Algebraic Models for the Free Loop Space and Differential Forms of a Manifold

by

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Abstract

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Our initial goal is to give a chain level description of the string topology loop product for a large class of spaces. This effort is described in two parts; the first uses Brown's theory of twisting cochains to obtain a model for the free loop space of a manifold and the second constructs a minimal model for the Frobenius algebra of differential forms of a manifold. The first part defines the loop product for closed, oriented manifolds and Poincar/'e Duality spaces. The second part is an attempt to understand the $Frob_\infty$ algebra of a manifold, with the idea of extending the methods in the first section to define the loop product for open manifolds.

Brown's theory of twisting cochains provides a chain model of a principal G bundle and its associated bundles. The free loop space is obtained by considering the path space fibration, and taking the associated bundle with the based loop space acting on itself by conjugation. Given a twisting cochain, then, we obtain a chain model of LM using Brown's theory. To

describe the chain-level loop product in this setting, we need a model for the intersection product in $C_*(M)$. For this, we use the cyclic C_∞ algebra structure on $H_*(M)$. Such a description would give a chain level description of the string topology loop product for open manifolds. We explain this in detail in Section 2.

Instead of using the cyclic C_∞ algebra, we could have used the Frobenius algebra structure. One would expect that the $Frob_\infty$ algebra can be used to show the necessary relations to define the loop product. Then given the $Frob_\infty$ algebra on $H_*(M)$ for an open manifold, we would have a chain level description of the loop product.

The purpose of Section 3 is to gain a better understanding of the $Frob_\infty$ algebra on $H^*(M)$. The Frobenius algebra on $H^*(M)$, induced by the wedge product and Poincar/'e Duality, is well understood; the structure on the level of forms inducing the Frobenius algebra is less well understood. We use the language of operads, dioperads, and properads and Koszul duality to give a definition of $Frob_\infty$ algebra. We also use descriptions of the transfer of structure using trees and integrating over cells in the moduli space of metrised ribbon graphs. When M is closed and oriented, these tools allow us to build a minimal model for the Frobenius algebra $\Omega^*(M)$ and to compare it with the cyclic C_∞ algebra $H^*(M)$.

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Chapter 1

Introduction

In Section 2, we use Brown's theory of twisting cochains to construct a chain level description of the free loop space of a closed, oriented manifold. In Section 3, we construct a minimal model for the Frobenius algebra of differential forms of a closed, oriented manifold. We outline the process in this section.

1.1 Twisting cochains and the free loop space

Brown's theory of twisting cochains, outlined in [B], provides a way to model the total space of a bundle in terms of the base and fiber. Given a principal bundle $G \to P \to M$ and a twisting cochain $\tau : C_*(M) \to C_*(G)$, Brown constructs a complex $(C_*(M) \otimes C_*(G), \partial_\tau)$ whose homology is isomorphic to $H_*(P)$. If Y is a G space and $Y \to P \times_G Y \to M$ is the associated bundle, then there is a complex $(C_*(M) \otimes C_*(Y), \partial_\tau)$ whose homology is isomorphic to $H_*(P \times_G Y)$. Quillen, in [Q], shows that when $Im(\tau) \subset Prim(H_*)$, the isomorphism is one of coalgebras. There is an extensive literature on twisting cochains due to their wide ranging applications. We have focused on these two results immediately related to this discussion.

In Section 2.2, we push Brown's theory to homotopy algebras. That is, given a C_∞ coalgebra C_*, a dg bialgebra H_*, and a twisting cochain $\tau : C_* \to H_*$ where $Im(\tau) \subset Prim(H_*)$, we define a twisted A_∞ coalgebra on $C_* \otimes H_*$. The twisted coalgebra structure is denoted $\{c_n^\tau : C_* \otimes H_* \to (C_* \otimes H_*)^{\otimes n}\}$. The twisted term in Brown's differential is described by applying the coproduct on C_*, then applying τ to one of the factors, and finally using the multiplication in H_*. The same idea is used for c_1^τ, except we use the higher homotopies $\{c_n : C_* \to C_*^{\otimes n}\}$ of the C_∞ coalgebra structure as well as the coproduct. We use the same process to obtain c_2^τ, except we use the maps $c_{n>2}$. And the process continues for all c_n^τ. If C_* is a strict differential graded coalgebra with $c_n = 0$ for $n > 2$, then the complex reduces to Brown's complex. For this reason, we denote c_1^τ by ∂_τ. The following theorem is proved in Section 2.2.

Theorem 1.1.1. *Let C_* be a C_∞ coalgebra, H_* a dg bialgebra, and τ :*

$C_* \to H_*$ *a twisting cochain such that* $Im(\tau) \subset Prim(H_*)$. *The maps* $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$ *define an* A_∞ *coalgebra on* $C_* \otimes H_*$.

We then define the conjugation action of H_* on itself. The action of a primitive element on H_* is both a derivation and a coderivation. If we go through the process of defining $\{c_n^\tau\}$ as above, except instead of using the multiplication in H_*, we use the conjugation action, the resulting maps also define an A_∞ coalgebra structure. Because the conjugation action involves the antipode map, we require H_* to be a dg Hopf algebra, as opposed to a dg bialgebra found in the Theorem 2.2.9.

Theorem 1.1.2. *Let* C_* *be a* C_∞ *coalgebra,* H_* *a dg Hopf algebra, and* $\tau : C_* \to H_*$ *a twisting cochain such that* $Im(\tau) \subset Prim(H_*)$. *The maps* $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$, *obtained using the conjugation action, define an* A_∞ *coalgebra on* $C_* \otimes H_*$.

Since the conjugation action is a derivation, if C_* also has a multiplication, it is reasonable to ask for an A_∞ algebra on $C_* \otimes H_*$. When C_* is a cyclic C_∞ coalgebra, there is a twisted A_∞ algebra on $C_* \otimes H_*$.

Theorem 1.1.3. *Let* C_* *be a cyclic* C_∞ *coalgebra,* H_* *be a Hopf algebra, and* $\tau : C_* \to H_*$ *be a twisting cochain with* $Im(\tau) \subset Prim(H_*)$. *The maps*

$\{\partial_\tau, m_2, m_3, \cdots\}$, *defined using the conjugation action in* H_*, *give* $C_* \otimes_\tau H_*$ *the structure of an* A_∞ *algebra.*

The A_∞ algebra and A_∞ coalgebra share the same differential ∂_τ, so they compute the same linear homology. We still do not know what the further compatibilities are.

Section 2.3 applies this work to the path space fibration $\Omega_b(M) \to P_b(M) \to M$. Since $\Omega_b(M)$ is homotopy equivalent to a topological group, we consider $P_b(M) \to M$ to be a principal bundle. The first step is to construct a twisting cochain $H_*(M) \to T(H_*(M)[-1])$, whose image is in $\mathcal{L}(H_*(M)[-1])$. We obtain such a map by considering the construction of a power series connection. Then we apply Theorem 2.2.9 to get an A_∞ coalgebra model of the based path space.

We also get a description of string topology operations from the path space fibration. Any group acts on itself by conjugation. The conjugate bundle is defined to be the associated bundle of a principal G bundle with respect to the conjugation action. The conjugate bundle obtained from the path space fibration is a model of the free loop space. Applying Theorem 2.2.17 gives an A_∞ coalgebra structure modeling the coalgebra on $H_*(LM)$ induced by the diagonal map. Applying Theorem 2.2.18 gives an A_∞ algebra structure modeling the algebra on $H_*(LM)$ given by the loop product.

The final section in the chapter applies the work in Section 2.2 to the case of a principal G bundle $G \to P \to M$, where G is a connected Lie group. The A_∞ coalgebra on $H_*(M) \otimes H_*(G)$ given by applying Theorem 2.2.9 can be expressed in terms of the characteristic classes of the bundle. We can also consider the conjugate bundle, denoted $Conj(P) \to M$. Then Theorem 2.2.17 gives an A_∞ coalgebra model for $H_*(Conj(P))$ and Theorem 2.2.18 gives an A_∞ algebra model.

Given a connection on $P \to M$, there is a map of bundles $P_b(M) \to P$, which induces a map on associated bundles with respect to the conjugation action $Conj(P_b(M)) \to Conj(P)$. Then the algebraic structures we get modeling the total space $Conj(P)$ are representations of algebraic structures on $Conj(P_b(M))$. In this way, we get representations of string topology.

1.2 Minimal model for the differential forms

In the second part of the thesis, we investigate the algebraic structure on the differential forms of a closed, oriented manifold $\Omega^*(M)$ and the induced structure on its cohomology $H^*(M)$. It is well known that $H^*(M)$ is a Frobenius algebra, i.e., there is a multiplication and comultiplication, such that the comultiplication is a map of modules. The structure on $\Omega^*(M)$

which induce Frobenius algebra is less well understood. We provide a model for this structure and show that when we transfer the maps to $H^*(M)$, we obtain a $Frob_\infty$ algebra. We then show that this $Frob_\infty$ algebra is a homotopy invariant of the manifold.

To do this, we recall in Section 3.1 the definition and properties of operads, dioperads, and properads. The main examples in this paper are

1. the cyclic C_∞ operad, which is used to describe the cyclic C_∞ algebra on $H^*(M)$ induced by the wedge product and Poincar/'e Duality,
2. the $Frob^0_\infty$ dioperad, which is used to describe the Frobenius algebra on $H^*(M)$,
3. and the $Frob_\infty$ properad, which models the Frobenius algebra structure as well but also includes operations with genus.

We will also need to discuss the notion of partial algebras. These objects were discussed by Wilson, [W], in the context of operads. We extend the discussion to include dioperads and properads.

In Section 3.2, we discuss methods of transferring algebraic structures from a chain complex to its homology as a way to build minimal models. Kontsevich and Soibelman show, given a contraction between an A_∞ algebra $(A_*, \{m_n^A\})$ and a chain complex B_*, how to define an A_∞ algebra structure

$\{m_n^B\}$ on B_*, [KS]. Cheng and Getzler, [CG], show that if A_* is a C_∞ algebra, then $\{m_n^B\}$ will be a C_∞ algebra structure. In a more general setting, Markl shows that for algebras A_* over a cofibrant replacement of a dg operad have the property that if there is a contraction between A_* and a chain complex B_*, then B_* is an algebra over the cofibrant replacement [M]. Sullivan, [S] and Granaker, [Gr], extend this discussion to dioperads and properads. We provide an explicit way to transfer a $Frob_\infty^0$ algebra, following the methods of [KS] and [CG].

With the abstract setting in place, the rest of this section focuses on constructing a minimal model for $\Omega^*(M)$ in two ways. First, we consider the partial Frobenius algebra on $Curr_*(M)$, using ideas from topological conformal field theory. This construction makes use of the heat kernel $\widehat{K}_t : Curr_*(M) \to Curr_*(M)$, which can be thought of as a contraction between $Curr_*(M)$ and itself. The differential forms $\Omega^*(M)$ sit inside $Curr_*(M)$ as smooth currents, and we show that the partial $Frob_\infty$ restricts to $\Omega^*(M)$. We apply the transfer to theorem to obtain a minimal model for $H^*(M)$.

The second approach for constructing a $Frob_\infty$ algebra is to compose functors

$$\{\text{cyclic operads}\} \to \{\text{dioperads}\} \to \{\text{properads}\}$$

to the map of cyclic operads C_∞operad $\to End(H^*(M))$ to obtain $Frob_\infty \to End(H^*(M))$. This will also be a $Frob_\infty$ algebra. We show in Section 3.5 that this $Frob_\infty$ algebra is the same as the previous one. Since $(H^*(M), \{m_n\}, PD)$ is known to be a homotopy invariant, this proves the claim.

Chapter 2

Free Loop Space

2.1 Background Material

Algebras and coalgebras are taken over $\mathbb{Q}$. Homology and cohomology are taken with coefficients in $\mathbb{Q}$.

2.1.1 Twisting Cochains

We first describe Brown's theory of twisting cochains in a purely algebraic setting. Let C_* be a differential graded coalgebra and A_* a differential graded algebra. Then $(Hom(C_*, A_*), \partial_C \otimes 1 + 1 \otimes \partial_A)$ is a differential graded algebra, and a twisting cochain is an element $\tau \in Hom(C_*, A_*)$ satisfying the Maurer Cartan equation

$$\partial_A \circ \tau - \tau \circ \partial_C + \tau \cdot \tau = 0.$$

The Maurer Cartan equation makes sense for any differential graded algebra, and a twisting cochain is a Maurer Cartan element in a differential graded algebra of the form $Hom(C_*, A_*)$. The tensor differential $\partial_C \otimes 1 + 1 \otimes \partial_A$ on $C_* \otimes A_*$ is twisted by adding a term

$$C_* \otimes A_* \overset{\Delta \otimes 1}{\to} C_* \otimes C_* \otimes A_* \overset{1 \otimes \tau \otimes 1}{\to} C_* \otimes A_* \otimes A_* \overset{1 \otimes m}{\to} C_* \otimes A_*.$$

We refer to this term as the twisted term, and ∂_τ is the sum of the tensor differential and twisted term. The coproduct on C_* defines a comodule on the tensor $C_* \otimes A_* \to C_* \otimes C_* \otimes A_*$. The coalgebra C_* is a comodule over itself.

Theorem 2.1.1. *[B] Let C_* be a coalgebra, A_* an algebra, and τ a twisting cochain. Then $(C_* \otimes A_*, \partial_\tau)$ is a chain complex. If $C_1 = 0$ and $\epsilon : A_* \to k$ is an augmentation, then $Id \otimes \epsilon : C_* \otimes A_* \to C_*$ is a map of comodules.*

Proof. In [[B], p. 229], ∂_τ is shown to square to zero. We give a diagrammatic proof of that $\partial^2 = 0$ in Remark 2.2.8.

The map $1 \otimes \epsilon$ obviously commutes with the comodule map, since the comodule map on $C_* \otimes A_*$ is given by the coproduct on C_* and the coproduct on C_* is the comodule structure for C_*. To show that $1 \otimes \epsilon$ commutes with the differential, it suffices to show that $1 \otimes \epsilon$ vanishes on the twisted term. To see this, note that ϵ is zero on any element of positive degree in A_*. Let

$c \otimes h \in C_* \otimes A_*$. If h is in positive degree, then the twisted term will have positive degree in the A_* factor and will map to zero under $1 \otimes \epsilon$. Consider $C_* \otimes 1$ in $C_* \otimes A_*$ and $\Delta(c) = \sum c_{(1i)} \otimes c_{(2i)}$. Since τ is a degree -1 map, $\tau(c_{(2i)})$ will have positive dimension for $|c_{(2i)}| > 1$ and be zero for $|c_{(2i)}| = 0$. Since $C_1 = 0$, $1 \otimes \epsilon$ will vanish on the twisted term.

□

We write $C_* \otimes_\tau A_*$ for the twisted complex $(C_* \otimes A_*, \partial_\tau)$.

This theory can be applied to principal bundles $G \to P \to M$. The chain complex $C_*(M)$ is a differential graded coalgebra, where the coproduct is induced by the diagonal map $M \to M \times M$. The group multiplication of G provides an algebra structure on $C_*(G)$. A twisting cochain is then a map $\tau : C_*(M) \to C_*(G)$ satisfying the Maurer Cartan equation.

The complex $(C_*(M) \otimes C_*(G), \partial_M \otimes Id + Id \otimes \partial_G)$ will not, in general, compute the homology of P. However, when we twist the differential by a suitable twisting cochain $\tau : C_*(M) \to C_*(G)$, the complex $(C_*(M) \otimes C_*(G), \partial_\tau)$ will compute $H_*(P)$.

Theorem 2.1.2 ([B], Theorem (4.2))**.** *The chain complex* $(C_*(M) \otimes C_*(G), \partial_\tau)$ *is chain equivalent to* $C_*(P)$.

The equivalence of the above theorem is of chain complexes and not of dg coalgebras, despite the fact that both complexes have coproducts. A further assumption is needed on τ to obtain an equivalence of dg coalgebras.

We return to the general setting. Let C_* be a dg coalgebra and H_* a dg bialgebra. The primitive elements $Prim(H_*) = \{h \in H_* | \Delta(h) = h \otimes 1 + 1 \otimes h\}$ is a Lie algebra whose universal enveloping algebra is H_*. The following lemma is a reformulation of Quillen ([Q], Appendix B).

Lemma 2.1.3. *Let $\tau : C_* \to H_*$ be a twisting cochain from a cocommutative coalgebra to a dg bialgebra. Then $(C_* \otimes H_*, \partial_\tau)$ is a differential graded coalgebra if and only if $Im(\tau) \subset Prim(H_*)$.*

Proof. To show that ∂_τ is a coderivation we need to show that

$$(\Delta_{C \otimes H})\partial_\tau = (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau)\Delta_{C \otimes H}.$$

The key is that multiplication by a primitive element is a coderivation. We give a diagrammatic proof in Remark 2.2.8. The reader can find the computation in ([Q], p. 289).

□

2.2 Algebraic Setting for Twisted Tensor Products

In this section, we extend the discussion of Brown's theory of twisting cochains to the homotopy algebra setting. Let $(C_*, \{c_n\})$ be a C_∞ coalgebra and H_* a strict dg bialgebra. Given a twisting cochain $\tau : C_* \to H_*$, we define a twisted A_∞ coalgebra structure on $C_* \otimes H_*$.

There are three properties that are used in Brown's setting. For C_* a strict dg coalgebra and A_* a strict dg algebra, the following properties are used.

1. $Hom(C_*, A_*)$ is a differential graded algebra.

2. twisting cochains $\tau : C_* \to A_*$ are in one to one correspondence with chain maps $\mathcal{F}(C_*) \to A_*$, where $\mathcal{F}$ is the cobar functor.

3. a twisting cochain $\tau \in Hom(C_*, A_*)$ defines a twisted differential on $C_* \otimes A_*$.

We address the analogs of these properties in the following subsections.

2.2.1 Maurer Cartan Equation in the Homotopy Algebra Setting

We review some definitions. An A_∞ algebra consists of a vector space V and maps $\{m_n : V[-1]^{\otimes n} \to V[-1]\}$ satisfying

$$\sum_{k=1}^{n} \sum_{j=0}^{n-1} m_{n-k+1} \circ (Id^{\otimes j} \otimes m_k \otimes Id^{n-j-k}) = 0.$$

The maps $\{m_n\}$ define a coderivation of square zero on $T(V[-1])$. The shuffle product is a map $T(V[-1]) \otimes T(V[-1]) \to T(V[-1])$. If m_n vanishes on the image of the shuffle product for every n, then $(V, \{m_n\})$ is a C_∞ algebra.

An A_∞ coalgebra and C_∞ coalgebra are the dual notions of A_∞ and C_∞ algebras. So V is an A_∞ coalgebra if there are maps $\{c_n : V[-1] \to V[-1]^{\otimes n}\}$ defining a derivation of square zero on $T(V[-1])$. If the unshuffle product $T(V[-1]) \to T(V[-1]) \otimes T(V[-1])$ vanishes on the image of each c_n, then $(V, \{c_n\})$ is a C_∞ coalgebra.

To deal with issues of convergence, we will make use of the completed tensor product. For a vector space V, let

$$\widehat{T}(V) = \prod_{i=0}^{\infty} V^{\otimes i}.$$

In our applications, V will be a finite dimensional vector space. So V has a unique topology making it a topological vector space. There is an induced topology on $\widehat{T}(V)$, known the inverse limit topology.

In order to say τ is a twisting cochain, the vector space $Hom(C_*, H_*)$ must have at least an A_∞ algebra structure. Moreover, we need the Lie algebra version of the Maurer Cartan equation, so we need an L_∞ algebra on $Hom(C_*, Prim(H_*))$.

Lemma 2.2.1. *Let $(C_*, \{c_n\})$ be a C_∞ coalgebra and A_* a differential graded algebra. The vector space $Hom(C_*, A_*)$ is an A_∞ algebra.*

Since $Hom(C_*, A_*) \cong C^* \otimes A_*$, the lemma is just the statement the tensor product of an A_∞ algebra with an associative algebra is an A_∞ algebra. We omit the proof, but define the maps m_n. Let

$$m_1^{Hom}(f) = \partial_A \circ f + (-1)^{|f|} f \circ \partial_C,$$

where $\partial_C = c_1$ of the C_∞ coalgebra structure. For $n > 1$, let

$$\begin{aligned} m_n^{Hom}(f_1, \cdots, f_n) : C_* &\rightarrow A_* \\ c &\mapsto m_A(f_1 \otimes \cdots \otimes f_n))c_n(c), \end{aligned}$$

where by m_A we mean multiply all the terms using multiplication of A_*. Since the multiplication in A_* is associative, m_n^{Hom} is well-defined.

The Maurer Cartan equation is then

$$\partial \circ \tau + \tau \circ \partial + m_2^{Hom}(\tau,\tau) + m_3^{Hom}(\tau,\tau,\tau) + m_4^{Hom}(\tau,\tau,\tau,\tau) + \cdots = 0.$$

Since we have an infinite sum, a note on convergence is in order. In our application, $A_* = \widehat{T}(H_*(M)[-1])$. The twisting cochain we construct will have the property that

$$Im(m_n(\tau, \cdots, \tau)) \subset (H_*(M)[-1])^{\otimes n}$$

So the infinite sum can be expressed as a finite sum in different tensors. This is well defined in the completed tensor product.

For the Lie version of the Maurer Cartan equation, we will need the following fact about L_∞ algebras.

The Koszul sign convention says that when two elements x and y of degree p and q are commuted, a sign of $(-1)^{pq}$ is obtained. For $x_1, \cdots, x_n$ and a permutation $\sigma \in S_n$, let $\epsilon(\sigma; x_1, \cdots x_n)$ be the sign so that

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1, \cdots, x_n) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)},$$

in the free graded commutative algebra $\bigwedge(x_1, \cdots, x_n)$. Let $\xi(\sigma) = sgn(\sigma) \cdot \epsilon(\sigma; x_1, \cdots, x_n)$.

Theorem 2.2.2 ([LM], Theorem 3.1)**.** *Let $(V, \{m_n\})$ be an A_∞ algebra.*

Then there is an L_∞ algebra on V given by symmetrizing m_n. That is, if

$$l_n(v_1, \cdots, v_n) = \sum_{\sigma \in S_n} \xi(\sigma) m_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

then $(V, \{l_n\})$ is an L_∞ algebra.

We denote the L_∞ algebra by $[V]$ to distinguish it from the A_∞ algebra V.

Lemma 2.2.3. *Let $(C_*, \{c_n\})$ be a C_∞ coalgebra and L_* be a differential graded Lie algebra. Then $Hom(C_*, L_*)$ is an L_∞ algebra.*

Proof. Our proof proceeds as follows. Let $U(L_*)$ be the universal enveloping algebra of L_*. The previous lemma shows that the space $Hom(C_*, U(L_*))$ is an A_∞ algebra, with structure maps $\{m_n\}$. Symmetrizing each m_n defines an L_∞ algebra, with structure maps denoted $\{l_n\}$. Let $c_n(x) = x_{n,1} \otimes \cdots \otimes x_{n,n}$. Then the L_∞ algebra is given by

$$l_n(f_1 \cdots f_n)(x) = \sum_{\sigma \in S_n} \xi(\sigma) f_1(x_{n,\sigma(1)}) \cdots f_n(x_{n,\sigma(n)}).$$

To prove the lemma, it suffices to show that the maps $\{l_n\}$ restricts to $Hom(C_*, L_*) \subset Hom(C_*, U(L_*))$.

Suppose $f_i \in Hom(C_*, L_*)$. This implies $\Delta(f_i(x)) = f_i(x) \otimes 1 + 1 \otimes f_i(x)$, where the coproduct is in $U(L_*)$. Since Δ is an algebra map, we see that

$$
\begin{aligned}
& \Delta \circ l_n(f_1, \cdots, f_n)(x) \\
= & \sum_{\sigma \in S_n} \Delta(f_1(x_{n,\sigma(1)})) \cdots \Delta(f_n(x_{n,\sigma(n)})) \\
= & \sum_{\sigma \in S_n} (f_1(x_{n,\sigma(1)} \otimes 1 + 1 \otimes f_1(x_{n,\sigma(1)}))) \cdots (f_n(x_{n,\sigma(n)}) \otimes 1 + 1 \otimes f_n(x_{n,\sigma(n)})) \\
= & \sum_{\sigma \in S_n} f_1(x_{n,\sigma(1)}) \cdots f_n(x_{n,\sigma(n)}) \otimes 1 + 1 \otimes f_1(x_{n,\sigma(1)}) \cdots f_n(x_{n,\sigma(n)}) \\
& + \sum_{\sigma \in S_n} \sum_j f_1(x_{n,\sigma(1)}) \cdots f(x_{n,\sigma(j)}) \otimes f(x_{n,\sigma(j+1)}) \cdots f_n(x_{(n,\sigma(n))}).
\end{aligned}
$$

We need to show that the cross terms cancel. The composition

$$C_* \stackrel{c_n}{\to} C_*^{\otimes n} \hookrightarrow T(C_*) \stackrel{\text{unshuffle}}{\to} T(C) \otimes T(C)$$

is zero by definition of a C_∞ coalgebra. Each permutation σ is an (i,j) unshuffle of some linear order of the $\{x_{n,i}\}$. For example, for S_3, the collection of all the $(2,1)$ unshufflings of $x_{3,1} \otimes x_{3,2} \otimes x_{3,3}$ and $x'_{3,1} \otimes x'_{3,2} \otimes x'_{3,3} = x_{3,2} \otimes x_{3,1} \otimes x_{3,3}$ exhausts all combinations of $x_{3,\sigma(1)} \otimes x_{3,\sigma(2)} \otimes x_{3,\sigma(3)}$.

The L_∞ algebra on $Hom(C_*, L_*)$ is then given by

$$l_n(f_1, \cdots f_n)(x) = \sum_{\sigma \in S_n} \xi(\sigma) f(x_{1,\sigma(1)}) \cdots f(x_{n,\sigma(n)}),$$

where the multiplications are in $U(L_*)$. □

Let A_* and B_* be two A_∞ algebras and $\{f_n : A_*^{\otimes n} \to B_*\}$ an A_∞ algebra morphism. Suppose the Maurer Cartan equation is well defined for A_* and

B_* (so either there are only finitely many maps defining the A_∞ algebra or a suitable notion of convergence of the infinite sum holds). Let $\tau \in A_*$ be a Maurer Cartan element. That is,

$$\partial_A \tau + m_2^A(\tau \otimes \tau) + m_3^A(\tau \otimes \tau \otimes \tau) + \cdots = 0.$$

The following well-known lemma shows how to obtain a Maurer Cartan element in B_* from τ and $\{f_n\}$.

Lemma 2.2.4. *Let A_*, B_* be two A_∞ algebras and $\{f_n : A_*^{\otimes n} \to B_*\}$ be an A_∞ algebra morphism between them. If τ is a Maurer Cartan element in A_* then*

$$\tau' = f(\tau) + f_2(\tau \otimes \tau) + \cdots + f_n(\tau^{\otimes n}) + \cdots$$

is a Maurer Cartan element in B_*.

2.2.2 Maurer Cartan Equation and Differential Graded Algebra Maps

The following lemmas will be used to construct twisting cochains.

