Order Number 9119691

# A class of (Q,R) inventory models with partial backorders, Poisson demands, and Erlang-distributed lead times

Woo, York Y., Ph.D.

City University of New York, 1991

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

A

# A Class of (Q,R) Inventory Models with Partial Backorders, Poisson Demands, and Erlang-Distributed Lead Times

by

YORK Y. WOO

A dissertation submitted to the Graduate Faculty in Business

in partial fulfillment of the requirements for

the degree of Doctor of Philosophy,

The City University of New York.

1991

This manuscript has been read and accepted for the Graduate Faculty in Business in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Dec. 6, 1990

Date

Georgios P. Sphicas

Chair of Examining Committee

Dec 11, 1990

Date

Donald Vredenburgh

Executive Officer

Dr. David Dannenbring

Dr. George O. Schneller

Dr. Nicholas C. Georgantzas

Supervisory Committee

The City University of New York

**ABSTRACT**

A Class of (Q,R) Inventory Models with Partial Backorders, Poisson Demands, and Erlang-Distributed Lead Times

by

York Y. Woo

Advisor: Professor Georghios P. Sphicas

The framework of analysis in this study is the stochastic (Q,R) model for a single inventory item. The inventory position, which is the level of inventory on hand plus on order minus backorders, is continuously monitored. Customers demand single units according to a Poisson process. When the position reaches the prespecified reorder level R, an order is placed with suppliers for a lot of size Q units. The lead time, defined as the time from placing the order until the shipment arrives, is a random variable with an Erlangian type of distribution.

We first consider such an inventory system with exponentially distributed lead times, which is a special case of Erlangian distributions. When a stockout occurs, some customers are willing to wait for backorders but the rest will take their business elsewhere. After a certain number of outstanding orders have been placed by the system, however, all new demands will be lost because the expected waiting time for backlogging will then be longer than customers' tolerance limits. An analytical model

exactly representing this system is derived by solving a truncated stochastic process. Under a special case, the model also represents another inventory system where the interarrival time between two successive customers has an Erlang distribution and an order is placed by the system whenever a unit is demanded. Some properties of the net inventory probability distribution are then presented with proof. Since solutions of the optimal inventory policy cannot be explicitly obtained, some experimentation is designed and performed in order to investigate the nature of this system. Numerical results of the sensitivity analysis are also reported with some discussion.

We then consider a continuous review inventory system with Erlang-distributed lead times where all customers will wait to be backlogged when the system is out of stock. Another analytical model which closely approximates this system is built by conditioning the probabilities of net inventory first on the inventory position and then on the lead time. Several properties of this model are then established. Closed-form expressions of both the net inventory probability and the average inventory cost function are also obtained. Sensitivity analysis is designed and conducted by using numerical examples to examine the effect of lead time variability on the optimal solution. Exponential lead time is studied as a special case, and then compared with the previous exact model when all unsatisfied demands can be backordered. Finally, we provide some suggestions for future research.

ACKNOWLEDGEMENTS

At the completion of this study, the author wants to express the deepest gratitude to Professor Georghios Sphicas, who planted the seed of this research, for his continuous inspiration and guidance. The author also gratefully acknowledges the other members of his dissertation committee, especially Dr. David Dannenbring and Dr. George Schneller, who provided many valuable comments on this manuscript.

The author would also like to thank Dr. Vredenburgh for his encouragement and support. Last but not least, the author wants to dedicate this work to his parents, King and Meider, for their endless love.

TABLE OF CONTENTS

LIST OF TABLES

## LIST OF FIGURES

# CHAPTER 1

# INTRODUCTION AND OVERVIEW

## 1. PROBLEM STATEMENT

Inventory control is a practical problem whenever certain commercial activities are undertaken. *Inventory* can be generally defined as: A quantity of goods or materials in the control of an enterprise and held for a time in a relatively idle or unproductive state, awaiting its intended use or sale. This definition suggests that the existence of an inventory reflects a temporary lull between two activities which can be termed the supply and demand processes. Customarily the supply process precedes and contributes goods to the inventory, while the demand process succeeds and depletes the same inventory. The inventory exists because the supply and demand processes differ in the rates at which they respectively provide or require stock.

Inventory serves an important role in any bussiness enterprise because it can provide the following functions:

1. Market exploitation. Anticipation of a future increase in selling price suggests a delay in disposing of stock on hand. Conversely, declining market prices are a motivation for creating a negative inventory such as intended backorders.

2. Protection against stockouts. To the extent that the supply or demand process fluctuates unpredictably, there is some risk of running out of stock and suffering the associated customer strife, disruption of operations, expediting costs, etc. Insurance against such stockouts is provided by so-called buffer stocks.
3. Operations smoothing. Demand fluctuation is usually synchronous with the season of the year or with the phenomenon of the business cycle. Although such fluctuation can be accommodated in other ways, such as changing production rates, there is usually the alternative of producing smoothly and storing for peak demands.
4. Lot-size economy. If the supply process involves shipment and receipt of goods from an outside supplier, the economy of larger-sized replenishment orders basically arises from lower shipping and delivery costs. In some cases, this may consist of the supplier's offer of a quantity discount on large orders. If the supply process is internal production, the economy accrues from fewer machine setups.
5. Control system economy. An often-overlooked purpose for carrying larger inventories is that less control effort is required. Inventory control systems are costly to design, implement, and maintain. A lower-cost, looser mode of control, with attendant larger stocks, may be justified.

The functions of inventory introduced above can be unified into a single, overall function. The purpose of maintaining inventories is to avoid the opportunity cost associated with not doing so. However, inventory maintenance is itself a costly activity. The cost of carrying stocks may include interests on inventory investment, storage and handling expenses, insurance policies against accidental losses, property taxes, etc. Therefore, the justification for holding stocks must be that the costs avoided exceed the carrying cost. Those costs which can be avoided by holding inventories are shortage costs, ordering costs, material costs, and systems costs.

Inventory control is one of the earliest developed areas in the fields of Industrial Engineering, Operations Research, Management Science, or Production and Operations Management. Inventory control can be interpreted as effort to achieve and maintain an economic balance between the costs incurred and the costs saved by holding material in stock. The control of inventory can be also viewed as control over availability for use of stocks. In general, the usual control mechanism involves regulation of the supply process. Accordingly, the predominant concern in this study will be the effectiveness of decisions regarding *when* and *how much* to order or produce to supply an inventory. However, the effectiveness of supply decisions is measured differently by different people, depending on their role in the organization. A common denominator is therefore needed to unify thought and promote communication among people with varying organizational goals. Costs are usually used to achieve this resolution. Therefore, the emphasis of inventory control is on designing cost-sensitive, cost-effective, and cost-minimizing inventory supply systems.

Modeling of supply behavior involves relating an actual supply to an ordering decision. The two quantities may differ in quantity or timing or both. Most inventory models in use today assume that the quantity supplied will be equal to that ordered. However, the timing of supply will generally lag behind the placement of an order for two reasons. An order for replenishment is supplied only after the necessary paperwork, materials handling, and transportation activities are completed. These delays are lumped together to form a delivery lag or lead time. In the model, lead time may be treated either as constant or as a random variable. The other reason why supply is not coincident in time with an ordering decision is that the supply rate may be finite. This is the case when the order is a production order rather than a purchase order. In this study, only the case of purchase orders will be considered.

While supply quantities are usually realistically modeled as known, future demand quantities frequently can not be. It is the degree of predictability which governs the modeling decision to treat future demands as known or unknown. If demands are quite predictable, it should be reasonable to treat them as known.

However, unpredictability is not to be confused with variability. Demands can be highly variable but nonetheless predictable. It is the *random* variation in demands which causes future demands to be unpredictable. Thus demands for future time periods are considered to be either known quantities or random variables. Since demand and lead time are usually the only quantities treated as random variables in an inventory model, such models will be called *deterministic* if demand and lead time are both treated as known. If one or both of demand and lead time are modeled as random variables, the model is called *probabilistic* or *stochastic*. In the case of stochastic models, demands for future periods in time and/or lead times are usually respectively assumed to be independently, identically distributed random variables with a known probability distribution.

Another important consideration in modeling inventory systems is the way in which a shortage affects demands. Demands which can not be satisfied when a stockout occurs are usually either backlogged or lost or a combination of both. Backlogged demands are assumed to be satisfied as soon as supply is available. Unsatisfied demands may also affect subsequent demands, as when dissatisfied customers take their business elsewhere or persuade their acquaintances to do likewise. Occasionally, however, unsatisfied demands may generate more demands than would have normally occurred. For example, demands for needed maintenance parts, if not provided, may cause additional events to occur which require more maintenance.

The framework of analysis in this study is the continuous-review stochastic (Q,R) model for a single inventory item. This model is also known as the lot-size reorder-point inventory model. It is presumed that information regarding the inventory level, including amount on hand and on order, is continually available at all times. In a practical sense, this assumption is considered valid if the time lag between the occurrence of a transaction and knowledge of its occurrence is negligibly small compared with the typical interval between successive demands. Here, two inventory levels commonly in use must be distinguished. *Net inventory* is usually defined as the level of on-hand inventory minus backorders while *inventory position* is the level of net inventory plus inventory on order.

It is therefore logical for such an inventory system to specify its operating policy so that an order is triggered whenever the stock position falls to a given level. In the inventory literature, that level is usually called R and referred to as the reorder point. The quantity to be ordered when an order is placed is fixed if it is assumed that customers demand the inventory item one at a time, and is usually called Q in the literature. The purpose of the analysis of (Q,R) models is therefore to find the optimal values for Q and R. The order quantity may be variable in a situation where customers demand various rather than one unit at a time, but those situations are beyond the scope of this study. A policy involving variable order quantities usually specifies that the inventory position be brought up to a level S whenever it falls below the level s, and is referred to as an (s,S) policy.

Another policy for operating an inventory system in a stochastic environment is known as the periodic-review policy, where the inventory level is observed only at equally spaced points in time. The choice between a continuous-review and a periodic-review ordering policy is one of the most fundamental decisions to be made in designing an inventory control system. The use of a periodic-review policy will result in higher holding and shortage costs and lower ordering and information control costs than the use of a continuous-review policy. However, due to the rapid growth of the use of electronic data processing systems to control inventories, the cost of handling inventory information for a continuous-review system is now no greater than that of a periodic-review system. Therefore, when this is combined with the fact that, exclusive of the review cost, a continuous-review system always results in a lower expected inventory cost per unit of time than a periodic-review system [Hadley and Whitin 1963, Sections 5-1 and 5-12], the reason for the trend to continuous review becomes clear.

## 2. OBJECTIVES AND SCOPE OF THE STUDY

This study attempts to investigate two single-item, continuous-review stochastic inventory systems so as to fill up some gaps in the literature and make some contributions to the current state of knowledge of such systems. The main objective is to develop two analytical models representing these systems, one exact and the other approximate. It is worthwhile to note here that the general trend in Management Science for recent years has been in the direction of finding "good enough" solutions, if they are easily obtainable, rather than exact ones, if they require a lot of effort and restrictive assumptions.

Properties and closed-form results are also derived for both models. Due to the complexity involved in the theoretical result, some numerical experimentation is designed and performed for each model in order to examine its nature and behavior. Finally, these two models are compared under some special cases by numerical examples. The ultimate goal of this study is to provide useful results for the design and implementation of real life inventory control systems.

The scope of this study is limited by some assumptions on the nature of problems under consideration. First of all, it is assumed for both inventory systems that demands arrive one at a time according to a Poisson process and lead times are independently identically distributed random variables with an Erlangian type of probability distribution. This also implies that the demand arrival rate and the service rate for replenishments are independent of the inventory level in the system.

The assumption of Erlang distribution for lead times is appropriate in many inventory problems. Consider that there are several phases in the process of replenishing orders and if each phase requires exponential time duration, the entire lead time would have an Erlang distribution. The method of stages allows one to model the Erlang process as a sequence of exponential processes. This, combined with the Poisson process generating demands, allows one to work with a completely Markovian model. Another reason for the Erlangian assumption is that the Erlang distribution provides a convenient family of distributions with a wide range of coefficients of variation between the constant case and the large variation of exponential case. Table 1.1 shows the first two moments of the probability distributions used in this study.

Table 1.1. Probability distributions used in this study

| Probability Distribution | Probability mass/ density function | Mean | Variance |
|---|---|---|---|
| Poisson with parameter $\lambda > 0$ | $\frac{\lambda^x}{x!} e^{-\lambda} \quad x = 0,1,2,\ldots$ | $\lambda$ | $\lambda$ |
| Exponential with parameter $\mu > 0$ | $\mu e^{-\mu t} \quad t \geq 0$ | $\frac{1}{\mu}$ | $\frac{1}{\mu^2}$ |
| Erlang with parameters $\mu > 0$ and $K = 1,2,3,\ldots$ | $\frac{(\mu t)^{K-1}}{(K-1)!} \mu e^{-\mu t} \quad t \geq 0$ | $\frac{K}{\mu}$ | $\frac{K}{\mu^2}$ |

Next, it is assumed for the first system that a constant fraction, which is independent of the backlog size, of customers are willing to wait for backorders when there is a stockout, but the remaining fraction will take their business elsewhere. Furthermore, those who will wait are assumed to have limited patience and only willing to wait a certain period of time for satisfaction. Therefore, after a certain number of outstanding orders have been placed by the system, all new demands will be lost and/or the management will not accept any new demand from customers, because the expected waiting time for backorders will then be longer than customers' tolerance limits. In other words, only a certain number of orders are allowed to be outstanding at all time. This last assumption allows one to truncate the Markov process into finite states.

The general justification for utilizing such a model would be that the management feels that customers are only willing to wait for a certain period of time for backlogging, after which their loyalty will be placed in jeopardy. It would naturally be assumed that the loss of this loyalty is in excess of merely losing a single sale. Therefore, the management wishes to utilize the limited customer patience in allowing only a certain length of queue to form. For example, only a portion of new car buyers will wait for a certain brand and/or model of cars from the dealer and the rest will turn to other brands or models when a stockout occurs. When backorders accumulate to a certain level, however, all buyers will go somewhere else or the dealer will advise new customers to do so because the expected waiting time for replenishments would be too long. Peterson and Silver [1985, pp. 253-254] provide a detailed discussion on the practicality of models with partial backorders.

However, for the second model, all customers are assumed to be willing to wait as long as necessary for backlogging when the system is out of stock. This is also one of the two assumptions frequently used in the inventory literature. The other common assumption is that all demands will be lost when a stockout occurs.

Finally, it is assumed for both models that cost elements are stationary, i.e., all inventory related costs do not change with time, and the system can reach steady state as the planning horizon approaches infinity. Furthermore, the system control cost is assumed to be independent of Q and R, and the purchase price of inventory item is assumed to be constant over time, i.e., it is independent of the order quantity. Therefore, two possible limitations in applying these models include the steady-state assumption, which does not allow for any variation in the mean demand and mean lead time, and the constant price assumption, which precludes quantity discounts.

## 3. RESEARCH METHODOLOGY

The purpose of analyzing (Q,R) models is to find the optimal values for the order quantity and reorder point. The optimal values of Q and R can be obtained by minimizing the average total inventory cost per time period. One procedure frequently used to determine the average cost per time period is to compute the expected cost per inventory cycle and then to multiply by the average number of cycles per period of time. This is a very powerful technique for developing the average cost expression. It can be used under a very wide variety of circumstances as illustrated by many studies in the literature.

Another different and interesting approach to formulate the average total cost per period of time, for the case where a Poisson process generates demands, is to use a Markov process discrete in space and continuous in time. Here, one first derives the state probabilities, and then uses them to obtain the average cost expression. While the Markovian type of analysis is useful in a variety of cases, it does not seem to have nearly the general applicability of the procedure mentioned above. The Markovian form of analysis, however, does have the advantage that it yields directly the state probabilities, which are not obtained directly from the other approach of generating the average cost.

Because of the Poisson demand assumption, both of our models in this study use the Markovian approach to formulate the average total cost per time period. The steady-state probabilities of net inventory level are derived directly from the analysis for both models. In the first model, the system of difference equations for a truncated Markov process is first formulated, and a system of recursive expressions is then obtained for the state probabilities of net inventory. Some properties of the net inventory distribution are also presented with theoretical proof. Under a special case, the model also represents another inventory system where the interarrival time between two successive customers has an Erlang distribution and an order is placed by the system whenever a unit is demanded. The average inventory cost per unit of time is formulated as a function of Q and R by using the state probabilities. Due to the complexity in the cost expression, some numerical experimentation is designed and conducted to investigate characteristics of the model.

In the second model, state probabilities of net inventory level are derived by conditioning them first on the inventory position and then on the lead time. This model is an extension to Erlang lead times of that of Hadley and Whitin [1963, Section 4-7], which is exact for Poisson demands and constant lead times. Because of the stochastic nature of lead times, our model is an approximate rather than exact representation of the real inventory system. Two different sets of closed-form expressions for the state probabilities and analytical properties of the distribution are obtained for the inventory level. Closed-form expression is formulated for the average cost function. Theoretical proof of the convexity of the cost function is attemped for the special case of exponential lead time. The relationship between optimal values of Q and R is also developed for this special case. A computational experimentation is designed and performed to show the nature of this model.

A simple two-dimensional search approach is used to locate the optimal values of order quantity and reorder point in the numerical experimentation. Finally, these two models are evaluated and compared under some special cases by the numerical results.

## 4. LITERATURE REVIEW

The continuous review policy in inventory control systems has been investigated by many researchers. Under this policy, an order quantity of Q is triggered whenever the inventory position falls to a prespecified level R. In the deterministic situation where demands and lead times are known and constant, the optimal operating policy can be determined without any difficulty [Johnson and Montgomery 1974, Section 2-2]. However, as demands and/or lead times become random variables, the exact optimal inventory policy can be obtained only under certain circumstances. Various approximations have been proposed for other situations where exact model is too difficult to formulate, resulting in different expressions of the expected average cost function and optimal solution. Therefore, the review of relevant literature will be reported into two categories, one of approximate stochastic (Q,R) models and the other of exact ones.

### 4.1. Approximate (Q,R) Models

Among the tremendous work reported in the inventory literature, many authors have studied the (Q,R) model with either complete backordering or complete lost sales under different assumptions [Hadley and Whitin 1963; Love 1979; Wagner 1969; Holt et al. 1960; etc.]. Due to difficulties arising from the stochastic nature of (Q,R) models, most of their work, to some extent, treats the inventory situation heuristically or approximately. One common assumption used in these approximate models is that at most one order is allowed to be outstanding at any point in time. Everyone of these models can then heuristically solve the inventory problem for any specified distribution of lead-time demand.

The heuristic treatment of expected on-hand inventory level in the complete backordering case differs among these approximate models. However, the expected average cost function can be expressed in the following general form:

$$C(Q,R) = \frac{A\lambda}{Q} + H\left[\frac{Q}{2} + R - \mu + \frac{\alpha}{2}B(R)\right] + \frac{\Pi\lambda}{Q}B(R) \ ;$$

where

$C(Q,R)$ = the expected average total inventory cost per time period for the order size $Q$ and reorder point $R$,

$A$ = ordering cost per order placed,

$H$ = carrying cost per unit held in stock per unit of time,

$\Pi$ = backordering cost of each unit backlogged,

$\lambda$ = the mean demand rate,

$\mu$ = the mean lead-time demand,

B(R) = the expected number of backorders per cycle given the reorder level R, i.e.,

$$B(R) = \int_R^{\infty} (x-R)\, f(x)\, dx \; ;$$

where f(.) is the probability density function of lead-time demand, and

$$\alpha = \begin{cases} 0 & \text{[Hadley and Whitin 1963, Section 4-2]} \\ 1 & \text{[Love 1979, Section 3-3]} \\ \mu/Q & \text{[Wagner 1969, pp. 821-831]} \end{cases}$$

It should be mentioned here that Wagner assumes constant lead time for his derivation while other models can be applied for both constant and stochastic lead times.

This average cost function is then partially differentiated with respect to Q and R by treating them as continuous variables. By setting the derivatives to zero, one can obtain two expressions in terms of optimal Q and R. Usually via the normal approximation of lead-time demand distribution, the optimal inventory policy can then be determined by iteratively applying these two expressions. Some researchers have conducted studies to evaluate the performance of these approximate models over that of some exact model [Gross and Ince 1975; Nahmias 1976; Chiou and Sphicas 1988]. For example, Chiou and Sphicas conclude that the three models mentioned above perform equally well in those cases they investigated, and suggest that any one of these models can be used for those cases with low or moderate lead-time demand.

When faced with a stockout, different customers react differently depending upon how it affects their respective businesses. Some are sensitive to the frequency of stockouts while others regard the number of backorders to be more important, In certain situations, such as spare parts, the crucial element is, however, the duration of stockout as it is directly responsible for the idle time of machines. It appears that probably the earliest models considering time-weighted backorders are due to Holt, Modigliani, Muth, and Simon [1960, pp. 226-227 and 249-251]. In their Model Two, by applying a heuristic treatment on inventory situation and assuming constant lead times, they obtain the expected average cost function as follows.

$$C(Q,R) = \frac{A\lambda}{Q} + H\left[\frac{Q}{2} + R - \mu + \frac{\mu}{Q}J(R)\right] + \frac{\hat{\Pi}\mu}{Q}J(R) \;;$$

where $\hat{\Pi}$ is the backordering cost per unit to be backlogged per time unit and

$$J(R) = \int_R^\infty \frac{(x-R)^2}{2x} f(x)\,dx \;.$$

The optimal values of Q and R can then be derived iteratively from expressions obtained by setting the partial derivatives of the cost function to zero. Das [1983] proves that this cost function is strictly convex with repect to Q and R, and provides a quadratic approach to approximate the optimal policy.

Federgruen and Zheng [1988] develop another form of the expected average cost function with time-weighted backordering cost for constant lead times as follows:

$$C(Q,R) = \frac{A\lambda}{Q} + \frac{1}{Q} \sum_{y=R+1}^{R+Q} G(y) ,$$

where, when t is the lead time, G(y) is the rate at which expected inventory costs accumulate at time $\tau$+t when the inventory position at time $\tau$ equals y. Let D, a random variable, denote the demand during lead time, then

$$G(y) = E[\ H(y-D)^+ + \hat{\Pi}(D-y)^+\ ] .$$

This cost function is valid as long as the inventory position at steady state is uniformly distributed and independent of the lead-time demand. They also present an efficient algorithm to locate the optimal inventory policy. Zheng [1989] further shows that the cost function is quite insensitive to Q and R at their optima, and argues that the simple, deterministic EOQ model can be used for stochastic cases without causing much cost increase.

For the lost-sales case, the approximate model not only treats the on-hand inventory heuristically but also ignores the portion of time during a cycle when the system is out of stock. The expected average cost function can be formulated as follows [Hadley and Whitin 1963, Section 4-3; Love 1979, Section 3-3; among others]:

$$C(Q,R) = \frac{A\lambda}{Q} + H\left[\frac{Q}{2} + R - \mu + B(R)\right] + \frac{L\lambda}{Q} B(R) ,$$

where L is the cost of each lost demand. Again, two expressions of optimal Q and R can be derived by setting partial derivatives of the cost function to zero. Given a lead-time demand distribution, the optimal Q and R can then be solved iteratively by these two expressions.

Approximate (Q,R) models which consider partial backordering have also been studied by some authors. Montgomery, Bazaraa, and Keswani [1973] analyzed a continuous review inventory system where, during the stockout period, a fraction of demands are backlogged while the remaining fraction are lost. Both deterministic and stochastic demands are considered. In the stochastic case, they develop an approximate model by linearly combining Hadley and Whitin's backordering and lost-sales models. Love [1979, Section 3-3] also builds a similar model by assigning the fractions of backlogged and lost demands as weights, repectively, to his complete backordering and lost-sales models. Based on Holt and others' heuristic treatment, Kim and Park [1985] consider a partial backordering inventory system where demand is stochastic and the backordering cost is assumed to be proportional to the length of time for which the backorder exists. By assuming that lead time is constant, they derive the necessary equations from which optimal Q and R can be calculated iteratively.

If not approximated by a normal distribution, these approximate models can also solve the inventory control problem for any specified lead-time demand distribution. In general, the nature of demand during lead time depends on assumptions about the size of each demand, the interarrival time between successive demands, and the lead time. Therefore, many authors have concentrated their attention on deriving the probability distribution of lead-time demand. For examples, Burgin [1972] studies the case of normal unit demads and Erlang lead times, and obtains a very complicated compound distribution for lead-time demand. Bagchi and Hayya [1984] modify the result by applying the so-called "post-integration truncation" on the compound distribution and derive a simpler expression for the lead-time demand distribution, which is a weighted average of a group of Erlang distributions. Das [1976] shows that after the post-integration truncation the demand during lead time for normal unit demands and exponential lead times is exponentially distributed.

Practical experience has shown that often a gamma distribution also constitutes an acceptable approximation of the lead-time demand [Burgin and Wild 1967; Burgin 1975; among others]. In two papers Tadikamalla [1978 and 1979] shows how the Weibull and lognormal distributions can be used to approximate the lead-time demand. Fortuin [1980] considers a single item (Q,R) inventory system where lead-time demand is assumed to have five different distributions, namely Gaussian, logistic, gamma, lognormal, and

Weibull. The expected shortage per cycle is calculated for all five distributions, leading to the result that there are very small numerical differences between these distributions provided that they are all nearly symmetric.

## 4.2. Exact (Q,R) Models

As mentioned earlier, exact continuous-review (Q,R) inventory models exist only under certain restrictive assumptions due to difficulties arising from the stochastic nature of demand and/or lead time. However, existing studies [for example, Chiou and Sphicas 1988] suggest that the exact model should be used for those cases with high lead-time demand and those cases with unrealistic cost condition. Zipkin [1986] also suggests that the approximate model may give misleading results when demand during lead time is highly variable because of volatility in the demand process and/or in the lead time, or when mean lead-time demand is large compared to the order quantity Q.

Almost all exact models assume that a Poisson process generates demands, because the Markovian property provided by this Poisson assumption can conveniently reduce much intractability in the derivation. Hadley and Whitin [1963, Section 4-7] consider such an inventory system where all unsatisfied demands will be backlogged and lead time is constant. They obtain steady-state probabilities of net inventory level, and hence expression of the average cost function, by conditioning those probabilities on inventory position. They also suggest in Section 4-12 that their model can be extended to stochastic lead time cases as long as there is never more than a single order outstanding. This provides a part of motivation for our second, approximate model.

Galliher, Morse, and Simond [1959] also study the backordering system for both constant and exponential lead times. In the case of constant lead time, they derive the same results as those obtained by Hadley and Whitin but in a different form. In the case of exponential lead time, they obtain rather complicated expressions for net inventory probabilities and some operating characteristcs of the system. Their results are one of the earliest exact formulations for stochastic (Q,R) models. The unmanageability in their results strongly motivates our attempt for the approximate model.

Gross and Harris [1973] analyze the case where the distribution of exponential lead time is allowed to depend on the level of unfilled demands. They model the inventory system under two different assumptions about how lead times depend on the number of outstanding orders. Posner and Yansouni [1972] study an inventory system where lead time is exponential and, during a stockout period, all customers will wait for backlogging but then cancel their demands if waiting time is too long. By allowing at most one order outstanding at any point in time, they derive expressions for net inventory probabilities and the expected cost function. Our first, exact model adopts this limited patience assumption but interprets it in a different way.

Sivazlian [1974] considers the backordering case of unit demand arrivals and general demand interarrival times. He shows that the steady-state distribution of inventory position is uniform on the set of integers between R+1 and R+Q and is independent of the distribution of demand interarrival times. Richards [1975] offers an extension of the same result to the case of random rather than unit demand size. In both cases of variable and unit demand sizes, the distribution of inventory position is independent of the nature of lead time.

Few exact stochastic (Q,R) models have been reported for the lost-sales case in the literature. Hadley and Whitin [1963, Section 4-11] treat the lost-sales case of unit Poisson demand and constant lead time with a maximum of one order outstanding at any point in time. They derive a set of expressions for on-hand inventory probabilities and average cost per unit time. They also suggest that in practice it is rather difficult to determine the optimal values of Q and R from the exact cost function and it is almost always true that the approximate model will be sufficient for most real world situations. Buchanan and Love [1985] consider the lost-sales case of Erlang-distributed lead times under the assumption that only one order can be outstanding at a given time. They obtain a system of expressions for on-hand inventory probabilities and formulate the average operating cost function. They also reduce their results to simpler ones for the special case of exponential lead times. Ravichandran [1984] studies the case of so-called "phase type" distributed lead times and unit Poisson demands. He also derives a set of expressions for net inventory probabilities by allowing at most one order to be outstanding.

