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ON THE GEOMETRY OF THE TEICHMÜLLER SPACE

by

NIKOLA LAKIC

A dissertation submitted to the Graduate Faculty in Mathematics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy, The City University of New York

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4/28/95 Frederick P. Gardiner

Date Chair of Examining Committee

4/28/95

Date Executive Officer

Frederick P. Gardiner

Dennis P. Sullivan

Linda Keen

Richard Sacksteder

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

ON THE GEOMETRY OF THE TEICHMÜLLER SPACE

by

NIKOLA LAKIC

Adviser: Professor Frederick P. Gardiner

Let $A(X)$ be the Banach space of integrable, holomorphic, quadratic differentials φ on a Riemann surface X. We characterize the points of $A(X)$ at which the norm is weak uniformly convex in terms of the infinitesimal form of Teichmüller's metric on QS mod S and we give a quantified version of this characterization. Sullivan's coiling property applies along any Beltrami line $[t|\varphi|/\varphi]$ for which φ is a point of weak uniform convexity and the amount of coiling is quantified by the quantified version of weak convexity. For a closed set J in $\mathbb{C}$, we let $A(J)$ be the Banach space of integrable functions in $\mathbb{C}$ which are holomorphic in the complement of J. We generalize Bers' approximation theorem by showing that rational functions with simple poles in J are dense in $A(J)$. Density is with respect to the L^1-norm over the whole complex plane, including J. Assume both X and Y are Riemann surfaces which are subsets of compact Riemann surfaces X_1 and Y_1, respectively, such that set $X_1 - X$ has infinitely many points. Finally we prove that the only surjective complex linear isometries between spaces of integrable holomorphic quadratic differentials on X and Y are the ones induced by conformal homeomorphisms and complex constants of modulus 1. As a corollary we conclude that if the Teichmüller space of X is biholomorphic to the Teichmüller space of Y, then X is quasiconformal to Y.

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CONTENTS

Chapter 1

Preliminaries

1.1 Teichmüller Space.

Let $\mathbb{D}$ be the unit disk and Γ be a torsion-free Fuchsian group acting on $\mathbb{D}$. Then $X = \mathbb{D}/\Gamma$ is a Riemann surface. We let $L^\infty(X)$ be the space of essentially bounded complex-valued measurable functions μ on $\mathbb{D}$, which satisfy $\mu(\gamma z)\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$ for every γ in Γ, and $M(X)$ be the open unit ball in $L^\infty(X)$. For any μ in $M(X)$, there exists a solution $f : \mathbb{D} \to \mathbb{D}$ of the Beltrami equation

$$f_{\overline{z}} = \mu f_z, \tag{1.1}$$

unique up to a postcomposition by a Möbius transformation. We let f^μ be the solution f of (1) normalized by $f(i) = i, f(1) = 1$ and $f(-1) = -1$.

Two elements μ_0 and μ_1 in $M(X)$ are equivalent if f^{μ_0} and f^{μ_1} coincide on $\partial\mathbb{D}$. The Teichmüller space $Teich(X)$ is $M(X)$ factored by this equivalence relation. Teichmüller's metric d_T is given by

$$d_T([0],[\mu]) = \inf \frac{1}{2}\log\frac{1+||\mu||_\infty}{1-||\mu||_\infty},$$

where the infimum is taken over all μ in the equivalence class $[\mu]$, and by declaring that right compositions

$$[\eta] \rightarrow [\frac{(f^\eta \circ f^\mu)_{\bar{z}}}{(f^\eta \circ f^\mu)_z}]$$

are isometries.

1.2 The Tangent Space to Teich(X).

A holomorphic quadratic differential φ on $X = \mathbb{D}/\Gamma$ is a holomorphic function $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ which satisfies $\varphi(\gamma z)\gamma'(z)^2 = \varphi(z)$ for every γ in Γ. $A(X)$ is the space of all holomorphic quadratic differentials φ on X which satisfy $||\varphi|| = \iint_\omega |\varphi| < \infty$ where ω is a fundamental domain for Γ.

The natural pairing:

$$(\mu,\varphi)_X = \iint_\omega \mu\varphi \ ; \quad \mu \in L^\infty(X), \varphi \in A(X)$$

induces a linear map P from $L^\infty(X)$ onto $A(X)^*$, the dual space of $A(X)$, defined by $P\mu(\varphi) = (\mu,\varphi)_X$. The kernel of P is $N = \{\mu \in L_\infty(X) | (\mu,\varphi)_X = 0 \text{ for every } \varphi \in A(X)\}$. We denote the equivalence class of an element μ in $L^\infty(X)$ modulo N by $[\mu]$.

$B(X)$, the tangent space to $Teich(X)$ at $[0]$, is the space $L_\infty(X)/N(X) = A(X)^*$. The infinitesimal form of Teichmüller's metric is

$$d_T([0],[t\mu]) = \sup |t \ Re \iint_\omega \mu\varphi| + O(t^2)$$

where the supremum is over all φ in the unit ball of $A(X)$ and the constant in $O(t^2)$ depends only on $||\mu||_\infty$, [G2].

A sequence φ_n in $A(X)$ is called a degenerating sequence if $||\varphi_n|| \leq 1$ for every n and if $\varphi_n \to 0$ uniformly on compact subsets of X. Let $B_0(X)$ be the space of all elements $[\mu]$ in $B(X)$ such that $(\mu, \varphi_n)_X$ tends to zero for every degenerating sequence φ_n in $A(X)$. It is known that the dual space of $B_0(X)$ is isometrically isomorphic to $A(X)$, [EG]. $B_0(X)$ is the closed subspace of $B(X)$, and the quotient norm makes $B(X)/B_0(X)$ into a Banach space.

1.3 The Universal Teichmüller Space.

Teichmüller space $T(\mathbb{D})$ can be viewed as QS mod G where G is the group of all orientation preserving Möbius homeomorphisms of the circle $\partial\mathbb{D}$, and QS is the class of all quasisymmetric homeomorphisms of the circle $\partial\mathbb{D}$. The boundary dilatation $BD(f)$ of a quasisymmetric homeomorphism f is obtained by looking at the infimum of all maximal dilatations of quasiconformal extensions of f to a neighborhood U of the boundary and taking the limit of these dilatations as U shrinks to the boundary. We call a quasisymmetric homeomorphism f symmetric, if $BD(f) = 1$.

Let S be the class of all symmetric homeomorphisms of $\partial\mathbb{D}$. The space QS mod S has its Teichmüller's metric $\overline{d}$ defined by

$$\overline{d}(Sf, Sg) = \frac{1}{2} \inf \log K(s_1 \circ f \circ g^{-1} \circ s_2^{-1})$$

where the infimum is over all s_1 and s_2 in S, [GS].

The infinitesimal form of Teichmüller's metric $\overline{d}$ is

$$\overline{d}(S, Sf^{t\mu}) = |t|\beta([\mu]) + O(t^2)$$

where

$$\beta([\mu]) = \sup_{(\varphi_n)} \limsup_{n\to\infty} |\iint \mu\varphi_n|$$

and the supremum is taken over all degenerating sequences φ_n in $A(\mathbf{D})$, ([G1]).

By lemma 5.1 in [GS],

$$\overline{d}(Sf, Sg) = \frac{1}{2}\log BD(f\circ g^{-1}).$$

Let $\overline{K_0}(f^\mu)$ be the infimum of the dilatations of f^μ restricted to U, over all annular neighborhoods U of the boundary of the unit disk and let $\overline{k_0}(f^\mu) = \frac{\overline{K_0}(f^\mu)-1}{\overline{K_0}(f^\mu)+1}$. We say that a Beltrami coefficient μ realizes its boundary dilatation if $BD(f) = \overline{K_0}(f^\mu)$, where f is the restriction of f^μ to the unit circle $\partial\mathbf{D}$.

In [G1] it is proved that a Beltrami coefficient μ realizes its boundary dilatation if, and only if,

$$\beta([\mu]) = \overline{k_0}(f^\mu).$$

That is the Hamilton-Reich-Strebel necessary and sufficient condition for extremality in QS mod S.

Also in [G1], Gardiner proved the following principle of Teichmüller contraction:

Assume $||\mu||_\infty = 1, 0 < k_1 < k_2 < 1$, and $\overline{d}(S, Sf^{k_1\mu}) \leq \lambda_1 d_p(0, k_1)$ where $\lambda_1 < 1$ and d_p is the Poincaré metric on the unit disk. Then there exists $\lambda < 1$ depending only on k_1, k_2 and λ_1 such that

$$\overline{d}(S, Sf^{k\mu}) \leq \lambda d_p(0, k)$$

for all μ with $||\mu||_\infty = 1$ and all k with $0 \leq k \leq k_2$.

This contracting property of the function $I_\mu(t) = [t\frac{\mu}{||\mu||_\infty}]$ is called a coiling property by Sullivan [S] because if the two points along the Beltrami line

$[t\mu]$ are not stretched apart as far as they can be by a certain factor, then throughout most of the line the distance between pairs of points is not great as it could be, by an amount depending on the given factor.

1.4 The Unit Sphere in A(X).

Let $S(X)$ be the set of all φ in $A(X)$ of norm one. If φ and ψ are in A(X) we let

$$< \varphi, \psi >= Re \iint_X \varphi \frac{|\psi|}{\psi}$$

be the (non-linear) pairing of φ and ψ. $< \varphi, \psi >$ is the first derivative of $f(t) = \iint_X |\psi + t\varphi|$ at $t = 0$.

We say that A(X) is weak uniformly convex, or non-flat, at a point φ in S(X), if $\psi_n \in S(X)$ and $< \psi_n, \varphi > \rightarrow 1$ imply $||\varphi - \psi_n|| \rightarrow 0$. A(X) is weak uniformly convex if it is weak uniformly convex at every φ in S(X). If $||\varphi - \psi|| = O((1- < \varphi, \psi >)^\alpha)$ for $\psi \in S(X)$, then A(X) is weak uniformly convex at $\varphi \in S(X)$ with exponent α. It is known that there are differentials in $S(\mathbb{D})$ where $A(\mathbb{D})$ is not weak uniformly convex ([M]), and that $A(\mathbb{D})$ is weak uniformly convex at $\varphi(z) = \frac{1}{\pi}$ with exponent $\frac{1}{2}$ ([Go]).

Chapter 2

Weak Uniform Convexity in $A(X)$.

2.1 A Criterion For Weak Uniform Convexity.

Define the infinitesimal functional $\beta(v)$ of $v \in B(X)$ as the supremum of

$$\limsup_{n\to\infty} |v(\varphi_n)|$$

over all degenerating sequences φ_n in A(X).

Note that if X is the unit disk, then β coincides with the previously defined differentiated form of Teichmüller's metric in QS mod S. Also, note that $v \in B_0(X)$ iff $\beta(v) = 0$. Therefore β defines a norm on the space $B(X)/B_0(X)$. We call β a degenerating norm.

Let $\varphi \in S(X)$ and $v = [\frac{|\varphi|}{\varphi}]$. If $\psi \in A(X)$ then $|v(\psi)| = |\iint \psi \frac{|\varphi|}{\varphi}| \leq ||\psi||$. Hence $||v|| = 1$. If $v(\psi) = ||v||$ for some $\psi \in S(X)$, then $v(\varphi + \psi) = 2||v||$, and thus $||\varphi + \psi|| = ||\varphi|| + ||\psi|| = 2$. That yields

$$|\varphi + \psi| \equiv |\varphi| + |\psi|,$$

$$|1 + \frac{\psi}{\varphi}| \equiv 1 + |\frac{\psi}{\varphi}|,$$

$$\frac{\psi}{\varphi} \geq 0.$$

Therefore, φ is the unique differential in the closed unit ball of A(X), where v takes value $||v||$.

Theorem 1 *Let $\varphi \in S(X)$ and $v = [\frac{|\varphi|}{\varphi}]$. Then $A(X)$ is weak uniformly convex at φ if and only if the degenerating norm of v is less than 1.*

PROOF. First suppose that $\beta(v) = 1$. Then there exists a degenerating sequence (φ_n) in A(X) such that $v(\varphi_n) = \iint \varphi_n \frac{|\varphi|}{\varphi}$ tends to 1. Hence,

$$< \varphi_n, \varphi >= Re \iint \varphi_n \frac{|\varphi|}{\varphi} \to 1.$$

However, by Lebesque's dominated convergence theorem,

$$||\varphi - \varphi_n|| - 1 = \iint (|\varphi - \varphi_n| - |\varphi_n|) \to \iint |\varphi| = 1.$$

Therefore $||\varphi - \varphi_n|| \to 2$, thus A(X) is flat at φ.

Now suppose that A(X) is flat at $\varphi \in S(X)$. Then there exists a sequence φ_n in S(X) so that $< \varphi_n, \varphi >\to 1$ and $||\varphi - \varphi_n|| \geq \epsilon > 0$. Since the family $\{\varphi_n\}$ is normal, some subsequence φ_{n_k} converges uniformly on compact sets to a differential $\psi \in A(X)$. It follows from Fatou's lemma that

$$||\psi|| \leq \lim_{k\to\infty} \inf ||\varphi_{n_k}|| = 1.$$

If $||\psi|| = 0$, then the sequence (φ_{n_k}) is degenerating, and $Re\ v(\varphi_{n_k}) =< \varphi_{n_k}, \varphi >\rightarrow 1$; thus $\beta(v) = 1$. If $0 < ||\psi|| < 1$, take $\psi_k = \frac{\varphi_{n_k} - \psi}{||\varphi_{n_k} - \psi||}$. Then, by Lebesque's dominated convergence theorem,

$$||\varphi_{n_k} - \psi|| - 1 = \iint (|\varphi_{n_k} - \psi| - |\varphi_{n_k}|) \rightarrow \iint -|\psi|.$$

Therefore, ψ_k tends to $\frac{0}{1-||\psi||} = 0$ uniformly on compact subsets of X. Furthermore,

$$1 \geq Re\ v(\psi_k) = \frac{Re\ v(\varphi_{n_k}) - Re\ v(\psi)}{||\varphi_{n_k} - \psi||} \rightarrow \frac{1 - Re\ v(\psi)}{1 - ||\psi||} \geq 1.$$

Hence, (ψ_k) is degenerating and $v(\psi_k) \rightarrow 1$; thus $\beta(v) = 1$.