Lemma 2.2.5. *Let C_* be an A_∞ coalgebra and A_* a dg associative algebra. There is a one to one correspondence between twisting cochains $\tau : C_* \to A_*$ and differential graded algebra maps $\tau_T : T(C_*[-1]) \to A_*$.*

Proof. Let $\partial^{T(C)} : T(C_*[-1]) \to T(C_*[-1])$ be the derivation of square zero given by the A_∞ coalgebra on C_*. Given a twisting cochain $\tau : C_* \to A_*$, let $\tau_T(c_1 \otimes \cdots \otimes c_n) = \tau(c_1) \cdots \tau(c_n)$. Then by construction, τ_T is an algebra map. It is a chain map, because

$$\begin{aligned} \partial^A(\tau(c)) &= \tau\partial^C(c) + m_A \circ (\tau \otimes \tau) \circ c_2(c) + m_A \circ \tau^{\otimes 3} \circ c_3(c) + \cdots \\ &= \tau_T \partial^{T(C)}(c), \end{aligned}$$

where the first equality is due to the Maurer Cartan equation for τ and the second equality is the definition of $\partial^{T(C)}$ in terms of the maps $c_n : C_*[-1] \to C_*[-1]^{\otimes n}$. Conversely, given a map of differential graded algebras $\tau_T : T(C_*) \to A_*$ restricting τ to C_* defines a twisting cochain.

□

Lemma 2.2.6. *Let C_* be a C_∞ coalgebra and H_* a Hopf algebra. There is a one to one correspondence between twisting cochains $\tau : C_* \to Prim(H_*)$ and differential graded Lie algebra maps $\mathcal{L}(C_*[-1]) \to Prim(H_*)$.*

Proof. This lemma is proved in the same way as that of the previous. Note that a C_∞ coalgebra defines a derivation of square zero on the free Lie algebra $\mathcal{L}(C_*[-1])$.

□

2.2.3 C_∞ coalg $\otimes_\tau$ bialg as an A_∞ coalgebra using left multiplication

Given a twisting cochain $\tau : C_* \to H_*$, we want to define a twisted A_∞ coalgebra structure on $C_* \otimes H_*$. First, we define the untwisted A_∞ coalgebra.

Lemma 2.2.7. *Let $(C_*, \{c_n\})$ be an A_∞ coalgebra and H_* be an algebra with a strictly coassociative comultiplication. Then $C_* \otimes H_*$ is an A_∞ coalgebra with structure maps*

$$c_n^{\otimes} = c_n \otimes \big((\Delta \otimes Id^{\otimes n-1}) \circ \cdots \circ \Delta\big) : C_* \otimes H_* \to (C_* \otimes H_*)^{\otimes n}.$$

Proof. The proof is straightforward, using the A_∞ coalgebra relations for C_* terms and that H_* is a strict coassociative coalgebra. □

Remark 2.2.8. Before we define an A_∞ coalgebra structure on $C_* \otimes H_*$, we return to the classical setting of Brown's twisting cochains. We introduce a graphical picture of ∂_τ and a graphical proof that $\partial_\tau^2 = 0$. This technique will be used to define the twisted A_∞ coalgebra later on. Let C_* be a differential graded coalgebra and H_* a differential graded bialgebra. Let τ be a twisting cochain and ∂_τ be the twisted differential.

To represent $\partial_\tau : C_* \otimes H_* \to C_* \otimes H_*$, we draw two vertical lines, one to represent C_* the other to represent H_*. We draw a horizontal dash to

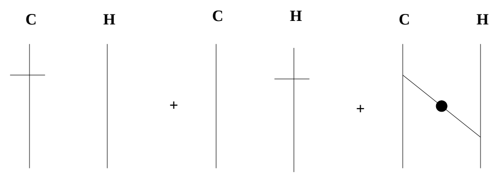

Figure 2.1: A graphical representation of $\partial_\tau = \partial_C \otimes 1 + 1 \otimes \partial_H + (1 \otimes m_A \otimes \tau \otimes 1)\Delta_C \otimes 1$. A vertical line with a dash represents the differential. The diagonal line with a vertex represents the map $\tau : C \to H$.

denote the differential. The twisting term applies the coproduct on C_* and τ to one of the factors. We represent the twisting cochain $\tau : C_* \to H_*$ by connecting the lines representing C_* and H_* with a line. The resulting vertex on C_* of valence three can be thought of as the coproduct and the vertex of valence three on H_* can be thought of as the product. We refer the reader to Figure 2.1 for a picture of ∂_τ.

We can prove that $\partial_\tau^2 = 0$ by analyzing the diagrams. The top row in Figure 2.2 are the terms that remain after canceling the terms in ∂_τ^2 that correspond to the tensor differential, which is well known to square to zero. Note that because ∂_C is a coderivation, the first and third terms in this row are equal to the first term in the second row of the figure. Similarly, since ∂_H is a derivation, the second and fourth terms on the first row equal the second term in the second row. The coassociativity of Δ_C and the associativity of

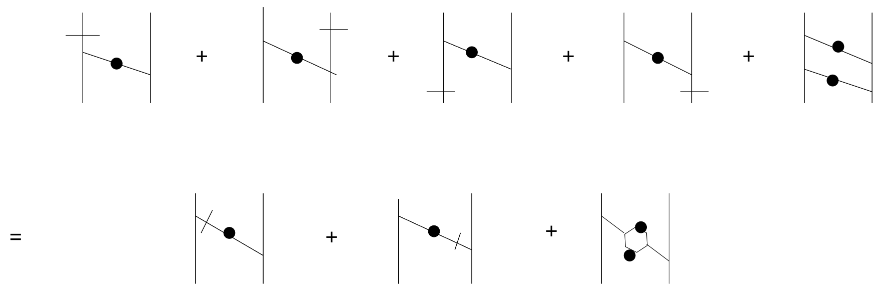

Figure 2.2: A graphical representation of $\partial_\tau^2 = 0$. The top row represents the five terms that remain in ∂_τ^2 when we cancel the terms corresponding to the tensor differential. The bottom row is zero because the middle lines represent the Maurer Cartan equation $\partial_H \tau + \tau \partial_C + \tau \cdot \tau$.

m_H imply the last term of the first row is equal to the last term of the second row. The bottom row then is equal to zero, because the middle lines describe the Maurer Cartan equation $\partial_H \tau + \tau \partial_C + \tau \cdot \tau$, which is zero by assumption.

There is a similar argument showing that if $Im(\tau) \subset Prim(H_*)$, then $(C_* \otimes H_*, \partial_\tau)$ is a differential graded coalgebra. The argument requires C_* to be a cocommutative coalgebra. We refer the reader to Figure 2.3.

We can now describe how to twist the A_∞ coalgebra. Let $\tau : C_* \to Prim(H_*)$ satisfy the Lie Maurer Cartan equation. First consider $c_1^{Hom} : C_* \otimes H_* \to C_* \otimes H_*$. As in the strict setting, there is a twisting term of the

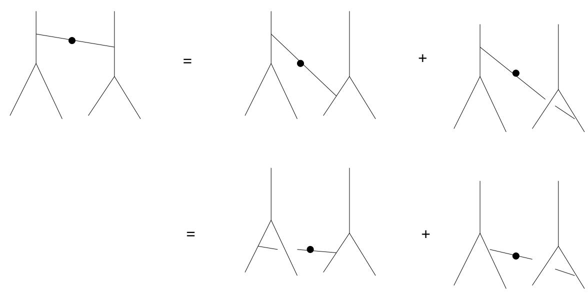

Figure 2.3: A graphical representation that ∂_τ is a coderivation of the coproduct of $C_* \otimes H_*$. The first equality is a result of the fact that multiplication by a primitive element is a coderivation. The second equality is a result of the coproduct in C_* being coassociative and cocommutative.

form

$$C_* \otimes H_* \overset{c_2}{\to} C_*^{\otimes 2} \otimes H_* \overset{1\otimes\tau\otimes 1}{\to} C_* \otimes H_*^{\otimes 2} \overset{1\otimes m_H}{\to} C_* \otimes H_*.$$

But this twisting only takes c_2 into account and ignores all of the higher c_n maps in the C_∞ coalgebra structure on C_*. To account for these maps, first apply c_n to C_* and apply $\tau^{\otimes n-1}$ to the last $n-1$ factors in $C_*^{\otimes n}$. Since $Im(\tau) \subset Prim(H_*)$, we can bracket these $n-1$ terms in all possible ways to get another primitive element. Then we multiply $Prim(H_*)$ and H_* terms.

To sum up, c_1^τ consists of terms

$$C_* \otimes H_* \stackrel{c_3 \otimes 1}{\rightarrow} C_*^{\otimes 3} \otimes H_* \stackrel{1\otimes\tau^{\otimes 2}\otimes 1}{\rightarrow} C_* \otimes H_*^{\otimes 2} \otimes H \stackrel{1\otimes[,]\otimes 1}{\rightarrow} C_* \otimes H_* \otimes H_* \stackrel{1\otimes m}{\rightarrow} C_* \otimes H_*$$

$$C_* \otimes H_* \stackrel{c_4 \otimes 1}{\rightarrow} C_*^{\otimes 4} \otimes H_* \stackrel{1\otimes\tau^{\otimes 3}\otimes 1}{\rightarrow} C_* \otimes H_*^{\otimes 3} \otimes H \stackrel{1\otimes[,]\otimes 1}{\rightarrow} C_* \otimes H_* \otimes H_* \stackrel{1\otimes m}{\rightarrow} C_* \otimes H_*$$

$$C_* \otimes H_* \stackrel{c_5 \otimes 1}{\rightarrow} C_*^{\otimes 5} \otimes H_* \stackrel{1\otimes\tau^{\otimes 4}\otimes 1}{\rightarrow} C_* \otimes H_*^{\otimes 4} \otimes H \stackrel{1\otimes[,]\otimes 1}{\rightarrow} C_* \otimes H_* \otimes H_* \stackrel{1\otimes m}{\rightarrow} C_* \otimes H_*$$

and continue for all n in this way. By $[,]$ for three or more terms, we mean

$$[x_1, \cdots, x_n] = \sum_{\sigma \in S_n} [x_{\sigma(1)}, [x_{\sigma(2)}, \cdots [x_{\sigma(n-1)}, x_{\sigma(n)}]]].$$

Note the similarity of the twisted terms with the L_∞ algebra on $Hom(C_*, Prim(H_*))$.

Since c_1^τ is an infinite sum, we need to address the issue of convergence in $C_* \otimes H_*$. In our application, $H_* = \widehat{T}(H_*(M)[-1])$, with the multiplication given by concatenation of tensors. Let $x \in C_* \otimes H_*(M)[-1]$. When c_n is used to twist the differential, the corresponding term in $c_1^\tau(x)$ will be an element in $C_* \otimes (H_*(M)[-1])^{\otimes n}$. Then c_1^τ consists of finite sums in different tensor products. So in the completed tensor product, $c_1^\tau(x)$ is well defined.

When C_* is a strict dg coalgebra, then c_1^τ is the same as the twisted differential ∂_τ in Brown's construction. So we write c_1^τ by ∂_τ.

The higher maps c_n can be twisted in the same manner as c_1. To twist $c_2 : C_* \otimes H_* \to C_*^{\otimes 2} \otimes H_*^{\otimes 2}$, we apply c_n for $n > 2$, then τ^{n-1} to the last $n-2$ factors of $C_*^{\otimes n}$, and bracketing these $n-2$ terms in all possible

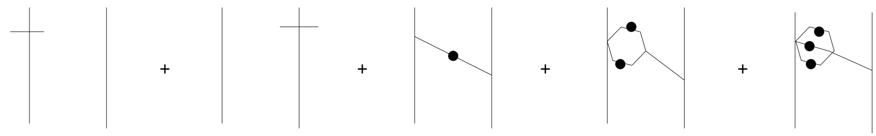

Figure 2.4: A graphical representation of ∂_τ. The terms are $\partial_C \otimes 1 + 1 \otimes \partial_H + (1 \otimes m)(1 \otimes \tau \otimes 1)c_2 \otimes 1 + (1 \otimes 1 \otimes \tau)c_3$

ways, multiplying the result with the element in H_*, and finally applying the coproduct in H_*. For $n = 3$, the process is the composition of

$$C_* \otimes H_* \stackrel{c_3 \otimes 1}{\to} C_*^{\otimes 3} \otimes H_* \stackrel{1^{\otimes 2} \otimes \tau \otimes 1}{\to} C_*^{\otimes 2} \otimes H_* \otimes H_* \stackrel{1 \otimes m}{\to} C_*^{\otimes 2} \otimes H_* \stackrel{1^{\otimes 2} \otimes \Delta}{\to} C_*^{\otimes 2} \otimes H_*^{\otimes 2}.$$

The resulting map is denoted $c_2^\tau : C_* \otimes H_* \to (C_* \otimes H_*)^{\otimes 2}$.

For $n > 3$, we must use the Lie bracket, and the composition of maps is

$$C_* \otimes H_* \stackrel{c_n \otimes 1}{\to} C_*^{\otimes n} \otimes H_* \stackrel{1^{\otimes 2} \otimes \tau^{\otimes n-2} \otimes 1}{\to} C_*^{\otimes 2} \otimes H^{\otimes n-2} \otimes H_*$$
$$\stackrel{1^{\otimes 2} \otimes [,]^{n-2} \otimes 1}{\to} 1 \otimes H_* \otimes H_* \stackrel{1 \otimes m}{\to} C_*^{\otimes 2} \otimes H_* \stackrel{1^{\otimes 2} \otimes \Delta}{\to} C_*^{\otimes 2} \otimes H_*^{\otimes 2}.$$

To show that $\{c_n^\tau\}$ defines an A_∞ coalgebra on $C_* \otimes H_*$, we use the diagrams as in Remark 2.2.8. For a picture of ∂_τ we refer the reader to Figure 2.4. For a picture of c_2^τ, we refer the reader to Figure 2.5. Since multiplying by a primitive element is a coderivation, we have some identities for the terms in c_2^τ. These identities are described in Figure 2.6.

We can now show that $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$ define an A_∞ coalgebra. The proof of the theorem uses a graphical approach.

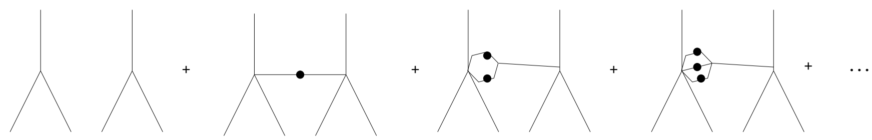

Figure 2.5: A graphical representation of c_2^τ.

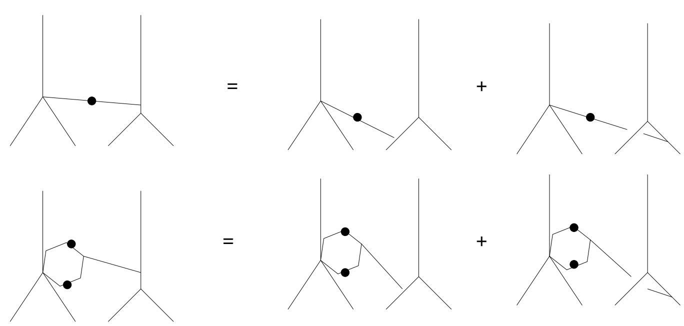

Figure 2.6: The above identities hold because $Im(\tau) \subset Prim(H_*)$ and multiplying by a primitive element is a coderivation. The same holds true for the other terms of c_2^τ and also for c_n^τ.

Theorem 2.2.9. *Let C_* be a C_∞ coalgebra, H_* a dg bialgebra, and $\tau : C_* \to H_*$ a twisting cochain such that $Im(\tau) \subset Prim(H_*)$. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$ define an A_∞ coalgebra on $C_* \otimes H_*$.*

Proof. We first show that ∂_τ is a differential. To show that $\partial_\tau^2 = 0$ we will show that expanding the terms yield many occurrences of the Maurer Cartan equation.

We list some of the terms of ∂_τ^2 in Figure 2.7. The fact that ∂_H is a derivation is expressed diagrammatically as in Figure 2.8. This relation can be used to add diagrams in the second and fourth rows of Figure 2.7. In place of the coderivation relations, we must use the C_∞ coalgebra relations for $(C_*, \{c_n\})$. The relation for $n = 3$ is expressed in Figure 2.9. We use these relations to add figures in the first and third rows of Figure 2.7. Some of the resulting diagrams will either cancel with diagrams in rows five or higher. The rest of the diagrams are shown in Figure 2.10. The Maurer Cartan equation is present in each row. Since τ is a twisting cochain, the sum to zero.

Next, we show that c_2^τ is a coderivation of ∂_τ. In Figure 2.11, the graphs representing $c_2^\tau \circ \partial_\tau$ are drawn and in Figure 2.12 the graphs representing $(\partial_\tau \otimes 1) \circ c_2^\tau$ are drawn. The graphs representing $(1 \otimes \partial_\tau) \circ c_2^\tau$ are the same

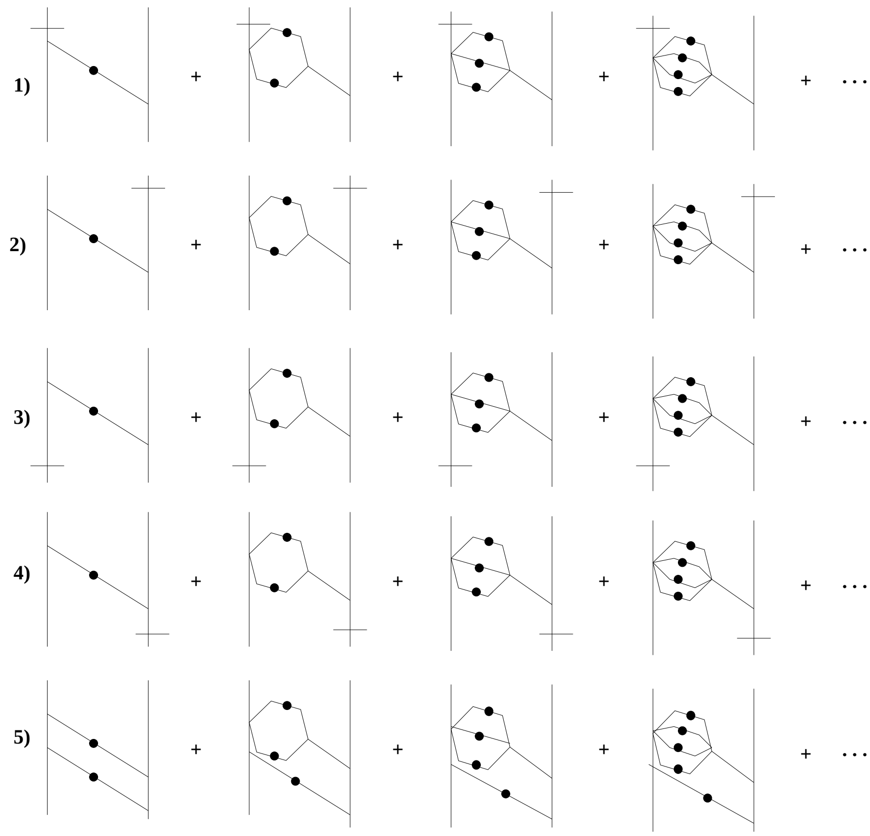

Figure 2.7: Some of the in terms of ∂_τ^2. We have left out the terms in the tensor part, as these are known to square to zero.

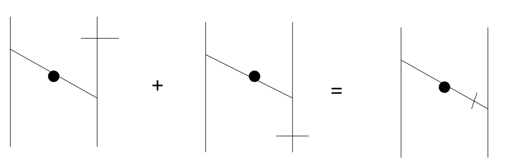

Figure 2.8: The equality here come from the fact that H_* is a differential graded algebra.

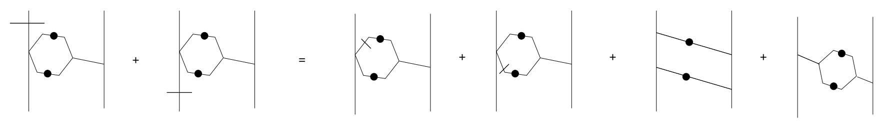

Figure 2.9: The equality here comes from the fact that $(C_*, \{c_n\})$ is a C_∞ coalgebra.

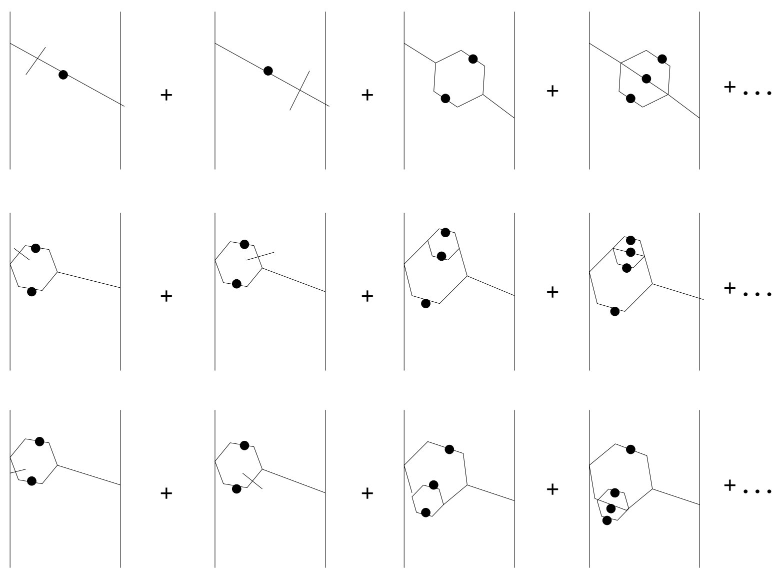

Figure 2.10: These remaining terms in $(\partial_\tau)^2$ sum to zero because $\partial_H \tau + \tau\partial_H + m_2^{Hom}(\tau,\tau) + m_3^{Hom}(\tau,\tau,\tau) + \cdots = 0$.

as the graphs representing $(\partial_\tau \otimes 1) \circ c_2^\tau$ except the graphs are connected by the right output edge of each tree as opposed to the left output edge.

Multiplication by a primitive element is a coderivation, which gives us identities expressed in Figure 2.6. This allows us to compare the graphs from $c_2^\tau \circ \partial_\tau$ with the graphs from $(\partial_\tau \otimes 1 + 1 \otimes \partial_\tau) \circ c_2^\tau$. Note that on the left hand side of each pairing, we have many compositions of the form $(1 \otimes \cdots \otimes c_j \otimes \cdots \otimes 1) \circ c_i$, where c_i, c_j are maps of the C_∞ coalgebra on C_*. The relations in the C_∞ coalgebra state that $\sum_{i+j+1=n}(1 \otimes \cdots \otimes c_j \otimes \cdots \otimes 1) \circ c_i = 0$. Noting which maps in our graphs appear in the sum and which graphs do not appear, we can apply the C_∞ coalgebra relation to obtain many identities. When this is done, we obtain graphs which involve the Maurer Cartan equation for τ, just as we did in showing $\partial_\tau^2 = 0$. Since τ is a twisting cochain, this sum is zero and c_2^τ is a coderivation of ∂_τ. In Figure 2.13 we organize the graphs in $c_2^\tau \circ \partial_\tau + (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau) \circ c_2^\tau$. The relations for the coalgebra structure on C_* state that the sum of these graphs are equal to the graphs in Figure 2.14. The sum of these graphs is zero, because of the Maurer Cartan equation.

The reader can see that this situation generalizes for $n > 2$. In each of these cases, we have many compositions involved in the C_∞ coalgebra

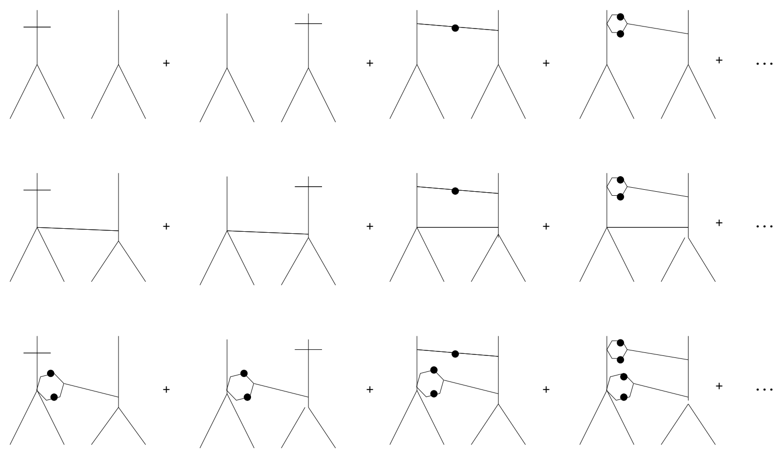

Figure 2.11: The graphs representing $c_2^\tau \circ \partial_\tau$.

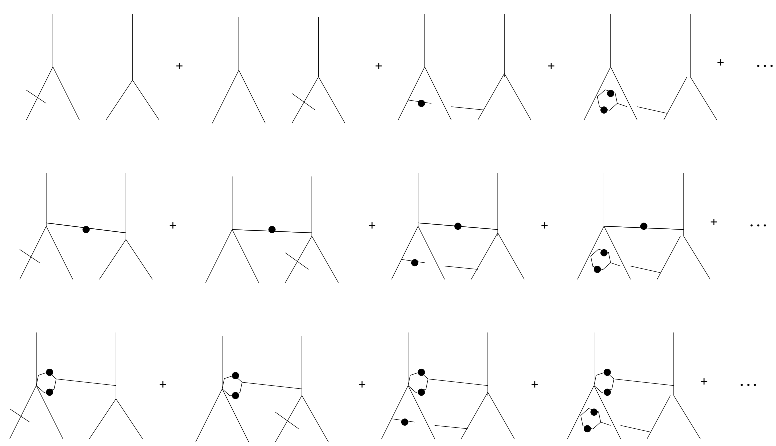

Figure 2.12: The graphs representing $\partial_\tau \otimes 1 \circ c_2^\tau$.

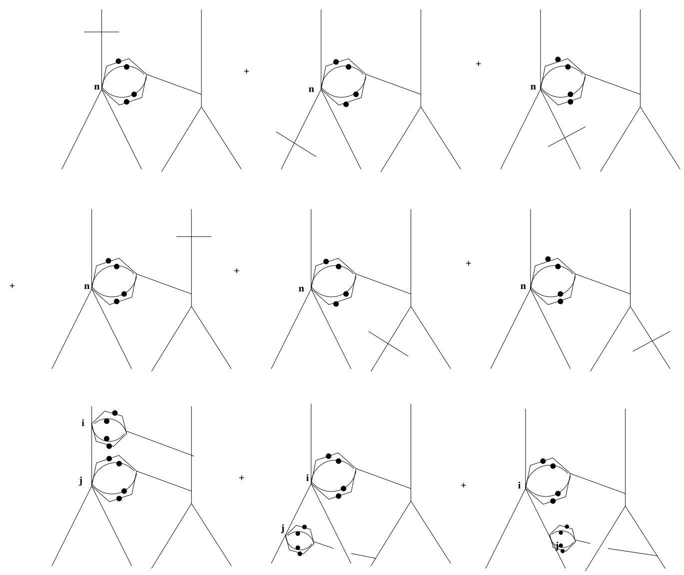

Figure 2.13: The graphs of $c_2^\tau \circ \partial_\tau + (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau)c_2^\tau$ organized to show how the C_∞ coalgebra on C_* is used.

relation for C_*. When we replace these graphs, using the coalgebra structure, we obtain graphs involving Maurer Cartan equation. We summarize the relation in Figure 2.15.