So far as we know, no exact (Q,R) model has been developed for the lost-sales case when more than one order is allowed to be outstanding except for the case where Q equals one. In certain real world situations, it is optimal to order inventory units one at a time as demanded. This can be true, for example, if the demand for the item is very low or the item is very expensive so that the cost of ordering is negligible compared with holding costs. The solution to the general (Q,R) model will indicate whether or not that Q = 1 is optimal. Sometimes, however, it is required by the supplier that the order quantity must be one. Since an order is placed each time there is a demand, the inventory position must remain constant. Denote the inventory position which is also the maximum inventory level by R, the problem is to determine the optimal value of R. This is usually referred to as the base stock (R) policy or as the continuous review (S-1,S) policy where S is the maximum inventory level. The special case of (Q,R) models for Q = 1 has been investigated by some authors for both complete backlogging and lost-sales cases.

Hadley and Whitin [1963, Section 4-13] study the backordering case for constant and exponential lead times and derive the same expression of net inventory probability for both cases as follows.

$$P(R-j) = \frac{(\lambda/\mu)^j}{j!} e^{-(\lambda/\mu)} \qquad j=0,1,2,\ldots;$$

where P(.) is the net inventory probability, R is the maximum inventory level, $\lambda$ is the mean demand rate, and $1/\mu$ is the lead time or mean lead time in the constant or exponential case, respectively. This is a very interesting result. They also prove an even more surprising result that, for the case where Q equals one, if demand is Poisson distributed and lead times are independent random variables, i.e., orders can cross each other, then this expression represents the net inventory probabilities for any lead time distribution with mean $1/\mu$. In other words, the net inventory probabilities and, hence, the optimal value of R are independent of the nature of lead time distribution as long as lead times are independent. Galliher et al. [1959] also analyze the backordering system for both cases of constant and exponential lead times and conclude the same results.

The lost-sales case for Q = 1 is also considered by Hadley and Whitin in their Section 4-13 under the assumption of exponential lead times. For this model, more than a single order is allowed to be outstanding at a given point in time. They derive the following truncated Poisson distribution for on-hand inventory probabilities by using the Markovian analysis:

$$P(R-j) = \frac{(\lambda/\mu)^j / j!}{\sum_{i=0}^{R} (\lambda/\mu)^i / i!} \qquad j=0,1,\ldots,R;$$

where $1/\mu$ is the mean lead time. They also point out that these probabilities have been shown to be independent of the nature of lead time distribution if lead times are independent and orders can cross. Unfortunately, however, those proofs do not include the case of constant lead times. Smith [1977] offers an approximate formula for this lost-sales case to locate the optimal maximum stock level R.

Recently, Moinzadeh [1989] treats the case of Q = 1 with unit Poisson demands, constant lead times, and partial backorders, and derives the following expressions for net inventory probabilities:

$$P(R-j) = \frac{(\lambda/\mu)^j}{j!} P(R) \qquad j=0,1,\ldots,R,$$

$$= \frac{\beta^{j-R}(\lambda/\mu)^j}{j!} P(R) \qquad j=R,R+1,R+2,\ldots, \text{ and}$$

$$P(R) = \left[ \sum_{i=0}^{R} \frac{(\lambda/\mu)^i}{i!} + \sum_{i=R+1}^{\infty} \frac{\beta^{i-R}(\lambda/\mu)^i}{i!} \right]^{-1} ;$$

where $1/\mu$ is the lead time and $\beta$ is the fraction of customers who will wait for backorders. He further points out that these probabilities are also valid for exponential lead times with mean $1/\mu$. We would like to provide an observation here that, because these results generalize those of Hadley and Whitin, the author indirectly proves that Hadley and Whitin's lost-sales case is also valid for constant lead times. During our review, it appears that no exact (Q,R) model which considers partial backordering has been reported in the literature except for the special case of Q = 1 mentioned above. This brings us to formulate our first, exact model for partial backorders.

Finally, several other authors have treated cases where customers demand variable units for an inventory item. For example, in the backordering case, Archibald and Silver [1978] solve the optimal policy for systems with exponential inter-demand times but general rather than unit demands and constant lead times. Sahin [1979] presents both the time-dependent and steady-state distributions of on-hand inventory and inventory position for cases operated under an (s,S) policy with general inter-demand times, general demands, and constant lead times. Galliher et al. [1959] also consider a backordering case where demands arrive according to a Poisson process, the number of units demanded follows a geometrical distribution, and the lead time is constant. They obtain expressions of net inventory probabilities for this so-called "stuttering Poisson" demand case.

Archibald [1981] studies the lost-sales case for discrete compound Poisson demands with constant lead times and at most one order outstanding. Altiok [1989] investigates a production/inventory system with discrete compound Poisson demands and phase-type processing times for both complete backordering and lost-sales cases. The inventory policy indicates that production starts when on-hand stock reaches a prespecified level and continues until the stock reaches its maximum level. He derives recursive expressions of net inventory probability for both cases. However, those models with variable rather than unit demands are beyond the scope of this study. **Table 1.2** summarizes those stochastic (Q,R) inventory models which are relevant to this study.

Table 1.2. Summary of relevant (Q,R) inventory models

| Model | Demand | Lead Time | $\beta^*$ | Outstanding Orders Allowed | Optimal |
|---|---|---|---|---|---|
| Holt et al. 1960 | General | Const. | 1 | 1 | Approx. |
| Wagner 1969 | General | Const. | 1 | 1 | Approx. |
| Federgruen & Zheng 1988 | General | Const. | 1 | 1 | Approx. |
| Hadley & Whitin 1963 | General | Gener. | 0 & 1 | 1 | Approx. |
| Kim & Park 1985 | General | Const. | [0,1] | 1 | Approx. |
| Love 1979 | General | Gener. | [0,1] | 1 | Approx. |
| Montgomery et al. 1973 | General | Gener. | [0,1] | 1 | Approx. |
| Hadley & Whitin 1963 | Poisson | Const. | 0 & 1 | 1 & ∞ | Exact |
| Galliher et al. 1959 | Poisson | Expon. | 1 | ∞ | Exact |
| Gross & Harris 1973 | Poisson | Expon. | 1 | ∞ | Exact |
| Posner & Yansouni 1972 | Poisson | Expon. | 1 | 1 | Exact |
| Woo-1 1990 | Poisson | Expon. | (0,1] | N | Exact |
| Buchanan & Love 1985 | Poisson | Erlang | 0 | 1 | Exact |
| Woo-2 1990 | Poisson | Erlang | 1 | ∞ | Approx. |
| Hadley & Whitin 1963† | Poisson | Gener. | 0 & 1 | R & ∞ | Exact |
| Moinzadeh 1989† | Poisson | Const. Expon. | [0,1] | ∞ | Exact |

* $\beta$ is the fraction of customers who will wait for backorders.

† For the case where Q = 1.

## 5. OVERVIEW

The framework of analysis in this study is the stochastic (Q,R) model for a single inventory item. The inventory position, which is the level of inventory on hand plus on order minus backorders, is continuously monitored. Customers demand single units according to a Poisson process. When the position reaches the prespecified reorder level R, an order is placed with suppliers for a lot of size Q units. The lead time, defined as the time from placing the order until the shipment arrives, is a random variable with an Erlangian type of distribution.

In Chapter 2, we first consider such an inventory system with exponentially distributed lead times, which is a special case of Erlangian distributions. When a stockout occurs, some customers are willing to wait for backorders but the rest will take their business elsewhere. After a certain number of outstanding orders have been placed by the system, however, all new demands will be lost because the expected waiting time for backlogging will then be longer than customers' tolerance limits. An analytical model exactly representing this system is derived by solving a truncated stochastic process. Under a special case, the model also represents another inventory system where the interarrival time between two successive customers has an Erlang distribution and an order is placed by the system whenever a unit is demanded. Some properties of the net inventory probability distribution are then presented with proof. Since solutions of the optimal inventory policy can not be explicitly obtained, some experimentation is designed and performed in Chapter 3 in order to investigate the nature of this system. Numerical results of the sensitivity analysis are also reported with some discussion.

In Chapter 4, we then consider a continuous review inventory system with Erlang-distributed lead times where all customers will wait to be backlogged when the system is out of stock. Another analytical model, which closely approximates this system, is built by conditioning the probabilities of net inventory first on the inventory position and then on the lead time. Several properties of this model are then established. Closed-form expressions of both the net inventory probability and the average inventory cost function are also obtained. Sensitivity analysis is designed and conducted in Chapter 5 by using numerical examples to examine the effect of lead time variability on the optimal solution. Exponential lead time is studied as a special case, and then compared with the previous exact model when all unsatisfied demands can be backordered. Finally, we provide some suggestions for future research in the end of that chapter.

# CHAPTER 2

# (Q,R) INVENTORY MODELS WITH PARTIAL BACKORDERS, POISSON DEMANDS, AND EXPONENTIAL LEAD TIMES

## 1. INTRODUCTION

Cost and operation of inventory depend a great deal on what happens to demands when the system is out of stock. Most of the existing inventory models assume that when a stockout occurs all customers either wait as long as necessary or not at all for the replenishment to arrive. Both of these assumptions are, however, extreme and they do not represent many realistic situations where backorders are partial. Consequently, it seems more practical to assume that a portion of customers will never cancel their demands and the other portion will cancel their demands when the delivery is not immediate. This behavioral feature of potential customers is the basic premise of our models.

In this chapter, we consider a single item continuous-review inventory system where demands arrive one at a time according to a Poisson process with average demand rate $\lambda$ and lead times are independently and identically distributed exponential random variables with mean $1/\mu$. The inventory position is continuously monitored, and an order will be triggered by the system for a lot of size Q units whenever the position reaches the reorder point R. The cost of placing the order regardless of the quantity ordered is A. A fraction, $\beta$, of customers are willing to wait for backorders when there is a stockout, but the remaining fraction will go to other vendors. Furthermore, those who will wait are assumed to have limited patience and only willing to wait a certain period of time for satisfaction. After a certain number of outstanding orders have been placed by the system, all new demands will be lost and/or the management will not accept any new demand from customers, because the expected waiting time for backorders will then be longer than customers' tolerance limits. In other words, there are only a limited number, N, of orders allowed to be outstanding at any point in time. The cost of backlogging one unit of demand is $\hat{\Pi}$ per unit time and a fixed cost of $\Pi$ is also associated with each backorder. If a demand is lost, a fixed cost of L is incurred. Finally, the cost of carrying one unit of inventory in stock is H per unit time.

The general justification for utilizing such a model would be that the management feels that customers are only willing to wait for a certain period of time for backlogging, after which their loyalty will be placed in jeopardy. It would naturally be assumed that the loss of this loyalty is in excess of merely losing a single sale. Therefore, the management wishes to utilize the limited customer patience in allowing only a certain length of queue to form. For example, only a portion of new car buyers will wait for a certain brand and/or model of cars from the dealer and the rest will turn to other brands or models when a stockout occurs. When backorders accumulate to a certain level, however, all buyers will go somewhere else or the dealer will advise new customers to do so because the expected waiting time for replenishments would be too long. A detailed discussion of the practicality of models with partial backorders is provided by Peterson and Silver [1985, pp. 253-254].

(Q,R) inventory models with Poisson demands and exponential lead times have been studied by several authors. Galliher, Morse, and Simond [1959] investigate such a system with complete backorders and obtain rather complicated expressions for net inventory probabilities and some operating characteristics of the system. Gross and Harris [1973] consider the backordering case where the distribution of exponential lead time is allowed to depend on the level of unsatisfied demands. They model the system under two different assumptions about how lead times depend on the number of outstanding orders. Posner and Yansouni [1972] study an inventory system where, during a stockout period, all customers will wait for backlogging but then cancel their demands if waiting time is too long. By allowing at most one order outstanding at any point in time, they derive expressions for net inventory probabilities and the expected cost function. Buchanan and Love [1985] analyze the lost-sales case of Erlang-distributed lead times under the assumption that only one order can be outstanding at a given time.

They obtain closed-form expressions of the on-hand inventory probability and the cost function for the exponential lead time as a special case.

Inventory models which consider partial backordering have also been studied by several authors. Montgomery, Bazaraa, and Keswani [1973] analyze a continuous-review inventory system where a fraction of demands can be backlogged and the remaining fraction is lost during a stockout period. Both cases of deterministic and stochastic demands are examined, although the stochastic case is treated heuristically. Love [1979, Section 3-3] also builds a similar stochastic model by linearly combining his complete backordering and lost-sales models. Kim and Park [1985] consider a partial backordering inventory system where demand is stochastic and the backlogging cost is assumed to be proportional to the length of time for which the backorder exists. By assuming that lead time is constant, they derive the necessary equations from which optimal Q and R can be calculated iteratively. Recently, Moinzadeh [1989] treats the partial backordering case of Q = 1 with unit Poisson demands and constant lead times, and obtains expressions for net inventory probabilities. He further points out that his results are also valid for the case of exponential lead times. It appears that no exact (Q,R) model which considers partial backlogging has been reported in the literature except for the special case of Q = 1 mentioned above. This provides us with a strong motivation for our models.

In next section, we derive an analytical model exactly representing the partial backordering inventory system by solving a truncated stochastic process. A set of recursive expressions for the net inventory probability are obtained. Several properties of the model are established and proved in Section 3. Examples of the net inventory probabilities are also derived when the maximum number of outstanding orders is small. In the case where not just some but all customers will wait for backlogging before a certain number of orders are outstanding, this model also represents another inventory system where the interarrival times between customers have an Erlang distribution and an order is placed by the system whenever a unit is demanded. In Section 4, we consider this special case and the continuous-review (S-1,S) model with Erlang demands just mentioned. Some properties are presented with proof. The inventory cost function and optimal policy will be discussed in the last section. Since closed-form solutions for the inventory policy cannot be obtained there, some numerical experimentation will be conducted in Chapter 3 in order to examine the nature of this inventory system.

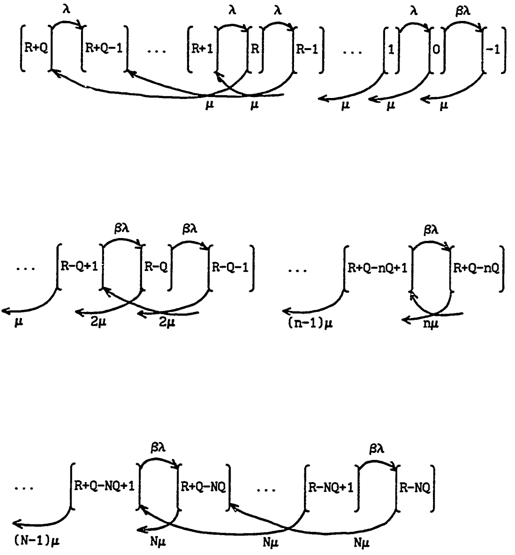
## 2. THE MODEL AND STEADY-STATE RESULTS

If we let X denote the net inventory level, then the set of possible ordering levels can be defined as that for which $X \in \{ R+Q-nQ \mid n=1,2,\ldots \}$. However, if N orders have been outstanding and $X = R+Q-(N+1)Q$, then all new demands will be lost and the last order will not be released until one of the outstanding orders arrives. Therefore, there are at most N orders outstanding at any point in time and $R-NQ \le X \le R+Q$. We also make the following assumptions:

1. Demands arrive one at a time according to a Poisson process with parameter $\lambda$.
2. Lead times are exponentially distributed with parameter $\mu$.
3. During a stockout period, a portion, $\beta$, of customers are willing to wait for backlogging, where $0 < \beta \le 1$.
4. Steady-state probabilities exist.
5. $Q > R > 0$.
6. The maximum number of outstanding orders N, $N \ge 1$, is a constant and prespecified by the inventory system.

The model can be formulated as a truncated Makovian process where the state is defined as the net inventory level. The rate diagram shown in **Figure 2.1** represents the state transition for this inventory process. Customer demands cause transitions to the right by one unit at rate $\lambda$ from state R+Q to 0 and at rate $\beta\lambda$ from state 0 until the inventory level reaches R-NQ, which is the last state. When the inventory level reaches the reorder point R+Q-nQ, a new order is triggered except at state R+Q-(N+1)Q, in which case the order will not be released until one currently outstanding order arrives. Arrival of an outstanding order causes a transition to the left by Q units at rate $n\mu$ if there are n orders outstanding. Transitions are therefore possible to the right when a demand occurs or to the left when an outstanding order arrives.

Figure 2.1. Rate diagram representing the partial backordering (Q,R) model



From Figure 2.1, the following system of Chapman-Kolmogorov difference equations representing the state transition can be obtained:

$$\lambda\ P(R+Q) = \mu\ P(R); \tag{2.1}$$

$$\lambda\ P(R+Q-j) = \lambda\ P[R+Q-(j-1)] + \mu\ P(R-j) \quad j=1,2,\ldots,Q-1; \tag{2.2}$$

$$(\lambda+\mu)\ P(R-j) = \lambda\ P[R-(j-1)] + 2\mu\ P(R-Q-j) \quad j=0,1,\ldots,R-1; \tag{2.3}$$

$$(\beta\lambda+\mu)\ P(0) = \lambda\ P(1) + 2\mu\ P(-Q); \tag{2.4}$$

$$(\beta\lambda+\mu)\ P(R-j) = \beta\lambda\ P[R-(j-1)] + 2\mu\ P(R-Q-j) \quad j=R+1,R+2,\ldots,Q-1; \tag{2.5}$$

$$(\beta\lambda+n\mu)\ P(R+Q-nQ-j) = \beta\lambda\ P[R+Q-nQ-(j-1)] + (n+1)\mu\ P(R-nQ-j) \quad j=0,1,\ldots,Q-1;\ n=2,3,\ldots,N-1; \tag{2.6}$$

$$(\beta\lambda+N\mu)\ P(R+Q-NQ) = \beta\lambda\ P(R+Q-NQ+1) + N\mu\ P(R-NQ); \tag{2.7}$$

$$(\beta\lambda+N\mu)\ P(R+Q-NQ-j) = \beta\lambda\ P[R+Q-NQ-(j-1)] \quad j=1,2,\ldots,Q-1; \tag{2.8}$$

$$N\mu\ P(R-NQ) = \beta\lambda\ P(R-NQ+1); \tag{2.9}$$

where P(.) = the steady-state probability of the associated net inventory level,

Q = the size of order quantity,

R = the reorder point in terms of inventory position,

$\lambda$ = the arrival rate of demands,

$\mu$ = the service rate of replenishments,

$\beta$ = the fraction of customers who will wait for backorders when a stockout occurs, and

N = the maximum number of outstanding orders allowed by the inventory system.

By using Equations (2.8) and (2.9), we can then derive that

$$P(R+Q-NQ-j) = \left[1+\frac{N}{\beta\rho}\right]^{Q-1-j}\left[\frac{N}{\beta\rho}\right] P(R-NQ) \qquad j=0,1,\ldots,Q-1; \qquad (2.10)$$

where $\rho$ is the utilization rate which equals $\lambda/\mu$. By (2.7) and (2.10),

$$P(R+Q-NQ+1) = \left[\left[1+\frac{N}{\beta\rho}\right]^{Q} - 1\right]\left[\frac{N}{\beta\rho}\right] P(R-NQ) . \qquad (2.11)$$

By (2.6),

$$P(R+Q-nQ-j) = \left[1+\frac{n}{\beta\rho}\right]^{Q-1-j} P(R-nQ+1)$$

$$- \left[\frac{n+1}{\beta\rho}\right] \sum_{i=j+1}^{Q-1} \left[1+\frac{n}{\beta\rho}\right]^{i-1-j} P(R-nQ-i) \qquad (2.12)$$

$$j=-1,0,\ldots,Q-2; \quad n=2,3,\ldots,N-1.$$

By (2.5),

$$P(R-j) = \left[1+\frac{1}{\beta\rho}\right]^{Q-1-j} P(R-Q+1) - \left[\frac{2}{\beta\rho}\right] \sum_{i=j+1}^{Q-1} \left[1+\frac{1}{\beta\rho}\right]^{i-1-j} P(R-Q-i)$$

$$j=R,R+1,\ldots,Q-2. \qquad (2.13)$$

By (2.4),

$$P(1) = \left[\beta+\frac{1}{\rho}\right] P(0) - \left[\frac{2}{\rho}\right] P(-Q) . \qquad (2.14)$$

By (2.3),

$$P(R-j) = \left[1+\frac{1}{\rho}\right]^{R-1-j} P(1) - \left[\frac{2}{\rho}\right] \sum_{i=j+1}^{R-1} \left[1+\frac{1}{\rho}\right]^{i-1-j} P(R-Q-i)$$

$$j=-1,0,\ldots,R-2. \qquad (2.15)$$

Finally, by (2.2),

$$P(R+Q-j) = P(R+1) - \left[\frac{1}{\rho}\right] \sum_{i=j+1}^{Q-1} P(R-i) \qquad j=0,1,\ldots,Q-2. \qquad (2.16)$$

Equations (2.10) to (2.16) are presented here only for the completeness of theoretical analysis, because it will be much easier to work through the state probabilities in the numerical experimentation by using Equations (2.1) to (2.9) than by evaluating the expressions in (2.10) to (2.16).

Note that so far we assume $Q > R > 0$ for deriving Equations (2.10) through (2.16). This assumption can be relaxed to just $Q > 1$. By adjusting the position of R, similar expressions as Equations (2.10) through (2.16) can be obtained for the case where $R \geq Q > 1$ and the case where $R \leq 0$ and $Q > 1$, repectively. Also note that we exclude the case of $\beta = 0$, because in that case all unfilled demands are lost and the setting of the model becomes different from that of backordering case. Under the assumption that $Q > R > 0$, at most one order can be outstanding at any time and the Markov process is truncated at the inventory state of zero in the lost-sales case. The system of balance equations becomes:

$$\lambda\ P(R+Q) = \mu\ P(R);$$

$$\lambda\ P(R+Q-j) = \lambda\ P[R+Q-(j-1)] + \mu\ P(R-j) \qquad j=1,2,\ldots,R;$$

$$\lambda\ P(R+Q-j) = \lambda\ P[R+Q-(j-1)] \qquad j=R+1,R+2,\ldots,Q-1;$$

$$(\lambda+\mu)\ P(R-j) = \lambda\ P[R-(j-1)] \qquad j=0,1,\ldots,R-1;$$

$$\mu\ P(0) = \lambda\ P(1);$$

where P(.) is the on-hand inventory probability. This complete lost-sales case with exponential lead times is treated by Buchanan and Love [1985] as a special case.

Equations (2.10) through (2.16) represent a set of recursive expressions for the steady-state net inventory probability in the partial backordering case. By working backwards, starting from P(R-NQ+1), P(R-NQ+2), ..., to P(R+Q), each of these probabilities can be expressed in terms of P(R-NQ). For instance, from (2.10) and (2.11), we have that

$$P(R+Q-NQ-j) = \left[1+\frac{N}{\beta\rho}\right]^{Q-1-j}\left[\frac{N}{\beta\rho}\right] P(R-NQ) \qquad j=0,1,\ldots,Q-1; \text{ and}$$

$$P(R+Q-NQ+1) = \left[\left[1+\frac{N}{\beta\rho}\right]^{Q} - 1\right]\left[\frac{N}{\beta\rho}\right] P(R-NQ) \;;$$

where $\rho = \lambda/\mu$. Now, from (2.12),

$$P[R+Q-(N-1)Q-j] = \left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j} P[R-(N-1)Q+1]$$

$$- \left[\frac{N}{\beta\rho}\right]\sum_{i=j+1}^{Q-1}\left[1+\frac{N-1}{\beta\rho}\right]^{i-1-j} P[R-(N-1)Q-i]$$

$$= \left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j}\left[\left[1+\frac{N}{\beta\rho}\right]^{Q} - 1\right]\left[\frac{N}{\beta\rho}\right] P(R-NQ)$$

$$- \left[\frac{N}{\beta\rho}\right]\sum_{i=j+1}^{Q-1}\left[1+\frac{N-1}{\beta\rho}\right]^{i-1-j}\left[1+\frac{N}{\beta\rho}\right]^{Q-1-i}\left[\frac{N}{\beta\rho}\right] P(R-NQ)$$

$$= \left[\left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N}{\beta\rho}\right]^{Q} + (N-1)\left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j}\right.$$

$$\left. - N\left[1+\frac{N}{\beta\rho}\right]^{Q-1-j}\right]\left[\frac{N}{\beta\rho}\right] P(R-NQ) \qquad j=-1,0,\ldots,Q-1;$$

$$
\begin{aligned}
P[R+Q-(N-2)Q-j] &= \left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j} P[R-(N-2)Q+1] \\
&\quad - \left[\frac{N-1}{\beta\rho}\right] \sum_{i=j+1}^{Q-1} \left[1+\frac{N-2}{\beta\rho}\right]^{i-1-j} P[R-(N-2)Q-i] \\
&= \left[\frac{N}{\beta\rho}\right] P(R-NQ) \Bigg[ \left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-1}{\beta\rho}\right]^{Q}\left[1+\frac{N}{\beta\rho}\right]^{Q} \\
&\quad + (N-1)\left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-1}{\beta\rho}\right]^{Q} - \left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N}{\beta\rho}\right]^{Q} \\
&\quad - (N-1)\left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N}{\beta\rho}\right]^{Q} + \frac{(N-1)(N-2)}{2}\left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j} \\
&\quad - (N-1)^{2}\left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j} + \frac{N(N-1)}{2}\left[1+\frac{N}{\beta\rho}\right]^{Q-1-j} \Bigg] \\
&\qquad j=-1,0,\ldots,Q-1;
\end{aligned}
$$

$$
\begin{aligned}
P[R+Q-(N-3)Q-j] &= \left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j} P[R-(N-3)Q+1] \\
&\quad - \left[\frac{N-2}{\beta\rho}\right] \sum_{i=j+1}^{Q-1} \left[1+\frac{N-3}{\beta\rho}\right]^{i-1-j} P[R-(N-3)Q-i] \\
&= \Bigg[ \left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-2}{\beta\rho}\right]^{Q}\left[1+\frac{N-1}{\beta\rho}\right]^{Q}\left[1+\frac{N}{\beta\rho}\right]^{Q} \\
&\quad + (N-1)\left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-2}{\beta\rho}\right]^{Q}\left[1+\frac{N-1}{\beta\rho}\right]^{Q} \\
&\quad - \left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-2}{\beta\rho}\right]^{Q}\left[1+\frac{N}{\beta\rho}\right]^{Q} \\
&\quad - \left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-1}{\beta\rho}\right]^{Q}\left[1+\frac{N}{\beta\rho}\right]^{Q}
\end{aligned}
$$

$$- (N-2)\left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-1}{\beta\rho}\right]^{Q}\left[1+\frac{N}{\beta\rho}\right]^{Q}$$

$$+\frac{(N-1)(N-2)}{2}\left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-2}{\beta\rho}\right]^{Q}$$

$$- (N-1)\left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-1}{\beta\rho}\right]^{Q} + \left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N}{\beta\rho}\right]^{Q}$$

$$- (N-1)(N-2)\left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N-1}{\beta\rho}\right]^{Q}$$

$$+ (N-2)\left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N}{\beta\rho}\right]^{Q}$$

$$+\frac{(N-1)(N-2)}{2}\left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{N}{\beta\rho}\right]^{Q}$$

$$+\frac{(N-1)(N-2)(N-3)}{6}\left[1+\frac{N-3}{\beta\rho}\right]^{Q-1-j}$$

$$-\frac{(N-1)(N-2)^2}{2}\left[1+\frac{N-2}{\beta\rho}\right]^{Q-1-j}+\frac{(N-1)^2(N-2)}{2}\left[1+\frac{N-1}{\beta\rho}\right]^{Q-1-j}$$

$$\left.-\frac{N(N-1)(N-2)}{6}\left[1+\frac{N}{\beta\rho}\right]^{Q-1-j}\right]\left[\frac{N}{\beta\rho}\right]P(R-NQ) \qquad j=-1,0,\ldots,Q-1;$$

..., and so on until the expression of P(R+Q) is derived. Due to normalization, P(R-NQ) can then be solved and so can other probabilities. The net inventory probabilities for the cases when N, the maximum number of outstanding orders, equals one and two will be obtained respectively in the next section.

## 3. SOME PROPERTIES OF THE MODEL

By applying Equations (2.10) through (2.16), the following property of the net inventory probability distribution can be derived.

**Proposition 2.1.**

$$\sum_{j=0}^{Q-1} P(R+Q-nQ-j) = \frac{\beta\rho}{n} P(R+Q-nQ+1) \qquad n=2,3,\ldots,N; \qquad (2.17)$$

$$\sum_{j=0}^{Q-1} P(R+Q-Q-j) = \rho\, P(R+Q-Q+1)\ ; \qquad (2.18)$$

where $\rho = \lambda/\mu$ and $\beta$ is the portion of customers who will wait for backorders.