If $||\psi|| = 1$, then $||\varphi_{n_k} - \psi|| \rightarrow 1 - ||\psi|| = 0$. Hence,

$$< \psi, \varphi >= Re \iint \psi \frac{|\varphi|}{\varphi} = \lim_{n \rightarrow \infty} Re \iint \varphi_{n_k} \frac{|\varphi|}{\varphi} = \lim_{n \rightarrow \infty} < \varphi_{n_k}, \varphi >= 1.$$

That implies $v(\psi) = ||v||$ and hence $\psi = \varphi$. Therefore $||\varphi_{n_k} - \varphi|| \rightarrow 0$; a contradiction. □

Corollary 1 *If $\varphi \in S(X)$ and $v = [\frac{|\varphi|}{\varphi}] \in B_0(X)$, then $A(X)$ is weak uniformly convex at φ.*

PROOF. Since $v \in B_0(X)$, we have $\beta(v) = 0 < 1$, and by theorem 1, A(X) is weak uniformly convex at φ. □

Corollary 2 *$A(\mathbb{D})$ is weak uniformly convex at each point $\varphi_n(z) = \frac{n+2}{2\pi} z^n$.*

PROOF. Let $v = [\frac{|\varphi_n|}{\varphi_n}]$. Then,

$$v(\psi) = \iint_D \psi \frac{|\varphi_n|}{\varphi_n} = \iint_D \psi(z) \frac{|z|^n}{z^n} dxdy =$$

$$= \int_0^1 \int_0^{2\pi} \frac{\psi(re^{it})}{e^{int}} r dr dt = \int_0^1 r^{n+1} (\frac{1}{i} \int_{\gamma_r} \frac{\psi(z)}{z^{n+1}} dz) dr$$

where γ_r is a circle with center at 0 and radius r. Hence,

$$v(\psi) = 2\pi \int_0^1 r^{n+1} \frac{\psi^{[n]}(0)}{n!} dr = 2\pi \frac{\psi^{[n]}(0)}{(n+2)n!}$$

where $\psi^{[n]}(z) = \frac{d^n}{dz^n}\psi(z)$. If ψ_k is degenerating then $\psi_k^{[n]}(0)$ tends to 0 as k tends to ∞. Thus $\beta(v) = 0$. □

Corollary 3 *If X is a Riemann surface of finite analytic type then $A(X)$ is weak uniformly convex.*

PROOF. In general $B(X) = A(X)^* = B_0(X)^{**}$. If $A(X)$ is finite dimensional, then $B_0(X) = B(X)$ and so $\beta(v) = 0$ for every v in $B(X)$. In particular, we apply theorem 1 to $v(\psi) = \iint \psi \frac{|\varphi|}{\varphi}$. □

Theorem 1 and corollaries 2 and 3 give the answers to questions in problems 1, 2 and 3 in [Go].

Corollary 4 *Let $\varphi \in S(\mathbb{D})$ and $\mu = \frac{|\varphi|}{\varphi}$. Then $A(\mathbb{D})$ is weak uniformly convex at φ iff $\overline{d}(Sf^{k\mu}, S) < d([k\mu], 0)$ for every $k \in (0,1)$. In other words, Sullivan's coiling property along the Beltrami line $[k\frac{|\varphi|}{\varphi}]$ is equivalent to weak uniform convexity of $A(\mathbb{D})$ at φ.*

PROOF. Combine theorem 1 and the Hamilton-Reich-Strebel necessary and sufficient condition for extremality in QS mod S. □

Note that the inequality $\overline{d}(Sf^{k\mu}, S) < d([k\mu], [0])$ means that the projection map $p : QS \to QS$ mod S decreases the distance between [0] and

$[k\mu]$. By the Strebel's frame mapping theorem, if the projection map p decreases the distance between $[0]$ and some point $[\eta]$ in QS, then there exists a constant k and a quadratic differential $\varphi \in S(\mathbb{D})$ such that $[\eta] = [k\frac{|\varphi|}{\varphi}]$. Therefore, corollary 4 shows that p decreases the distance between $[0]$ and some point $[\mu]$ in QS, if and only if there exists a constant k and a quadratic differential $\varphi \in S(\mathbb{D})$ such that $[\mu] = [k\frac{|\varphi|}{\varphi}]$ and $A(\mathbb{D})$ is weak uniformly convex at φ. In [XT], it is proved that if the quadratic differential $\varphi \in A(\mathbb{D})$ satisfies $|\arg(\varphi)(z)| \leq \theta$ for every $z \in \mathbb{D}$ with $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \arcsin\frac{1}{11}$, then p decreases the distance between $[0]$ and $[k\frac{|\varphi|}{\varphi}]$ for every k between 0 and 1. Therefore, if $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \arcsin\frac{1}{11}$, then $A(\mathbb{D})$ is weak uniformly convex at every point φ which satisfies $|\arg(\varphi)(z)| \leq \theta$ for every $z \in \mathbb{D}$.

2.2 Constant of Weak Uniform Convexity.

In this section we strengthen theorem 1 by introducing the constant of weak uniform convexity.

First we prove

Lemma 1 *Let $\varphi \in S(X)$, $v = [\frac{|\varphi|}{\varphi}]$, and let $D(\varphi, \delta)$ be the set of all ψ in S(X) with $< \psi, \varphi > \geq 1 - \delta$. Then the following are equivalent:*

(1) A(X) is flat at φ.

(2) There exist a constant $\epsilon > 0$ and a sequence (φ_n) in S(X) such that $< \varphi_n, \varphi > \rightarrow 1$ and $||\varphi_n - \varphi|| \geq \epsilon$.

(3) $\beta(v) = 1$.

(4) There exists a degenerating sequence (φ_n) such that $< \varphi_n, \varphi > \rightarrow 1$.

(5) There exists a sequence (φ_n) *in S(X) such that* $<\varphi_n, \varphi>\to 1$ *and* $||\varphi_n - \varphi|| \to 2$.

(6) $\sup_{\psi \in D(\varphi,\delta)} ||\varphi - \psi|| = 2$, *for every* $\delta > 0$.

PROOF. (1) is equivalent to (2) by the definition of weak uniform convexity. (1) is equivalent to (3) by theorem 1. (3) is equivalent to (4) by the definition of the degenerating norm β. To prove that (4) implies (5) suppose that (φ_n) is a degenerating sequence. Then, by Lebesque's dominated convergence theorem,

$$||\varphi_n - \varphi|| - 1 = \iint (|\varphi_n - \varphi| - |\varphi_n|) \to \iint |-\varphi| = 1.$$

Clearly, (5) implies (2) and (6).

Finally, letting $\delta \to 0$, we see that (6) implies (5). □

Definition 1 *We say that* $A(X)$ *is weak uniformly convex at* φ *in* $S(X)$ *with constant* δ, *if* $\sup_{\psi \in D(\varphi,\delta)} ||\varphi - \psi|| < 2$. $A(X)$ *is weak uniformly convex with constant* δ, *if it is weak uniformly convex with constant* δ *at every* $\varphi \in S(X)$.

It follows from lemma 1 that A(X) is weak uniformly convex at φ in S(X) if and only if there exists $\delta > 0$ such that A(X) is weak uniformly convex at φ with constant δ.

Theorem 2 *Let* $\varphi \in S(X), \delta > 0$, *and* $v = [\frac{|\varphi|}{\varphi}]$. *Then A(X) is weak uniformly convex at* φ *with constant* δ, *if and only if* $\beta(v) < 1 - \delta$.

PROOF. If $\beta(v) \geq 1 - \delta$, then there exists a degenerating sequence (φ_n) in S(X) such that $<\varphi_n, \varphi>= v(\varphi_n) \to \beta(v) \geq 1 - \delta$. Therefore, we can find nonnegative numbers c_n such that $c_n \to 0$ and $c_n \geq \frac{1-\delta-<\varphi_n,\varphi>}{\delta}$. Let

$\psi_n = \frac{\varphi_n + c_n\varphi}{||\varphi_n + c_n\varphi||}$. Then ψ_n is degenerating because $1 - c_n \leq ||\varphi_n + c_n\varphi|| \leq 1 + c_n$, φ_n is degenerating, and $c_n \to 0$. Therefore, by Lebesque's dominated convergence theorem, $||\psi_n - \varphi|| \to 2$. Furthermore,

$$< \psi_n, \varphi > = \frac{< \varphi_n, \varphi > + c_n}{||\varphi_n + c_n\varphi||} \geq \frac{< \varphi_n, \varphi > + c_n}{1 + c_n} \geq 1 - \delta.$$

Therefore, $\sup_{\psi \in D(\varphi,\delta)} ||\varphi - \psi|| = 2$ and A(X) is not weak uniformly convex at φ with constant δ.

To prove the converse, we assume that $\beta(v) < 1 - \delta$. Suppose that A(X) is not weak uniformly convex at φ with constant δ. Then, there exists a sequence φ_n in $D(\varphi, \delta)$ such that $||\varphi_n - \varphi|| \to 2$. Since the family $\{\varphi_n\}$ is normal, some subsequence of φ_n converges uniformly on compact sets to some $\psi \in A(X)$. Without loss of generality we can assume that φ_n converges uniformly on compact sets to ψ. We have,

$$1 = \lim(||\varphi_n - \varphi|| - 1) = \lim \iint (|\varphi_n - \varphi| - |\varphi_n|) = ||\psi - \varphi|| - ||\psi||.$$

Hence, $||\psi - \varphi|| = ||\psi|| + || - \varphi||$, thus $|\psi - \varphi| = |\psi| + | - \varphi|$. Therefore $\psi = -k\varphi$ with some constant $k \geq 0$. If $k = 0$ then φ_n is a degenerating sequence, and

$$\beta(v) \geq \overline{lim}|v(\varphi_n)| \geq \overline{lim} < \varphi_n, \varphi > \geq 1 - \delta, \text{ a contradiction.}$$

If $k = 1$ then, by Lebesque's dominated convergence theorem, $||\varphi_n - \psi|| \to 1 - ||\psi|| = 0$. Therefore,

$$< \varphi_n, \varphi > \to < \psi, \varphi > = -1, \text{ a contradiction.}$$

Finally, suppose that $0 < k < 1$. Let $\psi_n = \frac{\varphi_n - \psi}{||\varphi_n - \psi||}$. Then (ψ_n) is a degenerating sequence in S(X), and

$$Re\ v(\psi_n) = Re \frac{v(\varphi_n) - v(\psi)}{||\varphi_n - \psi||} \geq \frac{1 - \delta - Re\ v(\psi)}{||\varphi_n - \psi||} \to$$

$$\to \frac{1-\delta - Re\ v(\psi)}{1-||\psi||} = \frac{1-\delta+k}{1-k} \geq 1-\delta.$$

That yields $\beta(v) \geq 1-\delta$; a contradiction. □

The principle of Teichmüller contraction and theorem 2 improve corollary 4.

Corollary 5 *If $\varphi \in S(\mathbf{D})$, $0 < K < 1$, and if $A(\mathbf{D})$ is weak uniformly convex at φ with constant δ, then*

$$\overline{d}(S, Sf^{k\frac{|\varphi|}{\varphi}}) < (1-(1-K)^2\delta)d_P(0,k)$$

for every $0 < k \leq K$.

PROOF. Suppose that $0 < k \leq K$. Let $k_0 = \frac{BD(f^{k\frac{|\varphi|}{\varphi}})-1}{BD(f^{k\frac{|\varphi|}{\varphi}})+1}$. Then

$\overline{d}(S, Sf^{k\frac{|\varphi|}{\varphi}}) = d_P(0,k_0)$.

Since $A(\mathbf{D})$ is weak uniformly convex at φ with constant δ, $\beta([\frac{|\varphi|}{\varphi}]) < 1-\delta$ by theorem 2. By proposition 3.2 in [G1] we have

$$k - k_0 \geq (1-k)^2 k(1-\beta([\frac{|\varphi|}{\varphi}])) > (1-K)^2 k\delta.$$

Therefore $k_0 < k(1-(1-K)^2\delta)$.

Now consider function $l(c) = \log\frac{1+ac}{1-ac} - c\log\frac{1+a}{1-a}$. We have $l(0) = l(1) = 0$ and $l'(c) = \frac{2a}{1-a^2c^2} - \log\frac{1+a}{1-a}$. Therefore l' has exactly one zero on the interval $(0,1)$. Since $l'' > 0$, l is negative on $(0,1)$. Thus $\log\frac{1+ac}{1-ac} < c\log\frac{1+a}{1-a}$, for every $c \in (0,1)$.