□

2.2.4 C_∞ coalg $\otimes_\tau$ bialg as an A_∞ coalgebra using bracket action action

In the previous section, we used the twisting cochain and left multiplication in H_* to twist the A_∞ coalgebra structure on $C_* \otimes H_*$. In this section,

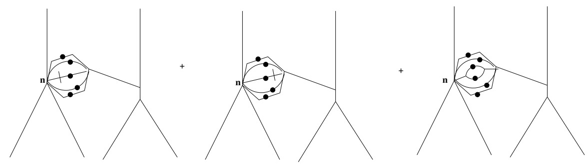

Figure 2.14: These graphs are equal to the graphs in Figure 2.13 using the C_∞ coalgebra on C_*. Note that these terms involve $\partial_H \tau + \tau \partial_C + \tau \cdot \tau + \tau \cdot \tau \cdot \tau + \cdots = 0$.

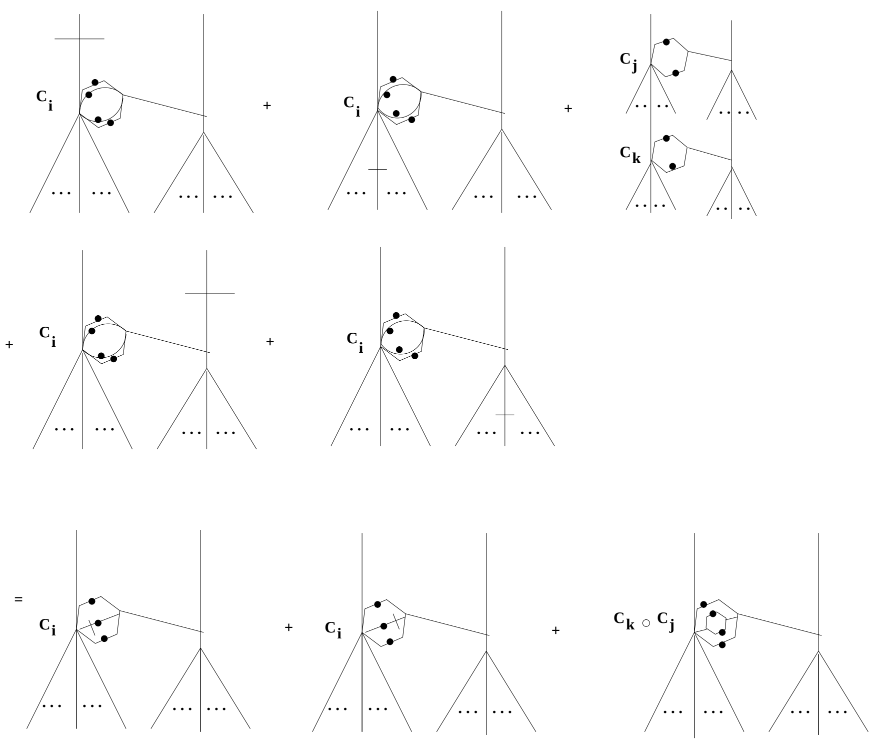

Figure 2.15: To show that c_n^τ form a coalgebra structure, use the relation above to get a sequence of graphs involving the Maurer Cartan equation. The equality is due to the fact that C_* is a C_∞ coalgebra.

we consider another action. For $a \in H_*$, the bracket action of a on H_* is defined by $[a, x] = ax - xa$. Note that $[a, -]$ is a derivation. If a is a primitive element, then $[a, -]$ is also a coderivation.

Given a twisting cochain $\tau : C_* \to H_*$ such that $Im(\tau) \subset Prim(H_*)$, we define a twisted A_∞ coalgebra structure on $C_* \otimes H_*$. The process is the same as the one defining the previous twisted A_∞ coalgebra, except we replace the multiplication in H_* with the bracket action. We use the same notation $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$ and so we will be explicit when to use left multiplication and when to use the bracket action.

Theorem 2.2.10. *Let C_* be a C_∞ coalgebra, H_* a dg bialgebra, and $\tau : C_* \to H_*$ a twisting cochain such that $Im(\tau) \subset Prim(H_*)$. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$, obtained from the bracket action, define an A_∞ coalgebra on $C_* \otimes H_*$.*

Proof. The only property of left multiplication used in the proof of Theorem 2.2.9 is that left multiplication by a primitive element is a coderivation. Since conjugation by a primitive element is a coderivation, the proof applies to this theorem as well.

□

2.2.5 Cyclic C_∞ coalg $\otimes_\tau$ bialg as an A_∞ algebra using bracket action

Sometimes a C_∞ coalgebra has extra structure on it, allowing one to define an algebra structure on $C_* \otimes H_*$. We consider the case when the coalgebra has a non-degenerate bilinear form that is compatible with the coalgebra structure, i.e., a cyclic C_∞ coalgebra. We review the relevant definitions.

A cyclic A_∞ algebra consists of a finite dimensional A_∞ algebra $(A_*, \{m_n\})$ and a non-degenerate bilinear form $\langle , \rangle : A_* \otimes A_* \to k$ such that

$$\langle m_n(x_1, \cdots, x_n), x_0 \rangle = (-1)^N \langle m_n(x_0, \cdots, x_{n-1}), x_n \rangle,$$

where $N = -1 + |x_0|(|x_1| + \cdots + |x_n|)$. The bilinear form defines an isomorphism between A and its dual. The maps m_n can then be viewed as elements in $A[-1]^{*\otimes n} \otimes A[-1] \cong A[-1]^{\otimes n+1}$.

Lemma 2.2.11. *Let $(A_*, \{m_n\}, \langle , \rangle)$ define a cyclic A_∞ algebra. Then $m_n \in A[-1]^{\otimes n+1}$ is cyclically invariant.*

Proof. Let $m_n = \sum x_1 \otimes \cdots \otimes x_{n+1} \in A[-1]^{\otimes n+1}$. It suffices to show that $x_1 \otimes \cdots \otimes x_{n+1} = x_2 \otimes \cdots \otimes x_{n+1} \otimes x_1$. This is seen to be the case by expressing $\langle , \rangle$ as an element in $A_* \otimes A_*$ and writing the conditions for a cyclic A_∞ algebra in terms of elements in the tensor algebra.

□

Viewing the maps $\{m_n\}$ as elements in the tensor and using the Koszul sign rule, one can determine the sign $(-1)^N$ found in the definition of a cyclic A_∞ algebra. We define a cyclic A_∞ coalgebra viewing c_n as cyclically invariant elements in the tensor product.

Definition 2.2.12. $(C_*, \{c_n\}, \langle , \rangle)$ is a **cyclic A_∞ coalgebra** if

1. C_* is finite dimensional
2. $(C_*, \{c_n\})$ is an A_∞ coalgebra,
3. $\langle , \rangle$ is a non-degenerate bilinear form,
4. the maps c_n when identified as elements $C_*^{\otimes n+1}$ using the bilinear form, are cyclically invariant.

The condition that C_* is finite dimensional implies that $\langle , \rangle$ defines an isomorphism between C_* and its dual C^*. A cyclic C_∞ coalgebra is defined in the obvious way. Given a cyclic C_∞ coalgebra C_*, the bilinear form and maps $\{c_n\}$ can be used to define a C_∞ algebra $\{m_n : C_*[-1]^{\otimes n} \to C_*[-1]\}$. So $C_* \otimes H_*$ has an A_∞ algebra structure given by combining the C_∞ algebra on C_* with the strict algebra structure on H_*. Does the twisting cochain $\tau : C_* \to H_*$ define a twisted A_∞ algebra on $C_* \otimes H_*$? We show that it does and unlike in the previous cases, we do not need to twist the higher maps. In Theorem 2.2.22, we prove the case when C_* is a strict cyclic coalgebra. Also, note that since bracketing is always a derivation, whether by a primitive element or not, we do not require $Im(\tau) \subset Prim(H_*)$. If $Im(\tau) \subset Prim(H_*)$ and H_* is a dg Hopf algebra, and not just a dg bialgebra, then the bracket action agrees with another action, which we call the conjugation action. We use this action in Theorem 2.2.18.

Theorem 2.2.13. *Let C_* be a cyclic C_∞ coalgebra, H_* be a dg bialgebra, and $\tau : C_* \to H_*$ be a twisting cochain. The maps $\{\partial_\tau, m_2, m_3, \cdots\}$ defined using the bracket action in H_* give $C_* \otimes_\tau H_*$ the structure of an A_∞ algebra.*

Proof. Since $\{\partial, m_2, m_3, \cdots\}$ defines an (untwisted) A_∞ algebra, it suffices to show that the twisted terms in ∂_τ all cancel. We first show that ∂_τ is a derivation of m_2,

$$\partial_\tau \circ m_2 = m_2 \circ (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau). \qquad (2.2.1)$$

We refer the reader to Figures 2.16 and 2.17 for graphs representing the LHS and RHS of equation (2.2.1). Since the bracket action is a derivation, the diagrams in Figure 2.16 are equal to the diagrams in Figure 2.18. We need to show that Figure 2.17 is equal to Figure 2.18.

The LHS of equation (2.2.1) has compositions $c_n \circ m_2 : C_*[-1]^{\otimes 2} \to C_*[-1]^{\otimes n}$. The maps on the RHS has compositions $(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes j}) \circ (c_n \otimes 1) :$

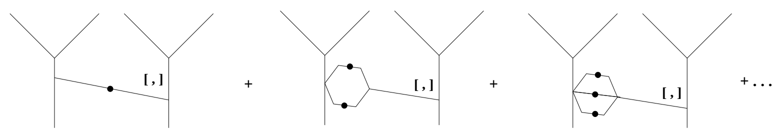

Figure 2.16: A graphical representation of $\partial_\tau \circ m_2$. The label $[,]$ is to remind the reader that the bracket action is applied on $T(H_*(M)[-1])$, and not the product in $T(H_*(M)[-1])$.

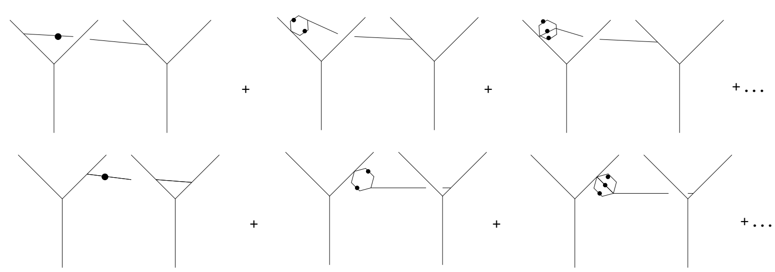

Figure 2.17: A graphical representation of $m_2 \circ (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau)$.

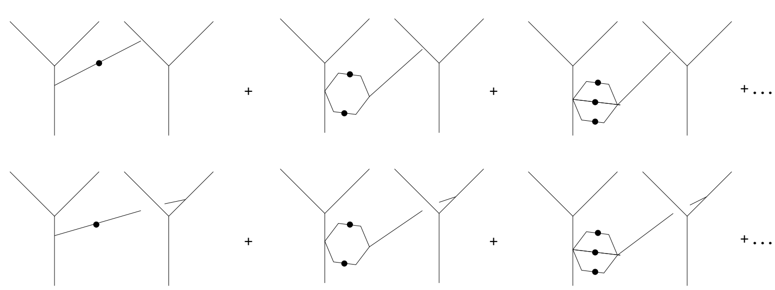

Figure 2.18: Because the bracket action is a derivation, these diagrams are equal to the one found in Figure 2.16.

$C_*[-1]^{\otimes 2} \to C_*[-1]^{\otimes n}$. We show these two maps are equal by writing the compositions as elements in $C_*[-1]^{\otimes n+2}$ and using Lemma 2.2.11.

The map c_n can be written as $\sum x_1 \otimes \cdots \otimes x_{n+1} \in C_*[-1]^{\otimes n+1}$ and m_2 as an $\sum y_1 \otimes y_2 \otimes y_3 \in C_*[-1]^{\otimes 3}$. Their composition $c_n \circ m_2$ is expressed as

$$\sum \langle x_1, y_3 \rangle x_2 \otimes \cdots x_n \otimes y_1 \otimes y_2 \in C_*[-1]^{\otimes 4}.$$

The composition on the RHS of the equation, $(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes j}) \circ (c_2 \otimes 1)$ is described in the same way except for a different pairing $\langle x_i, y_j \rangle$. However, since c_n and m_2 are cyclically invariant, the compositions are equal.

The higher compatibilities for the A_∞ algebra proceed in exactly the same way, with m_2 replaced by m_l.

□

Given the A_∞ algebra $C_* \otimes H_*$, we can symmetrize the maps to obtain an L_∞ algebra $([C_* \otimes H_*], \{\partial_\tau, l_2, l_3, \cdots\})$. This restricts to an L_∞ algebra structure on $C_* \otimes Prim(H_*)$.

Theorem 2.2.14. *Let $(C_* \otimes H_*, \{\partial_\tau, m_2, m_3, \cdots\})$ be the A_∞ algebra described in Theorem 2.2.13. Then $(C_* \otimes Prim(H_*), \{\partial_\tau, l_2, l_3, \cdots\})$, obtained by symmetrizing $\{m_n\}$, is an L_∞ algebra.*

Proof. Since C_* is finite dimensional, we can identify $C_* \otimes H_* \cong Hom(C^*, H_*)$, where C^* is a C_∞ coalgebra. Then the statement follows from Lemma 2.2.3. $\square$

This gives an A_∞ algebra structure on $C_* \otimes H_*$ and an L_∞ algebra structure on $C_* \otimes Prim(H_*)$. More can be said when C_* is a strict unital commutative algebra. In this situation, $C_* \otimes Prim(H_*)$ can be viewed as a Lie algebra over C_*. Its universal enveloping algebra over C_*, denoted $U_{C_*}(C_* \otimes Prim(H_*))$ is $C_* \otimes H_*$. Note if we take the universal enveloping algebra of $C_* \otimes Prim(H_*)$ (viewed as a Lie algebra over the ground field), we obtain $U(C_* \otimes H_*)$ which is not equal to $U_{C_*}(C_* \otimes Prim(H_*))$. We are not aware of the corresponding notion for C_∞ algebras to make the analogous statement. This seems to be a useful notion. We discuss the strict case in more detail in Theorem 2.2.23.

2.2.6 C_∞ coalg $\otimes_\tau$ Hopf alg as an A_∞ coalgebra using conjugation action

In our applications of the previous results, we would like to relate the twisted algebraic structures to the total space of some bundle. Let $G \to P \to M$ be a principal G bundle and $G \to Conj(P) \to M$ be the associated bundle with respect to the conjugation action. Note that $H_*(M)$ is a cyclic C_∞ coalgebra and $H_*(G)$ a bialgebra, and moreover, a Hopf algebra. Then given a suitable twisting cochain $\tau : H_*(M) \to H_*(G)$, we can form the twisted algebraic structures using the methods described above. The homology of the total space, $H_*(Conj(P))$ can be identified with linear homology of the twisted algebraic structures, that is homology the homology $H_*(M) \otimes H_*(G)$ with respect to ∂_τ. However, the argument uses Brown's theory of twisting cochains, which requires using the conjugation action. Because the conjugation action uses the inverse operation in G, the algebraic setup in this situation requires H_* to be a dg Hopf algebra. We will see that when $Im(\tau) \subset Prim(H_*)$, the conjugation action agrees with the bracket action.

Let H_* be a Hopf algebra. Denote the antipode map of H_* by $s : H_* \to H_*$. Given an element $a \in H_*$, we define the conjugation action of a on H_* by

$$\begin{aligned} conj_a : H_* &\to H_* \\ x &\mapsto \sum a_{(1i)} x s(a_{(2i)}). \end{aligned}$$

The homology of a topological group $H_*(G)$ is a Hopf algebra. The group acts on itself by conjugation, and so induces an action on $H_*(G)$. The following lemma shows that this action is the same as the conjugation action of the Hopf algebra.

Lemma 2.2.15. *Let G be a topological group. The conjugation action in G*

induces a map

$$\begin{aligned} H_*(G) \otimes H_*(G) &\to H_*(G) \\ a \otimes x &\mapsto \sum a_{(1i)} x s(a_{(2i)}). \end{aligned}$$

Proof. Conjugation is described by the composition

$$\begin{aligned} G \times G &\stackrel{Diag \times 1}{\to} G \times G \times G \stackrel{1 \times inv \times 1}{\to} G \times G \times G \to G \times G \times G \to G \\ (x, y) &\mapsto (x, x, y) \mapsto (x, x^{-1}, y) \mapsto (x, y, x^{-1}) \mapsto xyx^{-1}. \end{aligned}$$

The diagonal map in G induces the coproduct Δ on $H_*(G)$ and the inverse map in G induces the antipode s. This proves the lemma.

□

The following lemma shows that conjugation by a primitive element is a coderivation and a derivation.

Lemma 2.2.16. *Let H_* be a Hopf algebra.*

1. *Conjugation by a primitive element in a Hopf algebra is a coderivation.*
2. *Conjugation by a primitive element in a Hopf algebra is a derivation.*

Proof. 1. Let a be a primitive element of H_*. The antipode has to satisfy $m \circ (1 \otimes s) \circ \Delta(a) = 0$, which means $s(a) = -a$. Then $conj_a(x) =$

$\sum a_{(1i)} x s(a_{(2i)}) = ax - xa$. This is a coderivation because multiplying by a primitive element is a coderivation.

2. Let a be a primitive element. Then

$$\begin{aligned} conj_a(x) \cdot y + x \cdot conj_a(y) &= axy - xay + xay - xya \\ &= axy - xya \\ &= conj_a(xy). \end{aligned}$$

□

Theorem 2.2.17. *Let C_* be a C_∞ coalgebra, H_* a dg Hopf algebra, and $\tau : C_* \to H_*$ a twisting cochain such that $Im(\tau) \subset Prim(H_*)$. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$, obtained from the conjugation action, define an A_∞ coalgebra on $C_* \otimes H_*$.*

Proof. For $a \in Prim(H_*)$, the conjugation action, $conj_a$, and bracket action $[a,]$ agree. Since $Im(\tau) \subset Prim(H_*)$, the maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$ defined using the conjugation action are equal to the maps defined using the bracket action. The statement then follows from Theorem 2.2.10 □

Theorem 2.2.18. *Let C_* be a cyclic C_∞ coalgebra, H_* be a dg Hopf algebra, and $\tau : C_* \to H_*$ be a twisting cochain with $Im(\tau) \subset Prim(H_*)$. The maps*

$\{\partial_\tau, m_2, m_3, \cdots\}$ *defined using the conjugation action in* H_* *give* $C_* \otimes_\tau H_*$ *the structure of an* A_∞ *algebra.*

Proof. Since $Im(\tau) \subset Prim(H_*)$, the twisted A_∞ algebra structure defined using the conjugation action agrees with the twisted A_∞ algebra structure defined using the bracket action.. The proof then follows from Theorem 2.2.13. □

2.2.7 Addendum

The graphical approach taken above can obscure some sign issues. In this section, we show that ∂_τ is a differential without appealing to graphs. We also look at the strict (non-infinity) versions of the proofs, with the idea that this will also shed some light on the constructions.

We first show that the twisted differential ∂_τ is indeed a differential, by referencing the work of Chuang and Lazarev. In [CL2], a twisted A_∞ algebra is also defined, given a Maurer Cartan element. While their construction is different on the higher maps, it agrees with the twisted differential described in this paper.

Let C^* be an A_∞ algebra and A_* a strict dg associative algebra. Then $C^* \otimes A_*$ is an A_∞ algebra, and a twisting cochain is an element $\tau \in C^* \otimes A_*$

satisfying the Maurer Cartan equation

$$\partial_C\tau + \partial_H\tau + m_2(\tau,\tau) + m_3(\tau,\tau,\tau) + \cdots = 0.$$

The twisted differential is then

$$\partial_\tau(x) = \partial_C(x) + \partial_H(x) + m_2(\tau, x) + m_3(\tau,\tau,x) + \cdots .$$

This is related to our construction as follows. Let C_* be an A_∞ coalgebra, A_* a strict dga, and $\tau : C_* \to A_*$. We are only looking to define a differential, which is why we do not require a C_* coalgebra and a dg Hopf algebra. Then the A_∞ algebra C^* used above is the linear dual of the A_∞ coalgebra. The twisting cochain $\tau : C_* \to A_*$ can be viewed as an element in $C^* \otimes A_*$ satisfying the Maurer Cartan equation. The complex $C_* \otimes A_*$ is the A_*-dual of $C^* \otimes A_*$. The two definitions of the twisted differentials can then be related in this way.

Lemma 2.2.19. *Let C^* be an A_∞ algebra, A_* a differential graded algebra, and $\tau \in C^* \otimes A_*$ a twisting cochain. Then $\partial_\tau^2 = 0$.*

Proof. This is a special case of Theorem 2.6 (2)a in [CL2].

We write out some terms in $\partial_\tau^2(x)$. The elements $\tau, x \in C_* \otimes A_*$ can be written as $\tau = \sum \tau_C \otimes \tau_A$ and $x = \sum x_C \otimes x_A$. Then

$$\begin{aligned} \partial_\tau(x) &= (\partial_C x_C) \otimes x_A + (-1)^{|x_C|} x_C \otimes (\partial_A x_A) + m_2(\tau_C, x_C) \otimes \tau_A \cdot x_A \\ &\quad + m_3(\tau_C, \tau_C, x_C) \otimes \tau_A \cdot \tau_A \cdot x_A + \cdots , \end{aligned}$$

where we dropped the summation for ease of notation. Applying ∂_τ a second time yields compositions of the A_∞ algebra maps $\{m_n\}$. Using the relations for an A_∞ algebra and strict dg algebra, we obtain terms involving the Maurer Cartan equation for τ. The argument is similar to the one used to prove Theorem 2.2.9. □

Theorem 2.2.18 asserted the existence of a twisted A_∞ algebra on the tensor product. We review some definitions and then discuss the strict case of the theorem.

Definition 2.2.20. A Frobenius algebra structure on V consists of a commutative multiplication and a non-degenerate inner product such that

$$\langle a, bc \rangle = \langle ab, c \rangle.$$

Note that a Frobenius algebra is a cyclic C_∞ algebra with $m_n = 0$ for $n > 2$.

Using the non-degenerate inner product of a Frobenius algebra, one can turn the multiplication into a comultiplication. The multiplication and comultiplication satisfy a certain compatibility, which brings us to the notion of what some authors refer to as an open Frobenius algebra [CEG].

Definition 2.2.21. An open Frobenius algebra structure on V consists of a commutative multiplication and a cocommutative comultiplication such that the comultiplication is a map of bimodules. That is,

$$\Delta(ab) = \sum a_{(1i)} \otimes a_{(2i)}b = \sum ab_{(1i)} \otimes b_{(2i)}.$$

Abrams, [A], proved that unital Frobenius algebras and unital, counital open Frobenius algebras are equivalent.

Theorem 2.2.22. *Let C_* be a dg Frobenius algebra and H_* a dg bialgebra. Let $\tau : C_* \to H_*$ be a twisting cochain such that $Im(\tau) \subset Prim(H_*)$. Then $(C_* \otimes H_*, \partial_\tau)$ is a differential graded algebra.*

Proof. To prove the theorem, we need to show that the twisted term is a derivation. Let $a \otimes b, c \otimes d \in C_* \otimes H_*$, and let $conj_b : H_* \to H_*$ be the conjugation action by $b \in H_*$. Then we need to show that

$$(ac)_{(1i)} \otimes conj_{\tau(ac)_{(2i)}} bd = a_{(1i)}c \otimes (conj_{\tau(a_{(2i)})} b)d + ac_{(1i)} \otimes b(conj_{\tau(c_{(2i)})} d).$$

Since $Im(\tau) \subset Prim(H_*)$ and conjugating by a primitive element is a derivation, the LHS of the equation is

$$(ac)_{(1i)} \otimes conj_{\tau(ac)_{(2i)}} bd = (ac)_{(1i)} \otimes (conj_{\tau(ac)_{(2i)}} b)d + ac_{(1i)} \otimes b(conj_{\tau(ac)_{(2i)}} d).$$

We need to show that $(ac)_{(1i)} \otimes (ac)_{2i} = a_{(1i)} \otimes a_{(2i)}c = ac_{(1i)} \otimes c_{(2i)}$.

Note that this is the condition that the coproduct is a map of bimodules, i.e., an open Frobenius algebra. If we use the result that Frobenius algebras and open Frobenius algebras are equivalent, we are done.

We use another argument which follows the proof of Theorem 2.2.18. Using the non-degenerate inner product, we express the coproduct Δ as an element in $C_*^{\otimes 3}$. The multiplication $m_2 : C_* \otimes C_* \to C_*$ is obtained by dualizing the coproduct $C^* \otimes C^* \to C^*$ and using the isomorphism between C^* and C_*. So m_2 is represented by the same element in $C_*^{\otimes 3}$. Write this element as $m_2 = \Delta = \sum x_{(1i)} \otimes x_{(2i)} \otimes x_{(3i)} \in C_*^{\otimes 3}$.

We need to show that certain compositions of Δ and m_2 are equal. In writing the compositions of Δ and m_2, we use the subscript i to represent m_2 $(x_{(1i)} \otimes x_{(2i)} \otimes x_{(3i)})$ and the subscript j to represent Δ. Then compositions are then given by

$$\begin{aligned}
\Delta \circ m_2 &= \sum_{i,j} \langle x_{(3i)}, x_{(1j)} \rangle x_{(1i)} \otimes x_{(2i)} \otimes x_{(2j)} \otimes x_{(3j)} \\
(m_2 \otimes 1) \circ (\Delta \otimes 1) &= \sum_{i,j} \langle x_{(2j)}, x_{(1i)} \rangle x_{(1j)} \otimes x_{(3j)} \otimes x_{(2i)} \otimes x_{(3i)} \\
(m_2 \otimes 1) \circ (1 \otimes \Delta) &= \sum_{i,j} \langle x_{(3j)}, x_{(2i)} \rangle x_{(1j)} \otimes x_{(2j)} \otimes x_{(1i)} \otimes x_{(3i)}.
\end{aligned}$$

Since $m_2 = \Delta$ are cyclically invariant, we get the necessary equalities.

□

In our construction of a twisted A_∞ algebra structure on $C_* \otimes H_*$, we used a cyclic C_∞ coalgebra. A cyclic C_∞ algebra is the homotopy version of a Frobenius algebra. It should be possible to define a twisted A_∞ algebra using the homotopy version of an open Frobenius algebra. The Koszul Duality theory for dioperads, described in [G], and for properads, described in [V], provides a definition for such an object. The dioperad describing Lie bialgebras, denoted $BiLie$, and the dioperad describing open Frobenius algebras, denoted $BiLie^!$, are Koszul dual ([G] Corollary 5.10). So a resolution for $BiLie^!$ is obtained by taking the cobar dual of $BiLie$, denoted $D(BiLie)$, and an open $Frob_\infty$ algebra structure on V is a map of differential graded dioperads $D(BiLie) \to End(V)$, where $End(V)$ is the endomorphism dioperad.

The cohomology of a Poincar/'e Duality space is a cyclic C_∞ algebra. An open manifold is not a Poincar/'e Duality space, but its cohomology is an open Frobenius algebra. The constructions using cyclic C_∞ algebra would define string topology operations for Poincar/'e Duality spaces, and the constructions using open $Frob_\infty$ algebras would define string topology operations for open manfiolds.