**Proof:**

From Equations (2.10) and (2.11), it follows directly that

$$\sum_{j=0}^{Q-1} P(R+Q-NQ-j) = \frac{\beta\rho}{N} P(R+Q-NQ+1).$$

By using (2.12), for n=2,3,...,N-1,

$$\sum_{j=0}^{Q-1} P(R+Q-nQ-j) = P(R-nQ+1) + \sum_{j=0}^{Q-2} \left[ \left[1+\frac{n}{\beta\rho}\right]^{Q-1-j} P(R-nQ+1) - \left[\frac{n+1}{\beta\rho}\right] \sum_{i=j+1}^{Q-1} \left[1+\frac{n}{\beta\rho}\right]^{i-1-j} P(R-nQ-i) \right]$$

$$= P(R-nQ+1) \sum_{j=0}^{Q-1} \left[1+\frac{n}{\beta\rho}\right]^{Q-1-j} - \left[\frac{n+1}{\beta\rho}\right] \sum_{i=1}^{Q-1} P(R-nQ-i) \sum_{j=0}^{i-1} \left[1+\frac{n}{\beta\rho}\right]^{i-1-j}$$

$$= \left[\frac{\beta\rho}{n}\right] \left[\left[1+ \frac{n}{\beta\rho}\right]^{Q} - 1\right] P(R-nQ+1)$$

$$- \left[\frac{n+1}{\beta\rho}\right] \sum_{i=1}^{Q-1} \left[\frac{\beta\rho}{n}\right] \left[\left[1+ \frac{n}{\beta\rho}\right]^{i} - 1\right] P(R-nQ-i)$$

$$= \left[\frac{\beta\rho}{n}\right] \left[ \left[1+ \frac{n}{\beta\rho}\right]^{Q} P(R-nQ+1) - \left[\frac{n+1}{\beta\rho}\right] \sum_{i=0}^{Q-1} \left[1+ \frac{n}{\beta\rho}\right]^{i} P(R-nQ-i) \right]$$

$$- \left[\frac{\beta\rho}{n}\right] \left[ P(R-nQ+1) - \left[\frac{n+1}{\beta\rho}\right] \sum_{i=0}^{Q-1} P(R-nQ-i) \right]$$

$$= \left[\frac{\beta\rho}{n}\right] P(R+Q-nQ+1);$$

where the second to last step follows by simultaneously subtracting and adding one term of P(R-nQ), and the last step holds if

$$\sum_{j=0}^{Q-1} P[R+Q-(n+1)Q-j] = \left[\frac{\beta\rho}{n+1}\right] P[R+Q-(n+1)Q+1].$$

Therefore, we have proved Equation (2.17) by mathematical induction. Also note that, from (2.16) and (2.1),

$$P(R+Q) = P(R+1) - \left[\frac{1}{\rho}\right] \sum_{i=1}^{Q-1} P(R-i) \text{ and } P(R+Q) = \left[\frac{1}{\rho}\right] P(R).$$

This implies that

$$P(R+1) = \left[\frac{1}{\rho}\right] \sum_{i=0}^{Q-1} P(R-i),$$

which proves Equation (2.18). Equation (2.18) can also be proved by applying Equations (2.13), (2.14), and (2.15).

The interpretation of **Proposition 2.1** is that the sum of the net inventory probability set where there are n outstanding orders equals a constant times the probability of the last state of previous set, where the constant is equal to $\beta\rho/n$ for n=2,3,...,N and $\rho$ for n = 1. This property will be useful later in deriving the state probability of R-NQ. Another implication of **Proposition 2.1** is that the probability of n outstanding orders is equal to $\beta\rho/n$ times the probability of the last state when there are n-1 outstanding orders for n=2,3,...,N-1 and is equal to $\rho$ P(R+1) for n = 1. Because there are also N orders outstanding at the state of R-NQ, the probability of N outstanding orders equals the sum of $(\beta\rho/N)$ P(R+Q-NQ+1) and P(R-NQ). This leads us to:

**Proposition 2.2.**

$$E[M] = \rho\, P(R+1) + \beta\rho \sum_{n=2}^{N+1} P(R+Q-nQ+1) \;;$$

where E[M] is the expected number of outstanding orders, $\rho = \lambda/\mu$, $\beta$ is the fraction of customers who wait for backlogging, and N is the maximum number of outstanding orders.

**Proof:**

$$E[M] = \sum_{n=1}^{N} n\, \text{Prob( n orders outstanding )}$$

$$= \sum_{j=0}^{Q-1} P(R+Q-Q-j) + \sum_{n=2}^{N} n \sum_{j=0}^{Q-1} P(R+Q-nQ-j) + N\, P(R-NQ)$$

$$= \rho\, P(R+1) + \sum_{n=2}^{N} \beta\rho\, P(R+Q-nQ+1) + \beta\rho\, P(R-NQ+1)$$

$$= \rho\, P(R+1) + \beta\rho \sum_{n=2}^{N+1} P(R+Q-nQ+1);$$

where we apply Equations (2.17), (2.18), and (2.9) for the second to last step.

We are now ready for deriving some examples of the net inventory probability. First, when the maximum number of outstanding orders, N, equals one, the state probabilities of net inventory can be obtained from Equations (2.10) to (2.16) as follows.

$$P(R-j) = \left[1+\frac{1}{\beta\rho}\right]^{Q-1-j}\left[\frac{1}{\beta\rho}\right] P(R-Q) \qquad j=R,R+1,\ldots,Q-1;$$

$$P(R-j) = \left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[\frac{1}{\rho}\right] P(R-Q) \qquad j=0,1,\ldots,R-1;$$

$$P(R+Q-j) = P(R+1) - \left[\frac{1}{\rho}\right]\sum_{i=j+1}^{Q-1} P(R-i)$$

$$= \left[\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R} - \left[1+\frac{1}{\beta\rho}\right]^{Q-1-j}\right]\left[\frac{1}{\rho}\right] P(R-Q)$$

$$j=R-1,R,\ldots,Q-1;$$

$$P(R+Q-j) = P(R+1) - \left[\frac{1}{\rho}\right]\sum_{i=j+1}^{R-1} P(R-i)$$

$$= \left[\frac{1}{\rho}\right] P(R-Q) \left[\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R} - \left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\right] \qquad j=0,1,\ldots,R-1.$$

Now, by the result from **Proposition 2.1**,

$$\sum_{X=R-Q}^{R+Q} P(X) = P(R-Q) + \sum_{j=0}^{Q-1} P(R-j) + \sum_{j=0}^{Q-1} P(R+Q-j)$$

$$= P(R-Q) + \rho\, P(R+1) + \sum_{j=0}^{Q-1} P(R+Q-j)$$

$$= P(R-Q)\left[\beta + (1-\beta)\left[1+\frac{1}{\beta\rho}\right]^{Q-R} + \frac{Q}{\rho}\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\right];$$

therefore, because of normalization,

$$P(R-Q) = \left[\beta + (1-\beta)\left[1+\frac{1}{\beta\rho}\right]^{Q-R} + \frac{Q}{\rho}\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\right]^{-1}.$$

Next, when N = 2, we can obtain from (2.10) to (2.16) that

$$P(R-Q-j) = \left[1+\frac{2}{\beta\rho}\right]^{Q-1-j}\left[\frac{2}{\beta\rho}\right] P(R-2Q) \qquad j=0,1,\ldots,Q-1;$$

$$P(R-j) = \left[\left[1+\frac{1}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{2}{\beta\rho}\right]^{Q} + \left[1+\frac{1}{\beta\rho}\right]^{Q-1-j}\right.$$

$$\left. - 2\left[1+\frac{2}{\beta\rho}\right]^{Q-1-j}\right]\left[\frac{2}{\beta\rho}\right] P(R-2Q) \qquad j=R,R+1,\ldots,Q-1;$$

$$P(R-j) = \left[\left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q}\right.$$

$$+ \left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{1}{\beta\rho}\right]^{Q-R} - \frac{2}{2-\beta}\left[1+\frac{2}{\beta\rho}\right]^{Q-1-j}$$

$$\left. - \frac{2-2\beta}{2-\beta}\left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{2}{\beta\rho}\right]^{Q-R}\right]\left[\frac{2}{\rho}\right] P(R-2Q)$$

$$j=-1,0,\ldots,R-1;$$

$$P(R+Q-j) = \Bigg[ \left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}$$

$$- \frac{\beta}{2-\beta}\left[1+\frac{2}{\beta\rho}\right]^{Q} - \frac{2-2\beta}{2-\beta}\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{2}{\beta\rho}\right]^{Q-R}$$

$$- \left[1+\frac{1}{\beta\rho}\right]^{Q-1-j}\left[1+\frac{2}{\beta\rho}\right]^{Q} - \left[1+\frac{1}{\beta\rho}\right]^{Q-1-j}$$

$$+ \left[1+\frac{2}{\beta\rho}\right]^{Q-1-j} \Bigg] \left[\frac{2}{\rho}\right] P(R-2Q) \qquad j=R-1, R, \ldots, Q-1;$$

$$P(R+Q-j) = \Bigg[ \left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}$$

$$- \frac{\beta}{2-\beta}\left[1+\frac{2}{\beta\rho}\right]^{Q} - \frac{2-2\beta}{2-\beta}\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{2}{\beta\rho}\right]^{Q-R}$$

$$- \left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q}$$

$$- \left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{1}{\beta\rho}\right]^{Q-R} + \frac{\beta}{2-\beta}\left[1+\frac{2}{\beta\rho}\right]^{Q-1-j}$$

$$+ \frac{2-2\beta}{2-\beta}\left[1+\frac{1}{\rho}\right]^{R-1-j}\left[1+\frac{2}{\beta\rho}\right]^{Q-R} \Bigg] \left[\frac{2}{\rho}\right] P(R-2Q)$$

$$j=0, 1, \ldots, R-1.$$

Now,

$$\sum_{X=R-2Q}^{R+Q} P(X) = P(R-2Q) + \sum_{j=0}^{Q-1} P(R-Q-j) + \sum_{j=0}^{Q-1} P(R-j) + \sum_{j=0}^{Q-1} P(R+Q-j)$$

$$= P(R-2Q) + \frac{\beta\rho}{2} P(R-Q+1) + \rho\, P(R+1) + \sum_{j=0}^{Q-1} P(R+Q-j)$$

$$= P(R-2Q)\left\{\beta + (1-\beta)\left[2\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q} + 2\left[1+\frac{1}{\beta\rho}\right]^{Q-R} - 2\left[1+\frac{2}{\beta\rho}\right]^{Q-R} - \left[1+\frac{2}{\beta\rho}\right]^{Q}\right]\right.$$

$$\left. + \frac{2Q}{\rho}\left[\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R} - \frac{\beta}{2-\beta}\left[1+\frac{2}{\beta\rho}\right]^{Q} - \frac{2-2\beta}{2-\beta}\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{2}{\beta\rho}\right]^{Q-R}\right]\right\};$$

therefore,

$$P(R-2Q) = \left\{\beta + (1-\beta)\left[2\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q} + 2\left[1+\frac{1}{\beta\rho}\right]^{Q-R} - 2\left[1+\frac{2}{\beta\rho}\right]^{Q-R} - \left[1+\frac{2}{\beta\rho}\right]^{Q}\right]\right.$$

$$\left. + \frac{2Q}{\rho}\left[\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R}\left[1+\frac{2}{\beta\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{1}{\beta\rho}\right]^{Q-R} - \frac{\beta}{2-\beta}\left[1+\frac{2}{\beta\rho}\right]^{Q} - \frac{2-2\beta}{2-\beta}\left[1+\frac{1}{\rho}\right]^{R}\left[1+\frac{2}{\beta\rho}\right]^{Q-R}\right]\right\}^{-1}.$$

As we can see, the probability of the last state R-NQ decreases as the number of outstanding orders, N, increases. Therefore, when N is large enough, P(R-NQ) will approach zero and the effect of truncating the net inventory distribution becomes negligible. This will also be seen in the numerical results presented later in Chapter 3. Since the net inventory probability has been derived when N = 1 and when N = 2, the expected number of outstanding orders can then be obtained for these cases by Proposition 2.2.

When N = 1,

$$E[M] = \rho\, P(R+1) + \beta\rho\, P(R-Q+1)$$

$$= \left[1+ \frac{1}{\rho}\right]^{R}\left[1+ \frac{1}{\beta\rho}\right]^{Q-R} P(R-Q)\ .$$

When N = 2, the expected number of outstanding orders

$$E[M] = \rho\, P(R+1) + \beta\rho\, P(R-Q+1) + \beta\rho\, P(R-2Q+1)$$

$$= 2\left[\left[1+ \frac{1}{\rho}\right]^{R}\left[1+ \frac{1}{\beta\rho}\right]^{Q-R}\left[1+ \frac{2}{\beta\rho}\right]^{Q} + \left[1+ \frac{1}{\rho}\right]^{R}\left[1+ \frac{1}{\beta\rho}\right]^{Q-R}\right.$$

$$\left. - \frac{\beta}{2-\beta}\left[1+ \frac{2}{\beta\rho}\right]^{Q} - \frac{2-2\beta}{2-\beta}\left[1+ \frac{1}{\rho}\right]^{R}\left[1+ \frac{2}{\beta\rho}\right]^{Q-R}\right] P(R-2Q)\ .$$

Two degenerate cases respectively occur when R = 0 and when R = Q. In the case of N = 1, for example, if R = 0, then

$$P(-j) = \left[1+ \frac{1}{\beta\rho}\right]^{Q-1-j}\left[\frac{1}{\beta\rho}\right] P(-Q) \qquad j=0,1,\ldots,Q-1;$$

$$P(Q-j) = \left[\left[1+ \frac{1}{\beta\rho}\right]^{Q} - \left[1+ \frac{1}{\beta\rho}\right]^{Q-1-j}\right]\left[\frac{1}{\rho}\right] P(-Q) \qquad j=0,1,\ldots,Q-1;$$

$$P(-Q) = \left[\beta + \left[\frac{Q}{\rho} + 1 - \beta\right]\left[1+ \frac{1}{\beta\rho}\right]^{Q}\right]^{-1}.$$

When N = 1 and R = Q, the inventory will be truncated at state zero and the model is equivalent to the lost-sales one. The on-hand inventory probabilities become (see Equations (22) to (25) in Buchanan and Love [1985] for comparison):

$$P(Q-j) = \left[1+\frac{1}{\rho}\right]^{Q-1-j}\left[\frac{1}{\rho}\right] P(0) \qquad j=0,1,\ldots,Q-1;$$

$$P(2Q-j) = \left[\left[1+\frac{1}{\rho}\right]^{Q}-\left[1+\frac{1}{\rho}\right]^{Q-1-j}\right]\left[\frac{1}{\rho}\right] P(0) \qquad j=0,1,\ldots,Q-1;$$

$$P(0) = \left[1+\frac{Q}{\rho}\left[1+\frac{1}{\rho}\right]^{Q}\right]^{-1}.$$

# 4. THE SPECIAL CASE FOR $\beta = 1$

## 4.1. Steady-State Results

Sometimes, in real world, all customers of a certain inventory system will wait for backlogging but with limited patience when there is a stockout. For example, almost all new car owners are willing to wait for optional equipments and/or parts from the contracted dealer when a stockout occurs. But it is reasonable to assume that these customers have only limited patience so that they will all turn to other local auto parts shops or decide not to buy those optional equipments at all if the expected waiting time is longer than their tolerance limits. If this is the case, then the parameter $\beta$ equals 1.

When $\beta = 1$, Equations (2.10) through (2.16) from Section 2.1 can be simplified as the following:

$$P(R+Q-NQ-j) = \left[1+\frac{N}{\rho}\right]^{Q-1-j}\left[\frac{N}{\rho}\right] P(R-NQ) \qquad j=0,1,\ldots,Q-1;$$

$$P(R+Q-NQ+1) = \left[\left[1+\frac{N}{\rho}\right]^{Q}-1\right]\left[\frac{N}{\rho}\right] P(R-NQ) \;;$$

$$P(R+Q-nQ-j) = \left[1+\frac{n}{\rho}\right]^{Q-1-j} P(R-nQ+1) - \left[\frac{n+1}{\rho}\right]\sum_{i=j+1}^{Q-1}\left[1+\frac{n}{\rho}\right]^{i-1-j} P(R-nQ-i)$$

$$j=-1,0,\ldots,Q-2;\; n=1,2,\ldots,N-1;$$

$$P(R+Q-j) = P(R+1) - \left[\frac{1}{\rho}\right] \sum_{i=j+1}^{Q-1} P(R-i) \qquad j=0,1,\ldots,Q-2;$$

where $\rho = \lambda/\mu$ and N is the maximum number of outstanding orders placed by the inventory system. Equations (2.17) and (2.18) from **Proposition 2.1** can also be combined as

$$\sum_{j=0}^{Q-1} P(R+Q-nQ-j) = \frac{\rho}{n} P(R+Q-nQ+1) \qquad n=1,2,\ldots,N; \tag{2.19}$$

and **Proposition 2.2** becomes

$$E[M] = \rho \sum_{n=1}^{N+1} P(R+Q-nQ+1) \; , \tag{2.20}$$

where E[M] is the expected number of outstanding orders.

Now, when the maximum number of outstanding orders, N, equals one,

$$P(R-j) = \left[1+\frac{1}{\rho}\right]^{Q-1-j} \left[\frac{1}{\rho}\right] P(R-Q) \qquad j=0,1,\ldots,Q-1;$$

$$P(R+Q-j) = \left[ \left[1+\frac{1}{\rho}\right]^{Q} - \left[1+\frac{1}{\rho}\right]^{Q-1-j} \right] \left[\frac{1}{\rho}\right] P(R-Q)$$

$$j=0,1,\ldots,Q-1;$$

$$P(R-Q) = \left[ 1 + \frac{Q}{\rho} \left[1+\frac{1}{\rho}\right]^{Q} \right]^{-1} \; ; \text{ and}$$

$$E[M] = \left[1+\frac{1}{\rho}\right]^{Q} P(R-Q) \; .$$

When N = 2,

$$P(R-Q-j) = \left[1+\frac{2}{\rho}\right]^{Q-1-j} \left[\frac{2}{\rho}\right] P(R-2Q) \qquad j=0,1,\ldots,Q-1;$$

$$P(R-j) = \left[ \left[1+\frac{1}{\rho}\right]^{Q-1-j}\left[1+\frac{2}{\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{Q-1-j} - 2\left[1+\frac{2}{\rho}\right]^{Q-1-j} \right] \left[\frac{2}{\rho}\right] P(R-2Q) \qquad j=-1,0,\ldots,Q-1;$$

$$P(R+Q-j) = \left[ \left[1+\frac{1}{\rho}\right]^{Q}\left[1+\frac{2}{\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{Q} - \left[1+\frac{2}{\rho}\right]^{Q} - \left[1+\frac{1}{\rho}\right]^{Q-1-j}\left[1+\frac{2}{\rho}\right]^{Q} - \left[1+\frac{1}{\rho}\right]^{Q-1-j} + \left[1+\frac{2}{\rho}\right]^{Q-1-j} \right] \left[\frac{2}{\rho}\right] P(R-2Q) \qquad j=0,1,\ldots,Q-1;$$

$$P(R-2Q) = \left\{ 1 + \frac{2Q}{\rho} \left[ \left[1+\frac{1}{\rho}\right]^{Q}\left[1+\frac{2}{\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{Q} - \left[1+\frac{2}{\rho}\right]^{Q} \right] \right\}^{-1} \text{; and}$$

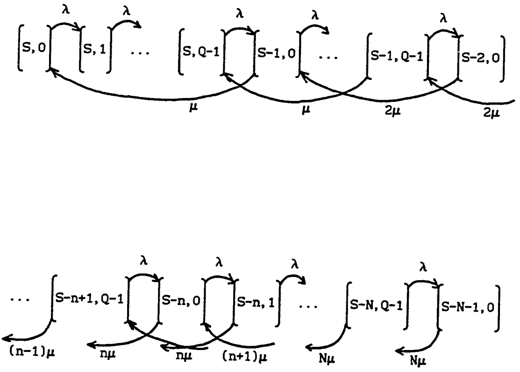
$$E[M] = 2\left[\left[1+\frac{1}{\rho}\right]^{Q}\left[1+\frac{2}{\rho}\right]^{Q} + \left[1+\frac{1}{\rho}\right]^{Q} - \left[1+\frac{2}{\rho}\right]^{Q}\right] P(R-2Q) \ .$$

## 4.2. (S-1,S) Models with Erlang Demands

In certain real world situations, it is either optimal or required to order units one at a time as demanded. This can be true, for example, if the demand is very low or the inventory item is very expensive. Now let us consider such an inventory system where the interarrival time between two successive customers follows an Erlang distribution with scale parameter $\lambda$ and phase parameter Q, and the lead time for replenishments is exponentially distributed with parameter $\mu$. The inventory position, which equals inventory on-hand plus on-order minus backorders, is continuously monitored and always kept at the level of S so that an order of size one will be triggered by the system whenever a unit is demanded. Furthermore, it is assumed that all customers are willing to wait for backlogging with limited patience when there is a stockout. After N orders have been outstanding, however, the management of this inventory system will accept only one more demand from customers in order to prevent the expected waiting time for backorders from being longer than customers' tolerance limits, but not release the last order until one of the outstanding orders arrives. In other words, there are only N orders allowed to be outstanding at any point in time.

This inventory model can also be represented by a queuing system where the interarrival time between customers has an Erlang distribution and the service time is an exponentially distributed random variable. Customers are served one at a time and there are N servers in the system. There is only one waiting space available, that is, customers will not enter the system when all servers have been occupied and one customer has already been waiting for services. The rate diagram representing the Markovian process for this inventory system is shown in **Figure 2.2**. The state (X,Y) is defined as the net inventory level X at current demand phase of Y. The existence of steady-state probabilities is also assumed.

Figure 2.2. Rate diagram representing the (S-1,S) model with Erlang demands



Note here that Q is the quantity of order size for the (Q,R) model but is the phase parameter of the Erlang demand distribution for the (S-1,S) model. With the following transformations for the net inventory probability, all results obtained in Section 4.1 are applicable to this new model:

$$P_{new}(S-n,j) = P(R+Q-nQ-j) \qquad j=0,1,\ldots Q-1;\ n=0,1,\ldots,N;$$

$$P_{new}(S-N-1,0) = P(R-NQ);$$

where P(X) = the steady-state probability of net inventory X for the backordering (Q,R) model considered previously,

$P_{new}(X,Y)$ = the steady-state probability of net inventory X when the demand process is at phase Y for the (S-1,S) model under consideration,

S = the maximum inventory level for the (S-1,S) model,

R = the reorder point for the (Q,R) model, and

N = the maximum number of outstanding orders allowed by both models.

From **Figure 2.2**, we can see that the net inventory probability for the (S-1,S) model is equal to the sum of probabilities of the same net inventory level at all different demand phases. In other words,

$$P_{new}(S-n) = \sum_{j=0}^{Q-1} P_{new}(S-n,j) \qquad n=0,1,\ldots,N;\ \text{and}$$

$$P_{new}(S-N-1) = P_{new}(S-N-1,0);$$

where $P_{new}(.)$ is the net inventory probability for the (S-1,S) model.

Hence, the following properties of the (S-1,S) model can easily be derived from Equations (2.19) and (2.20).

**Proposition 2.3.**

$$P_{new}(S-n) = \frac{\rho}{n} P_{new}(S-n+1,Q-1) \qquad n=1,2,\ldots,N;$$

where $\rho = \lambda/\mu$.

**Proof:**

For $n=1,2,\ldots,N$,

$$P_{new}(S-n) = \sum_{j=0}^{Q-1} P_{new}(S-n,j)$$

$$= \sum_{j=0}^{Q-1} P(R+Q-nQ-j)$$

$$= \frac{\rho}{n} P(R+Q-nQ+1)$$

$$= \frac{\rho}{n} P_{new}(S-n+1,Q-1) ,$$

where P(.) is the net inventory probability for the (Q,R) model.

Proposition 2.3 implies that, for $n=1,2,\ldots,N-1$, the probability of n orders outstanding for the (S-1,S) model is equal to $\rho/n$ times the probability of n-1 outstanding orders at the last demand phase. The probability that N orders are outstanding, however, equals the sum of $(\rho/N)\, P_{new}(S-N+1,Q-1)$ and $P_{new}(S-N-1,0)$, because there are also N orders outstanding when the net inventory is at the state of S-N-1. Now, from (2.20),

**Proposition 2.4.**

$$E[M] = \rho \sum_{n=0}^{N} P_{new}(S-n, Q-1) ,$$

where E[M] is the expected number of outstanding orders; and

$$E[I] = S - P_{new}(S-N-1, 0) - \rho \sum_{n=0}^{N} P_{new}(S-n, Q-1) ,$$

where E[I] is the expected net inventory level.

**Proof:**

From (2.20),

$$E[M] = \rho \sum_{n=1}^{N+1} P(R+Q-nQ+1)$$

$$= \rho \sum_{n=1}^{N+1} P_{new}(S-n+1, Q-1)$$

$$= \rho \sum_{n=0}^{N} P_{new}(S-n, Q-1) .$$

Now, the expected net inventory

$$E[I] = \sum_{n=0}^{N+1} (S-n)\, P_{new}(S-n)$$

$$= S \sum_{n=0}^{N+1} P_{new}(S-n) - \sum_{n=0}^{N+1} n\, P_{new}(S-n)$$

$$= S - \left[ \sum_{n=1}^{N} n\, P_{new}(S-n) + N\, P_{new}(S-N-1) \right] - P_{new}(S-N-1)$$

$$= S - P_{new}(S-N-1, 0) - \rho \sum_{n=0}^{N} P_{new}(S-n, Q-1) ;$$

where $\sum_{n=0}^{N+1} P_{new}(S-n) = 1$ and $\sum_{n=1}^{N} n\, P_{new}(S-n) + N\, P_{new}(S-N-1) = E[M]$.

The inventory position for this (S-1,S) model is S when the net inventory is at state S, S-1, ..., or S-N, and is S-1 when the net inventory level is S-N-1. Therefore, the expected inventory position is S - $P_{new}$(S-N-1). Since the order size is always one, the expected inventory on order is equal to the expected number of outstanding orders. By definition, the expected net inventory can also be derived by subtracting the expected inventory on order from the expected inventory position.

## 5. THE OPTIMAL INVENTORY POLICY

In order to locate the optimal order quantity and reorder point for the partial backordering (Q,R) model, we need to formulate the expected cost of operating this inventory system as a function of Q and R. Here, it is assumed that the purchase price of each inventory unit is independent of the order size, i.e., there are no quantity discounts offered by the supplier. Therefore, the purchasing cost becomes irrelevant in the inventory decision. In addition, let us define the following cost elements and other system parameters:

$C(Q,R)$ = the expected average inventory cost for the order size Q and reorder point R,

$A$ = fixed ordering cost per order placed,

$H$ = carrying cost per unit held in stock per unit time,

$\hat{\Pi}$ = backordering cost per unit to be backlogged per unit time,

$\Pi$ = fixed backordering cost per unit backlogged,

$L$ = fixed cost for each lost demand,

$\lambda$ = the mean arrival rate of demands,

$\mu$ = the mean service rate of replenishments,

$\beta$ = the fraction of customers who will wait for backorders when a stockout occurs, $0 < \beta \le 1$, and

$N$ = the maximum number of outstanding orders allowed by the inventory system, $N \ge 1$.