Let $C = 1-(1-K)^2\delta$. Then $\overline{d}(S, Sf^{k\frac{|\varphi|}{\varphi}}) = d_P(0,k_0) < d_P(0,Ck) < Cd_P(0,k)$. □

Corollary 6 *Let* $C < 1$, $\varphi \in S(X)$, *and* $v = [\frac{|\varphi|}{\varphi}] \in B_0(X)$. *Then* $A(X)$ *is weak uniformly convex at* φ *with constant* C.

Therefore,

(1) for every $C < 1$ *and every* n, $A(\mathbb{D})$ *is weak uniformly convex at* $\varphi_n(z) = \frac{n+2}{2\pi} z^n$ *with constant* C, *and*

(2) if $C < 1$ *and if* R *is a Riemann surface of finite analytic type, then* $A(R)$ *is weak uniformly convex with constant* C.

PROOF. Since $v \in B_0(X)$ we have $\beta(v) = 0$, and corollary 6 follows immediately from theorem 2. □

Chapter 3

Zygmund Bounded Functions on a Closed Set

3.1 Definitions.

In [G3], Gardiner defined the complex Zygmund space $\mathcal{Z}(\overline{\mathbb{C}})$ to be the set of complex functions V(z) defined on $\overline{\mathbb{C}}$ such that V induces a motion of cross ratios with bounded velocity measured in the Poincaré metric for the cross ratio of every possible quadruple of points selected from $\overline{\mathbb{C}}$, factored by quadratic polynomials.

In other words, $\mathcal{Z}(\overline{\mathbb{C}})$ is the set of functions V(z) satisfying

$$||V||_{cr} = \sup \ |\frac{MT}{LR}|\rho(\frac{MT}{LR})|V(LMRT)| < \infty \tag{3.1}$$

where $L = z_2 - z_1, M = z_3 - z_2, R = z_4 - z_3$ and $T = z_1 - z_4$, ρ is the infinitesimal form of the Poincaré metric in $\mathbb{C} - \{0, 1\}$,

$$V(LMRT) = \frac{V(z_2) - V(z_1)}{z_2 - z_1} - \frac{V(z_2) - V(z_3)}{z_2 - z_3} + \frac{V(z_3) - V(z_4)}{z_3 - z_4} - \frac{V(z_4) - V(z_1)}{z_4 - z_1}$$

and the supremum is over all combinations of four distinct points z_1, z_2, z_3 and z_4 in $\overline{\mathbf{C}}$. This norm makes $\mathcal{Z}(\overline{\mathbf{C}})$ into a Banach space.

Agard's formula

$$\rho(t)^{-1} = \frac{1}{\pi} \iint_{\mathbf{C}} \left| \frac{t(t-1)}{z(z-1)(z-t)} \right| dxdy,$$

shows that if $\mu \in L^\infty(\mathbf{C})$ and if V(t) is given by formula

$$V(t) = -\frac{1}{\pi} \iint_{\mathbf{C}} \mu(z)\left(\frac{1}{z-t} - \frac{t}{z-1} + \frac{t-1}{z}\right) dxdy \tag{3.2}$$

then $V \in \mathcal{Z}(\overline{\mathbf{C}})$ and $||V||_{cr} \leq ||\mu||_\infty$.

In [EG2], Earle and Gardiner defined the complex Zygmund space of a closed set J in $\overline{\mathbf{C}}$ as follows. Assume J is a closed set in $\overline{\mathbf{C}}$ that contains the point at infinity. Let $A(J)$ be the Banach space of functions $f(z)$, holomorphic in $\overline{\mathbf{C}} - J$ and measurable on $\mathbf{C}$ such that $||f|| = \iint_{\mathbf{C}} |f(z)| dxdy < \infty$. Let $\hat{\mathcal{Z}}(J)$ be the set of functions $V(z)$ defined on J such that

$$\pi \sup \left| \sum V(x_j) c_j \right| = ||V||_T < \infty \ ,$$

where the supremum is over all rational functions $\sum_{j=1}^n \frac{c_j}{z - x_j}$ in A(J) of norm 1; and let $\mathcal{Z}(J)$ be $\hat{\mathcal{Z}}(J)$ factored by the two dimensional subspace of affine functions.

Notice that every function V of the form (3), with $\mu \in L^\infty(\mathbf{C})$, belongs to $\mathcal{Z}(J)$. To prove the converse, assume that 0 and 1 are in J and let $V \in \mathcal{Z}(J)$ such that $V(0) = V(1) = 0$. Define $L(\varphi) = -\pi \sum_{j=1}^n V(x_j) c_j$ for every $\varphi = \sum_{j=1}^n \frac{c_j}{z - x_j}$ in A(J). Let $R(J)$ be the set of rational functions ψ which are holomorphic in $\mathbf{C}$ except for at most finitely many simple poles at points in J and for which $\iint_{\mathbf{C}} |\psi| < \infty$. Then, we have $|L(\varphi)| \leq ||V||_T ||\varphi||$ for every φ in $R(J)$. $R(J)$ is a subspace of $L^1(\mathbf{C})$, and by Hahn-Banach theorem we can extend L to a bounded linear functional $\hat{L}$ on $L^1(\mathbb{C})$. Therefore, there exists $\mu \in L^\infty(\mathbb{C})$ such that $\hat{L}(\varphi) = \iint_{\mathbb{C}} \mu\varphi$ for every $\varphi \in A(J)$.

Let

$$W(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\mu(\zeta)z(z-1)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta.$$

Then $W \in \mathcal{Z}(\overline{\mathbb{C}})$, $\overline{\partial}W = \mu$ in the generalized sense, and $Wz = Vz$ for every $z \in J$.

Therefore, every $V \in \mathcal{Z}(J)$ can be extended to a function $W \in \mathcal{Z}(\overline{\mathbb{C}})$ with bounded $\overline{\partial}$ derivative.

Also in [EG2], it is proved that the norms $||.||_T$ and $||.||_{cr}$ are equivalent when $J = \overline{\mathbb{C}}$.

3.2 The Pairing Between $\mathcal{Z}(J)$ and $A(J)$.

For $\varphi \in A(J)$ and V in $\mathcal{Z}(J)$ define a pairing $(\ ,\ ) : \mathcal{Z}(J) \times A(J) \to \mathbb{R}$ by

$$(V, \varphi) = Re \iint_{\mathbb{C}} \varphi \overline{\partial} \hat{V} \tag{3.3}$$

where $\hat{V}$ is any extension of V with bounded $\overline{\partial}$ derivative and with $\hat{V}(z) = o(z^2)$ as $z \to \infty$.

Theorem 3 *The pairing $(\ ,\ )$ between $A(J)$ and $\mathcal{Z}(J)$ given by (4) is well defined.*

PROOF. We have to show that the pairing does not depend on which extension of V is chosen. Assume that $V \in \mathcal{Z}(\overline{\mathbb{C}})$, such that V is identicaly equal 0 on J. Let $\varphi \in A(J)$ and let $(V, \varphi) = I_1 + I_2$, where

$$I_1 = \iint_{\mathbb{C}-J} \varphi \overline{\partial} V, \text{ and}$$

$$I_2 = \iint_J \varphi \overline{\partial} V.$$

Using the mollifier, Ahlfors showed that $I_1 = 0$. (See, for example, [G2], page 72).

To prove $I_2 = 0$ we have to show that $\overline{\partial} V(z) = 0$ for almost every $z \in J$. Let $\mu = \overline{\partial} V$. We may assume that set $P = \{z \in J | \mu(z) \neq 0\}$ has positive Lebesque measure. Therefore, there exists a point in P, which is a Lebesque point of μ and of the characteristic function of J. Without loss of generality we may assume that this point is 0. Let $z = re^{i\theta}$. If r is sufficiently small, then there exist points p and q in J, such that

$$|p - \frac{r}{3}e^{i\theta}| < \frac{r}{10} \text{ and } |q - \frac{2r}{3}e^{i\theta}| < \frac{r}{10}.$$

Letting $L = re^{i\theta} - 0, M = q - re^{i\theta}, R = p - q$, and $T = 0 - p$, we get

$$\frac{MT}{LR} = \frac{(p-0)(re^{i\theta} - q)}{(p-q)re^{i\theta}}, \text{ and}$$

$$V(LMRT) = \frac{V(p) - V(0)}{p - 0} - \frac{V(p) - V(q)}{p - q} + \frac{V(re^{i\theta}) - V(q)}{re^{i\theta} - q} - \frac{V(re^{i\theta}) - V(0)}{re^{i\theta} - 0}$$

Then, $|\frac{MT}{LR}|\rho(\frac{MT}{LR})|V(LMRT)| \leq ||V||_{cr} \leq ||\mu||_\infty$, and consequently $|V(z)| \leq C|z|$, for some constant C.

Now choose $\epsilon > 1$ such that $\frac{1}{\epsilon^2}|\mu(0)| > ||\mu||_\infty(1 - \frac{1}{\epsilon^2})$. Then take a smooth function h on $\mathbf{R}$ such that $h(x) = 0$ when $x > 1$, and $h(x) = 1$ when $x < 0$. Let

$$h_n(x) = h\left(\frac{x - \frac{1}{n\epsilon}}{\frac{1}{n} - \frac{1}{n\epsilon}}\right).$$

One verifies that

(1) $h_n(x) = 0$ for $x > \frac{1}{n}$,

(2) $h_n(x) = 1$ for $x < \frac{1}{n\epsilon}$, and

(3) $|h'_n(x)| \le Cn$ with some constant C,

provided that n is sufficiently large.

Let $H_n(z) = h_n(|z|)$. Then,

$$\frac{n^2}{\pi}\iint V\overline{\partial}H_n = \frac{n^2}{\pi}\iint \mu H_n.$$

We have

$$|\frac{n^2}{\pi}\iint V\overline{\partial}H_n| \le \frac{n^2}{\pi}|\iint_{\{|z|<\frac{1}{n}\}-J} const.| \le const.\frac{|\{|z|<\frac{1}{n}\}-J|}{|\{|z|<\frac{1}{n}\}|}$$

tends to 0 as n tends to ∞. Furthermore,

$$\limsup_{n\to\infty}|\frac{n^2}{\pi}\iint \mu H_n| = \limsup_{n\to\infty}(\frac{1}{\epsilon^2}\frac{(n\epsilon)^2}{\pi}|\iint_{\{|z|<\frac{1}{n\epsilon}\}}\mu H_n + \frac{n^2}{\pi}\iint_{\{\frac{1}{n\epsilon}<|z|<\frac{1}{n}\}}\mu H_n|) \ge$$

$$\ge \frac{1}{\epsilon^2}|\mu(0)| - ||\mu||_\infty(1-\frac{1}{\epsilon^2}).$$

This contradiction proves theorem 3. □

Corollary 7 *Let J be a closed set in the complex plane and let $R(J)$ be the set of rational functions ψ which are holomorphic in $\mathbb{C}$ except for at most finitely many simple poles at points in J and for which $\iint_{\mathbb{C}}|\psi| < \infty$. Then $R(J)$ is dense in $A(J)$ in the L^1-norm on $\mathbb{C}$.*

Remark: This corollary generalizes the Bers's approximation theorem, [B],[A], because the norm on $A(J)$ is given by integration over all of $\mathbb{C}$.

PROOF. If J is a finite set, then the theorem is obvious. If J is infinite, we may assume that 0,1 and ∞ are in J. Suppose that $R(J)$ is not dense in $A(J)$. Then there exists a non-zero bounded linear functional on $A(J)$ that vanishes at $R(J)$. By Hahn-Banach theorem it can be extended to a bounded linear functional on $L^1(\mathbb{C})$. By Riesz theorem, there exists L^∞ complex valued function $\mu(z)$ which satisfies the orthogonality condition

$$\iint \mu(z)\psi(z)dxdy = 0$$

for all ψ in $R(J)$. The potential function

$$V(z) = -\frac{z(z-1)}{\pi}\iint_{\mathbb{C}} \frac{\mu(\zeta)d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)}$$

satisfies $\overline{\partial}V = \mu$, where the derivative is taken in the distributional sense. Notice that $r(\zeta) = \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}$ belongs to $R(J)$ for every $z \in J$. Therefore $V(z) = 0$ for every $z \in J$. Theorem 3 implies that $Re \iint_{\mathbb{C}} \mu\varphi = (V, \varphi) = 0$ and $Im \iint_{\mathbb{C}} \mu\varphi = -(V, i\varphi) = 0$ for every $\varphi \in A(J)$. □

Corollary 8 *The dual space of $A(J)$ is isometrically isomorphic to the space $\mathcal{Z}(J)$.*

PROOF. This corollary follows immediately from theorem 3. □

Theorems 1 and 2 apply to $A(X)$, where X is a Riemann surface. Parallel theorems with parallel definitions and parallel proofs apply to $A(J)$, when J is arbitrary closed set in the plane with Lebesque measure 0. A sequence φ_n in $A(J)$ is called a degenerating sequence if $||\varphi_n|| \leq 1$ for every n and if φ_n tends to zero uniformly on compact subsets of $\mathbb{C} - J$. If $V \in \mathcal{Z}$ then $\beta(V) = \sup \limsup_{n\to\infty} |(V, \varphi_n)|$, where the supremum is taken over all degenerating sequences φ_n in $A(J)$, is called the degenerating norm.

Let S(J) be the unit sphere in A(J). We say that A(J) is weak uniformly convex at a point $\varphi \in S(J)$ with constant δ if $\sup_{\psi\in D(\varphi,\delta)} ||\varphi - \psi|| < 2$, where $D(\varphi, \delta)$ is the set of all ψ in S(J) with $([\frac{|\varphi|}{\varphi}], \psi) \geq 1 - \delta$. We obtain the following two theorems which are parallel to theorems 1 and 2.