Theorem 2.2.14 stated that the L_∞ algebra structure on $C_* \otimes H_*$ restricts to $C_* \otimes Prim(H_*)$. In the strict case, more can be said about the relation between the associative algebra $C_* \otimes H_*$ and the Lie algebra $C_* \otimes Prim(H_*)$. Let $U_{C_*}(C_* \otimes Prim(H_*))$ be the universal enveloping algebra of $C_* \otimes Prim(H_*)$ viewed as a Lie algebra over C_*. Recall, if A_* is an associative algebra, then $[A_*]$ is the Lie algebra obtained by symmetrizing the mulitplication.

Theorem 2.2.23. *The Lie bracket on $[C_* \otimes H_*]$ restricts to $C_* \otimes Prim(H_*)$. Moreover, if C_* is unital, $U_{C_*}(C_* \otimes Prim(H_*)) = C_* \otimes H_*$.*

Proof. We first show that the Lie bracket on $[C_* \otimes H_*]$ fixes $C_* \otimes Prim(H_*)$. This is a simple computation

$$\begin{aligned}
[a_1 \otimes b_1, a_2 \otimes b_2] &= a_1 a_2 \otimes b_1 b_2 - a_2 a_1 \otimes b_2 b_1 \\
&= a_1 a_2 \otimes (b_1 b_2 - b_2 b_1) \\
&= a_1 a_2 \otimes [b_1, b_2],
\end{aligned}$$

where the bracket is in $[H_*]$. Since $Prim(H_*)$ is a Lie subalgebra of $[H_*]$, this proves the claim.

For the second part, suppose C_* is unital. Then an element in $U_{C_*}(C_* \otimes Prim(H_*))$ can be re-written

$$(c_1 \otimes h_1) \otimes_{C_*} \cdots \otimes_{C_*} (c_n \otimes h_n) = (c_1 \cdots c_n \otimes h_1) \otimes_{C_*} (1 \otimes h_2) \otimes_{C_*} \cdots \otimes_{C_*} (1 \otimes h_n).$$

The claim then follows from the construction of the universal enveloping algebra as a quotient of the tensor algebra.

□

2.3 Application to Spaces

To describe string topology operations, we start with the path space fibration $\Omega_b(M) \to P_b(M) \to M$. The based loop space $\Omega_b(M)$ is homotopy equivalent to a topological group, so we view $\Omega_b(X)$ as a topological group and the path space fibration as a principal $\Omega_b(M)$ bundle. The group acts on itself by conjugation and the associated bundle with respect to this bundle, which we refer to as the conjugate bundle, is a model for the free loop space.

Lemma 2.3.1. *The conjugate bundle $\Omega_b(M) \to Conj(P_b(M)) \to M$ is equivalent to the free loop space bundle $\Omega_b(M) \to LM \to M$.*

Proof. The total space $Conj(P_b(M))$ is $P_b(M) \times_{\Omega_b(M)} \Omega_b(M)$. We define a bundle map from $Conj(P_b(M)) \to LM$. Let $[p, a]$ be an element in $P_b(M) \times_{\Omega_b(M)} \Omega_b(M)$ and choose a representative (p, a), where $p : [0, 1] \to M$ and $a : S^1 \to M$. Then consider the map $f : [p, a] \mapsto pap^{-1}$. This map is well defined since a different representative will be of the form $(pg, g^{-1}ag)$, which gets sent to

$$(pg)(g^{-1}ag)(pg)^{-1} = pap^{-1}.$$

If f maps fibers isomorphically onto fibers, then f will be a homeomorphism (see for example [MS], Lemma 2.3). Let $F_x(Conj)$ be the fiber of $ConjP_b(M)$ above the point $x \in M$. An element in the fiber is of the form $[p, a]$ where p is a path from b to x and a is a loop at b. Let $\alpha \in F_x(LM)$ be an element in the fiber of the free loop space bundle. Then letting p be any path from b to x and $a = p^{-1}\alpha p$, then $f[p, p^{-1}\alpha p] = \alpha$.

□

2.3.1 Power Series Connection

To apply the theorems proved in Section 2.2, we need to construct a twisting cochain. There are several different constructions available for this purpose. The commutative algebra structure on $\Omega^*(M)$ defines a C_∞ algebra on $H^*(M)$, (see [CG] for a description of how to transfer structure). The C_∞ algebra defines a derivation of square zero on $\mathcal{L}(H_*(M)[-1])$ and the inclusion $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$ defines a twisting cochain. Note that the C_∞ algebra on $H^*(M)$ is a minimal model for $\Omega^*(M)$. Kadeishvilli's Minimal Model Theorem, [K], provides another construction of a twisting cochain.

We choose to review the work of Chen [C] and Hain [H2] on power series connections, which gives an equivalent construction of the minimal model for $\Omega^*(M)$ as the one described above. A power series connection will be a twisting cochain from $H_*(M) \to \mathcal{L}(H_*(M)[-1])$ in slightly different terminology. The equivalence of Kadeishvilli's construction and Hain's construction is described in [Hue]. The construction is explicit and self contained, which is why we have chosen to include it.

Let M be a simply connected manifold. We introduce some notation. If L is a Lie algebra, let $I^2L = [L, L]$, and for $s > 2$, $I^sL = [L, I^{s-1}L]$. Also, for $w \in \Omega^*(M)$, let $J(w) = (-1)^{|w|}w$.

Hain, [H2], defines a *power series connection* to be a pair consisting of an element $\omega \in \Omega^*(M) \otimes \mathcal{L}(H_*(M;\mathbb{R})[-1])$ and derivation ∂ on $\mathcal{L}(M_*(X;\mathbb{R})[-1])$, such that

1. $\partial^2 = 0$

2. if $\omega \equiv \sum W_i X_i (\operatorname{mod} \Omega^*(M) \otimes I^2\mathcal{L}(H_*(M)[-1]))$, then W_i are closed forms whose cohomology classes form a basis for $H^*(M;\mathbb{R})$,

3. $\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega] = 0$.

The last condition for ω is referred to as the *twisting cochain condition.*

We go through Hain's construction of a power series connection, which requires the next lemma. The statement can be found in [H1], where a dual statement is proved.

Lemma 2.3.2 ([H1], Lemma 3.8)**.** *Let L be a graded Lie algebra and ∂ be a derivation of L such that $\partial(L) \subset [L, L]$. Suppose ω is an element of $\Omega^*(M) \otimes L$ such that*

1. $\omega \equiv \sum W_i X_i (mod\, \Omega^*(M) \otimes I^2 L)$*, where W_i are closed forms whose cohomology classes form a linear basis for $H^*(M)$,*
2. $\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega] \equiv 0 (mod\, \Omega^*(M) \otimes I^n L)$*,*

Then

1. $\partial^2 \equiv 0 (mod\, I^{n+1} L)$ *and*
2. $d(\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega]) \equiv 0 (mod\, \Omega^*(M) \otimes I^{n+1} L)$.

Theorem 2.3.3 ([H2]. Theorem 2.6)**.** *There exists a pair (ω, ∂) such that*

1. $\omega \in \Omega^*(M) \otimes \mathcal{L}(H_*(M)[-1])$*,*
2. *∂ is a derivation of $\mathcal{L}(H_*(M)[-1])$ of square zero*
3. $\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega] = 0$.

Proof. The proof can be found in [H2]. But we go over it, because this construction will be referred to later on. Let (X_i) be a basis of $H_*(M)$. Suppose (W_i) are closed forms in $\Omega^*(M)$ whose cohomology classes form a basis of $H^*(M)$ dual to (X_i). We construct ∂ and ω inductively and simultaneously. For ease of notation, let $L = \mathcal{L}(H_*(M)[-1])$.

The first step is to let

$$\begin{aligned} \omega_1 &= \sum_i W_i X_i \\ \partial_1 X_i &= 0 \text{ for all } i. \end{aligned}$$

Then the Maurer Cartan equation is partially satisfied,

$$\partial_1 \omega_1 + d\omega_1 - \frac{1}{2}[J\omega_1, \omega_1] \equiv 0 \pmod{\Omega^*(M) \otimes I^2 L}.$$

Now, suppose that ∂_r and ω_r for $r < s$ are defined so that

1. ∂_r is a derivation of L,
2. $\partial_{s-1} X_i \equiv \partial_r X_i \pmod{I^{r+1} L}$
3. $\omega_{s-1} \equiv \omega_r \pmod{\Omega^*(M) \otimes I^{r+1} L}$,
4. $\partial_r \omega_r + d\omega_r - \frac{1}{2}[J\omega_r, \omega_r] \equiv 0 \pmod{\Omega^*(M) \otimes I^{r+1} L}$.

We need to define ∂_s and ω_s to continue the induction step. By Lemma

2.3.2,

$$d\left(\partial_{s-1}\omega_{s-1} + d\omega_{s-1} - \frac{1}{2}[J\omega_{s-1}, \omega_{s-1}]\right) = 0.$$

But since the cohomology classes of (W_i) form a basis, we have the identity

$$\begin{aligned} & \partial_{s-1}\omega_{s-1} + d\omega_{s-1} - \frac{1}{2}[J\omega_{s-1}, \omega_{s-1}] \\ = & \sum_{i_1\cdots i_s}\left(\sum_i a_i^{i_1\cdots i_s} W_i + dW_{i_1\cdots i_s}\right)[X_{i_1}, [X_{i_2}, \cdots [X_{i_{s-1}}, X_{i_s}]]]. \end{aligned}$$

Then let

$$\begin{aligned} \omega_s &= \omega_{s-1} + \sum_{i_1\cdots i_s} W_{i_1\cdots i_s}[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{s-1}}, X_{i_s}]]] \\ \partial_s X_i &= \partial_{s-1}X_i + \sum_{i_1\cdots i_s} a_i^{i_1\cdots i_s}[X_{i_1}, [X_{i_2}, \cdots [X_{i_{s-1}}, X_{i_s}]]]. \end{aligned}$$

Looking at the Maurer Cartan equation modulo $\Omega^*(M) \otimes I^{s+1}L$,

$$\begin{aligned} & \partial_s\omega_s + d\omega_s - \frac{1}{2}[J\omega_s, \omega_s] \\ \equiv & \partial_{s-1}\omega_{s-1} + d\omega_{s-1} - \frac{1}{2}[J\omega_{s-1}, \omega_{s-1}] \\ & + \sum_i\left(\sum_{i_1\cdots i_s} a_i^{i_1\cdots i_s} W_i + dW_{i_1\cdots i_s}\right)[X_{i_1}, [X_{i_2}, \cdots [X_{i_{s-1}}, X_{i_s}]]] \\ \equiv & 0. \end{aligned}$$

This allows us to continue our induction. Define ω and ∂ by the equations

$$\begin{aligned} \partial X_i &\equiv \partial_s \pmod{I^{s+1}L} \\ \omega &\equiv \omega_s \pmod{\Omega^*(M) \otimes I^{s+1}L}. \end{aligned}$$

□

It is a result of rational homotopy theory that the homology of $(\mathcal{L}(H_*(M)[-1]), \partial)$ is isomorphic to $\pi_*(M) \otimes \mathbb{Q}$ and the homology of $(U(\mathcal{L}(H_*(M)[-1])), \partial)$ is isomorphic to $H_*(\Omega_b(M))$ as a Hopf algebra.

The twisting cochain will be the inclusion $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$. The power series connection defines the differential on $\mathcal{L}(H_*(M)[-1])$ to be used in the Maurer Cartan equation and the twisting cochain condition implies that the inclusion is indeed a twisting cochain. The power series connection also has the following consequence.

Theorem 2.3.4. *[GLS] The power series connection ω defines a dg coalgebra map $T(H^*(M)[1]) \to T(\Omega^*(M)[1])$. There is map $T(\Omega^*(M)[1]) \to T(H^*(M)[1])$ such that the composition of the two maps is homotopic to the identity on $T(\Omega^*(M)[1])$ and equal to the identity on $T(H^*(M)[1])$.*

Proof. The element ω defines a map $T(H^*(M)[1]) \to \Omega^*(M)$, using the adjunction between tensor and Hom. The twisting cochain condition on ω implies that the map satisfies the Maurer Cartan equation. The relations between power series connections and twisting cochains is described in [[GLS], Section 1.3]. Using the correspondence between twisting cochains and coalgebra maps then implies that extending the map as a coalgebra respects the differentials.

The second claim about the map $T(\Omega^*(M)[1]) \to T(H^*(M)[1])$ is a consequence of the map being a deformation retraction. This result can be found in [Mer].

□

2.3.2 A_∞ coalgebra modeling the homology of the principal path space

With a twisting cochain $H_*(M) \to \mathcal{L}(H_*(M)[-1])$ at our disposal, we can apply the theorems of Section 2.2 to the path space fibration and its conjugate bundle. This gives us three structures, a twisted A_∞ coalgebra on $H_*(M) \otimes T(H_*(M)[-1])$ modeling the coproduct on $H_*(P_b(M))$, a twisted A_∞ coalgebra on $H_*(M) \otimes T(H_*(M)[-1])$ with the conjugation action modeling $H_*(LM)$ modeling the coproduct on $H_*(LM)$, and a twisted A_∞ algebra on $H_*(M) \otimes T(H_*(M)[-1])$ modeling the loop product.

Theorem 2.3.5. *Let M be a simply connected manifold, $\Omega_b(M) \to P_b(M) \to M$ be the path space fibration, and $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$ be the twisting cochain given by the inclusion. Then $(H_*(M) \otimes T(H_*(M)[-1]), \{c_n^\tau\})$ defines an A_∞ coalgebra model $H_*(P)$.*

Proof. The diagonal map $M \to M \times M$ defines a C_∞ coalgebra on $H_*(M)$ and $T(H_*(M)[-1])$ is a Hopf algebra model for $H_*(\Omega_b(M))$. The theorem is then a consequence of Theorem 2.2.9.

□

2.3.3 A_∞ coalgebra modeling the homology of the free loop space

This brings us to defining operations in string topology. The tensor product $H_*(M) \otimes T(H_*(M)[-1])$ is an A_∞ coalgebra given by combining the C_∞ coalgebra on $H_*(M)$ and the strict associative algebra on $T(H_*(M)[-1])$. Using our twisting cochain, we twist the A_∞ coalgebra as described in Section 2.2.6.

Theorem 2.3.6. *Let $H_*(M)$ be a simply connected manifold. Consider the C_∞ coalgebra on $H_*(M)$, the Hopf algebra on $T(H_*(M)[-1])$, and the conjugation action on $T(H_*(M)[-1])$. The maps*

$$\begin{aligned} \partial_\tau : H_*(M) \otimes T(H_*(M)[-1]) &\rightarrow H_*(M) \otimes T(H_*(M)[-1]) \\ c_n^\tau : H_*(M) \otimes T(H_*(M)[-1]) &\rightarrow (H_*(M) \otimes T(H_*(M)[-1])^{\otimes n} \end{aligned}$$

define an A_∞ coalgebra. The linear homology, $(H_*(M) \otimes T(H_*(M)[-1]), \partial_\tau)$, is the homology of the free loop space of the manifold $H_*(LM)$.

Proof. The proof follows from the application of Theorem 2.2.9.

□

2.3.4 A_∞ algebra modeling the homology of the free loop space

The loop product in $H_*(LM)$, first described in [CS], is intuitively defined as combining the intersection product of $H_*(M)$ with loop concatenation in $H_*(\Omega_b(M))$. The set-up of twisted tensor products accommodates such a description. The tensor product $H_*(M) \otimes T(H_*(M)[-1])$ is an A_∞ algebra. The map

$$m_2 : (H_*(M) \otimes T(H_*(M)[-1]))^{\otimes 2} \to H_*(M) \otimes T(H_*(M)[-1])$$

is a combination of the intersection product and loop concatenation. However, its linear homology is not $H_*(LM)$ so it does not define an operation in $H_*(LM)$. For this we need to take the twisted differential ∂_τ. Unlike the coalgebra case, we do not need to twist the higher multiplication maps.

Theorem 2.3.7. *Let M be a simply connected manifold. Consider the cyclic C_∞ coalgebra on $H_*(M)$, the Hopf algebra on $T(H_*(M)[-1])$, and the conjugation action on $T(H_*(M)[-1])$. The maps*

$$\begin{aligned} \partial_\tau : H_*(M) \otimes T(H_*(M)[-1]) &\to H_*(M) \otimes T(H_*(M)[-1]) \\ m_n : (H_*(M) \otimes T(H_*(M)[-1]))^{\otimes n} &\to H_*(M) \otimes T(H_*(M)[-1]) \end{aligned}$$

define an A_∞ algebra on $H_*(M) \otimes T(H_*(M)[-1])$.

Proof. The proof is an application of Theorem 2.2.18. □

Example 2.3.8. Let $M = G$ be a connected Lie group and consider the path space fibration, $\Omega_b(G) \to P_b(G) \to G$. We claim that the conjugation action of $\Omega_b(G)$ is trivial, and so there is no twisting given by the twisting cochain $H_*(G) \hookrightarrow \mathcal{L}(H_*(G)[-1])$. Consequently, the string topology operations are given by the untwisted tensor $H_*(G) \otimes T(H_*(G)[-1])$.

To see that the conjugation action is trivial, recall that a Hopf algebra H_* is commutative if the Lie bracket on $Prim(H_*)$ is zero. In this case, the Hopf algebra is $H_*(\Omega_b(G))$. There is a homotopy equivalence, $\Omega_b(G) \cong \Omega_b^2(BG)$. The Lie bracket is the same as the Samelson bracket on $\pi_*(\Omega_b^2(BG))$ which is equal to the Whitehead bracket on $\pi_*(\Omega_b(BG))$. This bracket is zero because the Whitehead bracket is trivial on H-spaces. Since the multiplication is commutative, the conjugation action is trivial and there is no twisting coming from a twisting cochain. This computation agrees with that in [Hep]. In that paper, Hepworth uses the isomorphism between LG and $G \times \Omega_b(G)$ to determine the Batalin Vilkovisky algebra on $H_*(\Omega_b(G))$. Menichi, in [Me], investigates the BV structure on $H_*(\Omega_b^2(BG)) \otimes H_*(M)$, and also considers the case when $M = G$. In that paper, he constructs a BV algebra morphism $H_*(\Omega_b(G)) \to H_*(\Omega_b(G) \otimes H_*(M) \to H_*(LM)$.

The argument that the conjugation action is trivial can be applied to any manifold M that is an H-space.

2.4 Application to Principal G Bundles

We are interested in applying the results in Section 2.2 to the case of a principal G bundle $G \to P \to M$. This will turn out to be representations of the algebraic structures on $H_*(M) \otimes_\tau T(H_*(M)[-1])$ given in the previous section. Given a connection on a bundle $G \to P \to M$, we get a map of bundles $P_b(M) \to M$ to $P \to M$ in the following way. Choose a basepoint above the fiber in $P \to M$, and denote it by $e \in F_b(M)$. Then the fiber can be identified with G, and e is identified with the identity element. Using the lifting property for connections gives us maps

$$\begin{aligned} \Omega_b(M) &\to G \\ P_b(M) &\to P. \end{aligned}$$

The map $\Omega_b(M) \to G$ is often referred to as the holonomy map.

Lemma 2.4.1. *Let $G \to P \to M$ be a principal bundle with connection and $\Omega_b(M) \to P_b(M) \to M$ be the path space fibration. The diagram*

$$\begin{array}{ccc} P_b(M) & \longrightarrow & P \\ \downarrow & & \downarrow \\ M & \xrightarrow{Id} & M \end{array}$$

commutes. Furthermore, the map $P_b(M) \to P$ commutes with the $\Omega_b(M)$ action on $P_b(M)$ and the G action on P.

Proof. This first part is the definition of lifting paths. See ([KN], Proposition 3.2) for the second statement. $\square$

This bundle map induces a map on the conjugate bundles

$$\begin{array}{ccc} Conj(P_b(M)) & \longrightarrow & Conj(P) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M. \end{array}$$

An element in $Conj(P_b(M))$ is represented by an element $(p_t, \alpha) \in P_b(M) \times \Omega_b(M)$. The induced map is defined by taking a representative (p_t, α) and sending it by the map

$$(p_t, \alpha) \mapsto [\widetilde{p}_t(1), \widetilde{\alpha}] \in Conj(P) = P \times_G G.$$

A loop $\alpha \in \Omega_b(M)$ lifts to a path $\widetilde{\alpha}$ starting at $e \in F_b(M)$ and ending in $F_b(M)$. This path corresponds to an element in G. A path $p_t \in P_b(M)$ lifts to a path $\widetilde{p}_t$ in P starting at e. Then $p_t \mapsto \widetilde{p}_t(1) \in P$.

Proposition 2.4.2. *Let $G \to P \to M$ be a principal G bundle with connection and $\Omega_b(M) \to P_b(M) \to M$ be the path space fibration. The map*

$$\begin{array}{rcl} Conj(P_b(M)) & \to & Conj(P) \\ (p_t, \alpha) & \mapsto & (\widetilde{p}_t(1), \widetilde{\alpha}) \end{array}$$

is well defined and independent of choice of basepoint $e \in F_b(M)$.

Proof. Let $\beta \in \Omega_b(M)$. For the map to be well defined, $(\widetilde{p_t\beta}, \widetilde{\beta\alpha\beta^{-1}})$ and $(\widetilde{p}_1, \widetilde{\alpha})$ must be in the same equivalence class in $P \times_G G$. We see that conjugating $(\widetilde{p}_1, \widetilde{\alpha})$ by $\widetilde{\beta} \in G$ is $(\widetilde{p_t\beta}, \widetilde{\beta\alpha\beta^{-1}})$. So the map is well defined.

Choosing a different point $e' \in P_b(M)$ changes the map $P_b(M) \to P$ by the G action and changes the map $\Omega_b(M) \to G$ by a conjugation. In the conjugate bundle, the images belong to the same equivalence class. □

Given a bundle $G \to P \to M$, with G a connected Lie group, we look to construct a twisting cochain $\tau : H_*(M) \to H_*(G)$. Then using the methods in Section 2.2, we obtain various structures on $H_*(M) \otimes H_*(G)$ modeling $H_*(P)$. The twisting cochain will be in terms of the characteristic classes of the bundle.

Proposition 2.4.3. *([DP], p. 249) Let G be a Lie group and R a ring. The cohomology, $H^*(BG; R)$ is a polynomial R-algebra of finite type on generators of even degree.*

For $H_*(BG)$, we need a separate argument.

Lemma 2.4.4. *Let G be a connected Lie group. Then $H_*(BG)$ is a free commutative algebra.*

Proof. The classifying space BG is rationally equivalent to a product of Eilenberg Maclane spaces. Furthermore, since G is connected, the long exact sequence in homotopy groups of $G \to EG \to BG$, implies $\pi_1(BG) = 0$. The Eilenberg Maclane spaces here are then infinite loop spaces, and so BG is rationally an infinite loop space. This means $H_*(BG)$ is a Hopf algebra, which is commutative if the Lie bracket in $Prim(H_*(BG))$ is zero. This bracket is equivalent to the Whitehead bracket on $\pi_*(Y)$ where $\Omega^2_b(Y) = BG$. But Y is a loop space, since BG is, rationally, an infinite loop space. And the Whitehead bracket on H-spaces is zero.

Hopf algebras are self dual, so $H^*(BG)$ is a Hopf algebra and $H_*(BG)$ is the dual Hopf algebra. We see that $H_*(BG)$ is also a polynomial algebra.

□

2.4.1 Constructing the twisting cochain $H_*(M) \to H_*(G)$.

The power series connection $\omega \in \Omega^*(M) \otimes \mathcal{L}(H_*(M)[-1])$, constructed in Section 2.3.1 will be used once more. Theorem 2.3.4 defines a dg coalgebra map $T(H^*(M)[1]) \to T(\Omega^*(M)[1])$, which has an inverse $T(\Omega^*(M)[1]) \to T(H^*(M)[1])$.

Since G is a connected Lie group, $H^*(BG)$ is a polynomial algebra. This allows us to define maps from $H^*(BG)$ in terms of its polynomial generators. Let $\{p_i \in H^*(M\})$ be the characteristic classes of a bundle $G \to P \to M$. Then there is an algebra map $H^*(BG) \to \Omega^*(M)$ defined as follows. Let $\{P_i \in H^*(BG)\}$ be the polynomial generators which pullback to the characteristic classes $\{p_i\}$. Then define an algebra map by $P_i \mapsto \widehat{p_i}$, where $\widehat{p_i} \in \Omega^*(M)$ is a representative for p_i. Extend the map as an algebra map to all of $H^*(BG)$. The algebra map $H^*(BG) \to \Omega^*(M)$ defines a coalgebra map $T(H^*(BG)[1]) \to T(\Omega^*(M)[1])$.

So we have a coalgebra map $T(H^*(BG)[1]) \to T(\Omega^*(M) \to T(H^*(M)[1])$, which defines an algebra map $T(H_*(M)[-1]) \to T(H_*(BG)[-1])$. To this algebra map, there is a corresponding twisting cochain $H_*(M) \to T(H_*(BG)[-1])$. Since $T(H_*(BG)[-1])$ is a model for $\Omega_b(BG)$, which is homotopy equivalent to G, we could do our work with twisting cochains now.

To replace $T(H^*(BG)[1])$ with $H^*(G)$ we need to find a coalgebra map $H^*(G) \to T(H^*(BG)[1])$. Recall that $H^*(G)$ is generated by odd dimensional generators U_i. To each U_i there is a generator of $H^*(BG)$ one degree higher, which we denote by P_i. We define

$$\begin{aligned} f : H^*(G) &\to T(H^*(BG)[1]) \\ U_i &\mapsto P_i \\ U_{i_1}U_{i_2} &\mapsto P_{i_1} \otimes P_{i_2} + P_{i_2} \otimes P_{i_1} \end{aligned}$$

and extending the map as an algebra map. So $f(U_{i_1}\cdots U_{i_j}) = P_{i_1} \Cup \cdots \Cup P_{i_j}$, where $\Cup$ is the shuffle product.

Lemma 2.4.5. *The map $f : H^*(G) \to T(H^*(BG)[1])$ is a map of differential graded coalgebras. Therefore, f is a map of differential graded Hopf algebras.*

Proof. The coproduct on $H^*(G)$ is given by

$$\Delta_G(U_{i_1}U_{i_2}) = U_{i_1}U_{i_2} \otimes 1 + U_{i_1} \otimes U_{i_2} + U_{i_2} \otimes U_{i_1} + 1 \otimes U_{i_1}U_{i_2},$$

and extended so that Δ_G is an algebra map. The coproduct on $T(H^*(BG)[1])$ is given by deconcatenation,

$$\Delta(P_{i_1} \otimes \cdots \otimes P_{i_k}) \;=\; \sum_j P_{i_1} \otimes \cdots P_{i_j} \bigotimes P_{i_{j+1}} \otimes \cdots \otimes P_{i_k}.$$

The following computation shows that f is a coalgebra map,

$$\begin{aligned}
(f \otimes f) \circ \Delta(U_iU_j) &= (f \otimes f)(U_iU_j \otimes 1 + U_i \otimes U_j + U_j \otimes U_i + 1 \otimes U_iU_j) \\
&= (P_i \otimes P_j) \otimes 1 + (P_j \otimes P_i) \otimes 1 + P_i \otimes P_j \\
&\quad + P_j \otimes P_i + 1 \otimes (P_i \otimes P_j) + 1 \otimes (P_j \otimes P_i) \\
&= \Delta(P_i \otimes P_j + P_j \otimes P_i) \\
&= \Delta f(U_iU_j).
\end{aligned}$$

The differential on $H^*(G)$ is zero, so for f to be a chain map, f must map to cocycles in $T(H^*(BG)[1])$. We see that δ is zero on P_i. Then since

f maps to shuffle products of P_i and δ is a derivation with respect to the shuffle product, f maps to cocycles.