Note that, when the system is out of stock, $1-\beta$ portion of demands will be lost and the remaining portion can be backordered except at the last state R-NQ, where all customers will take their business elsewhere. Hence, during each unit of time, the following portion of demand will be lost:

$$\lambda \left[ (1-\beta) \sum_{X=R-NQ+1}^{0} P(X) + P(R-NQ) \right],$$

and the rest can either be satisfied or backlogged. Therefore, the number of orders placed per unit time is equal to

$$\frac{\lambda}{Q} \left[ 1 - (1-\beta) \sum_{X=R-NQ+1}^{0} P(X) - P(R-NQ) \right].$$

So, the expected inventory cost per unit time, which consists of ordering cost, carrying cost, linear and fixed backordering costs, and the cost of lost sales, can be formulated as:

$$C(Q,R) = \frac{A\lambda}{Q} \left[ 1 - (1-\beta) \sum_{X=R-NQ+1}^{0} P(X) - P(R-NQ) \right] + H \sum_{X=1}^{R+Q} X\, P(X)$$

$$+ \hat{\Pi} \sum_{X=R-NQ}^{-1} (-X)\, P(X) + \Pi\lambda\beta \sum_{X=R-NQ+1}^{0} P(X)$$

$$+ L\lambda \left[ (1-\beta) \sum_{X=R-NQ+1}^{0} P(X) + P(R-NQ) \right]$$

$$= \frac{A\lambda}{Q} + H \sum_{X=1}^{R+Q} X\, P(X) - \hat{\Pi} \sum_{X=R-NQ}^{-1} X\, P(X)$$

$$+ \lambda \left[ L - \frac{A}{Q} \right] \sum_{X=R-NQ}^{0} P(X) + \lambda\beta \left[ \frac{A}{Q} + \Pi - L \right] \sum_{X=R-NQ+1}^{0} P(X).$$

For the special case where $\beta = 1$, the cost function becomes:

$$C(Q,R) = \frac{A\lambda}{Q}[1 - P(R-NQ)] + H \sum_{X=1}^{R+Q} X\,P(X) + \hat{\Pi} \sum_{X=R-NQ}^{-1} (-X)\,P(X)$$

$$+ \Pi\lambda \sum_{X=R-NQ+1}^{0} P(X) + L\lambda\,P(R-NQ)$$

$$= \frac{A\lambda}{Q} + H \sum_{X=1}^{R+Q} X\,P(X) - \hat{\Pi} \sum_{X=R-NQ}^{-1} X\,P(X)$$

$$+ \Pi\lambda \sum_{X=R-NQ+1}^{0} P(X) + \lambda \left[L - \frac{A}{Q}\right] P(R-NQ) .$$

The steady-state net inventory probability for this model obtained in Section 2 can then be substituted into these cost functions. However, it appears impractical to prove analytically that the cost function is strictly convex with respect to Q and R for either the general or the special case. Furthermore, closed-form solutions of the optimal Q and R also cannot be obtained for either case. For each set of model parameters, a numerical search will then be needed for locating the optimal policy. Therefore, some numerical experimentation is required in order to examine the nature of this model and how it reacts to the change of system parameters. This will be explored in the next chapter.

# CHAPTER 3

## SENSITIVITY ANALYSIS FOR THE PARTIAL BACKORDERING MODEL

In Chapter 2 we derive several theoretical results for the partial backordering (Q,R) model with Poisson demands and exponential lead times. Due to the difficulty of obtaining closed-form solutions for the optimal inventory policy, some numerical experimentation is required to locate the optimal Q and R, and to examine how the model reacts with respect to system parameters. The design of experiments with a two-dimensional search procedure will be considered in the next section. In Section 2, numerical results will be reported for this partial backordering model followed by some discussion. Finally, our findings about this model will be summarized in Section 3.

## 1. EXPERIMENTAL DESIGN

Optimization of inventory policy is not difficult since the cost function has only two discrete decision variables. A simple two-dimensional search procedure is used to locate the optimal values of order quantity and reorder point. For each example, the value of R that minimizes inventory costs is found for any given Q, then the best value of Q which minimizes costs is found. Initially, the classic EOQ and mean lead-time demand are arbitrarily used as starting values for Q and R, respectively. Because strict convexity cannot be theoretically proved for the cost function, local optimality might exist. Therefore, the inventory cost is then examined on a small grid of feasible points centered around the best Q and R values found previously. If any improvement is detected, the search procedure will be restarted at the best of the grid points, otherwise the current Q and R are accepted as optimal.

Notice from Section 4.1 in Chapter 2 that, when $\beta = 1$, i.e., all unsatisfied demands will be backlogged when there are N or less orders outstanding, the probability distribution of net inventory level depends only upon the order quantity Q and is independent of the reorder point R. This provides an advantage in the search of optimal policy. The net inventory probability only has to be calculated once for each given Q, and the inventory cost can be evaluated by shifting the inventory level for different values of

R. Unfortunately, the property mentioned above does not hold true for the case where $0 < \beta < 1$. For partial backordering cases, the probability of net inventory must be calculated for each Q and R pair in order to evaluate the cost function.

Equations (2.10) through (2.16) from Chapter 2 represent a set of recursive expressions of net inventory probabilities, and can be used to calculate the probability distribution for each pair of Q and R. However, Equations (2.1) through (2.9) provide a simpler alternative way to compute the net inventory probability. If we define $P(R-NQ-j) = 0$ for $j=1,2,\ldots,Q-1$, then the following general relationships between probabilities can be derived from (2.1) through (2.9). When the net inventory level is R-NQ+1,

$$P(R-NQ+1) = \frac{N}{\beta\rho} P(R-NQ) \ ;$$

for other non-positive inventory states $R-NQ+2 \le X \le 0$,

$$P(X) = \left[1+ \frac{n}{\beta\rho}\right] P(X-1) - \frac{n+1}{\beta\rho} P(X-1-Q)$$

if X is among the group of states from R-nQ+2 to R+Q-nQ+1, except that

$$P(R+Q-NQ+1) = \left[1+ \frac{N}{\beta\rho}\right] P(R+Q-NQ) - \frac{N}{\beta\rho} P(R-NQ) \ ;$$

$$P(1) = \beta \left[\left[1+ \frac{n}{\beta\rho}\right] P(0) - \frac{n+1}{\beta\rho} P(-Q)\right] \quad \text{if } R-nQ+2 \le 1 \le R+Q-nQ+1;$$

for other positive inventory states $2 \le X \le R+Q$,

$$P(X) = \left[1 + \frac{n}{\rho}\right] P(X-1) - \frac{n+1}{\rho} P(X-1-Q)$$

if X is among the group of states from R-nQ+2 to R+Q-nQ+1; where $\rho$ is the ratio of mean demand rate $\lambda$ to mean rate of replenishments $\mu$, $\beta$ is the fraction of unfilled demands backlogged, and N is the maximum number of outstanding orders.

The net inventory probability distribution can be obtained more efficiently by this set of recursive expressions than by that of (2.10) to (2.16). First, each probability is expressed in terms of P(R-NQ), which is the probability of the last state. Then, P(R-NQ) can be calculated because of normalization and, hence, so can others. Note that we assume $Q > R > 0$ in order to derive (2.1) through (2.9). That assumption can be relaxed here to just $Q > 1$. As will be seen later in the numerical results, the value of R is allowed to be greater than or equal to Q or be negative in our examples.

With the probability distribution of net inventory level for each pair of Q and R, the expected average cost function can easily be evaluated for a given set of cost parameters. The following cost elements are used throughout the experimentation:

$A = 50$ \$/order,

$H = 1$ \$/unit in stock/unit time,

$\hat{\Pi} = 4$ \$/unit backordered/unit time,

$\Pi = 0$ \$/unit backordered, and

L = 3 \$/unit lost.

We first fix the mean lead time rate and let $(\lambda,\mu) = \{(100,8), (200,8), (400,8)\}$ to examine the effects of mean demand rate on the optimal inventory policy and cost. We then fix the mean demand rate and let $(\lambda,\mu) = \{(200,4), (200,2), (200,1)\}$ to see how lead time affects the optimal policy and cost.

We investigate the effect of the backordering fraction on the optimal solution by varying the value of $\beta$ as 1, 0.75, 0.5, and 0.25. The case of $\beta = 0$ is also considered for comparison purpose. The following cost function from Equation (29) of Buchanan and Love [1985] is used for the complete lost-sales case:

$$C(Q,R) = \frac{A\lambda}{Q} + \frac{1}{1 + Qs(1+s)^R}\left\{\left[L\lambda - \frac{A\lambda}{Q}\right]\right.$$
$$\left. + H\left[s(1+s)^R\left[QR + \frac{Q(Q+1)}{2}\right] - Q(1+s)^R + Q\right]\right\},$$

where $s = \mu/\lambda$. Finally, in order to see how the maximum number of outstanding orders affects the optimal decision, we let N = {1, 2, 3} in the cases where mean lead time rate equals 8 and N = {1, 2, 3, 4} in other cases except when $\beta = 0$, where only one outstanding order is allowed. Therefore, we totally examine 84 examples for our partial backlogging model and 6 examples for the complete lost-sales case. Numerical results of the experimentation will be presented in next section followed by some discussion.

## 2. NUMERICAL RESULTS AND DISCUSSION

Computer outputs of raw results are shown in Figure 3.1 to Figure 3.6. For each parameter set of mean demand rate $\lambda$, mean rate of replenishments $\mu$, and fraction backordered $\beta$, the optimal inventory policy and average cost are obtained for various numbers of outstanding orders. Several observations can then be offered. First of all, the optimal inventory cost decreases as the number of outstanding orders allowed increases. Because the cost of losing sales is high in our examples, allowing more outstanding orders will accommodate more demands and hence reduce the operating cost. Second, as mentioned in Section 3 of Chapter 2 the cumulative probability of the truncated tail is usually very small when N is large, and the effect of truncating the net inventory probability distribution becomes negligible as N, the number of outstanding orders allowed, increases. Also note that, for the set of cost parameters used in our examples, the effect of truncation can be ignored as long as demand during lead time, which equals the ratio of mean demand rate to mean lead time rate, is low. The effects of early truncation on both the optimal policy and cost are, however, significant when lead-time demand is relatively high. Table 3.1 shows the cost increase in percentages caused by allowing only one instead of a large number of orders to be outstanding.

Figure 3.1. Numerical results for the partial backordering model when $\lambda$ = 100 and $\mu$ = 8

```
THE MEAN DEMAND RATE IS     100
THE MEAN LEAD TIME RATE IS    8
ORDERING  COST  =   50 ($/ORDER)
CARRYING  COST  =    1 ($/UNIT/UNIT TIME)
BACKORDER COST1 =    4 ($/UNIT/UNIT TIME)
BACKORDER COST2 =    0 ($/UNIT BACKLOGGED)
LOST-SALE COST  =    3 ($/UNIT LOST)

THE FRACTION BACKORDERED IS  1.00

OUT. ORDERS        Q         R      TOTAL COST
     1           116       -11          93.14
     2           116       -11          93.13
     3           116       -11          93.13

THE FRACTION BACKORDERED IS  0.75

OUT. ORDERS        Q         R      TOTAL COST
     1           110         1        102.24
     2           110         1        102.23
     3           110         1        102.23

THE FRACTION BACKORDERED IS  0.50

OUT. ORDERS        Q         R      TOTAL COST
     1           109         6        107.30
     2           109         6        107.30
     3           109         6        107.30

THE FRACTION BACKORDERED IS  0.25

OUT. ORDERS        Q         R      TOTAL COST
     1           109        10        110.95
     2           109        10        110.95
     3           108        10        110.95

THE FRACTION BACKORDERED IS  0.00

OUT. ORDERS        Q         R      TOTAL COST
     1           109        12        113.81
```

**Figure 3.2.** Numerical results for the partial backordering model when $\lambda$ = 200 and $\mu$ = 8

```
THE MEAN DEMAND RATE IS      200
THE MEAN LEAD TIME RATE IS    8
ORDERING  COST   =    50 ($/ORDER)
CARRYING  COST   =     1 ($/UNIT/UNIT TIME)
BACKORDER COST1  =     4 ($/UNIT/UNIT TIME)
BACKORDER COST2  =     0 ($/UNIT BACKLOGGED)
LOST-SALE COST   =     3 ($/UNIT LOST)

THE FRACTION BACKORDERED IS  1.00

OUT. ORDERS        Q          R        TOTAL COST
     1            171        -10          136.43
     2            169         -9          136.29
     3            169         -9          136.29

THE FRACTION BACKORDERED IS  0.75

OUT. ORDERS        Q          R        TOTAL COST
     1            165          9          153.37
     2            164          9          153.35
     3            164          9          153.35

THE FRACTION BACKORDERED IS  0.50

OUT. ORDERS        Q          R        TOTAL COST
     1            163         19          163.27
     2            163         19          163.27
     3            163         19          163.27

THE FRACTION BACKORDERED IS  0.25

OUT. ORDERS        Q          R        TOTAL COST
     1            161         27          170.40
     2            161         27          170.40
     3            162         27          170.40

THE FRACTION BACKORDERED IS  0.00

OUT. ORDERS        Q          R        TOTAL COST
     1            161         32          176.01
```

**Figure 3.3.** Numerical results for the partial backordering model when $\lambda$ = 400 and $\mu$ = 8

```
THE MEAN DEMAND RATE IS     400
THE MEAN LEAD TIME RATE IS   8
ORDERING  COST  =   50 ($/ORDER)
CARRYING  COST  =    1 ($/UNIT/UNIT TIME)
BACKORDER COST1 =    4 ($/UNIT/UNIT TIME)
BACKORDER COST2 =    0 ($/UNIT BACKLOGGED)
LOST-SALE COST  =    3 ($/UNIT LOST)
```

THE FRACTION BACKORDERED IS 1.00

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 263 | -3 | 206.25 |
| 2 | 251 | -1 | 204.52 |
| 3 | 251 | -1 | 204.52 |

THE FRACTION BACKORDERED IS 0.75

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 251 | 32 | 238.73 |
| 2 | 247 | 33 | 238.30 |
| 3 | 247 | 33 | 238.30 |

THE FRACTION BACKORDERED IS 0.50

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 246 | 53 | 258.08 |
| 2 | 246 | 53 | 258.04 |
| 3 | 246 | 53 | 258.04 |

THE FRACTION BACKORDERED IS 0.25

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 245 | 67 | 272.14 |
| 2 | 245 | 67 | 272.14 |
| 3 | 245 | 67 | 272.14 |

THE FRACTION BACKORDERED IS 0.00

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 244 | 78 | 283.23 |

**Figure 3.4.** Numerical results for the partial backordering model when $\lambda$ = 200 and $\mu$ = 4

THE MEAN DEMAND RATE IS 200
THE MEAN LEAD TIME RATE IS 4
ORDERING COST = 50 ($/ORDER)
CARRYING COST = 1 ($/UNIT/UNIT TIME)
BACKORDER COST1 = 4 ($/UNIT/UNIT TIME)
BACKORDER COST2 = 0 ($/UNIT BACKLOGGED)
LOST-SALE COST = 3 ($/UNIT LOST)

THE FRACTION BACKORDERED IS 1.00

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 206 | 9 | 162.62 |
| 2 | 183 | 15 | 158.72 |
| 3 | 183 | 15 | 158.72 |
| 4 | 183 | 15 | 158.72 |

THE FRACTION BACKORDERED IS 0.75

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 193 | 27 | 176.55 |
| 2 | 184 | 29 | 175.17 |
| 3 | 182 | 30 | 175.17 |
| 4 | 184 | 29 | 175.17 |

THE FRACTION BACKORDERED IS 0.50

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 186 | 40 | 187.44 |
| 2 | 184 | 40 | 187.22 |
| 3 | 184 | 40 | 187.21 |
| 4 | 184 | 40 | 187.21 |

THE FRACTION BACKORDERED IS 0.25

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 182 | 50 | 196.67 |
| 2 | 182 | 50 | 196.67 |
| 3 | 183 | 50 | 196.67 |
| 4 | 182 | 50 | 196.67 |

THE FRACTION BACKORDERED IS 0.00

| OUT. ORDERS | Q | R | TOTAL COST |
|---|---|---|---|
| 1 | 180 | 58 | 204.73 |

Figure 3.5. Numerical results for the partial backordering model when $\lambda$ = 200 and $\mu$ = 2

```
THE MEAN DEMAND RATE IS      200
THE MEAN LEAD TIME RATE IS    2
ORDERING  COST  =    50 ($/ORDER)
CARRYING  COST  =     1 ($/UNIT/UNIT TIME)
BACKORDER COST1 =     4 ($/UNIT/UNIT TIME)
BACKORDER COST2 =     0 ($/UNIT BACKLOGGED)
LOST-SALE COST  =     3 ($/UNIT LOST)

THE FRACTION BACKORDERED IS  1.00

OUT. ORDERS       Q        R      TOTAL COST
     1          267       60        232.64
     2          155       97        200.73
     3          141      101        198.38
     4          141      101        198.34

THE FRACTION BACKORDERED IS  0.75

OUT. ORDERS       Q        R      TOTAL COST
     1          253       69        236.50
     2          138      116        213.83
     3          129      118        212.58
     4          129      118        212.57

THE FRACTION BACKORDERED IS  0.50

OUT. ORDERS       Q        R      TOTAL COST
     1          239       78        240.72
     2          127      127        222.34
     3          126      126        222.05
     4          126      126        222.04

THE FRACTION BACKORDERED IS  0.25

OUT. ORDERS       Q        R      TOTAL COST
     1          227       87        245.58
     2          128      130        230.09
     3          128      130        230.08
     4          128      130        230.08

THE FRACTION BACKORDERED IS  0.00

OUT. ORDERS       Q        R      TOTAL COST
     1          216       96        251.37
```

**Figure 3.6.** Numerical results for the partial backordering model when $\lambda$ = 200 and $\mu$ = 1

```
THE MEAN DEMAND RATE IS     200
THE MEAN LEAD TIME RATE IS   1
ORDERING  COST  =    50 ($/ORDER)
CARRYING  COST  =     1 ($/UNIT/UNIT TIME)
BACKORDER COST1 =     4 ($/UNIT/UNIT TIME)
BACKORDER COST2 =     0 ($/UNIT BACKLOGGED)
LOST-SALE COST  =     3 ($/UNIT LOST)

THE FRACTION BACKORDERED IS  1.00

OUT. ORDERS       Q         R        TOTAL COST
      1          208       208         325.13
      2          176       215         268.55
      3          134       228         249.44
      4          121       231         245.41

THE FRACTION BACKORDERED IS  0.75

OUT. ORDERS       Q         R        TOTAL COST
      1          208       208         325.13
      2          174       212         269.21
      3          123       231         254.06
      4          116       232         251.27

THE FRACTION BACKORDERED IS  0.50

OUT. ORDERS       Q         R        TOTAL COST
      1          208       208         325.13
      2          170       209         270.17
      3          116       232         257.79
      4          115       230         256.89

THE FRACTION BACKORDERED IS  0.25

OUT. ORDERS       Q         R        TOTAL COST
      1          208       208         325.13
      2          165       206         271.86
      3          115       230         262.25
      4          115       230         262.21

THE FRACTION BACKORDERED IS  0.00

OUT. ORDERS       Q         R        TOTAL COST
      1          276       141         316.45
```

Table 3.1. Cost increase when only one instead of a large number of outstanding orders are allowed*

| $\lambda$ | 100 | 200 | 400 | 200 | 200 | 200 |
|---|---|---|---|---|---|---|
| $\mu$ | 8 | 8 | 8 | 4 | 2 | 1 |
| $\beta$ | Cost increase (in percentages) | | | | | |
| 1 | 0.01 | 0.10 | 0.85 | 2.46 | 17.29 | 32.48 |
| 0.75 | 0.01 | 0.01 | 0.18 | 0.79 | 11.26 | 29.39 |
| 0.5 | 0.00 | 0.00 | 0.02 | 0.12 | 8.41 | 26.56 |
| 0.25 | 0.00 | 0.00 | 0.00 | 0.00 | 6.74 | 24.00 |

* $\lambda$ is the mean demand rate, $\mu$ is the mean lead time rate, and $\beta$ is the fraction of unfilled demands which can be backlogged.

Table 3.2. Cost increase when the optimal policy of N = 1 is used in the case of large N

| $\lambda$ | 100 | 200 | 400 | 200 | 200 | 200 |
|---|---|---|---|---|---|---|
| $\mu$ | 8 | 8 | 8 | 4 | 2 | 1 |
| $\beta$ | Cost increase (in percentages) | | | | | |
| 1 | 0.00 | 0.00 | 0.10 | 0.48 | 8.38 | 7.43 |
| 0.75 | 0.00 | 0.00 | 0.01 | 0.09 | 6.54 | 7.09 |
| 0.5 | 0.00 | 0.00 | 0.00 | 0.01 | 6.65 | 6.69 |
| 0.25 | 0.00 | 0.00 | 0.00 | 0.00 | 6.59 | 6.34 |

For example, when $(\lambda,\mu,\beta) = (200,1,1)$ (see Figure 3.6), the optimal average inventory costs when N = 1 and when N = 4 are 325.13 and 245.41, respectively. The cost increase, 32.48%, is calculated by dividing the difference between the two costs by the latter. As can also be seen from Figures 3.5 and 3.6, both the optimal policy and the cost when N = 1 differ significantly from those of large N in the cases where mean lead-time demand, which equals the ratio of mean demand rate $\lambda$ to mean lead time rate $\mu$, is greater than or equal to 100. This can be explained by the fact that penalty for each lost demand is usually higher than that for each demand backlogged or for each unit held in stock, as it is so in our examples. In the case of $\beta > 0$, some unsatisfied demands will be backordered if possible. Also note that any new demands arriving after N orders have been outstanding will be lost. So, when the demand during lead time is high, restricting the number of outstanding orders to one will require larger ordering quantity to reduce the frequency of stockouts and to accommodate more unsatisfied demands, and lower reorder level to compensate the high carrying cost caused by large Q. Overall, this will result in a higher maximum inventory level, which equals the sum of Q and R, and higher total cost.

Also note that the difference of optimal solutions between the cases of N = 1 and large N becomes smaller as $\beta$ approches zero. Because, in the case of small $\beta$, most unsatisfied demands are already lost, early truncation of net inventory distribution would have less impact on the situation than in the case of large $\beta$. Therefore, if possible, the management of a real world inventory system should allow at least two or three orders to be outstanding when demand during lead time is relatively high and/or most unsatisfied demands can be backlogged.

Now, when $\beta > 0$, it might be costly to use the optimal policy obtained by assuming only one outstanding order in the situation where actually more than one order is allowed to be outstanding. **Table 3.2** shows the cost increase in percentages when the optimal policy of $N = 1$ is applied to the case of large N. For example, in the case of $(\lambda,\mu) = (200,2)$ and $\beta = 0.25$ (see **Figure 3.5**), the policy of $(Q,R) = (227,87)$ is used for $N = 4$ and the inventory cost is calculated as 245.25. The difference between this cost and the optimal cost of $N = 4$, 230.08, is then divided by the latter to yield the percentage cost increase.

As we can see, the penalty of applying optimal policy of $N = 1$ to the system where more than one order is allowed to be outstanding becomes severe when lead-time demand is greater than or equal to 100. Most approximate models and some exact models in the literature allow only one order to be outstanding in the complete backordering case, and could hence yield biased results when applied in the system with moderate or high demand during lead time. However, the optimal policy of lost-sales case in our examples always satisfies the condition of $Q > R \geq 0$, and it is not possible to have more than one order outstanding under this condition when all unsatisfied demands will be lost during the stockout period. Therefore, it should be safe to assume that at most one order can be outstanding for the complete lost-sales case, as almost all authors do.

Table 3.3. Summarized results for the partial backordering model when mean lead time rate is fixed*

| λ | 100 | | | 200 | | | 400 | | |
|---|---|---|---|---|---|---|---|---|---|
| μ | 8 | | | 8 | | | 8 | | |
| β | Q | R | TC | Q | R | TC | Q | R | TC |
| 1 | 116 | -11 | 93.13 | 169 | -9 | 136.29 | 251 | -1 | 204.52 |
| 0.75 | 110 | 1 | 102.23 | 164 | 9 | 153.35 | 247 | 33 | 238.30 |
| 0.5 | 109 | 6 | 107.30 | 163 | 19 | 163.27 | 246 | 53 | 258.04 |
| 0.25 | 108 | 10 | 110.95 | 162 | 27 | 170.40 | 245 | 67 | 272.14 |
| 0 | 109 | 12 | 113.81 | 161 | 32 | 176.01 | 244 | 78 | 283.23 |

* λ is the mean demand rate, μ is the mean lead time rate, and β is the fraction of unfilled demands which can be backlogged.

Table 3.4. Summarized results for the partial backordering model when mean demand rate is fixed

| λ | 200 | | | 200 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|
| μ | 4 | | | 2 | | | 1 | | |
| β | Q | R | TC | Q | R | TC | Q | R | TC |
| 1 | 183 | 15 | 158.72 | 141 | 101 | 198.34 | 121 | 231 | 245.41 |
| 0.75 | 184 | 29 | 175.17 | 129 | 118 | 212.57 | 116 | 232 | 251.27 |
| 0.5 | 184 | 40 | 187.21 | 126 | 126 | 222.04 | 115 | 230 | 256.89 |
| 0.25 | 182 | 50 | 196.67 | 128 | 130 | 230.08 | 115 | 230 | 262.21 |
| 0 | 180 | 58 | 204.73 | 216 | 96 | 251.37 | 276 | 141 | 316.45 |

Table 3.3 and Table 3.4 respectively summarize the results when mean lead time rate is fixed and when mean demand rate is fixed, where the optimal policy and average inventory cost of large N are used for cases of $0 < \beta$ to eliminate the effect of truncation. Some observations can then be offered. First, for a given mean lead time rate, the optimal values of both Q and R as well as inventory cost for each case of $\beta$ monotonically increase as mean demand rate increases. This is because that increasing demand in an inventory system will require larger order quantity to keep the frequency of placing orders low and higher reorder level to prevent the system from being out of stock during the lead time, and will result in higher inventory costs. Note here that, although the total inventory cost increases, the average cost of each demand actually becomes lower when there are more demands. This might be explained by the effect of "economy of scale."

Next, when demand rate is fixed, the optimal reorder level becomes higher in order to accommodate more demands during lead time as lead time increases. The maximum inventory level and average cost also rise when lead time becomes longer. Since lead times of replenishments are somewhat controllable to an inventory system, they should be shortened as much as possible to reduce the operating cost. The management can design the operation according to the concept of "Just In Time" and eliminate the lead time if at all possible.

Finally, when $\beta > 0$, the optimal value of Q slightly decreases but remains in the same magnitude while the optimal value of R increses as $\beta$ decreases for each set of $\lambda$ and $\mu$. The maximum inventory level and average cost, however, always increase as $\beta$ becomes smaller for all values of $\beta$ including zero. When $\beta$ is getting smaller, more unsatisfied demands will be lost if the system is out of stock. Therefore, the inventory policy should have a higher inventory level to protect the system from stockouts, and will result in higher inventory costs.

## 3. CONCLUSION

As summary of the numerical result, we can conclude that, first, the effect of truncation on both the optimal policy and inventory cost will be negligible if lead-time demand is low and/or the number of outstanding orders allowed is large. Therefore, when demand during lead time is relatively high, the system should allow at least two or three orders to be outstanding in order to reduce the operating cost. Furthermore, the cost penalty caused by early truncation becomes smaller as the fraction of unsatisfied demands backlogged decreases. Allowing only one outstanding order would therefore have less impact on the inventory cost when more unsatisfied demands during the stockout period will be lost. Second, when lead-time demand is high, it could be quite costly in the backordering case to apply the optimal policy obtained by assuming only one outstanding order to the system where actually a large number of orders can be outstanding. It appears, however, that the assumption of only one order outstanding at any point in time might be legitimate for the lost-sales case.

Third, both of the maximum stock level and the reorder level as well as the inventory cost will increase if either demand rate or lead time increases. However, for a fixed lead time, the average cost of each demand becomes lower as demand rate increases. Since lead times are more controllable than demands, the management should seek ways to reduce the length of lead time as much as possible. Next, the inventory level and operating cost become monotonically higher when the fraction of unsatisfied demands backlogged is getting smaller. Therefore, it is obviously preferable for the system to accommodate as many demands as possible during the stockout period if penalty for losing sales is high.

Finally, although the inventory cost function of the partial backordering case is not theoretically proved to be convex, it behaves fairly well around the optimal values of Q and R. During our experimentation, local optimality is detected in only two cases, when $(\lambda,\mu,\beta) = (200,2,0.25)$ and $(200,1,0.25)$, respectively. Furthermore, only two local optimal points are found in these cases and they are both far away from the global optima. The cost function is also quite insensitive to the values of both Q and R when they approach the optima, i.e., the surface of inventory cost is very flat around the optimal Q and R values. This is generally the case for various inventory models and has been reported by other authors [for example, Zheng 1989].

# CHAPTER 4

# (Q,R) INVENTORY MODELS WITH COMPLETE BACKORDERS, POISSON DEMANDS, AND ERLANG-DISTRIBUTED LEAD TIMES

## 1. INTRODUCTION

In this chapter, we consider a single item (Q,R) inventory system where unit demands arrive according to a Poisson process with mean demand rate $\lambda$. The inventory position, which is the level of inventory on hand plus on order minus backorders, is continuously monitored and an order of size Q will be placed with suppliers by the system whenever the position reaches the reorder point R. The cost of placing the order regardless of the quantity ordered is A. The lead time, defined as the time from placing the order until the shipment arrives, is a random variable having an Erlang distribution of order K and with mean $K/\mu$. All customers are assumed to be willing to wait for backlogging when the system is out of stock. The cost of backlogging one unit of demand is $\hat{\Pi}$ per unit time and a fixed cost of $\Pi$ is also associated with each backorder occurred. The cost of carrying one unit of inventory in stock is H per unit time.

