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STRASSBERG, Helen Alene, 1940-
L-FUNCTIONS AND AUTOMORPHIC FORMS:
AN APPLICATION OF THE POISSON
SUMMATION FORMULA.

City University of New York, Ph.D., 1977
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

**L-Functions and Automorphic Forms:
An Application of the Poisson Summation Formula.**

by

HELEN STRASSBERG

A dissertation submitted to the Graduate Faculty
in Mathematics in partial fulfillment of the re-
quirements for the degree of Doctor of Philosophy,
The City University of New York

June 1977

This manuscript has been read and accepted
for the University Committee in Mathematics
in satisfaction of the dissertation require-
ment for the degree of Doctor of Philosophy.

May 17, 1977

Date

Howe Stacey
Chairman, Examining Committee

May 17, 1977

Date

Carlson T. Vazquez
Executive Officer

Alex Heller
Professor Alex Heller

Raymond T. Hoobler
Professor Raymond Hoobler

Linda Keen
Professor Linda Keen

Burton Randol
Professor Burton Randol

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ACKNOWLEDGEMENTS

It is a pleasure to thank Professor Jacquet for suggesting this problem and for supporting me throughout with his guidance and encouragement. His influence on my mathematical development has been tremendous.

I also wish to thank the faculty and students at the Graduate Center for their conversations and their curiosity, and Ms. Hutson for making the final stages of my graduate life easier with the skillful typing of this dissertation.

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§0. Introduction.

Dirichlet series associated to classical modular forms were first investigated systematically by Hecke. He showed that these series have an analytic continuation and functional equation, and he discovered that such a series has an Euler product expansion if and only if the associated form is an eigenfunction for a certain ring of operators (now called the Hecke operators). Much later, Weil completely characterized the Dirichlet series associated to a cusp form in terms of the analytic behavior of every series in a whole class of related Dirichlet series.

Dirichlet series associated to other types of automorphic forms have also been studied, notably the real analytic forms considered by Maass. More recently, Jacquet, Langlands, and others have studied the problem from the point of view of group representations. This approach has unified and considerably generalized the previous theories. Here, the automorphic forms are certain functions on $Gl(n,F)\backslash Gl(n,\underline{A})$, where \underline{A} is the ring of adeles of an algebraic number field F . Corresponding to the Hecke operators, there is an algebra $H(G)$ (the Hecke algebra) and its admissible representations on the space of automorphic forms. To each irreducible admissible representation π of $H(G)$ is associated a unique function $L(s,\pi)$, defined at first only for the real part of s large enough. For $Gl(2,\underline{A})$ Jacquet and Langlands proved that if π is such a representation which acts in the subspace of cusp forms, then $L(s,\pi)$ extends to an entire function which is bounded in vertical strips and satisfies a functional equation and they completely characterized the irreducible unitary representations which occur in the space of cusp forms in terms of a whole

set of L-functions related to $L(s, \pi)$. For $Gl(n, \underline{A})$, it is shown in [Godement-Jacquet] that if π is again such a representation which acts in the subspace of cusp forms, then $L(s, \pi)$ extends to an entire function that is bounded in vertical strips and satisfies a functional equation.

The aim of this paper is to extend some of these results beyond the subspace of cusp forms to the space of all automorphic forms. In §1 and §2 we summarize some information about admissible representations, the Hecke algebra, and the way in which the L-function is associated to an irreducible admissible representation. The main theorem (§4) is that if π occurs discretely in $V = L^2(G(F) \backslash G(\underline{A}), \omega)$, then $L(s, \pi)$ has an analytic continuation to the whole complex plane, has at most a finite number of poles, and satisfies a simple functional equation. In §5 we prove a similar theorem for admissible representations which occur in the space \mathcal{A} of automorphic forms on $G(F) \backslash G(\underline{A})$.

The primary tool in the proof is the Poisson summation formula applied to $\sum_{\gamma \in G(F)} \Phi(h\gamma g)$, where Φ belongs to $S(M)$, the space of Schwartz Bruhat functions on $M(n, \underline{A})$. There are difficulties in applying the formula because terms appear corresponding to the singular rational matrices. To avoid this, Φ will be chosen to lie in $S_0(M)$, the subspace of $S(M)$ consisting of functions Φ such that $\Phi(hxg) = \hat{\Phi}(hxg) = 0$ for all h and g in $G(\underline{A})$ and all rational matrices x with $\det x = 0$.

The appendix contains some of the same results in the language of Dirichlet series and Hecke operators.

Notations: If v is a place of F , we let $G_v = Gl(n, F_v)$. If v is a non-archimedean place of F and O_v is the ring of integers of F_v , define $K_v = Gl(n, O_v)$. If v is real, let $K_v = O(n, \mathbb{R})$

and if v is complex, let $K_v = U(n, \mathbb{C})$. In any case the Haar measure on K_v will be normalized so that the measure of K_v is 1.

The center $Z(\underline{A})$ of $G(\underline{A})$ will sometimes be identified to \mathbb{I} , the idele group of F .

§1. Admissible Representations.

If v is non-archimedean, define $H(G_v)$ to be the space of complex valued functions on G_v which are locally constant and compactly supported. This space is an algebra under convolution. A representation π of $H(G_v)$ on a complex vector space V is said to be admissible if (1) for every v in V there is an f in $H(G_v)$ so that $\pi(f)v = v$ and (2) for every open compact subgroup G' of G , the vectors of the form $\pi(f)v$, where $f \in H(G_v)$ is invariant on the left by G' , span a finite dimensional vector space. Given such a representation of $H(G_v)$ on V , there is a representation π of G_v on V related to the original representation by $\pi(f)v = \int_{G_v} f(g) \pi(g)v dg$ which is itself admissible in the following sense.

A representation of G_v on a complex vector space V is said to be admissible if (1) for every vector v in V the stabilizer of v in G_v is an open subgroup of G_v and (2) for every open compact subgroup G' of G , the space V' of vectors v in V stabilized by G' is finite dimensional.

There is a complete correspondence between admissible representations of G_v and $H(G_v)$. That is, a subspace is invariant under G_v if and only if it is invariant under $H(G_v)$ and an operator commutes with the action of G_v if and only if it commutes with the action of $H(G_v)$.

Any locally constant function on K_v can be regarded as an element of $H(G_v)$ by setting it equal to zero outside of K_v . In particular, if π_i is a finite family of inequivalent irreducible representations of K_v and $\xi_i(g) = \dim(\pi_i) \text{Tr } \pi_i(g^{-1})$, then ξ_i may be regarded as an element of $H(G_v)$. Functions of the form $\xi(g) = (\text{meas } K)^{-1} \sum \xi_i$ are idempotents in $H(G_v)$, and we refer to them as elementary idempotents.

If v is archimedean, let $H_1(G_v)$ be the space of infinitely differentiable compactly supported functions on G_v which are K -finite on both sides. Once a Haar measure has been chosen, the elements of H_1 may be regarded as measures, and then H_1 is an algebra under convolution. Consider the functions ξ on K which are finite sums of matrix elements of irreducible representations of K . These functions may be regarded as measures on G_v , and under convolution they form an algebra $H_2(G_v)$. Define $H(G_v) = H_1 + H_2$. It is also a convolution algebra. The elementary idempotents of H are defined as in the non-archimedean case.

A representation π of $H(G_v)$ on a complex vector space V is said to be admissible if

- 1) Every vector v in V is of the form

$$v = \sum_{1 \leq i \leq r} \pi(f_i) v_i$$

with f_i in H_1 and v_i in V .

- 2) For every elementary idempotent ξ the range of $\pi(\xi)$ is finite dimensional.
- 3) For every elementary idempotent ξ and every vector v in $\pi(\xi)V$, the map $f \rightarrow \pi(f)v$ of $\xi * H_1 * \xi$ into the finite

dimensional space $\pi(\xi)V$ is continuous.

Unlike the non-archimedean case, it is not possible in general to construct a representation of G_v itself on V .

Let F be a local field, archimedean or not. If π is a unitary continuous representation of $G(F)$ on a Hilbert space V , let V_0 be the subspace of K -finite vectors in V . If the measure f belongs to $H(G(F))$ then the operator $\pi(f) = \int f(g) \pi(g) dg$ leaves V_0 invariant. In this way, we get a representation π_0 of $H(G(F))$ on V_0 . Now V_0 is dense in V , so the class of π_0 completely determines the class of π . Assume that $(**)$: For every elementary idempotent ξ of $H(G(F))$ the range of $\pi(\xi)$ is finite dimensional. Then π_0 is admissible, and π is irreducible (in the topological sense) if and only if π_0 is irreducible (in the algebraic sense). The coefficients of the admissible representation π_0 are just the coefficients of π which are K -finite on both sides. If π is a unitary irreducible representation then it is known that π satisfies $(**)$ if F is archimedean or if π is square integrable, and it is a conjecture that it is always so.

From the local algebras $H(G_v)$ we can construct a global algebra $H(G)$ which is a restricted tensor product of the local ones. A representation of $H(G)$ on a complex vector space V is admissible if 1) every $v \in V$ can be written as a finite sum

$$\sum \pi(f_i)v_i \quad \text{with } v_i \in V \text{ and } f_i \in H_1(G).$$

- 2) For every elementary idempotent ξ the range of $\pi(\xi)$ is finite dimensional.
- 3) Let v_0 be an archimedean place, and suppose $\xi = \otimes \xi_v$ where

each ξ_v is an elementary idempotent of $H(G_v)$ and ξ_v is the normalized Haar measure on K_v for almost all v . Then if $\omega \in V$, the map $f_{v_0} \rightarrow \pi\left(f_{v_0} \otimes \left\{ \otimes_{v \neq v_0} \xi_v \right\}\right)\omega$ of $\xi_{v_0} H_{v_0} \xi_{v_0}$

into the finite dimensional space $\pi(\xi)V$ is continuous.