□

To replace $T(H_*(BG)[-1])$ with $H_*(G)$, we take the dual of the above map to get a differential graded algebra map $T(H_*(BG)[-1]) \to H_*(G)$. So given a twisting cochain $\tau : H_*(M) \to T(H_*(BG)[-1])$, composing maps defines a twisting cochain $H_*(M) \to T(H_*(BG)[-1]) \to H_*(G)$. Similarly, $H^*(G) \to H^*(M)$ is a twisting cochain obtained by composing the twisting cochain $T(H^*(BG)[1]) \to H^*(M)$ and the coalgebra map $H^*(G) \to T(H^*(BG)[1])$.

We summarize the construction of the twisting cochain and give a formula for it. Let $G \to P \to M$ be a principal G bundle, where G is a connected Lie group and M is a simply connected manifold. Let $\{P_i\}$ be the multiplicative basis for $H^*(BG)$ where $p_i \in H^*(M)$ is the pullback of $P_i \in H^*(BG)$. The elements P_i are even dimensional and correspond to an element $U_i \in H^*(G)$ such that $\{U_i\}$ form a basis for $H^*(G)$. The following coalgebra maps are composed

1. $T(\Omega^* M)[1]) \to T(H^*(M)[1])$

2. $T(H^*(BG)[1]) \to T(\Omega^*(M)[1])$

3. $H^*(G) \to T(H^*(BG)[1])$

to define a coalgebra map $H^*(G) \to T(H^*(M)[1])$ which corresponds to a twisting cochain $H^*(G) \to H^*(M)$. When this process is carried out, $\tau : H^*(G) \to H^*(M)$ is defined on generators by

$$\begin{aligned} H^*(G) &\to H^*(M) \\ U_i &\mapsto p_i, \end{aligned}$$

and zero on products of generators.

Proposition 2.4.6. *Consider the coalgebra structure on $H^*(G)$ given by group multiplication and the C_∞ algebra structure on $H^*(M)$ given by the cup product. Then the map $\tau : H^*(G) \to H^*(M)$ which on generators is $U_i \mapsto p_i$ and zero on products of generators is the twisting cochain coming from the twisting cochain $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$ given by the inclusion.*

Proof. There are no differentials on $H^*(G)$ and $H^*(M)$, and so it suffices to show that $m_n^{Hom}(\tau^{\otimes n}) = 0$ for each n. For $m_n^{Hom}(\tau^{\otimes n})$ to be possibly non-zero, we need to consider the product of n generators $U_{i_1} \cdots U_{i_n}$. We look at terms in $\Delta^n(U_{i_1} \cdots U_{i_n})$ of the form

$$\sum U_{i_{\sigma(1)}} \otimes \cdots \otimes U_{i_{\sigma(n)}}.$$

Then we apply τ to each factor and apply $m_n : H^*(M)^{\otimes n} \to H^*(M)$ of the C_∞ algebra. But each m_n vanishes on shuffle products, so it is zero on products of these terms.

$\square$

2.4.2 A_∞ coalgebra of $H_*(M) \otimes_\tau H_*(G)$ for a principal G-bundle

We can now define the twisted A_∞ coalgebra structure on $H_*(M) \otimes H_*(G)$. We use the dual of $\tau : H^*(G) \to H^*(M)$, to get a twisting cochain. The map is also denoted τ and is defined as

$$\begin{aligned} \tau : H_*(M) &\to H_*(G) \\ p_i^* &\mapsto U_i^*, \end{aligned}$$

is zero on products $p_{i_1}^* \cdots p_{i_n}^*$. Note that $U_i^* \in Prim(G)$, and $[U_{i_1}^*, U_{i_2}^*]$ is defined. The tensor differential on $H_*(M) \otimes H_*(G)$ is zero, so ∂_τ consists only of twisted terms. These terms are obtained by applying $\{c_n : H_*(M) \to H_*(M)^{\otimes n}\}$, applying τ to the last $n-1$ terms, bracketing the results, and then multiplying the resulting bracket with the element in $H_*(G)$. The higher coproducts $c_2^\tau, c_3^\tau. \cdots$ are defined in the same way.

Theorem 2.4.7. *Let $\{p\}$ be the characteristic classes of a G bundle $G \to P \to M$, where G is a connected Lie group and M a simply connected*

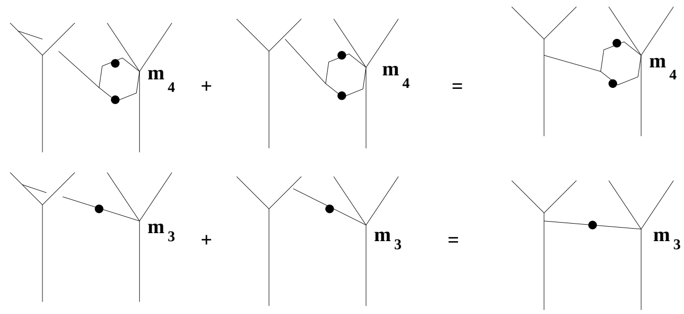

Figure 2.19: This identity is a consequence of the fact that $ker(\tau) \cup Prim(H_*) = H_*$. The figure is dual to Figure 2.6.

manifold. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \cdots\}$ *define an* A_∞ *coalgebra on* $H_*(M) \otimes H_*(G)$ *whose linear homology is isomorphic to* $H_*(P)$.

Proof. This is an application of Theorem 2.2.9.

□

The twisting cochain is more easily defined as $\tau : H^*(G) \to H^*(M)$, so the dual A_∞ algebra can be made more explicit. Note that if C_* is a C_∞ coalgebra, H_* a Hopf algebra, and a twisting cochain $C_* \to H_*$ has its image in the primitives, then its dual map $\tau : H^* \to C^*$ has the property that $ker(\tau) \cup Prim(H_*) = H_*$. This property of τ implies the derivation property dual to the statement that multiplying by a primitive element is a coderivation. It is described in Figure 2.19.

We define an A_∞ algebra on $H^*(G) \otimes H^*(M)$, where we view $H^*(G)$ as a

Hopf algebra and $H^*(M)$ as a C_∞ algebra. The map $\partial_\tau : H^*(G)\otimes H^*(M) \to H^*(G) \otimes H^*(M)$ is given by

$$\begin{aligned}
\partial_\tau(U_{i_1}\cdots U_{i_n}\otimes a) &= \sum_{\sigma\in S_n} U_{i_{\sigma(1)}}\cdots U_{i_{\sigma(n-1)}}\otimes m_2(p_{i_{\sigma(n)}}\otimes a) \\
&\quad + \sum_{\sigma\in S_n} U_{i_{\sigma(1)}}\cdots U_{i_{\sigma(n-2)}}\otimes m_3(p_{i_{\sigma(n-1)}}\otimes p_{i_{\sigma(n)}}\otimes a) \\
&\quad \vdots
\end{aligned}$$

The map $m_2^\tau : (H^*(G)\otimes H^*(M))^{\otimes 2} \to H^*(G)\otimes H^*(M)$ is given by

$$\begin{aligned}
& m_2^\tau(U_{i_1}\cdots U_{i_k}\otimes a, U_{i_{k+1}}\cdots U_{i_n}\otimes b) \\
= \; & U_{i_1}\cdots U_{i_n}\otimes m_2(a\otimes b) \\
& + \sum_{\sigma\in S_n} U_{i_{\sigma(1)}}\cdots U_{i_{\sigma(n-1)}}\otimes m_3(p_{i_{\sigma(n)}}\otimes a\otimes b) \\
& + \sum_{\sigma\in S_n} U_{i_{\sigma(1)}}\cdots U_{i_{\sigma(n-2)}}\otimes m_4(p_{i_{\sigma(n-1)}}\otimes p_{i_{\sigma(n)}}\otimes a\otimes b) \\
& \vdots
\end{aligned}$$

Proposition 2.4.8. *Let $\{p_i\}$ be the characteristic classes of a principal G bundle $P \to M$, with M simply connected and G a connected Lie group. The maps $\{\partial_\tau, m_2^\tau, \cdots\}$ define an A_∞ algebra on $H^*(G)\otimes H^*(M)$ whose linear cohomology is isomorphic to $H^*(P)$.*

Proof. This is the algebraic dual of Theorem 2.2.9. One can see that $\partial_\tau^2 = 0$

directly, as well.

$$
\begin{aligned}
& \partial_\tau^2(U_{i_1}\cdots U_{i_n}\otimes a) \\
= \quad & \sum_{\sigma'}\sum_{\sigma} U_{i_{\sigma'\sigma(1)}}\cdots U_{i_{\sigma'\sigma(n-2)}}\otimes m_2(p_{i_{\sigma'}}\otimes m_2(p_{i_{\sigma(n)}}\otimes a)) \\
& +\sum_{\sigma'}\sum_{\sigma} U_{i_{\sigma'\sigma(1)}}\cdots U_{i_{\sigma'\sigma(n-3)}}\otimes m_3(p_{i_{\sigma'\sigma(n-2)}}\otimes p_{i_{\sigma'\sigma(n-1)}}\otimes m_2(p_{i_{\sigma(n)}}\otimes a)) \\
& \vdots \\
& +\sum_{\sigma'}\sum_{\sigma} U_{i_{\sigma'\sigma(1)}}\cdots U_{i_{\sigma'\sigma(n-3)}}\otimes m_2(p_{i_{\sigma'(n-2)}}\otimes m_3(p_{i_{\sigma(n-1)}}\otimes p_{i_{\sigma(n)}}\otimes a)) \\
& +\sum_{\sigma'}\sum_{\sigma} U_{i_{\sigma'\sigma(1)}}\cdots U_{i_{\sigma'\sigma(n-4)}}\otimes m_3(p_{i_{\sigma'\sigma(n-3)}}\otimes p_{i_{\sigma'\sigma(n-2)}}\otimes m_3(p_{i_{\sigma(n-1)}}\otimes p_{i_{\sigma(n)}}\otimes a)) \\
& \vdots
\end{aligned}
$$

Note that on the $H^*(M)$ side of the tensor, there are compositions of m_i and m_j. The C_∞ algebra relation on $H^*(M)$ states that such sums will be zero.

For the higher identities, we use the identity in Figure 2.19 and follow the same argument that was made in Theorem 2.2.9. □

2.4.3 A_∞ coalgebra on $H_*(M)\otimes_\tau H_*(G)$ using conjugation action

The conjugation action of $H_*(G)$ on itself is trivial when G is a connected Lie group. This shows that there is no twisting needed for the A_∞ coalgebra on $H_*(M)\otimes H_*(G)$. That is, the coalgebra is given by $\{c_n\otimes\Delta^n\}$, where

$\{c_n\}$ is the C_∞ coalgebra given by the diagonal map and Δ^n is the n-fold composition of the coproduct on $H_*(G)$.

2.4.4 A_∞ algebra on $H_*(M) \otimes_\tau H_*(G)$ using conjugation action

Since the conjugation action is trivial, the A_∞ algebra on $H_*(M) \otimes H_*(G)$ is given by $\{m_n \otimes m_G\}$, with no twisting terms. Here, $\{m_n\}$ is the C_∞ algebra on $H_*(M)$ given by the intersection product and m_G is the associative multiplication in $H_*(G)$.

Chapter 3

Differential Forms

3.1 Operads, dioperads, and properads

3.1.1 Cyclic operads

We first review the definition of an operad. Let $\mathcal{C}$ be the category of $\mathbb{Z}$ graded vector spaces over a field k of characteristic zero.

Definition 3.1.1. An **operad** P in $\mathcal{C}$ consists of

1. objects $P(n)$ in $\mathcal{C}$ with $\mathbb{S}_n$ action for $n > 0$
2. for $m, n > 0$ and $1 \leq i \leq m$, morphisms

$$\circ_i : P(m) \otimes P(n) \to P(m+n-1),$$

3. a morphism $\eta : k \to P(1)$

satisfying the following associativity and equivariance conditions,

1. For compositions of $P(m) \otimes P(n) \otimes P(p)$, the following associative condition must be satisfied:

$$\circ_i(\circ_j) \otimes 1) = \begin{cases} \circ_{j+p-1}(\circ_i \otimes 1)(1 \otimes \tau), & \text{for } 1 \leq i < j-1, \\ \circ_j(1 \otimes \circ_{i-j+1}), & \text{for } j \leq i \leq j+n-1, \\ \circ_j(\circ_{i-n+1} \otimes 1)(1 \otimes \tau), & \text{for} j+n \leq i, \end{cases}$$

where $\tau : P(n) \otimes P(p) \to P(p) \otimes P(n)$ is the transposition given by the symmetry,

2. the operation is $\circ_i$ is equivariant in the sense that

$$\circ_i(\sigma\rho) = (\sigma \circ_i \rho)\circ_{\sigma(i)}$$

on $P(m) \otimes P(n)$ where $\sigma \in S(m)$, $\rho \in S(n)$, and $\sigma \circ_i \rho$ is a block permutation

A cyclic operad is an operad P such that the $\mathbb{S}_n$ action on $P(n)$ extends to an action on $\mathbb{S}_n^+$ satisfying associative and equivariance relations. Here, $\mathbb{S}_n^+$ is the permutation group on $\{0, 1, 2, \cdots, n\}$. The group $\mathbb{S}_n$ is viewed as the subgroup of permutations in $\mathbb{S}_n^+$ that fix 0. Let τ_n be the cyclic permutation $(0, 1, \cdots, n)$. Then an operad P is cyclic if the $\mathbb{S}_n^+$ action satisfies the following.

1. If $\eta : 1 \to P(1)$ is the unit, then the following diagram commutes

$$\begin{array}{ccc} P(1) & \xrightarrow{\tau_1} & P(1) \\ {\scriptstyle \eta}\uparrow & \nearrow{\scriptstyle \eta} & \\ 1 & & \end{array}$$

2. For each $m, n \geq 1$, the following diagram commutes

$$\begin{array}{ccc} P(m) \otimes P(n) & \xrightarrow{\circ_1} & P(m+n-1) \\ \downarrow{\scriptstyle \tau_m \otimes \tau_n} & & \\ P(m) \otimes P(n) & & \downarrow{\scriptstyle \tau_{m+n-1}} \\ \downarrow & & \\ P(n) \otimes P(m) & \xrightarrow{\circ_n} & P(m+n-1) \end{array}$$

If V is a finite dimensional vector space with a non-degenerate bilinear form $\langle , \rangle$, then $End(V)$ is a cyclic operad. The associative, commutative, and Lie operad are examples of cyclic operads, as shown in [GK].

3.1.2 Dioperads

Dioperads model structures on vector spaces with products and coproducts, where the composition rule glues one output to one input. In this way, genus is never created. Dioperads can model Frobenius algebras, since the relation that the coproduct be a bimodule map does not have genus and can be described by grafting one output of a tree with one input of another tree.

In contrast, dioperads cannot model bialgebras, since the requirement that the coproduct be an algebra map introduces genus.

Definition 3.1.2. A **dioperad** P in $\mathcal{C}$ consists of

1. objects $P(m,n)$ in $\mathcal{C}$ with an $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule structure for $m, n > 0$,

2. for $m_1, m_2, n_1, n_2 \geq 1$, $1 \leq i \leq n_1$, and $1 \leq j \leq m_2$, morphisms

$$ {}_i\circ_j : P(m_1, n_1) \otimes P(m_2, n_2) \to P(m_1 + m_2 - 1, n_1 + n_2 - 1), $$

3. a morphism $\eta : k \to P(1,1)$ such that

$$ {}_1 \circ_i (\eta \otimes Id) : k \otimes P(m,n) \tilde{\to} P(m,n), $$

and

$$ {}_j \circ_1 (Id \otimes \eta) : P(m,n)) \tilde{\to} P(m,n) $$

are the canonical isomorphisms.

The morphisms ${}_i\circ_j$ must respect the $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule actions. The conditions are described in [G].

The augmentation ideal of a dioperad P, denoted $\overline{P}$, is defined by $\overline{P}(1,1) = 0$ and $\overline{P}(m,n) = P(m,n)$ for $m + n \geq 3$.

If V is an object in $\mathcal{C}$, the *endomorphism dioperad* $End^0(V)$ is given by letting $End^0(V)(m,n)$ be the complex $Hom(V^{\otimes m}, V^{\otimes n})$. The superscript

0 indicates that only maps without genus are considered in the complex $Hom(V^{\otimes m}, V^{\otimes n})$ and distinguishes the endomorphism dioperad $End^0(V)$ from the endomorphism properad $End(V)$. An algebra over a dioperad P is a dioperad map $P \to End^0(V)$.

If V is an $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule, V^* is also an $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule. If $f \in V^*$ and $\sigma \in S^m$, then $\sigma f \in V^*$ is defined by $\sigma f(x) = f(\sigma(x))$. The S^n action on V^* is defined similarly.

There is an equivalent definition of a dioperad as a monoid in a monoidal category. We describe the category of two leveled trees. A tree will be a finite, directed tree, with m inputs and n outputs. If every vertex has valence at least three, then T is said to be reduced. Each tree T has a global flow- a direction on the edges signifying inputs and outputs. Let $Edge(T)$ be the set of edges of T, $edge(T)$ be the set of internal edges of T, $In(v)$ be the set of incoming edges at the vertex $v \in T$, and $Out(v)$ be the set of outgoing edges at v. We assume that $In(v)$ and $Out(v)$ are labelled by $\{1, 2, \cdots, |In(v)|\}$ and $\{1, 2, \cdots, |Out(v)|\}$. For a k- dimensional vector space V, denote by $Det(V)$ be the top exterior power of V. The global inputs and outputs will also be labelled. A 2-*leveled tree* is a directed tree such that the vertices are divided into a top level and bottom level. Denote the set of vertices in

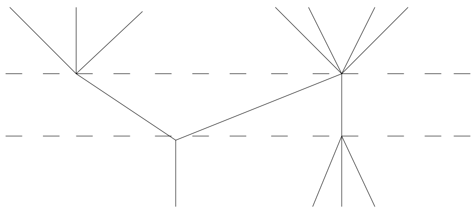

Figure 3.1: A 2 leveled tree.

the top level by N_1 and bottom level by N_2. We refer the reader to Figure 3.1. We let $\mathcal{T}$ be the category of graphs and $\mathcal{T}^2$ be the category of 2 leveled trees.

If $\{P(m,n)\}$ and $\{Q(m,n)\}$ are collections of two $(\mathbb{S}_m, \mathbb{S}_n)$-bimodules, their composition product is given by considering the collection of two leveled trees, identifying vertices of the top level with elements in $P(m,n)$ and vertices in the bottom level with elements in $Q(m,n)$. The composition product $\boxtimes$ of P and Q is given by

$$Q \boxtimes P = \left(\oplus_{g \in \mathcal{T}^2} \otimes_{v \in N_2} Q(|Out(v), In(v)) \otimes_k \otimes_{v \in N_1} P(|Out(v)|, |In(v)|)\right) / \sim,$$

where the equivalence is explained in Figure 3.2 .

The connected composition product $\boxtimes_c$ is obtained by restricting $\boxtimes$ to connected two leveled trees. The unit I for this monoidal product is

$$I(m,n) := \begin{cases} \mathbb{K} & \text{for } (m,n) = (1,1) \\ 0 & \text{otherwise.} \end{cases}$$

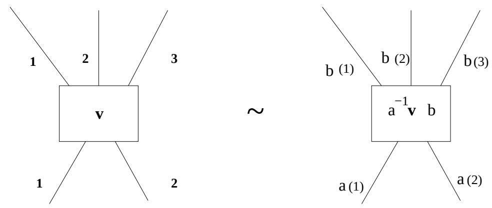

Figure 3.2: The equivalence relation in the definition $\boxtimes$.

In [G], another product is defined on the category of $\mathbb{S}$-bimodules, denoted $\boxdot$. It is obtained from $\boxtimes_c$ by restricting over saturated two level trees. A vertex $v \in T$ is saturated if every path from a leaf vertex to a root vertex passes through v. A tree is called saturated if every vertex is saturated. Then if P_1 and P_2 are two $\mathbb{S}$ bimodules, define $P_1 \boxdot P_2$ by

$$(P_1 \boxdot P_2)(m,n) = \bigoplus_{(T,l)} \otimes_{i=1}^{2} \otimes_{v \in l^{-1}} P_i(In(v), Out(v)), \tag{3.1.1}$$

where the sum is taken over all saturated 2 level labeled (m,n) trees (T,l). This product has a nice property with respect to Koszul duality, which we describe in Proposition 3.1.4 after defining the cobar complex.

Example 3.1.3. We describe the free dioperad. Let $E(m,n)$ is a collection of $(\mathbb{S}_m, \mathbb{S}_n)$ bimodules. Then for a tree T, let $E(T) = \otimes_{v \in T} E(In(v), Out(v))$. The free dioperad $\mathcal{F}(E)$ generated by E is given by

$$\mathcal{F}(E)(m,n) = \bigoplus_{(m,n) \text{ trees}} E(T)/\sim,$$

where the equivalence relation is the one described in the composition product $\boxtimes$ (see Figure 3.2).

Suppose $\{E(m,n)\}$ is an $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule, such that $E(m,n) = 0$ for all (m,n), except $E(2,1)$ and $E(1,2)$. Take the free dioperad generated by E. Let (R) be an ideal generated by $R(1,3) \subset \mathcal{F}(E)(1,3)$, $R(3,1) \subset \mathcal{F}(E)(3,1)$, and $R(2,2) \subset \mathcal{F}(2,2)$. Then the quotient dioperad $\mathcal{F}(E)/(R)$, denoted $\langle E; R\rangle$, is called a quadratic dioperad.

Let $P = \langle E; R\rangle$ be a quadratic operad. The quadratic dual of P is a dioperad $P^! = \langle E^\vee; R^\perp\rangle$, where $E^\vee(m,n) = E^* \otimes (Sgn_m \otimes Sgn_n)$ and $R^\perp(m,n)$ is the orthogonal complement of $R(m,n)$.

Let $Det(T)$ be the one dimensional vector space $Det(k^{Edge(T)})$ and $det(T) = Det(k^{edge(T)})$. Given a dioperad P, the *cobar complex* of $P(m,n)$, denoted $C(P)$ is a bicomplex $\mathcal{F}(\overline{P}^*(m,n))$ where the differential is the sum of the differential induced by the differential of P, denoted d', and the differential induced by edge contractions, denoted d''. Looking at d'' differential, we get a complex

$$\overline{P}(m,n)^* \stackrel{d''}{\to} \bigoplus_{\substack{|edge(T)|=1 \\ \text{reduced } (m,n) \text{ trees}}} \overline{P}^*(T) \otimes det(T) \stackrel{d''}{\to} \bigoplus_{\substack{|edge(T)|=2 \\ \text{reduced } (m,n) \text{ trees}}} \overline{P}^*(T) \otimes det(T) \stackrel{d''}{\to} \cdots,$$

where $\overline{P}(m,n)^*$ is placed in degree one.

The *cobar dual* $\mathbb{D}(P)$ is given by $\Lambda^{-1}(C(P))$. The chain complex with d'' is given by

$$\overline{P}(m,n)^{\vee} \stackrel{d''}{\to} \bigoplus_{\substack{|edge(T)|=1 \\ \text{reduced } (m,n) \text{ trees}}} \overline{P}^{*}(T) \otimes Det(T) \stackrel{d''}{\to} \bigoplus_{\substack{|edge(T)|=2 \\ \text{reduced } (m,n) \text{ trees}}} \overline{P}^{*}(T) \otimes Det(T) \stackrel{d''}{\to} \cdots,$$

where $\overline{P}(m,n)^{\vee}$ is put in degree $3-m-n$. The bimodule $\overline{P}(m,n)^{\vee}$ is $\overline{P}^{*} \otimes (Sgn_m \otimes Sgn_n)$.

Recall that a dioperad P is Koszul if there exists a dioperad Q such that $\mathbb{D}(Q) \to P$ is a quasimorphism. The dioperad Q is said to be Koszul dual to P, and is denoted by $P^!$.

Proposition 3.1.4 ([G], Proposition 5.9)**.** *Let $P = \langle E; R \rangle$ be a quadratic dioperad $A(n) := P(n,1)$ and $B(n) := P^{op}(n,1)$.*

1. *If $P(m,n) = A \boxdot B^{op}(m,n)$ for $(m,n) = (2,2)$, $(2,3)$, and $(3,2)$, then $P = A \boxdot B^{op}$. Similarly, if $P(m,n) = A^{op} \boxdot B(m,n)$ for $(m,n) = (2,2)$, $(2,3)$ and $(3,2)$, then $P = A^{op} \boxdot B$.*

2. *$P = A \boxdot B^{op}$ if and only if $P^! = B^{!op} \boxdot A^!$. Similarly, $P = A^{op} \boxdot B$ if and only if $P^! = B^! \boxdot A^{!op}$*

3. *If A and B are Koszul, and $P = A \boxdot B^{op}$, then P is Koszul. Similarly, if A and B are Koszul and $P = A^{op} \boxdot B$, then P is Koszul.*

3.1.3 Frobenius dioperad

The Frobenius dioperad, denoted $Frob^0$ is generated by an element in $m \in E(2,1)$ and $\Delta \in E(1,2)$, each with trivial $\mathbb{S}_2$ actions. The generators are required to satisfy the following relations:

$$(m\,{}_1\!\circ_1 m) - (m\,{}_1\!\circ_1 m)\sigma, (m\,{}_1\!\circ_1 m) - (m\,{}_1\!\circ_1 m)\sigma^2$$

$$(\Delta\,{}_1\!\circ_1 \Delta) - \sigma(\Delta\,{}_1\!\circ_1 \Delta), (\Delta\,{}_1\!\circ_1 \Delta) - \sigma^2(\Delta\,{}_1\!\circ_1 \Delta)$$

$$(\Delta\,{}_1\!\circ_1 m) - (m\,{}_1\!\circ_1 \Delta), (\Delta\,{}_1\!\circ_1 m) - (m\,{}_1\!\circ_2 \Delta),$$

$$(\Delta\,{}_1\!\circ_1 m) - (m\,{}_2\!\circ_1 \Delta), (\Delta\,{}_1\!\circ_1 m) - (m\,{}_2\!\circ_2 \Delta),$$

where $\sigma = (123) \in \mathbb{S}_3$. The superscript in $Frob^0$, which we use to distinguish it from properads, emphasizes that we are considering only genus zero operations. Note that $Frob^0(m,n)$ is one dimensional for each m and n.