Theorem 4 *Let $\varphi \in A(J)$ and $v = [\frac{|\varphi|}{\varphi}]$. Then $A(J)$ is weak uniformly convex at φ if and only if the degenerating norm of v is less then 1.*

Theorem 5 *Let $\varphi \in S(J), \delta > 0$, and $v = [\frac{|\varphi|}{\varphi}]$. Then $A(J)$ is weak uniformly convex at φ with constant δ, if and only if $\beta(v) < 1 - \delta$.*

3.3 The case J is a Circle or a Line.

Now we consider the case when J is a circle or a line in $\overline{\mathbb{C}}$. If B is a Möbius transformation transforming J_1 onto J_2, then B induces a natural isomorphism from $\mathcal{Z}(J_2)$ onto $\mathcal{Z}(J_1)$ by pullback; given $V \in \mathcal{Z}(J_2)$, $B^*V(z) = V(B(z))/B'(z)$. This mapping preserves vector fields of the form $(\alpha z^2 + \beta z + \gamma)\frac{\partial}{\partial z}$ and therefore B^* is well-defined on the space of vector fields on J factored by the three dimensional subspace of quadratic polynomials.

$\mathcal{Z}(\mathcal{J})$ has a real locus, $Z_{\text{real}}(J)$, consisting of those vector fields on J which point in direction tangent to J. Thus $V \in Z_{\text{real}}(\overline{\mathbb{R}})$ if V is real-valued and $V \in Z_{\text{real}}(\partial\mathbb{D})$ if $V(e^{i\theta})/ie^{i\theta}$ is real-valued. In [GS], Gardiner and Sullivan proved that either of these spaces is naturally identified with the tangent space to universal Teichmüller space. This fact was observed earlier by Reimann in [R]. Elements of $Z_{\text{real}}(\overline{\mathbb{R}})$ give infinitesimal generator for curves of quasisymmetric self-mappings of $\overline{\mathbb{R}}$ and similarly, $Z_{\text{real}}(\partial\mathbb{D})$ give infinitesimal generator for curves of quasisymmetric self-mappings of $\partial\mathbb{D}$. An element of $Z_{\text{real}}(\overline{\mathbb{R}})$ can be represented by a real-valued function $V(x)$ for which

$$\epsilon_V(t) = |\frac{V(x+t) + V(x-t) - 2V(x)}{t}| \tag{3.4}$$

is uniformly bounded for all t. This representation is unique up to the addition of a function of the form $ax + b$.

$Z_{\text{real}}(\partial\mathbb{D})$ is naturally dual to $A(\mathbb{D})$. This fact is somewhat suprising because, while $A(\mathbb{D})$ is complex Banach space, $Z_{\text{real}}(\partial\mathbb{D})$ appears to be only a real vector space. However, it has a natural complex structure coming from the Hilbert transform which is explained in [G3]. The natural isomorphism from $Z_{\text{real}}(\overline{\mathbb{R}})$ to the real Banach space $A(\mathbb{H})^*$, where $\mathbb{H}$ is the upper half plane, is given by the following recipe. Starting with V in $Z_{\text{real}}(\overline{\mathbb{R}})$ apply the Ahlfors-Beurling extension formula to obtain an extension $\hat{V}$ of V to the upper half plane with bounded $\overline{\partial}$ derivative in generalized sense. Then $\varphi \to Re \iint \varphi\overline{\partial}\hat{V}$ yields an element of $A(\mathbb{H})^*$ (see [G3]).

The predual space to $A(\mathbb{H})$ is realized by the closed subspace $z_{\text{real}}(\overline{\mathbb{R}})$ consisting of those V's in $Z_{\text{real}}(\overline{\mathbb{R}})$ for which $\epsilon_V(t)$ defined in (5) is bounded and which satisfy the further condition that $\epsilon_V(t) \to 0$ as $t \to 0$.

Let $p \in \mathbb{D}$. Define $v_p \in B(\mathbb{D}) = A(\mathbb{D})^*$ as point-evaluation at p; $v_p(\varphi) = c_p\varphi(p)$ for every $\varphi \in A(\mathbb{D})$, where c_p is a positive constant selected so that $||v_p|| = 1$. Since $\frac{1}{\pi}\iint|\varphi| \geq |\varphi(0)|$ for every holomorphic function φ in $\mathbb{D}$, we obtain $c_0 = \pi$. To obtain c_p let $\gamma(z) = \frac{z+p}{1+\overline{p}z}$. Then $\gamma'(0) = 1 - |p|^2 > 0$, thus $v_p(\psi) = v_0(\psi \circ \gamma \cdot \gamma'^2)$ for every ψ in $A(\mathbb{D})$. Therefore $c_p = \pi(1 - |p|^2)$. Furthermore $\varphi_0(z) = \frac{1}{\pi}$ is the only function in $S(\mathbb{D})$ for which $v_0(\varphi_0) = ||v_0|| = 1$. Therefore, $\varphi_p(z) = \varphi_0(\gamma^{-1}(z))(\gamma^{-1})'(z)^2 = \frac{(1-|p|^2)^2}{\pi(1-\overline{p}z)^4}$ is the only function in $S(\mathbb{D})$ for which $v_p(\varphi_p) = 1$.

We define $v_n \in B(\mathbb{D})$ by $v_n(\varphi) = c_n\varphi^{[n]}(0)$ for every $\varphi \in A(\mathbb{D})$, where c_n is a positive constant so that $||v_n|| = 1$. In corollary 2 we saw that $v_n \in B_0(\mathbb{D})$, $c_n = \frac{2\pi}{(n+2)n!}$ and $v_n(\varphi_n) = ||v_n|| = 1$ for $\varphi_n(z) = \frac{n+2}{2\pi}z^n$. Since $v_0 \in B_0(\mathbb{D})$ we have $v_p \in B_0(\mathbb{D})$ for every p in $\mathbb{D}$.

By Hahn-Banach theorem:

1) Linear functionals v_n span a dense subspace of $B_0(\mathbb{D})$ and

2) If (p_n) is a sequence in $\mathbf{D}$ with a limit point in $\mathbf{D}$ then (v_{p_n}) span a dense subspase of $B_0(\mathbf{D})$.

Similarly we can define point-evaluations for any Riemann surface X. For every point p choose a chart $z : U \to z(U)$ with $p \in U$ and define $v_p \in B(X)$ by $v_p(\psi) = c_p\psi(z(p))$ where $\psi(z(p))$ is the local expression for the quadratic differential ψ in terms of the local parameter z, and c_p is the constant chosen so that $||v_p|| = 1$. This linear functional depends on the choice of local parameter z. However, $ker(v_p)$ is defined independently of the local parameter. Therefore we can define v_p independently of this choice, up to a multiplication by a constant of modulus 1, if we stipulate that $||v_p|| = 1$.

Question : Which divergent sequences p_n in X have the property that v_{p_n} span a dense subspace of $B(X)$. Equivalently, which divergent sequences p_n in X have the property that $\varphi(p_n) = 0$ for every p_n and $\varphi \in A(X)$ imply that $\varphi(z) \equiv 0$?

As an illustration we find functions $V_n(x)$ and $U_n(x)$ in $Z_{\text{real}}(\overline{\mathbf{R}})$ that correspond to $Re\ v_n$ and $Im\ v_n$ under the natural isomorphism between $A(\mathbf{H})^*$ and $Z_{\text{real}}(\overline{\mathbf{R}})$ given above.

Let $\gamma(z) = \frac{z-i}{z+i}$. Then γ is conformal mapping from the upper half plane $\mathbf{H}$ onto the unit disk $\mathbf{D}$. Thus, we ought to find $\hat{V}_n : \overline{\mathbf{H}} \to \overline{\mathbf{H}}$ such that $\hat{V}_n(\mathbf{R}) \subset \mathbf{R}$, $\overline{\partial}\hat{V}_n = \gamma^*\mu_n$, and $\hat{V}_n(z) = O(z^2)$ as $z \to \infty$; where $\gamma^*\mu_n = \mu_n \circ \gamma \frac{\overline{\gamma'}}{\gamma'}$ and $\mu_n(z) = \frac{|z|^n}{z^n}$. Observe that if $V_n(z) = \frac{\hat{v}_n(\gamma(z))}{\gamma'(z)}$ then $\overline{\partial}\hat{v}_n = \mu_n$.

Hence, $\overline{\partial}\hat{v}_n = \frac{|z|^n}{z^n} = \frac{\overline{z}^{\frac{n}{2}}}{z^{\frac{n}{2}}}$; thus,

$$\hat{v}_n(z) = \frac{2}{n+2}\frac{\overline{z}^{\frac{n}{2}+1}}{z^{\frac{n}{2}}} + \text{ a conformal mapping .}$$

Therefore,

$$\hat{V}_n(z) = -\frac{i}{n+2}\frac{(\overline{z-i})^{\frac{n}{2}+1}(z+i)^{\frac{n}{2}+2}}{(\overline{z+i})^{\frac{n}{2}+1}(z-i)^{\frac{n}{2}}} + \text{ a conformal mapping } f(z).$$

Letting

$$f(z) = \frac{i}{n+2}\frac{(z-i)^{n+3}}{(z+i)^{n+1}},$$

we see that

$$|\hat{V}_n| \le \frac{1}{n+2}|z-i||z+i| + \frac{1}{n+2}\frac{|z-i|^{n+3}}{|z+i|^{n+1}} = O(z^2)$$

as $z \to \infty$. Furthermore,

$$\hat{V}_n(x) = 2\ Re\ (-\frac{i}{n+2}\frac{(x+i)^{n+3}}{(x-i)^{n+1}}) \text{ for x real.}$$

Therefore,

$$V_n(x) = \frac{2}{n+2}\frac{Im\ (x+i)^{2n+4}}{(x^2+1)^{n+1}} \text{ and}$$

$$U_n(x) = \frac{2}{n+2}\frac{Re\ (x+i)^{2n+4}}{(x^2+1)^{n+1}},$$

up to addition of a linear polynomial. So, for example, $V_0(x) = \frac{-8x}{x^2+1}$, and $U_0(x) = \frac{8}{x^2+1}$ up to addition of a linear polynomial.

Normalize $V_n(x)$ and $U_n(x)$ such that they vanish at 0 and at 1. Let $V \in Z_{\text{real}}(\overline{\mathbf{R}})$ such that $V(0) = 0$ and $V(1) = 1$. Functions $V_n(x)$ and $U_n(x)$, together span a subspace T of $Z_{\text{real}}(\overline{\mathbf{R}})$. T is dense in the weak star topology, because a holomorphic function on a domain with all derivatives at certain point equal to 0 is a zero function. Hence, some sequence (W_n) of linear combinations of V_n 's and U_n 's converges to V in the weak star topology, thus $R_x(W_n) \to R_x(V)$, where $R_x(t) = \frac{x(x-1)}{t(t-1)(t-x)}$, and x is arbitrary real number. It yields $W_n(x) \to V(x)$, for every real number x.

By Banach-Steinhaus theorem, there exists $M > 0$ such that $||W_n|| \le M$ for every n. That implies that every W_n satisfies $\epsilon \log \epsilon$ -modulus of continuity uniformly on compact sets. Therefore, $W_n(x)$ converges to $V(x)$ uniformly on compact subsets of $\mathbf{R}$ and we have proved the following theorem.

Theorem 6 *A linear span of functions V_n and U_n is dense in $Z_{\text{real}}(\overline{\mathbf{R}})$ in the topology of uniform convergence on compact subsets of* $\mathbf{R}$.

Let $0 < p < 1$.

$$\frac{||\varphi_p - \varphi_0||^2}{1- < \varphi_p, \varphi_0 >} = \frac{(\iint |\frac{(1-p^2)^2}{\pi(1-pz)^4} - \frac{1}{\pi}|)^2}{1 - \pi Re\ \varphi_p(0)} =$$

$$= \frac{(\iint |\frac{(1-p^2)^2}{(1-pz)^4} - 1|)^2}{\pi^2(1-(1-p^2)^2)} = \frac{(\iint |\frac{(2-p^2+p^2z^2-2pz)(2z-p-pz^2)}{(1-pz)^4}|)^2}{\pi^2(2-p^2)}.$$

Since

$$|\frac{(2-p^2+p^2z^2-2pz)(2z-p-pz^2)}{(1-pz)^4}| \leq \frac{24}{(1-p)^4},$$

by Lebesque's dominated convergence theorem,

$$\frac{||\varphi_p - \varphi_0||^2}{1- < \varphi_p, \varphi_0 >} \to \frac{(\iint_D |4z|)^2}{2\pi^2} > 0 \text{ as } p \to 0.$$

Now let $n \geq 1$ and let $\varphi_{n,p} = \varphi_n(\beta(z))\beta'(z)^2$ where $\beta(z) = \frac{z-p}{1-pz}$. We have $||\varphi_{n,p}|| = 1$ and

$$||\varphi_{n,p} - \varphi_n|| = \frac{n+2}{2\pi}||z^n - (1-p^2)^2\frac{(z-p)^n}{(1-pz)^{n+4}}|| =$$

$$= \frac{n+2}{2\pi}||\frac{npz^{n-1} - (n+4)pz^{n+1} + o(p)}{(1-pz)^{n+4}}|| = Cp + 0(p)$$

where $C = \frac{n+2}{2\pi}||z^{n-1}(1-(n+4)z^2)||$. Furthermore,

$$1- < \varphi_{n,p}, \varphi_n >= 1 - Re\ v_n(\varphi_{n,p}) =$$

$$= 1 - (1-p^2)^2\frac{1}{n!}((z-p)^n(1-pz)^{-n-4})^{[n]}(0) =$$

$$= 1 - (1-p^2)^2\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(((z-p)^n)^{[n-k]})(0)(((1-pz)^{-n-4})^{[k]})(0) =$$

$$= np^2(n+4) + 2p^2 + o(p^2).$$

Therefore,

$$\frac{||\varphi_{n,p} - \varphi_n||^2}{1 - <\varphi_{n,p}, \varphi_n>} \to \frac{C^2}{2 + n(n+4)} > 0 \text{ as } p \to 0.$$

Therefore $\frac{1}{2}$ is the best exponent of weak uniform convexity of $A(\mathbb{D})$ at $\varphi_0(z)$, and we proved the following theorem.