The assumption of Erlang distribution for lead times is appropriate in many inventory problems. Consider that there are several phases in the process of replenishing orders and if each phase requires exponential time duration, the entire lead time would have an Erlang distribution. In practice, the lead time distribution, when known, can often be fitted fairly well by a gamma distribution [Burgin and Wild 1967; Burgin 1975; among others]. The Erlang distribution is a special case of gamma distributions which has an integral number of phases. The method of stages allows one to model the Erlang process as a sequence of exponential processes. This, combined with the Poisson process generating demands, allows one to work with a completely Markovian model. Another reason for the Erlangian assumption is that the Erlang distribution provides a convenient family of distributions with a wide range of coefficients of variation between the constant case and the large variation of exponential case.

Continuous-review (Q,R) inventory models with unit Poisson demands and backorders have been studied by several authors. Hadley and Whitin [1963, Section 4-7] consider such an inventory system for constant lead times. They obtain steady-state probabilities of net inventory level, and hence expression of the average cost function, by conditioning those probabilities on inventory position. Galliher, Morse, and Simond [1959] also study the backordering system for both constant and exponential lead times. In the case of constant lead time, they derive the same results as those obtained by Hadley and Whitin but in a different form. In the case of exponential lead time, they obtain rather complicated expressions for net inventory probabilities and some operating characteristcs of the system. Their results are ones of the earliest exact formulations for stochastic (Q,R) models.

Gross and Harris [1973] analyze the case where the distribution of exponential lead time is allowed to depend on the level of unfilled demands. They model the inventory system under two different assumptions about how lead times depend on the number of outstanding orders. Posner and Yansouni [1972] study an inventory system where lead time is exponential and, during a stockout period, all customers will wait for backlogging but then cancel their demands if waiting time is too long. By allowing at most one order outstanding at any point in time, they obtain expressions for net inventory probabilities and the expected cost function.

In next section, we first derive a set of difference equations of the steady-state net inventory probability for the model under consideration. The inventory probability becomes, however, unmanageable when the order of Erlang lead time distribution is greater than one. An approximate model is therefore developed by conditioning the net inventory probability first on the inventory position and then on the lead time. Two different sets of closed-form expressions are obtained for the net inventory probability. The special case of exponential lead times is studied in Section 3. Several properties of the inventory distribution are established with proof.

In Section 4, we derive closed-form expressions of the expected net inventory, expected inventory on hand, expected backorders, and the probability that the system is out of stock for the general Erlang lead time case. The average inventory cost and the optimal policy are then considered in the last section. The proof of strict convexity of the cost function is attempted for the exponential lead time case. Since solutions of the optimal policy cannot be obtained explicitly, a numerical search is required to locate the optimal values of order size and reorder level for each set of system parameters. Some numerical experimentation will therefore be performed in Chapter 5 to investigate the nature of this model and the effect of lead time variability on the optimal solution.

## 2. BACKGROUND OF THE MODEL

For the inventory system under consideration, the following assumptions are made:

1. Demands arrive one at a time according to a Poisson process with parameter $\lambda$.
2. Lead times have an Erlang distribution with scale parameter $\mu$ and phase parameter K, where K is a positive integer greater than or equal to 1.
3. All customers will wait for backlogging when a stockout occurs.
4. Steady-state probabilities exist.
5. $Q \geq 1$.

According to the (Q,R) policy, an order will be placed by the system whenever the inventory level reaches R+Q-nQ, n=1,2,3,... Furthermore, let $[R+Q-nQ-j; i_{K-1}, i_{K-2}, \ldots, i_0]$, $j=0,1,\ldots,Q-1$ and $n=0,1,2,\ldots$, denote the state that the inventory level is R+Q-nQ-j and the number of outstanding orders currently in phase m is $i_m$, $0 \leq i_m \leq n$ and $m=0,1,\ldots,K-1$, for the Erlang lead time distribution of order K, $K \geq 1$. Note that the sum of all $i_m$'s is equal to n, the number of outstanding orders. Then, the model can be formulated as a Markovian process with infinite number of states, and all possible transitions into and out from a given net inventory state can be shown in Figure 4.1.

**Figure 4.1.** Steady-state transitions of the backordering model with Erlang lead times

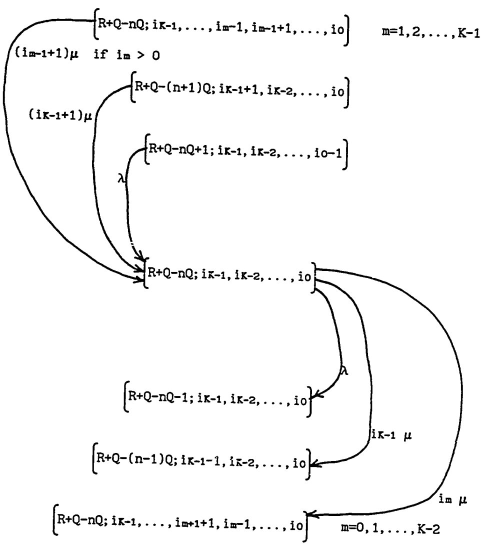


Figure 4.1. ( Continue... )

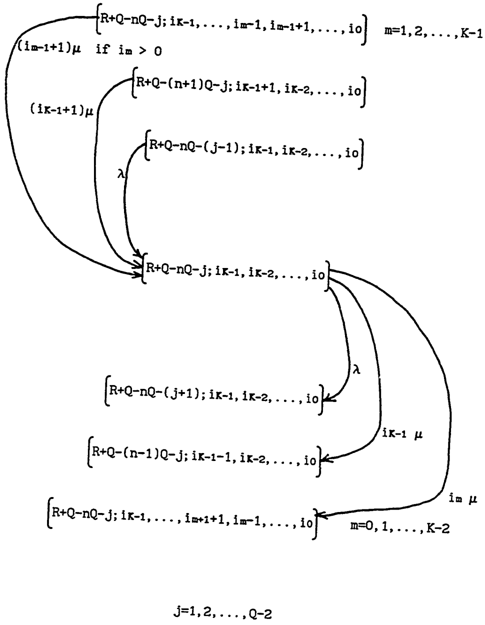
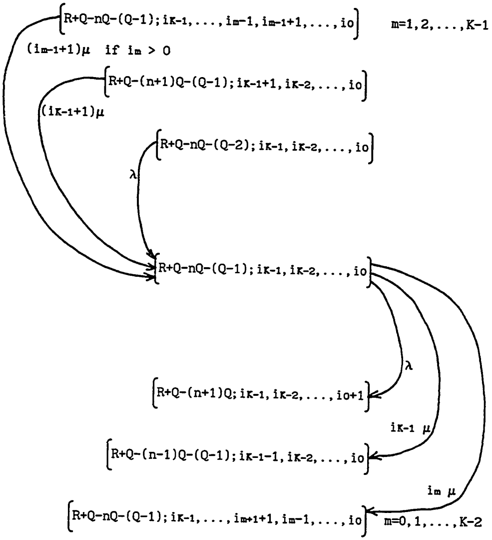


**Figure 4.1.** ( Continue... )



Therefore, from **Figure 4.1**, we can obtain the following balance equations for the net inventory probability:

$$\left[\lambda + \sum_{m=0}^{K-1} i_m \mu\right] P(R+Q-nQ; i_{K-1}, i_{K-2}, \ldots, i_0)$$

$$= \lambda\ P(R+Q-nQ+1; i_{K-1}, i_{K-2}, \ldots, i_0-1)$$

$$+ (i_{K-1}+1)\mu\ P[R+Q-(n+1)Q; i_{K-1}+1, i_{K-2}, \ldots, i_0]$$

$$+ \sum_{m \in S} (i_{m-1}+1)\mu\ P(R+Q-nQ; i_{K-1}, \ldots, i_m-1, i_{m-1}+1, \ldots, i_0)$$

$n=0,1,2,\ldots;$

$$\left[\lambda + \sum_{m=0}^{K-1} i_m \mu\right] P(R+Q-nQ-j; i_{K-1}, i_{K-2}, \ldots, i_0)$$

$$= \lambda\ P[R+Q-nQ-(j-1); i_{K-1}, i_{K-2}, \ldots, i_0]$$

$$+ (i_{K-1}+1)\mu\ P[R+Q-(n+1)Q-j; i_{K-1}+1, i_{K-2}, \ldots, i_0]$$

$$+ \sum_{m \in S} (i_{m-1}+1)\mu\ P(R+Q-nQ-j; i_{K-1}, \ldots, i_m-1, i_{m-1}+1, \ldots, i_0)$$

$j=1,2,\ldots,Q-1$ and $n=0,1,2,\ldots;$

where $P(.;.)$ is the steady-state probability of the associated inventory state and numbers of orders in various lead time phases, $S$ is such a set that $S = \{ m \mid m=1,2,\ldots,K-1 \text{ and } i_m > 0 \}$, and $K$ is the order of Erlang lead time distribution. Hence,

$$(\lambda + n\mu)\ P(R+Q-nQ; i_{K-1}, i_{K-2}, \ldots, i_0)$$

$$= \lambda\ P(R+Q-nQ+1; i_{K-1}, i_{K-2}, \ldots, i_0-1)$$

$$+ (i_{K-1}+1)\mu\ P[R+Q-(n+1)Q; i_{K-1}+1, i_{K-2}, \ldots, i_0]$$

$$+ \sum_{m \in S} (i_{m-1}+1)\mu\ P(R+Q-nQ; i_{K-1}, \ldots, i_m-1, i_{m-1}+1, \ldots, i_0)$$

$n=0,1,2,\ldots;$ and

$$(\lambda + n\mu)\, P(R+Q-nQ-j; i_{K-1}, i_{K-2}, \ldots, i_0)$$
$$= \lambda\, P[R+Q-nQ-(j-1); i_{K-1}, i_{K-2}, \ldots, i_0]$$
$$+ (i_{K-1}+1)\mu\, P[R+Q-(n+1)Q-j; i_{K-1}+1, i_{K-2}, \ldots, i_0]$$
$$+ \sum_{m \in S} (i_{m-1}+1)\mu\, P(R+Q-nQ-j; i_{K-1}, \ldots, i_m-1, i_{m-1}+1, \ldots, i_0)$$
$$j=1,2,\ldots,Q-1 \text{ and } n=0,1,2,\ldots;$$

because $\sum_{m=0}^{K-1} i_m = n.$

Now, for $j=0,1,\ldots,Q-1$ and $n=0,1,2,\ldots$, the probability of a given inventory level when $K \geq 1$

$$P(R+Q-nQ-j) = \sum_{i_{K-1}=0}^{n} \sum_{i_{K-2}=0}^{n-i_{K-1}} \cdots \sum_{i_0=0}^{n-\Sigma_{m=1}^{K-1} i_m} P(R+Q-nQ-j; i_{K-1}, i_{K-2}, \ldots, i_0).$$

As we can see, these net inventory probabilities for the exact model rapidly become unmanageable when K is greater than one. However, when K = 1, the above system of difference equations can be so reduced that

$$\lambda\, P(R+Q) = \mu\, P(R)\ ,$$
$$\lambda\, P(R+Q-j) = \lambda\, P[R+Q-(j-1)] + \mu\, P(R-j) \qquad j=1,2,\ldots,Q-1,$$
$$(\lambda+n\mu)\, P(R+Q-nQ-j) = \lambda\, P[R+Q-nQ-(j-1)] + (n+1)\mu\, P(R-nQ-j)$$
$$j=0,1,\ldots,Q-1 \text{ and } n=1,2,3,\ldots;$$

where P(.) is the steady-state probability of the associated net inventory level. Galliher et al. [1959] study this relatively simple special case for K = 1 but obtain rather complicated, non closed-form expressions for the inventory probability. This provides us with a strong motivation to formulate an approximate model for the general case where $K \geq 1$ and, hence, be able to derive some closed-form results for it.

Our model is an extension of that of Hadley and Whitin's to the case of Erlang-distributed lead times. In their model, Hadley and Whitin [1963, Section 4.7] make the same assumptions as those shown above except that they assume constant lead times for the replenishment. Since the reorder point R is expressed in terms of the inventory position and demands arrive one at a time, an order of size Q is placed and immediately bring the inventory position to R+Q when a demand arrives and reduces the position from R+1 to R. Therefore, the inventory position must have one of the values R+Q, R+Q-1, ..., R+1. To compute the steady-state net inventory probability, consider the system at any instant of time $\tau$ and at the time $\tau$-t, where t is the procurement lead time and constant. Note that everything on order at time $\tau$-t will have arrived in the system by time $\tau$ and nothing not on order at time $\tau$-t can have arrived in the system by time $\tau$. Thus, given that the inventory position was R+Q-j at time $\tau$-t, where j=0,1,...,Q-1, the probability that there are X units of net inventory at time $\tau$ is equal to the probability that R+Q-j-X units were demanded during the lead time t if R+Q-j-X $\geq$ 0 and is equal to zero otherwise.

Let P(.|t) denote the probability of the associated net inventory level at any instant of time $\tau$ given the lead time t. Then, by conditioning on the inventory position at time $\tau$-t, Hadley and Whitin derive that

$$P(X|t) = \sum_{j=0}^{R+Q-X} [\ P(\text{ R+Q-j-X units were demanded during time t })$$

$$P(\text{ inventory position was R+Q-j at time } \tau\text{-t })\ ]$$

$$= \frac{1}{Q} \sum_{i=0}^{R+Q-X} P(i;\lambda t) \qquad R+1 \le X \le R+Q; \qquad (4.1)$$

$$P(X|t) = \sum_{j=0}^{Q-1} [\ P(\text{ R+Q-j-X units were demanded during time t })$$

$$P(\text{ inventory position was R+Q-j at time } \tau\text{-t })\ ]$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} P(i;\lambda t) \qquad X \le R+1; \qquad (4.2)$$

where $P(.;\lambda t)$ is the probability mass function of a Poisson distribution with parameter $\lambda t$. Also note that, independent of the nature of lead times, steady states of the inventory position are uniformly distributed when demands arrive according to a Poisson process, each with a probability of 1/Q [Sivazlian 1974]. Same results as Equations (4.1) and (4.2) are also obtained by Galliher et al. [1959] in a different form.

But, Equations (4.1) and (4.2) only hold true when the lead time is a constant. Hadley and Whitin suggest that their results can be extended to the case of stochastic lead times as long as there is never more than a single order outstanding. Unfortunately, however, for (Q,R) models where the demand is Poisson-distributed and all demands will be backordered during a stockout period, it is not possible to specify rigorously that never more than a single order is outstanding. For any time interval of positive length, there is a positive probability that an arbitrarily large number of demands will occur in that time interval, and hence a positive probability that any given number of orders will be placed in the same interval. This in turn implies that there is a positive probability that any given number of orders can be outstanding at any point in time.

When more than one order can be outstanding and the lead time is a random variable, we would like to treat lead times as independent of each other, i.e., we would like to assume that the lead time for a given outstanding order is independent of lead times of the other orders. However, if this assumption is to be rigorously held true, then we must allow outstanding orders to cross, i.e., they need not be received in the same order in which they were placed by the system. Then, the statement claimed for deriving Equations (4.1) and (4.2), that everything on order at time $\tau$-t will have arrived in the system by time $\tau$ and nothing not on order at time $\tau$-t can have arrived in the system by time $\tau$ when the lead time for a given order is t, is no longer true because of the stochastic nature of lead times and the possibility of crossing orders.

Therefore, any extension of the results from Equations (4.1) and (4.2) to the case of stochastic lead times, such as our models, is just an approximation to the real system. How well these results will represent the true model depends on how small the probability that more than a single order is outstanding and the probability that orders will cross given that more than one order is outstanding are. It is often true in the real life situation that, even although two or more orders are outstanding at any point in time, the interval between the placing of orders is usually large enough so that there is essentially no interaction among orders. It can therefore be assumed to be a good approximation that the lead times are independent of each other as well as that orders do not cross. As will be illustrated in the second part of next chapter, our models closely approximate the exact ones when demand during lead time is relatively low.

## 3. THE APPROXIMATE MODEL AND STEADY-STATE RESULTS

By extending Hadley and Whitin's model to accommodate Erlang lead times, we can further condition the net inventory probability when the phase parameter of lead time distribution is K, P(X,K), on the lead time. By which,

$$P(X,K) = \int_0^\infty P(X|t)\ g(t;K,\mu)\ dt \qquad X \le R+Q, \qquad (4.3)$$

where $g(.;K,\mu)$ is the probability density function of a K-phase Erlang distribution with scale parameter $\mu$.

Therefore, by applying (4.1), (4.2), and (4.3), we can obtain that; for $R+1 \le X \le R+Q$,

$$P(X,K) = \int_0^\infty \left[ \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-1}}{(K-1)!} \mu\, e^{-\mu t}\ dt$$

$$= \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \int_0^\infty t^{i+K-1}\ e^{-(\lambda+\mu)t}\ dt$$

$$= \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \frac{(i+K-1)!}{(\lambda+\mu)^{i+K}}$$

$$= \frac{1}{Q} \sum_{i=0}^{R+Q-X} \binom{K-1+i}{i} \left[\frac{\lambda}{\lambda+\mu}\right]^i \left[\frac{\mu}{\lambda+\mu}\right]^K$$

$$= \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^K \sum_{i=0}^{R+Q-X} \binom{K-1+i}{i} \left[\frac{\rho}{1+\rho}\right]^i ; \qquad (4.4)$$

and, for $X \le R+1$,

$$P(X,K) = \int_0^\infty \left[ \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-1}}{(K-1)!} \mu e^{-\mu t} \, dt$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \int_0^\infty t^{i+K-1} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^i \mu^k}{i!(K-1)!} \frac{(i+K-1)!}{(\lambda+\mu)^{i+K}}$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \begin{bmatrix} K-1+i \\ i \end{bmatrix} \left[ \frac{\lambda}{\lambda+\mu} \right]^i \left[ \frac{\mu}{\lambda+\mu} \right]^K$$

$$= \frac{1}{Q} \left[ \frac{1}{1+\rho} \right]^K \sum_{i=R+1-X}^{R+Q-X} \begin{bmatrix} K-1+i \\ i \end{bmatrix} \left[ \frac{\rho}{1+\rho} \right]^i ; \qquad (4.5)$$

where $\rho = \lambda/\mu$ and K is the order of lead time distribution. For the above derivation, we apply the following identity that, when $X \geq 0$ and $Y > 0$,

$$\int_0^\infty t^X e^{-Yt} \, dt = \frac{X!}{Y^{X+1}} \int_0^\infty \frac{(Yt)^X}{X!} Y e^{-Yt} \, dt$$

$$= \frac{X!}{Y^{X+1}} ;$$

because the second integral integrates a Gamma density function and hence equals one.

If we make some transformations for the net inventory level as follows:

$R+Q-j = X$ for $j=0,1,\ldots,Q-1$ and $R+1 \le X \le R+Q$; and

$R-j = X$ for $j=-1,0,1,\ldots$ and $X \le R+1$;

then we can obtain from Equations (4.4) and (4.5) that

**Theorem 4.1.**

$$P(R+Q-j,K) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^K \sum_{n=0}^{j} \begin{bmatrix} K-1+n \\ n \end{bmatrix} \left[\frac{\rho}{1+\rho}\right]^n \tag{4.6}$$

$j=0,1,\ldots,Q-1;$

$$P(R-j,K) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^K \sum_{n=j+1}^{j+Q} \begin{bmatrix} K-1+n \\ n \end{bmatrix} \left[\frac{\rho}{1+\rho}\right]^n \tag{4.7}$$

$j=-1,0,1,\ldots;$

where $\rho = \lambda/\mu$, $K \ge 1$, and $P(.,K)$ is the probability of associated net inventory level when the Erlang lead time distribution is of order K.

Equations (4.6) and (4.7) represent a set of closed-form expressions for the net inventory probability. As we can see, they are both in the form of a polynomial of $[\rho/(1+\rho)]$. When the inventory level is in the range between R+Q, which is the maximum inventory level, and R+1, the order of that polynomial is equal to the number of units which have been depleted from the maximum level. However, when the inventory level is below R+1, the power of $[\rho/(1+\rho)]$ in that polynomial is between the range of the number of depleted units from inventory level R plus one and the number plus Q. Also note that, in these polynomials, terms with the same power of $[\rho/(1+\rho)]$ have the same coefficient for a given K.

Now, let us point out that there is an alternative way to derive Equations (4.6) and (4.7) and which is a lot more complicated than the approach shown above. This second approach and its results are shown in Appendix 1. Some results derived there will be useful later in other sections. Next, let us derive another set of expressions for the net inventory probability in a different form. We can obtain by applying results from Theorem 4.1 that; for K = 1,

$$P(R+Q-j,1) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^1 \sum_{n=0}^{j} \begin{bmatrix} 1-1+n \\ n \end{bmatrix} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[\frac{1}{1+\rho}\right] \sum_{n=0}^{j} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{j+1}\right] \qquad j=0,1,\ldots,Q-1, \qquad (4.8)$$

$$P(R-j,1) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^1 \sum_{n=j+1}^{j+Q} \begin{bmatrix} 1-1+n \\ n \end{bmatrix} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[\frac{1}{1+\rho}\right] \sum_{n=j+1}^{j+Q} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[\left[\frac{\rho}{1+\rho}\right]^{j+1} - \left[\frac{\rho}{1+\rho}\right]^{j+Q+1}\right]$$

$$= \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^{j+1}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right] \qquad j=-1,0,1,\ldots; \qquad (4.9)$$

for k = 2,

$$P(R+Q-j,2) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^2 \sum_{n=0}^{j} \binom{2-1+n}{n} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^2 \sum_{n=0}^{j} (n+1) \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[1 - (j+2)\left[\frac{\rho}{1+\rho}\right]^{j+1} + (j+1)\left[\frac{\rho}{1+\rho}\right]^{j+2}\right]$$

$$= \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{j+1} - (j+1)\frac{1}{1+\rho}\left[\frac{\rho}{1+\rho}\right]^{j+1}\right]$$

$j=0,1,\ldots,Q-1,$

$$P(R-j,2) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^2 \sum_{n=j+1}^{j+Q} \binom{2-1+n}{n} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^2 \sum_{n=j+1}^{j+Q} (n+1) \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[1 - (j+Q+2)\left[\frac{\rho}{1+\rho}\right]^{j+Q+1} + (j+Q+1)\left[\frac{\rho}{1+\rho}\right]^{j+Q+2}\right.$$
$$\left. - 1 + (j+2)\left[\frac{\rho}{1+\rho}\right]^{j+1} - (j+1)\left[\frac{\rho}{1+\rho}\right]^{j+2}\right]$$

$$= \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^{j+1}\left[1 + (j+1)\frac{1}{1+\rho}\right.$$
$$\left. - \left[\frac{\rho}{1+\rho}\right]^{Q} - (Q+j+1)\frac{1}{1+\rho}\left[\frac{\rho}{1+\rho}\right]^{Q}\right]$$

$j=-1,0,1,\ldots;$

for K = 3,

$$P(R+Q-j,3) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^3 \sum_{n=0}^{j} \begin{bmatrix} 3-1+n \\ n \end{bmatrix} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^3 \sum_{n=0}^{j} \frac{(n+2)(n+1)}{2} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\Bigg[1 - \frac{(j+3)(j+2)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+1}$$

$$+ (j+3)(j+1)\left[\frac{\rho}{1+\rho}\right]^{j+2} - \frac{(j+2)(j+1)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+3}\Bigg]$$

$$= \frac{1}{Q}\Bigg[1 - (j+2)\left[\frac{\rho}{1+\rho}\right]^{j+1} + (j+1)\left[\frac{\rho}{1+\rho}\right]^{j+2}$$

$$- \frac{(j+2)(j+1)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+1} + (j+2)(j+1)\left[\frac{\rho}{1+\rho}\right]^{j+2}$$

$$- \frac{(j+2)(j+1)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+3}\Bigg]$$

$$= \frac{1}{Q}\Bigg[1 - \left[\frac{\rho}{1+\rho}\right]^{j+1} - (j+1)\frac{1}{1+\rho}\left[\frac{\rho}{1+\rho}\right]^{j+1}$$

$$- \frac{(j+2)(j+1)}{2}\left[\frac{1}{1+\rho}\right]^2\left[\frac{\rho}{1+\rho}\right]^{j+1}\Bigg]$$

$j=0,1,\ldots,Q-1,$

$$P(R-j,3) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^3 \sum_{n=j+1}^{j+Q} \begin{bmatrix} 3-1+n \\ n \end{bmatrix} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^3 \sum_{n=j+1}^{j+Q} \frac{(n+2)(n+1)}{2} \left[\frac{\rho}{1+\rho}\right]^n$$

$$= \frac{1}{Q}\left[1 - \frac{(j+Q+3)(j+Q+2)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+Q+1}\right.$$

$$+ (j+Q+3)(j+Q+1)\left[\frac{\rho}{1+\rho}\right]^{j+Q+2} - \frac{(j+Q+2)(j+Q+1)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+Q+3}$$

$$- 1 + \frac{(j+3)(j+2)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+1}$$

$$\left. - (j+3)(j+1)\left[\frac{\rho}{1+\rho}\right]^{j+2} + \frac{(j+2)(j+1)}{2}\left[\frac{\rho}{1+\rho}\right]^{j+3}\right]$$

$$= \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^{j+1}\left[1 + (j+1)\frac{1}{1+\rho} + \frac{(j+2)(j+1)}{2}\left[\frac{1}{1+\rho}\right]^{2}\right.$$

$$- \left[\frac{\rho}{1+\rho}\right]^{Q} - (Q+j+1)\frac{1}{1+\rho}\left[\frac{\rho}{1+\rho}\right]^{Q}$$

$$\left. - \frac{(Q+j+2)(Q+j+1)}{2}\left[\frac{1}{1+\rho}\right]^{2}\left[\frac{\rho}{1+\rho}\right]^{Q}\right]$$

$j=-1,0,1,\ldots;$

where we use the following identities for the derivation:

$$\sum_{n=0}^{j} (n+1)\, q^{n} = \frac{1 - (j+2)\, q^{j+1} + (j+1)\, q^{j+2}}{(1-q)^{2}}, \qquad (4.10)$$

$$\sum_{n=0}^{j} (n+2)(n+1)\, q^{n} = \frac{1}{(1-q)^{3}}\left[2 - (j+3)(j+2)\, q^{j+1}\right.$$

$$\left. + 2(j+3)(j+1)\, q^{j+2} - (j+2)(j+1)\, q^{j+3}\right].$$

Therefore, from the result shown above, we can deduce the following theorem:

**Theorem 4.2.**

$$P(R+Q-j,K) = \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=0}^{K-1} \left[\begin{matrix} j+n \\ n \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{n}\right] \qquad (4.11)$$

$$j=0,1,\ldots,Q-1;$$

$$P(R-j,K) = \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^{j+1} \left[\sum_{n=0}^{K-1} \left[\begin{matrix} j+n \\ n \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{n} - \left[\frac{\rho}{1+\rho}\right]^{Q} \sum_{n=0}^{K-1} \left[\begin{matrix} Q+j+n \\ n \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{n}\right] \qquad (4.12)$$

$$j=0,1,2,\ldots;$$

where $\rho = \lambda/\mu$ and $K \geq 1$.

The proof of **Theorem 4.2** is shown in Appendix 2. Equations (4.11) and (4.12) represent a different set of closed-form expressions for the net inventory probability. Here, they are both in the form of a polynomial of $[1/(1+\rho)]$ with order of the number of lead time phases minus one. Unlike (4.6) and (4.7), however, the coefficient of terms with the same power for a given K in these polynomials is different from each other. One comment we can make about these two sets of expressions is that, when K is small, Equations (4.11) and (4.12) are easier to work with than (4.6) and (4.7) for the net inventory probability because of fewer terms.