From Proposition 3.1.4, we can see the following equality: $Frob^0 = Com^{op} \boxdot Com$. It is then Koszul, with Koszul dual $Lie \boxdot Lie^{op}$. Algebras over $Lie \boxdot Lie^{op}$ are Lie bialgebras, so we denote the dioperad by $BiLie^0$. The dioperad $BiLie^0$ is generated by elements $l \in E(2,1)$ and $\delta \in E(1,2)$

using the sign action of $\mathbb{S}_2$. Its relations are spanned by

$$(l\,{}_1\circ_1 l) + (l\,{}_1\circ_1 l)\sigma + (l\,{}_1\circ_1)\sigma^2$$

$$(\delta\,{}_1\circ_1 \delta) + \sigma(\delta\,{}_1\circ_1 \delta) + \sigma^2(\delta\,{}_1\circ_1 \delta)$$

$$\delta\,{}_1\circ_1 l) - (l\,{}_1\circ_1 \delta) - (l\,{}_1\circ_2 \delta) - (l\,{}_2\circ_1 \delta) - (l\,{}_2\circ_2 \delta).$$

3.1.4 Functor from cyclic operads to dioperads

Given a cyclic operad P, we show how to obtain a dioperad denoted $\mathcal{D}(P)$. To see why such a construction is possible, we consider the case of the cyclic endomorphism operad $End_{cyc}(V)$. If V is a finite dimensional vector space with a non-degenerate bilinear form $\langle , \rangle : V \otimes V \to k$, then $End(V)$ is a cylic operad. The $\mathbb{S}_m^+$ action is given by viewing a map $f : V^{\otimes m} \to V$ as

$$\begin{aligned} \hat{f} : V^{\otimes m+1} &\to k \\ (v_0, \cdots, v_m) &\mapsto \langle v_0, f(v_1, \cdots, v_m)\rangle. \end{aligned}$$

If f is viewed graphically as a tree with m whose leaves are labeled $1, \cdots, m$ and a root, then $\hat{f}$ is the same tree, with the root labeled by 0.

The maps $\circ_j : End(V)(m) \otimes End(V)(n) \to End(V)(m + n - 1)$ are defined by grafting the root of one tree to the edge with label j to the other

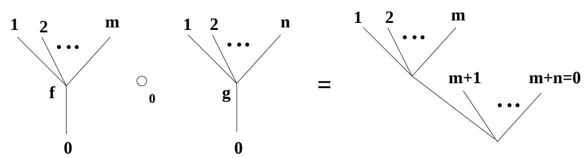

Figure 3.3: The result of grafting two trees along their root edges. Note that the final leaf, $m+n$ is relablled zero and designated to be the root.

tree. For $\hat{f} : V^m \to k$ and $\hat{g} : V^n \to k$, if $j \neq 0$, let $\hat{f} \circ_i \hat{g}$ be $\widehat{f \circ_i g}$; that is,

$$\begin{aligned} \hat{f} \circ_i \hat{g} : V^{m+n} &\to k \\ (v_0, \cdots, v_{m+n-1}) &\to \langle v_0, f \circ_i g(v_1, \cdots, v_{m+n-1}) \rangle. \end{aligned}$$

If the grafting occurs along a root edge, we define the composition rule as follows,

$$\begin{aligned} \hat{f} \circ_0 \hat{g} : V^{m+n} &\to k \\ (v_0, \cdots, v_{m+n-1}) &\mapsto \langle v_0, g(f(v_1, \cdots, v_m), v_{m+1}, \cdots, v_{m+n-1}) \rangle. \end{aligned}$$

The motivation for the definition can be seen from drawing the tree obtained by grafting to the root edge; we refer the reader to Figure 3.3.

To this cyclic operad, we associated a dioperad denoted $\mathcal{D}(End(V))$. Let $\mathcal{D}(End(V))(m,n) = End_{cyc}(V)(m+n-1)$. The $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule structure of $\mathcal{D}(End(V))(m,n)$ is given by viewing $\mathbb{S}_m$ and $\mathbb{S}_n$ as subgroups of $\mathbb{S}^+_{m+n-1}$. The group $\mathbb{S}_m$ is viewed as a subgroup of $\mathbb{S}^+_{m+n-1}$ by the usual identification (i.e. $1 \mapsto 1, 2 \mapsto 2, \cdots, m \mapsto m$). The group $\mathbb{S}_n$ is a subgroup of $\mathbb{S}^+_{m+n-1}$ given by the identification $1 \mapsto m+1, 2 \mapsto m+2, \cdots, n-1 \mapsto m+n-1, n \mapsto 0$.

To define the ${}_i\circ_j$ maps, we use the maps $\hat{f} \circ_j \hat{g}$ along with the τ_n. If f is a tree with m_1 inputs and n_1 outputs g is a tree with m_2 inputs and n_2 outputs, then $f \,{}_i\circ_j\, g$ is a tree with $m_1 + m_2 - 1$ inputs and $n_1 + n_2 - 1$ outputs obtained by grafting the i^{th} output of g with the j^{th} input of f. For $\hat{f}$ and $\hat{g}$, where the trees have only one output, we interpret the i^{th} output of $\hat{g}$ to be the root of $\hat{g}$ after τ_n^i has been applied to $\hat{g}$. We then apply τ_n^{-i} to undo the application. In summary, $\hat{f} \,{}_i\circ_j\, \hat{g} = \tau_n^{-i}(\hat{f} \circ_j \tau^i(\hat{g}))$.

This construction works for general cyclic operad, defining a functor

$$\mathcal{D} : \{\text{cyclic operads}\} \to \{\text{operads}\}.$$

Let P be a cyclic operad. Then define $\mathcal{D}(P)(m,n)$ to be the object $P(m+n-1)$. The $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule structure is inherited from the $\mathbb{S}^+_{n+m-1}$ structure on $P(m+n-1)$, using the same identifcations as above to view $\mathbb{S}_m$ and $\mathbb{S}_n$ as subgroups of $\mathbb{S}^+_{m+n-1}$. Since τ^i and $\circ_j$ maps are defined for any cyclic operad, the ${}_i\circ_j$ maps can be defined in the same way as above.

If we have a cyclic operad morphism $f : P \to Q$, there is a corresponding dioperad morphism $\mathcal{D}(P) \to \mathcal{D}(Q)$. The morphism is defined by sending

$\mathcal{D}(P)(m,n) = P(m+n-1)$ to $f(P(m+n-1)) \subset Q(m+n-1) = \mathcal{D}(Q)(m,n)$.

Proposition 3.1.5. *There exists an exact functor*

$$\{\textit{cyclic operads}\} \to \{\textit{dioperads}\}.$$

which sends the cyclic endomorphism operad to the endomorphism dioperad.

Proof. The functor is defined above. It is an exact functor because given an exact sequence

$$\cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots,$$

the terms in the induced sequence

$$\cdots \to \mathcal{D}(P_{i+1}) \to \mathcal{D}(P_i) \to \mathcal{D}(P_{i-1}) \to \cdots$$

are the same. The difference between P and $\mathcal{D}(P)$ lies in the actions and gluing compositions, but this is not used for exactness. So the induced sequence is exact as well.

For the claim about the cyclic endomorphism operad going to the endomorphism dioperad, note that $\mathcal{D}(End(V))(m,n)$ is the bimodule $Hom(V^{\otimes m+n-1}, V) \cong V^{\otimes n+m}$. Using the isomorphism between V and V^*, we can write this as $V^{*\otimes m} \otimes V^n \cong Hom(V^{\otimes m}, V^{\otimes n})$, so $\mathcal{D}(End(V))$ is at least a sub-dioperad of the endomorphism dioperad.

Again using Hom-Tensor adjunction and the isomorphism given by $\langle , \rangle$, an element in $Hom(V^{\otimes m}, V^{\otimes n})$ can be written as an element in $Hom(V^{\otimes m+n-1}, V)$. So $\mathcal{D}(End(V))(m,n)$ has all elements of the $(\mathbb{S}_m, \mathbb{S}_n)$ bimodule of the endomorphism dioperad $End(V)(m,n)$. □

Corollary 3.1.6. *Let V be a finite dimensional vector space. A cyclic C_∞ algebra structure on V defines a $Frob^0_\infty$ algebra structure on V.*

Proof. It is known that a cyclic commutative algebra is the same as a (unital and counital) Frobenius algebra, (see for instance [A]). One direction can be seen by proving that the cyclic commutative operad goes to the Frobenius dioperad.

Let Com be the commutative operad, where $Com(n)$ is the one dimensional space with trivial $\mathbb{S}_n$ action. This operad is generated by $Com(2)$ with the relation that all ways of repeatedly gluing $Com(2)$ with itself are equivalent. We see that $\mathcal{D}(Com)$ is generated by $\mathcal{D}(Com(2,1)$ and $\mathcal{D}(Comm)(1,2)$, with the relation that any way of obtaining an element in $\mathcal{D}(Com)(m,n)$ through gluing generators are equivalent. The Frobenius dioperad is described in the same way, so $\mathcal{D}(Com) = Frob^0$.

A C_∞ operad, denoted $(\Gamma(Com), \partial)$ is a cofibrant model for Com; that is, $\Gamma(Com)$ is a free operad, with a differential such that $(\Gamma(Com), \partial) \to Com$

is a quasi-isomorphism. Since $\mathcal{D}$ is an exact functor, when we apply $\mathcal{D}$ to $(\Gamma(Com), \partial) \to Com$, we obtain a quasi-isomophism $\mathcal{D}(\Gamma(Com), \partial) \to Frob^0$.

To show that $\mathcal{D}(\Gamma(Com))$ is a free dioperad, we use $\mathbb{D}\mathbb{D}(Com)$ as a model for $\Gamma(Com)$, where $\mathbb{D}$ is the cobar dual functor. We claim $\mathcal{D}(\mathbb{D}(Com))$ is isomorphic to $\mathbb{D}(\mathcal{D}(Com)) = \mathbb{D}(Frob^0)$. To see this, note that $\mathbb{D}(\mathcal{D})(i,j)$ consists of (i,j)-trees with vertices labeled by elements in Com and $\mathbb{D}(\mathcal{D}(Com))(i,j)$ consists of (i,j) trees with vertices labeled by elements in Com. Then we see that

$$\mathcal{D}\mathbb{D}\mathbb{D}(Com) = \mathbb{D}\mathbb{D}\mathcal{D}(Com) = \mathbb{D}\mathbb{D}(Frob^0)$$

which is a model for $Frob^0_\infty$ dioperad.

□

3.1.5 Properads

If we want our model of $Frob_\infty$ to include operations with genus, we need to consider properads. Merkulov and Vallette define a functor which allows us to do just that. Before describing the functor, some background on properads is in order.

Roughly speaking, a properad consists of the information of a dioperad along with composition rules that include genus. To describe such composi-

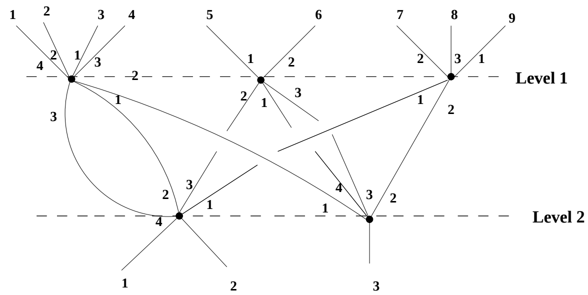

Figure 3.4: A two leveled graph.

tions, we make use of graphs (possibly with genus) as opposed to trees. We use the category of two leveled graphs.

Graphs will consist of a set of vertices V, edges E, and a global flow-direction on the edges signifying inputs and outputs. Let $In(v)$ and $Out(v)$ be the set of input edges and output edges at v. We assume that $In(v)$ and $Out(v)$ are labelled by $\{1, 2, \cdots, |In(v)|\}$ and $\{1, 2, \cdots, |Out(v)|\}$. The global inputs and outputs will also be labelled. A 2-*leveled graph* is a directed graph such that the vertices are divided into a top level and bottom level. Denote the set of vertices in the top level by N_1 and bottom level by N_2. We refer the reader to Figure 3.4. We let $\mathcal{G}$ be the category of graphs and $\mathcal{G}^2$ be the category of 2 leveled graphs.

The composition product $\boxtimes$ and connected composition product $\boxtimes_c$ is defined in the same way as it was for two-leveled trees, except we consider all two-leveled graphs. It is shown in [[V], Prop. 1.8] that $(S-\text{biMod}, \boxtimes_c, I)$ is a monoidal category. A *properad* is a monoid in the monoidal category $(S-\text{biMod}, \boxtimes_c, I)$. Alternatively, a properad P can be defined by giving an associative composition $\mu : P \boxtimes_c P \to P$ and a unit $\eta : I \to P$. A properad P is called *augmented* if there is a properad morphism $\epsilon : P \to I$. Let $\overline{P} = ker(\epsilon)$ and note that $P = \overline{P} \oplus I$. Let

$$(I \oplus \underbrace{\overline{P}}_{r}) \boxtimes_c (I \oplus \underbrace{\overline{P}}_{s})$$

denote the sub $\mathbb{S}$-bimodule of $P \boxtimes_c P$ generated by the compositions of s non-trivial elements on the top level with r non-trivial elements on the bottom level.

The endomorphism properad $End(V)$ is an example of a properad. Let $End(V)(m,n) = Hom(V^{\otimes n}, V^{\otimes m})$. The bimodule structure is given by permutation of the variables. The composition $End(V) \boxtimes End(V) \to End(V)$ is given by composition of linear maps.

There is a forgetful functor from Properads to $\mathbb{S}$-bimodules. The free properad functor $\mathcal{F} : \{\mathbb{S} - \text{bimod}\} \to \{\text{Properad}\}$ is the left adjoint to this functor. Theorem 2.3 in [V] gives an alternative description of the free properad.

Theorem 3.1.7 ([V], Thm 2.3)**.** *The free properad on an* $\mathbb{S}$*-bimodule* V *is* *given by the sum on connected graphs $\mathcal{G}$ with the vertices indexed by elements of V*

$$\mathcal{F}(V) = (\oplus_{g \in \mathcal{G}_c} \otimes_{v \in N} V(|Out(v), |In(v)|)) / \sim .$$

The composition μ comes from the composition of directed graphs.

The universal enveloping properad functor U : dioperads $\to$ properads is the left adjoint to the forgetful functor properads $\to$ dioperads. Let $F^0(V)$ be the free dioperad on V and $F(V)$ be the free properad on V.

Proposition 3.1.8 ([MV], Corollary 45)**.** *Let D be a dioperad defined by generators and relations $D = F^0(V)/(R)$, where (R) is the dioperad ideal generated by R. Then $U(D) = F(V)/(R)$, where (R) is the properadic ideal generated by R.*

The functor U is not exact, as is shown in [MV]. So even if a properad $P = F(V)/(R)$ is given by genus zero relations, it is generally not enough to resolve the dioperad $F^0(V)/(R)$ and apply U to the resolution to obtain a resolution for P as a properad. However, it is sometimes the case that this process provides a resolution. It is the case when P is a contractible properad.

Definition 3.1.9. A properad P is called **contractible** if it admits a model $(F(C), \partial^0) \tilde{\to} P$, where ∂^0 is the genus zero part of the free properad on C.

Proposition 3.1.10 ([MV], Proposition 48)**.** *Let $P = F(V)/(R)$ be a properad defined by genus zero relations, i.e. $R \subset F^0(V)$. The properad P is contractible if and only if the dioperad $D = F^0(V)/(R)$ admits a quasi-free dioperad resolution $(F^0(C), \partial^0) \tilde{\to} D$ such that the quasi-isomorphism is preserved by the universal enveloping properad functor U.*

Merkulov and Vallette also give a condition for determining when a properad is contractible. First a definition. Let P be a quadratic properad of the form $F(V,W)/(R \oplus D \oplus S)$, where $R \subset F^{(2)}(V)$ and $S \subset F^{(2)}(W)$, and

$$D \subset (I \oplus \underbrace{W}_{1}) \boxtimes_c (I \oplus \underbrace{V}_{1}) \bigoplus (I \oplus \underbrace{V}_{1}) \boxtimes_c (I \oplus \underbrace{W}_{1}).$$

Let $A := F(V)/(R)$ and $B := F(W)/(S)$. A morphism of $\mathbb{S}$-bimodules

$$\lambda : (I \oplus \underbrace{W}_{1}) \boxtimes_c (I \oplus \underbrace{V}_{1}) \to (I \oplus \underbrace{V}_{1}) \boxtimes_c (I \oplus \underbrace{W}_{1})$$

is called a *distributive law* if D is defined by the image of the map $(id, -\lambda)$ and the following morhpisms are injective

$$\underbrace{A}_{1} \boxtimes_c \underbrace{B}_{2} \to P \qquad (3.1.2)$$

$$\underbrace{A}_{2} \boxtimes_c \underbrace{B}_{1} \to P. \qquad (3.1.3)$$

Proposition 3.1.11 ([MV], Proposition 50)**.** *Let $D = F^0(V)/(R)$ be a binary Koszul dioperad defined by a distributive law such that V is finite* *dimensional. Then the associated properad $P := F(V)/(R)$ is Koszul and contractible.*

3.1.6 Frobenius properad

We use Propositions 3.1.10 and 3.1.11 to show that the Frobenius properad is contractible. Once this is done, we can apply the functor to the dioperad resolution of $Frob^0$ and obtain a properad resolution of $Frob$. The Frobenius properad is defined using the same generators and relations as the dioperad; the difference is that we now have graphs with genus. So the Frobenius properad is generated by an element in $m \in Frob(2,1)$ and $\Delta \in Frob(1,2)$, satisfying the same relation as the Frobenius dioperad.

Note that $Frob^0$ is defined by a distributive law. By Proposition 3.1.11, the Frobenius properad $Frob$ is contractible. We know that $BiLie^0$ and $Frob^0$ are Koszul dual dioperads, and so

$$\mathbb{D}(BiLie^0) \to Frob^0$$

is a resolution. By Proposition 3.1.10, the quasi-isomorphism is preserved when we apply the functor $U : \{\text{dioperads}\} \to \{\text{properads}\}$.

Lemma 3.1.12. *The Frobenius properad is contractible. So applying the* *composition of functors*

$$\{cyclic\ operads\} \xrightarrow{\mathcal{D}} \{dioperads\} \xrightarrow{U} \{properads\}$$

to a resolution of the cyclic commutative operad defines a properadic resolution of $Frob$*.*

Proof. Apply Propositions 3.1.10 and 3.1.11 to the case when $P = Frob$, $D = Frob^0$. The functor $\mathcal{D}$ is exact and U will preserve exactness since $Frob$ is contractible. □

3.1.7 Algebras over cyclic operads, dioperads, and properads

As mentioned earlier, operads, dioperads, and properads are used to model algebraic structures on vector spaces. If P is a cyclic operad (or dioperad or properad), then V is a P-algebra if there is a map of cyclic operads (or dioperads or properads)

$$P \to End(V).$$

Let P be a quadratic dg operad (or dioperad or properad) and $P^!$ be its quadratic dual. Then P is Koszul if $\mathbb{D}(P^!) \to P$ is a quasi-isomorphism. The interest in $\mathbb{D}(P^!)$ arises from the fact that P_∞ algebras are algebras over $\mathbb{D}(P^!)$. That is, a chain complex A_* is a P_∞ algebra if there exists a map of dg operads

$$\mathbb{D}(P^!) \to End(A_*).$$

Markl, in [M], shows that algebras over resolutions have properties that one would expect a homotopy algebra to satisfy. For example, if a P_∞ algebra A is chain equivalent to a chain complex B, then B is a P_∞ algebra and the chain maps in the equivalence extend to P_∞ algebra morphisms. Sullivan, in [S], discusses generalizations of this feature to include dioperads and properads.

We described two functors

1. $\mathcal{D}$: cyclic operads $\to$ dioperads,

2. U : dioperads $\to$ properads,

which will be used to construct $Frob^0_\infty$ and $Frob_\infty$ algebras. The first functor $\mathcal{D}$ is exact, so it sends resolutions to resolutions. Furthermore, if $\mathbb{D}(P^!) \to End^{cyc}(A_*)$ is a cyclic P_∞ algebra, then applying $\mathcal{D}$ to the cyclic operad map results in a map of dioperads $\mathcal{D}(\mathbb{D}(P^!)) \to End^0(A_*)$. Since $\mathcal{D}(\mathbb{D}(P^!))$ is a resolution for the dioperad $\mathcal{D}(P)$, the dioperad map defines a $\mathcal{D}(P)_\infty$ algebra structure on A_*.

The second functor is not exact in general, but in the cases we are interested in, it will be. We see that a dg dioperad map $\mathbb{D}(P^!) \to End^0(A_*)$

gets sent to a dg properad map $U(\mathbb{D}(P^!)) \to End(A_*)$. When U does preserve exactness, the dg properad map defines a $U(P)_\infty$ algebra structure on A_*. Lemma 3.1.12 shows that the Frobenius properad is contractible. So if we have a $Frob^0_\infty$ algebra structure $\mathbb{D}(BiLie^0) \to End^0(V)$, we can apply the functor to obtain a (properadic) $Frob_\infty$ algebra structure on V.

3.1.8 Partial algebras over cyclic operads, dioperads, and properads

We will be interested in situations when the algebraic operations on the vector space V have domains or ranges that are not $V^{\otimes n}$. These situations arise when looking at a coproduct on forms $\Delta_t : \Omega^*(M) \to \Omega^*(M \times M)$ or product on currents $\wedge_t : Curr_*(M \times M) \to Curr_*(M)$. We will call such objects partial algebra (coalgebra) structures, although there is a difference between these objects and the partial algebras found in [W].

Definition 3.1.13. A **domain** for a cochain complex (A, d) is a sequence of complexes $\{A_i\}$ such that

1. $A_1 = A$,

2. $A^{\otimes i}$ is a subcomplex of A_i,

3. there is an $\mathbb{S}_i$ action on A_i, such that for $i_1 + \cdots + i_k = i$, $A^{\otimes i_1} \otimes \cdots \otimes$

$A^{\otimes i_k}$ is an $\mathbb{S}_i$ invariant subspace of A_i,

4. $\mathbb{S}_i$ equivariant maps $A^{\otimes i} \hookrightarrow A_i$ and $A_i \to A^{\otimes i}$ whose compositions induce the identity map on cohomology,

Given a domain $\{A_i\}$ for (A, d), we can form partial endomorphism operad (or dioperad or properad). The definition of domain so far includes an $\mathbb{S}_n$ action on A_n. Let $\underline{End}(A)$ be the bimodule $\underline{End}(A)(m,n) = Hom(A_m, A_n)$. Given any $f : A_i \to A_j$, we have a map

$$\widehat{f} : A^{\otimes i} \hookrightarrow A_i \xrightarrow{f} A_j \to A^{\otimes j}.$$

So if $End(A)$ is an operad (or dioperad or properad), $\underline{End}(A)$ is an operad (or dioperad or properad) using the maps in (4) as above to define the composition maps. For example, for $f : A_{i_1} \to A_{j_1}$ and $g : A_{i_2} \to A_{j_2}$, let $f\,{}_i\!\circ_j g$ be defined by

$$A_{i_1+i_2-1} \to A^{\otimes i_1+i_2-1} \xrightarrow{\widehat{f}\,{}_i\circ_j \widehat{g}} A^{\otimes j_1+j_2-1} \to A_{j_1+j_2-1}.$$

We focus mainly on the partial endomorphism dioperad and properad. The partial endomorphism dioperad is denoted $\underline{End}(A)^0(m,n) = Hom(A_m, A_n)$, (where the superscript 0 indicates only genus zero operations are considered). If P is a dg dioperad, then A is a partial P-algebra if there is a map of dg dioperads, $P \to \underline{End}^0(A)$.

Remark 3.1.14. This definition of domain differs from the one found in [W]. In there, each A_i is a subcomplex of $A^{\otimes i}$, and the main example is the subcomplex of transversally intersecting chains in $C_*(M)$. The difference is mostly formal and the methods used in one situation translates in an obvious way to the other situation.

3.2 Transfer of structure and minimal models

3.2.1 Transferring A_∞ and C_∞ algebras by a contraction

Let A and B be cochain complexes over a field of characteristic zero. A *contraction* (f, g, H) consists of chain maps $f : A \to B$, $g : B \to A$, and a chain homotopy $H : A \to A[1]$ such that

1. $f \circ g = Id_B$,

2. $d_A H + H d_A = g \circ f - Id_A$.

Let A be a C_∞ algebra with maps $m_n^A : A[-1]^{\otimes n} \to A[-1]$. Then B can be given a C_∞ algebra structure such that g can be resolved to a C_∞ map. We describe the transfer map using trees. Let T be a planar tree with n inputs and one output. Associate to the input edges the map g and the output edge the map f. To each interior vertex of valence k associate the map m_k^A. To each interior edge associate the map H. Then T defines a map

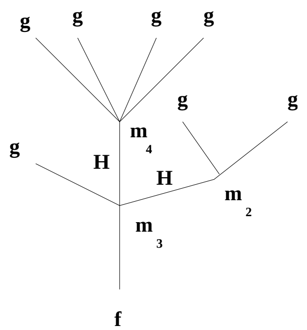

Figure 3.5: The tree above defines a map $m_n^T : B^{\otimes 7} \to B$ given by $f \circ m_3(Id \otimes H \otimes H) \circ (Id \otimes m_4 \otimes m_2) \circ g^{\otimes 7}$.

$m_n^T : B[-1]^{\otimes n} \to B[-1]$. We refer the reader to Figure 3.5 for an example of a map defined by a planar tree. Then m_n^B is defined as the sum

$$m_n^B = \sum_{\substack{\text{trees}\, T \\ \text{with n inputs}}} m_n^T.$$

We can define maps $G_n : B[-1]^{\otimes n} \to A[-1]$ in the same way as m_n^B, keeping all the same association of edges and vertices to maps, except associating the output edge with the homotopy H.

Theorem 3.2.1 ([CG], Theorem 12)**.** *Let (A, m_n^A) be a C_∞ algebra and (f, g, H) be a contraction between A and B. Then (B, m_n^B) is a C_∞ algebra and $\{G_n : B[-1]^{\otimes n} \to A[-1]\}$ is a C_∞ morphism.*

This method of transferring structure works for A_∞ and C_∞ coalgebras as well. The only difference is that trees with one input and n outputs are considered. Suppose $\{c_n : A[-1] \to A[-1]^{\otimes n}\}$ is an A_∞ coalgebra structure on A. Then $c_n^B : B[-1] \to B[-1]^{\otimes n}$ is defined as the sum of all maps c_n^T, where T is a tree with one input and n outputs.

Given a chain complex A with domain $\{A_i\}$ and a dg operad P, a partial P-algebra is then a map of dg operads $P \to \underline{End}(A)$. A partial A_∞ algebra structure on A with domain $\{A_n\}$ is then given by maps $m_n : A_n \to A$ satisfying the same compatibility relations.

We want to transfer partial A_∞ algebra structure by a contraction between A and B using the same methods described in this section. The definition of a domain allows us to do just that. To a corolla with i incoming edges and j outgoing edges, we associate a map $A_i \to A_j$. A general tree is obtained by grafting an outgoing edge of one corolla with an incoming edge of another corolla. We use the ${}_i\circ_j$ compositions, whose existence is assumed in the definition of complex to define the partial endomorphism operad $\underline{End}(A)$, to associate to an (n,m)-tree a map $A_n \to A_m$.