Theorem 7 *If n is a nonnegative integer and $\alpha > \frac{1}{2}$ then $A(\mathbb{D})$ is not weak uniformly convex at $\varphi_n(z) = \frac{n+2}{2\pi} z^n$ with exponent α.*

Question : Is $A(\mathbb{D})$ weak uniformly convex at points $\varphi_n(z) = \frac{n+2}{2\pi} z^n$ with exponent $\frac{1}{2}$?

Chapter 4

An Isometry Theorem for Quadratic Differentials on Riemann Surfaces of Finite Genus

4.1 Introduction

Let A(X) and A(Y) be the complex Banach spaces of integrable holomorphic quadratic differentials on Riemann surfaces X and Y. If α is a conformal mapping from Y onto X, and C is a complex constant of modulus one, then $C\alpha^* : A(X) \to A(Y)$ defined by $C\alpha^*(\varphi)(z) = C\varphi(\alpha(z))\alpha'(z)^2$ is complex linear isometry in the norms for A(X) and A(Y).

The main result of this chapter is the following theorem.

Isometry theorem *If the Riemann surfaces X and Y are subsets of* *compact Riemann surfaces X_1 and Y_1, respectively, such that set $X_1 - X$ has infinitely many points, then every linear isometry L from $A(X)$ onto $A(Y)$ comes from pull back by a conformal mapping α from Y onto X and multiplication by a complex constant C of modulus one, i.e. $L = C\alpha^*$.*

A Riemann surface is of finite analytic type (g,n) if it is obtained from a compact Riemann surface of genus g by deleting n points. The Riemann surface of finite analytic type is exceptional if (g,n) is equal to (0,3), (0,4), (1,1), (1,2) or (2,0). For the case of Riemann surface of type $(g, 0), g \geq 2$, the isometry theorem was proved by Royden ([R]). Using the same method of proof, Earle and Kra generalized the isometry theorem to the case of nonexceptional Riemann surfaces of finite analytic type. The method was to study the smoothness properties of the L^1-norm on A(X) and A(Y) and to consider the differentials in A(X) and A(Y) with zeroes of the highest possible order at the points in X and Y. This method does not generalize to the case where X is not of finite analytic type because there is no limit on the order of zero of a differential in A(X).

A Riemann surface is of finite genus if it can be holomorphically imbedded into a subset of a compact Riemann surface. We prove the isometry theorem in the case where X and Y are subsets of compact Riemann surfaces X_1 and Y_1 such that set $X_1 - X$ has infinitely many points. Therefore X and Y are Riemann surfaces of finite genus. A further condition imposed on X simply means that X is not a Riemann surface of finite analytic type, a case completely discussed in [EK]. Hence, we prove the isometry theorem precisely in the case when X is a Riemann surface of finite genus and infinite topological type and Y is a Riemann surface of finite genus. In the proof we use the smoothness of the L^1-norm on A(X) and A(Y), and we consider differentials in A(X) with at least a double zero at some fixed point in X.

In section 2 of this chapter it is shown that the integrability of the quadratic differential $\frac{\psi^2}{\varphi}$, where ψ and φ are in A(X), is an invariant for every linear isometry L from A(X) onto A(Y). That is obtained by proving that the integrability of $\frac{\psi^2}{\varphi}$ is equivalent to the existence of the second derivative of the function $f(t) = ||\varphi + t\psi||$ at $t = 0$, in the direction of both real and imaginary axises.

In section 3 we consider the case where X and Y are plane domains. We show that the image under L of the space of all quadratic differentials in A(X) with at least a double zero at some fixed point p in X is the space of all quadratic differentials in A(Y) with at least a double zero at some point q in Y. That implies that the image under L of the space of all differentials in A(X) that vanish at p is the space of all differentials in A(Y) that vanish at q. The function from Y to X that sends q to p is a conformal homeomorphism which together with a complex constant of modulus 1 realizes the given isometry.

We prove the isometry theorem in section 4, following the same steps from the plane domain case and using the standard theorems about the existence of certain meromorphic functions on Riemann surfaces.

Finally, in section 5 of this chapter we prove few corollaries including one about the existence of uncountably many non-biholomorphic Teichmüller spaces.

The isometry theorem was one of the three main tools in Royden's classification of all holomorphic automorphisms of the Teichmüller space of a compact Riemann surface of genus at least two ([R]). Royden showed that every holomorphic isomorphism F between Teichmüller spaces of compact Riemann surfaces of genus at least two is induced by a quasiconformal homeomorphism g; $F([f]) = [f \circ g]$. The starting point in Royden's proof is the fact that Teichmüller's metric is equal to Kobayashi's metric for finite-dimensional Teichmüller spaces. A consequence of this equality is that any holomorphic homeomorphism between Teichmüller spaces induces isometries on fibers of the tangent bundles. Gardiner proved that Teichmüller's and Kobayashi's metrics coincide on $Teich(X)$, for any Riemann surface X (See [G1]).

When X is compact except for finitely many punctures, Z(X), the tangent space to $Teich(X)$ is finite-dimensional, and therefore any isometry from Z(X) onto Z(Y) is induced by an isometry between their predual spaces A(X) and A(Y). Earle and Gardiner showed that an isometry from Z(X) onto Z(Y) is induced by an isometry from A(Y) onto A(X), for any Riemann surfaces X and Y ([EG]). If Riemann surfaces X and Y satisfy the hypotheses of the isometry theorem then every isometry L from A(X) onto A(Y) comes from pull back by a conformal mapping from Y onto X and multiplication by a complex constant of modulus one. Royden showed that the constant of modulus one is equal to one by using the fact that the action of the mapping class group on $Teich(X)$ is discontinous. Earle and Gardiner proved the corresponding result for infinite-dimensional Teichmüller spaces by considering Teichmüller disks and Jenkins-Strebel differentials ([EG]).

Therefore, the results in [EG] and the isometry theorem generalize Royden's classification.

Automorphism theorem *If the Riemann surfaces X and Y are subsets of compact Riemann surfaces X_1 and Y_1, respectively, such that set $X_1 - X$ has infinitely many points, then every holomorphic isomorphism from $Teich(X)$ onto $Teich(Y)$ is induced by a quasiconformal homeomorphism g from Y onto X . In particular, every biholomorphic self mapping of Teich(X)* *is induced by a quasiconformal self mapping of X.*

4.2 The Nonsmoothness of the Norm on Quadratic Differentials

Recall that if X is a Riemann surface then A(X) is the Banach space of all holomorphic quadratic differentials φ on X satisfying $||\varphi|| = \iint_X |\varphi| < \infty$.

Note that if φ and ψ are two differentials in $A(X)$, then $\frac{\psi^2}{\varphi}$ is also a quadratic differential on X.

In this section we use the smoothness properties of Teichmüller's metric to show that the integrability of a quadratic differential $\frac{\psi^2}{\varphi}$ is an invariant for any complex linear isometry L from A(X) onto A(Y). First we prove the following simple inequality for complex numbers.

Lemma 2 *For all complex numbers $z \neq 1$;* $|\frac{1-z}{|1-z|} - 1| \leq 4|z|$.

PROOF. Let $1 - z = re^{it}$.

If $r \geq \frac{1}{2}$, then $$(2r-1)\cos t \leq 2r - 1 \leq r^2,$$

$$-\cos t \leq r^2 - 2r\cos t,$$

$$-2\cos t \leq 2(r^2 - 2r\cos t,)$$

$$2 - 2\cos t \leq 2(r^2 - 2r\cos t + 1),$$

$$|e^{it} - 1|^2 \leq 2|1 - re^{it}|^2,$$

$$|\frac{1-z}{|1-z|} - 1|^2 \leq 2|z|^2.$$

If $0 \leq r \leq \frac{1}{2}$, then $|z| = |1 - re^{it}| \geq 1 - r \geq \frac{1}{2}$ and $|\frac{1-z}{|1-z|} - 1| = |e^{it} - 1| \leq 2$.

Therefore,

$$|\frac{1-z}{|1-z|}-1| \leq 4|z|$$

for every $z \neq 1$. □

Consider the real valued function $f(t) = ||\varphi + t\psi|| = \iint |\varphi + t\psi| dxdy$ where φ and ψ are elements of A(X), $||\varphi|| \neq 0$ and t is a real number.

Lemma 3 *Let X be an arbitrary Riemann surface. If $\varphi, \psi \in A(X)$ and $\frac{\psi^2}{\varphi} \in L^1(X)$, then $f(t) = ||\varphi + t\psi||$ has a second derivative at t=0.*

PROOF. One verifies that $f'(t) = Re \iint \psi \frac{|\varphi+t\psi|}{\varphi+t\psi}$ provided that $||\varphi + t\psi|| \neq 0$.

Therefore,

$$\frac{f'(t)-f'(0)}{t} = \iint Re(\frac{\psi}{t}(\frac{|\varphi+t\psi|}{\varphi+t\psi} - \frac{|\varphi|}{\varphi})) = \iint Re(\frac{\psi}{t}\frac{|\varphi|}{\varphi}(\frac{|1+t\frac{\psi}{\varphi}|}{1+t\frac{\psi}{\varphi}} - 1)).$$

An easy calculation shows that, except at the zeroes of $\varphi(z)$,

$$Re(\frac{\psi(z)}{t}\frac{|\varphi(z)|}{\varphi(z)}(\frac{|1+t\frac{\psi(z)}{\varphi(z)}|}{1+t\frac{\psi(z)}{\varphi(z)}} - 1))$$

converges to

$$\frac{Im^2(\psi(z)\overline{\varphi(z)})}{|\varphi(z)|^3} \quad \text{as } t \to 0.$$

$$\text{Also} \quad |Re(\frac{\psi}{t}\frac{|\varphi|}{\varphi}(\frac{|1+t\frac{\psi}{\varphi}|}{1+t\frac{\psi}{\varphi}} - 1))| \leq \frac{|\psi|}{|t|}4|t|\frac{|\psi|}{|\varphi|} \in L^1(X) \quad \text{by lemma 2.}$$

Therefore, by Lebesque's Dominated Convergence Theorem, $\lim_{n\to\infty} \frac{f'(t)-f'(0)}{t}$ exists and is equal to $\iint \frac{(Im(\psi\overline{\varphi}))^2}{|\varphi|^3}$. This proves lemma 3. □

Lemma 4 *Let X and Y be arbitrary Riemann surfaces. Suppose that L is a linear isometry from A(X) onto A(Y). Then, for every φ and ψ in A(X),*

$$\frac{\psi^2}{\varphi} \in L^1(X) \ \ \textit{iff} \ \ \frac{L(\psi)^2}{L(\varphi)} \in L^1(Y).$$

PROOF. Suppose that $\psi, \varphi \in A(X)$ and $\frac{\psi^2}{\varphi} \in L^1(X)$. For any subset S of Y, let $g_S(t) = \iint_S |L\varphi + tL\psi|$ and $h_S(t) = f(t) - g_S(t)$. By lemma 3 we know that $f(t)$ has a second derivative at $t = 0$. Take any point p in Y. Let m be the order of zero of $L\varphi$ at p, and k the order of zero of $L\psi$ at p. Let U be a small conformal disk with center at p such that $L\psi L\varphi$ has no zeroes in $\overline{U} - \{p\}$. Then, obviously, $h_U(t) = \iint_{U^c} |L\varphi + tL\psi|$ is a convex function. If $m - k \geq 2 + k$, then $g_U(t) - g_U(0) - tg'_U(0) = C\epsilon(t) + o(\epsilon(t))$ where $C > 0$, and $\epsilon(t) = |t|^{1+\frac{2+k}{m-k}}$ when $m - k > 2 + k$, and $\epsilon(t) = t^2 \log \frac{1}{|t|}$ when $m - k = 2 + k$ (See [G2]). Since h_U is convex, we have $h_U(t) - h_U(0) - th'_U(0) \geq 0$. This contradicts the existence of $f''(0)$; thus, $m - k < 2 + k$ and $\frac{(L\psi)^2}{L\varphi} \in L^1_{loc}(Y)$.

Now take any compact set $K \subset Y$.