## 4. THE SPECIAL CASE FOR K = 1

For the case where K = 1, i.e., the lead time is an exponentially distributed random variable, we can show by using (4.8) and (4.9) that

Expected Net Inventory

$$= \sum_{X=-\infty}^{R+Q} X\, P(X)$$

$$= \sum_{j=0}^{Q-1} (R+Q-j)\, P(R+Q-j) + \sum_{j=0}^{\infty} (R-j)\, P(R-j)$$

$$= \sum_{j=0}^{Q-1} (R+Q-j) \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{j+1} \right]$$

$$+ \sum_{j=0}^{\infty} (R-j) \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right]$$

$$= \frac{1}{Q} \sum_{j=0}^{Q-1} (R+Q-j) - (R+Q) \frac{1}{Q} \sum_{j=0}^{Q-1} \left[\frac{\rho}{1+\rho}\right]^{j+1}$$

$$+ \frac{1}{Q} \sum_{j=1}^{Q-1} j \left[\frac{\rho}{1+\rho}\right]^{j+1} + R \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] \sum_{j=0}^{\infty} \left[\frac{\rho}{1+\rho}\right]^{j+1}$$

$$- \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] \sum_{j=1}^{\infty} j \left[\frac{\rho}{1+\rho}\right]^{j+1}$$

$$= \frac{1}{Q} \frac{Q\,(R+Q+R+1)}{2} - (R+Q) \frac{1}{Q} (1+\rho) \left[ \frac{\rho}{1+\rho} - \left[\frac{\rho}{1+\rho}\right]^{Q+1} \right]$$

$$+ \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{2} \sum_{j=0}^{Q-2} (j+1) \left[\frac{\rho}{1+\rho}\right]^{j}$$

$$+ R \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] (1+\rho) \frac{\rho}{1+\rho}$$

$$- \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^Q \right] \left[\frac{\rho}{1+\rho}\right]^2 \sum_{j=0}^{\infty} (j+1) \left[\frac{\rho}{1+\rho}\right]^j$$

$$= \frac{Q+1}{2} + R - (R+Q)\, \rho \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^Q \right]$$

$$+ \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^2 (1+\rho)^2 \left[ 1 - Q \left[\frac{\rho}{1+\rho}\right]^{Q-1} + (Q-1) \left[\frac{\rho}{1+\rho}\right]^Q \right]$$

$$+ R\, \rho \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^Q \right]$$

$$- \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^Q \right] \left[\frac{\rho}{1+\rho}\right]^2 (1+\rho)^2$$

$$= \frac{Q+1}{2} + R + \rho^2 \frac{1}{Q} \left[ 1 - Q \frac{1+\rho}{\rho} \left[\frac{\rho}{1+\rho}\right]^Q + (Q-1) \left[\frac{\rho}{1+\rho}\right]^Q \right]$$

$$- \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^Q \right] \left[ (R+Q)\, \rho - R\, \rho + \rho^2 \right]$$

$$= \frac{Q+1}{2} + R + \rho \frac{1}{Q} \left[ \rho - Q\, (1+\rho) \left[\frac{\rho}{1+\rho}\right]^Q + (Q-1)\, \rho \left[\frac{\rho}{1+\rho}\right]^Q \right.$$

$$\left. - Q - \rho + Q \left[\frac{\rho}{1+\rho}\right]^Q + \rho \left[\frac{\rho}{1+\rho}\right]^Q \right]$$

$$= \frac{Q+1}{2} + R - \rho \; ;$$

P( out of stock )

$$= \sum_{X=-\infty}^{0} P(X)$$

$$= \sum_{j=R}^{\infty} P(R-j)$$

$$= \sum_{j=R}^{\infty} \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right]$$

$$= \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] \sum_{j=R}^{\infty} \left[\frac{\rho}{1+\rho}\right]^{j+1}$$

$$= \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] (1+\rho) \left[\frac{\rho}{1+\rho}\right]^{R+1}$$

$$= \rho \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{R} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right]$$

$$= \rho \; P(1) \; ;$$

Expected Backorders

$$= - \sum_{X=-\infty}^{-1} X \, P(X)$$

$$= \sum_{j=R+1}^{\infty} (j-R) \, P(R-j)$$

$$= \sum_{j=R+1}^{\infty} (j-R) \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right]$$

$$= \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] \left[ \sum_{j=R+1}^{\infty} j \left[\frac{\rho}{1+\rho}\right]^{j+1} - R \sum_{j=R+1}^{\infty} \left[\frac{\rho}{1+\rho}\right]^{j+1} \right]$$

$$= \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] \left[ \left[\frac{\rho}{1+\rho}\right]^{2} \sum_{j=R}^{\infty} (j+1) \left[\frac{\rho}{1+\rho}\right]^{j} - R \sum_{j=R+1}^{\infty} \left[\frac{\rho}{1+\rho}\right]^{j+1} \right]$$

$$= \frac{1}{Q} \left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \right] \left\{ \left[\frac{\rho}{1+\rho}\right]^{2} (1+\rho)^{2} \left[ 1 - 1 + (R+1) \left[\frac{\rho}{1+\rho}\right]^{R} - R \left[\frac{\rho}{1+\rho}\right]^{R+1} \right] - R\,(1+\rho) \left[\frac{\rho}{1+\rho}\right]^{R+2} \right\}$$

$$= \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right](1+\rho)\left[\frac{\rho}{1+\rho}\right]^{R+1}\left[\rho\,(R+1) - \rho\,R\,\frac{\rho}{1+\rho} - R\,\frac{\rho}{1+\rho}\right]$$

$$= \rho\,\frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right](1+\rho)\left[\frac{\rho}{1+\rho}\right]^{R+1}$$

$= \rho$ P( out of stock ) ;

Expected On-Hand Inventory

= Expected Net Inventory + Expected Backorders

$$= \frac{Q+1}{2} + R - \rho + \rho \text{ P( out of stock ) ;}$$

where we apply (4.10) for the derivation and P(.) is the net inventory probability when K = 1.

Furthermore, we assume that the reorder point, R, is greater than zero for deriving the above results except expected net inventory, where there is no restriction on R. In almost all real inventory systems, because the penalty for each backordering unit is usually greater than the cost of carrying one unit in stock, the reorder point is in terms of a positive inventory level so as to prevent the system from being out of stock during lead times of the replenishment. The assumption of $R > 0$ is, therefore, realistic. So, the following results can be summarized:

**Proposition 4.1.**

When the lead time is an exponential random variable,

$$\text{Expected Net Inventory} = \frac{Q+1}{2} + R - \rho \; ;$$

and, when R > 0,

$$\text{Expected On-Hand Inventory} = \frac{Q+1}{2} + R - \rho + \rho \, P(\text{out of stock}),$$

$$\text{Expected Backorders} = \rho \, P(\text{ out of stock }) \, ,$$

$$P(\text{ out of stock }) = \rho \, P(1) \; ;$$

where $\rho$ is the ratio of demand arrival rate $\lambda$ to service rate of the replenishment $\mu$ and P(1) is the probability that the inventory level is one.

In their paper, Galliher et al. [1959] investigate the exact model for the Exponential lead time case. Without having the closed-form expression for the net inventory probability, they, however, obtain the expected value and variance of the net inventory level. The expected net inventory obtained in **Proposition 4.1** is identical to that of theirs. However, the variance of net inventory in our model is different from that in their model. Galliher and others derive that

$$\text{Variance} = \frac{Q^2 - 1}{12} + \rho \left\{ \frac{\rho \, [\, 1 + (1/\rho) \,]^{Q+1} - \rho - (Q+1)}{[\, 1 + (1/\rho) \,]^{Q} - 1} \right\},$$

and we have that

$$\text{Variance} = \frac{Q^2 - 1}{12} + \rho\,(1 + \rho)\,.$$

Therefore, the difference between our result and theirs is

$$\Delta = \rho\,(1+\rho) - \rho \left\{ \frac{\rho\,[\,1 + (1/\rho)\,]^{Q+1} - \rho - (Q+1)}{[\,1 + (1/\rho)\,]^{Q} - 1} \right\}$$

$$= \rho \left\{ \frac{(1+\rho)\,[1+(1/\rho)]^{Q} - (1+\rho) - \rho\,[1+(1/\rho)]^{Q+1} + \rho + (Q+1)}{[\,1 + (1/\rho)\,]^{Q} - 1} \right\}$$

$$= \frac{\rho\,Q}{[\,1 + (1/\rho)\,]^{Q} - 1}$$

$$= \frac{Q\,\rho^{Q+1}}{(1+\rho)^{Q} - \rho^{Q}}\,.$$

Now, because

$$\lim_{\rho \to 0} \Delta = 0,$$

the difference decreases as $\rho$ decreases. Note here that $\rho$, which equals $\lambda/\mu$, is also the lead-time demand because $\lambda$ is the demand arrival rate and $1/\mu$ is the mean of exponential lead time. Therefore, when the lead time is exponentially distributed, the net inventory probability obtained earlier and, hence, our model closely approximate the exact ones as long as demand during lead time is low. Also note that, since $\Delta$ is strictly positive, the variance of our approximate net inventory probability distribution is always greater than that of the exact one.

When K = 1, several properties of the net inventory probability distribution in our model can also be established and proved as follows.

**Theorem 4.3.**

$$\sum_{i=j+1}^{\infty} P(R-i) = \rho\, P(R-j) \qquad j=-1,0,1,\ldots,$$

where $\rho$ is the ratio of demand arrival rate $\lambda$ to service rate of the replenishment $\mu$.

**Proof:**

From (4.9),

$$\sum_{i=j+1}^{\infty} P(R-i) = \sum_{i=j+1}^{\infty} \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{i+1} \left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right]$$

$$= \frac{1}{Q} \left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right] \sum_{i=j+1}^{\infty} \left[\frac{\rho}{1+\rho}\right]^{i+1}$$

$$= \frac{1}{Q} \left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right] (1+\rho) \left[\frac{\rho}{1+\rho}\right]^{j+2}$$

$$= \rho \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right]$$

$$= \rho\, P(R-j) .$$

The interpretation of **Theorem 4.3** is that, starting from R+1, the lead-time demand, $\rho$, times the probability of a certain inventory level is equal to the sum of probabilities of all levels less than that level.

**Theorem 4.4.**

$$P(R+1) > P(R+2) > \ldots > P(R+Q) \text{ , and} \quad (4.13)$$

$$P(R+1) > P(R) > P(R-1) > \ldots \quad (4.14)$$

**Proof:**

As we can see from (4.6), when the inventory level drops one unit between R+Q to R+1, the state probability increases by one term of $[\rho/(1+\rho)]$. Therefore, (4.13) holds true. Now, from (4.9), when the inventory level drops one unit below R+1, the probability decreases by being multiplied by $[\rho/(1+\rho)]$, which is strictly less than one. Thus, (4.14) is proved.

**Theorem 4.4** says that the state probability distribution of net inventory level has only one mode and the level of R+1 has the greatest state probability. In other words, the probability strictly increases when the inventory level depletes from R+Q to R+1 and then strictly decreases as the inventory level continues to drop below R+1. The result of (4.13) can be extended to the general case where $K \geq 1$ because (4.6), which is valid for the general case, is used for its proof. Unfortunately, however, (4.14) is not always true when $K \geq 2$.

## 5. SOME PROPERTIES OF THE MODEL

After having the expression of net inventory probabilities, we can then present several properties of the probability distribution for the general Erlang lead time case. First of all, from **Proposition A.1** in Appendix 1, we have that, for $K \geq 2$,

$$P(X,K) = \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1) \qquad X \leq R+Q-1.$$

Then, for $K \geq 2$, the expected net inventory

$$E[N,K] = \sum_{X=-\infty}^{R+Q} X\, P(X,K)$$

$$= (R+Q)\, P(R+Q,K) + \sum_{X=-\infty}^{R+Q-1} X\, P(X,K)$$

$$= (R+Q)\, P(R+Q,K) + \sum_{X=-\infty}^{R+Q-1} X \left[\frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1)\right]$$

$$= \frac{1}{1+\rho} (R+Q)\, P(R+Q,K-1) + \frac{1}{1+\rho} \sum_{X=-\infty}^{R+Q-1} X\, P(X,K-1)$$

$$+ \frac{\rho}{1+\rho} \sum_{X=-\infty}^{R+Q-1} X\, P(X+1,K)$$

$$= \frac{1}{1+\rho} \sum_{X=-\infty}^{R+Q} X\, P(X,K-1) + \frac{\rho}{1+\rho} \sum_{X=-\infty}^{R+Q} (X-1)\, P(X,K)$$

$$= \frac{1}{1+\rho} \sum_{X=-\infty}^{R+Q} X\, P(X,K-1) + \frac{\rho}{1+\rho} \sum_{X=-\infty}^{R+Q} X\, P(X,K)$$

$$- \frac{\rho}{1+\rho} \sum_{X=-\infty}^{R+Q} P(X,K)$$

$$= \frac{1}{1+\rho} E[N,K-1] + \frac{\rho}{1+\rho} E[N,K] - \frac{\rho}{1+\rho} ,$$

where the result of (A.5) in Appendix 1 is used for the derivation. Therefore, for $K \geq 2$,

$$\frac{1}{1+\rho} E[N,K] = \frac{1}{1+\rho} E[N,K-1] - \frac{\rho}{1+\rho} .$$

Hence,

$$\begin{aligned} E[N,K] &= E[N,K-1] - \rho \\ &= E[N,K-2] - \rho - \rho \\ &= \ldots \\ &= E[N,1] - (K-1)\,\rho \\ &= \frac{Q+1}{2} + R - \rho - (K-1)\,\rho \\ &= \frac{Q+1}{2} + R - K\,\rho , \end{aligned} \qquad (4.15)$$

where $K \geq 1$. Then, for $K \geq 2$, the expected on-hand inventory

$$\begin{aligned} E[H,K] &= \sum_{X=0}^{R+Q} X\,P(X,K) \\ &= (R+Q)\,P(R+Q,K) + \sum_{X=0}^{R+Q-1} X\,P(X,K) \\ &= (R+Q)\,P(R+Q,K) + \sum_{X=0}^{R+Q-1} X \left[ \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1) \right] \\ &= \frac{1}{1+\rho} (R+Q)\,P(R+Q,K-1) + \frac{1}{1+\rho} \sum_{X=0}^{R+Q-1} X\,P(X,K-1) \\ &\quad + \frac{\rho}{1+\rho} \sum_{X=0}^{R+Q-1} X\,P(X+1,K) \end{aligned}$$

$$= \frac{1}{1+\rho} \sum_{X=0}^{R+Q} X\, P(X,K-1) + \frac{\rho}{1+\rho} \sum_{X=1}^{R+Q} (X-1)\, P(X,K)$$

$$= \frac{1}{1+\rho} \sum_{X=0}^{R+Q} X\, P(X,K-1) + \frac{\rho}{1+\rho} \sum_{X=1}^{R+Q} X\, P(X,K)$$

$$- \frac{\rho}{1+\rho} \sum_{X=1}^{R+Q} P(X,K)$$

$$= \frac{1}{1+\rho} E[H,K-1] + \frac{\rho}{1+\rho} E[H,K] - \frac{\rho}{1+\rho} [\ 1 - P_{out,K}\ ]\ ,$$

where $P_{out,K}$ is the probability of being out of stock; therefore,

$$\frac{1}{1+\rho} E[H,K] = \frac{1}{1+\rho} E[H,K-1] - \frac{\rho}{1+\rho} [\ 1 - P_{out,K}\ ]\ .$$

So,

$$E[H,K] = E[H,K-1] - \rho + \rho\, P_{out,K}$$

$$= E[H,K-2] - \rho + \rho\, P_{out,K-1} - \rho + \rho\, P_{out,K}$$

$$= \ldots$$

$$= E[H,1] - (K-1)\,\rho + \rho \sum_{i=2}^{K} P_{out,i}$$

$$= \frac{Q+1}{2} + R - \rho + \rho\, P_{out,1} - (K-1)\,\rho + \rho \sum_{i=2}^{K} P_{out,i}$$

$$= \frac{Q+1}{2} + R - K\,\rho + \rho \sum_{i=1}^{K} P_{out,i}\ ,$$

where $K \geq 1$. Now, by defination, the expected backorders for the case where $K \geq 1$

$$E[B,K] = E[H,K] - E[N,K]$$

$$= \rho \sum_{i=1}^{K} P_{out,i}\ .$$

Next, for $K \geq 2$, the probability of being out of stock

$$P_{out,K} = \sum_{X=-\infty}^{0} P(X)$$

$$= \sum_{X=-\infty}^{0} \left[ \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1) \right]$$

$$= \frac{\rho}{1+\rho} \sum_{X=-\infty}^{1} P(X,K) + \frac{1}{1+\rho} \sum_{X=-\infty}^{0} P(X,K-1)$$

$$= \frac{\rho}{1+\rho} P(1,K) + \frac{\rho}{1+\rho} P_{out,K} + \frac{1}{1+\rho} P_{out,K-1} \ ;$$

therefore,

$$\frac{1}{1+\rho} P_{out,K} = \frac{1}{1+\rho} P_{out,K-1} + \frac{\rho}{1+\rho} P(1,K) \ .$$

Hence,

$$P_{out,K} = P_{out,K-1} + \rho\, P(1,K)$$

$$= P_{out,K-2} + \rho\, P(1,K-1) + \rho\, P(1,K)$$

$$= \ldots$$

$$= P_{out,1} + \rho \sum_{i=2}^{K} P(1,i)$$

$$= \rho \sum_{i=1}^{K} P(1,i) \ ,$$

where $K \geq 1$. Note that results from **Proposition 4.1** are used for the above derivation. Thus, the following proposition can be established.

**Proposition 4.2.**

When R > 0,

$$E[H,K] = \frac{Q+1}{2} + R - K\rho + \rho \sum_{i=1}^{K} P_{out,i} \ ,$$

$$E[B,K] = \rho \sum_{i=1}^{K} P_{out,i} \ , \text{ and}$$

$$P_{out,K} = \rho \sum_{i=1}^{K} P(1,i) \ ;$$

where $\rho = \lambda/\mu$, $K \geq 1$, and P(1,i) is the probability that the inventory level is one for the case where the lead time distribution is of order i.

By combining results from **Proposition 4.2** and (4.12), we can, hence, obtain several closed-form results for the net inventory probability distribution as follows.

**Theorem 4.5.**

$$E[N,K] = \frac{Q+1}{2} + R - K\rho \ ,$$

and, when R > 0,

$$E[H,K] = \frac{Q+1}{2} + R - K\rho$$

$$+ \rho^2 \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{K-1} \begin{bmatrix} K-n+1 \\ 2 \end{bmatrix} \begin{bmatrix} R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1} \begin{bmatrix} K-n+1 \\ 2 \end{bmatrix} \begin{bmatrix} Q+R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n \right] ,$$

$$E[B,K] = \rho^2 \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{K-1} \binom{K-n+1}{2} \binom{R-1+n}{n} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1} \binom{K-n+1}{2} \binom{Q+R-1+n}{n} \left[\frac{1}{1+\rho}\right]^n \right],$$

$$P_{out,K} = \rho \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{K-1} (K-n) \binom{R-1+n}{n} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1} (K-n) \binom{Q+R-1+n}{n} \left[\frac{1}{1+\rho}\right]^n \right];$$

where $\rho$ = the ratio of demand arrival rate $\lambda$ to service rate of each phase of the replenishment $\mu$,

$K$ = the order of Erlang lead time distribution, $K \geq 1$,

$E[N,K]$ = the expected value of net inventory level,

$E[H,K]$ = the expected value of on-hand inventory,

$E[B,K]$ = the expected value of backorders, and

$P_{out,K}$ = the probability of being out of stock.

**Proof:**

The first result is obtained from (4.15). Now, when $R > 0$,

$$P_{out,K} = \rho \sum_{i=1}^{K} P(1,i)$$

$$= \rho \sum_{i=1}^{K} \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{i-1} \binom{R-1+n}{n} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{i-1} \binom{Q+R-1+n}{n} \left[\frac{1}{1+\rho}\right]^n \right]$$

$$= \rho \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{K-1} \sum_{i=n+1}^{K} \begin{bmatrix} R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1} \sum_{i=n+1}^{K} \begin{bmatrix} Q+R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n \right]$$

$$= \rho \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{K-1} (K-n) \begin{bmatrix} R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1} (K-n) \begin{bmatrix} Q+R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n \right].$$

Hence,

$$E[B,K] = \rho \sum_{i=1}^{K} P_{out,i}$$

$$= \rho \sum_{i=1}^{K} \rho \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{i-1} (i-n) \begin{bmatrix} R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{i-1} (i-n) \begin{bmatrix} Q+R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n \right]$$

$$= \rho^2 \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{K-1} \begin{bmatrix} R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n \sum_{i=n+1}^{K} (i-n) - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1} \begin{bmatrix} Q+R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n \sum_{i=n+1}^{K} (i-n) \right]$$

$$= \rho^2 \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^R \left[ \sum_{n=0}^{K-1} \begin{bmatrix} K-n+1 \\ 2 \end{bmatrix} \begin{bmatrix} R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1} \begin{bmatrix} K-n+1 \\ 2 \end{bmatrix} \begin{bmatrix} Q+R-1+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^n \right].$$

To complete the proof, we use the fact that the expected on-hand inventory is the sum of expected net inventory and expected backorders to derive the expression for E[H,K].

Finally, one observation can be made about the net inventory probability. As we can see from either Equations (4.6) and (4.7) or (4.11) and (4.12) that state probabilities of net inventory level are independent of one of the decision variables, the reorder point R. That is, as long as the order quantity Q is fixed, the distribution of net inventory probabilities remains unchanged no matter what value the reorder point R has.

## 6. THE OPTIMAL INVENTORY POLICY

In order to locate the optimal order quantity and reorder point for this model, we need to formulate the expected cost of operating the inventory system as a function of Q and R. Here, it is assumed that the purchase price of each inventory unit is independent of the order size, i.e., there are no quantity discounts offered by the supplier. Thus, the purchasing cost of stock is irrelevant in determining the decision variables Q and R. In addition, let us define the following cost elements and other system parameters:

$C_K(Q,R)$ = the expected total inventory cost for the order size Q, $Q > 1$, and reorder point R, $R > 0$, when the Erlang lead time distribution is of order K, $K \geq 1$,

A = fixed ordering cost per order placed,

H = carrying cost per unit held in stock per unit time,

$\hat{\Pi}$ = backordering cost per unit to be backlogged per unit time,

$\Pi$ = fixed backordering cost for each unit backlogged,

$\lambda$ = the arrival rate of demands, and

$\mu$ = the service rate of each phase of the replenishment.

Then, the expected inventory cost per unit time, which consists of ordering cost, carrying cost, time-weighted backordering cost, and the penalty for backorders, can be formulated as

$$C_K(Q,R) = \frac{A\lambda}{Q} + H\,E[H,K] + \hat{\Pi}\,E[B,K] + \Pi\lambda\,P_{out,K}$$

$$= \frac{A\lambda}{Q} + H\left[\frac{Q+1}{2} + R - K\rho + E[B,K]\right] + \hat{\Pi}\,E[B,K] + \Pi\lambda\,P_{out,K}$$

$$= \frac{A\lambda}{Q} + \frac{HQ}{2} + HR + H\left[\frac{1}{2} - K\rho\right]$$

$$+ (H+\hat{\Pi})\,\rho^2\,\frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^R \left[\sum_{n=0}^{K-1}\left[\frac{K-n+1}{2}\right]\binom{R-1+n}{n}\left[\frac{1}{1+\rho}\right]^n\right.$$

$$\left. - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1}\left[\frac{K-n+1}{2}\right]\binom{Q+R-1+n}{n}\left[\frac{1}{1+\rho}\right]^n\right]$$

$$+ \Pi\lambda\,\rho\,\frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^R \left[\sum_{n=0}^{K-1}(K-n)\binom{R-1+n}{n}\left[\frac{1}{1+\rho}\right]^n\right.$$

$$\left. - \left[\frac{\rho}{1+\rho}\right]^Q \sum_{n=0}^{K-1}(K-n)\binom{Q+R-1+n}{n}\left[\frac{1}{1+\rho}\right]^n\right]; \tag{4.16}$$

where $\rho = \lambda/\mu$ and results from **Theorem 4.5** are applied.

Although the closed-form expression of expected average inventory cost is obtained above, it appears, however, impractical to prove analytically for the general case that the cost function is convex with respect to both Q and R. Furthermore, closed-form solutions for the optimal inventory policy can not be directly derived from (4.16). Therefore, for each set of model parameters, a numerical search is required for locating the optimal Q and R. Some numerical experimentation will be performed in next chapter to investigate the nature of this model and its reaction to the change of system parameters.

Now, when the lead time is an exponential random variable, i.e., K = 1, (4.16) can be reduced such that

$$C_1(Q,R) = \frac{A\lambda}{Q} + \frac{HQ}{2} + HR + H\left[\frac{1}{2} - \rho\right]$$
$$+ (H+\hat{\Pi})\,\rho^2\,\frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^R\left[1 - \left[\frac{\rho}{1+\rho}\right]^Q\right]$$
$$+ \Pi\lambda\,\rho\,\frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^R\left[1 - \left[\frac{\rho}{1+\rho}\right]^Q\right]$$
$$= \frac{A\lambda}{Q} + \frac{HQ}{2} + HR + H\left[\frac{1}{2} - \rho\right]$$
$$+ (H\rho+\hat{\Pi}\rho+\Pi\lambda)\rho\,\frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^R\left[1 - \left[\frac{\rho}{1+\rho}\right]^Q\right]$$
$$= \frac{A\lambda}{Q} + \frac{HQ}{2} + HR + \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^R - \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R+Q} + \beta\ ;$$

where $\alpha = (H\rho+\hat{\Pi}\rho+\Pi\lambda)\rho$ and $\beta = H\,(1/2 - \rho)$. Hence, treating Q and R as continuous variables, we can obtain the first and second partial derivatives of the cost function with respect to Q and R that

$$\frac{\partial C}{\partial Q} = -\frac{A\lambda}{Q^2} + \frac{H}{2} - \frac{\alpha}{Q^2}\left[\frac{\rho}{1+\rho}\right]^R + \frac{\alpha}{Q^2}\left[\frac{\rho}{1+\rho}\right]^{R+Q}$$
$$- \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R+Q}\ln\left[\frac{\rho}{1+\rho}\right]$$
$$= -\frac{A\lambda}{Q^2} + \frac{H}{2} - \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^R\left[\frac{1}{Q} - \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^Q + \left[\frac{\rho}{1+\rho}\right]^Q\ln\left[\frac{\rho}{1+\rho}\right]\right],$$

$$\frac{\partial C}{\partial R} = H + \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R} \ln\left[\frac{\rho}{1+\rho}\right] - \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R+Q} \ln\left[\frac{\rho}{1+\rho}\right]$$

$$= H + \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R} \ln\left[\frac{\rho}{1+\rho}\right]\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right],$$

$$\frac{\partial^2 C}{(\partial Q)^2} = \frac{2A\lambda}{Q^3} + \frac{2\alpha}{Q^3}\left[\frac{\rho}{1+\rho}\right]^{R} - \frac{2\alpha}{Q^3}\left[\frac{\rho}{1+\rho}\right]^{R+Q}$$

$$+ \frac{\alpha}{Q^2}\left[\frac{\rho}{1+\rho}\right]^{R+Q} \ln\left[\frac{\rho}{1+\rho}\right] + \frac{\alpha}{Q^2}\left[\frac{\rho}{1+\rho}\right]^{R+Q} \ln\left[\frac{\rho}{1+\rho}\right]$$

$$- \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R+Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2$$

$$= \frac{2A\lambda}{Q^3} + \frac{2\alpha}{Q^3}\left[\frac{\rho}{1+\rho}\right]^{R} - \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R+Q}\left\{\frac{2}{Q^2} - \frac{2}{Q}\ln\left[\frac{\rho}{1+\rho}\right] + \left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\right\},$$

$$\frac{\partial^2 C}{(\partial R)^2} = \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right],$$

$$\frac{\partial^2 C}{\partial Q\, \partial R} = -\frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R} \ln\left[\frac{\rho}{1+\rho}\right]\left[\frac{1}{Q} - \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^{Q} + \left[\frac{\rho}{1+\rho}\right]^{Q} \ln\left[\frac{\rho}{1+\rho}\right]\right].$$

As we can see, $\partial^2 C/(\partial R)^2 > 0$ for all $Q > 1$ and $R > 0$. However, the determinant of the Hessian matrix

$$\frac{\partial^2 C}{(\partial Q)^2}\frac{\partial^2 C}{(\partial R)^2} - \left[\frac{\partial^2 C}{\partial Q\, \partial R}\right]^2$$

$$= \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}\left\{\frac{2A\lambda}{Q^{3}}+\frac{2\alpha}{Q^{3}}\left[\frac{\rho}{1+\rho}\right]^{R}\right.$$

$$-\frac{2\alpha}{Q^{3}}\left[\frac{\rho}{1+\rho}\right]^{R+Q}+\frac{2\alpha}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{R+Q}\ln\left[\frac{\rho}{1+\rho}\right]$$

$$-\frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R+Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}-\frac{2A\lambda}{Q^{3}}\left[\frac{\rho}{1+\rho}\right]^{Q}$$

$$-\frac{2\alpha}{Q^{3}}\left[\frac{\rho}{1+\rho}\right]^{R+Q}+\frac{2\alpha}{Q^{3}}\left[\frac{\rho}{1+\rho}\right]^{R+2Q}$$

$$\left.-\frac{2\alpha}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{R+2Q}\ln\left[\frac{\rho}{1+\rho}\right]+\frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R+2Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}\right\}$$

$$-\left[\frac{\alpha}{Q}\right]^{2}\left[\frac{\rho}{1+\rho}\right]^{2R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}\left\{\frac{1}{Q^{2}}+\frac{1}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{2Q}\right.$$

$$+\left[\frac{\rho}{1+\rho}\right]^{2Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}-\frac{2}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{Q}$$

$$\left.+\frac{2}{Q}\left[\frac{\rho}{1+\rho}\right]^{Q}\ln\left[\frac{\rho}{1+\rho}\right]-\frac{2}{Q}\left[\frac{\rho}{1+\rho}\right]^{2Q}\ln\left[\frac{\rho}{1+\rho}\right]\right\}$$

$$= \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}\frac{2A\lambda}{Q^{3}}\left[1-\left[\frac{\rho}{1+\rho}\right]^{Q}\right]$$

$$+\left[\frac{\alpha}{Q}\right]^{2}\left[\frac{\rho}{1+\rho}\right]^{2R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}\left\{\frac{2}{Q^{2}}-\frac{2}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{Q}\right.$$

$$+\frac{2}{Q}\left[\frac{\rho}{1+\rho}\right]^{Q}\ln\left[\frac{\rho}{1+\rho}\right]-\left[\frac{\rho}{1+\rho}\right]^{Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}$$

$$-\frac{2}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{Q}+\frac{2}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{2Q}-\frac{2}{Q}\left[\frac{\rho}{1+\rho}\right]^{2Q}\ln\left[\frac{\rho}{1+\rho}\right]$$

$$+\left[\frac{\rho}{1+\rho}\right]^{2Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^{2}-\frac{1}{Q^{2}}-\frac{1}{Q^{2}}\left[\frac{\rho}{1+\rho}\right]^{2Q}$$

$$- \left[\frac{\rho}{1+\rho}\right]^{2Q} \left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2 + \frac{2}{Q^2}\left[\frac{\rho}{1+\rho}\right]^{Q}$$

$$- \frac{2}{Q}\left[\frac{\rho}{1+\rho}\right]^{Q} \ln\left[\frac{\rho}{1+\rho}\right] + \frac{2}{Q}\left[\frac{\rho}{1+\rho}\right]^{2Q} \ln\left[\frac{\rho}{1+\rho}\right] \Bigg\}$$

$$= \frac{2A\lambda\alpha}{Q^4}\left[\frac{\rho}{1+\rho}\right]^{R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right]$$

$$+ \left[\frac{\alpha}{Q}\right]^2\left[\frac{\rho}{1+\rho}\right]^{2R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\Bigg\{\frac{1}{Q^2} - \frac{2}{Q^2}\left[\frac{\rho}{1+\rho}\right]^{Q}$$

$$+ \frac{1}{Q^2}\left[\frac{\rho}{1+\rho}\right]^{2Q} - \left[\frac{\rho}{1+\rho}\right]^{Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\Bigg\}$$

$$= \frac{2A\lambda\alpha}{Q^4}\left[\frac{\rho}{1+\rho}\right]^{R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right]$$

$$+ \left[\frac{\alpha}{Q}\right]^2\left[\frac{\rho}{1+\rho}\right]^{2R}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\Bigg\{\frac{1}{Q^2}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{Q}\right]^2$$

$$- \left[\frac{\rho}{1+\rho}\right]^{Q}\left[\ln\left[\frac{\rho}{1+\rho}\right]\right]^2\Bigg\},$$

which cannot rigorously be proved strictly positive for all Q and R. Therefore, even for the special case K = 1, the convexity of the cost function cannot be analytically asserted.

