xi(\Gamma,or)$. The reason is that $-or$ can be obtained from or by changing (or reversing) the cyclic ordering at one vertex. It is enough to show one such reversion inflicts a sign on ξ. So suppose $(ijk) = (jki) = (kij)$ is the cyclic ordering at one given vertex and $c_{ij}^k = c_{jk}^i = c_{ki}^j$ is the structure constant associated with it. Reversing it would be the cyclic ordering $(ikj) = (kji) = (jik)$ at the vertex, and the structure constant associated with this new cyclic orientation would be $c_{ji}^k = c_{kj}^i = c_{ik}^j$, which is the opposite (in sign) of the previous structure constant.

Now consider chain α defined as following:

$$\alpha := \sum_{\text{all trivalent graphs in } GC^n} \frac{\xi(\Gamma, or)}{|Aut(\Gamma)|} (\Gamma, or)$$

α is a cycle in the top homology of GC^n, i.e., in $H_{2n-2}(GC^n)$.

To show that $d(\alpha) = 0$, first we notice that $d(\alpha)$ can be written as a linear combination of graphs, each of which with valence three at all but one vertex, which has valence four. Consider one such graph and call it Γ', and suppose its coefficient in $d(\alpha)$ is m, i.e., $d(\alpha) = m.\Gamma' +$ other terms. By blowing up the valence 4 vertex of Γ' and some analysis we will show $m = 0$. Since Γ' was one such arbitrary graph, this will prove that $d(\alpha) = 0$.

The reason is that for each labelling of Γ', the valence 4 vertex v seen below, can be blown up in three ways:

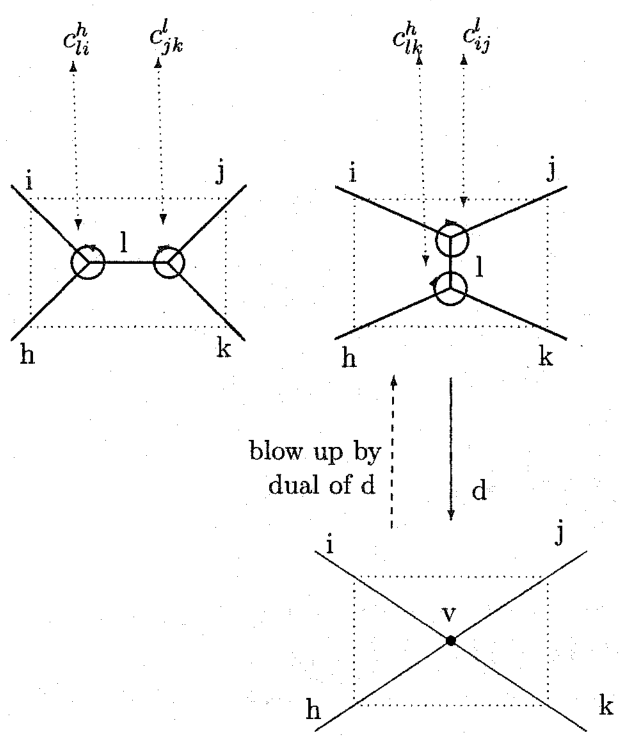


Figure 36. IHX or Jacobi, $\Sigma_l(c^l_{ij}c^h_{lk} + c^l_{jk}c^h_{li} + c^l_{ki}c^h_{lj}) = 0$, implies that α is a cycle.

Note that outside the dashed boxes the four graphs in the picture are identically the same (and labelled the same). Each of the three graphs can produce Γ' if the newly formed edge is collapsed. If we add up the contributions of $\frac{\xi(\Gamma_1,or)}{|Aut(\Gamma_1)|}$ (Γ_1, or) ,$\frac{\xi(\Gamma_2,or)}{|Aut(\Gamma_2)|}$ (Γ_2, or), and $\frac{\xi(\Gamma_3,or)}{|Aut(\Gamma_3)|}$ (Γ_3, or) to the coefficient of (Γ', or) we will get

$$\text{"something"}.[\sum_l (c^l_{ij}c^h_{lk} + c^l_{jk}c^h_{li} + c^l_{ki}c^h_{lj})]$$

which is 0 by the Jacobi. The "something" is the number corresponding to the identical parts of the three graphs outside of the dashed boxes and can be factored out.

Let us explain more carefully why we have to divide by the cardinality of automorphism group of the graph. To unravel this, we have to understand the meaning of $\xi(\Gamma, or)$ more deeply. ξ is a state sum, i.e., sum of many summands or terms of the form $\bigotimes_{\text{set of the vertices of graph }\Gamma} c_{ij}^k$. So if in any given labelling of the vertices of the graph Γ by numbers 1 through m, any two labels adjacent to one vertex are equal, by antisymmetry of structure constants that yield a zero summand in the state sum. For this reason many labelings contribute nothing. On the other hand if any labelling produces a non-zero number, then $Aut(\Gamma)$ can act on this and send it to another labelling which produces the same summand in the state sum. Thus any non-zero summand appears $|Aut(\Gamma)|$ times in the state sum of the graph Γ. So

$$\xi(\Gamma, or) = |Aut(\Gamma)|.[\sum_{\text{all labellings}} (\bigotimes_{\text{set of vertices of graph }\Gamma} c_{ij}^k)].$$

Each term inside the parenthesis is product of $(2n-2)$ c_{ij}^k's, thus the "something" above corresponds to the product of the $2n-4$ factors (vertices) of c_{ij}^k that lie outside the dashed boxes.

□

Appendix: Deformation Quantization

This section is just a few words about deformation quantization, formality conjecture and the odd commutative graph complex. A more complete note on these will appear later. The interested reader is recommended to see [3], [4], [5], [30].

In the words of Kontsevich [3]:

"This conjecture implies arbitrary smooth Poisson manifold can be formally quantized, and the equivalence class of the resulting algebra is canonically defined. In other terms, it means that non-commutative geometry of smooth spaces, is described by the semi-classical approximation"

In [3] he claims and gives evidence that if the formality conjecture for flat spaces $X = \mathbb{R}^d, d = 1, 2, 3, ...$ holds up to the $(n-1)$-st term in perturbation theory, then the possible obstruction to the formality on the n-th step comes from a cohomology class of the subcomplex of the odd graph complex in degree $(n, 2n-3)$, spanned by "good" graphs. Where a "good" graph (according to him) is a graph that is non-empty, connected, nonseparable, has no multiple edges, and has valence three or more at each vertex (nonseparability means that the complement of any vertex is connected). It is easy to see that the subcomplex spanned by good graphs is the same as $((GC_{\geq 3}/GC^L)/C_1', d)$ defined in 6.0.42, which we showed in fact is quasi isomorphic to the total complex.

In other words, if $H^{n,2n-3}(GC^{n-2}) = 0$ then the formality conjecture holds, where $(n, 2n-3) = (|V|, |E|)$ is the bigrading (with our notation of grading, by the number of vertices, this can be expressed as $H_n(GC^{n-2}) = 0$).

In 7.0.55 our calculation verifies this is true up to $n \leq 6$. Kontsevich later proved formality conjecture by other methods [4].

Remark 7.0.57. *I have an extended (but unedited) version of this manuscript, containing a few extra appendices explaining the relation of graph homology to other areas, such as moduli space of Riemann surfaces, automorphism and outer automorphism groups of free groups, Lie algebra homology, deformation theory, and Vassiliev knot invariants. The interested reader may contact me for the hopefully by then edited version of it, or better yet may directly find these various relations of graph complexes to the other areas at their (original) sources which have been provided in the bibliography.*

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