$\frac{(L\psi)^2}{L\varphi} \in L^1(K)$, and by the proof of lemma 3

$$g''_K(0) = \iint_K \frac{(Im(L\psi\overline{L\varphi}))^2}{|L\varphi|^3}.$$

Since h_K is convex,

$$\iint_K \frac{(Im(L\psi\overline{L\varphi}))^2}{|L\varphi|^3} \leq f''(0).$$

Letting $K \to Y$ we obtain

$$\iint_Y \frac{(Im(L\psi\overline{L\varphi}))^2}{|L\varphi|^3} \leq f''(0) < \infty \tag{4.1}$$

Now, instead of ψ, take $i\psi$. We have $\frac{(i\psi)^2}{\varphi} = -\frac{\psi^2}{\varphi} \in L^1(X)$, and so

$$\infty > \iint_Y \frac{(Im(iL\psi\overline{L\varphi}))^2}{|L\varphi|^3} = \iint_Y \frac{(Re(L\psi\overline{L\varphi}))^2}{|L\varphi|^3} \tag{4.2}$$

Adding (1) and (2), we see that

$$\infty > \iint_Y \frac{|L\psi\overline{L\varphi}|^2}{|L\varphi|^3} = \iint_Y \frac{|L\psi|^2}{|L\varphi|}.$$

Since L is invertible, that proves lemma 4. □

4.3 Plane Domain Case

Let X be a domain, and let C be a set in a complex plane with parameter z . Let $R(C)$ be the set of rational functions $r(z)$ which are holomorphic except for at most simple poles at points of C, and for which $||r|| = \iint_{\mathbf{C}} |r(z)|dxdy < \infty$.

We will frequently use an approximation theorem for A(X) due to Bers ([B]) and Ahlfors ([A]). We refer to this theorem as Bers' approximation theorem.

Theorem (Bers) If X is a domain in the complex plane and C is dense in X^c, the set complementary to X, then R(C) is dense in A(X).

Notice that if α is a conformal mapping from a plane domain Y onto a plane domain X, then $\alpha^* : A(X) \to A(Y)$ defined by $\alpha^*\varphi(z) = \varphi(\alpha(z))\alpha'(z)^2$ is complex linear isometry in the norms for A(X) and A(Y). Moreover if α is a Möbius transformation, then the set complementary to X is transformed onto a set complementary to Y.

In this subsection we prove:

Theorem 8 *If X and Y are two plane domains such that X^c is infinite and if $L : A(X) \to A(Y)$ is a linear invertible isometry, then there exist a constant C of modulus one and a conformal mapping β from Y onto X such that $L = C\beta^*$.*

Before beginning the proof of theorem 8 we state a definition and prove a lemma about rational functions which is used frequently in the proof .

Definition 2 *For any rational function* $R(z) = C\frac{(z-q_1)(z-q_2)\ldots(z-q_m)}{(z-p_1)(z-p_2)\ldots(z-p_n)}$, *the order of* R, *denoted by* $ord(R)$, *is equal to* $m - n$.

Lemma 5 *If* $R_1 \neq 0$ *and* $R_2 \neq 0$ *are two rational functions with no double zeroes in common, then there exist constants* C_1 *and* C_2 *such that*
(a) $C_1R_1 + C_2R_2$ *has no double zeroes,*
(b) *if* $R_1(p) = \infty$ *or* $R_2(p) = \infty$, *then* $C_1R_1(p) + C_2R_2(p) = \infty$, *and*
(c) $ord(C_1R_1 + C_2R_2) = max\{ord(R_1), ord(R_2)\}$.

PROOF. Suppose $(C_1, C_2) \neq 0$.

$C_1R_1(a)+C_2R_2(a) = 0$ and $C_1R_1'(a)+C_2R_2'(a) = 0$ imply that $R_1(a)R_2'(a)-R_1'(a)R_2(a) = 0$.

Consider $R(z) = R_1(z)R_2'(z) - R_1'(z)R_2(z)$. If $R \equiv 0$ then $R_1 = CR_2$ and we can take $C_1 = 0, C_2 = 1$.

Suppose that $R \neq 0$. Then, $A = \{a|R(a) = 0\}$ is a finite set.

If $R_1(a), R_2(a), R_1'(a)$ and $R_2'(a)$ are all equal to zero, then R_1 and R_2 have a common double zero at a.

If $a \in A$, and not both $R_1(a)$ and $R_2(a)$ are zero, then $C_1R_1(a) = -C_2R_2(a)$ is a linear condition for C_1 and C_2.

If $a \in A$, and not both $R_1'(a)$ and $R_2'(a)$ are zero, then $C_1R_1'(a) = -C_2R_2'(a)$ is a linear condition for C_1 and C_2.

Therefore, to satisfy condition (a), it is enough to take (C_1, C_2) from the set $\{(C_1, C_2)| \; (\forall a \in A) \quad C_1R_1(a) \neq -C_2R_2(a)$ or $C_1R_1'(a) \neq -C_2R_2'(a)\}$.

Since R_1 and R_2 have finitely many poles, to satisfy condition (b), we are only restricted by finitely many linear conditions for C_1 and C_2.

To satisfy (c), it is enough to take $C_1 \neq 0$ and $C_2 \neq 0$ such that $\alpha C_1 + \beta C_2 \neq 0$ where α and β are such that

$$R_1(z) = \alpha\frac{(z-a_1)...(z-a_n)}{(z-b_1)...(z-b_m)} \text{ and } R_2(z) = \beta\frac{(z-c_1)...(z-c_k)}{(z-d_1)...(z-d_j)}.$$

Therefore, to satisfy conditions (a), (b) and (c) we are only restricted by finitely many linear conditions for C_1 and C_2. □

Proof of theorem 8.

Since the proof of theorem 8 is rather long, it will be convenient to divide it into several steps.

Step I We may assume that 0 belongs to X and to Y, and that ∞ is a cluster point of X^c and of Y^c.

PROOF. The complement of X is an infinite set. That means that the boundary ∂X of X must contain at least one cluster point. Let p be a cluster point of ∂X, q an interior point of X, and $\alpha(z) = \frac{z-q}{z-p}$. Then, α^* is an isometry from $A(\alpha(X))$ onto $A(X)$, and $\alpha(X)$ contains 0 in its interior and ∞ on its boundary. □

Definition 3 *Let F be the space of all differentials in A(X) which have at least a double zero at 0, and V be the space of all differentials ψ in A(X) for which there exists a differential φ in F such that $\frac{\psi^2}{\varphi}$ is integrable in X;*

$$F = \{\varphi \in A(X) | \varphi(0) = 0 \text{ and } \varphi'(0) = 0\},$$

$$V = \bigcup_{\varphi \in F} \{\psi \in A(X) | \frac{\psi^2}{\varphi} \in L^1(X)\}.$$

Step II $\overline{V}$, the closure of V, is the set of all quadratic differentials in A(X) that vanish at 0 . Therefore, $Codimension(\overline{L(V)}) = 1$.

PROOF. If ψ and φ are in A(X), and φ belongs to F, and ψ does not vanish at 0 then $\frac{\psi^2}{\varphi}$ has a double pole at 0 and is not integrable. Therefore, $\overline{V}$ is contained in the set of qudratic differentials in A(X) which vanish at 0.

Conversely, assume that ψ is a rational function in A(X) and $\psi(0) = 0$. Take $p \in X^c$ such that p is not a pole of $\psi(z)$. Then $\varphi = \frac{z\psi(z)}{z-p}$ is in F and $\frac{\psi^2}{\varphi} = \frac{(z-p)\psi(z)}{z} \in L^1(X)$; thus, $\psi \in V$. By the Bers's approximation theorem, rational functions are dense in A(X); thus, $\overline{V} = \{\psi \in A(X) | \psi(0) = 0\}$. □

Step III $L(V) = \bigcup_{\varphi \in L(F)} \{\psi \in A(Y) | \frac{\psi^2}{\varphi} \in L^1(Y)\}$.

PROOF. Step III follows immediately from lemma 4. □

Step IV Every rational function R in L(F) with at least three poles and with $ord(R) \geq -5$ has a double zero.

PROOF. Suppose that some rational function R in L(F) has ord at least -5, poles in p_1, p_2 and p_3, and has no double zeroes .

If $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-p)} \in \overline{L(V)}$ for every $p \in Y^c - \{p_1, p_2, p_3, \infty\}$, then by the Bers' approximation theorem, $\overline{L(V)} = A(Y)$; a contradiction. Therefore, we may assume that $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-p)}$ is not in $\overline{L(V)}$ for some p in $Y^c - \{p_1, p_2, p_3, \infty\}$.

If p is a cluster point of Y^c, then there exists a neighborhood U of p such that $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-q)}$ is not in $\overline{L(V)}$ for all q in $U \bigcap Y^c$.

Take $a_1, a_2, a_3 \in U \bigcap Y^c$. Since $Codimension(L(F)) = 2$ we can find three complex constants α, β and γ, not all equal to zero, such that

$$M = \frac{\alpha z^2 + \beta z + \gamma}{(z-p_1)(z-p_2)(z-p_3)(z-a_1)(z-a_2)(z-a_3)} \in L(F).$$

Therefore, M has a pole at a_1, a_2 or a_3. Assume that M has a pole at a_1. Then, by lemma 5, we can find constants C_1 and C_2 such that $T = C_1 M + C_2 R$ has ord at least -5, poles at p_1, p_2, p_3 and a_1, and has no double zeroes. This implies $T \in L(F)$, and

$$\frac{\left(\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-a_1)}\right)^2}{T} \in L^1(Y),$$

which contradicts the fact that $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-a_1)}$ is not in $\overline{L(V)}$.

Therefore, we may assume that p is isolated in Y^c.

Now suppose that $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-q)} \in \overline{L(V)}$ for every q in $Y^c - \{p, p_1, p_2, p_3, \infty\}$. Then $R(Y^c - \{p, \infty\}) \subset \overline{L(V)}$, and by Bers' approximation theorem, $\overline{L(V)} = \{\psi \in A(Y) | \psi$ is holomorphic at p $\}$.

Let q_1, q_2, q_3 and q_4 be four distinct points in the complement of X. Let $\varphi_1 = \frac{1}{(z-q_1)(z-q_2)(z-q_3)(z-q_4)}$ and $\varphi_2 = z\varphi_1$. By step II, φ_2 is in $\overline{V}$ and φ_1 is not in $\overline{V}$. That implies that $L(\varphi_2)$ is in $\overline{L(V)}$, and $L(\varphi_1)$ is not in $\overline{L(V)}$; therefore, $L(\varphi_2)$ is holomorphic at p, and $L(\varphi_1)$ has a pole at p. This yields $\frac{(\varphi_1)^2}{\varphi_2} \in L^1(X)$, and $\frac{L(\varphi_1)^2}{L(\varphi_2)}$ has a double pole at p, contradicting lemma 3.

Therefore, we may also assume that $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-q)}$ is not in $\overline{L(V)}$ with q isolated in $Y^c - \{p_1, p_2, p_3, p, \infty\}$. Since $F \subset V$, neither $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-p)}$ nor $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-q)}$ is in L(F).

Take $r \in Y^c - \{\infty, p_1, p_2, p_3, p, q\}$. Since $Codimension(L(F)) = 2$, there exist three constants α, β and γ, not all equal to zero, such that

$$M = \frac{\alpha z^2 + \beta z + \gamma}{(z-p_1)(z-p_2)(z-p_3)(z-p)(z-q)(z-r)} \in L(F).$$

If $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-r)}$ is not in L(F), then M has a pole at p or at q. Assume that M has a pole at p. By lemma 5, we can find constants C_1 and C_2 such that $C_1M + C_2R$ has ord at least -5 , poles at p_1, p_2, p_3 and p, and has no double zeroes. But then $\frac{\left(\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-p)}\right)^2}{C_1M+C_2R} \in L_1(Y)$; thus, $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-p)}$ is in L(V); a contradiction.

Therefore $\frac{1}{(z-p_1)(z-p_2)(z-p_3)(z-r)} \in L(F)$ for all r in $Y^c - \{\infty, p_1, p_2, p_3, p, q\}$, and by Bers' approximation theorem, $L(F) = \{\psi \in A(Y) | \psi$ is holomorphic at p and q $\}$.

Now choose five distinct points q_1, q_2, q_3, q_4 and q_5 from the complement of X . $\varphi = \frac{1}{(z-q_1)\ldots(z-q_5)}$ is not in F; thus, $L(\varphi)$ is not in L(F). Suppose that $L(\varphi)$ has a pole at p. There exist constants α and β, not both equal to zero, such that $\alpha L(\varphi) + \beta L(z\varphi)$ is holomorphic at p. Also $z^2\varphi$ has a double zero at 0, so $L(z^2\varphi)$ is in L(F); thus, $L(z^2\varphi)$ is holomorphic at p. Choose a constant γ so that $\gamma z^2 + \beta z + \alpha$ has no double zeroes and has no zeroes in $\{q_1, q_2, q_3, q_4, q_5\}$. Then $\frac{\varphi^2}{(\gamma z^2 + \beta z + \alpha)\varphi} \in L^1(X)$, and $\frac{L(\varphi)^2}{\gamma L(z^2\varphi) + (\alpha L(\varphi) + \beta L(z\varphi))}$ has a double pole at p, contradicting lemma 4. □

Step V L(F) is the set of all differentials in A(Y) with the double zero at some point a in Y.

PROOF. We first choose five distinct points p_1, p_2, p_3, p_4 and p_5 in Y^c. Since $Codimension(L(F)) = 2$, there exist constants α, β and γ, not all equal to zero, such that $S = \frac{\alpha z^2 + \beta z + \gamma}{(z-p_1)\ldots(z-p_5)} \in L(F)$. By step IV, $S = \frac{\alpha(z-a)^2}{(z-p_1)\ldots(z-p_5)}$ for some complex number a. Suppose that R is a rational function in A(Y) with a double zero at a. Let $R(z) = (z-a)^{2+k}R_1(z)$ such that $k \geq 0$, and $R_1(a)$ is neither 0 nor ∞. Then, since $Codimension(L(F)) = 2$, $P(z) = R_1(z)[\alpha(z-a)^{2+k} + \beta(z-a) + \gamma] \in L(F)$ for some $(\alpha, \beta, \gamma) \neq 0$. If $P(a) \neq 0$ or $P'(a) \neq 0$, then, by lemma 5, we can find constants C_1 and C_2 such that $C_1 P + C_2 S$ has ord at least -5, poles at p_1, p_2 and p_3, and has no double zeroes. This contradicts step IV. Therefore $P(a) = 0$ and $P'(a) = 0$. This yields $\gamma = 0$ and $\beta = 0$, which means $R \in L(F)$. Therefore, every rational function in A(Y) with a double zero at a is in L(F).