Now, let us set the first derivatives equal to zero, thus,

$$- \frac{A\lambda}{Q^2} + \frac{H}{2} - \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^{R}\left[\frac{1}{Q} - \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^{Q}\right.$$

$$\left. + \left[\frac{\rho}{1+\rho}\right]^{Q}\ln\left[\frac{\rho}{1+\rho}\right]\right] = 0\ , \qquad (4.17)$$

$$H + \frac{\alpha}{Q}\left[\frac{\rho}{1+\rho}\right]^R \ln\left[\frac{\rho}{1+\rho}\right]\left[1 - \left[\frac{\rho}{1+\rho}\right]^Q\right] = 0 . \qquad (4.18)$$

From (4.18), we have that

$$\left[\frac{\rho}{1+\rho}\right]^R = \frac{-HQ}{\alpha \ln[\rho/(1+\rho)] \{1-[\rho/(1+\rho)]^Q\}} . \qquad (4.19)$$

By substituting (4.19) into (4.17), we can obtain that

$$-\frac{A\lambda}{Q^2} + \frac{H}{2} - \frac{-H}{\ln[\rho/(1+\rho)] \{1-[\rho/(1+\rho)]^Q\}}\left[\frac{1}{Q} - \frac{1}{Q}\left[\frac{\rho}{1+\rho}\right]^Q + \left[\frac{\rho}{1+\rho}\right]^Q \ln\left[\frac{\rho}{1+\rho}\right]\right] = 0 ,$$

and, hence,

$$-\frac{A\lambda}{Q^2} + \frac{H}{2} + \frac{H}{Q \ln[\rho/(1+\rho)]} + \frac{H [\rho/(1+\rho)]^Q}{\{1-[\rho/(1+\rho)]^Q\}} = 0 .$$

Then, it can be shown that

$$\left[\frac{\rho}{1+\rho}\right]^Q = \frac{(2A\lambda-HQ^2) \ln[\rho/(1+\rho)] - 2HQ}{(2A\lambda+HQ^2) \ln[\rho/(1+\rho)] - 2HQ}$$

$$= 1 - \frac{2HQ^2 \ln[\rho/(1+\rho)]}{(2A\lambda+HQ^2) \ln[\rho/(1+\rho)] - 2HQ} . \qquad (4.20)$$

The closed-form solution of R can then be derived in terms of Q from Equation (4.19) by taking the logarithm for both sides. This provides us with some convenience for the optimal policy search, because the optimal R for a given Q can easily be determined by (4.19). However, not only (4.20) is difficult to interpret, the solution of Q is also hard to obtain in a closed form. Therefore, some numerical examples for the case K = 1 will also be presented in the next chapter to show the nature of the model.

# CHAPTER 5

## SENSITIVITY ANALYSIS FOR THE COMPLETE BACKORDERING MODEL AND COMPARISON OF TWO MODELS

In Chapter 4 we obtain the closed-form expression of the expected average cost function for the complete backordering (Q,R) model with Poisson demands and Erlang lead times. Since explicit solutions of the optimal inventory policy cannot be derived, some numerical experimentation is required to locate the optimal values of Q and R, and to analyze the effect of changes in system parameters on the optimal solution. The design of a simple two-dimensional search procedure and several numerical results are presented in the next section. In Section 2, this approximate model will be compared with the exact model previously studied under some special conditions. Finally, conclusions about this study as well as some suggestions for future research will be summarized in the last section.

# 1. NUMERICAL RESULTS FOR THE COMPLETE BACKORDERING MODEL

Because closed-form expression has been obtained for the expected average inventory cost, the numerical evaluation of inventory costs is much easier in this model than that in the previous exact model, where net inventory probabilities must be calculated for each set of system parameters in order to evaluate the cost function. A simple two-dimensional search procedure can be used to locate the optimal values of Q and R for this backordering inventory system. For each example, the value of R that minimizes inventory costs is first found for any given Q, then the best value of Q which minimizes costs is found. Initially, the mean demand during lead time is arbitrarily used as starting point to search for R. Because the strict convexity property cannot be proved for the cost function, local optimality might exist. Actually, many local optima of Q are detected in each of our examples. Therefore, we search for the optimal Q exhaustively among a wide range of values centered around the classic EOQ. The best value of R found for each Q is, however, always optimal with respect to Q in all of our examples.

Note from the last section of Chapter 4 that, when $K = 1$, i.e., lead times are exponentially distributed, the optimal value of R with respect to any given Q can be explicitly obtained by taking the logarithm for both sides of Equation (4.19). This provides an advantage for the numerical search. Unfortunately, when $K > 1$, the procedure described earlier still must be applied to find the optimal R for each value of Q. Also note that we assume $R > 0$ in deriving the average inventory cost function for the general $K \geq 1$ case. As will be seen later, all of our numerical results satisfy that condition.

By using Equation (4.16) from Chapter 4, the cost function can easily be evaluated for each set of system parameters. The following cost elements are used throughout the experimentation:

$A = 50$ \$/order,

$H = 1$ \$/unit in stock/unit time,

$\hat{\Pi} = 4$ \$/unit backordered/unit time, and

$\Pi = 0$ \$/unit backordered.

We first fix the mean lead time rate as 4 and let the mean demand rate $\lambda$ be 200, 400, and 800, respectively, to examine the effects of demand rate on both the optimal policy and the inventory cost. We then fix the mean demand rate $\lambda$ as 200 and let the mean lead time rate vary as 1, 2, and 4 in order to see how lead time affects the optimal solution.

Note that, for Erlang lead times, the mean of lead time is $K/\mu$ and the variance of lead time equals $K/\mu^2$, where K and $\mu$ are the phase and scale parameters of the lead time distribution, respectively. Therefore, for each mean lead time rate $\mu/K$, the variance of lead time decreases as the order of lead time distribution K increases. For each lead time rate in our examples, we let K = {1, 2, 3, 4, 10, 100} in order to investigate the effect of lead time variability on the optimal inventory policy and cost. Thus, we totally examine 30 examples for this complete backlogging model. Numerical results of the optimal policy and inventory cost for these cases are shown in **Table 5.1** and **Table 5.2**.

Table 5.1. Numerical results for the complete backordering model when mean lead time rate is fixed*

| $\lambda$ | 200 | | | 400 | | | 800 | | |
|---|---|---|---|---|---|---|---|---|---|
| $\mu/K$ | 4 | | | 4 | | | 4 | | |
| K | Q | R | TC | Q | R | TC | Q | R | TC |
| 1 | 196 | 11 | 160.25 | 302 | 45 | 262.79 | 472 | 130 | 451.27 |
| 2 | 180 | 14 | 144.94 | 274 | 49 | 226.76 | 422 | 134 | 370.93 |
| 3 | 174 | 15 | 139.21 | 262 | 50 | 212.56 | 398 | 134 | 337.95 |
| 4 | 170 | 16 | 136.22 | 254 | 51 | 204.95 | 384 | 133 | 319.74 |
| 10 | 164 | 17 | 130.64 | 238 | 52 | 190.22 | 350 | 133 | 282.99 |
| 100 | 160 | 17 | 127.17 | 226 | 54 | 180.61 | 322 | 135 | 257.18 |

* $\lambda$ is the mean demand rate, $\mu/K$ is the mean lead time rate, and K is the number of phases for the Erlang lead time distribution.

Table 5.2. Numerical results for the complete backordering model when mean demand rate is fixed*

| λ | 200 | | | 200 | | | 200 | | |
|---|---|---|---|---|---|---|---|---|---|
| μ/K | 4 | | | 2 | | | 1 | | |
| K | Q | R | TC | Q | R | TC | Q | R | TC |
| 1 | 196 | 11 | 160.25 | 236 | 65 | 225.73 | 292 | 193 | 373.85 |
| 2 | 180 | 14 | 144.94 | 212 | 67 | 185.66 | 258 | 188 | 283.88 |
| 3 | 174 | 15 | 139.21 | 200 | 67 | 169.23 | 240 | 184 | 245.12 |
| 4 | 170 | 16 | 136.22 | 192 | 67 | 160.17 | 230 | 180 | 222.90 |
| 10 | 164 | 17 | 130.64 | 176 | 66 | 141.89 | 200 | 172 | 174.74 |
| 100 | 160 | 17 | 127.17 | 162 | 67 | 129.11 | 168 | 167 | 134.94 |

* λ is the mean demand rate, μ/K is the mean lead time rate, and K is the number of phases for the Erlang lead time distribution.

For a given mean lead time rate, the optimal ordering quantity Q and reorder level R, as well as the inventory cost for each case of K, monotonically increase as mean demand rate increases. This is because increasing demand in an inventory system will require a larger order quantity to keep the frequency of placing orders low and a higher reorder level to prevent the system from being out of stock during the lead time, and will result in higher total operating cost. However, the average cost of each demand satisfied becomes lower when there are more demands per unit time. Now, when mean demand rate is fixed, the optimal values of Q, R, and inventory cost for each case of K also monotonically increase as the mean of lead time becomes larger. Therefore, the management of an inventory system should seek ways to minimize the lead time of its replenishment orders so that the operating cost can be reduced.

Another more interesting observation can be made from Tables 5.1 and 5.2 is that, for the set of system parameters used, the inventory system always carries negative safety stock, which is the difference between the reorder level and the expected lead-time demand. For example, in the case where mean demand rate $\lambda$ and mean lead time rate $\mu/K$ are 800 and 4 respectively (see the last column of Table 5.1), the reorder inventory level R ranges from 130 to 135 for different cases of lead time phases K but the expected demand during lead time $K\lambda/\mu$ is always 200, which on average will result in a negative stock level at the end of each lead time. Therefore, our examples show that the safety stock level is not necessarily always positive.

As mentioned earlier, for any given mean, the variance of an Erlang lead time decreases as the order of its distribution K increases. In fact, the lead time becomes a constant equal to its mean and has zero variance when the number of phases approaches infinity. Therefore, for each set of rates for mean demand and mean lead time in our examples, the optimal operating cost decreases because of less lead time variability when the order of Erlang distribution increases. The optimal ordering quantity Q and the maximum inventory level R+Q also drops as the lead time variance becomes smaller.

Although the average cost function has a saw-toothed type of behavior with respect to Q and many local optima have been found during the experimentation, those local cost minima center around the global optimum in a strictly convex fashion. This kind of local optimality might be caused by the integrality restriction for the values of Q and R. If the values of R were allowed to be continuous, the cost function would have been strictly convex for the set of cost parameters we choose. The optimal value of R found for each Q is, however, global with respect to Q. Finally, like that in the previous partial backlogging model, the cost function for this model is also very insensitive to the values of both Q and R when they are around the optima.

## 2. COMPARISON OF TWO MODELS

We have so far studied two continuous-review inventory systems and respectively developed an analytical model for each of them. In the first system, only a fraction of unsatisfied demands will be backlogged and the rest will be lost when the system is out of stock; but all new demands will be lost after a limited number of replenishment orders have been outstanding. Furthermore, the lead time is assumed to be exponentially distributed. In the second system, however, all unfilled demands can be backordered during the stockout period and the lead time is a random variable with Erlang distribution. The model derived for the first system exactly represents that system but the second model only approximates the real inventory system.

In the case when all customers are willing to wait for backlogging and the number of outstanding orders allowed is large enough, the first inventory system could practically accommodate all unsatisfied demands because, as mentioned in Chapter 3, the effect of truncation becomes negligible if many orders are allowed to be outstanding. Therefore, for exponential lead times, the second model with complete backorders can be compared with the first model. In other words, when $\beta = 1$ and N is large in the first, exact model and when K = 1 in the second, approximate model, the two inventory systems become identical.

In order to compare the two models and to investigate the penalty incurred by using the approximate model, some numerical examples are examined. We let $\lambda = \{25, 50, 100, 200\}$ when $\mu = 1$, $\lambda = \{50, 100, 200, 400\}$ when $\mu = 2$, and $\lambda = \{100, 200, 400, 800\}$ when $\mu = 4$; where $\lambda$ is the mean demand rate and $\mu$ is the mean lead time rate. We further let the maximum number of outstanding orders, N, in the first model equal four so that the effect of truncating the net inventory level can be eliminated. Finally, the same set of cost parameters as that used in the previous section is applied for these examples.

The optimal policy and inventory cost of both models are obtained for each case of $\lambda$ and $\mu$. The cost penalty of using the approximate model is then calculated by first applying the optimal policy of the second model to the first model and then dividing the difference between the resulted cost and the true optimal cost by the latter. **Table 5.3** shows the optimal inventory policy and total operating cost of both models and the cost penalty in percentages for these 12 examples.

Table 5.3. Numerical results for the comparison of two models

| | | Model 1† $\beta = 1$ & $N = 4$ | | | Model 2‡ $K = 1$ | | | |
|---|---|---|---|---|---|---|---|---|
| $\lambda^*$ | $\mu^*$ | Q | R | TC | Q | R | TC | Error** (%) |
| 25 | 1 | 60 | 17 | 63.35 | 76 | 11 | 65.80 | 1.20 |
| 50 | 1 | 71 | 50 | 99.45 | 118 | 32 | 112.96 | 5.68 |
| 100 | 1 | 95 | 109 | 155.68 | 186 | 82 | 202.08 | 12.52 |
| 200 | 1 | 121 | 231 | 245.41 | 292 | 193 | 373.85 | 19.66 |
| 50 | 2 | 92 | 7 | 79.61 | 98 | 5 | 80.16 | 0.16 |
| 100 | 2 | 120 | 34 | 126.19 | 152 | 22 | 131.47 | 1.21 |
| 200 | 2 | 141 | 101 | 198.34 | 236 | 65 | 225.73 | 5.69 |
| 400 | 2 | 190 | 219 | 310.69 | 370 | 164 | 403.93 | 12.43 |
| 100 | 4 | 126 | -1 | 102.48 | 128 | 0 | 102.30 | 0.04 |
| 200 | 4 | 183 | 15 | 158.72 | 196 | 11 | 160.25 | 0.15 |
| 400 | 4 | 242 | 67 | 251.86 | 302 | 45 | 262.79 | 1.14 |
| 800 | 4 | 282 | 202 | 396.13 | 472 | 130 | 451.27 | 5.72 |

* $\lambda$ is the mean demand rate and $\mu$ is the mean lead time rate.

† All unsatisfied demands can be backlogged and up to four orders can be outstanding at any point in time.

‡ Lead times are exponentially distributed.

** The penalty of using the approximate model is calculated by applying the optimal policy of Model 2 to Model 1 and then dividing the difference between the resulted cost and the true optimal cost by the latter.

As we can see, for each set of mean demand rate $\lambda$ and mean lead time rate $\mu$, the optimal policy of the approximate model underestimates the reorder level R and overestimates the ordering quantity Q. The level of reorder point is underestimated because the second model is an extension of the constant lead time case to the Erlang lead time case and hence the stochastic nature of lead time has been overlooked. The ordering quantity is so overestimated that the frequency of placing orders can be reduced in order to compensate for the higher probability of stockout caused by low reorder level. This overall will result in higher maximum inventory level R+Q and higher total operating cost.

The cost penalty of using the approximate model is also obtained for each case of $\lambda$ and $\mu$. For example, when $\lambda$ = 200 and $\mu$ = 1, the ordering policy from the second model, (Q,R) = (292,193), is applied to the first model and the inventory cost is computed as 293.66. The difference between this cost and the true optimal cost 245.41 is then divided by the latter and yields the percentage cost increase 19.66%. As also implied in Section 3 of Chapter 4, the cost penalty monotonically decreases as the mean lead-time demand $\lambda/\mu$ decreases. To be more precisely, the cost penalty actually becomes smaller when the product of variances of demand and lead time, $\lambda/\mu^2$, is getting smaller.

For example, although the mean lead-time demands of $(\lambda,\mu)$ = (50,1), (200,2), and (800,4) are respectively 50, 100, and 200, the cost penalties for these cases are in the same magnitude because their values of $\lambda/\mu^2$ are all the same as 50. Many authors have reported in the literature that the approximate model can be used for the real inventory system as long as mean demand during lead time is relatively low. Our results indicate that how well an approximate model will represent the real system depends more on how variable the demand during lead time is. Zipkin [1986] also concludes the same relationship between the accuracy of an approximate model and the variability of lead-time demand.

# 3. CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

The framework of analysis in this study is the stochastic (Q,R) model for a single inventory item. The inventory position, which is the level of inventory on hand plus on order minus backorders, is continuously monitored. Customers demand single units according to a Poisson process. When the position reaches the prespecified reorder level R, an order is placed with suppliers for a lot of size Q units. The lead time, defined as the time from placing the order until the shipment arrives, is a random variable with an Erlangian type of distribution.

We first consider such an inventory system with exponentially distributed lead times, which is a special case of Erlangian distributions. When a stockout occurs, some customers are willing to wait for backorders but the rest will bring their business elsewhere. After a certain number of outstanding orders have been placed by the system, however, all new demands will be lost because the expected waiting time for backlogging will then be longer than customers' tolerance limits. An analytical model exactly representing this system is derived by solving a truncated stochastic process. Some properties of the net inventory probability distribution are then presented with proof. Since solutions of the optimal inventory policy cannot be explicitly obtained, some experimentation is designed and performed in order to investigate the nature of this system.

Numerical results of the first model show that, first, the effect of truncating the net inventory level on both the optimal policy and the total operating cost becomes negligible if lead-time demand is low and/or the number of outstanding orders allowed is large. Therefore, when demand during lead time is relatively high, the system should allow at least two or three orders to be outstanding in order to reduce the operating cost. Furthermore, the cost penalty caused by early truncation becomes smaller as the fraction of unsatisfied demands backlogged decreases. Allowing only one outstanding order would therefore have less impact on the inventory cost when more unsatisfied demands during the stockout period will be lost.

Second, when lead-time demand is high, it could be quite costly in the backordering case to apply the optimal policy obtained by assuming only one outstanding order to the system where actually a large number of orders can be outstanding. It appears, however, that the assumption of only one order outstanding at any point in time might be legitimate for the lost-sales case. Third, the optimal reorder level and maximum inventory level as well as the operating cost monotonically increase when the fraction of unsatisfied demands backlogged decreases. Therefore, it is obviously preferable for the system to accommodate as many demands as possible during the stockout period if the penalty for losing sales is high.

We then consider a continuous review inventory system with Erlang-distributed lead times where all customers will wait to be backlogged when the system is out of stock. Another analytical model which closely approximates this system is built by conditioning the probabilities of net inventory first on the inventory position and then on the lead time. Several properties of the model are then established. Closed-form expressions of both the net inventory probability and the average inventory cost function are also obtained. Sensitivity analysis is designed and conducted by some numerical examples to examine the effect of lead time variability on the optimal solution. For any given set of rates for mean demand and mean lead time, both the ordering quantity and the maximum inventory level as well as the operating cost monotonically decrease as the number of phases of the Erlang lead-time distribution increases because of less variability in the lead time.

Now, for both models, the maximum stock level and the reorder level as well as the inventory cost will increase if either demand rate or lead time increases. However, for a fixed lead time, the average cost of each demand decreases as demand rate increases. Since lead times are more controllable than demands, the management should seek ways to reduce the length of lead time as much as possible in order to minimize the operating cost. Finally, although it cannot be proved strictly convex, the inventory cost functions of both models behave fairly well and are quite insensitive to the values of the ordering quantity and reorder level when they approach the optima.

The case of exponential lead times is studied as a special case for the second model and then compared with the first, exact model when all unsatisfied demands can be backordered. The optimal policy of the approximate model always underestimates the reorder level and overestimates the ordering quantity because the stochastic nature of lead time is overlooked in that model. The second model, however, closely approximates the real inventory system as long as the product of variances of demand and lead time is relatively small. Since there is no exact model existing for the Erlang lead time case when unsatisfied demands can be backlogged, our second model provides a good approximation to solve the real inventory problem.

This study has extensively investigated two classes of (Q,R) inventory models and provided several important insights of them. One possible extension of this study is to linearly combine the complete backordering model with Erlang lead times with the lost-sales model of Buchanan and Love [1985] to accommodate the partial backordering case. In the case when all unsatisfied demands can be backlogged until a limited number of orders have been outstanding, the first model also represents another inventory system where the interarrival time between two successive customers has an Erlang distribution and an order is placed by the system whenever a unit is demanded. Some properties of this continuous-review (S-1,S) model have been derived in this study. The model can be further explored and extended to accommodate the partial backordering case by some adjustment. Finally, those models developed by this study can be widely applied to many real life inventory situations. How to integrate the results of these models into the real inventory system is a very important problem and also deserves further study.

# APPENDIX

## A.1. An Alternative Approach to Derive Theorem 4.1

From (4.1), (4.2), and (4.3), for $R+1 \le X \le R+Q-1$,

$$P(X,K) = \int_0^\infty \left[ \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-1}}{(K-1)!} \mu e^{-\mu t} \, dt$$

$$= \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \int_0^\infty t^{i+K-1} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \frac{i+K-1}{\lambda+\mu} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \sum_{i=1}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \frac{i}{\lambda+\mu} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$+ \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \frac{K-1}{\lambda+\mu} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{\lambda}{\lambda+\mu} \frac{1}{Q} \sum_{i=1}^{R+Q-X} \frac{\lambda^{i-1} \mu^K}{(i-1)!(K-1)!} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$+ \frac{\mu}{\lambda+\mu} \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{\lambda^i \mu^{K-1}}{i!(K-2)!} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{\lambda}{\lambda+\mu} \int_0^\infty \left[ \frac{1}{Q} \sum_{i=0}^{R+Q-X-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-1}}{(K-1)!} \mu\, e^{-\mu t} \, dt$$

$$+ \frac{\mu}{\lambda+\mu} \int_0^\infty \left[ \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-2}}{(K-2)!} \mu\, e^{-\mu t} \, dt$$

$$= \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1) \; ;$$

and for $X \le R$,

$$P(X,K) = \int_0^\infty \left[ \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-1}}{(K-1)!} \mu\, e^{-\mu t} \, dt$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \int_0^\infty t^{i+K-1} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^i \mu^K}{i!(K-1)!} \frac{i+K-1}{\lambda+\mu} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{\lambda}{\lambda+\mu} \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^{i-1} \mu^K}{(i-1)!(K-1)!} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$+ \frac{\mu}{\lambda+\mu} \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^i \mu^{K-1}}{i!(K-2)!} \int_0^\infty t^{i+K-2} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{\lambda}{\lambda+\mu} \int_0^\infty \left[ \frac{1}{Q} \sum_{i=R-X}^{R+Q-X-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-1}}{(K-1)!} \mu e^{-\mu t} \, dt$$

$$+ \frac{\mu}{\lambda+\mu} \int_0^\infty \left[ \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \frac{(\mu t)^{K-2}}{(K-2)!} \mu e^{-\mu t} \, dt$$

$$= \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1) \;;$$

where $\rho = \lambda/\mu$, $K \geq 2$, and $P(.,K)$ is the probability of associated net inventory level when the Erlang lead time distribution is of order K. For the above derivation, we utilize the integration by parts as follows:

$$\int_0^\infty t^X e^{-Yt} \, dt = \left[ \frac{t^X e^{-Yt}}{-Y} \Bigg|_0^\infty \right] + \frac{X}{Y} \int_0^\infty t^{X-1} e^{-Yt} \, dt$$

$$= \frac{X}{Y} \int_0^\infty t^{X-1} e^{-Yt} \, dt \;, \qquad \text{(A.1)}$$

where $X > 0$ and $Y > 0$. Therefore, we can obtain that

**Proposition A.1.**

For $K \geq 2$,

$$P(X,K) = \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1) \qquad X \leq R+Q-1. \qquad \text{(A.2)}$$

Note that Equation (A.2) holds true only for the case where $K \geq 2$.

When K = 1, for R+1 ≤ X ≤ R+Q-1,

$$P(X,1) = \int_0^{\infty} \left[ \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \mu e^{-\mu t} \, dt$$

$$= \frac{1}{Q} \sum_{i=0}^{R+Q-X} \frac{\lambda^i \mu}{i!} \int_0^{\infty} t^i \; e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \sum_{i=1}^{R+Q-X} \frac{\lambda^i \mu}{i!} \frac{i}{\lambda+\mu} \int_0^{\infty} t^{i-1} \; e^{-(\lambda+\mu)t} \, dt$$

$$+ \frac{1}{Q} \mu \int_0^{\infty} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{\lambda}{\lambda+\mu} \frac{1}{Q} \sum_{i=1}^{R+Q-X} \frac{\lambda^{i-1} \mu}{(i-1)!} \int_0^{\infty} t^{i-1} \; e^{-(\lambda+\mu)t} \, dt$$

$$+ \frac{1}{Q} \left[ \frac{\mu \, e^{-(\lambda+\mu)t}}{-(\lambda+\mu)} \Bigg|_0^{\infty} \right]$$

$$= \frac{\lambda}{\lambda+\mu} \int_0^{\infty} \left[ \frac{1}{Q} \sum_{i=0}^{R+Q-X-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \mu e^{-\mu t} \, dt$$

$$+ \frac{\mu}{\lambda+\mu} \frac{1}{Q}$$

$$= \frac{\rho}{1+\rho} P(X-1,1) + \frac{1}{1+\rho} \frac{1}{Q} \; ; \qquad \text{(A.3)}$$

and for $X \le R$,

$$P(X,1) = \int_0^\infty \left[ \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \mu e^{-\mu t} \, dt$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^i \mu}{i!} \int_0^\infty t^i \; e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^i \mu}{i!} \frac{i}{\lambda+\mu} \int_0^\infty t^{i-1} \; e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{\lambda}{\lambda+\mu} \frac{1}{Q} \sum_{i=R+1-X}^{R+Q-X} \frac{\lambda^{i-1} \mu}{(i-1)!} \int_0^\infty t^{i-1} \; e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{\lambda}{\lambda+\mu} \int_0^\infty \left[ \frac{1}{Q} \sum_{i=R-X}^{R+Q-X-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \right] \mu e^{-\mu t} \, dt$$

$$= \frac{\rho}{1+\rho} P(X+1,1); \qquad \text{(A.4)}$$

where $\rho = \lambda/\mu$. Here, again, we apply the result of (A.1) during the derivation. It can also be shown by continuously integrating by parts that

$$P(R+Q,K) = \int_0^\infty \left[ \frac{1}{Q} e^{-\lambda t} \right] \frac{(\mu t)^{K-1}}{(K-1)!} \mu e^{-\mu t} \, dt$$

$$= \frac{1}{Q} \frac{\mu^K}{(K-1)!} \int_0^\infty t^{K-1} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \frac{\mu^K}{(K-2)!} \frac{1}{\lambda+\mu} \int_0^\infty t^{K-2} e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \frac{\mu^K}{(K-3)!} \left[\frac{1}{\lambda+\mu}\right]^2 \int_0^\infty t^{K-3} e^{-(\lambda+\mu)t} \, dt$$

$$= \cdots$$

$$= \frac{1}{Q} \frac{\mu^K}{0!} \left[\frac{1}{\lambda+\mu}\right]^{K-1} \int_0^\infty e^{-(\lambda+\mu)t} \, dt$$

$$= \frac{1}{Q} \mu^K \left[\frac{1}{\lambda+\mu}\right]^{K-1} \left[ \frac{e^{-(\lambda+\mu)t}}{-(\lambda+\mu)} \Bigg|_0^\infty \right]$$

$$= \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^K, \qquad \text{(A.5)}$$

where $\rho = \lambda/\mu$ and $K \geq 1$.