Every irreducible admissible representation of $H(G)$ is factorizable. That is, $\pi = \otimes \pi_v$, where π_v is an irreducible admissible representation of $H(G_v)$, and, if ϵ_v is the normalized Haar measure of K_v , the range of $\pi_v(\epsilon_v)$ is one dimensional for almost all v . If π is unitary then each π_v is unitary.

§2. L-functions Associated to Admissible Representations.

If π is an admissible and irreducible representation of $H(G)$, the function $L(s, \pi)$ is associated to π in the following way. Let φ be a coefficient of π and let φ_v be its factor at the place v . Form the integral $Z(\varphi_v, \bar{\varphi}_v, s) = \int_{G_v} \varphi_v(g_v) \bar{\varphi}_v(g_v) |\det g_v|_v^{s + \frac{n-1}{2}} dg_v$,

with $\bar{\varphi}_v$ a Schwartz Bruhat function on $M(n, F_v)$. The integral is absolutely convergent if the real part of s is large enough. If v is non-archimedean, the set of all finite sums of such zeta integrals is a fractional ideal in $\mathbb{E}[X, X^{-1}]$, where $X = q^{-s}$ and q is the order of the residual field of F_v . This ideal has a unique generator $1/P_0(X)$ with $P_0(0) = 1$, and then $L(s, \pi_v) = 1/P_0(X)$.

If v is archimedean, take $\bar{\varphi}$ to be a Schwartz function of the following type. If $F_v = \mathbb{R}$, take $\bar{\varphi}(x) = P(x) \exp(-2\pi \text{Tr} x x^t)$ with P a polynomial on the real vector space $M(n, \mathbb{R})$. If $F_v = \mathbb{C}$, take $\bar{\varphi}(x) = P(x) \exp(-2\pi \text{Tr} x \bar{x}^t)$ with P a polynomial on the real vector

space $M(n, \mathbb{C})$. Let $G_1(s) = \pi^{-\frac{1}{2}s} \Gamma(\frac{1}{2}s)$ and $G_2(s) = (2\pi)^{1-s} \Gamma(s)$ where Γ is the gamma function. Define an Euler factor $E(s)$ to be a function of the form $P(s) \prod_i G_1(s+s_i) \prod_j G_2(s+s_j)$ if $F_v = \mathbb{R}$, and a function of the form $P(s) \prod_j G_2(s+s_j)$ if $F_v = \mathbb{C}$. P is a polynomial and s_i and s_j are some constants. Then there is a fixed Euler factor depending only on π_v such that the set of finite sums $\sum E^{-1}(s) Z(\varphi_i, \Phi_i, s)$ is a non-zero ideal in $\mathbb{C}[s]$. If P_0 is the unique generator with $P_0(0) = 1$, we define $L(s, \pi_v) = E(s) P_0(s)$. Note that at any place we can find a finite number of φ_i and Φ_i so that $\sum Z(\varphi_i, \Phi_i, s) = L(s, \pi_v)$.

If v is non-archimedean and π_v contains the trivial representation of K_v , then $L(s, \pi_v) = \prod_{i=1}^n (1 - \mu_i(\tilde{\omega}_v) |\tilde{\omega}_v|^{-s})^{-1}$ where μ_i are quasicharacters of F_v^\times and $\tilde{\omega}$ is a generator of the maximal ideal of the ring of integers of F_v . Let $\mu_i(\tilde{\omega}_v) = |\tilde{\omega}_v|^{s_{\mu_i}}$. The numbers s_{μ_i} depend on the spherical function attached to π_v . If π is unitary then each π_v is unitary, and it is known that their corresponding spherical functions are bounded independent of v . This implies that $\frac{1}{2}(n-1) < \operatorname{Re} s_{\mu_i} < \frac{1}{2}(n-1)$ for every μ_i and every v . All but a finite number of the π_v contain the trivial representation of K_v . Now we can define $L(s, \pi) = \prod_v L(s, \pi_v)$. Because of the bound on $\operatorname{Re} s_{\mu_i}$ the product converges absolutely for real part of s sufficiently large. If π is not necessarily unitary, as in §5, we will need another method to show the convergence of the product.

Let ψ be a non-trivial character of $F \backslash \underline{A}$, and ψ_v its local component at v . For Φ a Schwartz Bruhat function on $M(n, F_v)$ define $\hat{\Phi}(y) = \int \Phi(x) \psi_v(\operatorname{tr} xy^t) dx$, and let $\tilde{\pi}_v$ be the contragredient

of π_v . Then locally we have a functional equation

$$(2.1) \quad \frac{Z(\varphi_v, \bar{\Phi}_v, s)}{L(s, \pi_v)} \epsilon(s, \pi_v, \psi_v) = \frac{Z(\varphi_v^z, \bar{\Phi}_v^\wedge, 1-s)}{L(1-s, \tilde{\pi}_v)} .$$

Here $\varphi^z(g) = \varphi(g^{-1})$, and $\epsilon(s, \pi_v, \psi_v)$ is a non-zero constant times a power of q^{-s} if v is non-archimedean, and a non-zero constant times an exponential if v is archimedean. It follows that for $\varphi_v, \bar{\Phi}_v$ fixed, $Z(\varphi_v, \bar{\Phi}_v, s)$ is identically zero as a function of s if and only if $Z(\varphi_v^z, \bar{\Phi}_v^\wedge, 1-s)$ is identically zero.

§3. Continuation of Zeta Integrals.

We need the following convergence lemmas. The first is Lemma 12.5 in [Godement-Jacquet].

Lemma 3.1. If $\bar{\Phi} \in S(M)$, the integral $\int_{G(\underline{A})} \bar{\Phi}(g) |\det g|^s dg$ is absolutely

convergent if the real part of s is greater than n .

Lemma 3.2. For the real part of s sufficiently large, the integral

$$F(h, g, s) = \int_{Z(\underline{A})/Z(F)} \sum_{\gamma \in G(F)} \bar{\Phi}(h^{-1}\gamma g z) |\det z|^s dz |\det g|^s |\det h|^{-s}$$

converges absolutely and defines a holomorphic function of s . For each s and h the function is continuous in g , and as a function of h and g it belongs to $L^2(G_F Z_{\underline{A}} \backslash G_{\underline{A}}) \times L^2(G_F Z_{\underline{A}} \backslash G_{\underline{A}})$.

Proof: This is a direct consequence of Lemmas 11.8 and 11.9 in [Godement-Jacquet].

Now define $L^2(G(F) \backslash G(\underline{A}), \omega)$ to be the space of functions on $G(F) \backslash G(\underline{A})$ which transform under the center by $f(zg) = \omega(z)f(g)$, where ω is a character of $F \backslash \mathbb{I}$, and which are square integrable on

$G(\mathbb{F})Z(\underline{A})\backslash G(\underline{A})$. Throughout the rest of this section, V is the space $L^2(G(\mathbb{F})\backslash G(\underline{A}), \omega)$ and ρ is the representation of $G(\underline{A})$ on V given by right translations.

Proposition 3.3. If $f \in V$ and $\Phi \in S(M)$, then $\int_{G(\underline{A})} f(g) \Phi(g) |\det g|^s dg$

converges absolutely if the real part of s is large enough.

Proof: We may take f and Φ positive. Let σ be the real part of s . Then

$$\begin{aligned} \int f(g) \Phi(g) |\det g|^\sigma dg &\leq \int_{\underline{Z}_A G_{\mathbb{F}} \backslash \underline{G}_A} f(g) \int_{\underline{Z}_F \backslash \underline{Z}_A} \Phi(\gamma g z) |\det z|^\sigma dz |\det g|^\sigma dg \\ &\leq \|f\|_2^{\frac{1}{2}} \|F(e, \cdot, \sigma)\|_2^{\frac{1}{2}} \end{aligned}$$

which is finite for σ large enough.

Corollary 3.3.1. The integral $\int_{\substack{G(\underline{A}) \\ |\det g| \geq 1}} f(g) \Phi(g) |\det g|^s dg$ is absolutely

convergent for all s .

Proof: Again take f and Φ positive and σ equal to the real part of s . Then

$$0 \leq \int_{|\det g| \geq 1} f(g) \Phi(g) |\det g|^\sigma dg \leq \int f(g) \Phi(g) |\det g|^t dg < \infty$$

if t is large enough and $\sigma \leq t$.

Proposition 3.4. If the real part of s is large enough, and $\Phi \in S(M)$, the integral $Z(\rho, \Phi, s) = \int_{G(\underline{A})} \rho(g) \Phi(g) |\det g|^{s + \frac{n-1}{2}} dg$ converges absolutely

and defines an operator on V . There are choices of Φ for which this operator is not identically zero as a function of s .

Proof: We have

$$\begin{aligned} \|Z(\rho, \bar{\varphi}, s)f\|_2^2 &= \int_{\underline{Z}_A \underline{G}_F \backslash \underline{G}_A} \left| \int_{\underline{G}_A} f(hg) \bar{\varphi}(g) |\det g|^{s + \frac{n-1}{2}} dg \right|^2 dh \\ &= \|f\|_2^2 \left| \int_{\underline{G}_A} \bar{\varphi}(g) |\det g|^{s + \frac{n-1}{2}} dg \right|^2 . \end{aligned}$$

The right side is absolutely convergent for real part of s large enough, so that for s in some right half plane $Z(\rho, \bar{\varphi}, s)$ is a well

defined operator on V . Since $\|Z(\rho, \bar{\varphi}, s)\| = \left| \int_{\underline{G}(A)} \bar{\varphi}(g) |\det g|^{s + \frac{n-1}{2}} dg \right|$

there are always choices of $\bar{\varphi}$ for which the operator is not the zero operator.