Suppose that $a \in Y^c$. Then, since $Codimension(F) = 2$,

$$M = \frac{\alpha + \beta z + \gamma z^2}{(z-a)(z-p_1)...(z-p_4)} \in L(F) \text{ for some } (\alpha, \beta, \gamma) \neq 0.$$

By lemma 5, we can find constants C_1 and C_2 such that $C_1 M + C_2 S$ has ord at least -5, poles at p_1, p_2, p_3, and has no double zeroes, again contradicting step IV.

By Bers' approximation theorem and the fact that $Codimension(L(F)) = 2$, we have $L(F) = \{\psi \in A(Y) | \psi \text{ has a double zero at } a \}$. □

Step VI There exists a bijection β from Y onto X such that

$$\psi(\beta p) = 0 \text{ iff } L\psi(p) = 0, \text{ for every } p \text{ in } Y.$$

PROOF. Steps II, III and V imply that $\overline{L(V)} = \{\psi \in A(Y) | \psi(a) = 0\}$. Therefore, for every ψ in A(X),

$$\psi(0) = 0 \Leftrightarrow L(\psi)(a) = 0.$$

Step VI follows by applying previous steps to L^{-1}. □

Step VII β is conformal homeomorphism.

PROOF. Let $\varphi(z) = \frac{1}{(z-q_1)...(z-q_4)}$, $p \in Y$, and $\beta(p) = q$. Function $z \to \frac{z-q}{(z-q_1)...(z-q_4)}$ is in A(X) and has a zero at q. Hence,

$$L(z\varphi)(p) - qL(\varphi)(p) = 0.$$

Thus, $\beta(p) = \frac{L(z\varphi)(p)}{L(\varphi)(p)}$. Therefore, β is holomorphic. □

Step VIII $L = C\beta^*$ for some complex constant C of modulus one.

PROOF. Without loss of generality we can assume that X=Y and

$\beta \equiv identity$. If we let $\varphi(z) = \frac{1}{(z-q_1)\ldots(z-q_4)}$, then for every $\psi \in A(X)$ and every $p \in X$, quadratic differential $\psi - \frac{\psi(p)}{\varphi(p)}\varphi$ is in A(X) and has a zero at p. Therefore, by step VI, $L\psi(p) - \frac{\psi(p)}{\varphi(p)} L\varphi(p) = 0$. This yields $L\psi(p) = \psi(p)\frac{L\varphi(p)}{\varphi(p)}$. Let $f = \frac{L\varphi}{\varphi}$. Then $L\psi = f\psi \quad$ for all $\psi \in A(X)$.

The family $\{\psi, L\psi, L^2\psi, \ldots\}$ is normal. Therefore, $|f| \equiv 1$. □

4.4 Riemann Surfaces of Finite Genus

Assume both X and Y are Riemann surfaces which are subsets of compact Riemann surfaces X_1 and Y_1, respectively. Let L be a linear isometry from A(X) onto A(Y). If $X_1 - X$ is a finite set, then $A(X)$ is finite dimensional; thus, $A(Y)$ is also finite dimensional; so, $Y_1 - Y$ contains only finitely many points. In that case, all holomorphic automorphisms between Teichmüller spaces of X and Y are determined in [EK]. Therefore, we assume that $X_1 - X$ and $Y_1 - Y$ are infinite sets, which is equivalent to X and Y being of finite genus and infinite topological type.

An example for X is a Riemann surface of finite topological type with nonempty border.

Fix three points p_1, p_2 and p_3 in $X_1 - X$ and let $X_2 = X_1 - \{p_1, p_2, p_3\}$ Let $\pi : D \longrightarrow X_2$ be the universal covering map with a covering group Γ. Let $U = \pi^{-1}(X)$, and $\Theta : A(U) \longrightarrow A(U, \Gamma) = A(X)$ be a Poincaré Theta series. By Bers' approximation theorem, $R(U^c)$, the space of rational functions which are integrable over the complex plane and have all poles in U^c, is dense in $A(U)$. Since Θ is surjective (see [KR]), $\Theta(R(U^c))$ is dense in $\Theta(A(U)) = A(U, \Gamma) = A(X)$.

Every quadratic differential in $\Theta(R(U^c))$ is holomorphic in X_1 except for possibly finitely many poles in $X_1 - X$. Hence, for every φ in $\Theta(R(U^c))$, we can find a neighborhood U of X so that all poles of φ in U belong to $\overline{X} - X$. Denote $\Theta(R(U^c))$ by $R(X)$, and similarly define $R(Y)$.

In this section we prove the main result of the section:

Theorem 9 *If both X and Y are Riemann surfaces which are subsets of compact Riemann surfaces X_1 and Y_1, respectively , such that set $X_1 - X$ has infinitely many points, then every linear isometry from $A(X)$ onto $A(Y)$ comes from a pull-back by conformal mapping from Y onto X and multiplication by a complex constant of modulus one.*

In the proof of theorem 9 we use a theorem due to Noether about the existence of meromorphic functions with certain properties on a compact Riemann surface. We refer to this theorem as the "gap" theorem.

The "gap" theorem (Noether) *Let M be a compact Riemann surface of genus g. For any n points $p_1, p_2, p_3, \ldots, p_n$ in M with $n \geq 2g$, there exists a meromorphic function f on M such that Divisor(f) is a multiple of the divisor $-\chi_{p_1} - \chi_{p_2} - \ldots - \chi_{p_{n-1}} - \chi_{p_n}$ and Divisor(f) is not a multiple of the divisor $-\chi_{p_1} - \chi_{p_2} - \ldots - \chi_{p_{n-1}}$.*

The proof of the "gap" theorem can be found in [FK].

The proof of theorem 9 will follow the methods developed in section 4.3. Before we begin to prove theorem 9, we prove two preliminary lemmas. First we adapt lemma 5 to a new situation.

Lemma 6 *If $R_1, R_2 \in R(X)$ are non-zero meromorphic quadratic differentials with no double zeroes in common on $\overline{X}$, then there exist constants C_1 and C_2 such that* *(a)* $C_1R_1 + C_2R_2$ *has no double zeroes in* $\overline{X}$, *and*

(b) if $p \in \overline{X}$ *is a pole of* R_1 *or* R_2, *then* p *is a pole of* $C_1R_1 + C_2R_2$.

PROOF. To prove lemma 6 let $R = R_1R_2' - R_1'R_2$. Then R is a meromorphic abelian differential on a compact Riemann surface X_1, thus $A = \{a \in X_1 | P(a) = 0\}$ is a finite set provided that R_1 and R_2 are linearly independent, and the rest of the proof is the same as in lemma 5. □

The second preliminary lemma is a consequence of the "gap" theorem and lemma 6.

Lemma 7 *Let N be a compact surface and let S be a subset of N that contains infinitely many elements such that* $M = N - S$ *is a Riemann surface. Let* $s_1, s_2, \ldots, s_n \in S$ *and* $m \in M$. *Then there exists a meromorphic function* g *on N, such that*

(a) $g(m) = 0$,

(b) all poles of g *are simple and belong to* $S - \{s_1, s_2, \ldots, s_n\}$,

(c) g *has no zeroes in* $\{s_1, s_2, \ldots, s_n\}$, *and*

(d) g *has no double zeroes in N.*

PROOF. Since S is an infinite set, by the "gap" theorem, there exist two linearly independent meromorphic functions h_1 and h_2 on N which both satisfy (b). Let f be a non-trivial linear combination of h_1 and h_2 such that $f(m) = 0$. Let $\{a_1 = m, a_2, a_3, \ldots, a_k\}$ be the set of all zeroes of f in N. Let m_i be the order of zero of f at a_i for $i > 1$, and let $m_1 + 1$ be the order of zero of f at a_1. By the "gap" theorem, there exist meromorphic functions f_i on N such that f_i has a pole of order m_i at a_i , all poles of f_i in $N - \{a_i\}$ are simple and belong to $S - \{s_1, \ldots, s_n\}$, and such that f and f_i have no poles in common. Function ff_i satisfies (a) and (b), and $ff_i(a_i) \neq 0$, for $i > 1$, and

ff_1 has a simple zero at m. By induction, we can find constants $\alpha_1, \ldots, \alpha_n$ such that the meromorphic function $h = \alpha_1 ff_1 + \alpha_2 ff_2 + \ldots + \alpha_k ff_k$ has a simple zero at m and has no zeroes in $\{a_2, a_3, \ldots, a_k\}$. Now f and h have no double zeroes in common, and by the proof of lemma 6, some linear combination $g = \alpha f + \beta h$ satisfies (d) and (c). (Note that $g(s_i) \neq 0$ is equivalent to $\alpha f(s_i) + \beta h(s_i) \neq 0$, a linear condition for α and β). Since both f and h satisfy (a) and (b), g satisfies (a),(b),(c) and (d). □

Now we begin to prove theorem 9.

First we define spaces F and V in the same way as in section 4.3.

Definition 4 *Choose a point $b \in X$. Let*

$$F = \{\varphi \in A(X) | \varphi(b) = 0 \ and \ \varphi'(b) = 0\}$$

and

$$V = \bigcup_{\varphi \in F} \{\psi \in A(X) | \frac{\psi^2}{\varphi} \in L^1(X)\}.$$

Step I $\overline{V}$ is the space of all differentials in A(X) that vanish at b.

PROOF. If ψ does not vanish at b and φ is in F, then $\frac{\psi^2}{\varphi}$ has a double pole at b. That proves that $\overline{V}$ is the subset of the space of all differentials in A(X) that vanish at b.

To prove the converse suppose that $R \in R(X)$ such that $R(b) = 0$.

Set $X_1 - X$ is infinite; thus, by lemma 7, we can choose a meromorphic function g on X_1 such that

(a) $g(b) = 0$,

(b) all poles of g are simple and belong to $X_1 - X$,

(c) g and R have no poles in common,

(d) g has no double zeroes, and

(e) g has no zeroes at poles of R.

$\varphi = gR$ has a double zero at b, and all poles of g and R are simple, distinct, and outside X. Hence φ is in F. Futhermore, $\frac{R^2}{\varphi} = \frac{R}{g} \in L^1(X)$; thus, $R \in V$. Since $R(X)$ is dense in $A(X)$, we have $\overline{V} = \{\psi \in A(X) | \psi(b) = 0\}$. □

Step II $L(V) = \bigcup_{\varphi \in L(F)} \{\psi \in A(Y) | \frac{\psi^2}{\varphi} \in L^1(Y)\}$.

PROOF. Step II follows immediately from lemma 4. □

Step III Let p be an isolated point in $Y_1 - Y$. Then $\overline{L(V)} \neq \{\psi \in A(Y) | \psi$ is holomorphic at p $\}$.

PROOF. Suppose that $\overline{L(V)} = \{\psi \in A(Y) | \psi$ is holomorphic at p $\}$.

Let $\varphi \in R(X)$ such that $\varphi(b) \neq 0$. Such a differential exists by an easy consequence of the "gap" theorem. By lemma 7, we can choose a meromorphic function g on X_1 such that

(a) $g(b) = 0$,

(b) all poles of g are simple and belong to $X_1 - X$,

(c) g and φ have no poles in common,

(d) g has no double zeroes, and

(e) g has no zeroes at poles of φ.

Hence φ is not in $\overline{V}$ and $g\varphi$ is in $\overline{V}$, thus $L(\varphi)$ has a pole at p and $L(g\varphi)$ is holomorphic at p. Hence $\frac{\varphi^2}{g\varphi} = \frac{\varphi}{g} \in L^1(X)$, and $\frac{L(\varphi)^2}{L(g\varphi)}$ has a double pole at p, contradicting lemma 4. □

Now fix three points c_1, c_2, and c_3 in $Y_1 - Y$, and let $Q(Y)$ be the subset of $A(Y)$ consisting of all meromorphic quadratic differentials on Y_1, with simple poles at c_1, c_2, and c_3. By the proof of the Bers' approximation theorem, a linear span of differentials in $Q(Y)$ is dense in $A(Y)$.

Step IV Every R in $Q(Y) \bigcap L(F)$ has a double zero in $\overline{Y}$.

PROOF. Suppose that some $R \in Q(Y) \bigcap L(F)$ has no double zeroes in $\overline{Y}$.

If all poles (in Y_1) of some $\psi \in R(Y)$ belong to $\{c_1, c_2, c_3\}$, then $\frac{\psi^2}{R} \in L^1(Y)$; thus ψ belongs to $L(V)$. Let $Y_0 = Y_1 - \{c_1, c_2, c_3\}$. For every $p \in Y_0 - Y$, take $\psi_p \in A(Y_0 - \{p\}) - A(Y_0)$. The existence of such differential is guaranteed by the fact that the dimension of the space of integrable holomorphic quadratic differentials on a Riemann surface of finite analytic type (g, n) is $3g - 3 + n$, whenever $n > 2$; and it can be constructed using the Poincaré Theta series. If ψ_p is in $\overline{L(V)}$ for every $p \in Y_0 - Y$ then, $R(Y) \subset \overline{L(V)}$; a contradiction. Therefore ψ_p is not in $\overline{L(V)}$ for some $p \in Y_0 - Y$.