Now, by applying (A.2) recursively, we can obtain that; for $R+1 \leq X \leq R+Q-1$,

$$P(X,K) = \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1)$$

$$= \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} \left[\frac{\rho}{1+\rho} P(X+1,K-1) + \frac{1}{1+\rho} P(X,K-2)\right]$$

$$= \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho}\frac{\rho}{1+\rho} P(X+1,K-1)$$

$$+ \left[\frac{1}{1+\rho}\right]^2 \left[\frac{\rho}{1+\rho} P(X+1,K-2) + \frac{1}{1+\rho} P(X,K-3)\right]$$

$$= \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho}\frac{\rho}{1+\rho} P(X+1,K-1)$$

$$+ \left[\frac{1}{1+\rho}\right]^2 \frac{\rho}{1+\rho} P(X+1,K-2)$$

$$+ \left[\frac{1}{1+\rho}\right]^3 \left[\frac{\rho}{1+\rho} P(X+1,K-3) + \frac{1}{1+\rho} P(X,K-4)\right]$$

$$= \ldots$$

$$= \frac{\rho}{1+\rho} \sum_{i=2}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(X+1,i) + \left[\frac{1}{1+\rho}\right]^{K-1} P(X,1)$$

$$= \frac{\rho}{1+\rho} \sum_{i=2}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(X+1,i)$$

$$+ \left[\frac{1}{1+\rho}\right]^{K-1} \left[\frac{\rho}{1+\rho} P(X+1,1) + \frac{1}{1+\rho}\frac{1}{Q}\right]$$

$$= \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(X+1,i) + \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K}$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(X+1,i), \qquad \text{(A.6)}$$

where we use (A.3) for the third to last step and (A.5) for the last step; and for $X \le R$,

$$P(X,K) = \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1)$$

$$= \ldots$$

$$= \frac{\rho}{1+\rho} \sum_{i=2}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(X+1,i) + \left[\frac{1}{1+\rho}\right]^{K-1} P(X,1)$$

$$= \frac{\rho}{1+\rho} \sum_{i=2}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(X+1,i) + \left[\frac{1}{1+\rho}\right]^{K-1} \frac{\rho}{1+\rho} P(X+1,1)$$

$$= \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(X+1,i), \qquad (A.7)$$

where (A.4) is applied for the second to last step. Although (A.2) holds true only for $K \geq 2$, we can see, however, that (A.6) and (A.7) are also valid for $K = 1$ by comparing them with (A.3) and (A.4) respectively.

By repeatedly using (A.6), the following can then be obtained:

$$P(R+Q-1,K) = P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i),$$

$$P(R+Q-2,K) = P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q-1,m)$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q,m)$$

$$+ \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} \left[\frac{\rho}{1+\rho} \sum_{i=1}^{m} \left[\frac{1}{1+\rho}\right]^{m-i} P(R+Q,i)\right]$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=i}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q,m)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^{2} \sum_{i=1}^{K} \sum_{m=i}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^2 \sum_{i=1}^{K} (K-i+1) \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i),$$

$$P(R+Q-3,K) = P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q-2,m)$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q,m)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^2 \sum_{i=1}^{K} \sum_{m=i}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^3 \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i) \sum_{m=i}^{K} (m-i+1)$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^2 \sum_{i=1}^{K} (K-i+1) \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^3 \sum_{i=1}^{K} \frac{(K-i+2)(K-i+1)}{2} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{i=1}^{K} \begin{bmatrix} K-i+0 \\ 0 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^2 \sum_{i=1}^{K} \begin{bmatrix} K-i+1 \\ 1 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$+ \left[\frac{\rho}{1+\rho}\right]^3 \sum_{i=1}^{K} \begin{bmatrix} K-i+2 \\ 2 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i),$$

$\ldots,$

$$P(R+Q-j,K) = P(R+Q,K) + \sum_{n=1}^{j} \left[\frac{\rho}{1+\rho}\right]^n \sum_{i=1}^{K} \left[\begin{matrix} K-i+n-1 \\ n-1 \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$j=1,2,\ldots,Q-1; \qquad (A.8)$$

where $\rho = \lambda/\mu$, $K \geq 1$, and

$$\left[\begin{matrix} n \\ i \end{matrix}\right] = \frac{n!}{i!(n-i)!} .$$

Now that (A.8) is derived by mathematical deduction, it can easily be proved by mathematical induction as follows.

**Proof of Equation (A.8):**

Equation (A.8) is true for j = 1, because

$$P(R+Q-1,K) = P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$= P(R+Q,K) + \sum_{n=1}^{1} \left[\frac{\rho}{1+\rho}\right]^n \sum_{i=1}^{K} \left[\begin{matrix} K-i+n-1 \\ n-1 \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i).$$

From (A.6) we have that, for $j=2,3,\ldots,Q-1$,

$$P(R+Q-j,K) = P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P[R+Q-(j-1),m]$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q,m)$$

$$+ \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} \left[ \sum_{n=1}^{j-1} \left[\frac{\rho}{1+\rho}\right]^n \sum_{i=1}^{m} \left[\begin{matrix} m-i+n-1 \\ n-1 \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{m-i} P(R+Q,i) \right]$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q,m)$$

$$+ \sum_{n=1}^{j-1} \left[\frac{\rho}{1+\rho}\right]^{n+1} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i) \sum_{m=i}^{K} \begin{bmatrix} m-i+n-1 \\ n-1 \end{bmatrix}$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R+Q,m)$$

$$+ \sum_{n=1}^{j-1} \left[\frac{\rho}{1+\rho}\right]^{n+1} \sum_{i=1}^{K} \begin{bmatrix} K-i+n \\ n \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$= P(R+Q,K) + \frac{\rho}{1+\rho} \sum_{i=1}^{K} \begin{bmatrix} K-i+0 \\ 0 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$+ \sum_{n=2}^{j} \left[\frac{\rho}{1+\rho}\right]^{n} \sum_{i=1}^{K} \begin{bmatrix} K-i+n-1 \\ n-1 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i)$$

$$= P(R+Q,K)$$

$$+ \sum_{n=1}^{j} \left[\frac{\rho}{1+\rho}\right]^{n} \sum_{i=1}^{K} \begin{bmatrix} K-i+n-1 \\ n-1 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i);$$

where we apply the combinatorial identity, that

$$\sum_{m=0}^{n} \begin{bmatrix} m+i-1 \\ i-1 \end{bmatrix} = \begin{bmatrix} n+i \\ i \end{bmatrix}, \qquad \text{(A.9)}$$

for the proof so that

$$\sum_{m=i}^{K} \begin{bmatrix} m-i+n-1 \\ n-1 \end{bmatrix} = \sum_{m=0}^{K-i} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix} = \begin{bmatrix} K-i+n \\ n \end{bmatrix}.$$

Therefore, (A.8) is true for $j=1,2,\ldots,Q-1$ by induction.

Now, by applying (A.7) recursively, we can derive that

$$P(R,K) = \frac{\rho}{1+\rho} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+1,i),$$

$$P(R-1,K) = \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R,m)$$

$$= \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} \frac{\rho}{1+\rho} \sum_{i=1}^{m} \left[\frac{1}{1+\rho}\right]^{m-i} P(R+1,i)$$

$$= \left[\frac{\rho}{1+\rho}\right]^{2} \sum_{i=1}^{K} \sum_{m=i}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+1,i)$$

$$= \left[\frac{\rho}{1+\rho}\right]^{2} \sum_{i=1}^{K} \begin{bmatrix} K-i+1 \\ 1 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+1,i),$$

$$P(R-2,K) = \frac{\rho}{1+\rho} \sum_{m=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-m} P(R-1,m)$$

$$= \left[\frac{\rho}{1+\rho}\right]^{3} \sum_{i=1}^{K} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+1,i) \sum_{m=i}^{K} \begin{bmatrix} m-i+1 \\ 1 \end{bmatrix}$$

$$= \left[\frac{\rho}{1+\rho}\right]^{3} \sum_{i=1}^{K} \begin{bmatrix} K-i+2 \\ 2 \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+1,i),$$

$$\ldots,$$

$$P(R-j,K) = \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{i=1}^{K} \begin{bmatrix} K-i+j \\ j \end{bmatrix} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+1,i)$$

$$j=0,1,2,\ldots; \qquad \text{(A.10)}$$

where $\rho = \lambda/\mu$, $K \geq 1$, and (A.9) is used for the derivation. The proof of (A.10) by mathematical induction is similar to that of (A.8) and, hence, is omitted.

Having (A.8), (A.10), and (A.5), we are now ready for deriving the same results as Equations (4.6) and (4.7). By combining (A.5) and (A.8), we can have that

$$
\begin{aligned}
P(R+Q-j,K) &= P(R+Q,K) \\
&\quad + \sum_{n=1}^{j} \left[\frac{\rho}{1+\rho}\right]^{n} \sum_{i=1}^{K} \left[\begin{matrix} K-i+n-1 \\ n-1 \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{K-i} P(R+Q,i) \\
&= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K} \\
&\quad + \sum_{n=1}^{j} \left[\frac{\rho}{1+\rho}\right]^{n} \sum_{i=1}^{K} \left[\begin{matrix} K-i+n-1 \\ n-1 \end{matrix}\right] \left[\frac{1}{1+\rho}\right]^{K-i} \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^{i} \\
&= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K} \\
&\quad + \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K} \sum_{n=1}^{j} \left[\frac{\rho}{1+\rho}\right]^{n} \sum_{i=1}^{K} \left[\begin{matrix} K-i+n-1 \\ n-1 \end{matrix}\right] \\
&= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K} \left[ 1 + \sum_{n=1}^{j} \left[\begin{matrix} K-1+n \\ n \end{matrix}\right] \left[\frac{\rho}{1+\rho}\right]^{n} \right] \\
&= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K} \sum_{n=0}^{j} \left[\begin{matrix} K-1+n \\ n \end{matrix}\right] \left[\frac{\rho}{1+\rho}\right]^{n}
\end{aligned}
$$

$$j=0,1,\ldots,Q-1; \qquad \text{(A.11)}$$

where $P(.,K)$ is the probability of associated net inventory level when the Erlang lead time distribution is of order K. Note that, although (A.8) is true only for $j=1,2,\ldots Q-1$, (A.11) is also valid for $j = 0$ because, from (A.5),

$$
\begin{aligned}
P(R+Q-0,K) &= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K} \\
&= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K} \sum_{n=0}^{0} \left[\begin{matrix} K-1+n \\ n \end{matrix}\right] \left[\frac{\rho}{1+\rho}\right]^{n} .
\end{aligned}
$$

Then, from (A.10) and (A.11),

$$P(R-j,K) = \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{i=1}^{K} \binom{K-i+j}{j} \left[\frac{1}{1+\rho}\right]^{K-i} P(R+1,i)$$

$$= \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{i=1}^{K} \binom{K-i+j}{j} \left[\frac{1}{1+\rho}\right]^{K-i} \left[ \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^{i} \sum_{n=0}^{Q-1} \binom{i-1+n}{n} \left[\frac{\rho}{1+\rho}\right]^{n} \right]$$

$$= \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^{K} \sum_{n=0}^{Q-1} \left[\frac{\rho}{1+\rho}\right]^{n+j+1} \sum_{i=1}^{K} \binom{K-i+j}{j} \binom{i-1+n}{n}$$

$$= \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^{K} \sum_{n=0}^{Q-1} \binom{K-1+n+j+1}{n+j+1} \left[\frac{\rho}{1+\rho}\right]^{n+j+1}$$

$$= \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^{K} \sum_{n=j+1}^{j+Q} \binom{K-1+n}{n} \left[\frac{\rho}{1+\rho}\right]^{n}$$

$$j=-1,0,1,\ldots; \qquad \text{(A.12)}$$

where the combinatorial identity [Lovász 1979, p. 18], that

$$\sum_{k=0}^{m} \binom{u+k}{k} \binom{v-k}{m-k} = \binom{u+v+1}{m}, \qquad \text{(A.13)}$$

is utilized such that

$$\sum_{i=1}^{K} \binom{K-i+j}{j} \binom{i-1+n}{n} = \sum_{i=0}^{K-1} \binom{i+j}{j} \binom{K-i-1+n}{n}$$

$$= \sum_{i=0}^{K-1} \binom{j+i}{i} \binom{K-1+n-i}{K-1-i}$$

$$= \binom{j+K-1+n+1}{K-1}$$

$$= \binom{K-1+n+j+1}{n+j+1}.$$

Equation (A.10) is only true for j=0,1,2,..., but (A.12) is also valid for j = -1 because, from (A.11),

$$P[R-(-1),K] = P[R+Q-(Q-1),K]$$

$$= \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^K \sum_{n=-1+1}^{-1+Q} \binom{K-1+n}{n} \left[\frac{\rho}{1+\rho}\right]^n .$$

Finally, also note that both of (A.11) and (A.12) hold true for any positive integer K. As we can see, (A.11) and (A.12) are identical to Equations (4.6) and (4.7), respectively.

## A.2. Proof of Theorem 4.2

By applying (A.2) recursively, we can derive for $R+1 \le X \le R+Q-1$ and $K \ge 2$ that

$$P(X,K) = \frac{\rho}{1+\rho} P(X+1,K) + \frac{1}{1+\rho} P(X,K-1)$$

$$= \frac{\rho}{1+\rho}\left[\frac{\rho}{1+\rho} P(X+2,K) + \frac{1}{1+\rho} P(X+1,K-1)\right] + \frac{1}{1+\rho} P(X,K-1)$$

$$= \left[\frac{\rho}{1+\rho}\right]^2 \left[\frac{\rho}{1+\rho} P(X+3,K) + \frac{1}{1+\rho} P(X+2,K-1)\right]$$

$$+ \frac{\rho}{1+\rho}\,\frac{1}{1+\rho} P(X+1,K-1) + \frac{1}{1+\rho} P(X,K-1)$$

$$= \ldots$$

$$= \left[\frac{\rho}{1+\rho}\right]^j P(X+j,K) + \frac{1}{1+\rho}\sum_{i=1}^{j}\left[\frac{\rho}{1+\rho}\right]^{j-i} P(X+j-i,K-1),$$

where $1 \le j \le R+Q-X$. Therefore, if let $R+Q-j = X$, we can have that

$$P(R+Q-j,K) = \left[\frac{\rho}{1+\rho}\right]^j P(R+Q,K) + \frac{1}{1+\rho}\sum_{i=1}^{j}\left[\frac{\rho}{1+\rho}\right]^{j-i} P(R+Q-i,K-1)$$

$$j=1,2,\ldots,Q-1; \qquad \text{(A.14)}$$

where $\rho = \lambda/\mu$ and $K \ge 2$.

Now, Equation (4.11) is true for $K = 1$ because, from (4.8),

$$P(R+Q-j,1) = \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{j+1}\right]$$

$$= \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=0}^{1-1} \binom{j+n}{n} \left[\frac{1}{1+\rho}\right]^{n}\right]$$

$$j=0,1,\ldots,Q-1.$$

Then, from (A.14), for $K \geq 2$

$$P(R+Q-j,K) = \left[\frac{\rho}{1+\rho}\right]^{j} P(R+Q,K) + \frac{1}{1+\rho} \sum_{i=1}^{j} \left[\frac{\rho}{1+\rho}\right]^{j-i} P(R+Q-i,K-1)$$

$$= \left[\frac{\rho}{1+\rho}\right]^{j} P(R+Q,K) + \frac{1}{1+\rho} \sum_{i=1}^{j} \left[\frac{\rho}{1+\rho}\right]^{j-i} \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{i+1} \sum_{n=0}^{K-2} \binom{i+n}{n} \left[\frac{1}{1+\rho}\right]^{n}\right]$$

$$= \left[\frac{\rho}{1+\rho}\right]^{j} \frac{1}{Q} \left[\frac{1}{1+\rho}\right]^{K} + \frac{1}{Q}\,\frac{1}{1+\rho} \sum_{i=1}^{j} \left[\frac{\rho}{1+\rho}\right]^{j-i} - \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=0}^{K-2} \left[\frac{1}{1+\rho}\right]^{n+1} \sum_{i=1}^{j} \binom{i+n}{n}$$

$$= \frac{1}{Q}\left\{ \left[\frac{\rho}{1+\rho}\right]^{j} \left[\frac{1}{1+\rho}\right]^{K} + \left[1 - \left[\frac{\rho}{1+\rho}\right]^{j}\right] - \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=0}^{K-2} \left[\frac{1}{1+\rho}\right]^{n+1} \left[\binom{j+n+1}{n+1} - 1\right]\right\}$$

$$= \frac{1}{Q}\left[1 - \left[\frac{\rho}{1+\rho}\right]^{j} + \left[\frac{\rho}{1+\rho}\right]^{j} \left[\frac{1}{1+\rho}\right]^{K} - \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=0}^{K-2} \binom{j+n+1}{n+1} \left[\frac{1}{1+\rho}\right]^{n+1} + \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=0}^{K-2} \left[\frac{1}{1+\rho}\right]^{n+1}\right]$$

$$= \frac{1}{Q}\left\{ 1 - \left[\frac{\rho}{1+\rho}\right]^{j} + \left[\frac{\rho}{1+\rho}\right]^{j}\left[\frac{1}{1+\rho}\right]^{K} \right.$$

$$- \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=1}^{K-1} \binom{j+n}{n} \left[\frac{1}{1+\rho}\right]^{n}$$

$$\left. + \left[\frac{\rho}{1+\rho}\right]^{j} \left[ \frac{1}{1+\rho} - \left[\frac{1}{1+\rho}\right]^{K} \right] \right\}$$

$$= \frac{1}{Q}\left\{ 1 - \left[\frac{\rho}{1+\rho}\right]^{j} \left[ 1 - \frac{1}{1+\rho} \right] \right.$$

$$\left. - \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=1}^{K-1} \binom{j+n}{n} \left[\frac{1}{1+\rho}\right]^{n} \right\}$$

$$= \frac{1}{Q}\left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{n=0}^{K-1} \binom{j+n}{n} \left[\frac{1}{1+\rho}\right]^{n} \right]$$

$$j=0,1,\ldots,Q-1;$$

where we apply results from (A.5) and (A.9) for the proof. Equation (A.14) holds true only for $j=1,2,\ldots,Q-1$, but the result obtained above is also valid for $j = 0$ because

$$P(R+Q-0,K) = \frac{1}{Q}\left[\frac{1}{1+\rho}\right]^{K}$$

$$= \frac{1}{Q}\left[ 1 - \left[\frac{\rho}{1+\rho}\right]^{0+1} \sum_{n=0}^{K-1} \binom{0+n}{n} \left[\frac{1}{1+\rho}\right]^{n} \right].$$

Therefore, Equation (4.11) is true for $K \geq 1$ by induction.

Now, from (A.10), we have for $K \geq 1$ that

$$P(R-j,K) = \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{i=0}^{K-1} \binom{i+j}{j} \left[\frac{1}{1+\rho}\right]^{i} P(R+1,K-i)$$

$$= \left[\frac{\rho}{1+\rho}\right]^{j+1} \sum_{i=0}^{K-1} \binom{j+i}{i} \left[\frac{1}{1+\rho}\right]^{i} \frac{1}{Q} \Bigg[ 1 - \left[\frac{\rho}{1+\rho}\right]^{Q} \sum_{n=0}^{K-i-1} \binom{Q-1+n}{n} \left[\frac{1}{1+\rho}\right]^{n} \Bigg]$$

$$= \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \Bigg[ \sum_{i=0}^{K-1} \binom{j+i}{i} \left[\frac{1}{1+\rho}\right]^{i} - \left[\frac{\rho}{1+\rho}\right]^{Q} \sum_{i=0}^{K-1} \binom{j+i}{i} \sum_{n=0}^{K-i-1} \binom{Q-1+n}{n} \left[\frac{1}{1+\rho}\right]^{n+i} \Bigg]$$

$$= \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \Bigg[ \sum_{i=0}^{K-1} \binom{j+i}{i} \left[\frac{1}{1+\rho}\right]^{i} - \left[\frac{\rho}{1+\rho}\right]^{Q} \sum_{i=0}^{K-1} \binom{j+i}{i} \sum_{n=i}^{K-1} \binom{Q-1+n-i}{n-i} \left[\frac{1}{1+\rho}\right]^{n} \Bigg]$$

$$= \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \Bigg[ \sum_{n=0}^{K-1} \binom{j+n}{n} \left[\frac{1}{1+\rho}\right]^{n} - \left[\frac{\rho}{1+\rho}\right]^{Q} \sum_{n=0}^{K-1} \left[\frac{1}{1+\rho}\right]^{n} \sum_{i=0}^{n} \binom{j+i}{i} \binom{Q-1+n-i}{n-i} \Bigg]$$

$$= \frac{1}{Q} \left[\frac{\rho}{1+\rho}\right]^{j+1} \Bigg[ \sum_{n=0}^{K-1} \binom{j+n}{n} \left[\frac{1}{1+\rho}\right]^{n} - \left[\frac{\rho}{1+\rho}\right]^{Q} \sum_{n=0}^{K-1} \binom{Q+j+n}{n} \left[\frac{1}{1+\rho}\right]^{n} \Bigg]$$

$$j=0,1,2,\ldots;$$

where we use results from (4.11) and (A.13) for the second and the last steps, respectively. Therefore, Equation (4.12) is also true for $K \geq 1$ and, hence, the proof is completed.

**BIBLIOGRAPHY**


Altiok, T. 1989. (R,r) Production/Inventory Systems. *Opns. Res.* **37**, 266-276.

Archibald, B. C. 1981. Continuous Review (s,S) Policies with Lost Sales. *Mgmt. Sci.* **27**, 1171-1177.

Archibald, B. C., and E. A. Silver. 1978. (s,S) Policies under Continuous Review and Discrete Compound Poisson Demand. *Mgmt. Sci.* **24**, 899-909.

Bagchi, U., and J. C. Hayya. 1984. Demand during Lead Time for Normal Unit Demand and Erlang Lead Time. *J. Opl. Res. Soc.* 35, 131-135.

Buchanan, D. J., and R. F. Love. 1985. A (Q,R) Inventory Model with Lost Sales and Erlang-Distributed Lead Times. *Naval Res. Logist. Q.* **32**, 605-611.

Burgin, T. A. 1972. Inventory Control with Normal Demand and Gamma Lead Times. *Opl. Res. Q.* 23, 73-80.

Burgin, T. A. 1975. The Gamma Distribution and Inventory Control. *Opl. Res. Q.* 26, 507-525.

Burgin, T. A., and A. R. Wild. 1967. Stock Control: Experience and Usable Theory. *Opl. Res. Q.* 18, 35-52.

Chiou, S. C., and G. P. Sphicas. 1988. Relative Performance of Some Approximate Review Inventory Models. *SCIMA* 17, 53-62.

Das, C. 1976. Explicit Formulas for the Order Size and Reorder Point in Certain Inventory Problems. *Naval Res. Logist. Q.* 23, 25-30.

Das, C. 1983. (Q,r) Inventory Models with Time-Weighted Backorders. *J. Opl. Res. Soc.* **34**, 401-412.

Federgruen, A., and Y. S. Zheng. 1988. A Simple and Efficient Algorithm for Computing Optimal (r,Q) Policies in Continuous Review Inventory Systems. Working Paper, Wharton School, University of Pennsylvania, Philadelphia.

Fortuin, L. 1980. Five Popular Probability Density Functions: A Comparison in the Field of Stock-Control Models. *J. Opl. Res. Soc.* **31**, 937-942.

Galliher, H. P., P. M. Morse and M. Simond. 1959. Dynamics of Two Classes of Continuous-Review Inventory Systems. *Opns. Res.* 7, 362-384.

Gross, D., and C. M. Harris. 1973. Continuous-Review (s,S) Inventory Models with State-Dependent Leadtimes. *Mgmt. Sci.* 19, 567-574.

Gross, D., and J. F. Ince. 1975. A Comparison and Evaluation of Approximate Continuous Review Inventory Models. *Internat. J. Production Res.* 13, 9-13.

Hadley, G., and T. M. Whitin. 1963. *Analysis of Inventory Systems.* Prentice- Hall, Englewood Cliffs, NJ.

Holt, C. C., F. Modigliani, J. F. Muth and H. A. Simon. 1960. *Planning Production, Inventories, and Work Force.* Prentice-Hall, Englewood Cliffs, NJ.

Johnson, L. A., and D. C. Montgomery. 1974. *Operations Research in Production Planning, Scheduling, and Inventory Control.* Wiley, New York.

Kim, D. H., and K. S. Park. 1985. (Q,R) Inventory Model with a Mixture of Lost Sales and Time Weighted Backorders. *J. Opl.Res. Soc.* 36, 231-238.

Lovász, L. 1979. *Combinatorial Problems and Exercises.* Elsevier North-Holland, New York.

Love, S. F. 1979. *Inventory Control.* McGraw-Hill, New York.

Moinzadeh, K. 1989. Operating Characteristics of the (S-1,S) Inventory System with Partial Backorders and Constant Resupply Times. *Mgmt. Sci.* 35, 472-477.

Montgomery, D. C., M. S. Bazaraa and A. K. Keswani. 1973. Inventory Models with a Mixture of Backorders and Lost Sales. *Naval Res. Logist. Q.* 20, 255-263.

Nahmias, S. 1976. On the Equivalence of Three Approximate Continuous Review Inventory Models. *Naval Res. Logist. Q.* 23, 31-36.

Peterson, R., and E. A. Silver. 1985. *Decision Systems for Inventory Management and Production Planning.* 2nd Ed., Wiley, New York.

Posner, M. J. M., and B. Yansouni. 1972. A Class of Inventory Models with Customer Impatience. *Naval Res. Logist. Q.* 19, 483-492.

Ravichandran, N. 1984. A Note on (s,S) Inventory Policy. *AIIE Trans.* 16, 387-390.

Richards, F. R. 1975. Comments on the Distribution of Inventory Position in a Continuous-Review (s,S) Inventory System. *Opns. Res.* 23, 366-371.

Sahin, I. 1979. On the Stationary Analysis of Continuous Review (s,S) Inventory Systems with Constant Lead Times. *Opns. Res.* 27, 717-729.

Sivazlian, B. D. 1974. A Continuous Review (s,S) Inventory System with Arbitrary Interarrival Distribution between Unit Demand. *Opns. Res.* 22, 65-71.

Smith, S. A. 1977. Optimal Inventories for an (S-1,S) System with No Backorders. *Mgmt. Sci.* 23, 522-528.

Tadikamalla, P. R. 1978. Applications of the Weibull Distribution in Inventory Control. *J. Opl. Res. Soc.* 29, 77-83.

Tadikamalla, P. R. 1979. The Lognormal Approximation to the Lead Time Demand in Inventory Control. *OMEGA* 7, 553-556.

Wagner, H. M. 1969. *Principles of Operations Research with Applications to Managerial Decisions*. Prentice-Hall, Englewood Cliffs, NJ.

Zheng, Y. S. 1989. Is the EOQ still a Good Model when Demands are Stochastic? Working Paper, Wharton School, University of Pennsylvania, Philadelphia.

Zipkin, P. 1986. Inventory Service-Level Measures: Convexity and Approximation. *Mgmt. Sci.* 32, 975-981.