3.2.2 Transfer of partial $Frob^0_\infty$ algebra by a contraction

By definition of Koszul Duality, $\mathbb{D}(BiLie^0) \to Frob^0$ is a quasi-isomorphism. Then a vector space V is a $Frob^0_\infty$ algebra if there is a dioperad map

$\mathbb{D}(BiLie^0) \to End(V)$. The complex $\mathbb{D}(BiLie^0)$ is given by

$$\overline{Frob^0}(m,n)^* \xrightarrow{d''} \bigoplus_{\substack{|edge(T)|=1 \\ \text{reduced } (m,n) \text{ trees}}} \overline{Frob^0}^*(T) \otimes det(T) \xrightarrow{d''} \bigoplus_{\substack{|edge(T)|=2 \\ \text{reduced } (m,n) \text{ trees}}} \overline{Frob^0}^*(T) \otimes det(T) \xrightarrow{d''} \cdots .$$

The resulting trees have vertices labelled by elements in $Frob^0$. We can think of the left most term in this complex as (m,n) trees with no internal vertices. The differential is induced by contracting internal edges.

There are equivalent definitions C_∞ algebras, we look for an analogous situation with $Frob^0_\infty$. For our purposes, we look for analogous equivalence of the following two definitions of a C_∞ algebra structure on a vector space V:

1. a C_∞ algebra on V is a map of dg operads $\mathbb{D}(Lie) \to End(V)$

2. a C_∞ algebra on V is given by maps $\{m_n : V[-1]^{\otimes n} \to V[-1]\}$, which vanish on the image of the shuffle product $T(V[-1]) \otimes T(V[-1]) \to T(V[-1])$.

The second definition can be stated as saying a C_∞ algebra is an A_∞ algebra with the extra condition that the structure maps m_n vanish on the image of the shuffle product. It is closely related to the map of operads $Lie \hookrightarrow Ass \to Com$, and the fact that Lie and Com are Koszul.

We use the description of the Frobenius dioperad as $Com \boxdot Com^{op}$ to set up the analogous situation of C_∞ algebras in the dioperad setting. Denote the dioperad $Ass \boxdot Ass^{op}$ by $NC\,Frob^0$ since algebras over $Ass \boxdot Ass^{op}$ are a kind of non-commutative Frobenius algebra. A vector space V is an $NC\,Frob^0$ algebra if it has an associative product and a coassociative coproduct, Δ, such that

$$\Delta(ab) = a_{(1)} \otimes a_{(2)}b = ab_{(1)} \otimes b_{(2)}.$$

Denote the dioperad $Ass^{op} \boxdot Ass$ by ϵBi^0, since algebras over this dioperad are infinitesimal bialgebras. A vector space V is an infinitesimal bialgebra, or ϵ bialgebra for short, if it has an associative product and a coassociative coproduct such that

$$\Delta(ab) = a_{(1)} \otimes a_{(2)}b + (-1)^{|a|}ab_{(1)} \otimes b_{(2)}.$$

By Proposition 3.1.4, $NC\,Frob^0$ and ϵBi^0 are Koszul dual dioperads. A vector space V is an $NC\,Frob^0_\infty$ algebra if there is a dg dioperad map

$$\mathbb{D}(\epsilon Bi^0) \to End(V).$$

Such a map yields a collection of operations $\{m_{i,j} : V[-1]^{\otimes i} \to V[-1]^{\otimes j}\}$, such that for each i, j, the following equality holds,

$$[m_{i,j}, d] = \sum_{\substack{i'+i''=i-1 \\ j'+j''=j-1}} m_{i',j'}\ {}_a\circ_b\ m_{i'',j''}, \qquad (3.2.1)$$

where the sum is over all possible compositions ${}_a\circ_b$ in the endomorphism dioperad $End(V)$. When $j = 1$, the maps $\{m_{i,1}\}$ define an A_∞ algebra structure on V, and when $i = 1$, $\{m_{1,j}\}$ defines an A_∞ coalgebra. Now, suppose each $m_{i,j}$ vanish on the images of the shuffle product, and the composition of $m_{i,j}$ with the unshuffle product is zero. That is, the following compositions are zero:

$$T(V[-1]) \otimes T(V[-1]) \overset{\text{shuffle}}{\to} T(V[-1]) \overset{m_{i,j}}{\to} V^{\otimes j}, \quad (3.2.2)$$

$$V[-1]^{\otimes i} \overset{m_{i,j}}{\to} T(V[-1])T(V[-1]) \overset{\text{unshuffle}}{\to} T(V[-1]) \otimes T(V[-1]). \quad (3.2.3)$$

Then $\{m_{i,1}\}$ and $\{m_{1,j}\}$ define a C_∞ algebra and coalgebra structure. The other compatibilities define a commutative $Frob^0_\infty$ structure on V.

In Section 3.2.1, we showed how an A_∞ or C_∞ algebra can be transferred by a contraction. Then it was shown how to transfer a partial A_∞ algebra. In this section, we discuss how to transfer a $Frob^0_\infty$ algebra structure by a contraction.

Let (A_*, d) be a cochain complex, and suppose $\{m_{i,j} : A_*^{\otimes i} \to A_*^{\otimes j}\}$ is an $NC\,Frob^0_\infty$ algebra structure on A_*. Then given a contraction, with chain maps $f : A_* \to B_*$, $g : B_* \to A_*$, and chain homotopy $H : A_* \to B_*$, we can define maps $m^B_{i,j} : B_*^{\otimes i} \to B_*^{\otimes j}$ via trees. That is, given a tree T with i inputs and j outputs, associate to the input edges the map g, the output edges the map f, the interior edges the map H, and the vertices of valence (i', j') the map $m_{i',j'}$. Denote the resulting map by $m^T_{i,j} : B_*^{\otimes i} \to B_*^{\otimes j}$, and let $m^B_{i,j}$ be the sum of maps $m^T_{i,j}$ over all trees with i inputs and j outputs.

Lemma 3.2.2. *The maps $\{m^B_{i,j}\}$ define a NC $Frob^0_\infty$ algebra structure on B_*.*

Proof. We need to show that $dm^B_{i,j} + m^B_{i,j}d$ is equal to the sum of compositions over $m^B_{i',j}$ and $m^B_{i'',j''}$, where $i' + i'' = i - 1$ and $j' + j'' = j - 1$. The argument is similar to the one for transferring A_∞ algebras. Consider $m^T_{i,j}$ and take $[m_{i,j}, d]$. Using the fact that f and g are chain maps, and $(A, d, \{m_{i,j}\}$ is a $Frob^0_\infty$ algebra, we can move the differentials inside the graph, until it meets an interior edge. Then at the interior edge, we have $dH + Hd = g \circ f - Id_A$. The trees with Id_A cancel, because the trees we are summing over compositions of $m_{i',j'}$ and $m_{i'',j''}$, and we A_* is a $Frob^0_\infty$ algebra.

We are then left with trees with $g \circ f$. If we split the interior edge into two trees, with the map f associated to the upper tree and g associated to the lower tree, we see that the upper tree is exactly $m^{T'}_{i',j'}$ and the lower tree is $m^{T''}_{i'',j''}$. Since we are summing over all trees, in the sum we obtain $m^B_{i',j'}$ and $m^B_{i'',j''}$. This is what we wanted. □

Using the arguments in [CG], which showed that when $(A_*, \{m_n\})$ is a C_∞ algebra, then transferred structure on B_* is a C_∞ algebra, we have the corollary.

Corollary 3.2.3. *If* $(A_*, \{m_{i,j}\})$ *is a* $Frob_\infty^0$ *algebra, then* $(B_*, \{m_{i,j}^B\})$ *is a* $Frob_\infty^0$ *algebra.*

The argument in Lemma 3.2.2 applies to partial structures as well, provided that we can associate a tree to a map on the domain of the partial $Frob_\infty^0$ algebra. This will be the case when $A_* = \Omega^*(M)$ for $A_i = \Omega^*(M^{\times i})$ and $A_* = Curr_*(M)$ and $A_i = Curr_*(M^{\times i})$. We need to make use of kernels and the Dirac delta distributions to define the maps.

3.3 Topological conformal field theories

Let M be a compact Riemannian manifold. Then $\Omega^*(M)$ is a differential graded commutative algebra, so it is a C_∞ algebra. Let $Har^*(M)$ be the harmonic forms. We show there is a contraction (p, inc, H) where $inc : Har^*(M) \hookrightarrow \Omega^*(M)$ is the inclusion and $p : \Omega^*(M) \to Har^*(M)$ is the projection onto the harmonic forms using the Hodge decomposition.

We have the following operations

$$\begin{aligned} d : \Omega^*(M) &\to \Omega^{*+1}(M) \\ d^* : \Omega^*(M) &\to \Omega^{*-1}(M) \\ \Delta : \Omega^*(M) &\to \Omega^*(M), \end{aligned}$$

where $\Delta = [d, d^*]$. There is also an inner product $\langle \alpha, \beta \rangle = \int \alpha \wedge *\beta$. We can express $e^{-t\Delta}$ as an integral

$$e^{-t\Delta}\alpha(y) = \int K_t(x,y)\alpha(y),$$

where $K_t(x,y) \in \Omega^*(M \times M \times \mathbb{R}^+)$. The operator $e^{-t\Delta}$ is called the heat operator and $K_t(x,y)$ is called the heat kernel.

Definition 3.3.1. A **heat kernel** for Δ is an element

$$K \in C^\infty(\mathbb{R}^+) \otimes \Omega^*(M) \otimes \Omega^*(M)$$

such that the convolution operator $\widehat{K}_t : \Omega^*(M) \to \Omega^*(M)$ behaves like $e^{-t\Delta}$ where t is the coordinate on $\mathbb{R}^+$. This means that

$$\begin{aligned} \frac{d}{dt} K_t \alpha &= -\Delta K_t \alpha \\ \lim_{t \to 0} K_t \alpha &= \alpha. \end{aligned}$$

When $t = 0$, $e^{-t\Delta}$ is the identity and at $t = \infty$, $e^{-t\Delta}$ is the projection onto the harmonic forms. The chain map $\widehat{K}_t$ is chain homotopic to the identity. Let $L_t = d_x^* K_t \in \Omega^*(M \times M)$. Then the chain homotopy is an operator whose kernel is

$$\int_0^t d_x^* \widehat{K}_t.$$

We denote the chain homotopy by $\widehat{L}_t$.

Lemma 3.3.2. *Let A and B be operators given by kernels K and $K' \in \Omega^*(M \times M)$. Then,*

1. *A is a chain map if and only if K is a closed form,*
2. *A and B are chain homotopic if and only if K and K' are cohomologous.*

Proof. Suppose A is a chain map. Then

$$\begin{aligned} [A, d] &= 0 \\ d\int K(x,y)\alpha(y) - \int K(x,y)(d\alpha)(x) &= 0 \\ \int (dK(x,y)\alpha(y) + \int K(x,y)(d\alpha)(x) - \int K(x,y)(d\alpha)(x) &= 0 \\ \int (dK(x,y))\alpha(y) = 0, \end{aligned}$$

so $dK(x,y) = 0$. We can reverse this to get the converse.

To prove the second part, it is enough to consider the case when A is null-homotopic. Suppose $K(x,y)$ is exact. Then there is a form J, such that

$(d_x + d_y)J = K$. Let J be the kernel of some operator C. We need to show that $[d, C] = A$.

Since X is closed, Stokes's theorem implies $\int_{y\in X} d_y(J(x,y)\alpha)(y) = 0$. This is rewritten as

$$\int (d_y J(x,y))\alpha(y) = \int J(x,y) d_y \alpha(y).$$

Since $(d_y + d_x)J = K$, we see that

$$\begin{aligned}
\int d_y J(x,y)\alpha(y) + \int d_x J(x,y)\alpha(y) &= \int K(x,y)\alpha(y) \\
\int J(x,y) d_y \alpha(y) + \int d_x J(x,y)\alpha(y) &= A\alpha \\
C(d\alpha) + dC\alpha &= A\alpha,
\end{aligned}$$

which is what we wanted. Again, the converse statement is proved by reversing the order of equalities.

□

The natural setting for this work on minimal models is on currents. Let $Curr_*(M)$ be the linear dual of $\Omega^*(M)$. We will define a one parameter family of products and coproducts in which $\Omega^*(M)$ forms a sub-object.

Given $\alpha \in \Omega^*(M)$, let $[\alpha]$ be the current $[\alpha]\beta = \int_M \alpha \wedge \beta$. This defines an inclusion $\Omega^*(M) \hookrightarrow Curr_*(M)$. We say $[\alpha]$ is a smooth current.

Kernels can be used to define maps on currents. Let $K^i(x_1, \cdots, x_i)$ be an element in $\Omega^*(M^{\times i})$. Given a current C, let

$$\widehat{K}^{i,j}(C) = [C(x_j)K^i(x_1, \cdots, x_i)].$$

So $\widehat{K}^{i,j}(C)$ is a smooth current in $Curr_*(M^{\times i-1})$. The representing form is obtained by applying C to the j^{th} factor of $K^i \in \Omega^*(M^{\times i})$. Note that if $C = [\alpha]$, then $\widehat{K}^i[\alpha] = [\widehat{K}^i(\alpha)]$, so that the definition of $\widehat{K}^i$ on smooth currents agrees with the definition of $\widehat{K}^i$ on forms.

As they did on $\Omega^*(M)$, K_t and L_t define maps on $Curr_*(M)$. We keep the same notation, $\widehat{K}_t, \widehat{L}_t : Curr_*(M) \to Curr_*(M)$, and $d\widehat{L}_t + \widehat{L}_t d = \widehat{K}_t - Id_{Curr}$.

3.3.1 Cellular differential forms

We now define a form on the moduli space of metrised ribbon graphs. Let $\Gamma(m, n)$ be the set of graphs with m incoming vertices and n outgoing vertices, and let $\gamma \in \Gamma(m, n)$. We will use T to denote a tree in $\Gamma(m, n)$. An edge of γ is external if one of its vertices are external and is internal otherwise. To each edge e of γ, we assign a length $l(e) \in \mathbb{R}^+$. We impose the condition that a path which starts and ends at different outgoing external vertices has positive length. Denote the possible metrics on γ by $Met(\gamma)$. Note that $Met(\gamma)$ is $(\mathbb{R}^+)^{|e(\gamma)|}$ where $|e(\gamma)|$ is the number of edges of γ.

For each internal edge e of γ there is a map

$$Met(\gamma/e) \hookrightarrow Met(\gamma),$$

whose image is the set of all metrics on γ that take the value zero on e.

Definition 3.3.3. A **cellular differential form** $\omega \in \Omega^i_{cell}(\Gamma(m,n))$ assigns to each graph $\gamma \in \Gamma(m,n)$ a form $\omega_\gamma \in \Omega^i(Met(\gamma), \mathbb{C})$ such that

1. ω_γ is $Aut(\gamma)$ equivariant

2. for each internal edge e,

$$\omega_{\gamma/e} = \omega_\gamma|_{Met(\gamma/e)}.$$

There is an integration pairing

$$\int : C_i(\Gamma(m,n)) \otimes \Omega^i_{cell}(\Gamma(m,n)) \to \mathbb{C}. \tag{3.3.1}$$

We view $\Gamma(m,n)$ as a topological symmetric monoidal category as follows. The obejcts are the non-negative integers. The morphisms come from the continuous gluing maps

$$\Gamma(l,m) \times \Gamma(m,n) \to \Gamma(l,n).$$

And the monoidal structure comes from disjoint union,

$$\Gamma(m,n) \times \Gamma(s,t) \to \Gamma(m+s, n+t).$$

We wish to consider chains on $\Gamma(m,n)$ with coefficients in a local system, det. For $\gamma \in \Gamma(m,n)$, let $det(\gamma)$ be the relative cohomology group of γ,

$$det(\gamma) = \Pi^{n-\chi(\gamma)} det(H^*(\gamma, [n])).$$

With this background, we can define a form. Consider $\alpha = \alpha_1 \times \cdots \times \alpha_m \in \Omega^*(M^{\times m})$. For each edge e, let $\omega_e \in \Omega^*(Met(\gamma)) \otimes \Omega^*(M \times M)$ be

$$\omega_e = K_{l(e)} + dl(e)L_{l(e)}$$

if e is not an incoming external edge, and

$$(-1)^{p(\Omega^*(M)|\alpha|}(K_{l(e)} + dl(e)L_{l(e)})\alpha_i$$

if e is an incoming external edge with external vertex i.

To each edge e there is the two element set of half-edges $H(e)$. Furthermore, since the edges are directed, there is an isomorphism from $H(e)$ to $\{0,1\}$. This allows us to view ω_e as an element in $\Omega^*(Met(\gamma)) \otimes \Omega^*(M^{\times H(e)})$. Let $H(\gamma)$ be the set of half edges of γ. We denote the tensor product of all ω_e to get

$$\widetilde{K}_\gamma(\alpha) = \otimes_{e \in E(\gamma)} \omega_e \in \Omega^*(Met(\gamma)) \otimes \Omega^*(M^{\times H(\gamma)}).$$

For each vertex $v \in \gamma$, let $H_v(\gamma)$ be the set of half edges at γ. Since there is a cyclic order of the half edges, we can multiply the forms at the half-edges and integrate. This defines a map

$$Tr_v : \Omega^*(M^{\times H_v(\gamma)}) \to \Omega^*(M) \stackrel{\int_M}{\to} \mathbb{C}.$$

Applying Tr_v at each vertex $v \in V(\gamma) \setminus \{\text{outgoing vertex}\}$, then, defines an element

$$K_\gamma(\alpha) = \otimes_{v \in V(\gamma)\setminus\{v_{out}\}} Tr_v \widetilde{K}_\gamma(\alpha).$$

Lemma 3.3.4 ([Cos], Lemma 4.5.1)**.** *$K_\gamma(\alpha)$ is non-singular. There is a map*

$$K_\gamma : \Omega^*(M)^{\otimes n} \to \Omega^*(Met(\gamma)) \otimes \Omega^*(M)^{\otimes m}$$

that commutes with the differentials.

Lemma 3.3.5 ([Cos], Lemma 4.5.2)**.** *For all $\alpha \in \Omega^*(M^{\times m})$,*

$$K_\gamma(\alpha)|_{Met(\gamma/e)} = K_{\gamma/e}(\alpha).$$

Theorem 3.3.6 ([Cos], Theorem 4.5.4)**.** *There is a symmetric monoidal functor $C_*(\Gamma, det^{p(\Omega^*(M))}) \to Comp_{\mathbb{C}}$ sending*

$$\begin{aligned} n &\mapsto \Omega^*(M^{\times n}) \\ c &\mapsto \int_c K(\alpha), \end{aligned}$$

where c is a chain in $C_*(\Gamma(m,n), det^{p(\Omega^*(M))})$ and $\int_c$ is the integration pairing between $C_*(\Gamma(m,n))$ and $\Omega^*_{cell}(\Gamma(m,n))$.

3.3.2 Subspaces of the moduli space

We consider certain subpsaces of the moduli space of metrised graphs and the cellular differential forms K_γ of the graphs in this space. Associating the graphs with maps on $\Omega^*(M)$ and integrating over the subspaces will define the Frobenius algebra we need.

For example, consider the subspace of trees, i.e. genus zero graphs, with n inputs and one output. Furthermore, consider the metrics which take fixed values on the inputs and outputs, but varying between $[0, t]$ on the interior edges. For $n = 3$, there are three graphs and the moduli space of metrics is a one parameter family. The form K_T, where $T \in \Gamma(3, 1)$ in our subspace, is exact, since it is a one form on $\mathbb{R}$. The moduli space $\Gamma(3, 1)$ is described as the interval $[-t, t]$, where $s \in [-t, t]$ represents the metric which takes the value s on the interior edge of the binary tree, and where 0 represents the three corolla. See Figure 3.6 for a description. For $n = 4$, the moduli space is a pentagon. Continuing in this way, we obtain cells of the Stasheff associahedron.

We obtain a C_∞ algebra structure on $\Omega^*(M)$ by taking the cellular differential form K_γ and integrating over the above cells in the moduli space.

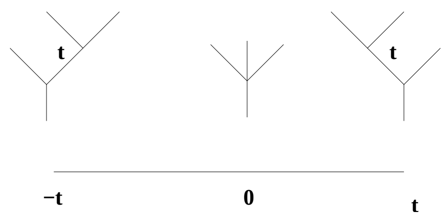

Figure 3.6: The subspace of the moduli space $\Gamma(3,1)$, where the metrics take the values in $[0,t]$ on the internal edge, can be identified with the interval $[-t,t]$. The metrised $(3,1)$ trees associated with the points $-t$, 0, and t in $[-t,t]$ are drawn above.

Integrating over these cells define maps m_n^t, for each n, given by

$$\begin{aligned} m_n^t : \Omega^*(M)^{\otimes n} &\rightarrow \Omega^*(M) \\ \alpha = \alpha_1 \otimes \cdots \otimes \alpha_n &\mapsto \int K_\gamma(\alpha). \end{aligned}$$

The map $m_2^t(\alpha,\beta) = \widehat{K}_t\alpha \wedge \widehat{K}_t\beta$ is strictly commutative but associative only up to homotopy. The homotopy is given by

$$m_3^t(\alpha,\beta,\gamma) = m_2^t(\widehat{L}_t m_2^t(\alpha,\beta),\gamma) + m_2^t(\alpha,\widehat{L}_t m_2^t(\beta,\gamma)).$$

The definitions of the higher maps proceed in much the same many as transfer of structure described in Section 3.2.1, where the contraction is given by $f = \widehat{K}_t$, $g = Id$, and chain homotopy $\widehat{L}_t$.

The form K_γ is exact, and so we can use Stokes's theorem to describe the boundary of m_n^t. The boundary consists of integrating over cells of one lower dimension, which can be described as compositions of m_i^t and m_j^t, where $i+j-1=n$. These are precisely the relations for an A_∞ algebra.

3.4 Constructing a partial $Frob_\infty^0$ algebra on $Curr_*(M)$

3.4.1 Double forms $\Omega^*(M \times M)$

We will briefly review some topics on $\Omega^*(M \times M)$. The reader can find the material in de Rahm's book [DR]. A differential form $\gamma \in \Omega^*(M \times M)$ can be viewed as a differential form in $\Omega^*(M)$ with values in $\Omega^*(M)$. That is, given local coordinates $(x_1, \cdots, x_n)$ for $U \subset M$ and $(y_1, \cdots, y_n)$ for $U' \in M$, γ can be expressed as the sum

$$\sum \beta_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where $\beta_{i_1 \cdots I_p}$ is a form that can be expressed as the sum

$$\sum c_{j_1 \cdots j_q} dy^{j_1} \wedge \cdots \wedge dy^{j_1},$$

where $c_{j_1 \cdots j_q}$ are real numbers. Then γ is a $p+q$ form, and can be expressed by

$$\gamma(x,y) = \sum c_{i_1 \cdots i_p j_1 \cdots j_q} (dx^{i_1} \wedge \cdots \wedge dx^{i_p})(dy^{j_1} \wedge \cdots \wedge dj^{j_q}).$$

The wedge product $\gamma(x,y) \wedge \gamma'(x,y)$ is defined in the obvious way, multiplying the dx terms and dy terms. We can also define the wedge product

$\gamma(x,y) \wedge \alpha(x)$, where $\alpha(x) \in \Omega^*(M)$, where we use ony the multiplication in the dx coordinates. Given $\alpha(x)$ and $\beta(y)$, we have the double form $\alpha(x)\beta(y) \in \Omega^*(M \times M)$, which we refer to as the cartesian product of the forms.

In $\Omega^*(M \times M)$, we can take the differential in a single coordinate. We denote it d_x or d_y, and we have the relations

$$d_x^2 = d_y^2 = 0$$

$$d_x d_y = d_y d_x.$$

Similarly, given a chain in one of the coordinates, we can integrate the form in that coordinate, $\int_{x \in c} \alpha(x,y)$, which defines a form in $\Omega^*(M)$.

3.4.2 Domain for $Curr_*(M)$

To define a partial structure on $Curr_*(M)$, we need to specify a domain first (defined in Section 3.2.1). Consider $\{Curr_*(M^{\times i})\}$. Note that $Curr_*(M)^{\otimes i} \hookrightarrow Curr_*(M^{\times i})$ is a quasi-isomorphism, since over field coefficients, $H_*(M)^{\otimes i} \cong H_*(M^{\times i})$. There is also an $\mathbb{S}_i$ module action, induced by permuting the factors $M^{\times i}$.

We need to define compositions of maps $Curr_*(M^{\times i}) \to Curr_*(M^{\times j})$, for various $i, j > 0$. This is achieved using kernels $\Omega^*(M^{\times n})$ and the Dirac delta distribution, denoted $\widehat{K}_0$. A distribution is a 0-current, i.e. the linear dual of smooth functions. It defines a map on forms by taking a local representation for a form and acting the distribution on the coordinate function. Currents are distribution-valued forms. Since distributions, in general, cannot be multiplied, they do not define maps on currents. We must be careful when using the Dirac delta distribution to define maps and make sure it is only applied to smooth distributions.

Given maps $m_{i_1,1} : Curr_*(M^{\times i_1}) \to Curr_*(M)$ and $m_{i_2,1} : Curr_*(M^{\times i_2}) \to Curr_*(M)$, we define a map $Curr_*(M^{i_1+i_2-1}) \to Curr_*(M)$ as follows. Apply the map m_{i_1} on the factors of $Curr_*(M^{\times i_1+i_2-1})$ as given in the composition and apply $\widehat{K}_t$ to the remaining factors. This will give us an element in $Curr_*(M^{\times i_2})$ to which we apply m_{i_2}. Note that the condition $t > 0$ is necessary, since K_0 is a distribution and cannot be applied to an arbitrary current. Since $\widehat{K}_t$ is chain homotopic to the identity, this change is manageable for our work on minimal models.

Similarly, we can define compositions of maps m_{1,j_1}, m_{1,j_2}. In our case, the maps $m_{1,i}$ will land in smooth currents, so we can apply the Dirac delta distribution to the factors.

To define the $\underline{Frob}^0_\infty$ algebra, we will have maps $m_{i,j} : Curr_*(M^{\times i}) \to$

$Curr_*(M^{\times j})$. These maps will be compositions of maps of the form $m_{i,1}$, $m_{1,j}$ and $\widehat{L}_t$. The definition of composition extends accordingly.

3.4.3 Partial $Frob_\infty^0$ algebra structure on $Curr_*(M)$

To define a $\underline{Frob}_\infty^0$ algebra structure on $Curr_*(M)$, we need to define a dg dioperad map

$$\mathbb{D}(BiLie^0) \to End^0(Curr_*(M)).$$

We first define the map on $BiLie^0 \subset \mathbb{D}(BiLie^0)$. Since $\mathbb{D}(BiLie^0)$ is a free dioperad, the map on $BiLie^0$ extends to $\mathbb{D}(BiLie^0)$. We then show that the map respects the differentials, which finishes the definition of the $\underline{Frob}_\infty^0$ structure.

To define a map $m_{i,j}^t : Curr_*(M^{\times i}) \to Curr_*(M^{\times j})$, consider all binary trees with i inputs and j outputs, with the equivalence relation as in the free dioperad construction. To each such tree there is a map

$$m_{i,j}^T : Curr_*(M)^{\otimes i} \to Curr_*(M^{\times j}),$$

obtained by identifying the input and output edges with $\widehat{K}_t$, the vertices with $\wedge$ or Δ_t as determined by the valence, the internal edges with $\widehat{L}_t$.