Proposition 3.5. Let f belong to V and $\bar{\varphi}$ to $S_0(M)$. Then

$$Z(f, \bar{\varphi}, s) = \int_{\underline{G}(A)} f(g) \bar{\varphi}(g) |\det g|^{s + \frac{n-1}{2}} dg \text{ has an analytic continuation to}$$

an entire function of s and satisfies $Z(f, \bar{\varphi}, s) = Z(f^\vee, \bar{\varphi}, 1-s)$.

Proof: We have

$$\begin{aligned} Z(f, \bar{\varphi}, s) &= \int_{|\det g| \geq 1} f(g) \bar{\varphi}(g) |\det g|^{s + \frac{n-1}{2}} dg + \int_{|\det g| \leq 1} f(g) \bar{\varphi}(g) |\det g|^{s + \frac{n-1}{2}} dg . \end{aligned}$$

The first integral is entire. The second converges absolutely for the real

part of s large enough, and equals $\int_{\substack{\underline{G}(F) \backslash \underline{G}(A) \\ |\det g| \leq 1}} f(g) \sum_{\underline{G}(F)} \bar{\varphi}(\gamma g) |\det g|^{s + \frac{n-1}{2}} dg$.

By the Poisson summation formula,

$$\sum_{\gamma \in G(F)} \hat{\Phi}(\gamma g) = \sum_{\gamma \in G(F)} \hat{\Phi}(\gamma g^z) |\det g|^{-n} + \sum_{\substack{x \in M(n, F) \\ \det x = 0}} \hat{\Phi}(xg^z) |\det g|^{-n} - \sum_{\substack{x \in M(n, F) \\ \det x = 0}} \hat{\Phi}(xg) .$$

Because $\hat{\Phi}$ lies in $S_0(M)$ the last two terms are zero. Since $\hat{\Phi}$ is also in $S_0(M)$ we may substitute the new sum $\sum \hat{\Phi}(\gamma g^z) |\det g|^{-n}$ for the original one and still have a convergent integral for the real part of s sufficiently large. The new integral is

$$\int_{\substack{G(F) \backslash G(A) \\ |\det g| \leq 1}} f(g) \sum \hat{\Phi}(\gamma g^z) |\det g|^{-n+s+\frac{n-1}{2}} = \int_{\substack{G(A) \\ |\det g| \geq 1}} f(g^z) \hat{\Phi}(g) |\det g|^{1-s+\frac{n-1}{2}} dg .$$

This integral is entire by Corollary 3.3.1. Finally,

$$Z(f, \hat{\Phi}, s) = \int_{|\det g| \geq 1} f(g) \hat{\Phi}(g) |\det g|^{s+\frac{n-1}{2}} dg + \int_{|\det g| \geq 1} f(g^z) \hat{\Phi}(g) |\det g|^{1-s+\frac{n-1}{2}} dg$$

and the assertion follows.

Corollary 3.5.1. Let $(h.f)(g) = f(hg)$. If $\hat{\Phi} \in S_0(M)$ the assertions of the theorem are true for $Z(h.f, \hat{\Phi}, s)$ with functional equation $Z(h.f, \hat{\Phi}, s) = Z((h.f)^z, \hat{\Phi}, 1-s)$.

Proof: The proof is the same as that of the Proposition.

Proposition 3.6. If $\hat{\Phi} \in S_0(M)$ and φ is any coefficient of ρ then

$$Z(\varphi, \hat{\Phi}, s) = \int_{G(A)} \varphi(g) \hat{\Phi}(g) |\det g|^{s+\frac{n-1}{2}} dg$$

is defined and holomorphic for

the real part of s large enough. It has an analytic continuation to the whole complex plane as an entire function of s , and satisfies $Z(\varphi, \hat{\Phi}, s) = Z(\varphi^z, \hat{\Phi}, 1-s)$.

Proof: Assume that Φ is positive and let σ be the real part of s .

Then

$$\begin{aligned} |Z(\varphi, \Phi, s)| &\leq \int_{G(\underline{A})} \int_{Z_{\underline{A}} G_{\underline{F}} \backslash G_{\underline{A}}} |f_1(hg) \overline{f_2(h)}| dh \Phi(g) |\det g|^{\sigma + \frac{n-1}{2}} dg \\ &\leq \|f_1\|_2^{\frac{1}{2}} \|f_2\|_2^{\frac{1}{2}} \int_{G_{\underline{A}}} \Phi(g) |\det g|^{\sigma + \frac{n-1}{2}} dg. \end{aligned}$$

This proves the first statement. The absolute convergence justifies the following computation.

$$\begin{aligned} Z(\varphi, \Phi, s) &= Z(\langle \rho(g)f_1, f_2 \rangle, \Phi, s) \\ &= \langle Z(h.f_1, \Phi, s), f_2 \rangle \\ &= \langle Z(h.f_1)^z, \hat{\Phi}, 1-s \rangle, f_2 \rangle \\ &= Z(\langle \rho(g^z)f_1, f_2 \rangle, \hat{\Phi}, 1-s) \\ &= Z(\varphi^z, \hat{\Phi}, 1-s). \end{aligned}$$

Note that φ^z is a coefficient of the regular representation on $L^2(G(\underline{F}) \backslash G(\underline{A}), \bar{\omega})$.

Corollary 3.6.1. For $\Phi \in S_0(M)$, the operator $A(s) = Z(\rho, \Phi, s)$ has an analytic continuation to an entire function of s with values in the Banach space \mathcal{B} of bounded linear operators on V .

Proof: By the argument on p.184 of [Godement-Jacquet], there is a majorization $|A(s)v, w| \leq c \|v\| \|w\|$ where c is a constant which is independent of v and w and independent of s as long as s remains in a strip of finite width. This implies that $A(s)v$ can be uniquely defined for every s and v .

By the Proposition, if $s \rightarrow s_0$, $|(A(s)-A(s_0))v, w| \rightarrow 0$ for every v and w , so that as a function of s , $A(s)v$ is continuous for every v . In fact, the convergence is independent of v if v lies in a bounded set, because

$$\begin{aligned} |(A(s)v - A(s_0)v), w|^2 &\leq \|v\|^2 \|w\|^2 \left| \int_{|\det g| \geq 1} \hat{\Phi}(g) \left\{ |\det g|^{s + \frac{n-1}{2}} - |\det g|^{s_0 + \frac{n-1}{2}} \right\} dg \right. \\ &\quad \left. + \int_{|\det g| \geq 1} \hat{\Phi}(g) \left\{ |\det g|^{1-s + \frac{n-1}{2}} - |\det g|^{1-s_0 + \frac{n-1}{2}} \right\} dg \right|^2. \end{aligned}$$

Therefore, $s \rightarrow s_0$ implies $\|A(s) - A(s_0)\| = \sup_{\|v\| \leq 1} |A(s)v - A(s_0)v| \rightarrow 0$,

so that $s \rightarrow A(s)$ is continuous in the norm topology.

The maps $B \rightarrow (Bv, w)$ from \mathcal{B} to \mathcal{E} form a total set in the space of continuous linear functionals on \mathcal{B} when \mathcal{B} is given the strong operator topology. Together with the continuity of $s \rightarrow A(s)v$ and Proposition 3.6, this implies that $s \rightarrow A(s)v$ is entire for every v .

Finally, since $s \rightarrow A(s)$ is continuous, we have that for every closed contour C in \mathcal{E} and every v , $\int_C A(s) ds \cdot v = \int_C A(s) v ds = 0$.

Therefore $s \rightarrow A(s)$ is entire.

Because of the functional equation of Proposition 3.6, the operator satisfies the functional equation $Z(\rho, \hat{\Phi}, s) = Z^t(\tilde{\rho}, \hat{\Phi}, 1-s)$ where the contragredient $\tilde{\rho}$ is the regular representation of $G(\underline{A})$ on $L^2(G(\underline{F}) \backslash G(\underline{A}), \bar{\omega})$.

§4. Continuation of L-functions.

Assume now that π is an irreducible admissible representation of

$H(G)$ on an algebraic subspace V of $L^2(G(F)\backslash G(\underline{A}), \omega)$, and assume that $\Phi \in S(M)$ is factorizable. That is, $\Phi = \prod_v \Phi_v$. If φ is any coefficient of π then $Z(\varphi, \Phi, s) = \prod_v Z(\varphi_v, \Phi_v, s)$. For all but a finite number of places $Z(\varphi_v, \Phi_v, s) = L(s, \pi_v)$, since φ_v is the spherical function associated to π_v and Φ_v is the characteristic function of K_v . At each of the remaining places there are a finite number of coefficients $\varphi_{i,v}$ and a finite number of functions $\Phi_{i,v} \in S(M_v)$ such that $\sum Z(\varphi_{i,v}, \Phi_{i,v}, s) = L(s, \pi_v)$. Then globally there are a finite number of φ_i and Φ_i such that

$$\sum Z(\varphi_i, \Phi_i, s) = L(s, \pi).$$

Now Φ_i will be modified to obtain a function Ψ_i belonging to $S_0(M)$. Choose distinct places v_1 and v_2 . Let Ψ_{v_1} and Ψ_{v_2} be Schwartz Bruhat functions whose supports lie in G_{v_1} and G_{v_2} respectively,

and for which $\int_{G_{v_1}} \varphi_{v_1}(g) \Psi_{v_1}(g) |\det g|_{v_1}^{\frac{s+n-1}{2}} dg = 1$ and