If p is a cluster point of $Y_1 - Y$, then take a sequence $p_n \in Y_0 - Y - \{p\}$ such that $p_n \to p$. Let $\Phi : D \to Y_0$ be the universal covering map with a covering group G. Let $Y_2 = Y_0 - \{p, p_1, p_2, p_3, \ldots\}$, $U = \Phi^{-1}(Y_2)$, $V = \Phi^{-1}(Y_2 \bigcup \{p\})$, and $\Theta : A(U) \to A(U, G) = A(Y_2)$ be a Poincaré Theta series. By the Bers' approximation theorem, $R(V^c)$ is dense in $A(U)$. Since Θ is surjective, $\Theta(R(V^c))$ is dense in $A(Y_2)$. Therefore, there exists a sequence (ψ_n) in $A(Y)$ such that $\psi_n \to \psi_p$, and ψ_n is holomorphic in Y_0 except for finitely many poles in $\{p_1, p_2, p_3, \ldots\}$. Without loss of generality, we can assume that ψ_n and ψ_m have no poles in common on Y_0 for $m \neq n$, and that ψ_n is not in $\overline{L(V)}$ for every n. Since $Codimension(F) = 2$, there exist constants α, β and γ, not all equal to 0, such that $M = \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3 \in L(F)$. Assume $\alpha \neq 0$. Then by lemma 6, we can find constants C_1 and C_2 such that $C_1 R + C_2 M$ has poles wherever ψ_1 has poles, and has no double zeroes in $\overline{Y}$. That yields $\frac{\psi_1^2}{C_1 R + C_2 M} \in L^1(Y)$; a contradiction that proves that p must be isolated in

$Y_1 - Y$.

Suppose that ψ_q is in $\overline{L(V)}$ for every $q \in Y_0 - Y - \{p\}$. Then, since $Codimension(\overline{V}) = 1$, we have $\overline{L(V)} = \{\psi \in A(Y) | \psi$ is holomorphic at p $\}$, which contradicts step III.

Therefore, ψ_q is not in $\overline{L(V)}$ for some q isolated in $Y_0 - Y - \{p\}$. This implies that neither ψ_q nor ψ_p is in $L(F)$. Let $\psi \in R(Y)$ be holomorphic at p and q. If ψ does not belong to $L(F)$, then since $Codimension(F) = 2$, there exist constants α, β and γ, not both α and β equal to zero, such that $M = \alpha\psi_p + \beta\psi_q + \gamma\psi_r \in L(F)$. Assume that $\alpha \neq 0$. Then M has a pole at p, and, by lemma 6, there exist constants C_1 and C_2 such that $T = C_1 M + C_2 R$ has poles at p, c_1, c_2, c_3 and has no double zeroes. But, then $\frac{\psi_p^2}{T} \in L^1(Y)$. This contradiction proves that every $\psi \in R(Y)$ which is holomorphic at p and q belongs to $L(F)$. Since $Codimension(F) = 2$ and $R(Y)$ is dense in $A(Y)$, we have

$$L(F) = \{\psi \in A(Y) | \psi \text{ is holomorphic at } p \text{ and } q\}.$$

If $\psi \in L(F)$, and φ is not in $L(F)$, then $\frac{\varphi^2}{\psi}$ has a double pole at p or q. Thus $L(F) = L(V)$; a contradiction.

Therefore, there exists $r \in Y_0 - Y - \{p, q\}$ such that ψ_r is not in $L(F)$. Since $Codimension(F) = 2$, □

Step V There exists a point a in Y such that every R in $L(F) \bigcap R(Y)$ has a double zero at a.

PROOF. Take a differential R in $L(F) \bigcap Q(Y)$. This is possible by changing c_1, c_2 and c_3 if necessary. Let $\{a_1, a_2, \ldots, a_n\}$ be the set of all double zeroes of R in $\overline{Y}$. If there exists a differential R_1 in $L(F) \bigcap R(Y)$ such that a_1 is not a double zero of R_1, then by the proof of lemma 6, there exist constants C_1 and C_2 such that the set of double zeroes of $C_1R_1 + C_2R$ in $\overline{Y}$ is a subset of $\{a_2, a_3, \ldots, a_n\}$, and that $C_1R_1 + C_2R$ belongs to $Q(Y)$. By induction, there exists $a \in \overline{Y}$ such that every quadratic differential in $L(F) \bigcap R(Y)$ has a double zero at a.

We now show that a must be in Y. Suppose not. Then $a \in \overline{Y} - Y$,thus there exists a quadratic differential ψ_a in $A(Y)$ which has a pole at a. By lemma 7, there exist differentials ψ and φ in $R(Y)$ such that $\psi(a) \neq 0$, $\varphi(a) = 0$, and $\varphi'(a) \neq 0$, Therefore, $Codimension(L(F)) = 2$ implies that some non-trivial linear combination of φ, ψ and ψ_a belongs to L(F), thus has a double zero at a; a contradiction. □

Step VI L(F) is the space of all quadratic differentials in A(Y) with the double zero at a.

PROOF. Take R in $R(Y)$ such that R has a double zero at a. Since $Codimension(F) = 2$, some linear combination of ψ, φ and R is in L(F), thus has a double zero at a. That implies that R is in $L(F)$. Since $R(Y)$ is dense in $A(Y)$, the set of all differentials in A(Y) with a double zero at a is a subset of L(F). Since $Codimension(F) = 2$, we have $L(F) = \{\psi \in A(Y) | \psi(a) = 0 \text{ and } \psi'(a) = 0\}$. □

Step VII $\overline{L(V)}$ is the set of all differentials in A(Y) that vanish at a.

PROOF. Step VII follows immediately from steps I, II and VI. □

Step VIII There exists a bijection β from X onto Y such that $\psi(p) = 0$ iff $L\psi(\beta(p)) = 0$ for every p in X and every ψ in A(X).

PROOF. To prove step VIII consider L^{-1} and observe that by lemma 7, for any two distinct points p and q in X, there exists a differential φ in A(X) such that $\varphi(p) = 0$ and $\varphi(q) \neq 0$. □

Step IX β is continuous.

PROOF. Suppose that β is not continuous. Then there exists a point p and a sequence (p_n) in X that converges to p such that $\beta(p_n)$ does not converge to $\beta(p)$. Since $\overline{Y}$ is compact, we can assume that $\beta(p_n)$ converges to a point q in $\overline{Y} - \{\beta(p)\}$.

By lemma 7, there exists a quadratic differential $\psi \in A(Y)$, and a function g meromorphic in Y_1 and holomorphic in Y such that $g\psi \in A(Y)$, $g(\beta(p)) \neq 0$, $\psi(\beta(p)) \neq 0$, and $g(q) = 0$. Then $\varphi_n = (g - g(\beta(p_n)))\psi$ belongs to A(Y) and vanishes at $\beta(p_n)$. Furthermore, φ_n converges to $g\psi$ in the L^1- norm. Therefore $||L^{-1}(\varphi_n) - L^{-1}(g\psi)|| \to 0$, $L^{-1}(\varphi_n)(p_n) = 0$, and $L^{-1}(g\psi)(p) \neq 0$, by step VIII. This contradicts the principle of argument. □

Step X β is holomorphic.

PROOF. Take any two linearly independent quadratic differentials φ and ψ in $A(X_1)$. Fix $p \in X$. Quadratic differential $\varphi(p)\psi - \psi(p)\varphi$ has a zero at p. Step VIII implies that $\varphi(p)L(\psi) - \psi(p)L(\varphi)$ has a zero at $\beta(p)$. Therefore $\frac{L\psi}{L\varphi}(\beta(p)) = \frac{\psi}{\varphi}(p)$ for almost every p. Since β is continuous and $\frac{L\psi}{L\varphi}$ is meromorphic, $\beta(p) = (\frac{L\psi}{L\varphi})^{-1} \circ (\frac{\psi}{\varphi})(p)$, locally, for almost every p; thus β is holomorphic. □

Step XI There exists a constant C of modulus one such that $L = C(\beta^{-1})^*$.

PROOF. By taking a geometric postcomposition, we can assume that $\beta = identity$.

The proof of step X implies that $L(\psi) = \psi\frac{L(\varphi)}{\varphi}$. Since $\{\psi, L(\psi), L(L(\psi)), \ldots\}$ is a normal family, $|\frac{L\varphi}{\varphi}| \equiv 1$. □

4.5 Some Applications

Theorem 10 *Let the Riemann surfaces X and Y be the subsets of compact Riemann surfaces X_1 and Y_1, respectively, such that set $X_1 - X$ has infinitely many points. If the cotangent spaces to the Teichmüller spaces of X and Y are isometric then the underlying Riemann surfaces X and Y are conformal.*

PROOF. Theorem 10 follows immediately from the isometry theorem. □

Any quasiconformal homeomorphism g of Y onto X induces a bijective map ρ_g from Teich(X) onto Teich(Y) by right translation; $\rho_g([f]) = [f \circ g]$. We call the bijection ρ_g a geometric isomorphism.

Theorem 11 *If the Riemann surfaces X and Y are subsets of compact Riemann surfaces X_1 and Y_1, respectively, such that set $X_1 - X$ has infinitely many points, then every holomorphic isomorphism from $Teich(X)$ onto $Teich(Y)$ is geometric.*

PROOF. Without loss of generality, we can assume that $F([id]) = [id]$. Since Kobayashi's metric is equal to Teichmüller's metric, F is an isometry. Therefore $F'([id])$ is a linear isometric map from Z(X), the tangent space of Teich(X) at $[id]$, onto Z(Y), the tangent space of Teich(Y) at $[id]$. This implies that $F'([id])$ is a dual map of an isometry G from A(Y) onto A(X) (see [EG]). Using theorem 9, we see that G is induced by a constant C and a conformal map from X onto Y. Therefore, without loss of generality, we may assume that X=Y and $F'([id]) = C\ Identity$. Earle and Gardiner showed that C=1 (See [EG]). Therefore, by the Cartan's uniqueness theorem (See [H]), $F = Identity$. □

Definition 5 *We call Teich(X) (conformally) equivalent to Teich(Y) if there exists a biholomorphic mapping from Teich(X) onto Teich(Y).*

Theorem 12 *There exist uncountably many non-equivalent infinite dimensional Teichmüller spaces.*

This theorem had also been discussed in [O].

PROOF. Suppose that $a_{i,1} = i$ and $a_{i,s+1} = 2^{a_{i,s}}$.

Let $X_k = \{0, \frac{1}{a_{k,1}}, \frac{1}{a_{k,2}}, \frac{1}{a_{k,3}}, \ldots\}^c$. Suppose that Teich(X_k) is equivalent to Teich(X_j) for two real numbers k and j with $k > j > 1$. Then by theorem 11, there exists a quasiconformal mapping g from X_k to X_j. By the extension property of quasiconformal mappings we can extend g to a quasiconformal homeomorphism h of the plane with h(0)=0.

Let $h(\frac{1}{a_{k,n}}) = \frac{1}{a_{j,f(n)}}$.

In the closed unit disk h is Hölder continous with some constant C and exponent α. Therefore $\frac{1}{a_{j,f(n)}} \leq C(\frac{1}{a_{k,n}})^\alpha$,

$$a_{j,f(n)} \geq \frac{1}{C}(a_{k,n})^\alpha,$$

$$\log a_{j,f(n)} \geq \alpha \log a_{k,n} - \log C,$$

$$\log a_{j,f(n)} \geq \frac{\alpha}{2} \log a_{k,n} \qquad \text{for large n},$$

$$a_{j,f(n)-1} \log 2 \geq \frac{\alpha}{2} a_{k,n-1} \log 2 \qquad \text{for large n},$$

$$a_{j,f(n)-1} \geq \frac{\alpha}{2} a_{k,n-1} \qquad \text{for large n} \tag{4.3}$$

Consider the function $l(x) = 2^{2^x} - 2^{x+1}$.

$l'(x) = 2^{2^x}(\log 2^x)2^x \log 2 - 2^{x+1} \log 2 =$

$= 2^x \log 2[2^{2^x} x \log 2 - 2] > 2 \log 2[4 \log 2 - 2] > 0$ for $x > 1$.

Therefore l is increasing on $(1, \infty)$.

Hence $2^{2^{a_{k,n}}} - 2^{a_{k,n}+1} \geq 2^{2^{a_{j,n}}} - 2^{a_{j,n}+1}$ for every n,

$a_{k,n+2} - 2a_{k,n+1} \geq a_{j,n+2} - 2a_{j,n+1}$ for every n,

$a_{k,n+2} - a_{j,n+2} \geq 2(a_{k,n+1} - a_{j,n+1})$ for every n.

Therefore, $a_{k,n} - a_{j,n} \to \infty$ as $n \to \infty$, and that implies : $\frac{a_{k,n}}{a_{j,n}} = 2^{a_{k,n-1} - a_{j,n-1}} \to \infty$ as $n \to \infty$.

Therefore,

$$\frac{\alpha}{2} a_{k,n-1} > a_{j,n-1} \quad \text{for large n} \tag{4.4}$$

(3) and (4) imply that there exists N such that $f(n) > n$ when $n \geq N$, but then $\{\frac{1}{a_{j,1}}, \frac{1}{a_{j,2}}, \ldots, \frac{1}{a_{j,N}}\} \subset \{h(\frac{1}{a_{k,1}}), h(\frac{1}{a_{k,2}}), \ldots, h(\frac{1}{a_{k,N-1}})\}$ which is a contradiction. □

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