Before showing that the maps $m_{i,j}^t$ defines a $\underline{Frob}_\infty^0$ algebra, we consider the cases when i or j is equal to one. When $j = 1$, we have a partial algebra structure. The multiplication $m^t_{2,1} : Curr_*(M \times M) \to Curr_*(M)$ is obtained as follows. Given a double current $A \times B \in Curr_*(M \times M)$, the heat kernel $\widehat{K}_t \times \widehat{K}_t$ is applied to $A \times B$ to obtained a smooth double current $[\alpha \times \beta]$. Then the representing forms are wedged, so that $m^t_{2,1}(A \times B) = [\alpha \wedge \beta]$. Note this product has degree $-d$, where d is the dimension of M. The higher maps $m^t_{i,1}$ are obtained by a process similar to the one found in Section 3.2.1, and as such defines a partial C_∞ algebra. Note that $m^t_{i,1}$ has degree $-(i-1)d+(i-2)$, since $i-1$ is the number of times the degree $-d$ operation is applied, and $i-2$ is the number of times $\widehat{L}_t$ is applied (internal edges).

We can continue the reasoning for $m^t_{1,j} : Curr_*(M) \to Curr_*(M^{\times j})$. The map $m^t_{1,2} : Curr_*(M) \to Curr_*(M \times M)$ is obtained by applying the $\Delta_* : Curr_*(M) \to Curr_*(M \times M)$ and then flowing the outputs for time t. The map is a degree zero. The higher maps $m^t_{1,j}$ is defined by compositions of $m^t_{1,2}$ and chain homotopy $\widehat{L}_t$. We then obtain a partial C_∞ coalgebra structure on $Curr_*(M)$. Note that $m^t_{1,j}$ has degree $j-2$, since $j-2$ is the number of interior edges, which counts the number of times $\widehat{L}_t$ is applied.

In general, $m^t_{i,j}$ is obtained by compositions of $m^t_{2,1}$, $m^t_{1,2}$, and $\widehat{L}_t$. This means that $m^t_{i,j}$ has degree $-(i-2)d + (i+j-3)$.

Theorem 3.4.1. *The maps $\{m^t_{i,j}\}$ define a $\underline{Frob}^0_\infty$ algebra structure on*

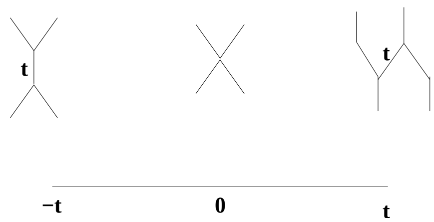

Figure 3.7: The cell in $\Gamma(2,2)$ obtained by considering metrics which take values in $[0,t]$ on the internal edge is identified with the interval.

$Curr_*(M)$.

Proof. To show that the necessary relations are satisfied, we view the maps as obtained by integrating over the moduli space. The map $m^t_{2,2}$ is obtained by considering the binary trees with two inputs and two outputs. There are two such trees, after identifying trees by the equivalence relation in the cobar complex. The two trees have one internal edge, and when the metric takes the value zero, the two trees are identified in the moduli space $\Gamma(2,2)$. The cell is identified with $[-t,t]$ as shown in Figure 3.7. Then $b_{2,2} = \int_{[-t,t]} K_\gamma$. The boundary of this map is is given by $(m_{2,1} \otimes 1) \circ (1 \otimes m_{1,2}) - m_{1,2} \circ m_{2,1}$ as desired.

Because the maps $m^t_{i,j}$ are defined by compositions of $m^t_{2,1}$, $m^t_{1,2}$, and $\widehat{L}_t$ and the relations for a $Frob_\infty$ algebra (3.2.1) require the boundary of $m^t_{i,j}$ to be compositions of lower operations, we can apply an inductive arguement to show the rest of relations are satisfied. □

We obtain a minimal model for $(Curr_*(M), \{m^t_{i,j}\})$ by considering the contraction $\widehat{K}_\infty : Curr_*(M) \to H_*(M)$, $inc : H_*(M) \to Curr_*(M)$, and $\widehat{L}_\infty$, where the inclusion is given by choosing the harmonic form α, such that the smooth current $[\alpha]$ represents the homology class. Then we can transfer the structure over to $H_*(M)$. This process defines a $Frob_\infty$ algebra structure on $H_*(M)$, since $H_*(M^{\times i}) \cong H_*(M)^{\otimes i}$.

For different $s, t > 0$, we have different $\underline{Frob}_\infty$ algebras on $Cur_*(M)$. But they induce the same $Frob_\infty$ algebra on $H_*(M)$. To see this, it suffices to show it for $m^t_{i,1}$ and $m^t_{1,j}$, since the rest of the maps are described in terms of these maps.

Lemma 3.4.2. *Let $s, t > 0$. Suppose $\{m^t_{i,1}\}$ and $\{m^s_{i,1}\}$ are the C_∞ algebra structures on $H_*(M)$ for the minimal model for $(Curr_*(M))$. The maps $m^t_{i,1}$ and $m^s_{i,1}$ are equal for each i. Similarly, the maps C_∞ coalgebra structure on $H_*(M)$ given by $\{m^t_{1,j}\}$ and $\{m^s_{1,j}\}$ are equal.*

Proof. The definition of $m^t_{i,1}$ and $m^s_{i,1}$ uses the inclusion $H_*(M) \hookrightarrow Curr_*(M)$, so we will be looking at smooth currents $[\alpha]$ where α is a harmonic form. The proof uses the following facts:

1. for $\alpha \in Har^*(M)$, $\widehat{K}_t(\alpha) = \alpha$, since $\alpha \in Ker(\Delta)$ and $\widehat{K}_t = e^{-t\Delta}$,

2. for $\alpha \in Har^*(M)$, $\widehat{L}_t(\alpha) = 0$, since

$$\widehat{L}_t(\alpha) = \int_0^t d^* \widehat{K}_u(x,y)(\alpha) du = \int_0^t d^*(\alpha) du = 0,$$

3. for an exact form $d\alpha$, $\widehat{L}_t(d\alpha)$ is exact, since

$$\widehat{L}_t(d\alpha) = \int_0^t d_x^* K_u(x,y) d_y(\alpha)(y) du = d\left(\int_o^t d^* K_t(x,y)\alpha(y)\right).$$

To show that $m^s_{i,1} : H_*(M)^{\otimes i} \to H_*(M)$ and $m^t_{i,1} : H_*(M)^{\otimes i} \to H_*(M)$ are equal, it suffices to show that the maps associated to the i-corollas are zero for $n > 2$.

Recall that $m^t_{i,1} : Curr_*(M^{\times n}) \to Curr_*(M)$ is defined by taking the sum over all binary trees with i inputs and one output. We start with $\alpha_1, \cdots, \alpha_i \in Har^*(M)$ and apply $\wedge_t$ and $\widehat{L}_t$ in the order prescribed by the binary tree. The first fact implies $\alpha_i \wedge_t \alpha_j = \alpha_i \wedge \alpha_j$. The wedge product is not harmonic, but since it is closed, it can be written as the sum of a harmonic form and exact form. Then the second and third facts imply $\widehat{L}_t(\alpha_i \wedge \alpha_j)$ is exact. So when we do perform the operations associated with the binary tree, we end up with an exact form. In cohomology, this map from $H_*(M)^{\otimes n} \to H_*(M)$ is zero, as desired.

The same reasoning applies to the coalgebra maps. □

3.4.4 Sub partial $Frob_\infty^0$ on $\Omega^*(M)$

The maps $m^t_{i,j} : Curr_*(M^{\times i}) \to Cur_*(M^{\times j})$ send smooth currents to smooth currents. This means that the smooth currents, $\Omega^*(M) \hookrightarrow Curr_*(M)$, are a sub-$\underline{Frob}_\infty^0$ algebra of $Curr_*(M)$. We wish to show a stronger statement. That when we dualize $\Omega^*(M)$ and its $\underline{Frob}_\infty$ algebra, we obtain the $\underline{Frob}_\infty$ algebra structure on $Curr_*(M)$.

This statement almost follows by construction. Note that $m^t_{2,1} : \Omega^*(M \times M) \to \Omega^*(M)$ is given by the composition

$$\Omega^*(M \times M) \stackrel{\widehat{K}_t \times \widehat{K}_t}{\to} \Omega^*(M \times M) \stackrel{Diag^*}{\to} \Omega^*(M).$$

Restricting to the subset $\Omega^*(M) \otimes \Omega^*(M)$, we see that $m^t_{2,1}(\alpha, \beta) = \widehat{K}_t(\alpha) \wedge \widehat{K}_t(\beta)$. When we take the linear dual of $m^t_{2,1}$, we obtain

$$Curr_*(M) \stackrel{Diag_*}{\to} Curr_*(M \times M) \stackrel{\widehat{K}_t \times \widehat{K}_t}{\to} Curr_*(M \times M),$$

which is exactly the definition of $m^t_{1,2} : Curr_*(M) \to Curr_*(M \times M)$. The main work was in defining the partial structures appropriately. It did not seem necessary to define a partial algebra structure on $\Omega^*(M)$, since it is possible to define an honest algebra structure. But this would not allow for an easy comparison to $Curr_*(M)$.

Proposition 3.4.3. *Let* $(Curr_*(M), \{m^t_{i,j}\})$ *and* $(\Omega^*(M), \{m^{t,\Omega}_{i,j}\})$ *be the* $\underline{Frob}^0_\infty$ *algebra structures defined above. Then the linear dual* $m^{t,\Omega*}_{i,j} : Curr_*(M^{\times j}) \to Curr_*(M^{\times i})$ *is equal to* $m^t_{j,i}$.

Proof. The proof is immediate from the constructions of the structure maps for the $\underline{Frob}^0_\infty$ algebra structures on $\Omega^*(M)$ and $Curr_*(M)$. Note that $m^{t,\Omega}_{i,j}$ is the restriction of $m^t_{i,j} : Curr_*(M^{\times i}) \to Curr_*(M^{\times j})$ to $\Omega^*(M^{\times i})$, viewing an element in $\Omega^*(M^{\times i})$ as a smooth current. But currents are covariant and differential forms are contravariant. So if we have a map $f_{i,j} : M^{\times i} \to M^{\times j}$, on forms we get an induced map $\Omega(M^{\times i}) \to \Omega^*(M^{\times j})$ and thinking of forms as smooth currents, we get the linear dual $\Omega^*(M^{\times j}) \to \Omega^*(M^{\times i})$. $\square$

3.4.5 Comparison with another $Frob_\infty$ algebra

In [W2], Wilson defines a minimal model on the Frobenius algebra of differential forms. For a resolution of $Frob$, he uses the cobar-bar construction, denoted $\Omega B(Frob)$. The set-up is different from ours, and so a few definitions are needed before we describe the construction.

A *coproperad* C is a comonoid in $(\mathbb{S} - bimod, \boxtimes_c, I)$. This data consists of two morphisms,

1. a morphism $\Delta : C \to C \boxtimes_c C$

2. a counit morphism $\eta : C \to I$.

In algebra, the linear maps from a coalgebra to an algebra is an algebra. Similarly, for a coproperad C and a properad P, the morphisms $Hom(C, P)$ is a properad. The properad multiplication is a map

$$Hom(C, P) \boxtimes_c Hom(C, P) \to Hom(C, P).$$

Suppose we have a two levelled graph, where the vertices in level one are labelled by maps $f_i : C \to P$ and the vertices in level two are labelled by maps $g_j : C \to P$. The product is then the composition

$$\mu : C \stackrel{\Delta}{\to} C \boxtimes_c C \stackrel{(f_1, \cdots, f_r; g_1 \cdots g_s)}{\to} P \boxtimes P_c \to P,$$

where the middle map is obtained by applying f_i to the i^{th} vertex in level one and g_j to the j^{th} vertex in level two. If C and P have differentials, $Hom(C, P)$ has a differential defined by pre and post-composing with the differentials,

$$D(f) = d_p \circ f - (-1)^{|f|} f \circ d_C.$$

A twisting morphism in $Hom^1(C, P)$ is a morphism $\tau : C \to P$ satisfiying the Maurer Cartan equation

$$D(\tau) + \mu(\tau, \tau) = 0.$$

Denote the set of twisting morphisms by $TW(C, P)$. The bifunctor $TW(-,-)$ can be represented on the left and the right using cobar and bar constructions, as we describe in Proposition 3.4.4.

The bar construction is a functor

$$B : \{\text{aug. dg properads}\} \to \{\text{coag. dg coproperads}\}.$$

For a properad P, $B(P)$ is the cofree coproperad $F^c(P)$. As an $\mathbb{S}$ bimodule, $F^c(P)$ is the same as the free properad $F(P)$. There are two terms for the differential. The bar differential, denoted d_B is the unique co-derivation that extends the partial product of P. The differential on P induces a differential on $B(P)$, which we denote by d_p.. The total differential is then the sum $d_B + d_P$.

The cobar construction is a functor

$$\Omega : \{\text{coag. dg coproperads}\} \to \{\text{aug. dg properads}\}.$$

For a coproperad C, $\Omega(C)$ is the free properad $F(C)$. Again, there are two terms for the differential. The cobar differential, denoted d_Ω, is the unique derivation that extens the partial coproduct. The total differential is then the sum $d_\Omega + d_P$. The following propisition relates twisting morphisms to the cobar and bar constructions.

Proposition 3.4.4 ([MV], Prop. 17)**.** *For every augmented dg properad and every coaugmented dg coproperad, there is a canonical one to one correspondence*

$$Mor_{dg\ properad}(\Omega(C), P) \cong TW(C, P) \cong Mor_{dg\ coproperad}(C, B(P)).$$

Using the bar construction, we can define the Koszul dual coproperad of a properad P. The coproperad $B(P)$ has a dual grading given by the number of vertices of the graph and the total grading of an element in $B(P)$. Let $B_{(s)}(P)^{(p)}$ denote the elements of degree p in $B(P)$ represented by graphs with s vertices labeled by elements in P. Then $d_B(B_{(s)}(P)^{(p)}) \subset B_{(p-1)}(P)^{(p)}$. The Koszul dual coproperad, denoted $P^{¡}$, is defined as

$$P^{¡}_{(p)} := H_{(p)}(B_*(P)^{(p)}, d_B.$$

The properad P is a *Koszul properad* if the natural inclusion $P^{¡} \hookrightarrow B(P)$ is a quasi-isomorphism. We can apply the cobar functor to the morphism, to obtain a map $\Omega(P^{¡}) \to \Omega B(P)$. There is a canonical quasi-isomorphism $\Omega B(P) \to P$, and so if P is a Koszul operad, there is a quasi-isomorphism $\Omega(P^{¡}) \to P$, which resembles the previous definition of Koszul duality. In fact, $P^{¡} \hookrightarrow B(P)$ is a quasi-isomorphism if and only if $\Omega(P^{¡}) \to P$ is a quasi-isomorphism, ([MV], Thm. 7.6).

The construction in [W2] proceeds as follows. First, it is shown that there is an inductive sequence of obstructions to defining a morphism of properads $\Omega B(P) \to Q$. Using Proposition 3.4.4, we see that the obstructions are given by defining a twisting morphism $B_{(s)}(P) \to Q$, and inducting over s.

Theorem 3.4.5 ([W2], Thm. 3)**.** *Let P be a dg properad with zero differential. Let Q be any proeprad. There is an inductive sequence of obstruction to defining a morphism of properads*

$$\phi : \Omega B(P) \to Q.$$

The next step to defining a $Frob_\infty$ structure on differential forms is to show that the obstructions vanish in the case $P = Frob$ and $Q = End(\Omega^*(M))$. This step is done in [W2].

Theorem 3.4.6 ([W2]. Thm. 4)**.** *There is a morphism of properads*

$$\Omega B(Frob) \to End(\Omega^*(M)),$$

inducing the Frobenius algebra structure on cohomology $H^*(M)$.

We now compare this construction of a $Frob_\infty$ algebra to the one in this paper. First, we restrict our attention to dioperads- since it is enough to show it in this case and then apply the functor from dioperads to properads. In the previous section, we defined a map $\mathbb{D}(BiLie^0) \to \underline{End}^0(\Omega^*(M))$. Let

$BiLie^{\#}$ be the codioperad dual to $BiLie^0$. Then our construction defines a map

$$\Omega(BiLie^{\#}) \to \underline{End}^0(\Omega^*(M)).$$

By Proposition 3.4.4, this is the same as a twisting morphism $BiLie^{\#} \to \underline{End}^0(\Omega^*(M))$. By Koszul duality, we have a quasi-isomorphism $BiLie^{\#} \hookrightarrow B(Frob^0)$. So we have a twisting morphism

$$BiLie^{\#} \hookrightarrow B(Frob^0) \to \underline{End}^0(\Omega^*(M)).$$

If we pass to cohomology, we get a twisting morphism $B(Frob^0) \to End^0(H^*(M))$.

We have shown that our construction of a $Frob^0_\infty$ algebra structure on $\Omega^*(M)$ can be obtained by following the obstruction theoretic approach in [W2]. However, there are choices involved in showing the obstruction vanishes. In particular, there is a coproduct $\Delta : \Omega^*(M) \to \Omega^*(M \times M)$ in [W2] which differs from $m^t_{1,2}$ for any value for t. The previous argument above and the uniqueness of minimal models implies that these two coproducts induce the same map on $H^*(M)$. We give a more direct proof of this fact.

The coproduct in [W2] is defined as follows. Let $T \in \Omega^n(M \times M)$ be the Thom class of the diagonal in $M \times M$. Recall that T is a closed form, whose cohomology class is Poincar/'e dual to $[M] \in H_n(M \times M)$. Also, note that $T = K_1$, the heat kernel at $t = 1$. Let $p_i : M \times M \to M$ be the projection onto the i^{th} factor. The coproduct is then defined as

$$\Delta : \Omega^*(M) \overset{(p_1^*+p_2^*)/2}{\to} \Omega^*(M \times M) \overset{\wedge T}{\to} \Omega^{*+n}(M \times M). \tag{3.4.1}$$

Note that the degree agrees with $m_{1,2}^{t=1}$. These maps are not equal on differential forms, but they induce the same maps on cohomology. To prove this, first recall the following "naturality" statement about the cap product, which can be found in ([H], Ch. 3.3). If $f : M \to N$, then the following diagram commutes

$$\begin{array}{ccc} H_i(M) \otimes H^j(M) & \xrightarrow{\cap} & H_{i-j}(M) \\ p_* \downarrow \quad\quad p^* \uparrow & & p_* \downarrow \\ H_i(N) \otimes H^j(N) & \xrightarrow{\cap} & H_{i-j}(N) \end{array} \tag{3.4.2}$$

Lemma 3.4.7. *The maps $\Delta^*, m_{1,2}^{1,*} : H^*(M) \to H^*(M \times M)$ agree.*

Proof. We consider the diagram in the case when we take $[M] \in H_n(M)$ to be the element in homology for the cap product. Let $[D] \in H_n(M \times M)$ be the diagonal in $M \times M$. For ease of notation, denote $(p_1^* + p_2^*)/2$ by p^* and similarly for p_*. Note that $p_*[D] = [M]$. Also, recall that $[D]$ is Poincar/'e dual to $[T] \in H^n(M \times M)$. Then for $a \in H^j(M)$, we have the following diagram:

$$\begin{array}{ccc} [D] \otimes p^* a & \longrightarrow & [D] \cap p^* a \\ \downarrow \quad \uparrow & & \downarrow \\ [M] \otimes \ a & \longrightarrow & [M] \cap a. \end{array}$$

Note that $[M] \cap a \in H_{n-j}(M)$ is Poincar/'e dual to a. Also, we can turn the rightmost arrow upwards using the map $Diag_*$. To get the the coproducts $m_{1,2}^{t,*}$ and Δ^*, we add a layer to the diagram to obtain the following

$$\begin{array}{ccc} [T]\otimes p^*a & \xrightarrow{\wedge T} & p^*a \wedge T \\ {\scriptstyle P.D._{M\times M}}\downarrow \quad \downarrow{\scriptstyle =} & & \uparrow{\scriptstyle P.D._{M\times M}^{-1}} \\ [D]\otimes p^*a & \xrightarrow{\cap} & [D]\cap p^*a \\ {\scriptstyle p_*}\downarrow \quad {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle Diag_*} \\ [M]\otimes\ a & \longrightarrow & [M]\cap a. \end{array} \tag{3.4.3}$$

Note that following a in the diagram

$$a \mapsto [M] \cap a \mapsto Diag_*([M] \cap a) \mapsto P.D._{M\times M}^{-1} Diag_*([M] \cap a)$$

is the definition of $m_{1,2}^{t,*}$. Also, following a in the diagram in the other direction

$$a \mapsto p^*(a) \mapsto p^*a \wedge T$$

is the definition of Δ^*. The commutativity of the diagram then proves the lemma. $\square$

Remark 3.4.8. In Section 3.4.4, we showed that $\Omega^*(M)$ was a sub-$\underline{Frob}_\infty^0$ algebra of $Curr_*(M)$. The coproduct $\Delta : \Omega^*(M) \to \Omega^*(M \times M)$ is not part of a sub-$\underline{Frob}_\infty^0$ algebra. We could define similar maps

$$Curr_*(M) \overset{Diag_*}{\to} Curr_*(M \times M) \overset{\cap T}{\to} Curr_*(M \times M),$$

where $\cap T$ is the cap product with the Thom class. In general, for $A \in Curr_i(M)$ and $\alpha \in \Omega^j(M)$, $A \cap \alpha \in Curr_{i-j}(M)$ acts on $\beta \in \Omega^{i-j}$ by

$$(A \cap \alpha)\beta := A(\alpha \wedge \beta).$$

In our situation, T is a form in $\Omega^n(M \times M)$ and the currents, being push-forwards of currents in $Curr_*(M)$ are in dimension $i \leq n$. So for dimension reasons, the coproduct will be zero except for $Curr_n(M)$.

3.5 Homotopy invariance of the $Frob_\infty$ algebra

We now have two ways of obtaining a minimal model for the Frobenius algebra on $Curr_*(M)$ and $\Omega^*(M)$. We can transfer the $\underline{Frob}_\infty$ algebra structures on $Curr_*(M)$ and $\Omega^*(M)$ given in the previous section to homology and cohomology by a contraction. The contraction is given by using the heat kernel for $t = \infty$ to project onto the harmonic forms. The chain map in the contraction is $\widehat{K}_\infty : \Omega^*(M) \to H^*(M)$ with chain homology $\widehat{L}_\infty$, and similarly for currents. The transfer of structure was described in Section 3.2.2.

The second way to obtain a minimal model is to consider the cyclic C_∞ algebra structure on $H^*(M)$. The C_∞ algebra is induced by the wedge product on differential forms ($\wedge$ or equivalently $\wedge_0$ in our notation), and the cyclic structure is given by Poincar/'e Duality. Applying the functor

$$\mathcal{D} : \text{cyclic operads} \to \text{dioperads}$$

to the map cyclic $C_\infty \to End(H^*(M))$ defines a $Frob_\infty$ algebra on $H^*(M)$, as described in Section 3.1.4. Since Poincar/'e duality and C_∞ algebra on $H^*(M)$ are homotopy invariants of M, showing that the two minimal models are the same proves the main claim of the chapter.

We break the proof up into four statements relating the various algebraic structures on currents and forms.

1. $(Curr_*(M), \{m^t_{i,j}\})$ is a $Frob^0_\infty$ algebra. The differential forms $\Omega^*(M)$ is a sub-algebra of $Curr_*(M)$.

2. The linear dual of $(\Omega^*(M), \{m^t_{i,j}\})$ is $(Curr_*(M), \{m^t_{i,j}\})$.

3. The minimal model for $(\Omega^*(M), \{m^t_{i,j}\})$ is a $\underline{Frob}^0_\infty$ structure on $H^*(M)$ and the minimal model for $(Curr_*(M), \{m^t_{i,j}\},)$ is a $\underline{Frob}^0_\infty$ structure on $H_*(M)$. These models are dual to each other.

4. The Poincar/'e Duality isomorphism from $H^*(M) \to H_*(M)$ identifies the structure maps $\{m^t_{i,j}\}$ in the minimal model for $H^*(M)$ with the

structure maps for $H_*(M)$. That is, the following diagram commutes

$$\begin{array}{ccc} H^*(M)^{\otimes i} & \xrightarrow{m^t_{i,j}} & H^*(M)^{\otimes j} \\ {\scriptstyle PD^{\otimes i}}\big\downarrow & & {\scriptstyle PD^{\otimes j}}\big\downarrow \\ H_*(M)^{\otimes i} & \xrightarrow{m^t_{i,j}} & H_*(M)^{\otimes j}. \end{array}$$

Statement (1) is a summary of the work in Sections 3.4.3 and 3.4.4, where we constructed $\underline{Frob}_\infty$ algebras. Statement (2) is proved in Proposition 3.4.3. Statement (3) is the same as (2) after passing to minimal models. That is, (3) is a corollary of 3.4.3. Finally, statement (4) follows from the fact that the cup product on cohomology and intersection product on homology are Poincar/'e dual operations.

Theorem 3.5.1. *The $Frob_\infty$ structure on $H^*(M)$ induced by TCFT on $\Omega^*(M)$ is a homotopy invariant of the manifold.*

Proof. We show that the $Frob_\infty$ structure on $H^*(M)$ is induced by the cyclic C_∞ on $H^*(M)$. The C_∞ structure on $H^*(M)$ is obtained by transferring the commutative algebra structure $(\Omega^*(M), \wedge)$ to $H^*(M)$. The cyclic structure is given by capping with the fundamental class $[M]$. Since the C_∞ structure and $\cap[M]$ are known to be homotopy invariants, showing that the two $Frob_\infty$ structures are the same proves the claim.

Lemma 3.4.2 shows that the maps $m_i : H^*(M)^{\otimes i} \to H^*(M)$ are equal to the maps $m^t_{i,1} : H^*(M)^{\otimes i} \to H^*(M)$ given by the minimal model for $(\Omega^*(M), \wedge_t)$. We write the maps m_i as $m_{i,1}$.

The dual of $m_{i,1} : H^*(M)^{\otimes i} \to H^*(M)$ is a map $c_{1,i} : H_*(M) \to H_*(M)^{\otimes i}$. By statement (3), $\{c_i\}$ defines the coalgebra maps in the $Frob^0_\infty$ structure on $H_*(M)$ coming from currents. Using Poincar/'e duality isomorphism between $H^*(M)$ and $H_*(M)$ and the maps $\{c_{1,i} : H_*(M) \to H_*(M)^{\otimes i}\}$, we obtain maps $\{m_{1,i} : H^*(M) \to H^*(M)^{\otimes i}\}$. By statement 4, these maps are the coalgebra maps in the $Frob_\infty$ structure on $H^*(M)$.

So far, we have shown that $m_{i,1} = m^t_{i,1}$ and $m_{1,j} = m^t_{1,j}$. We need to show that this is true for general map $m_{i,j} : H^*(M)^{\otimes i} \to H^*(M)^{\otimes j}$. Recall that $m^t_{i,j}$ is defined by compositions of $\{m^t_{i',1}\}$ and $\{m^t_{1,j'}\}$. Looking at the definition of the functor $\mathcal{D}$: {cyclic operads} $\to$ {dioperads}, we see that the same is true for for $m_{i,j}$. That is, $m_{i,j}$ is also defined by compositions of $\{m_{i',1}\}$ and $\{m_{1,j'}\}$. So we see that the two $Frob_\infty$ algebra structures on $H^*(M)$ are the same.

□

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