$\int_{G_{v_2}} \varphi_{v_2}(g) \Phi_{v_2}(g) |\det g|_{v_2}^{1-s+\frac{n-1}{2}} dg = 1$. To construct Ψ_{v_1} , for

instance, choose an h in G_{v_1} for which $\varphi_{v_1}(h) \neq 0$. There is a

compact subgroup K' of G_{v_1} such that $\int_K \varphi_{v_1}(hk) dk = \varphi_{v_1}(h)$. Take Ψ'

with its support in $h \cdot K'$ so that $I = \int_{K'} \Psi'(hk) dk$ is not zero. Then

$$\text{let } \Psi_{v_1}(g) = \frac{\Psi'(g) |\det g|_{v_1}^{-s-\frac{n-1}{2}}}{\varphi_{v_1}(h) \cdot I}.$$

Because there is a local functional equation (2.1) we have

$$Z(\varphi_{v_2}, \Psi_{v_2}, s) = \frac{L(s, \pi_{v_2}) \varepsilon^{-1}(s, \pi_{v_2})}{L(1-s, \tilde{\pi}_{v_2})}. \text{ Define } \Psi_i(g) =$$

$\prod_{v \neq v_1, v_2} \Phi_i(g_v) \cdot \Psi_{v_1}(g_{v_1}) \Psi_{v_2}(g_{v_2})$. Then $\Psi_i \in S_0(M)$ and

$$(4.1) \quad \sum_i \int \varphi_i(g) \Psi_i(g) |\det g|^{s + \frac{n-1}{2}} dg =$$

$$\frac{L(s, \pi)}{L(s, \pi_{v_1}) L(s, \pi_{v_2})} \cdot 1 \cdot \frac{L(s, \pi_{v_2}) \varepsilon^{-1}(s, \pi_{v_2})}{L(1-s, \tilde{\pi}_{v_2})}.$$

Each of the integrals in this expression defines an entire function of s , and $\varepsilon(s, \pi_{v_2})$ is also entire. Therefore

Proposition 4.2. If π is an irreducible admissible representation of $H(G)$ on an algebraic subspace V of $L^2(G(F) \backslash G(A), \omega)$ then

$$(4.2) \quad L(s, \pi) = \Xi(s) L(s, \pi_{v_1}) L(1-s, \tilde{\pi}_{v_2})$$

where $\Xi(s)$ is entire.

The construction of the functions Ψ_i presupposes that v_1 and v_2 are distinct, but in fact, if v is non-archimedean and π_v contains the trivial representation of K_v , then a relation of the form (4.2) is true for $v_1 = v_2$ as well. Maloletkin has constructed a function Ψ with the following properties:

$$1) \quad \Psi \in S_0(M_v)$$

$$2) \quad Z(\varphi_0, \Psi, s) = L(1-s, \tilde{\pi}_v)^{-1}$$

where φ_0 is the spherical function associated to π_v . Defining

$\Psi_i(g) = \prod_{v \neq v_1} \pi \bar{\phi}_i(g_v) \cdot \Psi(g_{v_1})$ and proceeding as before shows that

Proposition 4.3. If v is a finite place and π_v contains the trivial representation of K_v then $L(s, \pi) = \Xi(s) L(s, \pi_v) L(1-s, \tilde{\pi}_v)$ where $\Xi(s)$ is entire. It follows that

Theorem 4.4. If π is an irreducible admissible representation of $H(G)$ on an algebraic subspace V of $L^2(G(F) \backslash G(\underline{A}), \omega)$ then $L(s, \pi)$ has an analytic continuation to the whole complex plane. Any poles of $L(s, \pi)$ are poles of $L(s, \pi_{v_1}) L(1-s, \tilde{\pi}_{v_2})$ for every choice of distinct v_1 and v_2 , and poles of $L(s, \pi_v) L(1-s, \tilde{\pi}_v)$ whenever v is non-archimedean and π_v contains the trivial representation of K_v .

If v is archimedean and π_v contains the trivial representation of K_v , Andrianov has constructed a function $\Psi \in S_0(M)$ with the property that $Z(\varphi, \Psi, s) = c P(\varphi, s) L(s, \pi_v)$. P is a polynomial in s which depends on the spherical function φ , and which satisfies $P(\varphi, s) = P(\varphi^j, 1-s)$. Defining Ψ_i as before, we have $\sum Z(\varphi_i, \Psi_i, s) = c P(\varphi_v, s) L(s, \pi)$, so that

Proposition 4.5. If v is archimedean and π_v contains the trivial representation of K_v then $L(s, \pi) = \Xi(s) P^{-1}(\varphi_v, s)$ where $\Xi(s)$ is entire. There are at most a finite number of poles of $L(s, \pi)$, and any pole of $L(s, \pi)$ is a pole of $P^{-1}(\varphi_v, s)$.

Proposition 4.6. If π is as in Theorem 4.4, then $L(s, \pi)$ has at most a finite number of poles.

Proof: Let v be a finite place and let q_v be the number of elements in the residual field of F_v . Suppose that π_v contains the trivial representation of K_v . Then

$$L(s, \pi_v) = \prod_{i=1}^n L(s, \mu_{i,v}) = \prod_{i=1}^n \left(1 - q_v^{-s-s_{i,v}}\right)^{-1}$$

where $\mu_{i,v}(a) = |a|_v^{s_{i,v}}$ and $s_{i,v}$ is determined only modulo

$2i\pi/\log q_v$. It is sufficient to show that there is at most one common pole for $L(s, \mu_{v_1})$ and $L(s, \mu_{v_2})$ if $v_1 \neq v_2$. Let t be a common pole. Then $t = -s_1 + 2i\pi k/\log q_1 = -s_2 + 2i\pi m/\log q_2$. If t_1 is another such pole, $t_1 = t + 2i\pi k'/\log q_1 = t + 2i\pi m'/\log q_2$. Then $q_1^{m'} = q_2^{k'}$, so that $m' = k' = 0$.

Proposition 4.7. If π is as in Theorem 4.4, then $L(s, \pi)$ satisfies the functional equation $L(s, \pi) = \epsilon(s, \pi) L(1-s, \tilde{\pi})$.

Proof: By Proposition 3.6, $\sum Z(\varphi_i, \Psi_i, s) = \sum Z(\varphi_i^z, \hat{\Psi}_i, 1-s)$. Each $Z(\varphi_i^z, \hat{\Psi}_i, 1-s)$ is factorizable, and the local functional equations (2.1) show that

$$\sum Z(\varphi_i^z, \hat{\Psi}_i, 1-s) = \frac{\sum Z(\varphi_i, \Psi_i, s)}{L(s, \pi)} \epsilon(s, \pi) L(1-s, \tilde{\pi}).$$

Since the sum $\sum Z(\varphi_i, \Psi_i, s)$ is not identically zero, we may divide by it, and the functional equation follows.

If π is an irreducible admissible representation of $G(\underline{A})$ on $V \subset L^2(G(\underline{F}) \backslash G(\underline{A}), \omega)$, and χ is a character of $F^{\times} \backslash I$, the representation $\chi \otimes \pi$ defined by $\chi \otimes \pi(g) = \chi(\det g) \pi(g)$ is also an irreducible admissible representation on V . Its contragredient is $\chi^{-1} \otimes \tilde{\pi}$. From the proof of Proposition 4.2, it follows that $L(s, \chi \otimes \pi) = \Xi(s) L(s, \chi_1 \otimes \pi_1) L(1-s, \chi_2^{-1} \otimes \tilde{\pi}_2)$ for every choice of distinct places v_1 and v_2 . Now by Proposition 3.8 of [Jacquet-Langlands], if the orders of χ_1 and χ_2 are large enough, then both $L(s, \chi_1 \otimes \pi_1)$ and

$L(1-s, \chi_2^{-1} \otimes \tilde{\pi}_2)$ are 1, so we have the following:

Proposition 4.8. Let χ be a character of $F^X \backslash \mathbf{I}$, let π be as in Theorem 4.4, and let v_1 and v_2 be distinct places. Then if the local components χ_1 and χ_2 at v_1 and v_2 are sufficiently ramified, $L(s, \chi \otimes \pi)$ is entire.

In the following Proposition, $\pi(\mu, \nu)$ is as defined in [Jacquet-Langlands].

Proposition 4.9. If $n = 2$ and $L(s, \pi)$ has a pole then there are two quasicharacters μ, ν of $F^X \backslash \mathbf{I}$ so that for all but a finite number of places, π is equivalent to $\pi(\mu_v, \nu_v)$.

Proof: If π_v contains the trivial representation of K_v then there are quasicharacters $\mu_v(a) = |a|_v^{s_v}$ and $\nu_v(a) = |a|_v^{t_v}$ so that π_v is equivalent to $\pi(\mu_v, \nu_v)$. The numbers s_v and t_v are defined only modulo $2i\pi/\log q_v$. Since $\pi(\mu_v, \nu_v)$ is equivalent to $\pi(\nu_v, \mu_v)$, what we want to show is that representatives s_v and t_v can be chosen in such a way that $\{s_v, t_v\}$ is the same set for every v in S , the set of finite places for which π_v contains the trivial representation of K_v .

Suppose s_0 is a pole of $L(s, \pi)$. Then it is a pole of $L(s, \pi_{v_1}) L(1-s, \tilde{\pi}_{v_2})$ for every choice of finite places v_1 and v_2 in S . These places need not be distinct. Then $s_0 \in \{-s_1, -t_1, 1-s_2, 1-t_2\}$ for some representatives s_i and t_i . If $s_0 = -s_v$ for every choice of $v \in S$ we are done. In fact $\pi \left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g \right] = \omega(z) \pi(g)$ for every $z \in \mathbf{I}$, and locally for every $v \in S$, $\pi_v \left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g \right] = \mu_v(z) \nu_v(z) \pi_v(g)$. Therefore if $\omega(g) = |z|_A^{s_0}$, $\{s_v, t_v\} = \{-s_0, s_0 + s_0\}$ for every $v \in S$.

Suppose there is a place $v_1 \in S$ for which s_0 is not $-s_{v_1}$ or

$-t_{v_1}$. Then $s_0 = -s_v + 1$ or $-t_v + 1$ for every $v \in S$ because $s_0 \in \{-s_{v_1}, -t_{v_1}, 1-s_v, 1-t_v\}$ for every $v \in S$ including v_1 . In this case $\{s_v, t_v\} = \{1-s_0, s_0+s_0-1\}$ for every $v \in S$.

Corollary 4.9.1. Let π be an irreducible (admissible) representation of $H(G)$ on a subspace V of $L^2(G(F)\backslash G(\underline{A}), \omega)$. Then any pole of $L(s, \pi)$ is a pole of $L(s, \mu)$ for some grossencharacter μ .

Proof: Immediate consequence of the Proposition.

§5. Results for the Space of Automorphic Forms.

If \mathcal{A} is the space of automorphic forms (defined below), then $\mathcal{A} \cap L^2(G(F)\backslash G(\underline{A}), \omega)$ is dense in the discrete part of L^2 , and the continuous part of L^2 can be expressed in terms of the non- L^2 forms. In this section we will show that except for information about the poles, the results of §2-4 can be extended to irreducible admissible representations occurring in the space \mathcal{A} . Define

$$\|g\|_{\underline{A}}^* = \pi_v \|g_v\|_v^* = \pi_v \frac{\|g_v\|_v}{|\det g_v|_v} 1/n \quad \text{with}$$

$$\|g_v\|_v = \begin{cases} (\sum (g_{ij})^2)^{\frac{1}{2}} & \text{if } v \text{ is real} \\ (\sum g_{ij} \bar{g}_{ij})^{\frac{1}{2}} & \text{if } v \text{ is complex} \\ \sup |g_{ij}|_v & \text{if } v \text{ is non-archimedean.} \end{cases}$$

The product makes sense since almost all $\|g_v\|_v$ are 1. The function $\| \cdot \|_{\underline{A}}^*$ is bi-invariant for $K = \prod_v K_v$ and invariant for $Z(\underline{A})$.

Proposition 5.1. $\|g\|_{\underline{A}}^* \geq 1$ for all g in $G(\underline{A})$.

Proof: Let $g_v = k_1 a_v k_2$ with k_1, k_2 in K_v and a_v in the subgroup

of G_v consisting of diagonal matrices (a_1, \dots, a_n) with $|a_1| \geq \dots \geq |a_n|$. Then $\|g_v\|_v = \|a_v\|_v$. If v is archimedean, we may assume $|\det g_v|_v = 1$ because of the invariance by $Z(F_v)$. This implies that $|a_1| \geq 1$, and therefore that $\|g\|_{\mathbb{R}}^*$ and $\|g\|_{\mathbb{C}}^*$ are both greater than or equal to 1. If v is non-archimedean, we may suppose by the invariance by $Z(F_v)$ that $|a_1| = 1$, and $|a_i| \leq 1$ for $1 < i \leq n$. Then $\|g_v\|_v^* = \frac{1}{|a_2 \dots a_n|^{1/n}} \geq 1$.

The space \mathcal{A} of automorphic forms consists of the continuous functions on $G(F) \backslash G(\mathbb{A})$ with the following properties:

- 1) f is right K -finite.
- 2) f transforms according to a quasicharacter ω of $F^\times \backslash \mathbb{I}$ under translations by $Z(\mathbb{A})$.
- 3) The representation of $H(G)$ on the space $\{f * \varphi \mid \varphi \in H(G)\}$ is admissible.
- 4) For all g with $\|g\|_{\mathbb{A}}^*$ large enough $|\varphi(g)| \leq c (\|g\|_{\mathbb{A}}^*)^m$ for some constant c and integer m .

Proposition 5.2. For $\varphi \in \mathcal{A}$ and $\Phi \in S(M)$, $\int_{G(\mathbb{A})} \varphi(g) \Phi(g) |\det g|^s dg$

converges absolutely for real part of s large enough.

Proof: Let $I = \int_{G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi(g) \int_{Z_F \backslash Z_{\mathbb{A}}} \Phi(\gamma g z) |\det z|^s dz |\det g|^s dg$. A set of

representatives for $G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$ is the set of $g \in G_{\mathbb{A}}$ which can be written

$$g = \begin{pmatrix} a_1 & & & 0 \\ & \dots & & \\ & & a_{n-1} & \\ 0 & & & 1 \end{pmatrix} g' \quad \text{where } g' \text{ lies in some compact of } G_{\mathbb{A}}$$

and the diagonal entries satisfy $|a_i| \geq c |a_{i+1}|$ for some positive constant c .

If $|\det g|$ is large enough, say $\geq N$, then $\|g\|_{\mathbb{A}}^*$ is large enough to

apply the majorization for φ . This is so because the $|a_i|$ are all bounded away from zero. Break the integral into an integral I_+ for $|\det g| \geq N$ and an integral I_- for $|\det g| \leq N$. If $|\det g| \leq N$, the domain of integration is a compact, and the integrand is continuous in g if the real part of s is large enough, so that I_- converges absolutely. The integral I_+ is majorized by $I = c' \int_{G(\underline{A})} \|g\|^{*m} \varphi(g) |\det g|^\sigma dg$ if c' is taken large enough. Write

$G = K \cdot A \cdot K$ with A the subgroup of diagonal matrices. Then we may take $a = (a_1, \dots, a_n)$ in this decomposition with $|a_1| \geq \dots \geq |a_n|$, and we have $I \leq c' \int \|a\|^m \varphi'(a) |\det a|^{s-m/n} d^\times a$ where $\varphi'(a) = \int_{K \times K} \varphi(k_1 a k_2) dk_1 dk_2$

belongs to $S(M)$. Since $\|a\| \leq n|a_1|$ we have

$$I \leq c' \cdot n^m \int_{|a_1| \geq \dots \geq |a_n|} \varphi'(a_1, \dots, a_n) |a_1|^{m+s-\frac{m}{n}} |a_2 \dots a_n|^{s-\frac{m}{n}} d^\times a_1 \dots d^\times a_n .$$

Because φ' can be bounded by a product $\prod \varphi_i(a_i)$ of Schwartz functions on \mathbf{I} the integral can be done easily over successive variables. It is clear that it is finite when the real part of s is large enough, and that the convergence is uniform for s in a compact.

Proposition 5.3. If φ belongs to $S_0(M)$ and φ belongs to \mathcal{A} then $Z(h, \varphi, \varphi, s)$ has an analytic continuation to an entire function of s and satisfies the functional equation $Z(h, \varphi, \varphi, s) = Z(h, \varphi, \varphi, 1-s)$.

Proof: The proof is exactly as in Corollary 3.5.1.

For each φ in $S(M)$ there is an elementary idempotent ξ such that $\varphi(g) = \int \varphi(hg) \xi(h^{-1}) dh$. Substituting this in the integral $Z(h, \varphi, \varphi, s)$ and changing variables, we have

$$\begin{aligned}
(5.3) \quad \int_G \varphi(hg) \bar{\Phi}(g) |\det g|^{s + \frac{n-1}{2}} dg &= \int_G \left[\int_K \varphi(hkg) \xi(k) dk \right] \bar{\Phi}(g) |\det g|^{s + \frac{n-1}{2}} dg \\
&= \int f_h(g) \bar{\Phi}(g) |\det g|^{s + \frac{n-1}{2}} dg .
\end{aligned}$$

If π is an irreducible admissible representation of $H(G)$ on the subspace \mathcal{A}_1 of \mathcal{A} , and $\varphi \in \mathcal{A}_1$, then for each h , f_h is a coefficient of π .

Corollary 5.3.1. If f is a coefficient of π of the type constructed in (5.3), and $\bar{\Phi}$ belongs to $S_0(M)$, then $Z(f, \bar{\Phi}, s)$ is absolutely convergent for real part of s large enough, has an analytic continuation to an entire function of s , and satisfies $Z(f, \bar{\Phi}, s) = Z(f^\vee, \bar{\Phi}, 1-s)$.

Proof: Apply Proposition 5.3 to $Z(f_h, \bar{\Phi}, s) = Z(h \cdot \varphi, \bar{\Phi}, s)$ with $h = e$. These special zeta integrals will give most of the information about $L(s, \pi)$ that we were able to get in the unitary situation.

First, a special case. Suppose π contains the trivial representation of K . Then π_v contains the trivial representation of K_v for each v . If φ_v is the spherical function attached to π_v , and $\bar{\Phi}_v$ is the characteristic function of K_v for v non-archimedean, the function $\bar{\Phi}(x) = \exp(-\pi \operatorname{tr}(xx^t))$ for v real, and the function $\bar{\Phi}(x) = \exp(-2\pi \operatorname{tr}(x\bar{x}^t))$ for v complex, then

$$\int \varphi_v(g_v) \bar{\Phi}_v(g_v) |\det g_v|_v^{s + \frac{n-1}{2}} = L(s, \pi_v) .$$

Let $\varphi = \pi \varphi_v$ and $\bar{\Phi} = \pi \bar{\Phi}_v$.

Theorem 5.4. Let π be an irreducible admissible representation of

$G(\underline{A})$ on \mathcal{A} which contains the trivial representation of K . Define

$L(s, \pi) = \prod_v L(s, \pi_v)$. Then

- 1) The infinite product is absolutely convergent for the real part of s large enough.
- 2) $L(s, \pi)$ has an analytic continuation to the whole complex plane and satisfies $L(s, \pi) = \epsilon(s, \pi) L(1-s, \tilde{\pi})$.
- 3) $L(s, \pi)$ has at most a finite number of poles.

Proof: $Z(\varphi, \Phi, s)$ converges absolutely for real part of s sufficiently large, because the spherical function is a coefficient of the form (5.3).

Specifically, $\varphi(g) = \int_G f(kg) \xi(k^{-1}) dk$, where f is the unique right

K -invariant function in the space of π whose value at the identity is 1, and ξ is the characteristic function of K . It follows that $Z(\varphi, \Phi, s) = \prod_v Z(\varphi_v, \Phi_v, s) = \prod_v L(s, \pi_v)$ and that $\prod_v L(s, \pi_v)$ converges absolutely whenever $Z(\varphi, \Phi, s)$ does.

Now replace Φ by a function Ψ in $S_0(M)$ by the method of §4. Then just as before, we have the relation $L(s, \pi) = \Xi(s) L(s, \pi_1) L(1-s, \tilde{\pi}_2)$ where v_1 and v_2 are any two places (which can be the same if non-archimedean) and $\Xi(s)$ is an entire function which depends on v_1 and v_2 . This shows $L(s, \pi)$ has the desired continuation, and the same computations as in the L^2 case yield the other assertions.

More generally, let π be an irreducible admissible representation of $H(G)$ on a subspace \mathcal{A}_1 of \mathcal{A} . Consider the integrals $Z(f, \Phi, s)$ where f is a coefficient of the form constructed in (5.3). If Φ is factorizable, $Z(f, \Phi, s) = \prod_v Z(f_v, \Phi_v, s)$. At the non-archimedean places, the finite sums of integrals $Z(f_v, \Phi_v, s)$ of this type form an ideal contained in the ideal generated by such finite sums for arbitrary co-

efficients. The generator of the full ideal is $L(s, \pi_v)$, and therefore the generator of the subideal is $Q_v(X) \cdot L(s, \pi_v)$, where $Q(X)$ is a polynomial in $X = q_v^{-s}$ with $Q(0) = 1$. In fact $Q(X) | P_0(X)$. At the archimedean places the same phenomenon occurs. The generator of the full ideal is $E(s)P_0(s)$, and that of the sub-ideal is $E(s)P_0(s)Q_v(s)$, where Q is a polynomial with $Q(0) = 1$.

It follows that globally we can choose a finite number of coefficients φ_i and $\bar{\varphi}_i$ in $S(M)$ so that each $Z(\varphi_i, \bar{\varphi}_i, s)$ is of the form (5.3) and $\sum Z(\varphi_i, \bar{\varphi}_i, s) = \prod_{v \in S} L(s, \pi_v) \cdot \prod_{v \in S} Q_v(s)$, S being a finite set of places.

Theorem 5.5. Let π be an irreducible admissible representation of $H(G)$ on \mathcal{A} . Let $L(s, \pi) = \prod L(s, \pi_v)$. Then

- 1) The product is absolutely convergent for real part of s sufficiently large.
- 2) $L(s, \pi)$ has an analytic continuation to the whole complex plane and satisfies $L(s, \pi) = \epsilon(s, \pi) L(1-s, \tilde{\pi})$.

Proof: We have $\sum Z(\varphi_i, \bar{\varphi}_i, s) = \prod_{v \in S} L(s, \pi_v) \cdot \prod_{v \in S} Q_v$. The left side is

absolutely convergent for real part of s large enough, giving the convergence of the product.

Again replacing $\bar{\varphi}_i$ by ψ_i in $S_0(M)$ we have

$$\sum Z(\varphi_i, \psi_i, s) = L(s, \pi) \cdot L^{-1}(s, \pi_1) \cdot L^{-1}(1-s, \tilde{\pi}_2) \cdot \epsilon^{-1}(s, \pi_2) \prod_{\substack{v \in S \\ v \neq v_1, v_2}} Q_v.$$

The left side is entire, so

$$L(s, \pi) = \Xi(s) \cdot L(s, \pi_1) \cdot L(1-s, \tilde{\pi}_2) \cdot \prod_{\substack{v \in S \\ v \neq v_1, v_2}} Q_v^{-1}$$

which gives a continuation of $L(s, \pi)$ to the whole complex plane.

The functional equation follows as before.

APPENDIX

Let V be a real vector space of finite dimension n with lattices L and L_1 , where L_1 is a sublattice of L of index m . Let $\langle, \rangle: V \times V \rightarrow \mathbb{R}$ be a nondegenerate bilinear form, and let L^* and L_1^* be the lattices dual to L and L_1 . If Φ is a Schwartz function on V then the Fourier transform of Φ is $\hat{\Phi}(y) = \int_V \Phi(x) \psi\langle x, y \rangle dx$ with

ψ a fixed nontrivial additive character of \mathbb{R} and dx the Haar measure on V normalized so that $\hat{\hat{\Phi}}(g) = \Phi(-y)$. If Ψ is a function on L/L_1 then the Fourier transform of Ψ is a function on L_1^*/L^* defined by

$$\hat{\Psi}(y) = m^{-\frac{n}{2}} \sum_{x \in L/L_1} \Psi(x) \psi\left(\frac{\langle x, y \rangle}{m}\right). \text{ Extend } \Psi \text{ to all of } L \text{ by setting}$$

$\Psi(y)$ equal to the value of Ψ on the class of $y \bmod L_1$. Then we have the following version of the Poisson summation formula.

Lemma A.1. $\sum_{x \in L} \Phi(x) \Psi(x) = m^{-n/2} \sum_{x \in L^*} \hat{\Phi}(x/m) \hat{\Psi}(-x)$

Proof: $\sum_{x \in L} \Phi(x) \Psi(x) = \sum_{x \in L/L_1} \Psi(x) \sum_{y \in L} \Phi(x+my)$. Since

$$\begin{aligned} \int_V \Phi(x+my) \psi\langle y, z \rangle dy &= m^{-n} \psi\left(\frac{\langle x, -z \rangle}{m}\right) \int_V \Phi(y) \psi\left(\frac{\langle y, z \rangle}{m}\right) dy \\ &= m^{-n} \psi\left(\frac{\langle x, -z \rangle}{m}\right) \hat{\Phi}\left(\frac{z}{m}\right) \end{aligned}$$

we have

$$\begin{aligned} \sum_{x \in L} \Phi(x) \Psi(x) &= m^{-n} \sum_{z \in L^*} \hat{\Phi}(z/m) \sum_{x \in L/L_1} \Psi(x) \psi\left(\frac{\langle x, -z \rangle}{m}\right) \\ &= m^{-n/2} \sum_{z \in L^*} \hat{\Phi}(z/m) \hat{\Psi}(-z). \end{aligned}$$

We will use this lemma to prove the analytic continuation and functional equation for certain Dirichlet series attached to automorphic forms.

Let $G = \text{Gl}(n, \mathbb{E})$, $\Gamma = \text{Gl}(n, \mathbb{Z})$, $Z = \text{center of } G$, and $K = \text{O}(n, \mathbb{E})$. For the purposes of this appendix let an automorphic form for Γ be a smooth function on $\Gamma \backslash G$ with the following properties:

- 1) f is right K -finite and transforms according to a quasi character of \mathbb{E}^\times under translations by Z .
- 2) f satisfies some differential equation (which we do not need explicitly).
- 3) there are positive constants m, c such that $|f(g)| \leq c(\|g\|^*)^m$, the norm $\|\cdot\|^*$ being defined as in §5.

Let χ be a character of the multiplicative group of $\mathbb{Z}/m\mathbb{Z}$ which is primitive mod m and extend χ to \mathbb{Z} by setting $\chi(n) = 0$ if $(n, m) \neq 1$. If x is in $M(n, \mathbb{Z})$, write $\chi(x)$ for $\chi(\det x)$. Then χ may be regarded as a function on $M(n, \mathbb{Z}/m\mathbb{Z})$. In the definition of the Fourier transform take $\langle x, y \rangle$ to be $\text{tr}(xy^t)$. We use the following notations:

Σ is the sum over all x in $M(n, \mathbb{Z})$

Σ' is the sum restricted to those x in $M(n, \mathbb{Z})$ with non-zero determinant.

Lemma A.2. If ϕ is in $S(M)$ and χ is as above, then

$$\Sigma \phi(h^{-1}xg)\chi(x) = m^{-n^2/2} \Sigma \hat{\phi} \left[h^t \begin{pmatrix} m^{-1} & & 0 \\ & \cdot & \\ 0 & & m^{-1} \end{pmatrix} xg^t \right] \hat{\chi}(-x) |\det h|^n \cdot |\det g|^{-n}.$$

Proof: The lattice $M(n, \mathbb{Z})$ is self dual for the bilinear form $\text{tr}(xy^t)$, so this is a direct consequence of Lemma A.1 with $L/L_1 = M(n, \mathbb{Z}/m\mathbb{Z})$.

Lemma A.3. Let χ be a character of $\mathbb{Z}/m\mathbb{Z}^\times$ and $g'(\chi) =$

$$m^{1/2} \Sigma_{a \in \mathbb{Z}/m\mathbb{Z}} \chi(a) \psi(a/m), \text{ the corresponding Gaussian sum. Extend } \chi \text{ as}$$

above and view it as a function on $M(n, \mathbf{Z}/m\mathbf{Z})$. Then for $y \in M(n, \mathbf{Z}/m\mathbf{Z})$ we have

$$\hat{\chi}(y) = \begin{cases} \bar{\chi}(y) g^n(\chi) & \text{if } (\det y, m) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: First we show that $\hat{\chi}(e) = g^n(\chi)$. If $x \in M(n, \mathbf{Z}/m\mathbf{Z})$ is invertible, then x can be written uniquely in the form

$$x = \begin{bmatrix} a_{11} & & & a_{1n} \\ & \cdot & & \cdot \\ & & \cdot & \cdot \\ 0 & & & a_{nn} \end{bmatrix} \cdot \gamma$$

with $a_{ij} \in \mathbf{Z}/m\mathbf{Z}$ and $\gamma = (\gamma_{ij})$ in $P \cap \Gamma \backslash \Gamma$ where $\Gamma = \text{Sl}(n, \mathbf{Z}/m\mathbf{Z})$ and P is the subgroup of $\text{Gl}(n, \mathbf{Z}/m\mathbf{Z})$ consisting of upper triangular matrices. The decomposition is unique because x is invertible and $x = a \cdot \gamma = a_1 \cdot \gamma_1$ implies $a_1^{-1} \cdot a = \gamma_1 \cdot \gamma^{-1}$, so that $a_1^{-1} \cdot a$ belongs to both P and $P \cap \Gamma \backslash \Gamma$ and is consequently the identity.

Now

$$\hat{\chi}(e) = m^{-n^2/2} \sum_{x \in M(n, \mathbf{Z}/m\mathbf{Z})} \chi(x) \psi((\text{tr} x)/m).$$

Since $\chi(x) = 0$ when x is not invertible, we may assume that x has the above decomposition, so

$$\hat{\chi}(e) = m^{-n^2/2} \sum_{\substack{a_{ij} \in \mathbf{Z}/m\mathbf{Z} \\ \gamma \in P \cap \Gamma \backslash \Gamma}} \chi(a_{11} \dots a_{nn}) \psi \left[m^{-1} \text{tr} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \cdot & \cdot \\ & & \cdot & \cdot \\ 0 & & & a_{nn} \end{pmatrix} \begin{pmatrix} \gamma_{11} & \dots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \dots & \gamma_{nn} \end{pmatrix} \right]$$

$$= m^{-n^2/2} \sum_{\substack{a_{ii} \in \mathbf{Z}/m\mathbf{Z} \\ (\gamma_{ij}) \in P \cap \Gamma \backslash \Gamma}} \chi(a_{11} \dots a_{nn}) \psi \left[m^{-1} (a_{11} \gamma_{11} + \dots + a_{nn} \gamma_{nn}) \right].$$

$$\left\{ \prod_{\substack{1 \leq i < n \\ i < j}} a_{ij} \sum_{a_{ij} \in \mathbf{Z}/m\mathbf{Z}} \psi(m^{-1} a_{ij} \gamma_{ji}) \right\}$$

We need only sum for those γ such that $\gamma_{ji} = 0$ for all $i < j$, since otherwise at least one of the sums at the right is zero. When $\gamma_{ji} = 0$ for all $i < j$, each of the sums on the right has the value m , and the summation over $P \cap \Gamma \setminus \Gamma$ reduces to evaluation at the identity matrix. Therefore, $\hat{\chi}(e) = m^{-n/2} \sum_{a_{ii} \in \mathbb{Z}/m\mathbb{Z}} \chi(a_{11} \dots a_{nn}) \psi(m^{-1}(a_{11} + \dots + a_{nn})) = g^n(\chi)$. Now to prove the lemma, if y is invertible in $M(n, \mathbb{Z}/m\mathbb{Z})$ we have $\hat{\chi}(y) = m^{-n^2/2} \sum \chi(xy^z) \psi\left(\frac{\text{tr}x}{m}\right) = \chi^{-1}(y) g^n(\chi)$. Every y in $M(n, \mathbb{Z}/m\mathbb{Z})$, invertible or not, can be written

$$y = \gamma_1 \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \gamma_2$$

with γ_1 and γ_2 in $Sl(n, \mathbb{Z}/m\mathbb{Z})$, and then $\hat{\chi}(y) = \hat{\chi} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$.

Suppose that y is not invertible. If $\det y = 0$ we may assume $a_1 = 0$. Since χ is nontrivial there is an invertible r in $\mathbb{Z}/m\mathbb{Z}$ so that $\chi(r) \neq 1$. Then

$$\hat{\chi} \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \hat{\chi} \left[\begin{pmatrix} r & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right] = \bar{\chi}(r) \hat{\chi} \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

so that $\hat{\chi}(y) = 0$. If y is not invertible but $\det y$ is not zero, we may assume $a_i | a_{i+1}$. Then $(a_n, m) = m' \neq 1$. Set $m = m'd$ and $a_n = m'c$. Since χ is primitive mod m , there is a u in $\mathbb{Z}/m\mathbb{Z}$ such that $1+ud$ is invertible mod m and $\chi(1+ud) \neq 1$. Then

$$(1+ud)a_n \equiv a_n \pmod{m}$$

and

$$\hat{\chi} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \hat{\chi} \left[\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1+ud \end{pmatrix} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right] = \chi^{-1}(1+ud) \hat{\chi} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

so that $\hat{\chi}(y) = 0$.

Proposition A.4. Let f_1 and f_2 be in $L^2(\Gamma \backslash G, \omega)$ and $L^2(\Gamma \backslash G, \bar{\omega})$ respectively. Fix $\bar{\Phi}$ in $S(M)$, and let χ be a character as above.

Then

$$I(f_1, f_2, \bar{\Phi}, \chi, s) = m^{\frac{ns}{2}} \int_{Z \backslash G} dh \int_{\Gamma \backslash G} f_1(g) f_2(h) \Sigma' \bar{\Phi}(h^{-1} x g) \chi(x) |\det g|^s |\det h|^{-s} dg$$

defines a holomorphic function of s if the real part of s is sufficiently large. If $\chi^n(-1) \neq 1$, $I(f_1, f_2, \bar{\Phi}, \chi, s) = 0$.

Proof: Assume that $\bar{\Phi}$ is positive, and let σ be the real part of s .

$$|I(f_1, f_2, \bar{\Phi}, \chi, s)| \leq \|f_1\|_2^{\frac{1}{2}} \cdot \|f_2\|_2^{\frac{1}{2}} \cdot m^{\frac{n\sigma}{2}} \cdot \text{Vol}(\Gamma Z \backslash G) \cdot \|F(\dots, \sigma)\|_{L^2 \chi L^2}.$$

The right hand side is finite for σ large enough by Lemma 3.2, and convergence is uniform for s in a compact. For the second assertion, send x to $\begin{pmatrix} -1 & & \\ & \dots & \\ & & -1 \end{pmatrix} x$ and then send g to $\begin{pmatrix} -1 & & \\ & \dots & \\ & & -1 \end{pmatrix} g$. Since f_1 is invariant for $\begin{pmatrix} -1 & & \\ & \dots & \\ & & -1 \end{pmatrix}$, $I(f_1, f_2, \bar{\Phi}, \chi, s) = \chi^n(-1) I(f_1, f_2, \bar{\Phi}, \chi, s)$.

Theorem A.5. If χ is non-trivial, $I(f_1, f_2, \bar{\Phi}, \chi, s)$ has an analytic continuation to an entire function of s and satisfies the functional equation

$$I(f_1, f_2, \bar{\Phi}, \chi, s) = g^n(\chi) I(f_1^z, f_2^z, \bar{\Phi}, \chi, n-s).$$

Proof: Write I as the sum of an integral over $|\det g| \geq 1$ and an integral over $|\det g| \leq 1$. From the proof of Lemma 3.2, it follows that the first defines an entire function of s and the second converges absolutely if the real part of s is large enough. By application of the Poisson summation formula of Lemma A.3, the second equals (formally)

$$(A.5) \quad m^{\frac{n}{2}(s-n)} \int_{\Gamma Z \backslash G} dh \int_{\substack{\Gamma \backslash G \\ |\det g| \leq 1}} f_1(g) f_2(h) \Sigma' \hat{\Phi} \left[h^t \begin{pmatrix} m^{-1} & & \\ & \dots & \\ & & 0 \\ & & & m^{-1} \end{pmatrix} x g^z \right] \chi(-x) |\det g|^{s-n} |\det h|^{n-s} dh.$$

The terms for rank $x < n$ drop out because $\chi(x) = \hat{\chi}(x) = 0$ if $\det x = 0$.

Changing g into g^z shows that the integral (A.5) defines an entire function of s , which justifies the formal use of the summation formula. Substituting for $\hat{\chi}$ and making a change of variable we have

$$I(f_1, f_2, \hat{\Phi}, \chi, s) =$$

$$\frac{ns}{m^2} \int_{|\det g| \geq 1} dh \int f_1(g) f_2(h) \Sigma' \hat{\Phi}(h^{-1}xg) \chi(x) |\det g|^s |\det h|^{-s} dg$$

$$+ g^n(\chi) \chi^n(-1) m^{\frac{n}{2}(n-s)} \int_{|\det g| \geq 1} dh \int f_1(g^z) f_2(h^z) \Sigma' \hat{\Phi}(h^{-1}xg) \bar{\chi}(x) |\det g|^{n-s} |\det h|^{s-n} dg.$$

The functional equation follows from the fact that $g(\chi)g(\bar{\chi}) = \chi(-1)$.

An automorphic form is called a cusp form if $\int_{U \cap \Gamma \backslash U} f(ug) du = 0$ every

time U is the unipotent radical of a proper parabolic subgroup of G .

Corollary A.5.1. If χ is trivial and f_1 is a cusp form then the theorem is still true.

Proof: The only thing to show is that the terms for rank $x < n$ drop out when the Poisson summation formula is applied. These terms are

$$\int_{|\det g| \leq 1} dh \int f_1(g) f_2(h) \left\{ \sum_{\text{rk} x < n} \hat{\Phi} \left[h^t \begin{pmatrix} m^{-1} & & 0 \\ & \ddots & \\ 0 & & m^{-1} \end{pmatrix} x g^z \right] |\det g|^{-n} |\det h|^n \right.$$

$$\left. - \sum_{\text{rk} x < n} \hat{\Phi}(h^{-1}xg) \right\} |\det g|^s |\det h|^{-s} dg.$$

First we look at the terms for $\text{rk} x = 0$. We have a term with $\hat{\Phi}$ which is

$$-\hat{\Phi}(0) \cdot \int_{\Gamma\mathbb{Z}\backslash G} f_2(h) dh \int_{\Gamma\mathbb{Z}\backslash G} f_1(g) dg \int_0^1 |t|^s d^X t .$$

Since f_1 is a cusp form it is orthogonal to the constant functions, and so this term is zero. The term with $\hat{\Phi}$ is similarly found to be zero. Note that these terms are absolutely convergent for real part of s large enough, so it is legitimate to consider them separately.

If $\text{rk} x = n-j$, with $0 < \text{rk} x < n$, we can write $x = \begin{pmatrix} n-j & j \\ * & 0 \end{pmatrix} \cdot \gamma$

where $\gamma \in P_j \cap \Gamma \backslash \Gamma$, P_j being the parabolic subgroup of G with elements of the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}_j^{n-j}$. Call the set of $\begin{pmatrix} n-j & j \\ * & 0 \end{pmatrix}$ with entries in \mathbb{Z} and rank $n-j$ by X_j . Then we have:

$$\begin{aligned} & \int_{\Gamma\mathbb{Z}\backslash G} dh \int_{\Gamma\backslash G} f_1(g) f_2(h) \sum_{0 < \text{rk} x < n} \int_{\hat{\Phi}} [h^t x g^z] |\det g|^{s-n} d^X g \\ &= \sum_{j=1}^{n-1} \iint f_1(g) f_2(h) \sum_{x \in X_j} \sum_{\gamma \in P_j \cap \Gamma \backslash \Gamma} \int_{\hat{\Phi}} [h^t x \gamma g^z] |\det g|^{s-n} d^X g \\ &= \sum_{j=1}^{n-1} \int_{\Gamma\mathbb{Z}\backslash G} d^X h \int_{P_j^t \cap \Gamma \backslash G} f_1(g) f_2(h) \sum_{x \in X_j} \int_{\hat{\Phi}} [h^t x g^z] |\det g|^{s-n} d^X g . \end{aligned}$$

Now we may write $G = U_j M_j K$ where U_j is the unipotent radical of P_j^t and K is the standard maximal compact subgroup of G . The Haar measure has a corresponding decomposition, and we may calculate the integral over g in the form

$$\int_{U_j \cap \Gamma \backslash U_j} \int_{M_j \cap \Gamma \backslash M_j} \int_K f_1(umk) \sum_{x \in X_j} \int_{\hat{\Phi}} [h^t x u^z m^z k^z] |\det m|^{s-n} du d^X m dk .$$

Now $xu^z = x$ so the integral over u is $\int_{U_j \cap \Gamma \backslash U_j} f_1(umk) du$, and this is

zero since f_1 is a cusp form. The integral for $\bar{\phi}$ is done similarly. This shows that the integrals for $\text{rk} x < n$ are zero as desired, and gives immediately the functional equation $I(f_1, f_2, \bar{\phi}, s) = I(f_1^z, f_2^z, \hat{\bar{\phi}}, n-s)$.

Lemma A.6. Let $\bar{\phi}$ belong to $S(M)$ and let f_1 and f_2 be automorphic forms in $L^2(\Gamma \backslash G, \omega)$ and $L^2(\Gamma \backslash G, \bar{\omega})$ respectively. Then if $\varphi(g) = \int_{\Gamma \backslash G} f_1(hg) f_2(h) dh$ we have that $Z(\varphi, \bar{\phi}, s) = \int_G \varphi(g) \bar{\phi}(g) |\det g|^s dg$ converges

absolutely for real part of s sufficiently large.

Proof: The proof is the same as for Proposition 3.6.

Now let $S_k = \{x \in M(n, \mathbb{Z}) : |\det x| = k\}$, and define the Hecke operators T_k , which act on the space \mathcal{G} of automorphic forms, by $(T_k f)(h) = \sum_{x \in S_k / \Gamma} f(x^{-1}h)$. The T_k also act on $L^2(\Gamma \backslash G, \omega)$ by the same formula. These operators commute with each other. Now $V = \mathcal{G} \cap L^2(\Gamma \backslash G, \omega)$ is the direct sum of finite dimensional subspaces each of which is invariant for all the T_k . The adjoint operators T_k^* are given by $(T_k^* f)(h) = \sum_{x \in S_k / \Gamma} f(x^t h)$. The T_k^* commute with each other and with T_i

for every i , so that in particular each T_k is normal. Therefore there is a basis for $\mathcal{G} \cap L^2(\Gamma \backslash G, \omega)$ consisting of eigenfunctions for every T_k , and if f is a vector in V which is an eigenfunction for each T_k with eigenvalue a_k , then f is also an eigenfunction for every T_k^* with eigenvalue \bar{a}_k .

Theorem A.7. Let f_1 be an automorphic form in $L^2(\Gamma \backslash G, \omega)$ which is an

eigenfunction for each T_k with eigenvalue a_k . If χ has conductor $m > 1$,

the series $A(\chi, s) = m^{\frac{ns}{2}} \sum_{k \geq 1} \frac{a_k \chi(k)}{k^s}$ converges absolutely for the real

part of s sufficiently large, has an analytic continuation to an entire function of s , and satisfies the functional equation $A(\chi, s) Z(\varphi, \bar{\Phi}, s) = g^n(\chi) A^*(\bar{\chi}, n-s) Z(\varphi^z, \bar{\Phi}, n-s)$ with $A^*(\chi, s) = m^{ns/2} \sum_{k \geq 1} \bar{a}_k \chi(k) k^{-s}$ and with φ and $\bar{\Phi}$ as in Lemma A.6.

Proof: Let f_2 be an arbitrary automorphic form in $L^2(\Gamma \backslash G, \omega)$ and let

$$\begin{aligned} \varphi(g) &= \int_{\Gamma \backslash \mathbb{Z} \backslash G} f_1(hg) f_2(h) dh. \text{ We have } I(f_1, f_2, \bar{\Phi}, \chi, s) \\ &= m^{\frac{ns}{2}} \int \int f_1(g) f_2(h) \sum_{k \geq 1} \sum_{x \in S_k / \Gamma} \sum_{\gamma \in \Gamma} \chi(x) \bar{\Phi}(h^{-1} x \gamma g) |\det g|^s |\det h|^{-s} dg dh \\ &= m^{\frac{ns}{2}} \sum_{k \geq 1} \frac{\chi(k)}{k^s} \int \int \sum_{x \in S_k / \Gamma^1} f_1(x^{-1} g) f_2(h) \bar{\Phi}(h^{-1} g) |\det g|^s |\det h|^{-s} dg dh \\ &= A(\chi, s) Z(\varphi, \bar{\Phi}, s). \end{aligned}$$

We may choose f_2 and $\bar{\Phi}$ so that $Z(\varphi, \bar{\Phi}, s)$ is a nonzero constant.

Then the first two assertions follow from Proposition A.4 and Theorem A.5.

Now by Theorem A.5 $I(f_1, f_2, \bar{\Phi}, \chi, s) = g^n(\chi) I(f_1^z, f_2^z, \bar{\Phi}, \bar{\chi}, n-s)$,

and by a computation essentially the same as that above we have

$$I(f_1^z, f_2^z, \bar{\Phi}, \bar{\chi}, n-s) = A^*(\bar{\chi}, n-s) Z(\varphi^z, \bar{\Phi}, n-s).$$

The functional equation follows immediately.

Corollary A.7.1. The theorem is still true if χ is trivial and f_1 is a cusp form.

Proof: Immediate from Corollary A.5.1.

Corollary A.7.2. When $n = 2$, in the notations of the theorem

$$A(\chi, s) = A(\bar{\chi}, 2-s) .$$

Proof: If $n = 2$ there is a common set of representatives for

$\{x^t | x \in S_k/\Gamma\}$ and $\{x^{-1} | x \in S_k/\Gamma\}$, so that T_k is self-adjoint, and

$$a_k = \overline{a_k} .$$

Example: If π is the trivial representation of G then the coefficient

φ is a constant function and the eigenvalue a_k is the order of S_k/Γ .

If $(n, m) = 1$, the order of S_{nm}/Γ is the product of the orders of

S_n/Γ and S_m/Γ . (Follows from Proposition 3.16 [Shimura].) Therefore,

$$\sum a_k \chi(k) k^{-s} = \prod_p \left(\sum_{i=0}^{\infty} a_{p^i} \chi(p^i) p^{-is} \right)^{-1}$$

where p runs over all primes. For each p the sum on the right is

actually a rational expression in p^{-s} . Explicitly,

$$\sum_{i=0}^{\infty} a_{p^i} \chi(p^i) p^{-is} = (1 - \chi(p) p^{-s}) (1 - \chi(p) p^{1-s}) \dots (1 - \chi(p) p^{n-1-s}) .$$

(Follows from Theorem 3.21 [Shimura].) Therefore,

$$\sum a_k \chi(k) k^{-s} = \prod_{i=0}^{n-1} \prod_p (1 - \chi(p) p^{i-s})^{-1} ,$$

so that the series is actually the product of ordinary L -functions.

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